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Stabilizing Effect of the Magnetic Field and Large-Time Behavior of 2D Incompressible MHD System with Vertical Dissipation

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Abstract. The stabilizing and damping phenomenon of a background magnetic field on electrically conducting fluids has been observed in various physical experiments and numerical simulations. This paper establishes this observation as mathematically rigorous decay results on a 2D magnetohydrodynamic (MHD) system with only partial dissipation. Without the magnetic field, the fluid velocity obeys a 2D anisotropic Navier–Stokes equation and is not known to be stable in the Sobolev setting H^2 due to the potential double exponential growth of its H^2-norm in time. However, when coupled with the magnetic field in the MHD system concerned here, we show that the H^2-norm of any perturbation near a background magnetic field actually decays algebraically in time. This result demonstrates that the magnetic field indeed stabilizes and damps the electrically conducting fluids. Mathematically this result along with its proof offers a new and effective approach to the large-time behavior on partially dissipated systems of partial differential equations. Existing methods are mostly designed for systems with full dissipation and do not apply when the dissipation is anisotropic.

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1. Introduction

This paper intends to understand the stability problem and especially the precise large-time behavior on the perturbations near a background magnetic field governed by the incompressible magnetohydrodynakic (MHD) system. This study is partially motivated by a remarkable stabilizing phenomenon exhibited by electrically conducting fluids. Extensive physical experiments and numerical simulations have performed to understand the influence of the magnetic field on the bulk turbulence involving various electrically conducting fluids such as liquid metals. These experiments and simulations have observed a remarkable phenomenon that a background magnetic field can smooth and stabilize turbulent electrically conducting fluids (see, e.g., [1,2,6,11–13,21,22]).

We focus on a very special 2D incompressible MHD system with anisotropic dissipation,

$$\begin{cases} \partial_t u + u\cdot\nabla u + \nabla P = \nu\partial_{22}u + B\cdot\nabla B, \\ \partial_t B + u\cdot\nabla B + \eta B = B\cdot\nabla u, \\ \nabla\cdot u = \nabla\cdot B = 0, \end{cases} \tag{1.1}$$

where u represents the velocity field, P the total pressure and B the magnetic field, and ν and η denote the viscosity and the magnetic damping coefficients, respectively. The MHD systems, the center piece of the magnetohydrodynamics initiated by H. Alfvén [2], models electrically conducting fluid such as plasmas, liquid metals and electrolytes, and have a very wide range of applications in astrophysics, geophysics, cosmology and engineering (see, e.g., [4,13,35]). The MHD equations are also mathematically important. They not only share many crucial features with the Euler or the Navier–Stokes equations, but also exhibit many more fascinating characteristics such as various wave phenomena that the Euler or the Navier–Stokes equations lack.

Clearly, (1.1) admits a special class of steady-state solutions represented by the background magnetic field. Attention is focused on the steady-state solution

$$u^{(0)} = (0,0), \quad B^{(0)}(x) = e_1 = (1,0).$$

The perturbation (u, b) around this steady solution with $b = B - e_1$ obeys

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla P = \nu \partial_{22} u + b \cdot \nabla b + \partial_1 b, \\ \partial_t b + u \cdot \nabla b + \eta b = b \cdot \nabla u + \partial_1 u, \\ \nabla \cdot u = \nabla \cdot b = 0, \\ u(x,0) = u_0(x), \ b(x,0) = b_0(x). \end{cases} \tag{1.2}$$

The system (1.2) differs from the original system (1.1) by two extra terms, $\partial_1 b$ and $\partial_1 u$. As we shall see later, these two terms generated due to the background magnetic field play an important role in the stability properties of the perturbation as well as in the large-time behavior. These terms reflect the influence of the background magnetic field on the behavior of the fluids.

Our goal has been to understand the stability problem and the large-time behavior of solutions to (1.2). Due to the lack of the horizontal dissipation, these problems are not trivial. even when the magnetic field is identically zero, $b \equiv 0$, the velocity u satisfies the 2D anisotropic Navier–Stokes equation

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla P + \nu \partial_{22} u, \quad x \in \mathbb{R}^2, \, t > 0, \\ \nabla \cdot u = 0 \end{cases} \tag{1.3}$$

or, in terms of the vorticity $\omega = \nabla \times u$,

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = \nu \partial_{22} \omega, \quad x \in \mathbb{R}^2, \, t > 0, \\ u = \nabla^{\perp} \Delta^{-1} \omega := (-\partial_2, \partial_1) \Delta^{-1} \omega. \end{cases} \tag{1.4}$$

The stability problem on (1.4) in the Sobolev setting H^2 remains an open problem in the whole space case, although this problem in some other domains such as $\mathbb{R} \times \mathbb{T}$ has been resolved [17]. In the case of the whole space domain, the dissipation in one direction is insufficient to control the nonlinearity when we estimate the H^2-norm of u or the H^1-norm of ω. In fact, in the estimate of $\|\nabla \omega\|_{L^2}$,

$$\frac{d}{dt} \|\nabla \omega(t)\|_{L^2}^2 + 2\nu \|\partial_2 \nabla \omega(t)\|_{L^2}^2 = -2 \int \nabla \omega \cdot \nabla u \cdot \nabla \omega \, dx,$$

the nonlinear part contains four component terms

$$\begin{aligned} \text{Hard} &:= -\int_{\mathbb{R}^2} \nabla \omega \cdot \nabla u \cdot \nabla \omega \, dx \\ &= -\int_{\mathbb{R}^2} \partial_1 u_1 \, (\partial_1 \omega)^2 \, dx - \int_{\mathbb{R}^2} \partial_1 u_2 \, \partial_1 \omega \, \partial_2 \omega \, dx \\ &\quad - \int_{\mathbb{R}^2} \partial_2 u_1 \, \partial_1 \omega \, \partial_2 \omega \, dx - \int_{\mathbb{R}^2} \partial_2 u_2 \, (\partial_2 \omega)^2 \, dx \end{aligned} \tag{1.5}$$

and the first two terms in (1.5) do not admit any time-integrable upper bound. As a consequence, the best upper bound for the gradient of the vorticity $\|\nabla \omega(t)\|_{L^q}$ with $1 \le q \le \infty$ is double exponentially in time,

$$\|\nabla \omega(t)\|_{L^q} \le (\|\nabla \omega(0)\|_{L^q})^{e^{C\|\omega(0)\|_{L^\infty} t}}. \tag{1.6}$$

Indeed in the case of the 2D Euler equation in a unit disk, Kiselev and Sverak were able to construct an explicit vorticity solution whose gradient grows double exponentially [28]. Furthermore, classical approaches on the MHD well-posedness problem treat the magnetic field related terms as bad terms. As a consequence, the stability problem and large-time behavior concerned here in the classical framework appear to be hopeless.

The novel idea here is to treat the magnetic field related terms as good terms and to explore the smoothing and stabilizing effects of the magnetic field through coupling and interaction. In a previous work [20], the authors were successful in implementing this strategy to establish the stability of solutions to (1.2). For the sake of convenience of later references, we reproduce Theorem 1.3 from [20] here.

Theorem 1.1 [Theorem 1.3, [20]]. *Let $\nu > 0$ and $\eta > 0$. Consider (1.2) with the initial data $(u_0, b_0) \in H^2(\mathbb{R}^2)$, and $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. Then there exists a constant $\varepsilon = \varepsilon(\nu, \eta) > 0$ such that, if*

$$\|u_0\|_{H^2} + \|b_0\|_{H^2} \le \varepsilon,$$

then (1.2) has a unique global classical solution (u, b) satisfying, for any $t > 0$,

$$\|u(t)\|_{H^2}^2 + \|b(t)\|_{H^2}^2 + \int_0^t \left(\|\partial_1 u\|_{L^2}^2 + \|\partial_2 u\|_{H^2}^2 + \|b\|_{H^2}^2 \right) \, d\tau \le C\,\varepsilon^2$$

for some universal constant $C > 0$.

The goal of this paper is to give a precise account on the large-time behavior of these stable solutions. Clearly we need to continue to pursue the stabilizing and damping effect of the magnetic field. To do so, we combine the equations of u and b to derive an equivalent system of wave equations to reveal the stabilizing mechanism. We start by separating the linear terms in (1.2) from the nonlinear ones. Applying the Helmholtz-Leray projection operator

$$\mathbb{P} := I - \nabla \Delta^{-1} \nabla \cdot$$

to the velocity equation in (1.2), we eliminate the pressure to obtain

$$\partial_t u = \nu \partial_{22} u + \partial_1 b + N_1, \qquad N_1 = \mathbb{P}(-u \cdot \nabla u + b \cdot \nabla b). \tag{1.7}$$

By separating the linear terms from the nonlinear ones in (1.2), the equation of b can be written as

$$\partial_t b = -\eta b + \partial_1 u + N_2, \qquad N_2 = -u \cdot \nabla b + b \cdot \nabla u.$$

Thus, (1.2) can be written as

$$\begin{cases} \partial_t u = \nu \partial_{22} u + \partial_1 b + N_1, \\ \partial_t b = -\eta b + \partial_1 u + N_2, \\ \nabla \cdot u = \nabla \cdot b = 0, \\ u(x,0) = u_0(x), \; b(x,0) = b_0(x). \end{cases} \tag{1.8}$$

Differentiating (1.8) in time and making several substitutions, we find

$$\begin{cases} \partial_{tt} u - (\nu \partial_{22} - \eta) \partial_t u - (\partial_{11} u + \eta \nu \partial_{22} u) = N_3, \\ \partial_{tt} b - (\nu \partial_{22} - \eta) \partial_t b - (\partial_{11} b + \eta \nu \partial_{22} b) = N_4, \end{cases} \tag{1.9}$$

where N_3 and N_4 are given by

$$N_3 = (\partial_t + \eta) N_1 + \partial_1 N_2, \qquad N_4 = (\partial_t - \nu \partial_{22}) N_2 + \partial_1 N_1.$$

Surprisingly, both u and b are found to satisfy nonhomogeneous wave equations with exactly the same linear parts. Clearly, (1.9) exhibits much more regularization than its original counterpart in (1.2). The stabilizing and damping properties of (1.9) is a consequence of the background magnetic field and interactions within the MHD system. By exploiting these properties, we are able to establish the following theorem assessing the large-time behavior of the solutions of (1.2).

Theorem 1.2. *Let $\nu > 0$ and $\eta > 0$. Assume $(u_0, b_0) \in H^2 \cap L^1$ satisfies $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$, $\|(u_0, b_0)\|_{H^2 \cap L^1} \le \delta$ for sufficiently small $\delta > 0$. Let (u, b) be the corresponding solution obtained in Theorem 1.1, then, for a pure constant $c > 0$,*

$$\|(u(t), b(t))\|_{L^2} \le c\delta(1+t)^{-\frac{1}{2}}, \tag{1.10}$$

$$\|(\partial_1 u(t), \partial_1 b(t))\|_{L^2} \le c\delta(1+t)^{-\frac{1}{2}},$$

$$\|(\partial_2 u(t), \partial_2 b(t))\|_{L^2} \le c\delta(1+t)^{-1}. \tag{1.11}$$

In contrast to the potential double exponential growth rate in (1.6), Theorem 1.2 asserts that the solution of (1.2) actually decay algebraically in time. This result rigorously confirms the experimentally observed stabilizing and damping effect of the background magnetic field. The decay rates in (1.10) and (1.11) are the same as those for the fully dissipative heat equation, and reveal the stabilizing and damping effect of the magnetic field.

Theorem 1.2 is also mathematically important. It establishes the precise large-time behavior of a partially dissipated system. Many powerful classical methods designed for the large-time behavior of fully dissipated systems such as Schonbek's Fourier splitting scheme ([38–40]) may not apply to partially dissipated systems. The approach presented in this paper serves as a new method that work for some partially dissipated systems of partial differential equations.

Due to its physical applications and mathematical significance, the stability and large-time behavior problems on the MHD equations near a background magnetic field have recently attracted considerable interests. The stability problem on either the ideal MHD system or the fully dissipated MHD system with identical viscosity and resistivity has been thoroughly investigated and significant results have been obtained [2,3,7,23]. The requirement that the viscosity coefficient be the same as the resistivity coefficient comes from the use of the Elsässer variables. [44] allows these two coefficients to be slightly different. The paper of Lin et al. [31] initiated the study on the stability problem of the 2D MHD system with only velocity dissipation. By using the Lagrangian approach and controlling all quantities in terms of the trajectory, they were able to establish the desired stability. The work of Ren et al. [36] examined the stability and the large-time behavior simultaneously of the 2D MHD system without resistivity in an anisotropic Besov setting. The approach in [36] is Eulerian and establishes extensive anisotropic energy estimates. Instead of the velocity dissipation, Wu et al. [48] studied the stability of the 2D MHD system with only velocity damping and without resistivity. Their paper exploits the wave structure of the system. More recent studies on the MHD stability problem focuses on the anisotropic MHD systems. The paper of Boardman, Lin and Wu [5] deals with the stability problem on the 2D MHD system with the fluid vorticity satisfying an Euler-like equation. Wu and Zhu established the stability of the 3D anisotropic MHD system with velocity dissipation in two directions and the magnetic diffusion in only one direction [49]. We remark that there are substantial recent developments on the well-posedness and stability problems on the MHD systems and many other important results are also available (see, e.g., [8–10,14–16,18,19,24–27,29,30,32–34,37,41,43,46,47,49–53,55–59]). This list is by no means exhaustive.

We explain the main idea in the proof of Theorem 1.2. Clearly Theorem 1.2 can not be established via direct energy methods. Instead the approach here is to represent (1.2) in an integral form and then apply the bootstrapping argument. To convert (1.2) into an integral form, we first take the Fourier transform of (1.8) to obtain

$$\begin{cases} \partial_t \widehat{u} = -\nu\xi_2^2 \widehat{u} + i\xi_1 \widehat{b} + \widehat{N_1}, \\ \partial_t \widehat{b} = -\eta \widehat{b} + i\xi_1 \widehat{u} + \widehat{N_2}. \end{cases} \tag{1.12}$$

(1.12) can be written as a 2D system associated with a matrix A,

$$\partial_t \begin{pmatrix} \widehat{u} \\ \widehat{b} \end{pmatrix} = A \begin{pmatrix} \widehat{u} \\ \widehat{b} \end{pmatrix} + \begin{pmatrix} \widehat{N_1} \\ \widehat{N_2} \end{pmatrix}.$$

where

$$A = \begin{pmatrix} -\nu\xi_2^2 & i\xi_1 \\ i\xi_1 & -\eta \end{pmatrix}.$$

By Duhamel's principle,

$$\begin{pmatrix} \widehat{u}(t) \\ \widehat{b}(t) \end{pmatrix} = e^{At} \begin{pmatrix} \widehat{u}_0 \\ \widehat{b}_0 \end{pmatrix} + \int_0^t e^{A(t-\tau)} \begin{pmatrix} \widehat{N_1(\tau)} \\ \widehat{N_2(\tau)} \end{pmatrix} \, d\tau, \tag{1.13}$$

The fundamental solution matrix e^{At} can be made more explicit via the eigenvalues and eigenvectors of A. In fact, if λ_1 and λ_2 are the roots of the characteristic polynomial associated with A,

$$\lambda^2 + (\eta + \nu\xi_2^2)\lambda + \xi_1^2 + \nu\eta\xi_2^2 = 0$$

or

$$\lambda_1 = \frac{-(\eta + \nu\xi_2^2) - \sqrt{\Gamma}}{2}, \quad \lambda_2 = \frac{-(\eta + \nu\xi_2^2) + \sqrt{\Gamma}}{2}$$

with

$$\Gamma = (\eta + \nu\xi_2^2)^2 - 4(\xi_1^2 + \nu\eta\xi_2^2),$$

then e^{At} can be written explicitly as

$$e^{At} = \begin{pmatrix} \widehat{K_1} & \widehat{K_2} \\ \widehat{K_2} & \widehat{K_3} \end{pmatrix},$$

where

$$\begin{aligned} \widehat{K_1} &= \eta\frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} + \frac{\lambda_1 e^{\lambda_1 t} - \lambda_2 e^{\lambda_2 t}}{\lambda_1 - \lambda_2}, \\ \widehat{K_2} &= i\xi_1\frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}, \\ \widehat{K_3} &= \frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_1 - \lambda_2} - \eta\frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}. \end{aligned}$$

Thus we have converted (1.2) into the integral form

$$\begin{cases} \widehat{u}(t) = \widehat{K_1}\widehat{u_0} + \widehat{K_2}\widehat{b_0} + \int_0^t \widehat{K_1}(t-\tau)\widehat{N_1}(\tau) + \widehat{K_2}(t-\tau)\widehat{N_2}(\tau)\, d\tau, \\ \widehat{b}(t) = \widehat{K_2}\widehat{u_0} + \widehat{K_3}\widehat{b_0} + \int_0^t \widehat{K_2}(t-\tau)\widehat{N_1}(\tau) + \widehat{K_3}(t-\tau)\widehat{N_2}(\tau)\, d\tau. \end{cases} \tag{1.14}$$

More technical details are provided in Proposition 2.1.

