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Analysis of Navier–Stokes Models for Flows in Bidisperse Porous Media

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In memory of Professor Gabriela Kohr, with deep respect.

Abstract. Having in view a model proposed by Nield and Kuznetsov (Transp Porous Media 59:325–339, 2005; Transp Porous Media 96:495–499, 2013), we consider a more general system of coupled Navier–Stokes type equations in the incompressible case subject to the homogeneous Dirichlet condition in a bounded domain. We provide a deep theoretical analysis for large classes of equations and coupled systems of Navier–Stokes type with various non-homogeneous terms of reaction type. Existence results are obtained by using a variational approach making use of several fixed point principles and matrix theory.

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1. Introduction

The Stokes and Navier–Stokes systems play a main role in various areas of fluid mechanics, engineering, biology, chemistry, and there is a huge list of references concerning the mathematical analysis of related boundary value problems and of their applications. Among of them, we mention the books [4,5,9,13,24,25,29,41,42,45,46].

Let $N \geq 2$ and Ω be an open set in $\mathbb{R}^N$. Let $\eta, \kappa \geq 0$ and $\mu > 0$ be given constants. Let $\mathbf{u}$ and p be unknown vector and scalar fields. Let us assume that $\mathbf{f}$ is a given vector field defined on Ω. Then the equations

$$-\mu\Delta\mathbf{u} + \eta\mathbf{u} + \kappa\left(\mathbf{u}\cdot\nabla\right)\mathbf{u} + \nabla p = \mathbf{f},\ \operatorname{div}\,\mathbf{u} = 0\ \text{ in }\ \Omega \tag{1.1}$$

determine a *Navier–Stokes type system in the incompressible framework*. If $\eta = 0$ and $\kappa > 0$ we then obtain the well-known *Navier–Stokes system* in the incompressible case, while for $\eta = \kappa = 0$, (1.1) becomes the *Stokes system*, which is an Agmon-Douglis-Nirenberg elliptic and linear system (see, e.g., [13,16,29,45] for further details).

Extensions to a more general case of *anisotropic Stokes and Navier–Stokes systems* with L^∞-variable coefficients, and the analysis of various boundary value problems involving them can be consulted in [18–21] and the references therein.

Fabes et al. [12] used a layer potential approach in the analysis of the Dirichlet problem for the Stokes system on Lipschitz domains in the Euclidean setting (see also [10] for applications of the layer potential approach for strongly elliptic differential operators). Dindoš and Mitrea [11] proved the well-posedness in Sobolev and Besov spaces for the Dirichlet problem for the Stokes and Navier–Stokes systems with smooth coefficients in Lipschitz domains on compact Riemannian manifolds. Mitrea and Wright [27] obtained well-posedness results in Sobolev and Besov spaces for Dirichlet problems for the Stokes system

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with constant coefficients in Lipschitz domains in $\mathbb{R}^n$ for Dirichlet problems for the Stokes, Oseen and Navier–Stokes systems with constant coefficients in a non-solenoidal framework (see also the references therein).

Korobkov et al. [22] studied the flux problem in the theory of steady Navier–Stokes equations with constant coefficients and non-homogeneous boundary conditions. Amrouche and Rodríguez-Bellido [1] have proved the existence of a very weak solution for the non-homogeneous Dirichlet problem for the compressible Navier–Stokes system in a bounded domain of the class $C^{1,1}$ in $\mathbb{R}^3$.

Bulíček et al. [6] studied a boundary value problem with homogeneous Dirichlet condition associated with a system of nonlinear partial differential equations that generalize the classical fluid flow models of Stokes, Darcy, Forchheimer and Brinkman, by assuming that the viscosity and the drag coefficient depend on the shear rate and the pressure. The authors proved the existence of weak solutions to the problem under a minimal number of conditions, and analyzed relevant examples of viscosities and drag coefficients modeling real physical situations.

The authors in [18] analyzed in L^2-based Sobolev spaces, the non-homogeneous boundary value problems of Dirichlet-transmission type for the anisotropic Stokes and Navier–Stokes systems in a compressible framework in a bounded Lipschitz domain with a transversal Lipschitz interface in $\mathbb{R}^n$, $n \geq 2$ ($n = 2, 3$ for the nonlinear problems). They proved the existence of a weak solution to the Dirichlet problem and the Dirichlet-transmission problem for the nonlinear anisotropic Navier–Stokes system by using the Leray–Schauder fixed point theorem and various results and estimates from the linear case, as well as the Leray-Hopf inequality and some other norm inequalities. Explicit conditions for uniqueness of solutions to the nonlinear problems have been also provided. Mixed problems and mixed-transmission problems for the anisotropic Stokes and Navier–Stokes systems in bounded Lipschitz domains with transversal Lipschitz interfaces have been considered in [19] and analyzed from the variational point of view. The authors in [17] used a layer potential approach and the Leray–Schauder fixed point theorem and proved existence results for a nonlinear Neumann-transmission problem for the Stokes and Brinkman systems in L^p, Sobolev, and Besov spaces. Mazzucato and Nistor [26] obtained well-posedness and regularity results in Sobolev spaces for the linear elasticity equations in the anisotropic case with mixed boundary conditions on polyhedral domains.

1.1. Bidisperse Porous Media and Some Related Models

A bidisperse porous medium (BDPM) may be described as a standard porous medium in which the solid phase is replaced by another porous medium. Thus, a BDPM can be viewed as a medium composed of clusters of large particles that are agglomerations of small particles (cf. [8], see also [32,33]). The voids between the clusters are macro-pores and the voids within the clusters, which are much smaller in size, are micro-pores. We can then define the f-phase (the macro-pores) and the p-phase (the remainder of the structure). Bidisperse adsorbent or bidisperse capillary wicks in a heat pipe are practical applications related to bidisperse porous media. There are also various biological structures, such as bone regeneration scaffolds, that can be described in terms of bidisperse porous media (see also [33]).

Extending the Brinkman model for a monodisperse porous medium, Nield and Kuznetsov [30] considered a model to describe the steady-state momentum transfer in a BDPM by the following pair of coupled equations for the velocities $\mathbf{v}_f^*$ and $\mathbf{v}_p f^*$ in the macro and micro-pores,

$$\begin{cases} \mathbf{G} = \frac{\mu}{K_f}\mathbf{v}_f^* + \zeta(\mathbf{v}_f^* - \mathbf{v}_p^*) - \widetilde{\mu}_f \Delta^* \mathbf{v}_f^* \\ \mathbf{G} = \frac{\mu}{K_p}\mathbf{v}_p^* + \zeta(\mathbf{v}_p^* - \mathbf{v}_f^*) - \widetilde{\mu}_p \Delta^* \mathbf{v}_p^*, \end{cases}$$

where the asterisks denote dimensional variables, $\mathbf{G}$ is the negative of the applied pressure gradient, μ is the fluid viscosity, K_p and K_f are the permeabilities of the two phases, ζ is the coefficient for momentum transfer between the two phases, and $\widetilde{\mu}_f$ and $\widetilde{\mu}_p$ are the effective viscosities of the two phases (cf. also [32]). In the model proposed by Nield and Kuznetsov [30], the quadratic or Forchheimer terms $|\mathbf{v}_p^*|\mathbf{v}_p^*$ and $\mathbf{v}_f^*|\mathbf{v}_f^*$ have been neglected. This model and various extensions have been considered in many studies related to forced, natural and mixed convection. Among them, we mention [7,23,28,31,33,39,40,43] (see also the references therein). The thermal convection in an anisotropic bidisperse porous medium has been investigated in [44].

Nield and Kuznetsov [33] extended their linear model proposed in [30] by adding some semilinear terms, called the Forchheimer drag terms, as follows

$$\begin{cases} \mathbf{G} = \frac{\mu}{K_f}\mathbf{v}_f^* + \zeta(\mathbf{v}_f^* - \mathbf{v}_p^*) - \widetilde{\mu}_f \Delta^* \mathbf{v}_f^* + \frac{c_f \rho}{K_f^{1/2}} |\mathbf{v}_f^*| \mathbf{v}_f^* \\ \mathbf{G} = \frac{\mu}{K_p}\mathbf{v}_p^* + \zeta(\mathbf{v}_p^* - \mathbf{v}_f^*) - \widetilde{\mu}_p \Delta^* \mathbf{v}_p^* + \frac{c_p \rho}{K_p^{1/2}} |\mathbf{v}_p^*| \mathbf{v}_p^*, \end{cases}$$

where ρ is the density of the fluid, and c_f and c_p are the Forchheimer coefficients.

The Nield–Kuznetsov models described above are based on the same pressure gradient $-\mathbf{G}$ in both phases. Other models consider possible different pressures in the macro and micro phases. For instance, Straughan [44] having analyzed a model of thermal convection in an anisotropic bidispersive porous medium with permeability tensors in the macro and micro phases, considers different velocities $\mathbf{U}^f$ and $\mathbf{U}^p$ and different pressures p^f and p^p in the macro and micro-pores (see also [7] and the references therein for similar models of bidisperse porous media with different velocities and different pressures in macro and micro phases).

