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Large Time Behavior of Solutions to the 3D Rotating Navier–Stokes Equations

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Abstract. We consider the large time behavior of the solutions for the initial value problem of the Navier–Stokes equations with the Coriolis force in the three-dimensional whole space. We show the L^p temporal decay estimates with the dispersion effect of the Coriolis force for the global solutions. Moreover, we prove the large time asymptotic expansion of the solutions behaving like the first-order spatial derivatives of the integral kernel of the corresponding linear solution.

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1. Introduction

We consider the initial value problem for the 3D incompressible Navier–Stokes equations with the Coriolis force:

$$\begin{cases} \partial_t u - \Delta u + \Omega e_3 \times u + (u \cdot \nabla)u + \nabla p = 0 & x \in \mathbb{R}^3,\ t > 0, \\ \nabla \cdot u = 0 & x \in \mathbb{R}^3,\ t \geq 0, \\ u(x,0) = u_0(x) & x \in \mathbb{R}^3. \end{cases} \tag{1.1}$$

The unknowns $u = u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ and $p = p(x,t)$ denote the velocity field and the pressure of the fluid at the point $(x,t) \in \mathbb{R}^3 \times (0,\infty)$, respectively, while $u_0 = (u_{0,1}(x), u_{0,2}(x), u_{0,3}(x))$ is the initial velocity satisfying $\nabla \cdot u_0 = 0$. Here, e_3 denotes the unit vector $(0,0,1)$, and the term $\Omega e_3 \times u$ describes the Coriolis force with the Coriolis parameter $\Omega \in \mathbb{R}$.

The purpose of this paper is to study the large time behavior of global solutions to (1.1). In particular, we shall show the L^p temporal decay estimates and the asymptotic behaviors of solutions as t goes to infinity when the initial data u_0 is in $L^1(\mathbb{R}^3)$. More precisely, we shall prove that the unique global solution u to (1.1) satisfies

$$\|u(t)\|_{L^p} = o\left(t^{-\frac{3}{2}\left(1-\frac{1}{p}\right)}(1+|\Omega|t)^{-\left(1-\frac{2}{p}\right)}\right) \qquad (t \to \infty)$$

for $2 \leq p \leq p_*$ with some upper bound $2 < p_* < \infty$ [see (1.14)] when $u_0 \in L^1(\mathbb{R}^3)$ satisfies $\nabla \cdot u_0 = 0$. Moreover, if we further assume $|x|u_0 \in L^1(\mathbb{R}^3)$, we show that the global solution fulfills the temporal decay estimate

$$\|u(t)\|_{L^p} \leq Ct^{-\frac{1}{2}-\frac{3}{2}\left(1-\frac{1}{p}\right)}(1+|\Omega|t)^{-\left(1-\frac{2}{p}\right)}$$

for $t > 0$. We also give the asymptotic expansion of the solution behaving like the first-order spatial derivatives of the integral kernel of the corresponding linear solution as t goes to infinity.

Before stating our results, we first review the known results on the large time behavior of global solutions to (1.1). In the case $\Omega = 0$, the system (1.1) corresponds to the original incompressible Navier–Stokes equations in $\mathbb{R}^n$:

$$\begin{cases} \partial_t u - \Delta u + (u \cdot \nabla) u + \nabla p = 0 & x \in \mathbb{R}^n,\ t > 0, \\ \nabla \cdot u = 0 & x \in \mathbb{R}^n,\ t \geq 0, \\ u(x,0) = u_0(x) & x \in \mathbb{R}^n \end{cases} \tag{1.2}$$

with $n \geqslant 2$. Concerning the L^2 decay of weak solutions $u(t)$ to (1.2), Masuda [23] showed that $\|u(t)\|_{L^2} = o(1)$ as $t \to \infty$ for $u_0 \in (L^2 \cap L^n)(\mathbb{R}^n)$. Schonbek [27] and Kajikiya and Miyakawa [18] established the temporal decay estimates $\|u(t)\|_{L^2} \leq C t^{-\frac{n}{2}(\frac{1}{p} - \frac{1}{2})}$ for $t > 0$ when $u_0 \in (L^2 \cap L^p)(\mathbb{R}^n)$ with $1 \leq p < 2$. Wiegner [29] proved that if the initial data $u_0 \in L^2(\mathbb{R}^n)$ satisfies $\|e^{t\Delta} u_0\|_{L^2} \leq C(1+t)^{-\alpha}$ with some $\alpha \geq 0$, then weak solutions $u(t)$ to (1.2) have the decay estimate

$$\|u(t)\|_{L^2} \leq C(1+t)^{-\beta}, \quad \beta := \min\left\{\alpha,\ \frac{1}{2} + \frac{n}{4}\right\}. \tag{1.3}$$

In particular, if $u_0 \in L^2(\mathbb{R}^n)$ satisfies $(1+|x|)u_0 \in L^1(\mathbb{R}^n)$ then it holds $\|e^{t\Delta} u_0\|_{L^2} \leq C(1+t)^{-\frac{1}{2} - \frac{n}{4}}$ and (1.3) holds with the decay rate $\beta = \frac{1}{2} + \frac{n}{4}$.

For the L^p decay of the strong solution to (1.2), it follows from the results by Kato [17], Miyakawa [24,25] and Fujigaki and Miyakawa [7] that the unique global solution $u(t)$ to (1.2) satisfies the L^p temporal decay estimates

$$\|u(t)\|_{L^p} \leq C t^{-\frac{n}{2}(1 - \frac{1}{p})} \quad \text{and} \quad \lim_{t \to \infty} t^{\frac{n}{2}(1 - \frac{1}{p})} \|u(t)\|_{L^p} = 0 \qquad (1 \leq p \leq \infty) \tag{1.4}$$

if the divergence-free initial data $u_0 \in (L^1 \cap L^n)(\mathbb{R}^n)$ is small in $L^n(\mathbb{R}^n)$. Fujigaki and Miyakawa [7] showed the L^p decay estimate of the strong solution

$$\|u(t)\|_{L^p} \leq C t^{-\frac{1}{2} - \frac{n}{2}(1 - \frac{1}{p})} \qquad (1 \leq p \leq \infty,\ t > 0) \tag{1.5}$$

provided that u_0 is small in $L^n(\mathbb{R}^n)$ and satisfies $(1+|x|)u_0 \in L^1(\mathbb{R}^3)$. Furthermore, they [7] established the asymptotic expansion of the global solution $u(t)$ behaving like the first-order derivatives of the Gauss kernel: for $1 \leq p \leq \infty$

$$\begin{aligned} \lim_{t \to \infty} t^{\frac{1}{2} + \frac{n}{2}(1 - \frac{1}{p})} \Bigg\| u(t) &+ \sum_{j=1}^{n} \partial_j G_t(\cdot) \int_{\mathbb{R}^n} y_j u_0(y)\, dy \\ &+ \sum_{j=1}^{n} \partial_j \widetilde{G}_t(\cdot) \int_0^\infty \int_{\mathbb{R}^n} (u_j u)(y,s)\, dy\, ds \Bigg\|_{L^p} = 0. \end{aligned} \tag{1.6}$$

Here, $G_t(x) := (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$ is the Gauss kernel, and we set $\widetilde{G}_t := \mathcal{F}^{-1}[e^{-t|\xi|^2} P(\xi)]$, where $P(\xi) = (\delta_{jk} + \xi_j \xi_k / |\xi|^2)_{1 \leq j,k \leq n}$ is the Fourier multiplier matrix of the Helmholtz projection. We refer to [24–26,28] for the L^p temporal decay estimates of the global strong solutions to (1.2) when the initial data belongs to the Hardy spaces, the Besov spaces or the weighted Hardy spaces.

Next, we review the known results on the unique existence and the temporal decay estimates for global solutions to (1.1). Let $\mathbb{P}$ be the Helmholtz projection onto the divergence-free vector fields, and let J be the skew-symmetric constant matrix defined by

$$\mathbb{P} = (\delta_{jk} + R_j R_k)_{1 \leq j,k \leq 3}, \quad J = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{1.7}$$

respectively, where $R_j = -\partial_{x_j}(-\Delta)^{-\frac{1}{2}}$ is the Riesz transform for $j = 1,2,3$. Note that the Coriolis force in (1.1) can be written as $e_3 \times u = Ju$. Applying the Helmholtz projection $\mathbb{P}$ to (1.1), we have the following evolution equations on the velocity field u:

$$\begin{cases} \partial_t u - \Delta u + \Omega \mathbb{P} J \mathbb{P} u + \mathbb{P}(u \cdot \nabla) u = 0 & x \in \mathbb{R}^3,\ t > 0, \\ \nabla \cdot u = 0 & x \in \mathbb{R}^3,\ t \geq 0, \\ u(x,0) = u_0(x) & x \in \mathbb{R}^3. \end{cases} \tag{1.8}$$

Babin, Mahalov and Nicolaenko [2–4] considered the problem (1.8) in the periodic setting $\mathbb{T}^3$, and proved the global regularity of solutions for sufficiently large $|\Omega|$. Chemin, Desjardins, Gallagher and Grenier [5,6] proved that for the initial velocity $u_0 = v_0 + w_0 \in L^2(\mathbb{R}^2)^3 + H^{1/2}(\mathbb{R}^3)^3$, there exists a positive parameter $\omega_0 = \omega_0(u_0)$ such that for any $\Omega \in \mathbb{R}$ satisfying $|\Omega| \geq \omega_0$, the rotating Navier–Stokes equations (1.8) admits a unique global solution. Furthermore, it is shown in [5,6] that the unique global solution of (1.8) converges to that of the 2D Navier–Stokes equations with the initial data v_0 in the local-in-time norm $L^2_{loc}(0,\infty; L^q(\mathbb{R}^3))$ for $2 < q < 6$ as $|\Omega| \to \infty$. Hieber and Shibata [10] proved the global well-posedness of (1.8) for all $\Omega \in \mathbb{R}$ under the smallness condition on the initial data u_0 in $H^{\frac{1}{2}}(\mathbb{R}^3)$. They [10] also gave the temporal decay estimate $\|u(t)\|_{L^p} \leq Ct^{-\frac{3}{2}(\frac{1}{2}-\frac{1}{p})}$ for $3 < p < \infty$ and $t > 0$ with some constant $C = C(\|u_0\|_{H^{\frac{1}{2}}}, p) > 0$. See also [9,16,21] for the global well-posedness of (1.8) for small initial data in various scaling invariant spaces. In [15,20], it is shown that the system (1.8) possesses a unique global solution for the initial data in the scaling subcritical space $\dot{H}^s(\mathbb{R}^3)$ with $1/2 < s < 9/10$. More precisely, the authors in [20] established the linear decay estimate

$$\|\partial_x^\alpha e^{t(\Delta - \Omega \mathbb{P} J \mathbb{P})} u_0\|_{L^p} \leq C\|u_0\|_{L^r} t^{-\frac{|\alpha|}{2} - \frac{3}{2}(\frac{1}{r}-\frac{1}{p})}(1+|\Omega| t)^{-(1-\frac{2}{p})} \tag{1.9}$$

for $t > 0$ and $\Omega \in \mathbb{R}$ with $2 \leq p < \infty$, $1 < r \leq p'$ and $\alpha \in (\mathbb{N} \cup \{0\})^3$, and obtained the following result on the global existence of solutions:

Theorem 1.1 ([15,20]). *Suppose that s, q, and θ satisfy*

$$\frac{1}{2} < s < \frac{9}{10}, \quad \frac{1}{3} + \frac{s}{9} \leq \frac{1}{q} < \frac{7}{12} - \frac{s}{6}, \tag{1.10}$$

$$\frac{3}{2}\left(\frac{1}{2} - \frac{1}{q}\right) \leq \frac{1}{\theta} \leq \frac{5}{2}\left(\frac{1}{2} - \frac{1}{q}\right), \quad \frac{1}{2q} + \frac{s}{2} - \frac{1}{2} < \frac{1}{\theta} < \frac{5}{8} - \frac{3}{2q} + \frac{s}{4}. \tag{1.11}$$

Then, there exists a constant $C_* = C_*(s,q,\theta) > 0$ such that for any $u_0 \in \dot{H}^s(\mathbb{R}^3)^3$ with $\nabla \cdot u_0 = 0$ and $\Omega \in \mathbb{R} \setminus \{0\}$ satisfying

$$\|u_0\|_{\dot{H}^s(\mathbb{R}^3)} \leq C_* |\Omega|^{\frac{s}{2}-\frac{1}{4}}, \tag{1.12}$$

the Eq. (1.8) admits a unique global solution u in the class $C([0,\infty); \dot{H}^s(\mathbb{R}^3))^3 \cap L^\theta(0,\infty; \dot{H}^s_q(\mathbb{R}^3))^3$. Also, there exists a constant $C = C(s,q,\theta) > 0$ such that the global solution u satisfies

$$\|u\|_{L^\theta(0,\infty;\dot{H}^s_q)} \leq C|\Omega|^{-\frac{1}{\theta}+\frac{3}{2}(\frac{1}{2}-\frac{1}{q})}\|u_0\|_{\dot{H}^s} \tag{1.13}$$

for all $\Omega \in \mathbb{R} \setminus \{0\}$.

Ahn, Kim and Lee [1] extended Theorem 1.1 to the system (1.8) with the fractional Laplacian $(-\Delta)^\alpha$ for $1/2 < \alpha < 5/2$, and also derived the temporal decay estimates of solutions with the same decay rate as the linear solutions (1.9). In the case $\alpha = 1$, the L^p decay estimates obtained in [1] is written as

$$\|u(t)\|_{L^p} \leq C\|u_0\|_{L^r} t^{-\frac{3}{2}(\frac{1}{r}-\frac{1}{p})}(1+|\Omega|t)^{-(1-\frac{2}{p})},$$

for $t > 0$, where

$$\frac{1}{2} - \frac{1}{2}\left(\frac{1}{q} - \frac{s}{3}\right) \leq \frac{1}{p} \leq \frac{1}{q}, \quad \frac{1}{r} = \frac{1}{3} + \frac{1}{q} + \frac{1}{p} - \frac{s}{3}$$

and (s, q) are the exponents satisfying (1.10). Kim [19] considered the magnetohydrodynamics equations with the Coriolis force, and proved the global well-posedness and the temporal decay estimate for $u_0 \in (H^s \cap L^1)(\mathbb{R}^3)$ with $1/2 < s < 3/2$:

$$\|u(t)\|_{L^p} \leq Ct^{-\frac{3}{2}(1-\frac{1}{p})}(|\Omega|t)^{-(1-\frac{2}{p})}$$

with $0 < \gamma \leq s - \frac{1}{2}$ and $\max\left\{\frac{1}{4} + \frac{s}{6}, \frac{4-\gamma}{8+\gamma}\right\} < \frac{1}{p} \leq \frac{1}{2}$.

In this paper, we consider the L^p temporal decay estimate and the large time behavior of the global solution u to (1.8) constructed in Theorem 1.1 when the initial data u_0 is in $L^1(\mathbb{R}^3)$. We remark that the L^1-integrability implies $\int_{\mathbb{R}^3} u_0(y)\,dy = 0$, thanks to the divergence-free condition $\nabla \cdot u_0 = 0$. Hence, as is the case (1.4) for the original Navier–Stokes Eq. (1.2), it seems natural to expect that the L^p-norm of the global solution $u(t)$ decays faster than $t^{-\frac{3}{2}(1-\frac{1}{p})}(1+|\Omega|t)^{-(1-\frac{2}{p})}$ as $t \to \infty$. Our first result in this paper reads as follows:

Theorem 1.2. *Assume that the exponents s, q and θ satisfy (1.10)–(1.11), and that the exponent p satisfies*

$$\frac{1}{2} - \frac{1}{2}\left(\frac{1}{q} - \frac{s}{3}\right) \leq \frac{1}{p} \leq \frac{1}{2}. \tag{1.14}$$

Let $u_0 \in (\dot{H}^s \cap L^1)(\mathbb{R}^3)^3$ with $\nabla \cdot u_0 = 0$ and $\Omega \in \mathbb{R} \setminus \{0\}$ satisfy (1.12). Let $u \in C([0,\infty); \dot{H}^s(\mathbb{R}^3))^3 \cap L^\theta(0,\infty; \dot{H}^s_q(\mathbb{R}^3))^3$ be the unique global solution to (1.8) constructed in Theorem 1.1. Then, there exists a constant $C = C(p, \|u_0\|_{L^1}, \|u_0\|_{L^2}) > 0$ such that

$$\|u(t)\|_{L^p} \leq Ct^{-\frac{3}{2}(1-\frac{1}{p})}(1+|\Omega|t)^{-(1-\frac{2}{p})} \tag{1.15}$$

for all $t > 0$. Furthermore, it holds that

$$\lim_{t\to\infty} t^{\frac{3}{2}(1-\frac{1}{p})}(1+|\Omega|t)^{1-\frac{2}{p}}\|u(t)\|_{L^p} = 0. \tag{1.16}$$

We next address the asymptotic behavior of global solutions corresponding to (1.5) and (1.6) for the original Navier–Stokes Eq. (1.2) by Fujigaki and Miyakawa [7] when $(1+|x|)u_0 \in L^1(\mathbb{R}^3)$. In order to state our second result, we introduce some notation. Let $P(\xi)$ be the Fourier multiplier matrix of the Helmholtz projection $\mathbb{P}$ (1.7) defined by

$$\widehat{\mathbb{P}f}(\xi) = P(\xi)\widehat{f}(\xi), \quad P(\xi) := \left(\delta_{jk} - \frac{\xi_j\xi_k}{|\xi|^2}\right)_{1\leq j,k\leq 3} \tag{1.17}$$

for $\xi \in \mathbb{R}^3 \setminus \{0\}$. Let $A_\Omega := -\Delta + \Omega\mathbb{P}J\mathbb{P}$ be the linear operator associated with (1.8). It is known in [8,10] that the semigroup e^{-tA_Ω} generated by $-A_\Omega$ is given explicitly by

$$e^{-tA_\Omega}f = \mathcal{F}^{-1}\left[e^{-t|\xi|^2}\left\{\cos\left(\Omega\frac{\xi_3}{|\xi|}t\right)I + \sin\left(\Omega\frac{\xi_3}{|\xi|}t\right)R(\xi)\right\}\widehat{f}(\xi)\right] \tag{1.18}$$

for divergence-free vector fields $f \in L^2(\mathbb{R}^3)^3$. Here, I is the 3×3 identity matrix and $R(\xi)$ is the skew-symmetric matrix related to the Riesz transforms defined as

$$R(\xi) := \frac{1}{|\xi|}\begin{pmatrix} 0 & \xi_3 & -\xi_2 \\ -\xi_3 & 0 & \xi_1 \\ \xi_2 & -\xi_1 & 0 \end{pmatrix}, \quad \xi \in \mathbb{R}^3 \setminus \{0\}. \tag{1.19}$$