The next step is to extract the desired large-time decay estimates from the integral representation in (1.14). We use the bootstrapping argument (see, e.g., [42, p. 21]). As a preparation, we first derive suitable upper bounds for the kernel functions. Clearly the kernel functions are anisotropic and frequency dependent. By dividing the frequency space $\mathbb{R}^2$ into suitable subsets, we are able to obtain definite upper bounds for the kernel functions in each subset. The details are given in Proposition 2.2. To implement the bootstrapping argument, we make the ansatz

$$\begin{aligned} \|(u(t), b(t))\|_{L^2} &\le \tilde{c}\delta(1+t)^{-\frac{1}{2}}, \\ \|(\partial_1 u(t), \partial_1 b(t))\|_{L^2} &\le \tilde{c}\delta(1+t)^{-\frac{1}{2}}, \\ \|(\partial_2 u(t), \partial_2 b(t))\|_{L^2} &\le \tilde{c}\delta(1+t)^{-1}, \end{aligned} \tag{1.15}$$

where $\tilde{c}$ will be specified later. We show through the integral representation of $\widehat{u}$ and $\widehat{b}$ in (1.14) that

$$\begin{aligned} \|(u(t), b(t))\|_{L^2} &\le \frac{\tilde{c}}{2}\delta(1+t)^{-\frac{1}{2}}, \\ \|(\partial_1 u(t), \partial_1 b(t))\|_{L^2} &\le \frac{\tilde{c}}{2}\delta(1+t)^{-\frac{1}{2}}, \\ \|(\partial_2 u(t), \partial_2 b(t))\|_{L^2} &\le \frac{\tilde{c}}{2}\delta(1+t)^{-1}, \end{aligned} \tag{1.16}$$

with the coefficients being half of the corresponding ones in (1.15). Then the bootstrapping argument implies that (1.16) holds for all $1 \le t < \infty$. The process of establishing upper bounds in (1.16) is very long and tedious, and the details are presented in three subsections in Sect. 3. We just want to mention some of the technical points. Due to the higher decay rate for the vertical derivative than the horizontal one, efforts have been made throughout to replace the horizontal derivatives by the vertical ones. One way to do so is to make use of the divergence-free condition, $\nabla \cdot u = \nabla \cdot b = 0$. Another helpful way is to invoke the anisotropic type inequalities such as

$$\|f\|_{L^\infty(\mathbb{R}^2)} \leq C\|f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}} \|\partial_1 f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}} \|\partial_2 f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}} \|\partial_{12} f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}}.$$

These type of technicalities are used throughout the proof such as in (3.30) and many other places. The proof also employs many other helpful strategies such as dividing the time integral involving the nonlinear terms into two parts such as

$$\begin{aligned} &\int_0^t \|\widehat{K}_1(t-\tau)\, \widehat{u\cdot\nabla u}\|_{L^2(A_1)}\, d\tau \\ &= \int_0^{t/2} \|\widehat{K}_1(t-\tau)\, \widehat{u\cdot\nabla u}\|_{L^2(A_1)}\, d\tau + \int_{t/2}^t \|\widehat{K}_1(t-\tau)\, \widehat{u\cdot\nabla u}\|_{L^2(A_1)}\, d\tau. \end{aligned}$$

This division would help distinguish different properties of the integrand in different time intervals. The decay of the first piece relies on the kernel function while the decay of the second piece comes from the nonlinear term. We leave more technical details to Sect. 3.

The rest of this paper is divided into two main sections. Section 2 provides the details in the derivation of the integral representation (1.14). In addition, this section divides the frequency space $\mathbb{R}^2$ into suitable subdomains and establishes explicit upper bounds for the kernel functions in each subdomain. Section 3 presents the proof of Theorem 1.2 by applying the bootstrapping argument to (1.14). This is a very long and tedious process. For the sake of clarity, we divide this section into three subsections with each devoted to one of the inequalities in (1.16).

2. The Integral Representation and Bounds for the Kernels

This section details the derivation of the integral representation and establishes upper bounds for the kernel functions involved in the integral representation. These upper bounds will be used in the proof of Theorem 1.2. Proposition 2.1 and its proof are devoted to the integral representation while Proposition 2.2 focuses on the upper bounds for the kernel functions.

Proposition 2.1. *Let $\nu > 0$ and $\eta > 0$. Assume (u, b) is a solution of (1.2). Then (u, b) satisfies*

$$\begin{cases} \widehat{u}(t) = \widehat{K_1}\widehat{u_0} + \widehat{K_2}\widehat{b_0} + \int_0^t \widehat{K_1}(t-\tau)\widehat{N_1}(\tau) + \widehat{K_2}(t-\tau)\widehat{N_2}(\tau)\ d\tau, \\ \widehat{b}(t) = \widehat{K_2}\widehat{u_0} + \widehat{K_3}\widehat{b_0} + \int_0^t \widehat{K_2}(t-\tau)\widehat{N_1}(\tau) + \widehat{K_3}(t-\tau)\widehat{N_2}(\tau)\ d\tau, \end{cases} \tag{2.1}$$

where the kernel functions $\widehat{K}_1$ through $\widehat{K}_3$ are given by

$$\begin{aligned} \widehat{K_1} &= \eta\frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} + \frac{\lambda_1 e^{\lambda_1 t} - \lambda_2 e^{\lambda_2 t}}{\lambda_1 - \lambda_2} := \eta\, G_1 + G_2, \\ \widehat{K_2} &= i\xi_1 \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} := i\xi_1 G_1, \\ \widehat{K_3} &= \frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_1 - \lambda_2} - \eta\frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} := -\eta G_1 + G_3. \end{aligned}$$

with λ_1 and λ_2 being the roots of

$$\lambda^2 + (\eta + \nu\xi_2^2)\lambda + \xi_1^2 + \nu\eta\xi_2^2 = 0$$

or

$$\lambda_1 = \frac{-(\eta+\nu\xi_2^2)-\sqrt{\Gamma}}{2}, \quad \lambda_2 = \frac{-(\eta+\nu\xi_2^2)+\sqrt{\Gamma}}{2}, \quad \Gamma = (\eta+\nu\xi_2^2)^2 - 4(\xi_1^2+\nu\eta\xi_2^2),$$

and G_1, G_2 and G_3 given by

$$G_1 = \frac{e^{\lambda_1 t}-e^{\lambda_2 t}}{\lambda_1-\lambda_2}, \quad G_2 = \frac{\lambda_1 e^{\lambda_1 t}-\lambda_2 e^{\lambda_2 t}}{\lambda_1-\lambda_2}, \quad G_3 = \frac{\lambda_1 e^{\lambda_2 t}-\lambda_2 e^{\lambda_1 t}}{\lambda_1-\lambda_2}.$$

In the case when $\lambda_1 = \lambda_2$ or $\Gamma = 0$, the formulas of the kernel functions $\widehat{K}_1$ through $\widehat{K}_3$ are replaced by the corresponding limiting formulas

$$\begin{aligned}
\widehat{K_1} &= \eta \lim_{\lambda_2\to\lambda_1} G_1 + \lim_{\lambda_2\to\lambda_1} G_2 = \eta t\, e^{\lambda_1 t} + (1+\lambda_1 t)e^{\lambda_1 t},\\
\widehat{K_2} &= i\xi_1 t e^{\lambda_1 t},\\
\widehat{K_3} &= -\eta t\, e^{\lambda_1 t} + (1-\lambda_1 t)e^{\lambda_1 t}.
\end{aligned} \tag{2.2}$$

Proof. As explained in the introduction, any solution (u, b) of (1.2) would solve (1.13), namely

$$\begin{pmatrix} \widehat{u}(t) \\ \widehat{b}(t) \end{pmatrix} = e^{At}\begin{pmatrix} \widehat{u}_0 \\ \widehat{b}_0 \end{pmatrix} + \int_0^t e^{A(t-\tau)}\begin{pmatrix} \widehat{N_1(\tau)} \\ \widehat{N_2(\tau)} \end{pmatrix} d\tau \tag{2.3}$$

with

$$A = \begin{pmatrix} -\nu\xi_2^2 & i\xi_1 \\ i\xi_1 & -\eta \end{pmatrix}.$$

The characteristic polynomial of A is

$$\lambda^2 + (\eta+\nu\xi_2^2)\lambda + \xi_1^2 + \nu\eta\xi_2^2 = 0$$

and thus the eigenvalues of A are

$$\lambda_1 = \frac{-(\eta+\nu\xi_2^2)-\sqrt{\Gamma}}{2}, \quad \lambda_2 = \frac{-(\eta+\nu\xi_2^2)+\sqrt{\Gamma}}{2}, \quad \Gamma = (\eta+\nu\xi_2^2)^2 - 4(\xi_1^2+\nu\eta\xi_2^2),$$

The eigenvectors corresponding to λ_1 and λ_2 are given by

$$v^{(1)} = \begin{pmatrix} \eta+\lambda_1 \\ i\xi_1 \end{pmatrix}, \qquad v^{(2)} = \begin{pmatrix} \eta+\lambda_2 \\ i\xi_1 \end{pmatrix},$$

respectively. Therefore,

$$\begin{aligned}
A &= \begin{pmatrix} v^{(1)} & v^{(2)} \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} v^{(1)} & v^{(2)} \end{pmatrix}^{-1}.\\
e^{At} &= \frac{1}{i\xi_1(\lambda_1-\lambda_2)}\begin{pmatrix} \eta+\lambda_1 & \eta+\lambda_2 \\ i\xi_1 & i\xi_1 \end{pmatrix}\begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}\begin{pmatrix} i\xi_1 & -(\eta+\lambda_2) \\ -i\xi_1 & \eta+\lambda_1 \end{pmatrix}\\
&:= \begin{pmatrix} \widehat{K_1} & \widehat{K_2} \\ \widehat{K_2} & \widehat{K_3} \end{pmatrix}.
\end{aligned}$$

where

$$\begin{aligned}
\widehat{K_1} &= \eta\frac{e^{\lambda_1 t}-e^{\lambda_2 t}}{\lambda_1-\lambda_2} + \frac{\lambda_1 e^{\lambda_1 t}-\lambda_2 e^{\lambda_2 t}}{\lambda_1-\lambda_2},\\
\widehat{K_2} &= i\xi_1\frac{e^{\lambda_1 t}-e^{\lambda_2 t}}{\lambda_1-\lambda_2},\\
\widehat{K_3} &= \frac{\lambda_1 e^{\lambda_2 t}-\lambda_2 e^{\lambda_1 t}}{\lambda_1-\lambda_2} - \eta\frac{e^{\lambda_1 t}-e^{\lambda_2 t}}{\lambda_1-\lambda_2}.
\end{aligned}$$

To simplify the notation, we define

$$G_1 = \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}, \quad G_2 = \frac{\lambda_1 e^{\lambda_1 t} - \lambda_2 e^{\lambda_2 t}}{\lambda_1 - \lambda_2}, \quad G_3 = \frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_1 - \lambda_2}$$

and write

$$e^{At} = \begin{pmatrix} \widehat{K_1} & \widehat{K_2} \\ \widehat{K_2} & \widehat{K_3} \end{pmatrix} = \begin{pmatrix} G_2 + \eta G_1 & i\xi_1 G_1 \\ i\xi_1 G_1 & G_3 - \eta G_1 \end{pmatrix}. \tag{2.4}$$

Inserting (2.4) in (2.3) yields (2.1). In the case when $\lambda_1 = \lambda_2$, the associated eigenvector of A is

$$v^{(1)} = \begin{pmatrix} \eta + \lambda_1 \\ i\xi_1 \end{pmatrix}$$

and the general solution of $\partial_t \widehat{V} = A\widehat{V}$ is given by

$$a_1\, v^{(1)}\, e^{\lambda_1 t} + a_2\, (v^{(1)}\, t + \sigma) e^{\lambda_1 t},$$

where a_1 and a_2 are to be determined by the initial data, and σ solves

$$(A - \lambda_1 I)\sigma = v^{(1)}.$$

After some simple computation, we find

$$\sigma = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

We determine a_1 and a_2 by the initial data $\widehat{u}_0$ and $\widehat{b}_0$. This process leads to the kernel functions in (2.2) when $\lambda_1 = \lambda_2$. This completes the proof of Proposition 2.1. □

The next proposition provides upper bounds for the kernel functions $\widehat{K}_1$ through $\widehat{K}_3$. It is clear that the kernel functions depend on the Fourier frequency and are anisotropic. Consequently we need to divide the frequency space $\mathbb{R}^2$ into suitable subsets so that the behavior of these kernel functions are definite. Our decomposition will be based on the second eigenvalue,

$$\lambda_2 = \frac{-(\eta + \nu\xi_2^2) + \sqrt{\Gamma}}{2}, \qquad \Gamma = (\eta + \nu\xi_2^2)^2 - 4(\xi_1^2 + \nu\eta\xi_2^2).$$

A natural choice is to separate the domain where λ_2 behaves like $-\frac{1}{4}(\eta + \nu\xi_2^2)$ from the rest. In particular, this occurs if

$$\sqrt{\Gamma} \le \frac{1}{2}(\eta + \nu\xi_2^2) \quad \text{or} \quad \nu\eta\xi_2^2 + \xi_1^2 \ge \frac{3}{16}(\eta + \nu\xi_2^2)^2.$$

This explains the decomposition in the following proposition.

Proposition 2.2. *Let $\nu > 0$ and $\eta > 0$. We decompose $\mathbb{R}^2$ into two subsets A_1 and A_2 with*

$$A_1 = \{\xi \in \mathbb{R}^2,\ \nu\eta\xi_2^2 + \xi_1^2 \ge \frac{3}{16}(\eta + \nu\xi_2^2)^2\},$$
$$A_2 = \{\xi \in \mathbb{R}^2,\ \nu\eta\xi_2^2 + \xi_1^2 < \frac{3}{16}(\eta + \nu\xi_2^2)^2\}.$$

A_2 is further divided into A_{21} and A_{22} with

$$\begin{aligned} A_{21} &= \{\xi \in \mathbb{R}^2,\ \xi \in A_2,\ \nu\xi_2^2 \le \eta\}, \\ A_{22} &= \{\xi \in \mathbb{R}^2,\ \xi \in A_2,\ \nu\xi_2^2 > \eta\}. \end{aligned} \tag{2.5}$$

Then

(1) *For any $\xi \in A_1$, there is $c_0 > 0$ and $C > 0$ such that*

$$|\widehat{K_1}|,\ |\widehat{K_2}|,\ |\widehat{K_3}| \le C\, e^{-c_0(1+\xi_2^2)t}$$

(2) *For any* $\xi \in A_{21}$, *there is* $c_0 > 0$ *and* $C > 0$ *such that*

$$|\widehat{K_1}|,\ |\widehat{K_2}|,\ |\widehat{K_3}| \le C\left(e^{-c_0(1+\xi_2^2)t} + e^{-c_0|\xi|^2 t}\right).$$

(3) *For any* $\xi \in A_{22}$, *there is* $c_0 > 0$ *and* $C > 0$ *such that*

$$|\widehat{K_1}|,\ |\widehat{K_2}|,\ |\widehat{K_3}| \le C\left(e^{-c_0(1+\xi_2^2)t} + e^{-c_0(1+\frac{\xi_1^2}{\xi_2^2})t}\right).$$

Proof of Proposition 2.2. We start with the case when $\xi \in A_1$. For any $\xi \in A_1$,

$$\Gamma = (\eta+\nu\xi_2^2)^2 - 4(\nu\eta\xi_2^2+\xi_1^2) \le (\eta+\nu\xi_2^2)^2 - \frac{3}{4}(\eta+\nu\xi_2^2)^2 = \frac{1}{4}(\eta+\nu\xi_2^2)^2.$$

Therefore, either $\sqrt{\Gamma}$ is pure imaginary or $\sqrt{\Gamma} \le \frac{1}{2}(\eta+\nu\xi_2^2)$. Hence, the real parts $\Re(\lambda_1)$ and $\Re(\lambda_2)$ are bounded by

$$\Re(\lambda_1) \le -\frac{1}{2}(\eta+\nu\xi_2^2), \qquad \Re(\lambda_2) \le -\frac{1}{4}(\eta+\nu\xi_2^2).$$

To bound $\widehat{K_1}$, $\widehat{K_2}$ and $\widehat{K_3}$, we realize that they all involve only λ_1, λ_2 and G_1. In fact, since G_2 and G_3 can be written as

$$G_2 = \frac{\lambda_1 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t} + \lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_2 t}}{\lambda_1-\lambda_2} = e^{\lambda_2 t} + \lambda_1 G_1$$
$$G_3 = \frac{(\lambda_1-\lambda_2)e^{\lambda_2 t} + \lambda_2(e^{\lambda_2 t} - e^{\lambda_1 t})}{\lambda_1-\lambda_2} = e^{\lambda_2 t} - \lambda_2 G_1,$$

we have

$$\widehat{K}_1 = e^{\lambda_2 t} + \lambda_1 G_1 + \eta G_1, \quad \widehat{K_2} = i\xi_1 G_1, \quad \widehat{K_3} = e^{\lambda_2 t} - \lambda_2 G_1 - \eta G_1. \tag{2.6}$$

When $\Gamma > 0$, both λ_1 and λ_2 are real. Then the mean-value theorem implies that there is $\frac{1}{4} < a < 1$ such that

$$G_1 = \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1-\lambda_2} = t\, e^{-a(\eta+\nu\xi_2^2)t}.$$

When $\Gamma = 0$, $\lambda_1 = \lambda_2$ and (2.2) implies that G_1 is replaced by

$$G_1 = t\, e^{\lambda_1 t}.$$

When $\Gamma < 0$, both λ_1 and λ_2 are complex and

$$G_1 = -e^{-\frac{1}{2}(\eta+\nu\xi_2^2)t}\ \frac{i\sin(\frac{\sqrt{-\Gamma}}{2}t)}{\frac{\sqrt{-\Gamma}}{2}}$$

Therefore we always have

$$|G_1| \le t e^{-\frac{1}{4}(\eta+\nu\xi_2^2)t}.$$

We can check that λ_1 and λ_2 admit the following upper bound,

$$|\lambda_1|, |\lambda_2| \le \eta+\nu\xi_2^2.$$

In fact, if λ_1 is real, then

$$|\lambda_1| \le \frac{\eta+\nu\xi_2^2}{2} + \frac{\eta+\nu\xi_2^2}{4} = \frac{3}{4}(\eta+\nu\xi_2^2).$$

If λ_1 is complex-valued, then

$$|\lambda_1| \le \sqrt{\frac{(\eta+\nu\xi_2^2)^2 + \frac{1}{4}(\eta+\nu\xi_2^2)^2}{4}} = \frac{\sqrt{5}}{4}(\eta+\nu\xi_2^2).$$

It is then clear that λ_1 always satisfies

$$|\lambda_1| \le \eta + \nu\xi_2^2.$$

A similar argument leads to the bound $|\lambda_2| \le \eta + \nu\xi_2^2$. Then the upper bound $\widehat{K}_1$ follows easily from the definition of $\widehat{K}_1$ and the upper bound above for λ_1.