Having in view the model of Nield and Kuznetsov [33], where the steady-state momentum transfer is described by the previous semilinear system, and also the model of Straughan [44], we consider a more general nonlinear *coupled type Navier–Stokes system* arising in the analysis of fluid flows in bidisperse porous media. Thus, our paper is build around the following system

$$\begin{cases} -\mu_1 \Delta \mathbf{u}_1 + \eta_1 \mathbf{u}_1 + \kappa_1 (\mathbf{u}_1 \cdot \nabla) \mathbf{u}_1 + \nabla p_1 \\ \quad = \mathbf{h}_1 - \alpha_1 |\mathbf{u}_1|^{p-1} \mathbf{u}_1 - \gamma_1 (\mathbf{u}_1 - \mathbf{u}_2) \quad \text{in } \Omega \\ -\mu_2 \Delta \mathbf{u}_2 + \eta_2 \mathbf{u}_2 + \kappa_2 (\mathbf{u}_2 \cdot \nabla) \mathbf{u}_2 + \nabla p_2 \\ \quad = \mathbf{h}_2 - \alpha_2 |\mathbf{u}_2|^{p-1} \mathbf{u}_2 - \gamma_2 (\mathbf{u}_2 - \mathbf{u}_1) \quad \text{in } \Omega \\ \operatorname{div} \mathbf{u}_i = 0 \quad \text{in } \Omega, \quad i = 1, 2 \\ \mathbf{u}_i = 0 \quad \text{on } \partial\Omega, \quad i = 1, 2, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain ($N \leq 3$), $p \geq 1$, η_i, κ_i, $\alpha_i \geq 0$ and μ_i, $\gamma_i > 0$, $i = 1, 2$, are given constants whose meaning depends on the physical properties of fluid flow and porous medium, while $\mathbf{h}_i$, $i = 1, 2$, are given data in some Sobolev spaces.

In order to analyze this system, we provide a deep analysis of a homogeneous Dirichlet problem of more general coupled Navier–Stokes systems with various non-homogeneous terms of reaction type, and obtain existence results by using a variational approach combined with fixed point theorems, a technique already used for other classes of equations (see [34, Ch. 6], [35], [37, Chs.9-12], [38]).

The paper is structured as follows. First, we mention some well-known but useful results regarding the stationary Navies-Stokes equations in the incompressible case. The next section is devoted to the analysis of the homogeneous Dirichlet problem for the Navier–Stokes equations with reaction terms. We obtain existence results based on the Schauder fixed point theorem and the Leray–Schauder fixed point theorem. Uniqueness result can be also obtained by using the Banach contraction principle, by imposing additional conditions to the reaction terms. The third section is devoted to the analysis of a coupled system of Navier–Stokes equations. The last section is devoted to a coupled system that could describe a fluid flow in a bidisperse porous medium. We obtain related existence and uniqueness results that follow as consequences of the results obtained in the previous sections.

1.2. Stationary Navier–Stokes Type Equations

Let $N \leq 3$ and $\Omega \subset \mathbb{R}^N$ be an bounded domain.

Next we recall some well-known results about the system

$$\begin{cases} -\mu\Delta\mathbf{u}+\eta\mathbf{u}+\kappa\left(\mathbf{u}\cdot\nabla\right)\mathbf{u}+\nabla p=\mathbf{f} & \text{in } \Omega\\ \operatorname{div}\,\mathbf{u}=0 & \text{in } \Omega\\ \mathbf{u}=0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

where $\mu>0$, κ, $\eta\geq 0$ are given constants and $\mathbf{f}\in H^{-1}\left(\Omega\right)^N$ is a given distribution.

The variational form of system (1.2) with the unknown pair $(\mathbf{u},p)\in V\times L^2\left(\Omega\right)$ is

$$\mu\left(\mathbf{u},\mathbf{v}\right)_{H_0^1}+\eta\left(\mathbf{u},\mathbf{v}\right)_{L^2}+\kappa b\left(\mathbf{u},\mathbf{u},\mathbf{v}\right)-\left(p,\operatorname{div}\,\mathbf{v}\right)_{L^2}=\left(\mathbf{f},\mathbf{v}\right),\ \ \mathbf{v}\in H_0^1\left(\Omega\right)^N, \tag{1.3}$$

where

$$\left(\mathbf{u},\mathbf{v}\right)_{L^2}=\sum_{i=1}^N\int_\Omega u_iv_i,\qquad \left(\mathbf{u},\mathbf{v}\right)_{H_0^1}=\sum_{i=1}^N\int_\Omega\nabla u_i\cdot\nabla v_i$$

$$b\left(\mathbf{u},\mathbf{v},\mathbf{w}\right)=\sum_{i,j=1}^N\int_\Omega u_j\frac{\partial v_i}{\partial x_j}w_i,\qquad \left(\mathbf{f},\mathbf{v}\right)=\sum_{i=1}^N\left(f_i,v_i\right),$$

$$V=\left\{\mathbf{v}=\left(v_1,\ldots,v_N\right)\in H_0^1\left(\Omega\right)^N:\ \operatorname{div}\,\mathbf{v}=0\right\}.$$

For $\mathbf{u}\in V$, Eq. (1.3) gives

$$\mu\left(\mathbf{u},\mathbf{v}\right)_{H_0^1}+\eta\left(\mathbf{u},\mathbf{v}\right)_{L^2}+\kappa b\left(\mathbf{u},\mathbf{u},\mathbf{v}\right)=\left(\mathbf{f},\mathbf{v}\right),\quad \mathbf{v}\in V. \tag{1.4}$$

Once a solution $\mathbf{u}\in V$ to (1.4) is found, the pressure $p\in L^2\left(\Omega\right)$ is guaranteed by De Rham's Theorem (cf., e.g., [45, Proposition 1.1, Chapter 1], [15, Theorem 2.3, Chapter 1], see also [2], [3, Theorem 2.1]).

On $H_0^1\left(\Omega\right)^N$ consider the inner product and norm

$$\left(\mathbf{u},\mathbf{v}\right)_0:=\int_\Omega\left(\mu\nabla\mathbf{u}\cdot\nabla\mathbf{v}+\eta\mathbf{u}\cdot\mathbf{v}\right),\quad \left|\mathbf{u}\right|_0^2:=\int_\Omega\left(\mu\left|\nabla\mathbf{u}\right|^2+\eta\left|\mathbf{u}\right|^2\right),$$

which when applied to the subspace V will be denoted by $\left(\cdot,\cdot\right)_V$ and $\left|\cdot\right|_V$, respectively. Then the embedding constants and the corresponding inequalities for the inclusions $V\subset L^2\left(\Omega\right)^N\subset V'$ are

$$\left|\mathbf{u}\right|_{L^2}\leq\frac{1}{\sqrt{\mu\lambda_1+\eta}}\left|\mathbf{u}\right|_V\quad\left(\mathbf{u}\in V\right),\quad \left|\mathbf{h}\right|_{V'}\leq\frac{1}{\sqrt{\mu\lambda_1+\eta}}\left|\mathbf{h}\right|_{L^2}\quad\left(\mathbf{h}\in L^2\left(\Omega\right)^N\right), \tag{1.5}$$

where λ_1 is the first eigenvalue of $-\Delta$ with respect to the homogeneous Dirichlet problem. Indeed, knowing that

$$\left|\mathbf{u}\right|_{L^2}\leq\frac{1}{\sqrt{\lambda_1}}\left|\mathbf{u}\right|_{H_0^1}\quad\left(\mathbf{u}\in H_0^1\left(\Omega\right)^N\right),\quad \left|\mathbf{h}\right|_{H^{-1}}\leq\frac{1}{\sqrt{\lambda_1}}\left|\mathbf{h}\right|_{L^2}\quad\left(\mathbf{h}\in L^2\left(\Omega\right)^N\right),$$

we have

$$\left|\mathbf{u}\right|_V^2=\mu\left|\mathbf{u}\right|_{H_0^1}^2+\eta\left|\mathbf{u}\right|_{L^2}^2\geq\left(\mu\lambda_1+\eta\right)\left|\mathbf{u}\right|_{L^2}^2, \tag{1.6}$$

whence the first inequality in (1.5). Based on (1.6), the second inequality is obtained as follows:

$$\left|\mathbf{h}\right|_{V'}=\sup_{\mathbf{v}\in V}\frac{\left|\left(\mathbf{h},\mathbf{v}\right)\right|}{\left|\mathbf{v}\right|_V}\leq\left|\mathbf{h}\right|_{L^2}\sup_{\mathbf{v}\in V}\frac{\left|\mathbf{v}\right|_{L^2}}{\left|\mathbf{v}\right|_V}\leq\frac{1}{\sqrt{\mu\lambda_1+\eta}}\left|\mathbf{h}\right|_{L^2}.$$