By the Duhamel principle, the system (1.8) can be transformed into the following integral equation:

$$u(t) = e^{-tA_\Omega}u_0 - \int_0^t e^{-(t-\tau)A_\Omega}\mathbb{P}\nabla \cdot (u \otimes u)(\tau)\,d\tau, \quad t > 0. \tag{1.20}$$

Now, we define the functions $H_\Omega(\xi,t)$ and $\widetilde{H}_\Omega(\xi,t)$ as

$$H_\Omega(\xi,t) := \cos\left(\Omega\frac{\xi_3}{|\xi|}t\right) I + \sin\left(\Omega\frac{\xi_3}{|\xi|}t\right) R(\xi), \tag{1.21}$$

$$\widetilde{H}_\Omega(\xi,t) := \cos\left(\Omega\frac{\xi_3}{|\xi|}t\right) P(\xi) + \sin\left(\Omega\frac{\xi_3}{|\xi|}t\right) R(\xi), \tag{1.22}$$

for $\xi \in \mathbb{R}^3 \setminus \{0\}$ and $t \geq 0$. Then, we set

$$K_\Omega(x,t) := \mathcal{F}^{-1}\left[e^{-t|\xi|^2} H_\Omega(\xi,t)\right](x), \quad \widetilde{K}_\Omega(x,t) := \mathcal{F}^{-1}\left[e^{-t|\xi|^2} \widetilde{H}_\Omega(\xi,t)\right](x) \tag{1.23}$$

for $x \in \mathbb{R}^3$ and $t \geq 0$. Note that the functions $K_\Omega(x,t)$ and $\widetilde{K}_\Omega(x,t)$ are the integral kernel of the linear semigroup e^{-tA_Ω} and $e^{-tA_\Omega}\mathbb{P}$, respectively, and there hold

$$e^{-tA_\Omega}u_0 = K_\Omega(\cdot,t) * u_0, \quad e^{-tA_\Omega}\mathbb{P}f = \widetilde{K}_\Omega(\cdot,t) * f.$$

We set the function space $L^1_1(\mathbb{R}^3)$ of the initial data as

$$L^1_1(\mathbb{R}^3) := \left\{ f \in L^1(\mathbb{R}^3) \;\middle|\; |x|f \in L^1(\mathbb{R}^3) \right\}.$$

Our second result on the asymptotic behavior of global solutions to (1.8) for the initial data $u_0 \in L^1_1(\mathbb{R}^3)^3$ reads as follows:

Theorem 1.3. *Assume that the exponents s, q and θ satisfy (1.10)–(1.11), and that the exponent p satisfies*

$$\frac{1}{2} - \frac{1}{2}\left(\frac{1}{q} - \frac{s}{3}\right) \leq \frac{1}{p} \leq \frac{1}{2}.$$

Let $u_0 \in (\dot{H}^s \cap L^1_1)(\mathbb{R}^3)^3$ with $\nabla \cdot u_0 = 0$ and $\Omega \in \mathbb{R}\setminus\{0\}$ satisfy (1.12). Let $u \in C([0,\infty);\dot{H}^s(\mathbb{R}^3))^3 \cap L^\theta(0,\infty;\dot{H}^s_q(\mathbb{R}^3))^3$ be the unique global solution to (1.8) constructed in Theorem 1.1. Then, there exists a constant $C = C(p, \||x|u_0\|_{L^1}, \|u_0\|_{L^2}) > 0$ such that

$$\|u(t)\|_{L^p} \leq Ct^{-\frac{1}{2}-\frac{3}{2}(1-\frac{1}{p})}(1+|\Omega|t)^{-(1-\frac{2}{p})} \tag{1.24}$$

for all $t > 0$. Furthermore, it holds that

$$\begin{aligned} &\lim_{t\to\infty} t^{\frac{1}{2}+\frac{3}{2}(1-\frac{1}{p})}(1+|\Omega|t)^{1-\frac{2}{p}} \Bigg\| u(t) + \sum_{j=1}^{3} \partial_j K_\Omega(\cdot,t) \int_{\mathbb{R}^3} y_j u_0(y)\,dy \\ &+ \sum_{j=1}^{3} \partial_j \widetilde{K}_\Omega(\cdot,t) \int_0^\infty \int_{\mathbb{R}^3} (u_j u)(y,s)\,dy\,ds \\ &+ \Omega \sum_{j=1}^{3} \int_0^{\frac{t}{2}} \int_0^1 \partial_j \mathbb{P} J \mathbb{P} \widetilde{K}_\Omega(\cdot, t-\tau s)\,d\tau \int_{\mathbb{R}^3} s(u_j u)(y,s)\,dy\,ds \Bigg\|_{L^p} = 0. \end{aligned} \tag{1.25}$$

Let us give several remarks on Theorem 1.3. In the case $\Omega = 0$, we see by (1.21), (1.22) and (1.23) that $H_0(\xi,t) = I$, $\widetilde{H}_0(\xi,t) = P(\xi)$ and

$$K_0(x,t) = G_t(x)I, \quad \widetilde{K}_0(x,t) = \mathcal{F}^{-1}[e^{-t|\xi|^2}P(\xi)](x) = \widetilde{G}_t(x).$$

Hence the asymptotic expansion (1.25) in Theorem 1.3 corresponds to (1.6) for the original Navier–Stokes equations by [7]. Next, we remark that it follows from Lemma 3.1 and (5.35) that

$$\|\partial_x^\alpha K_\Omega(\cdot,t)\|_{L^p} + \|\partial_x^\alpha \widetilde{K}_\Omega(\cdot,t)\|_{L^p} \leq Ct^{-\frac{|\alpha|}{2}-\frac{3}{2}(1-\frac{1}{p})}(1+|\Omega|t)^{-(1-\frac{2}{p})},$$

and then

$$\begin{aligned} &\left\| \int_0^{\frac{t}{2}} \int_0^1 \partial_j \mathbb{P} J \mathbb{P} \widetilde{K}_\Omega(\cdot, t-\tau s)\,d\tau \int_{\mathbb{R}^3} s(u_j u)(y,s)\,dy\,ds \right\|_{L^p} \\ &\leq Ct^{-\frac{1}{2}-\frac{3}{2}(1-\frac{1}{p})}(1+|\Omega|t)^{-(1-\frac{2}{p})} \end{aligned} \tag{1.26}$$

by the L^2 decay estimate $\|u(t)\|_{L^2} \le C(1+t)^{-\frac{5}{4}}$ in Lemma 4.2 (2). Hence the functions appeared in (1.25) would be expected to be the leading terms of the global solutions $u(t)$ to (1.8) as $t \to \infty$.

Finally, let us mention the proof of the asymptotic behavior (1.25) and give the comparisons with the previous studies. In [7], the authors applied the mean value theorem to the Gauss kernel $\widetilde{G}_t(x-y,t-s)$ with respect to both the space and the time variables, and proved the asymptotic expansion (1.6) for the solutions to (1.2). Ishige, Kawakami and Kobayashi [14] established a general method to show the higher-order asymptotic expansion of solutions for various nonlinear parabolic equations (see also [11–13]). The authors [14] introduced the operator $P_k(t)$ having the cancellation property $\int_{\mathbb{R}^n} x^\alpha [P_k(t)f](x)dx = 0$ of moments for all $\alpha \in (\mathbb{N}\cup\{0\})^n$ with $|\alpha| \le k$, and obtained the higher-order asymptotic expansion of solutions without using the time derivatives of the integral kernel. In our situation for (1.8), since it holds

$$\partial_t \widetilde{K}_\Omega(x,t) = \Delta \widetilde{K}_\Omega(x,t) - \Omega \mathbb{P} J \mathbb{P} \widetilde{K}_\Omega(x,t)$$

and $\mathbb{P}J\mathbb{P}$ is the Fourier multiplier of the 0-th order, we see that the time differentiation does not give a faster decay than the original kernel $\widetilde{K}_\Omega$. Also, since the kernel $\widetilde{K}_\Omega(\cdot,t)$ does not belong to $L^1(\mathbb{R}^n)$ because of the fact that $\xi_3/|\xi|$ in (1.22) is not continuous at $\xi = 0$, it seems difficult to consider the moment conditions in [14]. For the proof of Theorem 1.3, we adapt the arguments in [1,7,29] with the correction term $\Omega\mathbb{P}J\mathbb{P}\widetilde{K}_\Omega(x,t)$, and show the asymptotic behavior (1.25) by using the L^2 temporal decay estimates and the space-time integrability of the solution u in $L^\theta(0,\infty;\dot{H}^s_q(\mathbb{R}^3))$.

This paper is organized as follows. In Sect. 2, we prepare several function spaces and recall the known results on linear estimates. In Sect. 3, we show the temporal decay estimates and the asymptotics of the linear solutions. In Sect. 4, we prove the L^p decay estimates (1.15) and (1.24) for the nonlinear solutions. In Sect. 5, we present the proof of the nonlinear asymptotic behaviors (1.16) and (1.25) for the global solution to (1.8).

2. Preliminaries

In this section, we introduce the definitions of several function spaces, and recall the known results on the linear estimates for the semigroup e^{-tA_Ω}.

Let $\mathscr{S}(\mathbb{R}^3)$ be the Schwartz space of all rapidly decreasing infinitely differentiable functions on $\mathbb{R}^3$, and let $\mathscr{S}'(\mathbb{R}^3)$ be the set of all tempered distributions. The Fourier transform and the inverse Fourier transform of $\varphi \in \mathscr{S}(\mathbb{R}^3)$ are defined by

$$\mathcal{F}[\varphi](\xi) = \widehat{\varphi}(\xi) = \int_{\mathbb{R}^3} e^{-ix\cdot\xi}\varphi(x)\,dx, \quad \mathcal{F}^{-1}[\varphi](x) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{ix\cdot\xi}\varphi(\xi)\,d\xi$$

for $\xi, x \in \mathbb{R}^3$, respectively. Also, $\mathscr{P}(\mathbb{R}^3)$ denotes the set of all polynomials in $\mathbb{R}^3$.

Definition 2.1. (i) Let $s \in \mathbb{R}$ and $1 \le p \le \infty$. The homogeneous Sobolev space $\dot{H}^s_p(\mathbb{R}^3)$ is defined by

$$\dot{H}^s_p(\mathbb{R}^3) := \left\{ f \in \mathscr{S}'(\mathbb{R}^3)/\mathscr{P}(\mathbb{R}^3) \;\middle|\; \|f\|_{\dot{H}^s_p} < \infty \right\},$$
$$\|f\|_{\dot{H}^s_p} := \left\| \mathcal{F}^{-1}\left[|\xi|^s \widehat{f}(\xi) \right] \right\|_{L^p}.$$

(ii) For $s \in \mathbb{R}$, the homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^3)$ is defined by

$$\dot{H}^s(\mathbb{R}^3) := \dot{H}^s_2(\mathbb{R}^3), \quad \|f\|_{\dot{H}^s} := \left\| \mathcal{F}^{-1}\left[|\xi|^s \widehat{f}(\xi) \right] \right\|_{L^2} = \frac{1}{(2\pi)^{\frac{3}{2}}} \left\| |\xi|^s \widehat{f}(\xi) \right\|_{L^2}.$$

Next, we recall the definition of the Littlewood–Paley decomposition. Let $\varphi_0 \in \mathscr{S}(\mathbb{R}^3)$ satisfy the following properties:

$$0 \le \widehat{\varphi_0}(\xi) \le 1 \quad \text{for all } \xi \in \mathbb{R}^3, \quad \operatorname{supp}\widehat{\varphi_0} \subset \left\{ \xi \in \mathbb{R}^3 \;\middle|\; 1/2 \le |\xi| \le 2 \right\},$$
$$\text{and} \quad \sum_{j\in\mathbb{Z}} \widehat{\varphi_j}(\xi) = 1 \quad \text{for } \xi \in \mathbb{R}^3 \setminus \{0\},$$

where $\varphi_j(x) := 2^{3j}\varphi_0(2^j x)$. Then, we define the Littlewood–Paley operators $\{\Delta_j\}_{j\in\mathbb{Z}}$ by $\Delta_j f := \varphi_j * f$ for $f \in \mathscr{S}'(\mathbb{R}^3)$. Also, we put $\widehat{\psi}(\xi) := 1 - \sum_{j=1}^{\infty} \widehat{\varphi_j}(\xi)$ for $\xi \in \mathbb{R}^3$.

Definition 2.2. (i) Let $s \in \mathbb{R}$ and $1 \le p, q \le \infty$. The homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R}^3)$ is defined by

$$\dot{B}^s_{p,q}(\mathbb{R}^3) := \left\{ f \in \mathscr{S}'(\mathbb{R}^3)/\mathscr{P}(\mathbb{R}^3) \mid \|f\|_{\dot{B}^s_{p,q}} < +\infty \right\},$$
$$\|f\|_{\dot{B}^s_{p,q}} := \left\| \left\{ 2^{sj} \|\Delta_j f\|_{L^p} \right\}_{j\in\mathbb{Z}} \right\|_{\ell^q(\mathbb{Z})}.$$

(ii) Let $s \in \mathbb{R}$ and $1 \le p, q \le \infty$. The inhomogeneous Besov space $B^s_{p,q}(\mathbb{R}^3)$ is defined by

$$B^s_{p,q}(\mathbb{R}^3) := \left\{ f \in \mathscr{S}'(\mathbb{R}^3) \mid \|f\|_{B^s_{p,q}} < +\infty \right\},$$
$$\|f\|_{B^s_{p,q}} := \|\psi * f\|_{L^p} + \left\| \left\{ 2^{sj} \|\Delta_j f\|_{L^p} \right\}_{j\in\mathbb{N}} \right\|_{\ell^q(\mathbb{N})}.$$

For $s > 0$, it is known that the norm equivalence

$$c_1\|f\|_{B^s_{p,q}} \le \|f\|_{L^p} + \|f\|_{\dot{B}^s_{p,q}} \le c_2\|f\|_{B^s_{p,q}} \tag{2.1}$$

holds for $1 \le p, q \le \infty$ with some positive constants c_1 and c_2.

Finally, we recall the known results on the linear estimates for the semigroup e^{-tA_Ω}. We set

$$\mathcal{G}_\pm(\tau)[f](x) := \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{ix\cdot\xi} e^{\pm i\tau\frac{\xi_3}{|\xi|}} \widehat{f}(\xi)\, d\xi, \quad x \in \mathbb{R}^3,\ \tau \in \mathbb{R}. \tag{2.2}$$

Then, the linear semigroup e^{-tA_Ω} generated by the linear operator $A_\Omega = -\Delta + \Omega\mathbb{P}J\mathbb{P}$ is explicitly written as

$$\begin{aligned} e^{-tA_\Omega} f &= \mathcal{F}^{-1}\left[e^{-t|\xi|^2} \left\{ \cos\left(\Omega\frac{\xi_3}{|\xi|}t\right) I\widehat{f}(\xi) + \sin\left(\Omega\frac{\xi_3}{|\xi|}t\right) R(\xi)\widehat{f}(\xi) \right\} \right] \\ &= \frac{1}{2}\mathcal{G}_+(\Omega t)e^{t\Delta}(I + \mathcal{R})f + \frac{1}{2}\mathcal{G}_-(\Omega t)e^{t\Delta}(I - \mathcal{R})f \end{aligned} \tag{2.3}$$

for $f \in L^2(\mathbb{R}^3)^3$ with $\nabla \cdot f = 0$, where

$$\mathcal{R} = \begin{pmatrix} 0 & R_3 & -R_2 \\ -R_3 & 0 & R_1 \\ R_2 & -R_1 & 0 \end{pmatrix} \tag{2.4}$$

and $R_j = -\partial_{x_j}(-\Delta)^{-\frac{1}{2}}$ is the Riesz transform. See [8,10,15] for the derivation of the explicit formula (2.3) of the semigroup e^{-tA_Ω}.

The dispersion estimate for $\mathcal{G}_\pm(\tau)$ was obtained in [20].

Lemma 2.3 ([20, Lemma 2.2]). *For $2 \le p \le \infty$, there exists a positive constant $C = C(p)$ such that*

$$\|\mathcal{G}_\pm(\tau)f\|_{\dot{B}^s_{p,q}} \le C(1 + |\tau|)^{-(1-\frac{2}{p})}\|f\|_{\dot{B}^{s+3(1-\frac{2}{p})}_{p',q}}$$

for all $\tau \in \mathbb{R}$, $s \in \mathbb{R}$, $1 \le q \le \infty$ and $f \in \dot{B}^{s+3(1-\frac{2}{p})}_{p',q}(\mathbb{R}^3)$ with $1/p + 1/p' = 1$.

We end this section by recalling the L^q-L^p smoothing estimates for the linear semigruops e^{-tA_Ω} and $e^{t\Delta}$.

Lemma 2.4 ([10, Proposition 2.4]). *Let $\alpha \in (\mathbb{N} \cup \{0\})^3$ and $1 \le q \le 2 \le p \le \infty$. Then, there exists a constant $C = C(\alpha, p, q) > 0$ such that*

$$\|\partial_x^\alpha e^{-tA_\Omega} f\|_{L^p} \le Ct^{-\frac{|\alpha|}{2}-\frac{3}{2}(\frac{1}{q}-\frac{1}{p})}\|f\|_{L^q}$$

for all $\Omega \in \mathbb{R}$, $t > 0$ and $f \in L^q(\mathbb{R}^3)^3$ with $\nabla \cdot f = 0$.