Using the simple fact that $\rho e^{-C_1\rho} \le C_2$ for any $\rho \ge 0$ and $C_1 > 0$ and suitable $C_2 > 0$, we have, for $c_0 > 0$ and $C > 0$,

$$|\widehat{K_1}| \le |e^{\lambda_2 t}| + |\lambda_1 G_1| + \eta |G_1| \le e^{-\frac{1}{4}(\eta+\nu\xi_2^2)t} + 2(\eta+\nu\xi_2^2)te^{-\frac{1}{4}(\eta+\nu\xi_2^2)t} \le Ce^{-c_0(1+\xi_2^2)t}$$
$$|\widehat{K_3}| \le |e^{\lambda_2 t}| + |\lambda_2 G_1| + \eta |G_1| \le e^{-\frac{1}{4}(\eta+\nu\xi_2^2)t} + 2(\eta+\nu\xi_2^2)te^{-\frac{1}{4}(\eta+\nu\xi_2^2)t} \le Ce^{-c_0(1+\xi_2^2)t}.$$

To bound $\widehat{K_2}$, we divide the consideration into two cases:

$$\frac{|\xi_1|}{|\sqrt{\Gamma}|} \le 1 \quad \text{and} \quad \frac{|\xi_1|}{|\sqrt{\Gamma}|} > 1.$$

When $\frac{|\xi_1|}{|\sqrt{\Gamma}|} \le 1$, we write, due to $\lambda_1 - \lambda_2 = -\sqrt{\Gamma}$,

$$|\widehat{K_2}| \le \frac{|\xi_1|}{|\sqrt{\Gamma}|}(|e^{\lambda_1 t}| + |e^{\lambda_2 t}|) \le e^{-\frac{1}{2}(\eta+\nu\xi_2^2)t} + e^{-\frac{1}{4}(\eta+\nu\xi_2^2)t} \le C\, e^{-\frac{1}{4}(1+\xi_2^2)t}.$$

When $\frac{|\xi_1|}{|\sqrt{\Gamma}|} > 1$, then

$$|(\eta+\nu\xi_2^2)^2 - 4(\nu\eta\xi_2^2 + \xi_1^2)| \le \xi_1^2$$

which is equivalent to

$$0 \le (\eta+\nu\xi_2^2)^2 - 4(\nu\eta\xi_2^2 + \xi_1^2) \le \xi_1^2 \tag{2.7}$$

or

$$0 \le 4(\nu\eta\xi_2^2 + \xi_1^2) - (\eta+\nu\xi_2^2)^2 \le \xi_1^2. \tag{2.8}$$

Clearly, (2.7) implies

$$-(\eta+\nu\xi_2^2)^2 \le -4(\nu\eta\xi_2^2 + \xi_1^2) \le -4\xi_1^2$$

while (2.8) yields

$$-(\eta+\nu\xi_2^2)^2 \le -4(\nu\eta\xi_2^2 + \xi_1^2) + \xi_1^2 \le -3\xi_1^2.$$

In either case, we have, for $c > 0$

$$-(\eta+\nu\xi_2^2) \le -c\,|\xi_1|.$$

Therefore,

$$\begin{aligned} |\widehat{K_2}| &\le |\xi_1| te^{-a(\eta+\nu\xi_2^2)t} = |\xi_1| te^{-\frac{a}{2}(\eta+\nu\xi_2^2)t} e^{-\frac{a}{2}(\eta+\nu\xi_2^2)t} \\ &\le |\xi_1| te^{-\frac{a}{2}c|\xi_1|t} e^{-\frac{a}{2}(\eta+\nu\xi_2^2)t} \le Ce^{-c_0(1+\xi_2^2)t} \quad \text{for } c_0 > 0 \text{ and } C > 0. \end{aligned}$$

We now turn to the case when $\xi \in A_2$. For $\xi \in A_2$,

$$\lambda_1 \le -\frac{1}{2}(\eta+\nu\xi_2^2).$$

By $\Gamma = (\eta+\nu\xi_2^2)^2 - 4(\nu\eta\xi_2^2 + \xi_1^2) \le (\eta+\nu\xi_2^2)^2$,

$$\begin{aligned} \lambda_2 &= \frac{-(\eta+\nu\xi_2^2) + \sqrt{\Gamma}}{2} = \frac{(-(\eta+\nu\xi_2^2) + \sqrt{\Gamma})(-(\eta+\nu\xi_2^2) - \sqrt{\Gamma})}{-2((\eta+\nu\xi_2^2) + \sqrt{\Gamma})} \\ &= \frac{2(\nu\eta\xi_2^2 + \xi_1^2)}{-(\eta+\nu\xi_2^2 + \sqrt{\Gamma})} \le \frac{-2(\nu\eta\xi_2^2 + \xi_1^2)}{2(\eta+\nu\xi_2^2)} = -\frac{\nu\eta\xi_2^2 + \xi_1^2}{\eta+\nu\xi_2^2}. \end{aligned} \tag{2.9}$$

Since $\Gamma = (\eta+\nu\xi_2^2)^2 - 4(\nu\eta\xi_2^2+\xi_1^2) \geq (\eta+\nu\xi_2^2)^2 - \frac{3}{4}(\eta+\nu\xi_2^2)^2 \geq \frac{1}{4}(\eta+\nu\xi_2^2)^2$, we obtain $\sqrt{\Gamma} \geq \frac{1}{2}(\eta+\nu\xi_2^2)$. It follows that

$$|G_1| = \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\sqrt{\Gamma}} \leq \frac{2}{\eta+\nu\xi_2^2}\left(e^{-\frac{1}{2}(\eta+\nu\xi_2^2)t} + e^{-\frac{\nu\eta\xi_2^2+\xi_1^2}{\eta+\nu\xi_2^2}t}\right). \tag{2.10}$$

Furthermore, for $\xi \in A_2$, we have

$$\xi_1^2 \leq \frac{3}{16}(\eta+\nu\xi_2^2)^2 \quad \text{or} \quad \frac{|\xi_1|}{\eta+\nu\xi_2^2} \leq \frac{\sqrt{3}}{4}$$

and thus

$$|\widehat{K_2}| \leq |\xi_1||G_1| \leq C\left(e^{-\frac{1}{2}(\eta+\nu\xi_2^2)t} + e^{-\frac{\nu\eta\xi_2^2+\xi_1^2}{\eta+\nu\xi_2^2}t}\right).$$

In addition, by (2.6) and the bound $|\lambda_1| \leq \eta+\nu\xi_2^2$,

$$\begin{aligned}|\widehat{K_1}| &\leq e^{-\frac{\nu\eta\xi_2^2+\xi_1^2}{\eta+\nu\xi_2^2}t} + 4\frac{\eta+\nu\xi_2^2}{\eta+\nu\xi_2^2}\left(e^{-\frac{1}{2}(\eta+\nu\xi_2^2)t} + e^{-\frac{\nu\eta\xi_2^2+\xi_1^2}{\eta+\nu\xi_2^2}t}\right)\\ &\leq C\left(e^{-\frac{1}{2}(\eta+\nu\xi_2^2)t} + e^{-\frac{\nu\eta\xi_2^2+\xi_1^2}{\eta+\nu\xi_2^2}t}\right).\end{aligned}$$

$|\widehat{K_3}|$ admits the same upper bound. By further using the definitions of A_{21} and A_{22} in (2.5), we obtain the desired upper bounds. This completes the proof of Proposition 2.2. □

3. Proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2. The framework of the proof is the bootstrapping argument. The proof involves the estimates of many terms and is a long and tedious process. It will be divided into three subsections after we present several tool lemmas.

We need several basic tool lemmas. The first one provides the $L^p - L^q$ estimate for a general fractional Laplacian heat operator $e^{\nu t\Lambda^\alpha}$. The fractional Laplacian operator Λ^α with $\alpha \in \mathbb{R}$ is defined via the Fourier transform

$$\widehat{\Lambda^\alpha f}(\xi) = |\xi|^\alpha \widehat{f}(\xi).$$

The proof of this $L^p - L^q$ estimate can be found in many references (see, e.g., [45]).

Lemma 3.1. *Let $\alpha > 0$, $\beta \geq 0$ and $1 \leq p \leq q \leq \infty$. There is a constant $C > 0$ such that for $t > 0$,*

$$\|\Lambda^\beta e^{-c_0\Lambda^\alpha t} f\|_{L^q(\mathbb{R}^d)} \leq C\, t^{-\frac{\beta}{\alpha} - \frac{d}{\alpha}(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p(\mathbb{R}^d)}.$$

The next lemma presents an 1D Sobolev inequality involving fractional derivatives. This 1D inequality is at the core of many higher dimensional anisotropic Sobolev inequalities. The proof of this lemma can be found in [54].

Lemma 3.2. *Assume that f is in $L^q(\mathbb{R})$,*

$$\|f\|_{L^q(\mathbb{R})} \leq C\|f\|_{L^2}^{1-\frac{1}{s}(\frac{1}{2}-\frac{1}{q})}\|\Lambda^s f\|_{L^2}^{\frac{1}{s}(\frac{1}{2}-\frac{1}{q})},$$

where $2 \leq q \leq \infty$ and $\frac{1}{s}(\frac{1}{2} - \frac{1}{q}) \leq 1$.

Anisotropic Sobolev inequalities have become a necessary tool in the study of anisotropic equations. The next lemma states a 2D anisotropic inequality, which can be seen as a consequence of the previous lemma.

Lemma 3.3. *The following estimates hold when the right-hand sides are all bounded.*

$$\|f\|_{L^\infty(\mathbb{R}^2)} \le C\|f\|_{L^2(\mathbb{R}^2)}^{\frac14}\|\partial_1 f\|_{L^2(\mathbb{R}^2)}^{\frac14}\|\partial_2 f\|_{L^2(\mathbb{R}^2)}^{\frac14}\|\partial_{12} f\|_{L^2(\mathbb{R}^2)}^{\frac14}.$$

For the convenience of later reference, we also provide two standard inequalities. The first one is a Sobolev inequality while the second one is a calculus inequality on the fractional derivative of a product.

Lemma 3.4. *Assume that $f \in L^q(\mathbb{R}^2)$ with $2 < q < \infty$. Then*

$$\|f\|_{L^q} \le C\|f\|_{L^2}^{\frac{2}{q}}\|\nabla f\|_{L^2}^{1-\frac{2}{q}}.$$

Lemma 3.5. *For any $s > 0$, then for all $f, g \in H^s \cap L^\infty$, and we have the estimates*

$$\|\Lambda^s(fg)\|_{L^p} \le C\left(\|\Lambda^s f\|_{L^{p_1}}\|g\|_{L^{p_2}} + \|f\|_{L^{p_3}}\|\Lambda^s g\|_{L^{p_4}}\right),$$

where $\frac1p = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$. and $p, p_2, p_3 \in (1, \infty)$. In particular,

$$\|\Lambda^s(fg)\|_{L^2} \le C\left(\|\Lambda^s f\|_{L^2}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|\Lambda^s g\|_{L^2}\right).$$

We are ready to prove Theorem 1.2.

Proof of Theorem 1.2. We prove Theorem 1.2 by the bootstrapping argument. We make the ansatz, for $1 \le t < T$,

$$\begin{aligned}
\|(u(t), b(t))\|_{L^2} &\le \tilde{c}\delta(1+t)^{-\frac12},\\
\|(\partial_1 u(t), \partial_1 b(t))\|_{L^2} &\le \tilde{c}\delta(1+t)^{-\frac12},\\
\|(\partial_2 u(t), \partial_2 b(t))\|_{L^2} &\le \tilde{c}\delta(1+t)^{-1}.
\end{aligned} \tag{3.1}$$

where $\tilde{c}$ will be specified later. We show through the integral representation of $\widehat{u}$ and $\widehat{b}$ in (1.14) that

$$\begin{aligned}
\|(u(t), b(t))\|_{L^2} &\le \frac{\tilde{c}}{2}\delta(1+t)^{-\frac12},\\
\|(\partial_1 u(t), \partial_1 b(t))\|_{L^2} &\le \frac{\tilde{c}}{2}\delta(1+t)^{-\frac12},\\
\|(\partial_2 u(t), \partial_2 b(t))\|_{L^2} &\le \frac{\tilde{c}}{2}\delta(1+t)^{-1}.
\end{aligned} \tag{3.2}$$

Since the coefficients in (3.2) are just half of those in (3.1), the bootstrapping argument then implies (3.2) holds for all $1 \le t < \infty$.