Also $V\subset H_0^1\left(\Omega\right)^N\subset H^{-1}\left(\Omega\right)^N\subset V'$ and

$$\left|\mathbf{h}\right|_{V'}=\sup_{\mathbf{v}\in V}\frac{\left|\left(\mathbf{h},\mathbf{v}\right)\right|}{\left|\mathbf{v}\right|_V}\leq\sup_{\mathbf{v}\in H_0^1(\Omega)^N}\frac{\left|\left(\mathbf{h},\mathbf{v}\right)\right|}{\left|\mathbf{v}\right|_V}=\left|\mathbf{h}\right|_{H^{-1}}\quad\left(\mathbf{h}\in H^{-1}\left(\Omega\right)^N\right). \tag{1.7}$$

Recall that the trilinear functional $b:V\times V\times V\to\mathbb{R}$ satisfies

$$b\left(\mathbf{u},\mathbf{v},\mathbf{w}\right)+b\left(\mathbf{u},\mathbf{w},\mathbf{v}\right)=0\qquad\left(\mathbf{u},\mathbf{v},\mathbf{w}\in V\right),$$

$$\left|b\left(\mathbf{u},\mathbf{v},\mathbf{w}\right)\right|\leq M\left|\mathbf{u}\right|_V\left|\mathbf{v}\right|_V\left|\mathbf{w}\right|_V,$$

where $M > 0$ is a constant depending on μ and η. Also, using the Galerkin method, one can prove that for every $\mathbf{f} \in V'$, Eq. (1.4) has at least one solution $\mathbf{u} \in V$ (see, e.g., [13,45]).

Uniqueness: For every $\mathbf{f} \in V'$, with $|\mathbf{f}|_{V'} < 1/(\kappa M)$, Eq. (1.4) has at most one solution $\mathbf{u} \in V$. Indeed, if $\mathbf{u}_1, \mathbf{u}_2 \in V$ are solutions and we let $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$, then using (1.4) one has

$$\begin{aligned} 0 &= (\mathbf{u}, \mathbf{v})_V + \kappa b(\mathbf{u}_1, \mathbf{u}_1, \mathbf{v}) - \kappa b(\mathbf{u}_2, \mathbf{u}_2, \mathbf{v}) \\ &= (\mathbf{u}, \mathbf{v})_V + \kappa (b(\mathbf{u}_1, \mathbf{u}_1, \mathbf{v}) - b(\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}) + b(\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}) - b(\mathbf{u}_2, \mathbf{u}_2, \mathbf{v})) \\ &= (\mathbf{u}, \mathbf{v})_V + \kappa (b(\mathbf{u}_1, \mathbf{u}, \mathbf{v}) + b(\mathbf{u}, \mathbf{u}_2, \mathbf{v})), \end{aligned}$$

which for $\mathbf{v} = \mathbf{u}$, since $b(\mathbf{u}_1, \mathbf{u}, \mathbf{u}) = 0$, gives

$$|\mathbf{u}|_V^2 = -\kappa b(\mathbf{u}, \mathbf{u}_2, \mathbf{u}) \le \kappa M |\mathbf{u}|_V^2 |\mathbf{u}_2|_V .$$

On the other hand for $\mathbf{u}_2$, taking in (1.4) $\mathbf{v} = \mathbf{u} = \mathbf{u}_2$, one has

$$|\mathbf{u}_2|_V^2 = (\mathbf{f}, \mathbf{u}_2) \le |\mathbf{f}|_{V'} |\mathbf{u}_2|_V .$$

Then

$$|\mathbf{u}|_V^2 (1 - \kappa M |\mathbf{f}|_{V'}) \le 0,$$

which for $|\mathbf{f}|_{V'} < 1/(\kappa M)$ yields $|\mathbf{u}|_V = 0$, that is $\mathbf{u}_1 = \mathbf{u}_2$. Denote the unique solution by $\mathbf{u}_\mathbf{f}$.

Thus we may define the *solution operator*

$$S : D_0 \to V, \quad \mathbf{f} \longmapsto \mathbf{u}_\mathbf{f}.$$

Here $D_0 := \{\mathbf{f} \in V' : \ \kappa M |\mathbf{f}|_{V'} < 1\}$. In addition, since the trilinear functional b satisfies

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) + b(\mathbf{u}, \mathbf{w}, \mathbf{v}) = 0,$$

one has $b(\mathbf{u}, \mathbf{u}, \mathbf{u}) = 0$, whence taking in (1.4) $\mathbf{v} = \mathbf{u} = S(\mathbf{f})$ we see that

$$|S(\mathbf{f})|_V \le |\mathbf{f}|_{V'} . \tag{1.8}$$

Also using the linearity of b in each of its variables gives

$$\begin{aligned} & b(\mathbf{u}, \mathbf{u}, \mathbf{w}) - b(\mathbf{v}, \mathbf{v}, \mathbf{w}) \\ &\quad = b(\mathbf{u}, \mathbf{u}, \mathbf{w}) - b(\mathbf{v}, \mathbf{u}, \mathbf{w}) + b(\mathbf{v}, \mathbf{u}, \mathbf{w}) - b(\mathbf{v}, \mathbf{v}, \mathbf{w}) \\ &\quad = b(\mathbf{u} - \mathbf{v}, \mathbf{u}, \mathbf{w}) + b(\mathbf{v}, \mathbf{u} - \mathbf{v}, \mathbf{w}) . \end{aligned}$$

Then

$$\begin{aligned} |S(\mathbf{f}) - S(\mathbf{g})|_V^2 &= (\mathbf{f} - \mathbf{g}, S(\mathbf{f}) - S(\mathbf{g})) \\ &\quad + \kappa b(S(\mathbf{g}), S(\mathbf{g}), S(\mathbf{f}) - S(\mathbf{g})) \\ &\quad - \kappa b(S(\mathbf{f}), S(\mathbf{f}), S(\mathbf{f}) - S(\mathbf{g})) \\ &= (\mathbf{f} - \mathbf{g}, S(\mathbf{f}) - S(\mathbf{g})) \\ &\quad + \kappa b(S(\mathbf{g}) - S(\mathbf{f}), S(\mathbf{g}), S(\mathbf{f}) - S(\mathbf{g})) \\ &\quad + \kappa b(S(\mathbf{f}), S(\mathbf{g}) - S(\mathbf{f}), S(\mathbf{f}) - S(\mathbf{g})) . \end{aligned}$$

Hence

$$\begin{aligned} |S(\mathbf{f}) - S(\mathbf{g})|_V^2 &\le |\mathbf{f} - \mathbf{g}|_{V'} |S(\mathbf{f}) - S(\mathbf{g})|_V \\ &\quad + \kappa M |S(\mathbf{f}) - S(\mathbf{g})|_V^2 (|S(\mathbf{g})|_V + |S(\mathbf{f})|_V) \\ &\le |\mathbf{f} - \mathbf{g}|_{V'} |S(\mathbf{f}) - S(g)|_V \\ &\quad + \kappa M (|\mathbf{f}|_{V'} + |\mathbf{g}|_{V'}) |S(\mathbf{f}) - S(\mathbf{g})|_V^2 . \end{aligned}$$

As a result, if $|\mathbf{f}|_{V'}, |\mathbf{g}|_{V'} \le \rho$, then

$$(1 - 2\kappa M \rho) |S(\mathbf{f}) - S(\mathbf{g})|_V \le |\mathbf{f} - \mathbf{g}|_{V'} . \tag{1.9}$$

Therefore, if $1-2\kappa M\rho>0$, i.e., $\rho<\frac{1}{2\kappa M}$, then the *operator S is Lipschitz continuous on the ball of V' centered at the origin and of radiusρ*. Note that in the case $\kappa=0$, inequality (1.9) shows that the solution operator S is Lipschitz continuous on the entire space V'. Let us introduce a notation for the Lipschitz constant, namely

$$L(\rho):=(1-2\kappa M\rho)^{-1}.$$

Note that, in the case of the Stokes system, one has $\kappa=\eta=0$ and thus S is well-defined and Lipschitz continuous on the whole space V'.

1.3. Notions of Vector Analysis in Fixed Point Theory

In this paper we assume that the reader is familiar with Banach contraction principle and Schauder and Leray–Schauder fixed point theorems. However, we consider it useful to present some less known elements of vector analysis in fixed point theory. For more details we refer the reader to [36].

A square matrix $\mathcal{A}$ with nonnegative entries is said to be *convergent to zero* if its power $\mathcal{A}^k$ tends to the zero matrix as k tends to infinity. This property is equivalent to the fact that the spectral radius of $\mathcal{A}$ is less than one, and also to the property that $\mathcal{J}-\mathcal{A}$ is invertible and its inverse also has nonnegative entries (here $\mathcal{J}$ is the unit matrix).