Lemma 2.5 ([20, Lemma 3.2], [19, Lemma 2.5]). *Let $s \geq 0$, $2 \leq p < \infty$ and $1 \leq q \leq p'$. Then, there exists a constant $C = C(s,p,q) > 0$ such that*

$$\|e^{-tA_\Omega} f\|_{\dot{H}^s_p} \leq C t^{-\frac{s}{2} - \frac{3}{2}(\frac{1}{q} - \frac{1}{p})} (1 + |\Omega| t)^{-(1-\frac{2}{p})} \|f\|_{L^q}$$

for all $\Omega \in \mathbb{R}$, $t > 0$ and $f \in L^q(\mathbb{R}^3)^3$ with $\nabla \cdot f = 0$.

Lemma 2.6 ([22, Lemma 2.2]). *For $-\infty < s_0 \leq s_1 < \infty$, there exists a positive constant $C = C(s_0, s_1)$ such that*

$$\|e^{t\Delta} f\|_{\dot{B}^{s_1}_{p,q}} \leq C t^{-\frac{1}{2}(s_1 - s_0)} \|f\|_{\dot{B}^{s_0}_{p,q}}$$

for all $t > 0$, $1 \leq p, q \leq \infty$ and $f \in \dot{B}^{s_0}_{p,q}(\mathbb{R}^3)$.

3. Linear Decay Estimates and Asymptotics

In this section, we shall establish the temporal decay estimates for the linear solution $e^{-tA_\Omega} u_0$ when $u_0 \in L^1(\mathbb{R}^3)$ or $(1 + |x|) u_0 \in L^1(\mathbb{R}^3)$. Furthermore, we obtain the asymptotic profile of the linear solution $e^{-tA_\Omega} u_0$ as $t \to \infty$.

3.1. Linear Decay Estimates

Let us set

$$H_\Omega(\xi, t) := \cos\left(\Omega \frac{\xi_3}{|\xi|} t\right) I + \sin\left(\Omega \frac{\xi_3}{|\xi|} t\right) R(\xi), \quad K_\Omega(x,t) := \mathcal{F}^{-1}\left[e^{-t|\xi|^2} H_\Omega(\xi, t)\right](x), \tag{3.1}$$

where $R(\xi)$ is the skew-symmetric matrix defined by (1.19). Then, it follows from (2.3) that the linear solution $e^{-tA_\Omega} u_0$ can be written as

$$e^{-tA_\Omega} u_0(x) = K_\Omega(\cdot, t) * u_0(x) = \int_{\mathbb{R}^3} K_\Omega(x - y, t) u_0(y)\, dy. \tag{3.2}$$

We firstly show the L^p estimates for the integral kernel $K_\Omega(\cdot, t)$. Let $G_t(x)$ be the Gauss kernel in $\mathbb{R}^3$, which is defined as

$$G_t(x) := \frac{1}{(4\pi t)^{\frac{3}{2}}} e^{-\frac{|x|^2}{4t}}, \quad t > 0,\ x \in \mathbb{R}^3.$$

Note that it holds $\widehat{G_t}(\xi) = e^{-t|\xi|^2}$ and $G_1 \in \mathscr{S}(\mathbb{R}^3)$.

Lemma 3.1. *For $2 \leq p \leq \infty$ and $\alpha \in (\mathbb{N} \cup \{0\})^3$, there exists a positive constant $C = C(p, \alpha)$ such that*

$$\|\partial_x^\alpha K_\Omega(\cdot, t)\|_{L^p} \leq C t^{-\frac{|\alpha|}{2} - \frac{3}{2}(1 - \frac{1}{p})} (1 + |\Omega| t)^{-(1 - \frac{2}{p})} \tag{3.3}$$

for all $\Omega \in \mathbb{R}$ and all $t > 0$.

Proof. Since $H_\Omega(\xi, t)$ is homogeneous of degree 0 in ξ, the change of variable $\xi \mapsto \frac{\xi}{\sqrt{t}}$ gives

$$\begin{aligned}
\partial_x^\alpha K_\Omega(x,t) &= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} (i\xi)^\alpha e^{ix\cdot\xi} e^{-t|\xi|^2} H_\Omega(\xi, t)\, d\xi \\
&= \frac{t^{-\frac{|\alpha|}{2} - \frac{3}{2}}}{(2\pi)^3} \int_{\mathbb{R}^3} e^{ix \cdot \frac{\xi}{\sqrt{t}}} (i\xi)^\alpha e^{-|\xi|^2} H_\Omega(\xi, t)\, d\xi \\
&= t^{-\frac{|\alpha|}{2} - \frac{3}{2}} \mathcal{F}^{-1}\left[\widehat{\partial_x^\alpha G_1}(\xi) H_\Omega(\xi, t)\right]\left(\frac{x}{\sqrt{t}}\right).
\end{aligned} \tag{3.4}$$

Moreover, we directly calculate as

$$
\begin{aligned}
&\mathcal{F}^{-1}\left[\widehat{\partial_x^\alpha G_1}(\xi)H_\Omega(\xi,t)\right]\left(\frac{x}{\sqrt{t}}\right)\\
&= \int_{\mathbb{R}^3} e^{i\frac{x}{\sqrt{t}}\cdot\xi}\left\{\frac{1}{2}(e^{i\Omega\frac{\xi_3}{|\xi|}t}+e^{-i\Omega\frac{\xi_3}{|\xi|}t})I+\frac{1}{2i}(e^{i\Omega\frac{\xi_3}{|\xi|}t}-e^{-i\Omega\frac{\xi_3}{|\xi|}t})R(\xi)\right\}\widehat{\partial_x^\alpha G_1}(\xi)\,\frac{d\xi}{(2\pi)^3}\\
&= \frac{1}{2}\int_{\mathbb{R}^3} e^{i\frac{x}{\sqrt{t}}\cdot\xi}e^{i\Omega\frac{\xi_3}{|\xi|}t}(I-iR(\xi))\widehat{\partial_x^\alpha G_1}(\xi)\,\frac{d\xi}{(2\pi)^3}\\
&\quad+\frac{1}{2}\int_{\mathbb{R}^3} e^{i\frac{x}{\sqrt{t}}\cdot\xi}e^{-i\Omega\frac{\xi_3}{|\xi|}t}(I+iR(\xi))\widehat{\partial_x^\alpha G_1}(\xi)\,\frac{d\xi}{(2\pi)^3}\\
&= \frac{1}{2}\mathcal{G}_+(\Omega t)[(I+\mathcal{R})\partial_x^\alpha G_1]\left(\frac{x}{\sqrt{t}}\right)+\frac{1}{2}\mathcal{G}_-(\Omega t)[(I-\mathcal{R})\partial_x^\alpha G_1]\left(\frac{x}{\sqrt{t}}\right),
\end{aligned}
\tag{3.5}
$$

where $\mathcal{R}$ is defined in (2.4). Therefore, it follows from (3.4) to (3.5) that

$$
\begin{aligned}
&\|\partial_x^\alpha K_\Omega(\cdot,t)\|_{L^p}\\
&\leqslant \frac{t^{-\frac{|\alpha|}{2}-\frac{3}{2}\left(1-\frac{1}{p}\right)}}{2}\left\{\|\mathcal{G}_+(\Omega t)[(I+\mathcal{R})\partial_x^\alpha G_1]\|_{L^p}+\|\mathcal{G}_-(\Omega t)[(I-\mathcal{R})\partial_x^\alpha G_1]\|_{L^p}\right\}.
\end{aligned}
\tag{3.6}
$$

We first consider the case $2\le p<\infty$. It follows from Lemma 2.3 and the continuous embedding $\dot{B}^0_{p,2}(\mathbb{R}^3)\hookrightarrow L^p(\mathbb{R}^3)$ that

$$
\begin{aligned}
\|\mathcal{G}_\pm(\Omega t)[(I\pm\mathcal{R})\partial_x^\alpha G_1]\|_{L^p} &\leqslant C\,\|\mathcal{G}_\pm(\Omega t)[(I\pm\mathcal{R})\partial_x^\alpha G_1]\|_{\dot{B}^0_{p,2}}\\
&\leqslant C(1+|\Omega|t)^{-(1-\frac{2}{p})}\,\|(I\pm\mathcal{R})\partial_x^\alpha G_1\|_{\dot{B}^{3(1-\frac{2}{p})}_{p',2}}\\
&\leqslant C(1+|\Omega|t)^{-(1-\frac{2}{p})}\,\|G_1\|_{\dot{B}^{|\alpha|+3(1-\frac{2}{p})}_{p',2}}.
\end{aligned}
\tag{3.7}
$$

Since $G_1\in\mathscr{S}(\mathbb{R}^3)$, we see by (2.1) that

$$
\|G_1\|_{\dot{B}^{|\alpha|}_{2,2}}\simeq\|G_1\|_{\dot{H}^{|\alpha|}}<\infty\quad\text{and}\quad\|G_1\|_{\dot{B}^{|\alpha|+3(1-\frac{2}{p})}_{p',2}}\leqslant C\,\|G_1\|_{B^{|\alpha|+3(1-\frac{2}{p})}_{p',2}}<\infty
\tag{3.8}
$$

for $p=2$ and $2<p<\infty$, respectively. For the case $p=\infty$, we have by Lemma 2.3 and the continuous embedding $\dot{B}^0_{\infty,1}(\mathbb{R}^3)\hookrightarrow L^\infty(\mathbb{R}^3)$ that

$$
\begin{aligned}
\|\mathcal{G}_\pm(\Omega t)[(I\pm\mathcal{R})\partial_x^\alpha G_1]\|_{L^\infty} &\leqslant C\,\|\mathcal{G}_\pm(\Omega t)[(I\pm\mathcal{R})\partial_x^\alpha G_1]\|_{\dot{B}^0_{\infty,1}}\\
&\leqslant C(1+|\Omega|t)^{-1}\,\|(I\pm\mathcal{R})\partial_x^\alpha G_1\|_{\dot{B}^3_{1,1}}\\
&\leqslant C(1+|\Omega|t)^{-1}\,\|G_1\|_{\dot{B}^{|\alpha|+3}_{1,1}}
\end{aligned}
\tag{3.9}
$$

and $\|G_1\|_{\dot{B}^{|\alpha|+3}_{1,1}}\le C\,\|G_1\|_{B^{|\alpha|+3}_{1,1}}<\infty$. Therefore, we obtain from (3.6) to (3.9) that

$$
\|\partial_x^\alpha K_\Omega(\cdot,t)\|_{L^p}\le Ct^{-\frac{|\alpha|}{2}-\frac{3}{2}(1-\frac{1}{p})}(1+|\Omega|t)^{-(1-\frac{2}{p})}
$$

for all $t>0$. This completes the proof of Lemma 3.1. □

Applying Lemma 3.1, we show the following L^1-L^p temporal decay estimates for the linear semigroup e^{-tA_Ω}.

Lemma 3.2. (1) *For $2\le p\le\infty$ and $\alpha\in(\mathbb{N}\cup\{0\})^3$, there exists a positive constant $C=C(p,\alpha)$ such that*

$$
\|\partial_x^\alpha e^{-tA_\Omega}f\|_{L^p}\le Ct^{-\frac{|\alpha|}{2}-\frac{3}{2}(1-\frac{1}{p})}(1+|\Omega|t)^{-(1-\frac{2}{p})}\|f\|_{L^1}
\tag{3.10}
$$

for all $t>0$, $\Omega\in\mathbb{R}$ and $f\in L^1(\mathbb{R}^3)^3$ satisfying $\nabla\cdot f=0$.

(2) *For $2 \le p \le \infty$ and $\alpha \in (\mathbb{N} \cup \{0\})^3$, there exists a positive constant $C = C(p, \alpha)$ such that*

$$\|\partial_x^\alpha e^{-tA_\Omega} f\|_{L^p} \le C t^{-\frac{|\alpha|+1}{2} - \frac{3}{2}(1-\frac{1}{p})} (1 + |\Omega| t)^{-(1-\frac{2}{p})} \||x| f\|_{L^1} \tag{3.11}$$

for all $t > 0$, $\Omega \in \mathbb{R}$ and $f \in L^1(\mathbb{R}^3)^3$ satisfying $\nabla \cdot f = 0$ and $|x| f \in L^1(\mathbb{R}^3)^3$.

Remark 3.3. We remark that the temporal decay estimate (3.10) for $f \in L^1(\mathbb{R}^3)$ has already been shown by Kim [19, Lemma 2.5]. Here, we shall give an alternative proof by using Lemma 3.1.

Proof of Lemma 3.2. (1) Applying the Hausdorff-Young inequality and Lemma 3.1, we have by (3.2) that

$$\|\partial_x^\alpha e^{-tA_\Omega} f\|_{L^p} \le \|\partial_x^\alpha K_\Omega(\cdot, t)\|_{L^p} \|f\|_{L^1} \le C t^{-\frac{|\alpha|}{2} - \frac{3}{2}(1-\frac{1}{p})} (1 + |\Omega| t)^{-(1-\frac{2}{p})} \|f\|_{L^1}.$$

(2) Since $f \in L^1(\mathbb{R}^3)^3$ and $\nabla \cdot f = 0$, we see that it holds

$$\widehat{f}(0) = \int_{\mathbb{R}^3} f(y)\, dy = 0. \tag{3.12}$$

Then, applying the mean value theorem, the Minkowski inequality and Lemma 3.1, we obtain

$$\begin{aligned}
\|\partial_x^\alpha e^{-tA_\Omega} f\|_{L^p} &= \left\| \int_{\mathbb{R}^3} \partial_x^\alpha K_\Omega(\cdot - y, t) f(y)\, dy \right\|_{L^p} \\
&= \left\| \int_{\mathbb{R}^3} \{\partial_x^\alpha K_\Omega(\cdot - y, t) - \partial_x^\alpha K_\Omega(\cdot, t)\} f(y)\, dy \right\|_{L^p} \\
&= \left\| \int_{\mathbb{R}^3} \sum_{j=1}^3 \int_0^1 \partial_{x_j} \partial_x^\alpha K_\Omega(\cdot - \theta y, t)(-y_j)\, d\theta f(y)\, dy \right\|_{L^p} \\
&\le \sum_{j=1}^3 \int_{\mathbb{R}^3} \int_0^1 \|\partial_{x_j} \partial_x^\alpha K_\Omega(\cdot - \theta y, t)\|_{L^p}\, d\theta |y_j| |f(y)|\, dy \\
&= \sum_{j=1}^3 \|\partial_{x_j} \partial_x^\alpha K_\Omega(\cdot, t)\|_{L^p} \int_{\mathbb{R}^3} |y_j| |f(y)|\, dy \\
&\le C t^{-\frac{|\alpha|+1}{2} - \frac{3}{2}(1-\frac{1}{p})} (1 + |\Omega| t)^{-(1-\frac{2}{p})} \||x| f\|_{L^1}.
\end{aligned}$$

This completes the proof of Lemma 3.2. □

3.2. Linear Asymptotics

In this subsection, we shall show the following asymptotic profiles of the linear solution $e^{-tA_\Omega} u_0$ as t goes to infinity.

Theorem 3.4. (1) *Suppose that u_0 satisfies $u_0 \in L^1(\mathbb{R}^3)^3$ and $\nabla \cdot u_0 = 0$. Then, for $2 \le p \le \infty$, it holds that*

$$\lim_{t \to \infty} (1 + |\Omega| t)^{1-\frac{2}{p}} t^{\frac{3}{2}(1-\frac{1}{p})} \|e^{-tA_\Omega} u_0\|_{L^p} = 0. \tag{3.13}$$

(2) *Let $m \in \mathbb{N}$. Suppose that u_0 satisfies $(1 + |x|)^m u_0 \in L^1(\mathbb{R}^3)^3$ and $\nabla \cdot u_0 = 0$. Then, for $2 \le p \le \infty$, it holds that*

$$\begin{aligned}
&\lim_{t \to \infty} (1 + |\Omega| t)^{1-\frac{2}{p}} t^{\frac{m}{2} + \frac{3}{2}(1-\frac{1}{p})} \left\| e^{-tA_\Omega} u_0 - \sum_{1 \leqslant |\alpha| \leqslant m} \frac{(-1)^{|\alpha|}}{\alpha!} (\partial_x^\alpha K_\Omega)(\cdot, t) \int_{\mathbb{R}^3} y^\alpha u_0(y)\, dy \right\|_{L^p} \\
&= 0.
\end{aligned} \tag{3.14}$$

Proof. (1) Since $u_0 \in L^1(\mathbb{R}^3)^3$ and $\nabla \cdot u_0 = 0$, it holds $\int_{\mathbb{R}^3} u_0(y)\,dy = 0$ and then

$$e^{-tA_\Omega}u_0(x) = \int_{\mathbb{R}^3} \{K_\Omega(x-y,t) - K_\Omega(x,t)\}u_0(y)\,dy.$$

Similarly to (3.4) and (3.5), we have

$$K_\Omega(x,t) = \frac{t^{-\frac{3}{2}}}{2} \sum_{\sigma\in\{\pm\}} \mathcal{G}_\sigma(\Omega t)\,[(I+\sigma\mathcal{R})G_1]\left(\frac{x}{\sqrt{t}}\right). \tag{3.15}$$

Therefore, we see that

$$\begin{aligned} &\|e^{-tA_\Omega}u_0\|_{L^p} \\ &\le \frac{t^{-\frac{3}{2}}}{2} \sum_{\sigma\in\{\pm\}} \int_{\mathbb{R}^3} \Bigg\| \mathcal{G}_\sigma(\Omega t)\,[(I+\sigma\mathcal{R})G_1]\left(\frac{\cdot - y}{\sqrt{t}}\right) \\ &\quad - \mathcal{G}_\sigma(\Omega t)\,[(I+\sigma\mathcal{R})G_1]\left(\frac{\cdot}{\sqrt{t}}\right)\Bigg\|_{L^p} |u_0(y)|\,dy \\ &= \frac{t^{-\frac{3}{2}(1-\frac{1}{p})}}{2} \sum_{\sigma\in\{\pm\}} \int_{\mathbb{R}^3} \Bigg\| \mathcal{G}_\sigma(\Omega t)\,[(I+\sigma\mathcal{R})G_1]\left(\cdot - \frac{y}{\sqrt{t}}\right) \\ &\quad - \mathcal{G}_\sigma(\Omega t)\,[(I+\sigma\mathcal{R})G_1]\,(\cdot)\Bigg\|_{L^p} |u_0(y)|\,dy \\ &= \frac{t^{-\frac{3}{2}(1-\frac{1}{p})}}{2} \sum_{\sigma\in\{\pm\}} \int_{\mathbb{R}^3} \left\| \mathcal{G}_\sigma(\Omega t)\left[(I+\sigma\mathcal{R})\left\{G_1(\cdot - yt^{-\frac{1}{2}}) - G_1(\cdot)\right\}\right]\right\|_{L^p} |u_0(y)|\,dy. \end{aligned} \tag{3.16}$$

In the case $p=2$, we have (3.16) that

$$t^{\frac{3}{4}}\|e^{-tA_\Omega}u_0\|_{L^2} \le C\int_{\mathbb{R}^3} \left\|G_1(\cdot - yt^{-\frac{1}{2}}) - G_1(\cdot)\right\|_{L^2} |u_0(y)|\,dy.$$

Hence the desired result (3.13) follows from the dominated convergence theorem.