The main efforts are devoted to the inequalities in (3.2). This process involves the estimates of many terms and is very long. For the sake of clarity, we divide the rest of this section into three subsections with each subsection devoted to one of the inequalities in (3.2). □

3.1. Estimates of $\|(u(t), b(t))\|_{L^2}$

This subsection proves the first inequality in (3.2). To estimate $\|(u(t), b(t))\|_{L^2(\mathbb{R}^2)}$, we estimate it in the three subdomains A_1, A_{21} and A_{22} defined in Proposition 2.2. By (1.14),

$$\begin{aligned}
\|\widehat{u}(t)\|_{L^2(A_1)} \le{}& \|\widehat{K_1}(t)\widehat{u_0}\|_{L^2(A_1)} + \|\widehat{K_2}(t)\widehat{b_0}\|_{L^2(A_1)} + \int_0^t \|\widehat{K_1}(t-\tau)\widehat{N_1}(\tau)\|_{L^2(A_1)}\, d\tau\\
&+ \int_0^t \|\widehat{K_2}(t-\tau)\widehat{N_2}(\tau)\|_{L^2(A_1)}\, d\tau\\
:={}& I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

By Part (1) in Proposition 2.2,

$$I_1 = \|\widehat{K_1}(t)\widehat{u_0}\|_{L^2(A_1)} \le C\|e^{-c_0(1+\xi_2^2)t}\widehat{u_0}\|_{L^2(A_1)}$$

$$\leq Ce^{-c_0t}\|\widehat{u_0}\|_{L^2(\mathbb{R}^2)} \leq C(1+t)^{-\frac{1}{2}}\|u_0\|_{L^2}, \tag{3.3}$$

where we have used $e^{-c_0t} \leq C(1+t)^{-\frac{1}{2}}$ for $t \geq 0$. Similarly,

$$I_2 = \|\widehat{K_2}(t)\widehat{b_0}\|_{L^2(A_1)} \leq C(1+t)^{-\frac{1}{2}}\|b_0\|_{L^2}. \tag{3.4}$$

Noticing that $\widehat{N_1} = (I - \frac{\xi\otimes\xi}{|\xi|^2})(-\widehat{u\cdot\nabla u} + \widehat{b\cdot\nabla b})$ and using the boundedness of the Riesz transform on L^2, we have

$$\begin{aligned}
I_3 &= \int_0^t \|\widehat{K_1}(t-\tau)\widehat{N_1}(\tau)\|_{L^2(A_1)}\, d\tau \\
&\leq C\int_0^t \left(\|\widehat{K_1}\widehat{u\cdot\nabla u}\|_{L^2(A_1)} + \|\widehat{K_1}\widehat{b\cdot\nabla b}\|_{L^2(A_1)}\right) d\tau \\
&= I_{3,1} + I_{3,2}.
\end{aligned}$$

$I_{3,1}$ is further decomposed into two parts,

$$\begin{aligned}
I_{3,1} &\leq C\int_0^{t/2} \|\widehat{K_1}(t-\tau)\widehat{u\cdot\nabla u}(\tau)\|_{L^2(A_1)}\, d\tau \\
&\quad + C\int_{t/2}^t \|\widehat{K_1}(t-\tau)\widehat{u\cdot\nabla u}(\tau)\|_{L^2(A_1)}\, d\tau \\
&= I_{3,1,1} + I_{3,1,2}.
\end{aligned}$$

By Proposition 2.2, Hölder's inequality and Ladyzhenskaya's inequality,

$$\begin{aligned}
I_{3,1,1} &\leq C\int_0^{t/2} e^{-c_0(t-\tau)}\|u\cdot\nabla u(\tau)\|_{L^2}\, d\tau \leq Ce^{-c_0\frac{t}{2}}\int_0^{t/2}\|u\|_{L^4}\|\nabla u\|_{L^4}\, d\tau \\
&\leq Ce^{-\frac{c_0}{2}t}\int_0^{t/2}\|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}^{1/2}\, d\tau \\
&\leq Ce^{-\frac{c_0}{2}t}\int_0^{t/2}(\tilde{c}\delta(1+\tau)^{-\frac{1}{2}})^{\frac{1}{2}}(\tilde{c}\delta(1+\tau)^{-\frac{1}{2}})(c\delta)^{\frac{1}{2}}\, d\tau \\
&\leq C\tilde{c}^{3/2}\delta^2 e^{-\frac{c_0}{2}t}\int_0^{t/2}(1+\tau)^{-3/4}\, d\tau \leq C\tilde{c}^{3/2}\delta^2 e^{-\frac{c_0 t}{2}}(1+t)^{1/4} \\
&\leq C\tilde{c}^{3/2}\delta^2(1+t)^{-1/2},
\end{aligned}$$

where we have used the ansatz in (3.1) and the fact that $\|u\|_{H^2} \leq c\delta$. In addition, in the last step, we have used $e^{-\frac{c_0 t}{2}} \leq C(1+t)^{-3/4}$ for $C > 0$. We estimate $I_{3,1,2}$.

$$\begin{aligned}
I_{3,1,2} &= C\int_{t/2}^t \|\widehat{K_1}(t-\tau)\widehat{u\cdot\nabla u}(\tau)\|_{L^2(A_1)}\, d\tau \\
&\leq C\int_{t/2}^t \|e^{-c_0(1+\xi_2^2)(t-\tau)}\widehat{u\cdot\nabla u}(\xi,\tau)\|_{L^2(A_1)}\, d\tau \\
&\leq C\int_{t/2}^t e^{-c_0(t-\tau)}\|e^{-c_0\xi_2^2(t-\tau)}\widehat{u\cdot\nabla u}(\xi,\tau)\|_{L^2(A_1)}\, d\tau \\
&\leq C\int_{t/2}^t e^{-c_0(t-\tau)}(t-\tau)^{-\frac{1}{4}}\|u(\tau)\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{3}{2}}\, d\tau \\
&\leq C\int_{t/2}^t e^{-c_0(t-\tau)}(t-\tau)^{-\frac{1}{4}}\tilde{c}^2\delta^2(1+\tau)^{-\frac{1}{4}-\frac{3}{4}}\, d\tau \leq C\tilde{c}^2\delta^2(1+t)^{-1},
\end{aligned} \tag{3.5}$$

where we have used $\int_{t/2}^{t} e^{-c_0(t-\tau)}(t-\tau)^{-\frac{1}{4}}\ d\tau = C$ for $C>0$ in the last inequality of (3.5), and invoked the following estimate in the fourth inequality of (3.5),

$$\begin{aligned}
\|e^{-c_0\xi_2^2(t-\tau)}\widehat{u\cdot\nabla u}(\xi,\tau)\|_{L^2(A_1)}^2 &= \|\|e^{-c_0\xi_2^2(t-\tau)}\widehat{u\cdot\nabla u}(\xi,\tau)\|_{L^2_{\xi_2}}\|_{L^2_{\xi_1}}^2 \\
&= \int\int |e^{-c_0\xi_2^2(t-\tau)}\widehat{u\cdot\nabla u}(\xi,\tau)|^2\ d\xi_2\ d\xi_1 \\
&\le \int\int |e^{-c_0\xi_2^2(t-\tau)}|^2\ d\xi_2 \|\widehat{u\cdot\nabla u}(\xi,\tau)\|_{L^\infty_{\xi_2}}^2\ d\xi_1 \\
&= \int\int (t-\tau)^{-\frac{1}{2}} e^{-c_0\eta_2^2} d\eta_2 \|\widehat{u\cdot\nabla u}(\xi,\tau)\|_{L^\infty_{\xi_2}}^2\ d\xi_1 \\
&= C(t-\tau)^{-\frac{1}{2}} \int \|\widehat{u\cdot\nabla u}(\xi,\tau)\|_{L^\infty_{\xi_2}}^2\ d\xi_1 \\
&\le C(t-\tau)^{-\frac{1}{2}} \|u\cdot\nabla u\|_{L^1_{x_2}L^2_{x_1}}^2 \\
&\le C(t-\tau)^{-\frac{1}{2}} \|\|u\|_{L^2_{x_2}}\|\nabla u\|_{L^2_{x_2}}\|_{L^2_{x_1}}^2 \\
&\le C(t-\tau)^{-\frac{1}{2}} \|u\|_{L^2_{x_2}L^\infty_{x_1}}^2 \|\nabla u\|_{L^2}^2 \\
&\le C(t-\tau)^{-\frac{1}{2}} \|u\|_{L^2}\|\partial_1 u\|_{L^2}\|\nabla u\|_{L^2}^2.
\end{aligned}$$

Similarly,

$$I_{3,2} \le C(\tilde{c}^2 + \tilde{c}^{3/2})\delta^2(1+t)^{-\frac{1}{2}}.$$

Therefore, for a constant $C>0$,

$$I_3 \le C(\tilde{c}^2 + \tilde{c}^{3/2})\delta^2(1+t)^{-\frac{1}{2}}. \tag{3.6}$$

By invoking N_2 in (1.7) and going through a very similar process, we have

$$I_4 \le C(\tilde{c}^2 + \tilde{c}^{3/2})\delta^2(1+t)^{-\frac{1}{2}}. \tag{3.7}$$

Combining (3.3), (3.4), (3.6) and (3.7) yields

$$\|\widehat{u(t)}\|_{L^2(A_1)} \le C(1+t)^{-\frac{1}{2}}(\|u_0\|_{L^2} + \|b_0\|_{L^2}) + C\tilde{c}^2\delta^2(1+t)^{-\frac{1}{2}}. \tag{3.8}$$

We now turn to $\|(u(t),b(t))\|_{L^2(A_{21})}$. By (1.14),

$$\begin{aligned}
\|\widehat{u}(t)\|_{L^2(A_{21})} &\le \|\widehat{K_1}(t)\widehat{u_0}\|_{L^2(A_{21})} + \|\widehat{K_2}(t)\widehat{b_0}\|_{L^2(A_{21})} \\
&\quad + \int_0^t \|\widehat{K_1}(t-\tau)\widehat{N_1}(\tau)\|_{L^2(A_{21})}\ d\tau \\
&\quad + \int_0^t \|\widehat{K_2}(t-\tau)\widehat{N_2}(\tau)\|_{L^2(A_{21})}\ d\tau \\
&:= J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

By Part (2) in Proposition 2.2 and Lemma 3.1,

$$\begin{aligned}
J_1 = \|\widehat{K_1}(t)\widehat{u_0}\|_{L^2(A_{21})} &\le C\|e^{-c_0(1+\xi_2^2)t}\widehat{u_0}\|_{L^2(A_{21})} + C\|e^{-c_0|\xi|^2 t}\widehat{u_0}\|_{L^2(A_{21})} \\
&\le Ce^{-c_0 t}\|\widehat{u_0}\|_{L^2(\mathbb{R}^2)} + C\|e^{c_0\Delta t}u_0\|_{L^2(\mathbb{R}^2)} \\
&\le C(1+t)^{-\frac{1}{2}}\|u_0\|_{L^2(\mathbb{R}^2)} + Ct^{-\frac{2}{2}(1-\frac{1}{2})}\|u_0\|_{L^1(\mathbb{R}^2)} \\
&\le C(1+t)^{-\frac{1}{2}}\|u_0\|_{L^2\cap L^1}.
\end{aligned}$$

where we have used $e^{-c_0 t} \le C(1+t)^{-\frac{1}{2}}$ for $t \ge 0$. Similarly,

$$J_2 \le C(1+t)^{-\frac{1}{2}}\|b_0\|_{L^2\cap L^1}.$$

By Proposition 2.2,

$$\begin{aligned} J_3 &= \int_0^t \|\widehat{K_1}(t-\tau)\widehat{N_1}(\tau)\|_{L^2(A_{21})}\ d\tau \\ &\le C\int_0^t \|e^{-c_0(1+\xi_2^2)(t-\tau)}|\widehat{u\cdot\nabla u}|(\xi,\tau)\|_{L^2(\mathbb{R}^2)}\ d\tau \\ &\quad + C\int_0^t \|e^{-c_0(1+\xi_2^2)(t-\tau)}|\widehat{b\cdot\nabla b}|(\xi,\tau)\|_{L^2(\mathbb{R}^2)}\ d\tau \\ &\quad + C\int_0^t \|e^{-c_0|\xi|^2(t-\tau)}|\widehat{u\cdot\nabla u}|(\xi,\tau)\|_{L^2(\mathbb{R}^2)}\ d\tau \\ &\quad + C\int_0^t \|e^{-c_0|\xi|^2(t-\tau)}|\widehat{b\cdot\nabla b}|(\xi,\tau)\|_{L^2(\mathbb{R}^2)}\ d\tau \\ &= J_{3,1}+J_{3,2}+J_{3,3}+J_{3,4}. \end{aligned}$$

By (3.6), for $C>0$,

$$J_{3,1}+J_{3,2} \le C(\tilde{c}^2+\tilde{c}^{3/2})\delta^2(1+t)^{-\frac{1}{2}}.$$

We further decompose $J_{3,3}$ as

$$\begin{aligned} J_{3,3} &= C\int_0^t \|e^{-c_0|\xi|^2(t-\tau)}|\widehat{u\cdot\nabla u}|(\xi,\tau)\|_{L^2(\mathbb{R}^2)}\ d\tau \\ &= C\int_0^{t/2} \|e^{-c_0|\xi|^2(t-\tau)}|\widehat{u\cdot\nabla u}|(\xi,\tau)\|_{L^2(\mathbb{R}^2)}\ d\tau \\ &\quad + C\int_{t/2}^t \|e^{-c_0|\xi|^2(t-\tau)}|\widehat{u\cdot\nabla u}|(\xi,\tau)\|_{L^2(\mathbb{R}^2)}\ d\tau \\ &= J_{3,3,1}+J_{3,3,2}. \end{aligned}$$

By Lemma 3.1, and the ansatz (3.1),

$$\begin{aligned} J_{3,3,1} &= C\int_0^{t/2} \|e^{-c_0|\xi|^2(t-\tau)}|\xi||\widehat{u\otimes u}|(\xi,\tau)|\|_{L^2}\ d\tau \\ &\le C\int_0^{t/2} (t-\tau)^{-\frac{1}{2}-\frac{2}{2}(1-\frac{1}{2})}\|u\otimes u\|_{L^1}\ d\tau \\ &\le C(t/2)^{-1}\int_0^{t/2} \|u\otimes u\|_{L^1}\ d\tau \\ &\le Ct^{-1}\int_0^{t/2} \|u\|_{L^2}^2 d\tau \\ &\le Ct^{-1}\int_0^{t/2} (\tilde{c}\delta(1+\tau)^{-\frac{1}{2}})^2 d\tau \\ &\le C\tilde{c}^2\delta^2 t^{-1}\ln(1+t/2) \le C(\sigma)\tilde{c}^2\delta^2 t^{-1+\sigma}, \end{aligned}$$

where we have used $t^{-\sigma}\ln(1+t/2)\le C(\sigma)$ for $\sigma>0$ and for all $t\ge 1$. By Lemma 3.1, the ansatz (3.1) and Hölder's inequality,

$$\begin{aligned} J_{3,3,2} &= C\int_{t/2}^t \|e^{-c_0|\xi|^2(t-\tau)}|\widehat{u\cdot\nabla u}|(\xi,\tau)|\|_{L^2}\ d\tau \\ &\le C\int_{t/2}^t (t-\tau)^{-\frac{2}{2}(1-\frac{1}{2})}\|u\cdot\nabla u\|_{L^1}\ d\tau \end{aligned}$$

$$\begin{aligned}
&\le C\int_{t/2}^{t}(t-\tau)^{-\frac{1}{2}}\|u(\tau)\|_{L^2}\|\nabla u(\tau)\|_{L^2}\ d\tau\\
&\le C\int_{t/2}^{t}(t-\tau)^{-\frac{1}{2}}\tilde{c}\delta(1+\tau)^{-\frac{1}{2}}\tilde{c}\delta(1+\tau)^{-\frac{1}{2}}\ d\tau\\
&\le C\tilde{c}^2\delta^2\int_{t/2}^{t}(t-\tau)^{-\frac{1}{2}}(1+\tau)^{-1}\ d\tau\\
&\le C\tilde{c}^2\delta^2(1+t/2)^{-1}\int_{t/2}^{t}(t-\tau)^{-\frac{1}{2}}\ d\tau\\
&\le C\tilde{c}^2\delta^2(1+t)^{-1}(t/2)^{1/2}\\
&\le C\tilde{c}^2\delta^2(1+t)^{-\frac{1}{2}}.
\end{aligned}$$

$J_{3,4}$ admits the same upper bound as $J_{3,3}$,

$$J_{3,4}\le C(\sigma)\tilde{c}^2\delta^2 t^{-1+\sigma}+C\tilde{c}^2\delta^2(1+t)^{-\frac{1}{2}}.$$

J_4 admits the same bound as J_3. By taking σ sufficiently small, say $\sigma<\frac{1}{2}$, we have

$$\|\widehat{u}(t)\|_{L^2(A_{21})}\le C(1+t)^{-\frac{1}{2}}\left(\|u_0\|_{L^2\cap L^1}+\|b_0\|_{L^2\cap L^1}\right)+C(\tilde{c}^2+\tilde{c}^{3/2})\delta^2(1+t)^{-\frac{1}{2}}. \tag{3.9}$$

We estimate $\|\widehat{u}\|_{L^2(A_{22})}$. By (1.14),

$$\begin{aligned}
\|\widehat{u}(t)\|_{L^2(A_{22})}&\le\|\widehat{K_1}(t)\widehat{u_0}\|_{L^2(A_{22})}+\|\widehat{K_1}(t)\widehat{b_0}\|_{L^2(A_{22})}\\
&\quad+\int_0^t\|\widehat{K_2}(t-\tau)\widehat{N_1}(\tau)\|_{L^2(A_{22})}\ d\tau\\
&\quad+\int_0^t\|\widehat{K_2}(t-\tau)\widehat{N_2}(\tau)\|_{L^2(A_{22})}\ d\tau\\
&:=M_1+M_2+M_3+M_4.
\end{aligned}$$

By Part (3) in Proposition 2.2,

$$\begin{aligned}
M_1&=\|\widehat{K_1}(t)\widehat{u_0}\|_{L^2(A_{22})}\\
&\le C\|e^{-c_0(1+\xi_2^2)t}\widehat{u_0}\|_{L^2(A_{22})}+\|e^{-c_0(1+\frac{\xi_1^2}{\xi_2^2})t}\widehat{u_0}\|_{L^2(A_{22})}\\
&\le C\|e^{-c_0(1+\xi_2^2)t}\widehat{u_0}\|_{L^2(\mathbb{R}^2)}+\|e^{-c_0(1+\frac{\xi_1^2}{\xi_2^2})t}\widehat{u_0}\|_{L^2(\mathbb{R}^2)}\\
&\le Ce^{-c_0t}\|u_0\|_{L^2}\le C(1+t)^{-\frac{1}{2}}\|u_0\|_{L^2}.
\end{aligned}$$