A matrix $\mathcal{A}=[a_{ij}]_{i,j=1,2}$ of size two is convergent to zero if and only if

$$\text{tr } \mathcal{A}<\min\{2,\ 1+\det \mathcal{A}\}. \tag{1.10}$$

The property of being convergent to zero of a matrix $\mathcal{A}$ is useful to pass from a matrix inequality of the form $(\mathcal{J}-\mathcal{A})\,\mathbf{u}\le\mathbf{v}$, where $\mathbf{u},\mathbf{v}$ are column vectors and the inequality is understood on components, to the inequality $\mathbf{u}\le(\mathcal{J}-\mathcal{A})^{-1}\mathbf{v}$, without change of inequality. The notion is even more important since it extends to matrices the situation on real numbers $0\le a<1$, asked on the Lipschitz constant in Banach contraction theorem. More exactly we have the following result, a special case of the more general *Perov's fixed point theorem*:

Let $k\ge1$ be a given integer, $(X,|.|_X)$ be a Banach space, and $X^k:=\underbrace{X\times\cdots\times X}_{\text{k}-\text{times}}$. If $D\subset X^k$ is closed, and $T:D\to D$, $T=(T_1,\cdots,T_k)$ is a mapping satisfying the matrix condition

$$\begin{bmatrix}|T_1(\mathbf{u})-T_1(\mathbf{v})|_X\\ \vdots\\ |T_k(\mathbf{u})-T_k(\mathbf{v})|_X\end{bmatrix}\le\mathcal{A}\begin{bmatrix}|u_1-v_1|_X\\ \vdots\\ |u_k-v_k|_X\end{bmatrix}$$

for all $\mathbf{u}=(u_1,\cdots,u_k)$, $\mathbf{v}=(v_1,\cdots,v_k)\in D$ and some matrix $\mathcal{A}$, which is convergent to zero, then T has a unique fixed point $\mathbf{u}\in D$, i.e., $T_i(\mathbf{u})=u_i$ for $i=1,\ldots,k$.

2. Navier–Stokes Type Equations with Reaction Terms

Consider now the problem

$$\begin{cases}-\mu\Delta\mathbf{u}+\eta\mathbf{u}+\kappa(\mathbf{u}\cdot\nabla)\,\mathbf{u}+\nabla p=\mathbf{h}+F(\mathbf{u}) & \text{in } \Omega\\ \text{div }\mathbf{u}=0 \quad \text{in } \Omega\\ \mathbf{u}=0 \quad \text{on } \partial\Omega\end{cases} \tag{2.1}$$

where $\mathbf{h}\in H^{-1}(\Omega)^N$ and $F:H_0^1(\Omega)^N\to H^{-1}(\Omega)^N$. The problem can be reduced to the fixed point equation in V,

$$\mathbf{u}=S(\mathbf{h}+F(\mathbf{u})) \tag{2.2}$$

having in mind that the solution operator S is defined and Lipschitz continuous on the open ball of $H^{-1}(\Omega)^N$ centered at the origin and of radius $1/(2\kappa M)$.

Theorem 2.1. *Assume that* $F : H_0^1(\Omega)^N \to H^{-1}(\Omega)^N$, $F(0) = 0$ *and*

$$|F(\mathbf{u}) - F(\mathbf{v})|_{H^{-1}} \leq a\,|\mathbf{u} - \mathbf{v}|_0 \quad \left(|\mathbf{u}|_0, |\mathbf{v}|_0 < \frac{1}{2\kappa M}\right) \tag{2.3}$$

for some constant $a < 1$. *Then for each* $\mathbf{h} \in H^{-1}(\Omega)^N$ *with*

$$|\mathbf{h}|_{H^{-1}} < \frac{(1-a)^2}{2\kappa M}, \tag{2.4}$$

Equation (2.2) *has a unique solution* $\mathbf{u} \in V$ *such that*

$$|\mathbf{u}|_V < \frac{1-a}{2\kappa M}. \tag{2.5}$$

Proof. First, we consider the case $\kappa > 0$. Let ε be arbitrarily closed to $1/a$ with $1 < \varepsilon < 1/a$, such that

$$|\mathbf{h}|_{H^{-1}} \leq (1-a)\,\frac{\varepsilon - 1}{2\kappa M\varepsilon}.$$

Denote $R = R(\varepsilon) := \dfrac{\varepsilon - 1}{2\kappa M\varepsilon}$. Thus $|\mathbf{h}|_{H^{-1}} \leq (1-a)\,R$. Clearly $R < \frac{1-a}{2\kappa M} < \frac{1}{2\kappa M}$, hence inequality (2.3) holds for all $\mathbf{u}, \mathbf{v}$ in the closed ball B_R of $(V, |.|_V)$ centered at the origin and of radius R. In addition, for $\mathbf{u} \in B_R$, one has

$$\begin{aligned} |\mathbf{h} + F(\mathbf{u})|_{V'} &\leq |\mathbf{h} + F(\mathbf{u})|_{H^{-1}} \leq |\mathbf{h}|_{H^{-1}} + a\,|\mathbf{u}|_V \\ &\leq |\mathbf{h}|_{H^{-1}} + aR \leq (1-a)\,R + aR = R < \frac{1}{2\kappa M}. \end{aligned} \tag{2.6}$$

Consequently the operator

$$T(\mathbf{u}) := S(\mathbf{h} + F(\mathbf{u}))$$

is well defined in B_R and for $|\mathbf{u}|_V, |\mathbf{v}|_V \leq R$, using (1.9) one has

$$\begin{aligned} |T(\mathbf{u}) - T(\mathbf{v})|_V &\leq L(R)\,|F(\mathbf{u}) - F(\mathbf{v})|_{V'} \leq L(R)\,|F(\mathbf{u}) - F(\mathbf{v})|_{H^{-1}} \\ &\leq aL(R)\,|\mathbf{u} - \mathbf{u}|_V\,. \end{aligned}$$

Since $aL(R) < 1$ (which can be checked easily), we have that T is a contraction on B_R. In addition from (1.8) and (2.6) we have

$$|T(\mathbf{u})|_V \leq |\mathbf{h} + F(\mathbf{u})|_{H^{-1}} \leq R, \tag{2.7}$$

which proves that $T(B_R) \subset B_R$. Thus the Banach contraction principle applies and gives the existence and uniqueness of solution $\mathbf{u}$ in B_R, where $R = R(\varepsilon) = \dfrac{1 - \varepsilon^{-1}}{2\kappa M}$. Since $R(\varepsilon) \to \frac{1-a}{2\kappa M}$ as $\varepsilon \to \frac{1}{a}$, we obtain the existence and uniqueness of solution $\mathbf{u}$ satisfying (2.5). This closes the proof in the case $\kappa > 0$.

In the case $\kappa = 0$, the operator T is a contraction on the entire space V and hence the solution exists and is unique in V. □

For the next result, instead of the Lipschitz continuity, we only assume a linear growth of F.

Theorem 2.2. *Assume that* $F : H_0^1(\Omega)^N \to H^{-1}(\Omega)^N$ *is completely continuous, satisfies the growth condition*

$$|F(\mathbf{u})|_{H^{-1}} \leq a\,|\mathbf{u}|_V \quad \left(|\mathbf{u}|_0 < \frac{1}{2\kappa M}\right) \tag{2.8}$$

with some constant $a > 0$ *and* $\mathbf{h}$ *is as in* (2.4). *Then Eq.* (2.2) *has at least one solution* $\mathbf{u} \in V$ *satisfying* (2.5).

Proof. As above, T maps the ball B_R of $(V, |\cdot|_V)$ into itself. In addition, the operator T is completely continuous. The result follows from Schauder's fixed point theorem. □

A better result can be derived from the Leray–Schauder fixed point theorem, without the linear growth condition on F.