Next we consider the case $2<p<\infty$. Applying the embedding $\dot{B}^0_{p,2}(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$ and Lemma 2.3 to (3.16), we have

$$\begin{aligned} &\|e^{-tA_\Omega}u_0\|_{L^p} \\ &\le Ct^{-\frac{3}{2}(1-\frac{1}{p})} \sum_{\sigma\in\{\pm\}} \int_{\mathbb{R}^3} \left\| \mathcal{G}_\sigma(\Omega t)\left[(I+\sigma\mathcal{R})\left\{G_1(\cdot - yt^{-\frac{1}{2}}) - G_1(\cdot)\right\}\right]\right\|_{\dot{B}^0_{p,2}} |u_0(y)|\,dy \\ &\le Ct^{-\frac{3}{2}(1-\frac{1}{p})}(1+|\Omega|t)^{-(1-\frac{2}{p})} \int_{\mathbb{R}^3} \left\|G_1(\cdot - yt^{-\frac{1}{2}}) - G_1(\cdot)\right\|_{\dot{B}^{3(1-\frac{2}{p})}_{p',2}} |u_0(y)|\,dy. \end{aligned} \tag{3.17}$$

Here, since it holds

$$G_1(x-yt^{-\frac{1}{2}}) - G_1(x) = e^{\frac{\Delta}{2}}\left[G_{\frac{1}{2}}(\cdot - yt^{-\frac{1}{2}}) - G_{\frac{1}{2}}(\cdot)\right](x),$$

it follows from Lemma 2.6 and the embedding $L^{p'}(\mathbb{R}^3) \hookrightarrow \dot{B}^0_{p',2}(\mathbb{R}^3)$ that

$$\begin{aligned} \left\|G_1(\cdot - yt^{-\frac{1}{2}}) - G_1(\cdot)\right\|_{\dot{B}^{3(1-\frac{2}{p})}_{p',2}} &\le C\left(\frac{1}{2}\right)^{-\frac{3}{2}(1-\frac{2}{p})} \left\|G_{\frac{1}{2}}(\cdot - yt^{-\frac{1}{2}}) - G_{\frac{1}{2}}(\cdot)\right\|_{\dot{B}^0_{p',2}} \\ &\le C\left\|G_{\frac{1}{2}}(\cdot - yt^{-\frac{1}{2}}) - G_{\frac{1}{2}}(\cdot)\right\|_{L^{p'}}. \end{aligned} \tag{3.18}$$

Hence we have by (3.17) and (3.18) that

$$t^{\frac{3}{2}(1-\frac{1}{p})}(1+|\Omega|t)^{1-\frac{2}{p}}\|e^{-tA_\Omega}u_0\|_{L^p} \le C\int_{\mathbb{R}^3} \left\|G_{\frac{1}{2}}(\cdot - yt^{-\frac{1}{2}}) - G_{\frac{1}{2}}(\cdot)\right\|_{L^{p'}} |u_0(y)|\,dy.$$

Then, the dominated convergence theorem yields (3.13) for $2 < p < \infty$.

Finally, we treat the case $p = \infty$. Similarly to (3.17), we apply the embedding $\dot{B}^0_{\infty,1}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$ and Lemma 2.3 to (3.16) to obtain

$$\|e^{-tA_\Omega}u_0\|_{L^\infty} \le Ct^{-\frac{3}{2}}(1+|\Omega|t)^{-1}\int_{\mathbb{R}^3}\left\|G_1(\cdot - yt^{-\frac{1}{2}}) - G_1(\cdot)\right\|_{\dot{B}^3_{1,1}} |u_0(y)|\,dy.$$

Note that it holds

$$\|f\|_{\dot{B}^3_{1,1}} \le C\|f\|_{B^3_{1,1}} \le C\|f\|_{B^4_{1,\infty}} \simeq C\|(1-\Delta)^2 f\|_{B^0_{1,\infty}} \le C\|(1-\Delta)^2 f\|_{L^1}.$$

Hence we have by the dominated convergence theorem that

$$\begin{aligned}&t^{\frac{3}{2}}(1+|\Omega|t)\|e^{-tA_\Omega}u_0\|_{L^\infty}\\ &\quad\le C\int_{\mathbb{R}^3}\left\|(1-\Delta)^2G_1(\cdot - yt^{-\frac{1}{2}}) - (1-\Delta)^2G_1(\cdot)\right\|_{L^1}|u_0(y)|\,dy \to 0\end{aligned}$$

as $t \to \infty$. This completes the proof of Theorem 3.4 (1).

(2) By the Taylor theorem, we have

$$\begin{aligned}&K_\Omega(x-y,t) - K_\Omega(x,t)\\ &\quad= \sum_{1\le|\alpha|\le m-1}\frac{\partial_x^\alpha K_\Omega(x,t)}{\alpha!}(-y)^\alpha + \sum_{|\alpha|=m}\frac{m}{\alpha!}\int_0^1(1-\theta)^{m-1}\partial_x^\alpha K_\Omega(x-\theta y,t)\,d\theta(-y)^\alpha\\ &\quad= \sum_{1\le|\alpha|\le m}\frac{(-1)^{|\alpha|}}{\alpha!}\partial_x^\alpha K_\Omega(x,t)y^\alpha\\ &\qquad+ \sum_{|\alpha|=m}\frac{m}{\alpha!}\int_0^1(1-\theta)^{m-1}\left\{\partial_x^\alpha K_\Omega(x-\theta y,t) - \partial_x^\alpha K_\Omega(x,t)\right\}d\theta(-y)^\alpha.\end{aligned} \tag{3.19}$$

Then, since it holds $\int_{\mathbb{R}^3}u_0(y)\,dy = 0$, we have by (3.19) that

$$\begin{aligned}&e^{-tA_\Omega}u_0(x)\\ &\quad= \int_{\mathbb{R}^3}\{K_\Omega(x-y,t) - K_\Omega(x,t)\}u_0(y)\,dy\\ &\quad= \sum_{1\le|\alpha|\le m}\frac{(-1)^{|\alpha|}}{\alpha!}\partial_x^\alpha K_\Omega(x,t)\int_{\mathbb{R}^3}y^\alpha u_0(y)\,dy\\ &\qquad+ \sum_{|\alpha|=m}\frac{m}{\alpha!}\int_{\mathbb{R}^3}\int_0^1(1-\theta)^{m-1}\left\{\partial_x^\alpha K_\Omega(x-\theta y,t) - \partial_x^\alpha K_\Omega(x,t)\right\}d\theta(-y)^\alpha u_0(y)\,dy,\end{aligned}$$

which yields

$$\begin{aligned}&\left\|e^{-tA_\Omega}u_0 - \sum_{1\le|\alpha|\le m}\frac{(-1)^{|\alpha|}}{\alpha!}\partial_x^\alpha K_\Omega(\cdot,t)\int_{\mathbb{R}^3}y^\alpha u_0(y)\,dy\right\|_{L^p}\\ &\quad\le C_m\sum_{|\alpha|=m}\int_{\mathbb{R}^3}\int_0^1\|\partial_x^\alpha K_\Omega(\cdot-\theta y,t) - \partial_x^\alpha K_\Omega(\cdot,t)\|_{L^p}\,|y|^m|u_0(y)|\,d\theta dy.\end{aligned} \tag{3.20}$$

Similarly to (3.4) and (3.5), we have

$$\partial_x^\alpha K_\Omega(x,t) = \frac{1}{2}t^{-\frac{|\alpha|}{2}-\frac{3}{2}}\sum_{\sigma\in\{\pm\}}\mathcal{G}_\sigma(\Omega t)\left[(I+\sigma\mathcal{R})\partial_x^\alpha G_1\right]\left(\frac{x}{\sqrt{t}}\right). \tag{3.21}$$

Therefore, similarly to (3.16), we obtain by (3.20) and (3.21) that

$$\begin{aligned} &t^{\frac{m}{2}+\frac{3}{2}(1-\frac{1}{p})}\left\| e^{-tA_\Omega}u_0 - \sum_{1\le|\alpha|\le m}\frac{(-1)^{|\alpha|}}{\alpha!}\partial_x^\alpha K_\Omega(\cdot,t)\int_{\mathbb{R}^3} y^\alpha u_0(y)\,dy\right\|_{L^p} \\ &\le C_m \sum_{|\alpha|=m}\sum_{\sigma\in\{\pm\}}\int_{\mathbb{R}^3}\int_0^1 \left\|\mathcal{G}_\sigma(\Omega t)\left[(I+\sigma\mathcal{R})\left\{\partial_x^\alpha G_1(\cdot - y\theta t^{-\frac{1}{2}}) - \partial_x^\alpha G_1(\cdot)\right\}\right]\right\|_{L^p} \\ &\quad \times |y|^m|u_0(y)|\,d\theta dy. \end{aligned} \tag{3.22}$$

Then, since $|y|^m|u_0(y)|$ is in $L^1(\mathbb{R}^3_y\times(0,1)_\theta)$, we can apply the exactly same arguments as Theorem 3.4 (1) to (3.22), and obtain the desired asymptotics result. □

4. Nonlinear Decay Estimates

In this section, we adapt the ideas in [1,29], and show the L^p temporal decay estimates for the global solution u to (1.8).

4.1. L^2-decay Estimates

Lemma 4.1. *Suppose that the exponents s,q,θ and the initial data $u_0\in(\dot{H}^s\cap L^1)(\mathbb{R}^3)^3$ satisfy the asumptions in Theorem 1.1, and let $u\in C([0,\infty);\dot{H}^s(\mathbb{R}^3))^3$ be the unique global solution to (1.8) constructed in Theorem 1.1. Then, there exists a absolute constant $C>0$ such that*

$$\begin{aligned} &\|u(t)\|_{L^2}^2 e^{\int_0^t g(r)^2dr} \\ &\quad\le \|u_0\|_{L^2}^2 + C\int_0^t g(s)^2 e^{\int_0^s g(r)^2dr}\left\{\|e^{s\Delta}u_0\|_{L^2}^2 + g(s)^5\left(\int_0^s \|u(r)\|_{L^2}^2dr\right)^2\right\}ds \end{aligned}$$

for all $t\ge 0$ and all bounded positive function $g\in C([0,\infty);(0,\infty))$.

Proof. We remark that the inequality in Lemma 4.1 was obtained by Wiegner [29, (2.1)] for global weak solutions to the original Navier–Stokes equations. Since $u_0\in(\dot{H}^s\cap L^1)(\mathbb{R}^3)^3$, we see by the Sobolev embedding $\dot{H}^s(\mathbb{R}^3)\hookrightarrow L^q(\mathbb{R}^3)$ with $\frac{1}{q}=\frac{1}{2}-\frac{s}{3}$ and the interpolation inequality that

$$\|u_0\|_{L^2}\le \|u_0\|_{L^1}^{\frac{2s}{3+2s}}\|u_0\|_{L^q}^{\frac{3}{3+2s}} \le C\|u_0\|_{L^1}^{\frac{2s}{3+2s}}\|u_0\|_{\dot{H}^s}^{\frac{3}{3+2s}}. \tag{4.1}$$

Hence the solution u also belongs to $C([0,\infty);L^2(\mathbb{R}^3))^3$. Since there hold

$$\langle \Omega\mathbb{P}J\mathbb{P}u,u\rangle_{L^2}=0,\quad \left|\widehat{e^{-tA_\Omega}u_0}(\xi)\right| = \left|\widehat{e^{t\Delta}u_0}(\xi)\right|,$$

we can apply the same argument as [29, (2.1)] and obtain the desired inequality. □

Lemma 4.2. *Suppose that the exponents s,q,θ and the initial data $u_0\in(\dot{H}^s\cap L^1)(\mathbb{R}^3)^3$ satisfy the asumptions in Theorem 1.1, and let $u\in C([0,\infty);\dot{H}^s(\mathbb{R}^3))^3$ be the unique global solution to (1.8) constructed in Theorem 1.1.*

(1) *There exists a positive constant $C=C(\|u_0\|_{L^1},\|u_0\|_{L^2})$ such that*

$$\|u(t)\|_{L^2}\le C(1+t)^{-\frac{3}{4}}$$

for all $t>0$.

(2) *Assume further that* $|x|u_0 \in L^1(\mathbb{R}^3)^3$. *Then, there exists a positive constant* $C = C(\||x|u_0\|_{L^1}, \|u_0\|_{L^2})$ *such that*

$$\|u(t)\|_{L^2} \le C(1+t)^{-\frac{5}{4}}$$

for all $t > 0$.

Proof. We follow the same argument as [29]. Firstly, suppose that there hold

$$\|e^{t\Delta}u_0\|_{L^2} \le C_1(1+t)^{-\frac{\alpha_0}{2}}, \quad \|u(t)\|_{L^2} \leqslant C_2(1+t)^{-\frac{\beta}{2}} \tag{4.2}$$

with some $\alpha_0 > 0$ and $0 \le \beta < 1$. Then, take an exponent α and a function $g(t)$ so that

$$\alpha > \max\left\{\alpha_0,\, \frac{1}{2} + 2\beta\right\}, \quad g(t) = \frac{\alpha^{\frac{1}{2}}}{(1+t)^{\frac{1}{2}}}.$$

Then, we see that $\int_0^t g(r)^2 dr = \log(1+t)^\alpha$, and Lemma 4.1 and (4.2) yield

$$\begin{aligned}
&\|u(t)\|_{L^2}(1+t)^\alpha \\
&\le \|u_0\|_{L^2}^2 + C\int_0^t \frac{\alpha}{1+s}(1+s)^\alpha \left\{\|e^{s\Delta}u_0\|_{L^2}^2 + \frac{\alpha^{\frac{5}{2}}}{(1+s)^{\frac{5}{2}}}\left(\int_0^s \|u(r)\|_{L^2}^2\, dr\right)^2\right\} ds \\
&\le \|u_0\|_{L^2}^2 + C\alpha\int_0^t (1+s)^{\alpha-1}\left\{C_1^2(1+s)^{-\alpha_0} + \frac{\alpha^{\frac{5}{2}}C_2^4}{(1+s)^{\frac{5}{2}}}\left(\int_0^s (1+r)^{-\beta}\, dr\right)^2\right\} ds \\
&\le \|u_0\|_{L^2}^2 + CC_1^2\alpha\int_0^t (1+s)^{\alpha-\alpha_0-1}\, ds + \frac{CC_2^4\alpha^{\frac{7}{2}}}{(1-\beta)^2}\int_0^t (1+s)^{\alpha-(\frac{1}{2}+2\beta)-1}\, ds \\
&\le \|u_0\|_{L^2}^2 + \frac{CC_1^2\alpha}{\alpha-\alpha_0}(1+t)^{\alpha-\alpha_0} + \frac{CC_2^4\alpha^{\frac{7}{2}}}{(1-\beta)^2\left\{\alpha - (\frac{1}{2}+2\beta)\right\}}(1+t)^{\alpha-(\frac{1}{2}+2\beta)}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
\|u(t)\|_{L^2}^2 &\le \|u_0\|_{L^2}^2(1+t)^{-\alpha} + CC_1^2(1+t)^{-\alpha_0} + CC_2^4(1+t)^{-(\frac{1}{2}+2\beta)} \\
&\le \left\{\|u_0\|_{L^2} + C(C_1^2 + C_2^4)\right\}(1+t)^{-\widetilde{\beta}},
\end{aligned} \tag{4.3}$$

where

$$\widetilde{\beta} = \min\left\{\alpha_0,\, \frac{1}{2} + 2\beta\right\}. \tag{4.4}$$

(1) Let $u_0 \in (\dot{H}^s \cap L^1)(\mathbb{R}^3)^3$. Note that it holds $u_0 \in L^2(\mathbb{R}^3)^3$ by (4.1). Hence it follows from the smoothing estimates for the heat semigroup that

$$\|e^{t\Delta}u_0\|_{L^2} \le Ct^{-\frac{3}{4}}\|u_0\|_{L^1}, \quad \|e^{t\Delta}u_0\|_{L^2} \le \|u_0\|_{L^2},$$

which yield

$$\|e^{t\Delta}u_0\|_{L^2} \le C(\|u_0\|_{L^1} + \|u_0\|_{L^2})(1+t)^{-\frac{3}{4}} \tag{4.5}$$

for all $t \ge 0$. Moreover, taking the L^2 inner product of (1.8) with $u(t)$ gives

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^2 = 0.$$

Hence we have the energy equality:

$$\|u(t)\|_{L^2}^2 + 2\int_0^t \|\nabla u(\tau)\|_{L^2}^2\, d\tau = \|u_0\|_{L^2}^2, \tag{4.6}$$

which yields the estimate $\|u(t)\|_{L^2} \le \|u_0\|_{L^2}$. Therefore, the estimates (4.2) hold for $\alpha_0 = 3/2$ and $\beta = 0$, and then we have by (4.3) and (4.4) that

$$\|u(t)\|_{L^2}^2 \le C(1+t)^{-\frac{1}{2}}$$

with some constant $C = C(\|u_0\|_{L^1}, \|u_0\|_{L^2}) > 0$. Again, applying (4.3) and (4.4) with $\alpha_0 = 3/2$ and $\beta = 1/2$, we obtain

$$\|u(t)\|_{L^2}^2 \le C(1+t)^{-\frac{3}{2}} \tag{4.7}$$

with some constant $C = C(\|u_0\|_{L^1}, \|u_0\|_{L^2}) > 0$.