Similarly,

$$M_2\le C(1+t)^{-\frac{1}{2}}\|b_0\|_{L^2}.$$

By Proposition 2.2,

$$
\begin{aligned}
M_3 &= \int_0^t \|\widehat{K_1}(t-\tau)\widehat{N_1}(\tau)\|_{L^2(A_{22})}\, d\tau \\
&\le C\int_0^t \|(e^{-c_0(1+\xi_2^2)(t-\tau)} + e^{-c_0(1+\frac{\xi_1^2}{\xi_2^2})(t-\tau)})(|\widehat{u\cdot\nabla u}(\tau)| + |\widehat{b\cdot\nabla b}(\tau)|)\|_{L^2}\, d\tau \\
&\le C\int_0^t e^{-c_0(t-\tau)}\left(\|u\cdot\nabla u(\tau)\|_{L^2} + \|b\cdot\nabla b(\tau)\|_{L^2}\right)\, d\tau \\
&\le C\int_0^{t/2} e^{-c_0(t-\tau)}\left(\|u\cdot\nabla u\|_{L^2} + \|b\cdot\nabla b\|_{L^2}\right)\, d\tau \\
&\quad + C\int_{t/2}^t e^{-c_0(t-\tau)}\left(\|u\cdot\nabla u\|_{L^2} + \|b\cdot\nabla b\|_{L^2}\right)\, d\tau = M_{3,1} + M_{3,2}.
\end{aligned}
$$

We set

$$M_{3,1} = M_{3,1,1} + M_{3,1,2}.$$

For $2 < q < \infty$ and $\tilde q$ satisfying $\frac{1}{q} + \frac{1}{\tilde q} = \frac{1}{2}$, we have, by Lemma 3.4,

$$\|u\|_{L^q} \le C\|u\|_{L^2}^{\frac{2}{q}}\|\nabla u\|_{L^2}^{1-\frac{2}{q}}, \qquad \|\nabla u\|_{L^{\tilde q}} \le C\|\nabla u\|_{L^2}^{1-\frac{2}{q}}\|\Delta u\|_{L^2}^{\frac{2}{q}}. \tag{3.10}$$

and thus

$$
\begin{aligned}
M_{3,1,1} &\le Ce^{-\frac{c_0}{2}t}\int_0^{t/2}\|u\cdot\nabla u\|_{L^2(\mathbb{R}^2)}\, d\tau \le Ce^{-\frac{c_0}{2}t}\int_0^{t/2}\|u\|_{L^q}\|\nabla u\|_{L^{\tilde q}}\, d\tau \\
&\le Ce^{-\frac{c_0}{2}t}\int_0^{t/2}\|u\|_{L^2}^{\frac{2}{q}}\|\nabla u\|_{L^2}^{2(1-\frac{2}{q})}\|\Delta u\|_{L^2}^{\frac{2}{q}} d\tau \\
&\le Ce^{-\frac{c_0}{2}t}\int_0^{t/2}(\tilde c\delta(1+\tau)^{-1/2})^{2/q}(\tilde c\delta(1+\tau)^{-\frac{1}{2}})^{2(1-2/q)}(c\delta)^{2/q}\, d\tau \\
&\le Ce^{-\frac{c_0}{2}t}\tilde c^{2-2/q}\delta^2\int_0^{t/2}(1+\tau)^{-1+1/q}d\tau \le C\tilde c^{2-2/q}\delta^2(1+t)^{-1/2},
\end{aligned}
$$

where we have used $(1+t)^{\frac{1}{q}}e^{-\frac{c_0}{2}t} \le C(1+t)^{-1/2}$. Similarly, $M_{3,1,2}$ obeys the same bound,

$$M_{3,1} \le C\tilde c^{2-2/q}\delta^2(1+t)^{-1/2}.$$

$M_{3,2}$ is naturally divided into two parts,

$$M_{3,2} \le C\int_{t/2}^t e^{-c_0(t-\tau)}\left(\|u\cdot\nabla u\|_{L^2} + \|b\cdot\nabla b\|_{L^2}\right)\, d\tau := M_{3,2,1} + M_{3,2,2}.$$

By Hölder's inequality and (3.10),

$$
\begin{aligned}
M_{3,2,1} &\le C\int_{t/2}^t e^{-c_0(t-\tau)}\|u(\tau)\|_{L^q}\|\nabla u(\tau)\|_{L^{\tilde q}}\, d\tau \\
&\le C\int_{t/2}^t e^{-c_0(t-\tau)}\|u\|_{L^2}^{\frac{2}{q}}\|\nabla u\|_{L^2}^{2(1-\frac{2}{q})}\|\Delta u\|_{L^2}^{\frac{2}{q}} d\tau \\
&\le C\int_{t/2}^t e^{-c_0(t-\tau)}(\tilde c\delta(1+\tau)^{-1/2})^{2/q}(\tilde c\delta(1+\tau)^{-\frac{1}{2}})^{2(1-2/q)}(c\delta)^{2/q}\, d\tau \\
&\le C\tilde c^{2-2/q}\delta^2(1+t/2)^{-1+1/q}\int_{t/2}^t e^{-c_0(t-\tau)}\, d\tau \\
&\le C\tilde c^{2-2/q}\delta^2(1+t/2)^{-1+1/q}.
\end{aligned}
$$

By taking $q=3$, we obtain

$$M_{3,2,1} \leq C\tilde{c}^{4/3}\delta^2(1+t)^{-2/3}.$$

$M_{3,2,2}$ admits the same bound,

$$M_3 \leq C\tilde{c}^2\delta^2(1+t)^{-1/2} + C\tilde{c}^{4/3}\delta^2(1+t)^{-2/3}.$$

Similarly, M_4 obeys the same upper bound. Therefore,

$$\begin{aligned}\|\widehat{u}(t)\|_{L^2(A_{22})} &\leq C(1+t)^{-\frac{1}{2}}(\|u_0\|_{L^2}+\|b_0\|_{L^2})\\ &\quad + C\tilde{c}^2\delta^2(1+t)^{-1/2} + C\tilde{c}^{4/3}\delta^2(1+t)^{-2/3}.\end{aligned} \tag{3.11}$$

By (3.8), (3.9) and (3.11),

$$\begin{aligned}\|u(t)\|_{L^2} &\leq C_1(1+t)^{-\frac{1}{2}}\|(u_0,b_0)\|_{L^1\cap L^2}\\ &\quad + C_2\tilde{c}^2\delta^2(1+t)^{-\frac{1}{2}} + C_3\tilde{c}^{3/2}\delta^2(1+t)^{-\frac{1}{2}} + C_4\tilde{c}^{4/3}\delta^2(1+t)^{-2/3}.\end{aligned} \tag{3.12}$$

Therefore, if we choose $\tilde{c}$ and δ satisfying

$$C_1 \leq \frac{\tilde{c}}{8}, \qquad C_2\tilde{c}\delta \leq \frac{1}{32}, \qquad C_3\tilde{c}^{\frac{1}{2}}\delta \leq \frac{1}{32}, \qquad C_4\tilde{c}^{\frac{1}{3}}\delta \leq \frac{1}{16},$$

then (3.12) implies

$$\begin{aligned}\|u(t)\|_{L^2} &\leq \frac{\tilde{c}}{8}\delta(1+t)^{-\frac{1}{2}} + \frac{\tilde{c}}{16}\delta(1+t)^{-\frac{1}{2}} + \frac{\tilde{c}}{16}\delta(1+t)^{-\frac{1}{2}}\\ &= \frac{\tilde{c}}{4}\delta(1+t)^{-\frac{1}{2}}.\end{aligned} \tag{3.13}$$

Similarly, $\|b\|_{L^2}$ obeys the same bound. Therefore,

$$\|(u(t),b(t))\|_{L^2} \leq \frac{\tilde{c}}{2}\delta(1+t)^{-\frac{1}{2}}.$$

This completes the proof of the first inequality in (3.2).

3.2. Estimates of $\|(\partial_2 u(t), \partial_2 b(t))\|_{L^2}$

The goal of this subsection is to prove the third inequality in (3.2), namely

$$\|(\partial_2 u(t), \partial_2 b(t))\|_{L^2} \leq \frac{\tilde{c}}{2}\delta(1+t)^{-1}.$$

Applying ∂_2 to (1.14),

$$\begin{cases}\widehat{\partial_2 u}(t) = \widehat{K_1}\widehat{\partial_2 u_0} + \widehat{K_2}\widehat{\partial_2 b_0} + \displaystyle\int_0^t \widehat{K_1}(t-\tau)\widehat{\partial_2 N_1}(\tau) + \widehat{K_2}(t-\tau)\widehat{\partial_2 N_2}(\tau)\, d\tau\\ \widehat{\partial_2 b}(t) = \widehat{K_2}\widehat{\partial_2 u_0} + \widehat{K_3}\widehat{\partial_2 b_0} + \displaystyle\int_0^t \widehat{K_2}(t-\tau)\widehat{\partial_2 N_1}(\tau) + \widehat{K_3}(t-\tau)\widehat{\partial_2 N_2}(\tau)\, d\tau.\end{cases} \tag{3.14}$$

We estimate $\|\widehat{\partial_2 u}\|_{L^2(A_1)}$, $\|\widehat{\partial_2 u}\|_{L^2(A_{21})}$ and $\|\widehat{\partial_2 u}\|_{L^2(A_{22})}$. We start with $\|\widehat{\partial_2 u}\|_{L^2(A_1)}$. By (3.14),

$$\begin{aligned}\|\widehat{\partial_2 u}(t)\|_{L^2(A_1)} &\leq \|\widehat{K_1}(t)\widehat{\partial_2 u_0}\|_{L^2(A_1)} + \|\widehat{K_2}(t)\widehat{\partial_2 b_0}\|_{L^2(A_1)}\\ &\quad + \int_0^t \|\widehat{K_1}(t-\tau)\widehat{\partial_2 N_1}(\tau)\|_{L^2(A_1)}\, d\tau\\ &\quad + \int_0^t \|\widehat{K_2}(t-\tau)\widehat{\partial_2 N_2}(\tau)\|_{L^2(A_1)}\, d\tau\\ &:= O_1 + O_2 + O_3 + O_4.\end{aligned}$$

By Proposition 2.2,

$$O_1 \le \|e^{-c_0(1+\xi_2^2)t}\widehat{\partial_2 u_0}\|_{L^2(\mathbb{R}^2)} \le e^{-c_0 t}\|\partial_2 u_0\|_{L^2} \le C\delta(1+t)^{-1},$$

where we have used $(1+t)e^{-c_0 t} \le C$. Similarly,

$$O_2 \le C\delta(1+t)^{-1}.$$

O_3 is naturally decomposed into two parts,

$$\begin{aligned} O_3 &\le \int_0^t \|\widehat{K_1}(t-\tau)\widehat{\partial_2(u\cdot\nabla u)}(\tau)\|_{L^2(A_1)}\, d\tau + \int_0^t \|\widehat{K_1}(t-\tau)\widehat{\partial_2(b\cdot\nabla b)}(\tau)\|_{L^2(A_1)}\, d\tau \\ &= O_{3,1} + O_{3,2}. \end{aligned}$$

We further write

$$\begin{aligned} O_{3,1} &= \int_0^{t/2} \|\widehat{K_1}(t-\tau)\widehat{\partial_2(u\cdot\nabla u)}(\tau)\|_{L^2(A_1)}\, d\tau + \int_{t/2}^t \|\widehat{K_1}(t-\tau)\widehat{\partial_2(u\cdot\nabla u)}(\tau)\|_{L^2(A_1)}\, d\tau \\ &= O_{3,1,1} + O_{3,1,2}. \end{aligned}$$

By Ladyzhenskaya's inequality, Proposition 2.2 and Lemma 3.3,

$$\begin{aligned} O_{3,1,1} &\le \int_0^{t/2} e^{-c_0(t-\tau)}\|\partial_2(u\cdot\nabla u)\|_{L^2}\, d\tau \\ &\le e^{-\frac{c_0}{2}t}\int_0^{t/2} (\|\partial_2 u\|_{L^4}\|\nabla u\|_{L^4} + \|u\|_{L^\infty}\|\partial_2\nabla u\|_{L^2})\, d\tau \\ &\le Ce^{-\frac{c_0}{2}t}\int_0^{t/2} \|\partial_2 u\|_{L^2}^{\frac12}\|\Delta u\|_{L^2}\|\nabla u\|_{L^2}^{\frac12} + \|u\|_{L^2}^{\frac14}\|\partial_1 u\|_{L^2}^{\frac14}\|\partial_2 u\|_{L^2}^{\frac14}\|\Delta u\|^{\frac54}\, d\tau \\ &\le Ce^{-\frac{c_0}{2}t}\int_0^{t/2} \tilde{c}(1+\tau)^{-\frac34}\delta^2 + \tilde{c}^{\frac34}(1+\tau)^{-\frac12}\delta^2\, d\tau \\ &\le C\tilde{c}\delta^2(1+t)^{-1} + C\tilde{c}^{\frac34}\delta^2(1+t)^{-1}, \end{aligned} \tag{3.15}$$

where we used $e^{-\frac{c_0}{2}t}(1+t)^\gamma \le C(\gamma)$ for any $\gamma > 0$. To bound $O_{3,1,2}$, we write the norm in $O_{3,1,2}$ from the frequency space to be in the physical space, and then use Hölder's inequality, Lemma 3.1 and Lemma 3.2 to obtain

$$\begin{aligned} O_{3,1,2} &= \int_{t/2}^t e^{-c_0(t-\tau))}\|e^{-c_0\xi_2^2(t-\tau)}\widehat{\partial_2(u\cdot\nabla u)}\|_{L^2}\, d\tau \\ &\le \int_{t/2}^t e^{-c_0(t-\tau))}\Big\|\|\Lambda_2 e^{-c_0\Lambda_2^2(t-\tau)}(u\cdot\nabla u)\|_{L^2_{x_2}}\Big\|_{L^2_{x_1}}\, d\tau \\ &\le C\int_{t/2}^t e^{-c_0(t-\tau)}(t-\tau)^{-\frac34}\Big\|\|u\cdot\nabla u\|_{L^1_{x_2}}\Big\|_{L^2_{x_1}}\, d\tau \\ &\le C\int_{t/2}^t e^{-c_0(t-\tau)}(t-\tau)^{-\frac34}\Big\|\|u\|_{L^2_{x_2}}\|\nabla u\|_{L^2_{x_2}}\Big\|_{L^2_{x_1}}\, d\tau \\ &\le C\int_{t/2}^t e^{-c_0(t-\tau)}(t-\tau)^{-\frac34}\|u\|_{L^2_{x_2}L^\infty_{x_1}}\|\nabla u\|_{L^2}\, d\tau \\ &\le C\int_{t/2}^t e^{-c_0(t-\tau)}(t-\tau)^{-\frac34}\|u\|_{L^2}^{\frac12}\|\partial_1 u\|_{L^2}^{\frac12}\|\nabla u\|_{L^2}\, d\tau \\ &\le C\tilde{c}^2\delta^2(1+t)^{-1}, \end{aligned} \tag{3.16}$$

where we have used $\int_{t/2}^{t} e^{-c_0(t-\tau)}(t-\tau)^{-\frac{3}{4}}\, d\tau < \infty$. Since $O_{3,2}$ admits the same bound as $O_{3,1}$,

$$O_3 \le C\tilde{c}\delta^2(1+t)^{-1} + C\tilde{c}^{\frac{3}{4}}\delta^2(1+t)^{-1} + C\tilde{c}^2\delta^2(1+t)^{-1}.$$

O_4 obeys the same bound as O_3. Therefore,

$$\begin{aligned}\|\widehat{\partial_2 u}(t)\|_{L^2(A_1)} \le{}& C\delta(1+t)^{-1} + C\tilde{c}\delta^2(1+t)^{-1} + C\tilde{c}^{\frac{3}{4}}\delta^2(1+t)^{-1}\\ &+ C\tilde{c}^2\delta^2(1+t)^{-1}.\end{aligned}\tag{3.17}$$

Next we bound $\|\widehat{\partial_2 u}\|_{L^2(A_{21})}$. By (3.14),

$$\begin{aligned}\|\widehat{\partial_2 u}(t)\|_{L^2(A_{21})} \le{}& \|\widehat{K_1}(t)\widehat{\partial_2 u_0}\|_{L^2(A_{21})} + \|\widehat{K_2}(t)\widehat{\partial_2 b_0}\|_{L^2(A_{21})}\\ &+ \int_0^t \|\widehat{K_1}(t-\tau)\widehat{\partial_2 N_1}(\tau)\|_{L^2(A_{21})}\, d\tau\\ &+ \int_0^t \|\widehat{K_2}(t-\tau)\widehat{\partial_2 N_2}(\tau)\|_{L^2(A_{21})}\, d\tau\\ :={}& P_1 + P_2 + P_3 + P_4.\end{aligned}$$