Theorem 2.3. *Assume that the operator* $F : H_0^1(\Omega)^N \to H^{-1}(\Omega)^N$ *is completely continuous and there exists* $c < 1$ *such that*

$$(F(\mathbf{u}), \mathbf{u}) \le c\,|\mathbf{u}|_0^2 \quad \left(\mathbf{u} \in H_0^1(\Omega)^N\right). \tag{2.9}$$

Then for each $R > 0$ *satisfying*

$$(1-c)\,R + \sigma < \frac{1}{2\kappa M},$$

where $\sigma = \sup_{\mathbf{u}\in B_R} |F(\mathbf{u})|_{H^{-1}}$ *and* $B_R = \{\mathbf{u} \in V : \ |\mathbf{u}|_V \le R\}$, *Eq.* (2.2) *has at least one solution* $\mathbf{u} \in B_R$ *for every* $\mathbf{h} \in H^{-1}(\Omega)^N$ *with*

$$|\mathbf{h}|_{H^{-1}} \le (1-c)\,R. \tag{2.10}$$

Moreover, any solution $\mathbf{u}$ *satisfies*

$$|\mathbf{u}|_V \le \frac{1}{1-c}\,|\mathbf{h}|_{H^{-1}}. \tag{2.11}$$

Proof. For $\mathbf{u} \in B_R$, one has

$$|\mathbf{h} + F(\mathbf{u})|_{H^{-1}} \le |\mathbf{h}|_{H^{-1}} + \sigma \le (1-c)\,R + \sigma < \frac{1}{2\kappa M}.$$

Hence T is well-defined and continuous on the closed ball B_R of V. Moreover, the Lipschitz continuity of the solution operator S implies that S is a bounded operator (it maps bounded sets into bounded sets) which together with the complete continuity of F implies that the operator T is also completely continuous on B_R. (Recall that $T(\mathbf{u}) := S(\mathbf{h} + F(\mathbf{u}))$.)

Next we show that the Leray–Schauder condition holds, namely

$$|\mathbf{u}|_V = R \ \Rightarrow\ T(\mathbf{u}) \ne \lambda\mathbf{u} \ \text{ for all } \lambda > 1.$$

Assume the contrary, i.e. $T(\mathbf{u}) = \lambda\mathbf{u}$ for some $\mathbf{u}$ with $|\mathbf{u}|_V = R$ and $\lambda > 1$. Then as in (1.4) we obtain the variational equation

$$\lambda\mu\,(\mathbf{u},\mathbf{v})_{H_0^1} + \lambda\eta\,(\mathbf{u},\mathbf{v})_{L^2} + \lambda^2\kappa b\,(\mathbf{u},\mathbf{u},\mathbf{v}) = (\mathbf{h} + F(\mathbf{u}), \mathbf{v}), \quad \mathbf{v} \in V.$$

Choosing $\mathbf{v} = \mathbf{u}$ in the above equation and using inequalities (1.7) and (2.9) one obtains

$$\begin{aligned} |\mathbf{u}|_V^2 < \lambda\,|\mathbf{u}|_V^2 &= (\mathbf{h} + F(\mathbf{u}), \mathbf{u}) \\ &\le |\mathbf{h}|_{V'}\,|\mathbf{u}|_V + c\,|\mathbf{u}|_V^2 \le |\mathbf{h}|_{H^{-1}}\,|\mathbf{u}|_V + c\,|\mathbf{u}|_V^2 . \end{aligned}$$

Since $|\mathbf{u}|_V = R$, it follows that

$$R < |\mathbf{h}|_{H^{-1}} + cR \le (1-c)\,R + cR = R,$$

a contradiction. Thus, the Leray–Schauder fixed point theorem guarantees the existence of a solution.

To prove (2.11), assume that $\mathbf{u} \in V$ is a solution of Eq. (2.2). Then, as above, we find

$$|\mathbf{u}|_V \le |\mathbf{h}|_{H^{-1}} + c\,|\mathbf{u}|_V,$$

whence the conclusion. □

Remark 2.4. (a) If $F : L^2(\Omega)^N \to L^2(\Omega)^N$ and there exists a constant $a_0 > 0$ such that

$$|F(\mathbf{u}) - F(\mathbf{v})|_{L^2} \le a_0\,|\mathbf{u} - \mathbf{v}|_{L^2} \quad \left(\mathbf{u}, \mathbf{v} \in L^2(\Omega)^N\right),$$

then using twice Poincaré's inequality gives

$$|F(\mathbf{u})-F(\mathbf{v})|_{H^{-1}} \leq \frac{1}{\sqrt{\lambda_1}}|F(\mathbf{u})-F(\mathbf{V})|_{L^2} \leq \frac{a_0}{\sqrt{\lambda_1}}|\mathbf{u}-\mathbf{v}|_{L^2}$$
$$\leq \frac{a_0}{\sqrt{\lambda_1(\mu\lambda_1+\eta)}}|\mathbf{u}-\mathbf{v}|_V$$

for all $\mathbf{u},\mathbf{v}\in V$ and so condition (2.3) holds with $a=a_0/\sqrt{\lambda_1(\mu\lambda_1+\eta)}$.

(b) Analogously, if $F: L^2(\Omega)^N \to L^2(\Omega)^N$ and

$$|F(\mathbf{u})|_{L^2} \leq a_0 |\mathbf{u}|_{L^2} \qquad \left(\mathbf{u}\in L^2(\Omega)^N\right),$$

then condition (2.8) holds with $a=a_0/\sqrt{\lambda_1(\mu\lambda_1+\eta)}$.

(c) If $F: L^p(\Omega)^N \to L^q(\Omega)^N$ is continuous and either $p<2^*,\ q\geq (2^*)'$, or $p\leq 2^*,\ q>(2^*)'$, then F is completely continuous from $H_0^1(\Omega)^N$ to $H^{-1}(\Omega)^N$. This follows from the continuous (compact) embeddings $H_0^1(\Omega)\subset L^p(\Omega)$ for $p\leq 2^*\ (p<2^*)$, and $L^q(\Omega)\subset H^{-1}(\Omega)$ for $q\geq (2^*)'\ \left(q>(2^*)'\right)$. In particular, if $F: L^2(\Omega)^N\to L^2(\Omega)^N$ is continuous, then it is completely continuous from $H_0^1(\Omega)^N$ to $H^{-1}(\Omega)^N$.

Recall that for $N>2$, $2^*=\frac{2N}{N-2}$, while for $N=2$, $2^*=+\infty$. In addition, $\frac{1}{2^*}+\frac{1}{(2^*)'}=1$.

3. A Coupled System of Two Navier–Stokes Type Equations

Consider now the system

$$\begin{cases} -\mu_i\Delta\mathbf{u}_i+\eta_i\mathbf{u}_i+\kappa_i(\mathbf{u}_i\cdot\nabla)\mathbf{u}_i+\nabla p_i = \mathbf{h}_i+F_i(\mathbf{u}) & \text{in } \Omega \\ \operatorname{div}\mathbf{u}_i=0 \quad \text{in } \Omega \\ \mathbf{u}_i=0 \quad \text{on } \partial\Omega \quad (i=1,2) \end{cases} \tag{3.1}$$

where $\mathbf{u}$ stands for the pair $(\mathbf{u}_1,\mathbf{u}_2)$, $\mathbf{h}_i\in H^{-1}(\Omega)^N$ and $F_i: H_0^1(\Omega)^{2N}\to H^{-1}(\Omega)^N\ (i=1,2)$. The problem can be reduced to the fixed point equation in V^2

$$\mathbf{u}_i = S_i(\mathbf{h}_i+F_i(\mathbf{u})),\quad i=1,2, \tag{3.2}$$

where $\mathbf{u}=(\mathbf{u}_1,\mathbf{u}_2)\in V^2$, and S_i stands for the solution operator corresponding to $\mathbf{u}_i\in V$. (Here S_i is the solution operator corresponding to system (3.1) with fixed i, compare also with (2.1).)

Denote by $(\cdot,\cdot)_i$ and $|\cdot|_i$ the inner product $(\cdot,\cdot)_0$ and norm $|\cdot|_0$ corresponding to μ_i and η_i. Also denote by M_i the constant M with respect to the norm $|\cdot|_i$ of V.

Assume that the following conditions hold:

(H1) for $i=1,2$, one has $F_i: H_0^1(\Omega)^{2N}\to H^{-1}(\Omega)^N,\ F_i(0)=0$ and

$$|F_i(\mathbf{u})-F_i(\mathbf{v})|_{H^{-1}} \leq a_{i1}|\mathbf{u}_1-\mathbf{v}_1|_1+a_{i2}|\mathbf{u}_2-\mathbf{v}_2|_2,$$
$$\forall\, \mathbf{u}_j,\ \mathbf{v}_j\in H_0^1(\Omega)^N \text{ such that } |\mathbf{u}_j|_j, |\mathbf{v}_j|_j<\theta_j,\ j=1,2,$$

where a_{i1},a_{i2} are nonnegative constants such that $a_{i1}+a_{i2}<1$, and $\theta_j:=1/(2\kappa_j M_j)$.

(H2) there exist $\varepsilon_i>1$, $i=1,2$, such that

$$\varepsilon_1 a_{11}+\varepsilon_2 a_{22} < \min\{2,\ 1+\varepsilon_1\varepsilon_2(a_{11}a_{22}-a_{12}a_{21})\}. \tag{3.3}$$

(H3) $\mathbf{h}_i\in H^{-1}(\Omega)^N$ and

$$|\mathbf{h}_i|_{H^{-1}} \leq (1-a_{i1}-a_{i2})R \quad (i=1,2),$$

where

$$R:=\min\{R_1,\ R_2\},\quad R_i:=\frac{\varepsilon_i-1}{2\kappa_i\varepsilon_i M_i}. \tag{3.4}$$

The next result extends Theorem 2.1 to system (3.1).