(2) Assume further that $|x|u_0 \in L^1(\mathbb{R}^3)^3$. Then, since there hold

$$\|e^{t\Delta}u_0\|_{L^2} \le Ct^{-\frac{5}{4}} \||x|u_0\|_{L^1}, \quad \|e^{t\Delta}u_0\|_{L^2} \le \|u_0\|_{L^2},$$

we have

$$\|e^{t\Delta}u_0\|_{L^2} \le C(\||x|u_0\|_{L^1} + \|u_0\|_{L^2})(1+t)^{-\frac{5}{4}} \tag{4.8}$$

for all $t \ge 0$. Hence, we see that the estimates (4.2) hold for $\alpha_0 = 5/2$ and $\beta = 0$. Then, (4.3) and (4.4) give

$$\|u(t)\|_{L^2}^2 \le C(1+t)^{-\frac{1}{2}}$$

with some constant $C = C(\||x|u_0\|_{L^1}, \|u_0\|_{L^2}) > 0$. Then, similarly to (4.7), we have

$$\|u(t)\|_{L^2}^2 \le C(1+t)^{-\frac{3}{2}} \tag{4.9}$$

for all $t \ge 0$. Here, we remark that (4.9) gives

$$\int_0^\infty \|u(r)\|_{L^2}^2 \, dr \le C \int_0^t \frac{1}{(1+r)^{\frac{3}{2}}} \, dr \le C < \infty. \tag{4.10}$$

Now, take an exponent α and a function $g(t)$ so that

$$\alpha > \frac{5}{2}, \quad g(t) = \frac{\alpha^{\frac{1}{2}}}{(1+t)^{\frac{1}{2}}}.$$

Then, it follows from Lemma 4.1, (4.8) and (4.10) that

$$\begin{aligned}
&\|u(t)\|_{L^2}(1+t)^\alpha \\
&\le \|u_0\|_{L^2}^2 + C\int_0^t \frac{\alpha}{1+s}(1+s)^\alpha \left\{ \|e^{s\Delta}u_0\|_{L^2}^2 + \frac{\alpha^{\frac{5}{2}}}{(1+s)^{\frac{5}{2}}} \left(\int_0^s \|u(r)\|_{L^2}^2 \, dr \right)^2 \right\} ds \\
&\le \|u_0\|_{L^2}^2 + C\alpha \int_0^t (1+s)^{\alpha - \frac{7}{2}} \, ds \\
&\le \|u_0\|_{L^2}^2 + C(1+t)^{\alpha - \frac{5}{2}},
\end{aligned}$$

which yields

$$\begin{aligned}
\|u(t)\|_{L^2}^2 &\le \|u_0\|_{L^2}^2 (1+t)^{-\alpha} + C(1+t)^{-\frac{5}{2}} \\
&\le C(1+t)^{-\frac{5}{2}}
\end{aligned}$$

for all $t \geqslant 0$. This completes the proof of Lemma 4.2.

□

4.2. L^p-decay Estimates

In this subsection, we adapt the arguments in [1] and show the L^p temporal decay estimates for the solution u to (1.8). For $1 \le p \le \infty$ and $j = 0, 1$, we put

$$\|u\|_{X_j^p(t)} := \sup_{0<\tau\le t} \tau^{\frac{j}{2} + \frac{3}{2}(1-\frac{1}{p})} (1 + |\Omega|\tau)^{1-\frac{2}{p}} \|u(\tau)\|_{L^p}, \quad t > 0.$$

Lemma 4.3. *Suppose that the exponents s, q and θ satisfy (1.10)–(1.11), and that the exponent p satisfies*

$$\frac{1}{2} - \frac{1}{2}\left(\frac{1}{q} - \frac{s}{3}\right) \le \frac{1}{p} \le \frac{1}{q}\left(< \frac{1}{2}\right). \tag{4.11}$$

Let $C_* > 0$ be the constant in (1.12). Then, there exists a constant $0 < C_{**} \le C_*$ such that for any $u_0 \in \dot{H}^s(\mathbb{R}^3)^3$ with $\nabla \cdot u_0 = 0$ and $\Omega \in \mathbb{R} \setminus \{0\}$ satisfying

$$\|u_0\|_{\dot{H}^s(\mathbb{R}^3)} \le C_{**}|\Omega|^{\frac{s}{2}-\frac{1}{4}}, \tag{4.12}$$

the unique global solution $u \in C([0,\infty); \dot{H}^s(\mathbb{R}^3))^3 \cap L^\theta(0,\infty; \dot{H}^s_q(\mathbb{R}^3))^3$ constructed in Theorem 1.1 satisfies

$$\int_{\frac{t}{2}}^t \|e^{-(t-\tau)A_\Omega}\mathbb{P}\nabla\cdot(u\otimes u)(\tau)\|_{L^p}\,d\tau \le \frac{1}{2}\|u\|_{X^p_j(t)} t^{-\frac{j}{2}-\frac{3}{2}(1-\frac{1}{p})}(1+|\Omega|t)^{-(1-\frac{2}{p})}$$

for all $t > 0$ and $j = 0, 1$.

Proof. Let us set

$$\frac{1}{q_s} = \frac{1}{q} - \frac{s}{3}, \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q_s}.$$

Then, we see that the exponents p and r satisfy $1 < r \le p' < 2 < p < \infty$. Applying Lemma 2.5 and the embedding $\dot{H}^s_q(\mathbb{R}^3) \hookrightarrow L^{q_s}(\mathbb{R}^3)$ gives

$$\begin{aligned}
&\int_{\frac{t}{2}}^t \|e^{-(t-\tau)A_\Omega}\mathbb{P}\,\mathrm{div}(u\otimes u)(\tau)\|_{L^p}\,d\tau \\
&\le C\int_{\frac{t}{2}}^t \{1+|\Omega|(t-\tau)\}^{-(1-\frac{2}{p})}(t-\tau)^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{r}-\frac{1}{p})}\|(u\otimes u)(\tau)\|_{L^r}\,d\tau \\
&\le C\int_{\frac{t}{2}}^t \{1+|\Omega|(t-\tau)\}^{-(1-\frac{2}{p})}(t-\tau)^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{q}-\frac{s}{3})}\|u(\tau)\|_{L^p}\|u(\tau)\|_{L^{q_s}}\,d\tau \\
&\le C\int_{\frac{t}{2}}^t \{1+|\Omega|(t-\tau)\}^{-(1-\frac{2}{p})}(t-\tau)^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{q}-\frac{s}{3})}\|u(\tau)\|_{\dot{H}^s_q} \\
&\qquad \times \tau^{-\frac{j}{2}-\frac{3}{2}(1-\frac{1}{p})}(1+|\Omega|\tau)^{-(1-\frac{2}{p})}\tau^{\frac{j}{2}+\frac{3}{2}(1-\frac{1}{p})}(1+|\Omega|\tau)^{1-\frac{2}{p}}\|u(\tau)\|_{L^p}\,d\tau \\
&\le C\|u\|_{X^p_j(t)}(1+|\Omega|t)^{-(1-\frac{2}{p})}t^{-\frac{j}{2}-\frac{3}{2}(1-\frac{1}{p})} \\
&\qquad \times \int_{\frac{t}{2}}^t \{1+|\Omega|(t-\tau)\}^{-(1-\frac{2}{p})}(t-\tau)^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{q}-\frac{s}{3})}\|u(\tau)\|_{\dot{H}^s_q}\,d\tau.
\end{aligned} \tag{4.13}$$

Here, let us set

$$h_\Omega(t) := (1+|\Omega|t)^{-(1-\frac{2}{p})}t^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{q}-\frac{s}{3})}, \quad t > 0.$$

The direct calculation gives that

$$\begin{aligned}
\|h_\Omega\|_{L^{\theta'}(0,\infty)} &= |\Omega|^{\frac{1}{2}+\frac{3}{2}(\frac{1}{q}-\frac{s}{3})-\frac{1}{\theta'}}\left(\int_0^\infty \frac{1}{\tau^{\theta'\{\frac{1}{2}+\frac{3}{2}(\frac{1}{q}-\frac{s}{3})\}}}\frac{1}{(1+\tau)^{\theta'(1-\frac{2}{p})}}\,d\tau\right)^{\frac{1}{\theta'}} \\
&= C|\Omega|^{\frac{1}{\theta}+\frac{3}{2q}-\frac{s}{2}-\frac{1}{2}} < \infty.
\end{aligned} \tag{4.14}$$

Indeed, since $s > 1/2$, the assumption (1.11) on θ implies

$$\frac{1}{\theta} < \frac{5}{8} - \frac{3}{2q} + \frac{s}{4} < \frac{1}{2} - \frac{3}{2q} + \frac{s}{2}.$$

This yields $\theta'\{\frac{1}{2}+\frac{3}{2}(\frac{1}{q}-\frac{s}{3})\}<1$. Also, by $1/p \le 1/q$ and (1.11), we have

$$\frac{1}{\theta} > \frac{1}{2q}+\frac{s}{2}-\frac{1}{2} \ge \frac{2}{p}-\frac{3}{2q}+\frac{s}{2}-\frac{1}{2},$$

which implies $\theta'\{\frac{1}{2}+\frac{3}{2}(\frac{1}{q}-\frac{s}{3})+(1-\frac{2}{p})\}>1$. Therefore, it follows from the Hölder inequality, (1.13) and (4.14) that

$$\begin{aligned}
&\int_{\frac{t}{2}}^{t}\{1+|\Omega|(t-\tau)\}^{-(1-\frac{2}{p})}(t-\tau)^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{q}-\frac{s}{3})}\|u(\tau)\|_{\dot H^s_q}\,d\tau\\
&\quad\le \|h_\Omega\|_{L^{\theta'}(0,\infty)}\|u\|_{L^\theta(t/2,t;\dot H^s_q)}\\
&\quad\le C|\Omega|^{\frac{1}{\theta}+\frac{3}{2q}-\frac{s}{2}-\frac{1}{2}}\cdot|\Omega|^{-\frac{1}{\theta}+\frac{3}{4}(1-\frac{2}{q})}\|u_0\|_{\dot H^s}\\
&\quad= C|\Omega|^{-\frac{1}{2}(s-\frac{1}{2})}\|u_0\|_{\dot H^s}.
\end{aligned}\tag{4.15}$$

Hence we obtain from (4.13) to (4.15) that

$$\begin{aligned}
&\int_{\frac{t}{2}}^{t}\|e^{-(t-\tau)A_\Omega}\mathbb{P}\,\mathrm{div}(u\otimes u)(\tau)\|_{L^p}\,d\tau\\
&\quad\le C|\Omega|^{-\frac{1}{2}(s-\frac{1}{2})}\|u_0\|_{\dot H^s}\|u\|_{X^p_j(t)}t^{-\frac{j}{2}-\frac{3}{2}(1-\frac{1}{p})}(1+|\Omega|t)^{-(1-\frac{2}{p})}\\
&\quad\le \frac{1}{2}\|u\|_{X^p_j(t)}t^{-\frac{j}{2}-\frac{3}{2}(1-\frac{1}{p})}(1+|\Omega|t)^{-(1-\frac{2}{p})}
\end{aligned}$$

by taking $C_{**}<\frac{1}{2C}$ in (4.12). This completes the proof of Lemma 4.3. ◻

Lemma 4.4. *Suppose that the exponents s,q,θ and the initial data $u_0\in(\dot H^s\cap L^1)(\mathbb{R}^3)^3$ satisfy the asumptions in Theorem 1.1, and let $u\in C([0,\infty);\dot H^s(\mathbb{R}^3))^3$ be the unique global solution to (1.8) constructed in Theorem 1.1. Then, for $2\le p\le\infty$, there exists a constant $C=C(p,\|u_0\|_{L^1},\|u_0\|_{L^2})>0$ such that*

$$\int_0^{\frac{t}{2}}\|e^{-(t-\tau)A_\Omega}\mathbb{P}\nabla\cdot(u\otimes u)(\tau)\|_{L^p}\,d\tau \le Ct^{-\frac{j}{2}-\frac{3}{2}(1-\frac{1}{p})}(1+|\Omega|t)^{-(1-\frac{2}{p})}$$

for all $t>0$ and $j=0,1$.

Proof. We first consider the case $j=0$. It follows from (3.10) in Lemma 3.2 that

$$\begin{aligned}
&\int_0^{\frac{t}{2}}\|e^{-(t-\tau)A_\Omega}\mathbb{P}\nabla\cdot(u\otimes u)(\tau)\|_{L^p}\,d\tau\\
&\quad\le C\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{3}{2}(1-\frac{1}{p})}\{1+|\Omega|(t-\tau)\}^{-(1-\frac{2}{p})}\|(u\cdot\nabla u)(\tau)\|_{L^1}\,d\tau\\
&\quad\le Ct^{-\frac{3}{2}(1-\frac{1}{p})}(1+|\Omega|t)^{-(1-\frac{2}{p})}\int_0^{\frac{t}{2}}\|u(\tau)\|_{L^2}\|\nabla u(\tau)\|_{L^2}\,d\tau.
\end{aligned}\tag{4.16}$$

Then, since $u_0\in L^1(\mathbb{R}^3)^3$, the L^2 decay estimate in Lemma 4.2 (1) and the energy equality (4.6) give that

$$\begin{aligned}
\int_0^{\frac{t}{2}}\|u(\tau)\|_{L^2}\|\nabla u(\tau)\|_{L^2}\,d\tau &\le C\int_0^{\frac{t}{2}}\frac{1}{(1+\tau)^{\frac{3}{4}}}\|\nabla u(\tau)\|_{L^2}\,d\tau\\
&\le C\left(\int_0^\infty\frac{d\tau}{(1+\tau)^{\frac{3}{2}}}\right)^{\frac{1}{2}}\left(\int_0^\infty\|\nabla u(\tau)\|_{L^2}^2\,d\tau\right)^{\frac{1}{2}}\\
&\le C<\infty
\end{aligned}\tag{4.17}$$

with some constant $C=C(\|u_0\|_{L^1},\|u_0\|_{L^2})>0$. Then, (4.16) and (4.17) yield the desired estimates for $j=0$.

In the case $j=1$, we have by (3.10) in Lemma 3.2 and the L^2 decay estimate in Lemma 4.2 (1) that

$$\begin{aligned}
&\int_0^{\frac{t}{2}} \|e^{-(t-\tau)A_\Omega}\mathbb{P}\nabla\cdot(u\otimes u)(\tau)\|_{L^p}\,d\tau \\
&\le C\int_0^{\frac{t}{2}} (t-\tau)^{-\frac12-\frac32(1-\frac1p)}\{1+|\Omega|(t-\tau)\}^{-(1-\frac2p)}\|(u\otimes u)(\tau)\|_{L^1}\,d\tau \\
&\le Ct^{-\frac12-\frac32(1-\frac1p)}(1+|\Omega|t)^{-(1-\frac2p)}\int_0^{\frac{t}{2}}\|u(\tau)\|_{L^2}^2\,d\tau \\
&\le Ct^{-\frac12-\frac32(1-\frac1p)}(1+|\Omega|t)^{-(1-\frac2p)}\int_0^{\frac{t}{2}}\frac{1}{(1+\tau)^{\frac32}}\,d\tau \\
&\le Ct^{-\frac12-\frac32(1-\frac1p)}(1+|\Omega|t)^{-(1-\frac2p)}
\end{aligned}$$

with some constant $C=C(p,\|u_0\|_{L^1},\|u_0\|_{L^2})>0$. This completes the proof of Lemma 4.4. □

We are ready to give the proof of the L^p-time decay estimates.

Theorem 4.5. *Suppose that the exponents s,q and θ satisfy (1.10)–(1.11), and that the exponent p satisfies*

$$\frac12-\frac12\left(\frac1q-\frac s3\right)\le\frac1p\le\frac12. \tag{4.18}$$

Let $u_0\in(\dot H^s\cap L^1)(\mathbb{R}^3)^3$ with $\nabla\cdot u_0=0$ and $\Omega\in\mathbb{R}\setminus\{0\}$ satisfy (4.12), and let $u\in C([0,\infty);\dot H^s(\mathbb{R}^3))^3\cap L^\theta(0,\infty;\dot H^s_q(\mathbb{R}^3))^3$ be the unique global solution to (1.8) constructed in Theorem 1.1.

1. *There exists a positive constant $C=C(p,\|u_0\|_{L^1},\|u_0\|_{L^2})$ such that*
$$\|u(t)\|_{L^p}\le Ct^{-\frac32(1-\frac1p)}(1+|\Omega|t)^{-(1-\frac2p)} \tag{4.19}$$
for all $t>0$.
2. *Assume further that $|x|u_0\in L^1(\mathbb{R}^3)^3$. Then, there exists a positive constant $C=C(p,\||x|u_0\|_{L^1},\|u_0\|_{L^2})$ such that*
$$\|u(t)\|_{L^p}\le Ct^{-\frac12-\frac32(1-\frac1p)}(1+|\Omega|t)^{-(1-\frac2p)} \tag{4.20}$$
for all $t>0$.