By Part (2) in Proposition 2.2 and Lemma 3.1,

$$\begin{aligned}P_1 &= \|\widehat{K_1}(t)\widehat{\partial_2 u_0}\|_{L^2(A_{21})}\\ &\le C\|e^{-c_0(1+\xi_2^2)t}\widehat{\partial_2 u_0}\|_{L^2(A_{21})} + C\|e^{-c_0|\xi|^2 t}\widehat{\partial_2 u_0}\|_{L^2(A_{21})}\\ &\le C\|e^{-c_0(1+\xi_2^2)t}\widehat{\partial_2 u_0}\|_{L^2(A_{21})} + C\|\xi e^{-c_0|\xi|^2 t}\widehat{u_0}\|_{L^2(A_{21})}\\ &\le Ce^{-c_0 t}\|\widehat{\partial_2 u_0}\|_{L^2(\mathbb{R}^2)} + C\|e^{-c_0\Lambda^2 t}\Lambda u_0\|_{L^2(\mathbb{R}^2)}\\ &\le C(1+t)^{-1}\|\partial_2 u_0\|_{L^2(\mathbb{R}^2)} + Ct^{-1}\|u_0\|_{L^1(\mathbb{R}^2)}\\ &\le C(1+t)^{-1}\|u_0\|_{H^1\cap L^1} \le C\delta(1+t)^{-1}.\end{aligned}\tag{3.18}$$

where we have used $e^{-c_0 t} \le C(1+t)^{-1}$ for $t \ge 0$. Similarly,

$$P_2 \le C(1+t)^{-1}\|b_0\|_{H^1\cap L^1} \le C\delta(1+t)^{-1}.$$

We rewrite P_3 as

$$\begin{aligned}P_3 &= \int_0^t \|\widehat{K_1}(t-\tau)\widehat{\partial_2 N_1}(\tau)\|_{L^2(A_{21})}\, d\tau\\ &\le C\int_0^t \|e^{-c_0(1+\xi_2^2)(t-\tau)}\widehat{\partial_2(u\cdot\nabla u)}\|_{L^2(\mathbb{R}^2)}\, d\tau\\ &\quad + C\int_0^t \|e^{-c_0(1+\xi_2^2)(t-\tau)}\widehat{\partial_2(b\cdot\nabla b)}\|_{L^2(\mathbb{R}^2)}\, d\tau\\ &\quad + C\int_0^t \|e^{-c_0|\xi|^2(t-\tau)}\widehat{\partial_2(u\cdot\nabla u)}\|_{L^2(\mathbb{R}^2)}\, d\tau\\ &\quad + C\int_0^t \|e^{-c_0|\xi|^2(t-\tau)}\widehat{\partial_2(b\cdot\nabla b)}\|_{L^2(\mathbb{R}^2)}\, d\tau\\ &= P_{3,1} + P_{3,2} + P_{3,3} + P_{3,4}.\end{aligned}$$

$P_{3,1}$ can be bounded similarly as O_3,

$$P_{3,1} \le C\tilde{c}\delta^2(1+t)^{-1} + C\tilde{c}^{\frac{3}{4}}\delta^2(1+t)^{-1} + C\tilde{c}^2\delta^2(1+t)^{-1}.\tag{3.19}$$

$P_{3,2}$ admits the same bound as the one for $P_{3,1}$. To bound $P_{3,3}$, we divide it into two parts,

$$\begin{aligned} P_{3,3} &= C\int_0^{t/2} \|e^{-c_0|\xi|^2(t-\tau)}\widehat{\partial_2(u\cdot\nabla u)}\|_{L^2(\mathbb{R}^2)}\, d\tau \\ &\quad + C\int_{t/2}^{t} \|e^{-c_0|\xi|^2(t-\tau)}\widehat{\partial_2(u\cdot\nabla u)}\|_{L^2(\mathbb{R}^2)}\, d\tau \\ &= P_{3,3,1} + P_{3,3,2}. \end{aligned}$$

By Lemma 3.1 and Hölder's inequality,

$$\begin{aligned} P_{3,3,1} &\le C\int_0^{t/2} \|e^{-c_0|\xi|^2(t-\tau)}\widehat{\partial_2(u\cdot\nabla u)}\|_{L^2(\mathbb{R}^2)}\, d\tau \\ &\le C\int_0^{t/2} \||\xi_2\xi|e^{-c_0|\xi|^2(t-\tau)}|\widehat{u\otimes u}|\|_{L^2(\mathbb{R}^2)}\, d\tau \\ &\le C\int_0^{t/2} \||\xi|^2 e^{-c_0|\xi|^2(t-\tau)}|\widehat{u\otimes u}|\|_{L^2(\mathbb{R}^2)}\, d\tau \\ &\le C\int_0^{t/2} (t-\tau)^{-\frac{3}{2}}\|u\otimes u\|_{L^1}\, d\tau \le C(t/2)^{-\frac{3}{2}}\int_0^{t/2} \|u\|_{L^2}^2\, d\tau \\ &\le Ct^{-\frac{3}{2}}\int_0^{t/2} (\tilde{c}\delta(1+\tau)^{-\frac{1}{2}})^2\, d\tau \le C\tilde{c}^2\delta^2 t^{-\frac{3}{2}}\ln(1+t/2) \\ &\le C\tilde{c}^2\delta^2(1+t)^{-1}. \end{aligned}$$

where we have used $t^{-\frac{1}{2}}\ln(1+t/2) \le C$ for all $t \ge 1$. By Lemma 3.1, Lemma 3.4 and Hölder's inequality,

$$\begin{aligned} P_{3,3,2} \le& C\int_{t/2}^{t} \|e^{-c_0|\xi|^2(t-\tau)}\widehat{\partial_2(u\cdot\nabla u)}\|_{L^2(\mathbb{R}^2)}\, d\tau \\ &\le C\int_{t/2}^{t} \||\xi|e^{-c_0|\xi|^2(t-\tau)}\widehat{\partial_2(u\otimes u)}\|_{L^2}\, d\tau \\ &\le C\int_{t/2}^{t} (t-\tau)^{-\frac{1}{2}-\frac{2}{2}(\frac{1}{q}-\frac{1}{2})}\|\partial_2(u\otimes u)\|_{L^q} d\tau \\ &\le C\int_{t/2}^{t} (t-\tau)^{-\frac{1}{q}}\|\partial_2 u\|_{L^2}\|u\|_{L^r} d\tau \qquad (1<q<2, \quad r>2) \\ &\le C\int_{t/2}^{t} (t-\tau)^{-\frac{1}{q}}\|\partial_2 u\|\|u\|_{L^2}^{\frac{2}{r}}\|\nabla u\|_{L^2}^{1-\frac{2}{r}}\, d\tau \\ &\le C\int_{t/2}^{t} (t-\tau)^{-\frac{1}{q}}(\tilde{c}\delta(1+\tau)^{-1})(\tilde{c}\delta(1+\tau)^{-\frac{1}{2}})\, d\tau \\ &\le C\tilde{c}^2\delta^2(1+t)^{-\frac{3}{2}}\int_{t/2}^{t} (t-\tau)^{-\frac{1}{q}}\, d\tau \\ &\le C\tilde{c}^2\delta^2(1+t)^{-\frac{3}{2}}t^{1-\frac{1}{q}} \le C\tilde{c}^2\delta^2(1+t)^{-\frac{1}{2}-\frac{1}{q}} \le C\tilde{c}^2\delta^2(1+t)^{-1}. \end{aligned}$$

Therefore,

$$P_{3,3} \le C\tilde{c}^2\delta^2(1+t)^{-1}. \tag{3.20}$$

Similarly, $P_{3,4}$ obeys the same bound. By (3.19) and (3.20),

$$P_3 \le C\tilde{c}\delta^2(1+t)^{-1} + C\tilde{c}^{\frac{3}{4}}\delta^2(1+t)^{-1} + C\tilde{c}^2\delta^2(1+t)^{-1}. \tag{3.21}$$

Furthermore, P_4 admits the same bound as P_3 in (3.21). By collecting all the bounds for P_1, P_2, P_3 and P_4 from (3.18) to (3.21), we obtain

$$\|\widehat{\partial_2 u}(t)\|_{L^2(A_{21})} \leq C\delta(1+t)^{-1} + C\tilde{c}\delta^2(1+t)^{-1} + C\tilde{c}^{\frac{3}{4}}\delta^2(1+t)^{-1} + C\tilde{c}^2\delta^2(1+t)^{-1}. \tag{3.22}$$

Next we estimate $\|\widehat{\partial_2 u}\|_{L^2(A_{22})}$. By (3.14),

$$\begin{aligned}
\|\widehat{\partial_2 u}(t)\|_{L^2(A_{22})} &\leq \|\widehat{K_1}(t)\widehat{\partial_2 u_0}\|_{L^2(A_{22})} + \|\widehat{K_2}(t)\widehat{\partial_2 b_0}\|_{L^2(A_{22})} \\
&\quad + \int_0^t \|\widehat{K_1}(t-\tau)\widehat{\partial_2 N_1}(\tau)\|_{L^2(A_{22})}\, d\tau \\
&\quad + \int_0^t \|\widehat{K_2}(t-\tau)\widehat{\partial_2 N_2}(\tau)\|_{L^2(A_{22})}\, d\tau \\
&:= Q_1 + Q_2 + Q_3 + Q_4.
\end{aligned}$$

By Part (3) in Proposition 2.2,

$$\begin{aligned}
Q_1 &= \|\widehat{K_1}(t)\widehat{\partial_2 u_0}\|_{L^2(A_{22})} \\
&\leq C\|e^{-c_0(1+\xi_2^2)t}\widehat{\partial_2 u_0}\|_{L^2(A_{22})} + \|e^{-c_0(1+\frac{\xi_1^2}{\xi_2^2})t}\widehat{\partial_2 u_0}\|_{L^2(A_{22})} \\
&\leq C\|e^{-c_0(1+\xi_2^2)t}\widehat{\partial_2 u_0}\|_{L^2(\mathbb{R}^2)} + \|e^{-c_0(1+\frac{\xi_1^2}{\xi_2^2})t}\widehat{\partial_2 u_0}\|_{L^2(\mathbb{R}^2)} \\
&\leq Ce^{-c_0 t}\|\partial_2 u_0\|_{L^2} \leq C(1+t)^{-1}\|\partial_2 u_0\|_{L^2} \leq C\delta(1+t)^{-1}.
\end{aligned} \tag{3.23}$$

Similarly, Q_2 admits the same bound, namely,

$$Q_2 \leq C(1+t)^{-1}\|\partial_2 b_0\|_{L^2} \leq C\delta(1+t)^{-1}. \tag{3.24}$$

The bounds in Proposition 2.2 are not sufficient for estimating Q_3 and Q_4, so we drive some alternative upper bounds. Recall that

$$A_{22} = \{\xi \in \mathbb{R}^2,\ \nu\eta\xi_2^2 + \xi_1^2 < \frac{3}{16}(\eta+\nu\xi_2^2)^2,\ \nu\xi_2^2 > \eta\},$$

and G_2 and G_3 can be rewritten as

$$\begin{aligned}
G_2 &= \frac{\lambda_2 e^{\lambda_2 t} - \lambda_1 e^{\lambda_1 t}}{\lambda_2 - \lambda_1} = \frac{\lambda_2(e^{\lambda_2 t} - e^{\lambda_1 t}) + (\lambda_2-\lambda_1)e^{\lambda_1 t}}{\lambda_2 - \lambda_1} = e^{\lambda_1 t} + \lambda_2 G_1. \\
G_3 &= \frac{\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t}}{\lambda_2 - \lambda_1} = \frac{\lambda_1(e^{\lambda_1 t} - e^{\lambda_2 t}) + (\lambda_2-\lambda_1)e^{\lambda_1 t}}{\lambda_2 - \lambda_1} = e^{\lambda_1 t} - \lambda_1 G_1.
\end{aligned}$$

Furthermore, by the statement of Proposition 2.1,

$$\widehat{K_1} = e^{\lambda_1 t} + \lambda_2 G_1 + \eta G_1, \qquad \widehat{K_2} = i\xi_1 G_1, \qquad \widehat{K_3} = e^{\lambda_1 t} - \lambda_1 G_1 - \eta G_1.$$

By (2.9) and (2.10), we obtain the new upper bounds for $\widehat{K_1}$ and $\widehat{K_2}$,

$$\begin{aligned}
|\widehat{K_1}| &\leq e^{-c_0(1+\xi_2^2)t} + C(\frac{\nu\eta\xi_2^2+\xi_1^2}{\eta+\nu\xi_2^2} + \eta)|G_1| \\
&\leq e^{-c_0(1+\xi_2^2)t} + \frac{2C}{\eta+\nu\xi_2^2}\left(\frac{\nu\eta\xi_2^2+\xi_1^2}{\eta+\nu\xi_2^2} + \eta\right)\left(e^{-\frac{1}{2}(\eta+\nu\xi_2^2)t} + e^{-\frac{\nu\eta\xi_2^2+\xi_1^2}{\eta+\nu\xi_2^2}t}\right),
\end{aligned} \tag{3.25}$$

$$|\widehat{K_2}| \leq \frac{|2\xi_1|}{\eta+\nu\xi_2^2}\left(e^{-\frac{1}{2}(\eta+\nu\xi_2^2)t} + e^{-\frac{\nu\eta\xi_2^2+\xi_1^2}{\eta+\nu\xi_2^2}t}\right). \tag{3.26}$$

To bound Q_3, we first decompose it as

$$Q_3 = \int_0^t \|\widehat{K_1}(t-\tau)\widehat{\partial_2 N_1}(\tau)\|_{L^2(A_{22})}\, d\tau$$

$$
\begin{aligned}
&= \int_0^{t/2} \|\widehat{K_1}(t-\tau)\widehat{\partial_2 N_1}(\tau)\|_{L^2(A_{22})}\, d\tau + \int_{t/2}^{t} \|\widehat{K_1}(t-\tau)\widehat{\partial_2 N_1}(\tau)\|_{L^2(A_{22})}\, d\tau \\
&:= Q_{3,1} + Q_{3,2}.
\end{aligned}
$$

Invoking the upper bounds in Part (3) in Proposition 2.2 and further dividing $Q_{3,1}$ into four parts, we can show via similar techniques as for $O_{3,1,1}$ in (3.15) that

$$
\begin{aligned}
Q_{3,1} &= \int_0^{t/2} \|\widehat{K_1}(t-\tau)\widehat{\partial_2 N_1}(\tau)\|_{L^2(A_{22})}\, d\tau \\
&\le C\int_0^{t/2} \|e^{-c_0(1+\xi_2^2)(t-\tau)}\widehat{\partial_2(u\cdot\nabla u)}\|_{L^2(\mathbb{R}^2)}\, d\tau \\
&\quad + C\int_0^{t/2} \|e^{-c_0(1+\xi_2^2)(t-\tau)}\widehat{\partial_2(b\cdot\nabla b)}\|_{L^2(\mathbb{R}^2)}\, d\tau \\
&\quad + C\int_0^{t/2} \|e^{-c_0((1+\frac{\xi_1^2}{\xi_2^2})(t-\tau)}\widehat{\partial_2(u\cdot\nabla u)}\|_{L^2(\mathbb{R}^2)}\, d\tau \\
&\quad + C\int_0^{t/2} \|e^{-c_0(1+\frac{\xi_1^2}{\xi_2^2})(t-\tau)}\widehat{\partial_2(b\cdot\nabla b)}\|_{L^2(\mathbb{R}^2)}\, d\tau \\
&\le C\int_0^{t/2} e^{-c_0(t-\tau)}(\|\partial_2(u\cdot\nabla u)\|_{L^2} + \|\partial_2(b\cdot\nabla b)\|_{L^2})\, d\tau \\
&\le Ce^{-\frac{c_0}{2}t}\int_0^{t/2} (\|\partial_2(u\cdot\nabla u)\|_{L^2} + \|\partial_2(b\cdot\nabla b)\|_{L^2})d\tau \\
&\le C\tilde{c}\delta^2(1+t)^{-1} + C\tilde{c}^{\frac{3}{4}}\delta^2(1+t)^{-1}.
\end{aligned} \tag{3.27}
$$

To bound $Q_{3,2}$, we use the new bounds in (3.25) and (3.26). By Hölder's inequality and (3.25),

$$
\begin{aligned}
Q_{3,2} &\le C\int_{t/2}^{t} \|e^{-c_0(1+\xi_2^2)(t-\tau)}|\widehat{\partial_2(u\cdot\nabla u)}|\|_{L^2} + \|e^{-c_0(1+\xi_2^2)(t-\tau)}|\widehat{\partial_2(b\cdot\nabla b)}|\|_{L^2}\, d\tau \\
&+ C\int_{t/2}^{t} \left\| \left(\frac{\nu\eta\xi_2^2+\xi_1^2}{(\eta+\nu\xi_2^2)^2}+1\right) e^{-c_0(1+\xi_2^2)(t-\tau)} \left(|\widehat{\partial_2(u\cdot\nabla u)}| + |\widehat{\partial_2(b\cdot\nabla b)}|\right)\right\|_{L^2} d\tau \\
&+ C\int_{t/2}^{t} \left\| \frac{1}{\eta+\nu\xi_2^2}\left(\frac{\nu\eta\xi_2^2+\xi_1^2}{\eta+\nu\xi_2^2}+\eta\right) e^{-\frac{\nu\eta\xi_2^2+\xi_1^2}{\eta+\nu\xi_2^2}(t-\tau)} \left(|\widehat{\partial_2(u\cdot\nabla u)}| + |\widehat{\partial_2(b\cdot\nabla b)}|\right)\right\|_{L^2} d\tau \\
&:= Q_{3,2,1} + Q_{3,2,2} + Q_{3,2,3}.
\end{aligned}
$$