Theorem 3.1. *Under assumptions* (H1)-(H3), *system* (3.2) *has a unique solution* $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2) \in V^2$ *such that*

$$|\mathbf{u}_i|_i \le R, \quad i = 1, 2.$$

Proof. First note that the operator $T_i(\mathbf{u}) := S_i(\mathbf{h}_i + F_i(\mathbf{u}))$ is well-defined on

$$D_R := \left\{\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2) \in V^2 : \ |\mathbf{u}_i|_i \le R, \quad i = 1, 2\right\}. \tag{3.5}$$

Indeed, using (H1) one has

$$\begin{aligned} |\mathbf{h}_i + F_i(\mathbf{u})|_{H^{-1}} &\le |\mathbf{h}_i|_{H^{-1}} + |F_i(\mathbf{u})|_{H^{-1}} \\ &\le |\mathbf{h}_i|_{H^{-1}} + a_{i1}|\mathbf{u}_1|_1 + a_{i2}|\mathbf{u}_2|_2 \\ &\le (1 - a_{i1} - a_{i2})R + a_{i1}R + a_{i2}R \\ &= R \le R_i = \frac{\varepsilon_i - 1}{2\kappa_i \varepsilon_i M_i} \\ &< \frac{1}{2\kappa_i M_i}. \end{aligned}$$

Also if $T = (T_1, T_2)$, then $T(D_R) \subset D_R$. Indeed, if $\mathbf{u} \in D_R$, then according to (1.8) and the last estimate,

$$|T_i(\mathbf{u})|_i = |S_i(\mathbf{h}_i + F_i(\mathbf{u}))|_i \le |\mathbf{h}_i + F_i(\mathbf{u})|_{H^{-1}} \le R,$$

that is $T(\mathbf{u}) \in D_R$. Next, again from (H1), we have

$$|T_i(\mathbf{u}) - T_i(\mathbf{v})|_i \le L_i(R_i)\left(a_{i1}|\mathbf{u}_1 - \mathbf{v}_1|_1 + a_{i2}|\mathbf{u}_2 - \mathbf{v}_2|_2\right), \quad i = 1, 2,$$

where

$$L_i(R_i) = \frac{1}{1 - 2\kappa_i M_i R_i}, \quad i = 1, 2.$$

These inequalities can be put under the vector form

$$\begin{bmatrix} |T_1(\mathbf{u}) - T_1(\mathbf{v})|_1 \\ |T_2(\mathbf{u}) - T_2(\mathbf{v})|_2 \end{bmatrix} \le \mathcal{M} \begin{bmatrix} |\mathbf{u}_1 - \mathbf{v}_1|_1 \\ |\mathbf{u}_2 - \mathbf{v}_2|_2 \end{bmatrix}$$

in terms of the matrix

$$\mathcal{M} = \left[L_i(R_i)\, a_{ij}\right]_{i,j=1,2}.$$

Here, a direct computation based on the expression of R_i given by (3.4) shows that

$$L_i(R_i) = \varepsilon_i,$$

and hence $\mathcal{M} = [\varepsilon_i a_{ij}]_{i,j=1,2}$. Moreover, inequality (3.3), in view of (1.10), implies that $\mathcal{M}$ is a matrix that converges to zero, equivalently, whose spectral radius is less than one. Therefore Perov's fixed point theorem applies to T and guarantees that T has in D_R a unique fixed point.

Again we recall that if $\kappa_i = 0$ for some $i \in \{1, 2\}$, one has that $R_i = +\infty$. Moreover, if $\kappa_1 = \kappa_2 = 0$, then $D_R = V^2$. □

Remark 3.2. (a) A sufficient condition for *(H2)* to hold is that the matrix $\mathcal{A} := [a_{ij}]_{i,j=1,2}$ is convergent to zero, i.e.

$$a_{11} + a_{22} < \min\{2, \ 1 + a_{11}a_{22} - a_{12}a_{21}\}. \tag{3.6}$$

Indeed, the strict inequality in (3.6) remains true if we insert, as condition (3.3) requires, the coefficients $\varepsilon_1, \varepsilon_2 > 1$ sufficiently closed to 1.

(b) A sufficient condition for (3.6) to hold is that $a_{i1} + a_{i2} < 1$ for $i = 1, 2$. Indeed, summing up the two inequalities gives $\sum_{i,j=1}^{2} a_{ij} < 2$, whence one has even more $a_{11} + a_{22} < 2$. In addition, the inequality $a_{11} + a_{22} < 1 + a_{11}a_{22} - a_{12}a_{21}$ also holds in view of its equivalent form $a_{12}a_{21} < (1 - a_{11})(1 - a_{22})$ and of the assumptions $a_{12} < 1 - a_{11}$ and $a_{21} < 1 - a_{22}$.

Remark 3.3. In the case of the Stokes type equations, when $\kappa_1 = \kappa_2 = 0$, under condition (**H1**) with $a_{i1} + a_{i2} < 1$ for $i = 1, 2$, problem (3.1) has a unique solution for every $(\mathbf{h}_1, \mathbf{h}_2) \in H^{-1}(\Omega)^{2N}$.

If instead of the Lipschitz condition (H1) and of condition (H3) we consider the more relaxed hypotheses:

(H1*) for $i = 1, 2$, $F_i : H_0^1(\Omega)^{2N} \to H^{-1}(\Omega)^N$ is completely continuous and

$$|F_i(\mathbf{u})|_{H^{-1}} \le a_{i1} |\mathbf{u}_1|_1 + a_{i2} |\mathbf{u}_2|_2 \quad \left(|u_j|_j < \theta_j,\ j = 1, 2\right);$$

(H3*) $1 > a_{i1} + a_{i2}$, $\mathbf{h}_i \in H^{-1}(\Omega)^N$ and

$$|\mathbf{h}_i|_{H^{-1}} < (1 - a_{i1} - a_{i2})\,\theta \quad (i = 1, 2), \tag{3.7}$$

where

$$\theta := \min\{\theta_1,\ \theta_2\},$$

then we can prove the existence of at least one solution.

Theorem 3.4. *Under assumptions* (H1*) *and* (H3*), *system* (3.2) *has at least one solution* $u = (\mathbf{u}_1, \mathbf{u}_2) \in V^2$ *such that*

$$|\mathbf{u}_i|_i < \theta \quad (i = 1, 2).$$

Proof. In view of the strict inequalities (3.7), we may chose a number R with $0 < R < \theta$ such that

$$|\mathbf{h}_i|_{H^{-1}} \le (1 - a_{i1} - a_{i2})\,R, \qquad i = 1, 2.$$

Then, as in the proof of Theorem 3.1, the operators T_i are well-defined and for $T = (T_1, T_2)$ and D_R given by (3.5) with R as above, one has $T(D_R) \subset D_R$. Here in addition T is completely continuous, hence Schauder's fixed point theorem applies and gives the conclusion. □

The next result is the version for systems of Theorem 2.3.

Theorem 3.5. *Assume that the operators* $F_i : H_0^1(\Omega)^{2N} \to H^{-1}(\Omega)^N$ $(i = 1, 2)$ *are completely continuous and there exist* $c_i,\ d_i \ge 0$ *with* $c_i + d_i < 1$ *such that that for each* i,

$$(F_i(\mathbf{u}), \mathbf{u}_i) \le c_i |\mathbf{u}_i|_i^2 + d_i |\mathbf{u}_1|_1 |\mathbf{u}_2|_2, \quad \mathbf{u} \in H_0^1(\Omega)^{2N}. \tag{3.8}$$

Then for each $R > 0$ *satisfying*

$$\sigma_i(R) < \theta_i - (1 - c_i - d_i)\,R, \tag{3.9}$$

where $\sigma_i(R) := \sup_{\mathbf{u} \in D_R} |F_i(\mathbf{u})|_{H^{-1}}$, *system* (3.2) *has at least one solution* $\mathbf{u} \in D_R$ *for every* $(\mathbf{h}_1, \mathbf{h}_2) \in H^{-1}(\Omega)^{2N}$ *with*

$$|\mathbf{h}_i|_{H^{-1}} \le (1 - c_i - d_i)\,R. \tag{3.10}$$

Moreover, for any solution $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2)$ *one has the matrix estimate*

$$\begin{bmatrix} |\mathbf{u}_1|_1 \\ |\mathbf{u}_2|_2 \end{bmatrix} \le (\mathcal{I} - \mathcal{A})^{-1} \begin{bmatrix} |\mathbf{h}_1|_{H^{-1}} \\ |\mathbf{h}_2|_{H^{-1}} \end{bmatrix}, \tag{3.11}$$

where

$$\mathcal{A} = \begin{bmatrix} c_1 & d_1 \\ d_2 & c_2 \end{bmatrix}$$

and $\mathcal{I}$ *is the unit matrix.*

Proof. We follow the same ideas as in the proof of Theorem 2.3. Here the Leray–Schauder theorem applies on the set D_R.