Proof. (1) Let us first consider the case that the exponent p satisfies

$$\frac12-\frac12\left(\frac1q-\frac s3\right)\le\frac1p\le\frac1q. \tag{4.21}$$

Then, it follows from (3.10) in Lemmas 3.2, 4.3 and 4.4 with $j=0$ that

$$\begin{aligned}
\|u(t)\|_{L^p}&\le\|e^{-tA_\Omega}u_0\|_{L^p}+\left(\int_0^{\frac t2}+\int_{\frac t2}^t\right)\|e^{-(t-\tau)A_\Omega}\mathbb{P}\nabla\cdot(u\otimes u)(\tau)\|_{L^p}\,d\tau \\
&\le t^{-\frac32(1-\frac1p)}(1+|\Omega|t)^{-(1-\frac2p)}\left\{C+\frac12\|u\|_{X_0^p(t')}\right\}
\end{aligned} \tag{4.22}$$

for all $0<t\le t'$ with some positive constant $C=C(p,\|u_0\|_{L^1},\|u_0\|_{L^2})$. Hence we have $\|u\|_{X_0^p(t')}\le 2C$, which yields the desired estimate (4.19). For the exponent p satisfying

$$\frac1q\le\frac1p\le\frac12,$$

we take the exponent $\eta\in[0,1]$ so that $\frac1p=\frac\eta2+\frac{1-\eta}{q}$. Then, by the interpolation, Lemma 4.2 (1) and (4.19) for q, we obtain for all $t>0$

$$\begin{aligned}
\|u(t)\|_{L^p}&\le\|u(t)\|_{L^2}^\eta\|u(t)\|_{L^q}^{1-\eta} \\
&\le Ct^{-\frac32(1-\frac1p)}(1+|\Omega|t)^{-(1-\frac2p)}.
\end{aligned}$$

(2) Firstly, consider the case that the exponent p satisfies (4.21). Similarly to (4.22), we have by (3.11) in Lemmas 3.2, 4.3 and 4.4 with $j = 1$ that

$$\begin{aligned}\|u(t)\|_{L^p} &\le \|e^{-tA_\Omega}u_0\|_{L^p} + \left(\int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t\right)\|e^{-(t-\tau)A_\Omega}\mathbb{P}\nabla\cdot(u\otimes u)(\tau)\|_{L^p}\,d\tau \\ &\le t^{-\frac{1}{2}-\frac{3}{2}(1-\frac{1}{p})}(1+|\Omega|t)^{-(1-\frac{2}{p})}\left\{C+\frac{1}{2}\|u\|_{X_1^p(t')}\right\} \qquad (4.23)\end{aligned}$$

for all $0 < t \le t'$ with some positive constant $C = C(p, \||x|u_0\|_{L^1}, \|u_0\|_{L^2})$. This implies $\|u\|_{X_1^p(t')} \le 2C$, and we obtain the desired estimate (4.20). In the case $\frac{1}{q} \le \frac{1}{p} \le \frac{1}{2}$, we take the exponent $\eta \in [0, 1]$ so that $\frac{1}{p} = \frac{\eta}{2} + \frac{1-\eta}{q}$. We obtain by the interpolation, Lemma 4.2 (2) and (4.20) for q that

$$\begin{aligned}\|u(t)\|_{L^p} &\le \|u(t)\|_{L^2}^{\eta}\|u(t)\|_{L^q}^{1-\eta} \\ &\le Ct^{-\frac{1}{2}-\frac{3}{2}(1-\frac{1}{p})}(1+|\Omega|t)^{-(1-\frac{2}{p})}\end{aligned}$$

for all $t > 0$. This completes the proof of Theorem 4.5.

□

5. Nonlinear Asymptotics

We are now ready to give the proofs of Theorems 1.2 and 1.3. Firstly, we shall show Theorem 1.2.

Proof of Theorem 1.2. The temporal decay estimate (1.15) is already shown in Theorem 4.5. Hence it suffices to show the asymptotic behavior (1.16).

By the Duhamel formula (1.20), we have

$$\begin{aligned}u(t) &= e^{-tA_\Omega}u_0 - \left(\int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t\right)e^{-(t-\tau)A_\Omega}\mathbb{P}\nabla\cdot(u\otimes u)(\tau)\,d\tau \\ &=: I_1 + I_2 + I_3. \qquad (5.1)\end{aligned}$$

For I_1, it follows from (3.13) in Theorem 3.4 that

$$\lim_{t\to\infty} t^{\frac{3}{2}(1-\frac{1}{p})}(1+|\Omega|t)^{1-\frac{2}{p}}\|I_1(t)\|_{L^p} = 0. \qquad (5.2)$$

Concerning I_2, Lemma 4.4 with $j = 1$ gives that

$$t^{\frac{3}{2}(1-\frac{1}{p})}(1+|\Omega|t)^{1-\frac{2}{p}}\|I_2(t)\|_{L^p} \le Ct^{-\frac{1}{2}} \to 0 \qquad (5.3)$$

as $t \to \infty$. Hence it remains to show the estimate for $I_3(t)$. Firstly, assume that the exponent p satisfies

$$\frac{1}{2} - \frac{1}{2}\left(\frac{1}{q} - \frac{s}{3}\right) \le \frac{1}{p} \le \frac{1}{q}. \qquad (5.4)$$

This is the same condition as (4.11) in Lemma 4.3. Put

$$\frac{1}{q_s} = \frac{1}{q} - \frac{s}{3}, \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q_s}.$$

Then, the exponents p and r satisfy $2 < p < \infty$ and $1 < r \le p'$. It follows from Lemma 2.5, the embedding $\dot{H}^s_q(\mathbb{R}^3) \hookrightarrow L^{q_s}(\mathbb{R}^3)$ and (4.19) in Theorem 4.5 that

$$\begin{aligned}
\|I_3(t)\|_{L^p} &\le \int_{\frac{t}{2}}^t \|e^{-(t-\tau)A_\Omega}\mathbb{P}\operatorname{div}(u\otimes u)(\tau)\|_{L^p}\,d\tau \\
&\le C\int_{\frac{t}{2}}^t \{1+|\Omega|(t-\tau)\}^{-(1-\frac{2}{p})}(t-\tau)^{-\frac12-\frac32(\frac1r-\frac1p)}\|(u\otimes u)(\tau)\|_{L^r}\,d\tau \\
&\le C\int_{\frac{t}{2}}^t \{1+|\Omega|(t-\tau)\}^{-(1-\frac{2}{p})}(t-\tau)^{-\frac12-\frac32(\frac1q-\frac s3)}\|u(\tau)\|_{L^p}\|u(\tau)\|_{L^{q_s}}\,d\tau \\
&\le Ct^{-\frac32(1-\frac1p)}(1+|\Omega|t)^{-(1-\frac2p)} \\
&\quad\times\int_{\frac{t}{2}}^t \{1+|\Omega|(t-\tau)\}^{-(1-\frac{2}{p})}(t-\tau)^{-\frac12-\frac32(\frac1q-\frac s3)}\|u(\tau)\|_{\dot{H}^s_q}\,d\tau.
\end{aligned} \tag{5.5}$$

Here, similarly to (4.14) and (4.15), we have

$$\begin{aligned}
&\int_{\frac{t}{2}}^t \{1+|\Omega|(t-\tau)\}^{-(1-\frac{2}{p})}(t-\tau)^{-\frac12-\frac32(\frac1q-\frac s3)}\|u(\tau)\|_{\dot{H}^s_q}\,d\tau \\
&\quad\le C|\Omega|^{\frac1\theta+\frac{3}{2q}-\frac s2-\frac12}\|u\|_{L^\theta(t/2,t;\dot{H}^s_q)}
\end{aligned} \tag{5.6}$$

Since u belongs to $L^\theta(0,\infty;\dot{H}^s_q(\mathbb{R}^3))^3$, we have by (5.5) and (5.6) that

$$t^{\frac32(1-\frac1p)}(1+|\Omega|t)^{1-\frac2p}\|I_3(t)\|_{L^p} \le C|\Omega|^{\frac1\theta+\frac{3}{2q}-\frac s2-\frac12}\left(\int_{\frac t2}^t \|u(\tau)\|_{\dot{H}^s_q}^\theta\,d\tau\right)^{\frac1\theta} \to 0 \tag{5.7}$$

as $t\to\infty$. This completes the proof of (1.16) when the exponent p satisfies (5.4).

Next, we shall consider the estimate for $I_3(t)$ for $p=2$. We follow the argument in [7], and set

$$v(t) := -\int_\tau^t e^{-(t-s)A_\Omega}\mathbb{P}\nabla\cdot(u\otimes u)(s)\,ds, \quad t>\tau.$$

We see that $v(t) = u(t) - e^{-(t-\tau)A_\Omega}u(\tau)$, and $v(t)$ should solve

$$\begin{cases} \partial_t v - \Delta v + \Omega\mathbb{P}J\mathbb{P}v = -\mathbb{P}(u\cdot\nabla u) & x\in\mathbb{R}^3,\ t>\tau, \\ v(\tau) = 0 & x\in\mathbb{R}^3. \end{cases} \tag{5.8}$$

Taking the L^2-inner product of (5.8) with v, we have

$$\frac12\frac{d}{dt}\|v(t)\|_{L^2}^2 + \|\nabla v(t)\|_{L^2}^2 = -\langle (u\cdot\nabla)u(t),\, v(t)\rangle_{L^2}.$$

Since $v(t) = u(t) - e^{-(t-\tau)A_\Omega}u(\tau)$, the integration by parts and the divergence-free condition give that

$$\langle (u\cdot\nabla)u(t),\, v(t)\rangle_{L^2} = -\langle (u\cdot\nabla)v(t),\, u(t)\rangle_{L^2} = -\langle (u\cdot\nabla)v(t),\, e^{-(t-\tau)A_\Omega}u(\tau)\rangle_{L^2}.$$

Hence we have

$$\frac12\frac{d}{dt}\|v(t)\|_{L^2}^2 + \|\nabla v(t)\|_{L^2}^2 = \langle (u\cdot\nabla)v(t),\, e^{-(t-\tau)A_\Omega}u(\tau)\rangle_{L^2}. \tag{5.9}$$

Note that Lemmas 2.4 and 4.2 (1) yield

$$\begin{aligned}
\|e^{-(t-\tau)A_\Omega}u(\tau)\|_{L^\infty} &\le C(t-\tau)^{-\frac34}\|u(\tau)\|_{L^2} \\
&\le C(t-\tau)^{-\frac34}(1+\tau)^{-\frac34}.
\end{aligned} \tag{5.10}$$

Also, we remark that it holds

$$\|u(t)\|_{L^2} \le C(1+t)^{-\frac34} \le Ct^{-\frac34} \le C(t-\tau)^{-\frac34} \tag{5.11}$$

by Lemma 4.2 (1). Hence we have from (5.9), (5.10) and (5.11)

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|v(t)\|_{L^2}^2 + \|\nabla v(t)\|_{L^2}^2 &\le \|u(t)\|_{L^2}\|\nabla v(t)\|_{L^2}\|e^{-(t-\tau)A_\Omega}u(\tau)\|_{L^\infty} \\
&\le C\|\nabla v(t)\|_{L^2}(t-\tau)^{-\frac{3}{2}}(1+\tau)^{-\frac{3}{4}} \\
&\le \frac{1}{2}\|\nabla v(t)\|_{L^2}^2 + C(t-\tau)^{-3}(1+\tau)^{-\frac{3}{2}},
\end{aligned}
$$

which yields

$$
\frac{d}{dt}\|v(t)\|_{L^2}^2 + \|\nabla v(t)\|_{L^2}^2 \le C(t-\tau)^{-3}(1+\tau)^{-\frac{3}{2}}. \tag{5.12}
$$

Let $\rho > 0$ be a positive parameter to be chosen later. It follows from the Plancherel theorem that

$$
\begin{aligned}
\|\nabla v\|_{L^2}^2 &= \int_{\mathbb{R}^3} |\xi|^2|\widehat{v}(\xi)|^2\,d\xi \\
&\ge \int_{|\xi|>\sqrt{\rho}} |\xi|^2|\widehat{v}(\xi)|^2\,d\xi \\
&\ge \rho\int_{|\xi|>\sqrt{\rho}} |\widehat{v}(\xi)|^2\,d\xi \\
&= \rho\|v\|_{L^2}^2 - \rho\int_{|\xi|\le\sqrt{\rho}} |\widehat{v}(\xi)|^2\,d\xi.
\end{aligned}
$$

This and (5.12) imply that

$$
\frac{d}{dt}\|v(t)\|_{L^2}^2 + \rho\|v(t)\|_{L^2}^2 \le \rho\int_{|\xi|\le\sqrt{\rho}} |\widehat{v}(\xi)|^2\,d\xi + C(t-\tau)^{-3}(1+\tau)^{-\frac{3}{2}}. \tag{5.13}
$$

Here, since

$$
\widehat{v}(\xi,t) = -\int_\tau^t e^{-(t-s)|\xi|^2}H_\Omega(\xi,t-s)P(\xi)\left\{(i\xi)\cdot\widehat{u\otimes u}(\xi,s)\right\}ds,
$$

we have

$$
\begin{aligned}
|\widehat{v}(\xi,t)| &\le C|\xi|\int_\tau^t \|\widehat{u\otimes u}(s)\|_{L^\infty}\,ds \\
&\le C|\xi|\int_\tau^t \|u\otimes u(s)\|_{L^1}\,ds \\
&\le C|\xi|\int_\tau^t \|u(s)\|_{L^2}^2\,ds.
\end{aligned}
$$

Therefore, we see that

$$
\begin{aligned}
\int_{|\xi|\le\sqrt{\rho}} |\widehat{v}(\xi)|^2\,d\xi &\le C\left(\int_\tau^t \|u(s)\|_{L^2}^2\,ds\right)^2\int_{|\xi|\le\sqrt{\rho}}|\xi|^2\,d\xi \\
&\le C\rho^{\frac{5}{2}}\left(\int_\tau^t \|u(s)\|_{L^2}^2\,ds\right)^2.
\end{aligned} \tag{5.14}
$$

Substituting (5.14) into (5.13) gives that

$$
\frac{d}{dt}\|v(t)\|_{L^2}^2 + \rho\|v(t)\|_{L^2}^2 \le C\rho^{\frac{7}{2}}\left(\int_\tau^t \|u(s)\|_{L^2}^2\,ds\right)^2 + C(t-\tau)^{-3}(1+\tau)^{-\frac{3}{2}} \tag{5.15}
$$

for $0 < \tau < t$. Now, we set

$$
\rho = \rho(t) = m(t-\tau)^{-1}, \quad m > \frac{7}{2}.
$$

Then, the inequality (5.15) can be written as

$$\begin{aligned}&\frac{d}{dt}\|v(t)\|_{L^2}^2 + m(t-\tau)^{-1}\|v(t)\|_{L^2}^2\\&\quad\le Cm^{\frac{7}{2}}(t-\tau)^{-\frac{7}{2}}\left(\int_\tau^t \|u(s)\|_{L^2}^2\,ds\right)^2 + C(t-\tau)^{-3}(1+\tau)^{-\frac{3}{2}}.\end{aligned}\tag{5.16}$$

It follows from (5.16) that

$$\begin{aligned}&\frac{d}{dt}\left\{(t-\tau)^m\|v(t)\|_{L^2}^2\right\}\\&\quad=(t-\tau)^m\left\{\frac{d}{dt}\|v(t)\|_{L^2}^2 + m(t-\tau)^{-1}\|v(t)\|_{L^2}^2\right\}\\&\quad\le Cm^{\frac{7}{2}}(t-\tau)^{m-\frac{7}{2}}\left(\int_\tau^t \|u(s)\|_{L^2}^2\,ds\right)^2 + C(t-\tau)^{m-3}(1+\tau)^{-\frac{3}{2}}.\end{aligned}$$

Hence we have

$$\begin{aligned}&(t-\tau)^m\|v(t)\|_{L^2}^2\\&\quad\le Cm^{\frac{7}{2}}\int_\tau^t (s-\tau)^{m-\frac{7}{2}}\left(\int_\tau^s \|u(r)\|_{L^2}^2\,dr\right)^2 ds + C(1+\tau)^{-\frac{3}{2}}\int_\tau^t (s-\tau)^{m-3}\,ds\\&\quad\le Cm^{\frac{7}{2}}(t-\tau)^{m-\frac{7}{2}}\int_\tau^t\left(\int_\tau^s \|u(r)\|_{L^2}^2\,dr\right)^2 ds + C(1+\tau)^{-\frac{3}{2}}(t-\tau)^{m-2},\end{aligned}$$

which yields

$$\|v(t)\|_{L^2}^2 \le C(t-\tau)^{-\frac{7}{2}}\int_\tau^t\left(\int_\tau^s \|u(r)\|_{L^2}^2\,dr\right)^2 ds + C(1+\tau)^{-\frac{3}{2}}(t-\tau)^{-2}.\tag{5.17}$$

Now, we put $\tau = t/2$, then

$$v(t) := -\int_{\frac{t}{2}}^t e^{-(t-s)A_\Omega}\mathbb{P}\nabla\cdot(u\otimes u)(s)\,ds = I_3(t)$$

and we have by (5.17) that

$$\|I_3(t)\|_{L^2}^2 \le Ct^{-\frac{7}{2}}\int_{\frac{t}{2}}^t\left(\int_{\frac{t}{2}}^s \|u(r)\|_{L^2}^2\,dr\right)^2 ds + C(1+t)^{-\frac{3}{2}}t^{-2}.\tag{5.18}$$

Here, it follows from Lemma 4.2 (1) that

$$\begin{aligned}\int_{\frac{t}{2}}^t\left(\int_{\frac{t}{2}}^s \|u(r)\|_{L^2}^2\,dr\right)^2 ds &\le \frac{t}{2}\left(\int_{\frac{t}{2}}^\infty \|u(r)\|_{L^2}^2\,dr\right)^2\\&\le Ct\left(\int_{\frac{t}{2}}^\infty \frac{1}{(1+r)^{\frac{3}{2}}}\,dr\right)^2\\&\le C\frac{t}{1+t}\le C<\infty.\end{aligned}\tag{5.19}$$

Hence by (5.18) and (5.19), we obtain

$$\|I_3(t)\|_{L^2}^2 \le Ct^{-\frac{7}{2}} + C(1+t)^{-\frac{3}{2}}t^{-2},$$

which yields

$$t^{\frac{3}{4}}\|I_3(t)\|_{L^2} \le Ct^{-1} \to 0\tag{5.20}$$

as $t\to\infty$. This gives the proof of (1.16) for $p=2$.