We rewrite $Q_{3,2,1}$ into two parts,

$$
\begin{aligned}
Q_{3,2,1} &= C\int_{t/2}^{t} \|e^{-c_0(1+\xi_2^2)(t-\tau)}|\widehat{\partial_2(u\cdot\nabla u)}|\|_{L^2}\, d\tau \\
&\quad + C\int_{t/2}^{t} \|e^{-c_0(1+\xi_2^2)(t-\tau)}|\widehat{\partial_2(b\cdot\nabla b)}|\|_{L^2}\, d\tau \\
&= Q_{3,2,1,1} + Q_{3,2,1,2}.
\end{aligned}
$$

Following similar estimates as those for $O_{3,1,2}$ in (3.16), we have

$$
Q_{3,2,1,1} \le C\tilde{c}^2\delta^2(1+t)^{-1}.
$$

Clearly, $Q_{3,2,1,2}$ admits the same bound,

$$
Q_{3,2,1} \le C\tilde{c}^2\delta^2(1+t)^{-1}. \tag{3.28}
$$

For $\xi \in A_{22}$, we have $\frac{\nu\eta\xi_2^2+\xi_1^2}{(\eta+\nu\xi_2^2)^2} < \frac{3}{16}$. By (3.28),

$$\begin{aligned}
Q_{3,2,2} &\le C\int_{t/2}^{t}\left\|\left(\frac{\nu\eta\xi_2^2+\xi_1^2}{(\eta+\nu\xi_2^2)^2}+1\right)e^{-c_0(1+\xi_2^2)(t-\tau)}|\widehat{\partial_2(u\cdot\nabla u)}|\right\|_{L^2}\,d\tau\\
&\quad + C\int_{t/2}^{t}\left\|\left(\frac{\nu\eta\xi_2^2+\xi_1^2}{(\eta+\nu\xi_2^2)^2}+1\right)e^{-c_0(1+\xi_2^2)(t-\tau)}|\widehat{\partial_2(b\cdot\nabla b)}|\right\|_{L^2}\,d\tau\\
&\le C\int_{t/2}^{t}\|e^{-c_0(1+\xi_2^2)(t-\tau)}\widehat{\partial_2(u\cdot\nabla u)}\|_{L^2}\,d\tau\\
&\quad + C\int_{t/2}^{t}\|e^{-c_0(1+\xi_2^2)(t-\tau)}\widehat{\partial_2(b\cdot\nabla b)}\|_{L^2}\,d\tau\\
&\le CQ_{3,2,1} \le C\tilde{c}^2\delta^2(1+t)^{-1}.
\end{aligned}$$

$Q_{3,2,3}$ can be further rewritten as

$$\begin{aligned}
Q_{3,2,3} &= C\int_{t/2}^{t}\left\|\frac{1}{\eta+\nu\xi_2^2}\left(\frac{\nu\eta\xi_2^2+\xi_1^2}{\eta+\nu\xi_2^2}+\eta\right)e^{-\frac{\nu\eta\xi_2^2+\xi_1^2}{\eta+\nu\xi_2^2}(t-\tau)}|\widehat{\partial_2(u\cdot\nabla u)}|\right\|_{L^2}\,d\tau\\
&\quad + C\int_{t/2}^{t}\left\|\frac{1}{\eta+\nu\xi_2^2}\left(\frac{\nu\eta\xi_2^2+\xi_1^2}{\eta+\nu\xi_2^2}+\eta\right)e^{-\frac{\nu\eta\xi_2^2+\xi_1^2}{\eta+\nu\xi_2^2}(t-\tau)}|\widehat{\partial_2(b\cdot\nabla b)}|\right\|_{L^2}\,d\tau\\
&= Q_{3,2,3,1}+Q_{3,2,3,2}.
\end{aligned}$$

We first estimate $Q_{3,2,3,1}$,

$$\begin{aligned}
Q_{3,2,3,1} &\le C\int_{t/2}^{t}\left\|\frac{|\xi|^2}{(1+\xi_2^2)^2}e^{-c_0\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)}|\widehat{\partial_2(u\cdot\nabla u)}|\right\|_{L^2}\,d\tau\\
&\quad + C\int_{t/2}^{t}\left\|\frac{|\xi|}{1+\xi_2^2}e^{-c_0\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)}\widehat{u\cdot\nabla u}\right\|_{L^2}\,d\tau\\
&= Q_{3,2,3,1,1}+Q_{3,2,3,1,2}.
\end{aligned}$$

The process of controlling $Q_{3,2,3,1,1}$ is tedious, so we first estimate $Q_{3,2,3,1,2}$. By Lemma 3.3,

$$\begin{aligned}
Q_{3,2,3,1,2} &\le C\int_{t/2}^{t}\left\|\frac{|\xi|}{\sqrt{1+\xi_2^2}}(t-\tau)^{\frac12}(t-\tau)^{-\frac12}e^{-c_0\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)}\widehat{u\cdot\nabla u}\right\|_{L^2}\,d\tau\\
&\le C\int_{t/2}^{t}(t-\tau)^{-\frac12}\|e^{-\frac{c_0}{4}(1+\frac{\xi_1^2}{\xi_2^2})(t-\tau)}\widehat{u\cdot\nabla u}\|_{L^2}\,d\tau\\
&\le C\int_{t/2}^{t}(t-\tau)^{-\frac12}e^{-\frac{c_0}{4}(t-\tau)}\|u\cdot\nabla u\|_{L^2}\,d\tau\\
&\le C\int_{t/2}^{t}(t-\tau)^{-\frac12}e^{-\frac{c_0}{4}(t-\tau)}\|u\|_{L^\infty}\|\nabla u\|_{L^2}\,d\tau\\
&\le C\int_{t/2}^{t}(t-\tau)^{-\frac12}e^{-\frac{c_0}{4}(t-\tau)}\|u\|_{L^2}^{\frac14}\|\partial_1 u\|_{L^2}^{\frac14}\|\partial_2 u\|_{L^2}^{\frac14}\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}^{\frac14}\,d\tau\\
&\le C\int_{t/2}^{t}(t-\tau)^{-\frac12}e^{-\frac{c_0}{4}(t-\tau)}(\tilde{c}\delta(1+\tau)^{-\frac12})^{\frac32}(\tilde{c}\delta(1+\tau)^{-1})^{\frac14}(c\delta)^{\frac14}d\tau\\
&\le C\tilde{c}^2\delta^2(1+t/2)^{-1} \le C\tilde{c}^2\delta^2(1+t)^{-1},
\end{aligned}$$

where we have used the facts that $\int_{t/2}^{t} (t-\tau)^{-\frac{1}{2}} e^{-\frac{c_0}{4}(t-\tau)}\, d\tau < \infty$ and $\gamma\, e^{-\frac{c_0}{2}\gamma^2} \le C$ or more explicitly

$$\frac{|\xi|}{\sqrt{1+\xi_2^2}} (t-\tau)^{\frac{1}{2}}\, e^{-\frac{c_0}{2}\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)} \le C.$$

As (3.1) indicates, the decay rates associated with the horizontal and the vertical derivatives are different. To bound $Q_{3,2,3,1,1}$ properly, we need to distinguish the horizontal derivative from the vertical one. By $\nabla\cdot u = 0$, we write

$$\partial_2(u\cdot\nabla u) = \partial_1\partial_2(uu_1) + \partial_2\partial_2(uu_2)$$

and divide $Q_{3,2,3,1,1}$ into two parts,

$$\begin{aligned}
Q_{3,2,3,1,1} &\le C\int_{t/2}^{t} \left\| \frac{|\xi|^2}{(1+\xi_2^2)^2} e^{-c_0\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)} |\widehat{\partial_1\partial_2(uu_1) + \partial_2\partial_2(uu_2)}| \right\|_{L^2} d\tau \\
&\le C\int_{t/2}^{t} \left\| \frac{|\xi|^2}{(1+\xi_2^2)^2} e^{-c_0\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)} |\widehat{\partial_1\partial_2(uu_1)}| \right\|_{L^2} d\tau \\
&\quad + C\int_{t/2}^{t} \left\| \frac{|\xi|^2}{(1+\xi_2^2)^2} e^{-c_0\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)} |\widehat{\partial_2\partial_2(uu_2)}| \right\|_{L^2} d\tau \\
&= Q_{3,2,3,1,1,1} + Q_{3,2,3,1,1,2}.
\end{aligned}$$

Since $\xi \in A_{22}$, we have $|\xi|^2 \le C(1+\xi_2^2)^2$. By Lemmas 3.5 and 3.3,

$$\begin{aligned}
Q_{3,2,3,1,1,1} \le& \int_{t/2}^{t} \Bigg\| \left(\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)\right)^{\sigma} \left(\frac{|\xi|}{(1+\xi_2^2)}\right)^{2-\sigma} \\
&\times |\xi_1|^{1-\sigma}(t-\tau)^{-\sigma} e^{-c_0\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)} |\widehat{\partial_2(uu_1)}| \Bigg\|_{L^2} d\tau \\
\le& C(\sigma)\int_{t/2}^{t} \left\| (t-\tau)^{-\sigma}|\xi_1|^{1-\sigma} e^{-\frac{c_0}{4}(1+\frac{\xi_1^2}{\xi_2^2})(t-\tau)} |\widehat{\partial_2(uu_1)}| \right\|_{L^2} d\tau \\
\le& C(\sigma)\int_{t/2}^{t} (t-\tau)^{-\sigma} e^{-\frac{c_0}{4}(t-\tau)} \left\| \widehat{\Lambda_1^{1-\sigma}(\partial_2(uu_1))} \right\|_{L^2} d\tau \\
\le& C(\sigma)\int_{t/2}^{t} (t-\tau)^{-\sigma} e^{-\frac{c_0}{4}(t-\tau)} \|u\|_{L^2}^{\frac{\sigma}{3}} \|\nabla u\|_{L^2}^{\frac{\sigma}{3}} \|\Delta u\|_{L^2}^{1-\frac{2\sigma}{3}} \|\partial_2 u\|_{L^2}\, d\tau \\
&+ C(\sigma)\int_{t/2}^{t} (t-\tau)^{-\sigma} e^{-\frac{c_0}{4}(t-\tau)} \|\partial_2 u\|_{L^2}^{\sigma} \|\partial_1\partial_2 u\|_{L^2}^{1-\sigma} \\
&\times \|\partial_2 u\|_{L^2}^{\frac{1}{4}} \|\partial_1 u\|_{L^2}^{\frac{1}{4}} \|u\|_{L^2}^{\frac{1}{4}} \|\Delta u\|^{\frac{1}{4}}\, d\tau \\
\le& C(\sigma)\int_{t/2}^{t} (t-\tau)^{-\sigma} e^{-\frac{c_0}{4}(t-\tau)} (\tilde{c}\delta(1+\tau)^{-\frac{1}{2}})^{\frac{2\sigma}{3}} (C\delta)^{1-\frac{2\sigma}{3}} \tilde{c}\delta(1+\tau)^{-1}\, d\tau \\
&+ C(\sigma)\int_{t/2}^{t} (t-\tau)^{-\sigma} e^{-\frac{c_0}{4}(t-\tau)} (\tilde{c}\delta(1+\tau)^{-1})^{\sigma+\frac{1}{4}} (C\delta)^{\frac{5}{4}-\sigma} (\tilde{c}\delta(1+\tau)^{-\frac{1}{2}})^{\frac{1}{2}}\, d\tau \\
\le& C(\sigma)\tilde{c}^{\frac{2\sigma}{3}+1}\delta^2(1+t)^{-1-\frac{\sigma}{3}} + C(\sigma)\tilde{c}^{\sigma+\frac{3}{4}}\delta^2(1+t)^{-\sigma-\frac{1}{2}} \\
\le& C(\sigma)\tilde{c}^{\frac{3}{2}}\delta^2(1+t)^{-1},
\end{aligned}$$

where we have set $\sigma = \frac{3}{4}$, and used $\int_{t/2}^{t} (t-\tau)^{-\sigma} e^{-\frac{c_0}{4}(t-\tau)}\, d\tau < \infty$, and

$$\frac{|\xi|}{1+\xi_2^2} \le C(\sigma), \qquad \left(\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)\right)^{\sigma} e^{-\frac{c_0}{2}\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)} \le C(\sigma). \tag{3.29}$$

In addition, we have used the following upper bound on $\|\Lambda_1^{1-\sigma}(\partial_2(uu_1))\|$ in the fourth inequality above, by Lemma 3.5,

$$\begin{aligned} \|\Lambda_1^{1-\sigma}(\partial_2(uu_1))\|_{L^2} &\le \|\Lambda_1^{1-\sigma}(\partial_2 u u_1)\|_{L^2} + \|\Lambda_1^{1-\sigma}(u\partial_2 u_1)\|_{L^2} \\ &\le C\|\partial_2 u\|_{L^2}\|\Lambda_1^{1-\sigma}u_1\|_{L^\infty} + C\|\Lambda_1^{1-\sigma}\partial_2 u\|_{L^2}\|u_1\|_{L^\infty} \\ &\quad + C\|\partial_2 u_1\|_{L^2}\|\Lambda_1^{1-\sigma}u\|_{L^\infty} + C\|\Lambda_1^{1-\sigma}\partial_2 u_1\|_{L^2}\|u\|_{L^\infty} \\ &\le C\|u\|_{L^2}^{\frac{\sigma}{3}}\|\nabla u\|_{L^2}^{\frac{\sigma}{3}}\|\Delta u\|_{L^2}^{1-\frac{2\sigma}{3}}\|\partial_2 u\|_{L^2} \\ &\quad + C\|\partial_2 u\|_{L^2}^{\sigma}\|\partial_1\partial_2 u\|_{L^2}^{1-\sigma}\|\partial_2 u\|_{L^2}^{\frac14}\|\partial_1 u\|_{L^2}^{\frac14}\|u\|_{L^2}^{\frac14}\|\Delta u\|_{L^2}^{\frac14}. \end{aligned} \tag{3.30}$$

Similarly, by (3.29),

$$\begin{aligned} Q_{3,2,3,1,1,2} &\le \int_{t/2}^{t} \Bigg\| \left(\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)\right)^{\sigma}\left(\frac{|\xi|}{(1+\xi_2^2)}\right)^{2-\sigma} \\ &\qquad \times |\xi_2|^{1-\sigma}(t-\tau)^{-\sigma} e^{-c_0\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)}\widehat{|\partial_2(uu_2)|}\Bigg\|_{L^2} d\tau \\ &\le C(\sigma)\int_{t/2}^{t} \left\|(t-\tau)^{-\sigma}|\xi_2|^{1-\sigma} e^{-\frac{c_0}{4}(1+\frac{\xi_1^2}{\xi_2^2})(t-\tau)}\widehat{|\partial_2(uu_2)|}\right\|_{L^2} d\tau \\ &\le C(\sigma)\int_{t/2}^{t} (t-\tau)^{-\sigma} e^{-\frac{c_0}{4}(t-\tau)} \left\|\widehat{\Lambda_2^{1-\sigma}(\partial_2(uu_2))}\right\|_{L^2} d\tau \\ &\le C(\sigma)\int_{t/2}^{t} (t-\tau)^{-\sigma} e^{-\frac{c_0}{4}(t-\tau)}\|u\|_{L^2}^{\frac{\sigma}{3}}\|\nabla u\|_{L^2}^{\frac{\sigma}{3}}\|\Delta u\|_{L^2}^{1-\frac{2\sigma}{3}}\|\partial_2 u\|_{L^2}\, d\tau \\ &\quad + C(\sigma)\int_{t/2}^{t} (t-\tau)^{-\sigma} e^{-\frac{c_0}{4}(t-\tau)}\|\partial_2 u\|_{L^2}^{\sigma}\|\partial_2\partial_2 u\|_{L^2}^{1-\sigma} \\ &\quad \times \|\partial_2 u\|_{L^2}^{\frac14}\|\partial_1 u\|_{L^2}^{\frac14}\|u\|_{L^2}^{\frac14}\|\Delta u\|^{\frac14}\, d\tau \\ &\le C(\sigma)\tilde{c}^{\frac32}\delta^2(1+t)^{-1}, \end{aligned}$$

where we have set $\sigma = \frac{3}{4}$, and used (3.30) and the following estimate

$$\begin{aligned} \|\Lambda_2^{1-\sigma}(\partial_2(uu_2))\|_{L^2} &\le C\|u\|_{L^2}^{\frac{\sigma}{3}}\|\nabla u\|_{L^2}^{\frac{\sigma}{3}}\|\Delta u\|_{L^2}^{1-\frac{2\sigma}{3}}\|\partial_2 u\|_{L^2} \\ &\quad + C\|\partial_2 u\|_{L^2}^{\sigma}\|\partial_2\partial_2 u\|_{L^2}^{1-\sigma}\|\partial_2 u\|_{L^2}^{\frac14}\|\partial_1 u\|_{L^2}^{\frac14}\|u\|_{L^2}^{\frac14}\|\Delta u\|_{L^2}^{\frac14}. \end{aligned}$$