For $\mathbf{u} \in D_R$, using (3.9) one has

$$\begin{aligned}|\mathbf{h}_i + F_i(\mathbf{u})|_{H^{-1}} &\le |\mathbf{h}_i|_{H^{-1}} + |F_i(\mathbf{u})|_{H^{-1}} \le |\mathbf{h}_i|_{H^{-1}} + \sigma_i(R) \\ &\le (1 - c_i - d_i) R + \sigma_i(R) < \frac{1}{2\kappa_i M_i} = \theta_i.\end{aligned}$$

Hence T is well-defined and continuous on the closed subset D_R of V^2. Moreover, the Lipschitz continuity of S_i implies that S_i is a bounded operator (maps bounded sets into bounded sets) which together with the complete continuity of F_i implies that $T = (T_1, T_2)$, $T_i(\mathbf{u}) = S_i(\mathbf{h}_i + F_i(\mathbf{u}))$, is completely continuous on D_R. Next we show that the Leray–Schauder condition holds.

Assume the contrary, i.e. $T(\mathbf{u}) = \lambda \mathbf{u}$ for some $\mathbf{u} \in D_R$ with $|\mathbf{u}_1|_1 = R$ or $|\mathbf{u}_2|_2 = R$ and some $\lambda > 1$. Assume that $|\mathbf{u}_1|_1 = R$. Then multiplying by $\mathbf{u}_1$ the equation

$$-\lambda\mu_1 \Delta \mathbf{u}_1 + \lambda\eta_1 \mathbf{u}_1 + \lambda^2 \kappa_1 (\mathbf{u}_1 \cdot \nabla) \mathbf{u}_1 + \nabla p_1 = \mathbf{h}_1 + F_1(\mathbf{u}),$$

and using (1.7) gives

$$|\mathbf{u}_1|_1^2 < \lambda |\mathbf{u}_1|_1^2 = (\mathbf{h}_1 + F_1(\mathbf{u}), \mathbf{u}_1) \le |\mathbf{h}_1|_{H^{-1}} |\mathbf{u}_1|_1 + (F_1(\mathbf{u}), \mathbf{u}_1). \tag{3.12}$$

Furthermore,using (3.8) we obtain

$$(F_1(\mathbf{u}), \mathbf{u}_1) \le c_1 |\mathbf{u}_1|_1^2 + d_1 |\mathbf{u}_1|_1 |\mathbf{u}_2|_2 \le (c_1 + d_1) R^2.$$

Hence

$$R^2 < |\mathbf{h}_1|_{H^{-1}} R + (c_1 + d_1) R^2.$$

Similarly, in case that $|\mathbf{u}_2|_2 = R$ we derive

$$R^2 < |\mathbf{h}_2|_{H^{-1}} R + (c_2 + d_2) R^2.$$

Hence in view of (3.10), we arrive to the contradiction $R < R$.

To prove (3.11) we start with

$$|\mathbf{u}_i|_i^2 = (\mathbf{h}_i + F_i(\mathbf{u}), \mathbf{u}_i) \le |\mathbf{h}_i|_{H^{-1}} |\mathbf{u}_i|_i + c_i |\mathbf{u}_i|_i^2 + d_i |\mathbf{u}_1|_1 |\mathbf{u}_2|_2$$

which give

$$\begin{aligned}|\mathbf{u}_1|_1 &\le |\mathbf{h}_1|_{H^{-1}} + c_1 |\mathbf{u}_1|_1 + d_1 |\mathbf{u}_2|_2, \\ |\mathbf{u}_2|_2 &\le |\mathbf{h}_2|_{H^{-1}} + c_2 |\mathbf{u}_2|_2 + d_2 |\mathbf{u}_1|_1.\end{aligned}$$

These can be put together under the matrix form

$$(\mathcal{I} - \mathcal{A}) \begin{bmatrix} |\mathbf{u}_1|_1 \\ |\mathbf{u}_2|_2 \end{bmatrix} \le \begin{bmatrix} |\mathbf{h}_1|_{H^{-1}} \\ |\mathbf{h}_2|_{H^{-1}} \end{bmatrix}.$$

From the assumption $c_i + d_i < 1$ $(i = 1, 2)$ we easily can check that matrix $\mathcal{A}$ is convergent to zero, which guarantees that $\mathcal{I} - \mathcal{A}$ has an inverse whose entries are nonnegative. Thus the multiplication by $(\mathcal{I} - \mathcal{A})^{-1}$ does not change inequality and yields (3.11). □

Remark 3.6. Simple computation of $(\mathcal{I} - \mathcal{A})^{-1}$ shows that estimate (3.11) means explicitly:

$$\begin{aligned}|\mathbf{u}_1|_1 &\le \frac{(1 - c_2) |\mathbf{h}_1|_{H^{-1}} + d_1 |\mathbf{h}_2|_{H^{-1}}}{(1 - c_1)(1 - c_2) - d_1 d_2}, \\ |\mathbf{u}_2|_2 &\le \frac{d_2 |\mathbf{h}_1|_{H^{-1}} + (1 - c_1) |\mathbf{h}_2|_{H^{-1}}}{(1 - c_1)(1 - c_2) - d_1 d_2}.\end{aligned}$$

Remark 3.7. In the case of the Stokes type equations, when $\kappa_1 = \kappa_2 = 0$, under condition (3.8) with the coefficients c_i, d_i such that the matrix $\mathcal{A}$ is convergent to zero, the problem has a unique solution for every $(\mathbf{h}_1, \mathbf{h}_2) \in H^{-1}(\Omega)^{2N}$. Indeed, in this case the operator T is defined and completely continuous on the whole space V^2 and (3.11) gives the *a priori* bounds of solutions.

4. Navier–Stokes Type Model for Fluid Flow in Bidisperse Porous Media

Now we come back to the specific model of Navier–Stokes type for bidisperse porous media. Thus we consider the following system

$$\begin{cases} -\mu_1\Delta\mathbf{u}_1+\eta_1\mathbf{u}_1+\kappa_1\left(\mathbf{u}_1\cdot\nabla\right)\mathbf{u}_1+\nabla p_1 \\ \qquad =\mathbf{h}_1-\ \alpha_1\left|\mathbf{u}_1\right|^{p-1}\mathbf{u}_1-\gamma_1\left(\mathbf{u}_1-\mathbf{u}_2\right)\quad\text{in}\ \ \Omega \\ -\mu_2\Delta\mathbf{u}_2+\eta_2\mathbf{u}_2+\kappa_2\left(\mathbf{u}_2\cdot\nabla\right)\mathbf{u}_2+\nabla p_2 \\ \qquad =\mathbf{h}_2-\ \alpha_2\left|\mathbf{u}_2\right|^{p-1}\mathbf{u}_2\ -\gamma_2\left(\mathbf{u}_2-\mathbf{u}_1\right)\quad\text{in}\ \ \Omega \\ \text{div}\ \mathbf{u}_i=0\quad\text{in}\ \ \Omega,\quad i=1,2 \\ \mathbf{u}_i=0\quad\text{on}\ \ \partial\Omega,\qquad i=1,2, \end{cases} \tag{4.1}$$

where $1\leq p<2^*-1$ $(N\leq 3)$. Here for

$$F_1\left(\mathbf{u}\right)=-\ \alpha_1\left|\mathbf{u}_1\right|^{p-1}\mathbf{u}_1\ \ -\gamma_1\left(\mathbf{u}_1-\mathbf{u}_2\right)$$
$$F_2\left(\mathbf{u}\right)=-\alpha_2\left|\mathbf{u}_2\right|^{p-1}\mathbf{u}_2-\gamma_2\left(\mathbf{u}_2-\mathbf{u}_1\right)\ ,$$

we have $F_i:H_0^1\left(\Omega\right)^{2N}\to L^{2^*/p}\left(\Omega\right)^N$. Indeed, from $H_0^1\left(\Omega\right)\subset L^{2^*}\left(\Omega\right)$ we have $\left|\mathbf{u}\right|^{p-1}\mathbf{u}\in L^{2^*/p}\left(\Omega\right)^N$.

Wishing to prove the existence of solutions we shall guarantee that all the assumptions of Theorem 3.5 can be satisfied.

First, since $p<2^*-1$, $2^*/p>(2^*)'$, and so one has $L^{2^*/p}\left(\Omega\right)\subset H^{-1}\left(\Omega\right)$ compactly. It follows that F_i is completely continuous from $H_0^1\left(\Omega\right)^{2N}$ to $H^{-1}\left(\Omega\right)^N$.