Finally, for the exponent p satisfying

$$\frac{1}{q} \le \frac{1}{p} \le \frac{1}{2},$$

we take $\eta \in [0,1]$ such that $\frac{1}{p} = \frac{\eta}{2} + \frac{1-\eta}{q}$. Interpolating (5.7) and (5.20), we obtain

$$\begin{aligned} & t^{\frac{3}{2}(1-\frac{1}{p})}(1+|\Omega|t)^{1-\frac{2}{p}}\|I_3(t)\|_{L^p} \\ & \le \left\{ t^{\frac{3}{4}}\|I_3(t)\|_{L^2}\right\}^{\eta} \left\{ t^{\frac{3}{2}(1-\frac{1}{q})}(1+|\Omega|t)^{1-\frac{2}{q}}\|I_3(t)\|_{L^q}\right\}^{1-\eta} \to 0 \end{aligned}$$

as $t \to \infty$. This completes the proof of Theorem 1.2. □

We finally present the proof of Theorem 1.3.

Proof of Theorem 1.3. We have already shown the time decay estimate (1.24) in Theorem 4.5. Hence it remains to prove the asymptotic behavior (1.25).

Let us decompose

$$-\int_0^t e^{-(t-s)A_\Omega}\mathbb{P}\nabla\cdot(u\otimes u)(s)\,ds =: J_1 + J_2, \tag{5.21}$$

where

$$\begin{aligned} J_1 &:= -\int_0^{\frac{t}{2}} e^{-(t-s)A_\Omega}\mathbb{P}\nabla\cdot(u\otimes u)(s)\,ds, \\ J_2 &:= -\int_{\frac{t}{2}}^t e^{-(t-s)A_\Omega}\mathbb{P}\nabla\cdot(u\otimes u)(s)\,ds. \end{aligned}$$

We first consider the estimate for J_2. Let us treat the case that the exponent p satisfies

$$\frac{1}{2} - \frac{1}{2}\left(\frac{1}{q} - \frac{s}{3}\right) \le \frac{1}{p} \le \frac{1}{q}. \tag{5.22}$$

Note that (5.22) is the same assumption as (4.11) in Lemma 4.3. Setting

$$\frac{1}{q_s} = \frac{1}{q} - \frac{s}{3}, \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q_s},$$

we see that there hold $2 < p < \infty$ and $1 < r \le p'$. Hence we can apply Lemma 2.5, and it follows from the embedding $\dot{H}^s_q(\mathbb{R}^3) \hookrightarrow L^{q_s}(\mathbb{R}^3)$, (4.20) in Theorem 4.5 and (5.6) that

$$\begin{aligned} \|J_2(t)\|_{L^p} &\le \int_{\frac{t}{2}}^t \|e^{-(t-\tau)A_\Omega}\mathbb{P}\,\mathrm{div}(u\otimes u)(\tau)\|_{L^p}\,d\tau \\ &\le C\int_{\frac{t}{2}}^t \{1+|\Omega|(t-\tau)\}^{-(1-\frac{2}{p})}(t-\tau)^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{r}-\frac{1}{p})}\|(u\otimes u)(\tau)\|_{L^r}\,d\tau \\ &\le C\int_{\frac{t}{2}}^t \{1+|\Omega|(t-\tau)\}^{-(1-\frac{2}{p})}(t-\tau)^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{q}-\frac{s}{3})}\|u(\tau)\|_{L^p}\|u(\tau)\|_{L^{q_s}}\,d\tau \\ &\le Ct^{-\frac{1}{2}-\frac{3}{2}(1-\frac{1}{p})}(1+|\Omega|t)^{-(1-\frac{2}{p})} \\ &\quad \times \int_{\frac{t}{2}}^t \{1+|\Omega|(t-\tau)\}^{-(1-\frac{2}{p})}(t-\tau)^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{q}-\frac{s}{3})}\|u(\tau)\|_{\dot{H}^s_q}\,d\tau \\ &\le C|\Omega|^{\frac{1}{\theta}+\frac{3}{2q}-\frac{s}{2}-\frac{1}{2}}t^{-\frac{1}{2}-\frac{3}{2}(1-\frac{1}{p})}(1+|\Omega|t)^{-(1-\frac{2}{p})}\|u\|_{L^\theta(\frac{t}{2},t;\dot{H}^s_q)}. \end{aligned} \tag{5.23}$$

Since $u \in L^\theta(0,\infty;\dot{H}^s_q(\mathbb{R}^3))^3$, we have by (5.23) that

$$t^{\frac{1}{2}+\frac{3}{2}(1-\frac{1}{p})}(1+|\Omega|t)^{1-\frac{2}{p}}\|J_2(t)\|_{L^p} \le C|\Omega|^{\frac{1}{\theta}+\frac{3}{2q}-\frac{s}{2}-\frac{1}{2}}\left(\int_{\frac{t}{2}}^t \|u(\tau)\|^\theta_{\dot{H}^s_q}\,d\tau\right)^{\frac{1}{\theta}} \to 0 \tag{5.24}$$

as $t \to \infty$.

Next, let us consider the L^2 estimate for J_2. Set

$$v(t) := -\int_\tau^t e^{-(t-s)A_\Omega}\mathbb{P}\nabla\cdot(u\otimes u)(s)\,ds, \quad t>\tau.$$

Note that we now assume $(1+|x|)u_0 \in L^1(\mathbb{R}^3)^3$, and then it follows from Lemmas 2.4 and 4.2 (2) that

$$\|e^{-(t-\tau)A_\Omega}u(\tau)\|_{L^\infty} \le C(t-\tau)^{-\frac{3}{4}}\|u(\tau)\|_{L^2} \le C(t-\tau)^{-\frac{3}{4}}(1+\tau)^{-\frac{5}{4}} \tag{5.25}$$

and

$$\|u(t)\|_{L^2} \le C(1+t)^{-\frac{5}{4}} \le Ct^{-\frac{5}{4}} \le C(t-\tau)^{-\frac{5}{4}} \tag{5.26}$$

We apply the same argument as in the proof of Theorem 1.2 by using (5.25) and (5.26) instead of (5.10) and (5.11), respectively. Then, similarly to (5.15), we have

$$\frac{d}{dt}\|v(t)\|_{L^2}^2 + \rho\|v(t)\|_{L^2}^2 \le C\rho^{\frac{7}{2}}\left(\int_\tau^t \|u(s)\|_{L^2}^2\,ds\right)^2 + C(t-\tau)^{-4}(1+\tau)^{-\frac{5}{2}} \tag{5.27}$$

for $0<\tau<t$ and $\rho>0$. Here we set

$$\rho = \rho(t) = m(t-\tau)^{-1}, \quad m>4.$$

Substituting this $\rho(t)$ into (5.27) gives

$$\begin{aligned}&\frac{d}{dt}\|v(t)\|_{L^2}^2 + m(t-\tau)^{-1}\|v(t)\|_{L^2}^2\\ &\quad\le Cm^{\frac{7}{2}}(t-\tau)^{-\frac{7}{2}}\left(\int_\tau^t \|u(s)\|_{L^2}^2\,ds\right)^2 + C(t-\tau)^{-4}(1+\tau)^{-\frac{5}{2}}.\end{aligned} \tag{5.28}$$

Then, it follows from (5.28) that

$$\begin{aligned}&\frac{d}{dt}\left\{(t-\tau)^m\|v(t)\|_{L^2}^2\right\}\\ &\quad= (t-\tau)^m\left\{\frac{d}{dt}\|v(t)\|_{L^2}^2 + m(t-\tau)^{-1}\|v(t)\|_{L^2}^2\right\}\\ &\quad\le Cm^{\frac{7}{2}}(t-\tau)^{m-\frac{7}{2}}\left(\int_\tau^t \|u(s)\|_{L^2}^2\,ds\right)^2 + C(t-\tau)^{m-4}(1+\tau)^{-\frac{5}{2}},\end{aligned}$$

which yields

$$\begin{aligned}&(t-\tau)^m\|v(t)\|_{L^2}^2\\ &\quad\le Cm^{\frac{7}{2}}\int_\tau^t (s-\tau)^{m-\frac{7}{2}}\left(\int_\tau^s \|u(r)\|_{L^2}^2\,dr\right)^2 ds + C(1+\tau)^{-\frac{5}{2}}\int_\tau^t (s-\tau)^{m-4}\,ds\\ &\quad\le Cm^{\frac{7}{2}}(t-\tau)^{m-\frac{7}{2}}\int_\tau^t\left(\int_\tau^s \|u(r)\|_{L^2}^2\,dr\right)^2 ds + C(1+\tau)^{-\frac{5}{2}}(t-\tau)^{m-3}.\end{aligned}$$

Hence we have

$$\|v(t)\|_{L^2}^2 \le C(t-\tau)^{-\frac{7}{2}}\int_\tau^t\left(\int_\tau^s \|u(r)\|_{L^2}^2\,dr\right)^2 ds + C(1+\tau)^{-\frac{5}{2}}(t-\tau)^{-3}. \tag{5.29}$$

Here, we see by Lemma 4.2 (2) that

$$\begin{aligned}\int_\tau^t \left(\int_\tau^s \|u(r)\|_{L^2}^2\,dr\right)^2 ds &\le (t-\tau)\left(\int_\tau^\infty \|u(r)\|_{L^2}^2\,dr\right)^2\\ &\le C(t-\tau)\left(\int_\tau^\infty \frac{1}{(1+r)^{\frac{5}{2}}}\,dr\right)^2\\ &\le C\frac{t-\tau}{(1+\tau)^3}. \end{aligned} \tag{5.30}$$

Thus, substituting (5.30) into (5.29) gives

$$\|v(t)\|_{L^2}^2 \le C(t-\tau)^{-\frac{5}{2}}(1+\tau)^{-3} + C(t-\tau)^{-3}(1+\tau)^{-\frac{5}{2}}. \tag{5.31}$$

Now, let us set $\tau = t/2$. Then,

$$v(t) := -\int_{\frac{t}{2}}^t e^{-(t-s)A_\Omega}\mathbb{P}\nabla\cdot(u\otimes u)(s)\,ds = J_2(t).$$

Hence we have by (5.31) that

$$\|J_2(t)\|_{L^2}^2 \le Ct^{-\frac{5}{2}}(1+t)^{-3} + Ct^{-3}(1+t)^{-\frac{5}{2}},$$

which yields

$$t^{\frac{5}{4}}\|J_2(t)\|_{L^2} \le C(1+t)^{-\frac{3}{2}} + Ct^{-\frac{1}{4}}(1+t)^{-\frac{5}{4}} \to 0 \tag{5.32}$$

as $t\to\infty$.

For the case that the exponent p satisfying

$$\frac{1}{q} \le \frac{1}{p} \le \frac{1}{2},$$

we take $\eta\in[0,1]$ such that $\frac{1}{p} = \frac{\eta}{2} + \frac{1-\eta}{q}$. Interpolating (5.24) and (5.32) gives

$$\begin{aligned}&t^{\frac{1}{2}+\frac{3}{2}(1-\frac{1}{p})}(1+|\Omega|t)^{1-\frac{2}{p}}\|J_2(t)\|_{L^p}\\ &\quad\le \left\{t^{\frac{5}{4}}\|J_2(t)\|_{L^2}\right\}^\eta \left\{t^{\frac{1}{2}+\frac{3}{2}(1-\frac{1}{q})}(1+|\Omega|t)^{1-\frac{2}{q}}\|J_2(t)\|_{L^q}\right\}^{1-\eta} \to 0\end{aligned}$$

as $t\to\infty$. Hence we obtain that for p satisfying $\frac{1}{2}-\frac{1}{2}\left(\frac{1}{q}-\frac{s}{3}\right) \le \frac{1}{p} \le \frac{1}{2}$, there holds

$$\lim_{t\to\infty} t^{\frac{1}{2}+\frac{3}{2}(1-\frac{1}{p})}(1+|\Omega|t)^{1-\frac{2}{p}}\|J_2(t)\|_{L^p} = 0. \tag{5.33}$$

Next, let us consider the estimates for J_1. Let us rewrite

$$J_1 = -\sum_{j=1}^3 \int_0^{\frac{t}{2}}\int_{\mathbb{R}^3} \partial_j\widetilde{K}_\Omega(x-y,t-s)(u_ju)(y,s)\,dy\,ds.$$

Then, we decompose J_1 as

$$\begin{aligned}
&J_1 + \sum_{j=1}^{3} \partial_j \widetilde{K}_\Omega(x,t) \int_0^\infty \int_{\mathbb{R}^3} (u_j u)(y,s)\, dy\, ds \\
&= \sum_{j=1}^{3} \partial_j \widetilde{K}_\Omega(x,t) \int_{\frac{t}{2}}^\infty \int_{\mathbb{R}^3} (u_j u)(y,s)\, dy\, ds \\
&\quad - \sum_{j=1}^{3} \int_0^{\frac{t}{2}} \int_{\mathbb{R}^3} \left\{ \partial_j \widetilde{K}_\Omega(x-y,t-s) - \partial_j \widetilde{K}_\Omega(x,t-s) \right\} (u_j u)(y,s)\, dy\, ds \\
&\quad - \sum_{j=1}^{3} \int_0^{\frac{t}{2}} \int_{\mathbb{R}^3} \left\{ \partial_j \widetilde{K}_\Omega(x,t-s) - \partial_j \widetilde{K}_\Omega(x,t) \right\} (u_j u)(y,s)\, dy\, ds \\
&=: J_{11} + J_{12} + J_{13}.
\end{aligned} \tag{5.34}$$

Let us firstly consider the estimate for J_{11}. Similarly to Lemma 3.1, we see that for $2 \le p \le \infty$ and $\alpha \in (\mathbb{N} \cup \{0\})^3$ there exists a positive constant $C = C(p,\alpha)$ such that it holds

$$\left\| \partial_x^\alpha \widetilde{K}_\Omega(\cdot,t) \right\|_{L^p} \le C t^{-\frac{|\alpha|}{2} - \frac{3}{2}(1-\frac{1}{p})} (1+|\Omega|t)^{-(1-\frac{2}{p})} \tag{5.35}$$

for all $\Omega \in \mathbb{R}$ and for all $t > 0$. Then, we have by (5.35) and Lemma 4.2 (2) that for $2 \le p \le \infty$

$$\begin{aligned}
\|J_{11}(t)\|_{L^p} &\le \sum_{j=1}^{3} \left\| \partial_j \widetilde{K}_\Omega(\cdot,t) \right\|_{L^p} \int_{\frac{t}{2}}^\infty \int_{\mathbb{R}^3} |(u_j u)(y,s)|\, dy\, ds \\
&\le C t^{-\frac{1}{2} - \frac{3}{2}(1-\frac{1}{p})} (1+|\Omega|t)^{-(1-\frac{2}{p})} \int_{\frac{t}{2}}^\infty \|u(s)\|_{L^2}^2\, ds \\
&\le C t^{-\frac{1}{2} - \frac{3}{2}(1-\frac{1}{p})} (1+|\Omega|t)^{-(1-\frac{2}{p})} \int_{\frac{t}{2}}^\infty \frac{1}{(1+s)^{\frac{5}{2}}}\, ds,
\end{aligned}$$

which yields

$$t^{\frac{1}{2} + \frac{3}{2}(1-\frac{1}{p})} (1+|\Omega|t)^{1-\frac{2}{p}} \|J_{11}(t)\|_{L^p} \le C(1+t)^{-\frac{3}{2}} \to 0 \quad (t \to \infty). \tag{5.36}$$

Next, we consider the estimate for J_{12}. Similarly to (3.4), (3.5) and (3.21), we see that

$$\begin{aligned}
\partial_j \widetilde{K}_\Omega(x,t) &= t^{-2} \mathcal{F}^{-1} \left[\widehat{\partial_j G_1}(\xi) \widetilde{H}_\Omega(\xi,t) \right] \left(\frac{x}{\sqrt{t}} \right) \\
&= \frac{1}{2} t^{-2} \sum_{\sigma \in \{\pm\}} \mathcal{G}_\sigma(\Omega t) \left[(\mathbb{P} + \sigma \mathcal{R}) \partial_j G_1 \right] \left(\frac{x}{\sqrt{t}} \right)
\end{aligned} \tag{5.37}$$

for $j = 1, 2, 3$. Hence we have

$$
\begin{aligned}
&\left\|\partial_j \widetilde{K}_\Omega(\cdot - y, t-s) - \partial_j \widetilde{K}_\Omega(\cdot, t-s)\right\|_{L^p} \\
&\quad \le \frac{1}{2}(t-s)^{-2} \sum_{\sigma\in\{\pm\}} \left\| \mathcal{G}_\sigma(\Omega(t-s)) \left[(\mathbb{P}+\sigma\mathcal{R})\partial_j G_1\right] \left(\frac{\cdot - y}{\sqrt{t-s}}\right)\right. \\
&\qquad \left. - \mathcal{G}_\sigma(\Omega(t-s)) \left[(\mathbb{P}+\sigma\mathcal{R})\partial_j G_1\right] \left(\frac{\cdot}{\sqrt{t-s}}\right) \right\|_{L^p} \\
&\quad = \frac{(t-s)^{-\frac{1}{2}-\frac{3}{2}(1-\frac{1}{p})}}{2} \sum_{\sigma\in\{\pm\}} \left\| \mathcal{G}_\sigma(\Omega(t-s)) \left[(\mathbb{P}+\sigma\mathcal{R})\partial_j G_1\right] \left(\cdot - \frac{y}{\sqrt{t-s}}\right)\right. \\
&\qquad \left. - \mathcal{G}_\sigma(\Omega(t-s)) \left[(\mathbb{P}+\sigma\mathcal{R})\partial_j G_1\right] (\cdot) \right\|_{L^p} \\
&\quad \le \frac{(t-s)^{-\frac{1}{2}-\frac{3}{2}(1-\frac{1}{p})}}{2} \\
&\qquad \times \sum_{\sigma\in\{\pm\}} \left\| \mathcal{G}_\sigma(\Omega(t-s)) \left[(\mathbb{P}+\sigma\mathcal{R}) \left\{\partial_j G_1\left(\cdot - \frac{y}{\sqrt{t-s}}\right) - \partial_j G_1(\cdot)\right\}\right]\right\|_{L^p}.
\end{aligned} \tag{5.38}
$$