Secondly, if for some $R>0$ we take $\left|\mathbf{u}_1\right|_1,\ \left|\mathbf{u}_2\right|_2\leq R$, then we have

$$\begin{aligned}\left|F_1\left(\mathbf{u}\right)\right|_{H^{-1}}&\leq\alpha_1\left|\left|\mathbf{u}_1\right|^{p-1}\mathbf{u}_1\right|_{H^{-1}}+\gamma_1\left(\left|\mathbf{u}_1\right|_{H^{-1}}+\left|\mathbf{u}_2\right|_{H^{-1}}\right)\\&\leq\alpha_1c_1\left|\mathbf{u}_1\right|_{L^{2^*}}^p+\frac{\gamma_1}{\sqrt{\lambda_1}}\left(\left|\mathbf{u}_1\right|_{L^2}+\left|\mathbf{u}_2\right|_{L^2}\right)\\&\leq\alpha_1\widetilde{c_1}R^p+2\gamma_1\widetilde{c_2}R,\end{aligned}$$

where c_1 is the embedding constant of the inclusion $L^{2^*/p}\left(\Omega\right)\subset H^{-1}\left(\Omega\right)$ and $\widetilde{c_1},\ \widetilde{c_2}$ take into account the embedding constants for $V\subset L^{2^*}\left(\Omega\right)^N$ and $V\subset L^2\left(\Omega\right)^N$, respectively. Similarly

$$\left|F_2\left(\mathbf{u}\right)\right|_{H^{-1}}\leq\alpha_2\widetilde{c_1}R^p+2\gamma_2\widetilde{c_2}R.$$

Therefore

$$\sigma_i\left(R\right)=\sup_{\mathbf{u}\in D}\left|F_i\left(\mathbf{u}\right)\right|_{H^{-1}}\leq\alpha_i\widetilde{c_1}R^p+2\gamma_i\widetilde{c_2}R\quad\left(i=1,2\right).$$

Third, we have

$$\begin{aligned}\left(F_1\left(\mathbf{u}\right),\mathbf{u}_1\right)&=\Big(-\ \alpha_1\left|\mathbf{u}_1\right|^{p-1}\mathbf{u}_1\ \ -\gamma_1\left(\mathbf{u}_1-\mathbf{u}_2\right),\ \mathbf{u}_1\Big)\leq\gamma_1\left(\mathbf{u}_2,\mathbf{u}_1\right)\\&\leq\gamma_1\left|\mathbf{u}_1\right|_{L^2}\left|\mathbf{u}_2\right|_{L^2}\leq C\gamma_1\left|\mathbf{u}_1\right|_1\left|\mathbf{u}_2\right|_2\leq C\gamma_1R^2,\end{aligned}$$

where

$$C=\frac{1}{\sqrt{\left(\lambda_1\mu_1+\eta_1\right)\left(\lambda_1\mu_2+\eta_2\right)}},\tag{4.2}$$

and analogously

$$\left(F_2\left(\mathbf{u}\right),\mathbf{u}_2\right)\leq C\gamma_2R^2.$$

Thus, in this example, one has $c_i=0$, $d_i=C\gamma_i$, $i=1,2$, and condition (3.9) holds if

$$\left(1-C\gamma_i\right)R+\alpha_i\widetilde{c_1}R^p+2\gamma_i\widetilde{c_2}R<\theta_i.\tag{4.3}$$

Therefore Theorem 3.5 yields the following existence result.

Theorem 4.1. *Let the coefficients μ_i, α_i, $\gamma_i > 0$ and κ_i, $\eta_i \geq 0$ be given with $\gamma_i < 1/C$, $i = 1, 2$, where C is the constant given by (4.2). Then for each $R > 0$ satisfying (4.3), system (4.1) has at least one solution $\mathbf{u} \in D_R$ for every $(\mathbf{h}_1, \mathbf{h}_2) \in H^{-1}(\Omega)^{2N}$ with*

$$|\mathbf{h}_i|_{H^{-1}} \leq (1 - C\gamma_i) R. \tag{4.4}$$

Moreover, for any solution $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2) \in V^2$, one has

$$|\mathbf{u}_1|_1 \leq \frac{|\mathbf{h}_1|_{H^{-1}} + C\gamma_1 |\mathbf{h}_2|_{H^{-1}}}{1 - C^2\gamma_1\gamma_2}, \quad |\mathbf{u}_2|_2 \leq \frac{C\gamma_2 |\mathbf{h}_1|_{H^{-1}} + |\mathbf{h}_2|_{H^{-1}}}{1 - C^2\gamma_1\gamma_2}.$$

Notice that condition (4.3) is fulfilled if R is sufficiently small and that it can be chosen like this, according to (4.4), provided that $|\mathbf{h}_i|_{H^{-1}}$ are small enough. Thus we have

Corollary 4.2. *Let the coefficients μ_i, α_i, $\gamma_i > 0$ and κ_i, $\eta_i \geq 0$ be given with $\gamma_i < 1/C$. Then problem (4.1) has solutions for each $(\mathbf{h}_1, \mathbf{h}_2) \in H^{-1}(\Omega)^{2N}$ with small $|\mathbf{h}_i|_{H^{-1}}$.*

If now $\mathbf{h}_i$ are given arbitrarily, then (4.4) is fulfilled by some large enough R. Under this number R, condition (4.3) holds provided that θ_i are large (equivalently, κ_i are small, that is system (4.1) is close to a Stokes type system). Thus we have

Corollary 4.3. *For every coefficients μ_i, α_i, $\gamma_i > 0$ and κ_i, $\eta_i \geq 0$ with $\gamma_i < 1/C$, and every $(\mathbf{h}_1, \mathbf{h}_2) \in H^{-1}(\Omega)^{2N}$, problem (4.1) has solutions provided that κ_i are small enough.*

We note that in the case of the Stokes equations, when $\kappa_1 = \kappa_2 = 0$, the above theorem guarantees the existence of a solution for every $(\mathbf{h}_1, \mathbf{h}_2) \in H^{-1}(\Omega)^{2N}$, under the only condition that $\gamma_i < 1/C$ for $i = 1, 2$. Recall that $1/C = \sqrt{(\lambda_1\mu_1 + \eta_1)(\lambda_1\mu_2 + \eta_2)}$. However, a better result is the following:

Theorem 4.4. *Let $\kappa_1 = \kappa_2 = 0$. Then problem (4.1) has a solution for every $(\mathbf{h}_1, \mathbf{h}_2) \in H^{-1}(\Omega)^{2N}$ provided that*

$$\gamma_1\gamma_2 < (\lambda_1\mu_1 + \eta_1)(\lambda_1\mu_2 + \eta_2). \tag{4.5}$$

Proof. In this case, the operator T being well-defined and completely continuous on the whole space V^2, it remains to guarantee the *a priori* boundedness of the solutions of the equations $T(\mathbf{u}) = \lambda\mathbf{u}$ for $\lambda \geq 1$. For any such a solution, one has

$$|\mathbf{u}_i|_i^2 \leq |\mathbf{h}_i|_{H^{-1}} |\mathbf{u}_i|_i + C\gamma_i |\mathbf{u}_1|_1 |\mathbf{u}_2|_2, \quad i = 1, 2.$$

These can be put under a matrix form

$$\begin{bmatrix} |\mathbf{u}_1|_1 \\ |\mathbf{u}_2|_2 \end{bmatrix} \leq \begin{bmatrix} |\mathbf{h}_1|_{H^{-1}} \\ |\mathbf{h}_2|_{H^{-1}} \end{bmatrix} + \begin{bmatrix} 0 & C\gamma_1 \\ C\gamma_2 & 0 \end{bmatrix} \begin{bmatrix} |\mathbf{u}_1|_1 \\ |\mathbf{u}_2|_2 \end{bmatrix}.$$

The involved square matrix $\mathcal{A}$ is convergent to zero if $\gamma_1\gamma_2 < 1/C^2$, which is our assumption (4.5). Thus the above matrix inequality is equivalent to

$$\begin{bmatrix} |\mathbf{u}_1|_1 \\ |\mathbf{u}_2|_2 \end{bmatrix} \leq (\mathcal{I} - \mathcal{A})^{-1} \begin{bmatrix} |\mathbf{h}_1|_{H^{-1}} \\ |\mathbf{h}_2|_{H^{-1}} \end{bmatrix}$$

which gives the desired a priori bounds. The result follows from the Leray–Schauder fixed point theorem. □

Remark 4.5. If in (4.1), $\alpha_1 = \alpha_2 = 0$, under some suitable conditions on $\mathbf{h}_1$ and $\mathbf{h}_2$, one can obtain an existence and uniqueness result by using Banach contraction principle and a similar argument as in [19].

Finally we note that our analysis can be developed in order to treat some models of fluid flows in tridisperse porous media. Moreover, the analysis can be adapted to treat some models with variable coefficients in the anisotropic case. For a numerical approach related to such models we refer the reader to the paper [14] and the references therein.

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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