For p satisfying $\frac{1}{2} - \frac{1}{2}\left(\frac{1}{q} - \frac{s}{3}\right) \le \frac{1}{p} \le \frac{1}{2}$, similarly to (3.17) and (3.18), it follows from the embedding $\dot{B}^0_{p,2}(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$, Lemmas 2.3, 2.6 and the embedding $L^{p'}(\mathbb{R}^3) \hookrightarrow \dot{B}^0_{p',2}(\mathbb{R}^3)$ that

$$
\begin{aligned}
&\left\| \mathcal{G}_\sigma(\Omega(t-s)) \left[(\mathbb{P}+\sigma\mathcal{R}) \left\{\partial_j G_1\left(\cdot - \frac{y}{\sqrt{t-s}}\right) - \partial_j G_1(\cdot)\right\}\right]\right\|_{L^p} \\
&\quad \le C \left\| \mathcal{G}_\sigma(\Omega(t-s)) \left[(\mathbb{P}+\sigma\mathcal{R}) \left\{\partial_j G_1\left(\cdot - \frac{y}{\sqrt{t-s}}\right) - \partial_j G_1(\cdot)\right\}\right]\right\|_{\dot{B}^0_{p,2}} \\
&\quad \le C \{1+|\Omega|(t-s)\}^{-(1-\frac{2}{p})} \left\| \partial_j G_1\left(\cdot - \frac{y}{\sqrt{t-s}}\right) - \partial_j G_1(\cdot)\right\|_{\dot{B}^{3(1-\frac{2}{p})}_{p',2}} \\
&\quad \le C \{1+|\Omega|(t-s)\}^{-(1-\frac{2}{p})} \left(\frac{1}{2}\right)^{-\frac{1}{2}-\frac{3}{2}(1-\frac{2}{p})} \left\| G_{\frac{1}{2}}\left(\cdot - \frac{y}{\sqrt{t-s}}\right) - G_{\frac{1}{2}}(\cdot)\right\|_{\dot{B}^0_{p',2}} \\
&\quad \le C \{1+|\Omega|(t-s)\}^{-(1-\frac{2}{p})} \left\| G_{\frac{1}{2}}\left(\cdot - \frac{y}{\sqrt{t-s}}\right) - G_{\frac{1}{2}}(\cdot)\right\|_{L^{p'}}.
\end{aligned} \tag{5.39}
$$

Hence we have by (5.38) and (5.39)

$$
\begin{aligned}
&\|J_{12}(t)\|_{L^p} \\
&\quad \le \sum_{j=1}^{3} \int_0^{\frac{t}{2}} \int_{\mathbb{R}^3} \left\|\partial_j \widetilde{K}_\Omega(\cdot - y, t-s) - \partial_j \widetilde{K}_\Omega(\cdot, t-s)\right\|_{L^p} |(u_j u)(y,s)|\, dy\, ds \\
&\quad \le C \int_0^{\frac{t}{2}} \int_{\mathbb{R}^3} (t-s)^{-\frac{1}{2}-\frac{3}{2}(1-\frac{1}{p})} \{1+|\Omega|(t-s)\}^{-(1-\frac{2}{p})} \\
&\qquad \times \left\| G_{\frac{1}{2}}\left(\cdot - \frac{y}{\sqrt{t-s}}\right) - G_{\frac{1}{2}}(\cdot)\right\|_{L^{p'}} |u(y,s)|^2\, dy\, ds \\
&\quad \le C t^{-\frac{1}{2}-\frac{3}{2}(1-\frac{1}{p})} (1+|\Omega|t)^{-(1-\frac{2}{p})} \\
&\qquad \times \int_0^{\frac{t}{2}} \int_{\mathbb{R}^3} \left\| G_{\frac{1}{2}}\left(\cdot - \frac{y}{\sqrt{t-s}}\right) - G_{\frac{1}{2}}(\cdot)\right\|_{L^{p'}} |u(y,s)|^2\, dy\, ds.
\end{aligned}
$$

Here, let $R > 0$ be a positive parameter to be chosen later, and let t satisfy $t > 2R$. We decompose

$$\begin{aligned} &t^{\frac{1}{2}+\frac{3}{2}(1-\frac{1}{p})} (1+|\Omega|t)^{1-\frac{2}{p}} \|J_{12}(t)\|_{L^p} \\ &\quad \le C \left(\int_0^R + \int_R^{\frac{t}{2}} \right) \int_{\mathbb{R}^3} \left\| G_{\frac{1}{2}} \left(\cdot - \frac{y}{\sqrt{t-s}} \right) - G_{\frac{1}{2}}(\cdot) \right\|_{L^{p'}} |u(y,s)|^2 \, dy \, ds. \end{aligned} \tag{5.40}$$

Since $|u(y,s)|^2 \in L^1(\mathbb{R}^3_y \times (0,R)_s)$ by Lemma 4.2 (2), it follows from the dominated convergence theorem that

$$\lim_{t\to\infty} \int_0^R \int_{\mathbb{R}^3} \left\| G_{\frac{1}{2}} \left(\cdot - \frac{y}{\sqrt{t-s}} \right) - G_{\frac{1}{2}}(\cdot) \right\|_{L^{p'}} |u(y,s)|^2 \, dy \, ds = 0. \tag{5.41}$$

For the second term in (5.40), we have by Lemma 4.2 (2) that

$$\begin{aligned} &\int_R^{\frac{t}{2}} \int_{\mathbb{R}^3} \left\| G_{\frac{1}{2}} \left(\cdot - \frac{y}{\sqrt{t-s}} \right) - G_{\frac{1}{2}}(\cdot) \right\|_{L^{p'}} |u(y,s)|^2 \, dy \, ds \\ &\quad \le 2\|G_{\frac{1}{2}}\|_{L^{p'}} \int_R^{\frac{t}{2}} \|u(s)\|_{L^2}^2 \, ds \\ &\quad \le C \int_R^{\frac{t}{2}} \frac{1}{(1+s)^{\frac{5}{2}}} \, ds \le \frac{C}{(1+R)^{\frac{3}{2}}}. \end{aligned} \tag{5.42}$$

Then, for any $\varepsilon > 0$, take a large $R = R_\varepsilon > 0$ so that $(1+R)^{-\frac{3}{2}} \le \varepsilon$. Then, it follows from (5.41) to (5.42) that

$$\limsup_{t\to\infty} t^{\frac{1}{2}+\frac{3}{2}(1-\frac{1}{p})} (1+|\Omega|t)^{1-\frac{2}{p}} \|J_{12}(t)\|_{L^p} \le C\varepsilon,$$

which yields

$$\lim_{t\to\infty} t^{\frac{1}{2}+\frac{3}{2}(1-\frac{1}{p})} (1+|\Omega|t)^{1-\frac{2}{p}} \|J_{12}(t)\|_{L^p} = 0. \tag{5.43}$$

Finally, we consider the estimate for J_{13} in (5.34). We firstly remark that there hold

$$-P(\xi)JP(\xi) = \frac{\xi_3}{|\xi|} R(\xi), \quad P(\xi)R(\xi) = R(\xi)P(\xi) = R(\xi), \quad R(\xi)^2 = -P(\xi)$$

and then

$$\begin{aligned} &- P(\xi)JP(\xi) \left\{ \cos\left(\Omega \frac{\xi_3}{|\xi|} t \right) P(\xi) + \sin\left(\Omega \frac{\xi_3}{|\xi|} t \right) R(\xi) \right\} \\ &\quad = \frac{\xi_3}{|\xi|} \left\{ -\sin\left(\Omega \frac{\xi_3}{|\xi|} t \right) P(\xi) + \cos\left(\Omega \frac{\xi_3}{|\xi|} t \right) R(\xi) \right\}. \end{aligned}$$

Hence we have

$$\partial_t \widetilde{K}_\Omega(x,t) = \Delta \widetilde{K}_\Omega(x,t) - \Omega \mathbb{P} J \mathbb{P} \widetilde{K}_\Omega(x,t). \tag{5.44}$$

Then it follows from the mean value theorem and (5.44) that

$$\begin{aligned} J_{13} &= -\sum_{j=1}^{3} \int_0^{\frac{t}{2}} \int_{\mathbb{R}^3} \left\{ \partial_j \widetilde{K}_\Omega(x, t-s) - \partial_j \widetilde{K}_\Omega(x,t) \right\} (u_j u)(y,s)\, dy\, ds \\ &= \sum_{j=1}^{3} \int_0^{\frac{t}{2}} \int_0^1 \partial_j \partial_t \widetilde{K}_\Omega(x, t-\tau s)\, d\tau \int_{\mathbb{R}^3} s(u_j u)(y,s)\, dy\, ds \\ &= \sum_{j=1}^{3} \int_0^{\frac{t}{2}} \int_0^1 \partial_j \Delta \widetilde{K}_\Omega(x, t-\tau s)\, d\tau \int_{\mathbb{R}^3} s(u_j u)(y,s)\, dy\, ds \\ &\quad - \Omega \sum_{j=1}^{3} \int_0^{\frac{t}{2}} \int_0^1 \partial_j \mathbb{P} J \mathbb{P} \widetilde{K}_\Omega(\cdot, t-\tau s)\, d\tau \int_{\mathbb{R}^3} s(u_j u)(y,s)\, dy\, ds \\ &=: K_1 - K_2. \end{aligned} \tag{5.45}$$

By (5.35) and Lemma 4.2 (2), we have

$$\begin{aligned} \|K_1(t)\|_{L^p} &\le \sum_{j=1}^{3} \int_0^{\frac{t}{2}} \int_0^1 \left\| \partial_j \Delta \widetilde{K}_\Omega(\cdot, t-\tau s) \right\|_{L^p} d\tau \int_{\mathbb{R}^3} s|(u_j u)(y,s)|\, dy\, ds \\ &\le C \int_0^{\frac{t}{2}} \int_0^1 (t-\tau s)^{-\frac{3}{2} - \frac{3}{2}(1-\frac{2}{p})} \left\{ 1 + |\Omega|(t-\tau s) \right\}^{-(1-\frac{2}{p})} s \|u(s)\|_{L^2}^2\, d\tau\, ds \\ &\le C t^{-\frac{3}{2} - \frac{3}{2}(1-\frac{2}{p})} \left(1 + |\Omega| t\right)^{-(1-\frac{2}{p})} \int_0^{\frac{t}{2}} \frac{s}{(1+s)^{\frac{5}{2}}}\, ds \\ &\le C t^{-\frac{3}{2} - \frac{3}{2}(1-\frac{2}{p})} \left(1 + |\Omega| t\right)^{-(1-\frac{2}{p})}. \end{aligned} \tag{5.46}$$

Hence (5.45) and (5.46) yield

$$t^{\frac{1}{2} + \frac{3}{2}(1-\frac{2}{p})} \left(1 + |\Omega| t\right)^{1-\frac{2}{p}} \|J_{13}(t) + K_2(t)\|_{L^p} \le \frac{C}{t} \to 0 \qquad (t \to \infty). \tag{5.47}$$

Now, the Duhamel formula (1.20) and the decompositions (5.21), (5.34), (5.45) gives

$$\begin{aligned} &u(t) + \sum_{j=1}^{3} \partial_j K_\Omega(\cdot, t) \int_{\mathbb{R}^3} y_j u_0(y)\, dy + \sum_{j=1}^{3} \partial_j \widetilde{K}_\Omega(\cdot, t) \int_0^\infty \int_{\mathbb{R}^3} (u_j u)(y,s)\, dy\, ds \\ &\quad + \Omega \sum_{j=1}^{3} \int_0^{\frac{t}{2}} \int_0^1 \partial_j \mathbb{P} J \mathbb{P} \widetilde{K}_\Omega(\cdot, t-\tau s)\, d\tau \int_{\mathbb{R}^3} s(u_j u)(y,s)\, dy\, ds \\ &= \left\{ e^{-tA_\Omega} u_0 + \sum_{j=1}^{3} \partial_j K_\Omega(\cdot, t) \int_{\mathbb{R}^3} y_j u_0(y)\, dy \right\} + J_2 + J_{11} + J_{12} + (J_{13} + K_2). \end{aligned}$$

Therefore, we obtain the desired asymptotic behavior (1.25) by (3.14) in Theorem 3.4, (5.33), (5.36), (5.43) and (5.47). This completes the proof of Theorem 1.3. □

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

References

[1] Ahn, J., Kim, J., Lee, J.: Coriolis effect on temporal decay rates of global solutions to the fractional Navier–Stokes equations. Math. Ann. **383**, 259–289 (2022)
[2] Babin, A., Mahalov, A., Nicolaenko, B.: Regularity and integrability of 3D Euler and Navier–Stokes equations for rotating fluids. Asymptot. Anal. **15**, 103–150 (1997)
[3] Babin, A., Mahalov, A., Nicolaenko, B.: Global regularity of 3D rotating Navier–Stokes equations for resonant domains. Indiana Univ. Math. J. **48**, 1133–1176 (1999)
[4] Babin, A., Mahalov, A., Nicolaenko, B.: 3D Navier–Stokes and Euler equations with initial data characterized by uniformly large vorticity. Indiana Univ. Math. J. **50**, 1–35 (2001)
[5] Chemin, J.-Y., Desjardins, B., Gallagher, I., Grenier, E.: Anisotropy and dispersion in rotating fluids, Stud. Math. Appl., 31, North-Holland, Amsterdam, 171–192 (2002)
[6] Chemin, J.-Y., Desjardins, B., Gallagher, I., Grenier, E.: Mathematical geophysics, Oxford Lecture Series in Mathematics and its Applications, 32, The Clarendon Press, Oxford University Press, Oxford, xii+250 (2006)
[7] Fujigaki, Y., Miyakawa, T.: Asymptotic profiles of nonstationary incompressible Navier–Stokes flows in the whole space. SIAM J. Math. Anal. **33**, 523–544 (2001)
[8] Giga, Y., Inui, K., Mahalov, A., Matsui, S.: Navier–Stokes equations in a rotating frame in $\mathbb{R}^3$ with initial data nondecreasing at infinity. Hokkaido Math. J. **35**, 321–364 (2006)
[9] Giga, Yoshikazu, Inui, K., Mahalov, A., Saal, J.: Uniform global solvability of the rotating Navier–Stokes equations for nondecaying initial data. Indiana Univ. Math. J. **57**, 2775–2791 (2008)
[10] Hieber, M., Shibata, Y.: The Fujita–Kato approach to the Navier-Stokes equations in the rotational framework. Math. Z. **265**, 481–491 (2010)
[11] Ishige, K., Ishiwata, M., Kawakami, T.: The decay of the solutions for the heat equation with a potential. Indiana Univ. Math. J. **58**, 2673–2707 (2009)
[12] Ishige, K., Kawakami, T.: Refined asymptotic profiles for a semilinear heat equation. Math. Ann. **353**, 161–192 (2012)
[13] Ishige, K., Kawakami, T.: Asymptotic expansions of solutions of the Cauchy problem for nonlinear parabolic equations. J. Anal. Math. **121**, 317–351 (2013)
[14] Ishige, K., Kawakami, T., Kobayashi, K.: Asymptotics for a nonlinear integral equation with a generalized heat kernel. J. Evol. Equ. **14**, 749–777 (2014)
[15] Iwabuchi, T., Takada, R.: Global solutions for the Navier–Stokes equations in the rotational framework. Math. Ann. **357**, 727–741 (2013)
[16] Iwabuchi, T., Takada, R.: Global well-posedness and ill-posedness for the Navier–Stokes equations with the Coriolis force in function spaces of Besov type. J. Funct. Anal. **267**, 1321–1337 (2014)
[17] Kato, T.: Strong L^p-solutions of the Navier–Stokes equation in R^m, with applications to weak solutions. Math. Z. **187**(4), 471–480 (1984)
[18] Kajikiya, R., Miyakawa, M.: On L^2 decay of weak solutions of the Navier–Stokes equations in R^n. Math. Z. **192**, 135–148 (1986)
[19] Kim, J.: Rotational effect on the asymptotic stability of the MHD system. J. Differential Equations **319**, 288–311 (2022)

[20] Koh, Y., Lee, S., Takada, R.: Dispersive estimates for the Navier–Stokes equations in the rotational framework. Adv. Differential Equations **19**, 857–878 (2014)
[21] Konieczny, P., Yoneda, T.: On dispersive effect of the Coriolis force for the stationary Navier–Stokes equations. J. Differential Equations **250**, 3859–3873 (2011)
[22] Kozono, H., Ogawa, T., Taniuchi, Y.: Navier–Stokes equations in the Besov space near L^∞ and BMO. Kyushu J. Math. **57**, 303–324 (2003)
[23] Masuda, K.: Weak solutions of Navier–Stokes equations. Tohoku Math. J. (2) **36**, 623–646 (1984)
[24] Miyakawa, T.: Hardy spaces of solenoidal vector fields, with applications to the Navier–Stokes equations. Kyushu J. Math. **50**, 1–64 (1996)
[25] Miyakawa, T.: Application of Hardy space techniques to the time-decay problem for incompressible Navier–Stokes flows in R^n. Funkcial. Ekvac. **41**, 383–434 (1998)
[26] Okabe, T., Tsutsui, Y.: Navier–Stokes flow in the weighted Hardy space with applications to time decay problem. J. Differ. Equ. **261**, 1712–1755 (2016)
[27] Schonbek, M.E.: Large time behaviour of solutions to the Navier–Stokes equations. Comm. Partial Differ. Equ. **11**, 733–763 (1986)
[28] Tsutsui, Y.: An application of weighted Hardy spaces to the Navier–Stokes equations. J. Funct. Anal. **266**, 1395–1420 (2014)
[29] Wiegner, M.: Decay results for weak solutions of the Navier–Stokes equations on R^n. J. London Math. Soc. (2) **35**, 303–313 (1987)

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