Therefore,

$$Q_{3,2,3,1} \le C\tilde{c}^2\delta^2(1+t)^{-1} + C(\sigma)\tilde{c}^{\frac32}\delta^2(1+t)^{-1}.$$

Similarly, $Q_{3,2,3,2}$ admits the same bound. Collecting the bounds for $Q_{3,2,1}, Q_{3,2,2}$ and $Q_{3,2,3}$ yields

$$Q_{3,2} \le C\tilde{c}^2\delta^2(1+t)^{-1} + C(\sigma)\tilde{c}^{\frac32}\delta^2(1+t)^{-1}. \tag{3.31}$$

Combining the estimates for $Q_{3,1}$ and $Q_{3,2}$ in (3.27) and (3.31) respectively, we obtain

$$Q_3 \le (\tilde{c} + \tilde{c}^{\frac34} + \tilde{c}^2 + \tilde{c}^{\frac32})C\delta^2(1+t)^{-1}. \tag{3.32}$$

Next we bound Q_4. By (3.26), we rewrite Q_4 as

$$\begin{aligned} Q_4 &= \int_0^t \|\widehat{K_2}(t-\tau)\widehat{\partial_2 N_2}(\tau)\|_{L^2(A_{22})}\, d\tau \\ &= \int_0^{t/2} \|\widehat{K_2}(t-\tau)\widehat{\partial_2 N_2}(\tau)\|_{L^2(A_{22})}\, d\tau + \int_{t/2}^t \|\widehat{K_2}(t-\tau)\widehat{\partial_2 N_2}(\tau)\|_{L^2(A_{22})}\, d\tau \\ &:= Q_{4,1} + Q_{4,2}. \end{aligned}$$

By Part (3) in Proposition 2.2 and by (3.27), $Q_{4,1}$ obeys the same bound as $Q_{3,1}$, namely,

$$Q_{4,1} \le C\tilde{c}\delta^2(1+t)^{-1} + C\tilde{c}^{\frac{3}{4}}\delta^2(1+t)^{-1}. \tag{3.33}$$

Since the bound for $\widehat{K_2}$ in (3.26) is not the same as the bound for $\widehat{K_1}$ in (3.25), we need to estimate $Q_{4,2}$ differently from $Q_{3,2}$.

$$\begin{aligned} Q_{4,2} &\le C\int_{t/2}^t \left\| \frac{|\xi_1|}{1+\xi_2^2} e^{-c_0(1+\xi_2^2)(t-\tau)} \left(|\widehat{\partial_2(u\cdot\nabla b)}| + |\widehat{\partial_2(b\cdot\nabla u)}| \right) \right\|_{L^2} d\tau \\ &\quad + C\int_{t/2}^t \left\| \frac{|\xi_1|}{1+\xi_2^2} e^{-\frac{\nu\eta\xi_2^2+\xi_1^2}{\eta+\nu\xi_2^2}(t-\tau)} \left(|\widehat{\partial_2(u\cdot\nabla b)}| + |\widehat{\partial_2(b\cdot\nabla u)}| \right) \right\|_{L^2} d\tau \\ &:= Q_{4,2,1} + Q_{4,2,2}. \end{aligned}$$

Since $\xi \in A_{22}$, $|\xi|^2 \le C\,(1+\xi_2^2)^2$. By the same process as in (3.16),

$$\begin{aligned} Q_{4,2,1} &\le C\int_{t/2}^t \|e^{-c_0(1+\xi_2^2)(t-\tau)}\widehat{\partial_2(u\cdot\nabla b)}\|_{L^2}\, d\tau \\ &\quad + C\int_{t/2}^t \|e^{-c_0(1+\xi_2^2)(t-\tau)}\widehat{\partial_2(b\cdot\nabla u)}\|_{L^2}\, d\tau \\ &\le C\tilde{c}^2\delta^2(1+t)^{-1}. \end{aligned}$$

We further decompose $Q_{4,2,2}$ into two parts,

$$\begin{aligned} Q_{4,2,2} &\le C\int_{t/2}^t \left\| \frac{|\xi_1|}{1+\xi_2^2} e^{-\frac{\nu\eta\xi_2^2+\xi_1^2}{\eta+\nu\xi_2^2}(t-\tau)} |\widehat{\partial_2(u\cdot\nabla b)}| \right\|_{L^2} d\tau \\ &\quad + C\int_{t/2}^t \left\| \frac{|\xi_1|}{1+\xi_2^2} e^{-\frac{\nu\eta\xi_2^2+\xi_1^2}{\eta+\nu\xi_2^2}(t-\tau)} |\widehat{\partial_2(b\cdot\nabla u)}| \right\|_{L^2} d\tau \\ &:= Q_{4,2,2,1} + Q_{4,2,2,2}. \end{aligned}$$

As before, we write $\partial_2(u\cdot\nabla b) = \partial_1\partial_2(bu_1) + \partial_2\partial_2(bu_2)$ and thus decompose $Q_{4,2,2,1}$ into two parts,

$$\begin{aligned} Q_{4,2,2,1} &\le C\int_{t/2}^t \left\| \frac{|\xi_1|}{1+\xi_2^2} e^{-c_0\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)} |\widehat{\partial_1\partial_2(bu_1) + \partial_2\partial_2(bu_2)}| \right\|_{L^2} d\tau \\ &\le C\int_{t/2}^t \left\| \frac{|\xi_1|}{1+\xi_2^2} e^{-c_0\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)} |\widehat{\partial_1\partial_2(bu_1)}| \right\|_{L^2} d\tau \\ &\quad + C\int_{t/2}^t \left\| \frac{|\xi_1|}{1+\xi_2^2} e^{-c_0\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)} |\widehat{\partial_2\partial_2(bu_2)}| \right\|_{L^2} d\tau \\ &= Q_{4,2,2,1,1} + Q_{4,2,2,1,2}. \end{aligned}$$

The first part $Q_{4,2,2,1,1}$ can be bounded by

$$Q_{4,2,2,1,1} \le \int_{t/2}^t \left\| \left(\frac{|\xi|^2}{1+\xi_2^2}(t-\tau) \right)^{\frac{1+\sigma}{2}} |\xi_1|^{1-\sigma}(t-\tau)^{-\frac{1+\sigma}{2}} e^{-c_0\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)} |\widehat{\partial_2(bu_1)}| \right\|_{L^2} d\tau$$

$$
\begin{aligned}
&\le C(\sigma)\int_{t/2}^{t}\left\|\,|\xi_1|^{1-\sigma}(t-\tau)^{-\frac{1+\sigma}{2}}e^{-\frac{c_0}{4}(1+\frac{\xi_1^2}{\xi_2^2})(t-\tau)}|\widehat{\partial_2(bu_1)}|\right\|_{L^2}\,d\tau\\
&\le C(\sigma)\int_{t/2}^{t}(t-\tau)^{-\frac{1+\sigma}{2}}e^{-\frac{c_0}{4}}(t-\tau)\left\|\widehat{\Lambda_1^{1-\sigma}(\partial_2(bu_1))}\right\|_{L^2}\,d\tau\\
&\le C(\sigma)\int_{t/2}^{t}(t-\tau)^{-\frac{1+\sigma}{2}}e^{-\frac{c_0}{4}}(t-\tau)\|u\|_{L^2}^{\frac{\sigma}{3}}\|\nabla u\|_{L^2}^{\frac{\sigma}{3}}\|\Delta u\|_{L^2}^{1-\frac{2\sigma}{3}}\|\partial_2 b\|_{L^2}\,d\tau\\
&\quad+C(\sigma)\int_{t/2}^{t}(t-\tau)^{-\frac{1+\sigma}{2}}e^{-\frac{c_0}{4}(t-\tau)}\|\partial_2 b\|_{L^2}^{\sigma}\|\partial_1\partial_2 b\|_{L^2}^{1-\sigma}\|\partial_2 u\|_{L^2}^{\frac14}\\
&\quad\times\|\partial_1 u\|_{L^2}^{\frac14}\|u\|_{L^2}^{\frac14}\|\Delta u\|_{L^2}^{\frac14}\,d\tau\\
&\quad+C(\sigma)\int_{t/2}^{t}(t-\tau)^{-\frac{1+\sigma}{2}}e^{-\frac{c_0}{4}}(t-\tau)\|b\|_{L^2}^{\frac{\sigma}{3}}\|\nabla b\|_{L^2}^{\frac{\sigma}{3}}\|\Delta b\|_{L^2}^{1-\frac{2\sigma}{3}}\|\partial_2 u_1\|_{L^2}\,d\tau\\
&\quad+C(\sigma)\int_{t/2}^{t}(t-\tau)^{-\frac{1+\sigma}{2}}e^{-\frac{c_0}{4}}(t-\tau)\|\partial_2 u_1\|_{L^2}^{\sigma}\|\partial_1\partial_2 u_1\|_{L^2}^{1-\sigma}\\
&\quad\times\|\partial_2 b\|_{L^2}^{\frac14}\|\partial_1 b\|_{L^2}^{\frac14}\|b\|_{L^2}^{\frac14}\|\Delta b\|_{L^2}^{\frac14}\,d\tau\\
&\le C(\sigma)\int_{t/2}^{t}(t-\tau)^{-\frac{1+\sigma}{2}}e^{-\frac{c_0}{4}(t-\tau)}(\tilde c\delta(1+\tau)^{-\frac12})^{\frac{2\sigma}{3}}(C\delta)^{1-\frac{2\sigma}{3}}\tilde c\delta(1+\tau)^{-1}\,d\tau\\
&\quad+C(\sigma)\int_{t/2}^{t}(t-\tau)^{-\frac{1+\sigma}{2}}e^{-\frac{c_0}{4}(t-\tau)}(\tilde c\delta(1+\tau)^{-1})^{\sigma+\frac14}(C\delta)^{\frac54-\sigma}(\tilde c\delta(1+\tau)^{-\frac12})^{\frac12}\,d\tau\\
&\le C(\sigma)\tilde c^{\frac{2\sigma}{3}+1}\delta^2(1+t)^{-1-\frac{\sigma}{3}}+C(\sigma)\tilde c^{\sigma+\frac34}\delta^2(1+t)^{-\sigma-\frac12}\\
&\le C\tilde c^{\frac32}\delta^2(1+t)^{-1},
\end{aligned}
$$

where we set $\sigma=\frac34$, and have used $\int_{t/2}^{t}(t-\tau)^{-\frac{1+\sigma}{2}}e^{-\frac{c_0}{4}(t-\tau)}\,d\tau<\infty$ and

$$
\left(\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)\right)^{\frac{1+\sigma}{2}}e^{-\frac{c_0}{2}\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)}\le C(\sigma).
$$

In addition, we have also used the following upper bound on $\|\Lambda_1^{1-\sigma}(\partial_2(bu_1))\|$, by Lemma 3.5,

$$
\begin{aligned}
\|\Lambda_1^{1-\sigma}(\partial_2(bu_1))\|_{L^2}&\le\|\Lambda_1^{1-\sigma}(\partial_2 bu_1)\|_{L^2}+\|\Lambda_1^{1-\sigma}(b\partial_2 u_1)\|_{L^2}\\
&\le C\|\partial_2 b\|_{L^2}\|\Lambda_1^{1-\sigma}u_1\|_{L^\infty}+C\|\Lambda_1^{1-\sigma}\partial_2 b\|_{L^2}\|u_1\|_{L^\infty}\\
&\quad+C\|\partial_2 u_1\|_{L^2}\|\Lambda_1^{1-\sigma}b\|_{L^\infty}+C\|\Lambda_1^{1-\sigma}\partial_2 u_1\|_{L^2}\|b\|_{L^\infty}\\
&\le C\|u\|_{L^2}^{\frac{\sigma}{3}}\|\nabla u\|_{L^2}^{\frac{\sigma}{3}}\|\Delta u\|_{L^2}^{1-\frac{2\sigma}{3}}\|\partial_2 b\|_{L^2}\\
&\quad+C\|\partial_2 b\|_{L^2}^{\sigma}\|\partial_1\partial_2 b\|_{L^2}^{1-\sigma}\|\partial_2 u\|_{L^2}^{\frac14}\|\partial_1 u\|_{L^2}^{\frac14}\|u\|_{L^2}^{\frac14}\|\Delta u\|_{L^2}^{\frac14}\\
&\quad+C\|b\|_{L^2}^{\frac{\sigma}{3}}\|\nabla b\|_{L^2}^{\frac{\sigma}{3}}\|\Delta b\|_{L^2}^{1-\frac{2\sigma}{3}}\|\partial_2 u_1\|_{L^2}\\
&\quad+C\|\partial_2 u_1\|_{L^2}^{\sigma}\|\partial_1\partial_2 u_1\|_{L^2}^{1-\sigma}\|\partial_2 b\|_{L^2}^{\frac14}\|\partial_1 b\|_{L^2}^{\frac14}\|b\|_{L^2}^{\frac14}\|\Delta b\|_{L^2}^{\frac14}.
\end{aligned}
$$

Similarly,

$$
Q_{4,2,2,1,2}\le C\tilde c^{\frac32}\delta^2(1+t)^{-1}.
$$

Since $Q_{4,2,2,2}$ obeys the same bound as $Q_{4,2,2,1}$, we find

$$
Q_{4,2,2}\le C(\sigma)\tilde c^{\frac32}\delta^2(1+t)^{-1}.
$$

Therefore,

$$Q_{4,2} \leq C\tilde{c}^2\delta^2(1+t)^{-1} + C\tilde{c}^{\frac{3}{2}}\delta^2(1+t)^{-1}. \tag{3.34}$$

Putting (3.33) and (3.34) together yields

$$Q_4 \leq (\tilde{c} + \tilde{c}^{\frac{3}{4}} + \tilde{c}^2 + \tilde{c}^{\frac{3}{2}})C\delta^2(1+t)^{-1}. \tag{3.35}$$

Combining (3.23), (3.24), (3.32) and (3.35), we have

$$\|\widehat{\partial_2 u}(t)\|_{L^2(A_{22})} \leq C\delta(1+t)^{-1} + (\tilde{c} + \tilde{c}^{\frac{3}{4}} + \tilde{c}^{\frac{3}{2}})C\delta^2(1+t)^{-1} \tag{3.36}$$

Collecting the estimates in (3.17), (3.22) and (3.36), we find

$$\|\widehat{\partial_2 u}(t)\|_{L^2} \leq C_1\delta(1+t)^{-1} + \tilde{c}C_2\delta^2(1+t)^{-1} + (\tilde{c}^{\frac{3}{4}} + \tilde{c}^2 + \tilde{c}^{\frac{3}{2}})C_3\delta^2(1+t)^{-1}.$$

If we choose $\tilde{c}$ and δ satisfying

$$C_1 \leq \frac{\tilde{c}}{8}, \qquad C_2\delta \leq \frac{1}{16}, \qquad (\tilde{c}^{\frac{3}{4}} + \tilde{c}^2 + \tilde{c}^{\frac{3}{2}})C_3\delta \leq \frac{\tilde{c}}{16},$$

then we obtain

$$\begin{aligned}\|\partial_2 u(t)\|_{L^2} &\leq \frac{\tilde{c}}{8}\delta(1+t)^{-1} + \frac{\tilde{c}}{16}\delta(1+t)^{-1} + \frac{\tilde{c}}{16}\delta(1+t)^{-1} \\ &= \frac{\tilde{c}}{4}\delta(1+t)^{-1}.\end{aligned}$$

The same upper bound holds for $\|\partial_2 b\|_{L^2}$. Thus we have obtained

$$\|(\partial_2 u(t), \partial_2 b(t))\|_{L^2} \leq \frac{\tilde{c}}{2}\delta(1+t)^{-1}.$$

This completes the proof of the third inequality in (3.2).

3.3. Estimates of $\|(\partial_1 u(t), \partial_1 b(t))\|_{L^2}$

This subsection establishes the second inequality in (3.2), namely

$$\|(\partial_1 u(t), \partial_1 b(t))\|_{L^2} \leq \frac{\tilde{c}}{2}\delta(1+t)^{-\frac{1}{2}}.$$

Applying ∂_1 to (1.14) yields

$$\begin{cases}\widehat{\partial_1 u}(t) = \widehat{K_1}\widehat{\partial_1 u_0} + \widehat{K_2}\widehat{\partial_1 b_0} + \displaystyle\int_0^t \widehat{K_1}(t-\tau)\widehat{\partial_1 N_1}(\tau) + \widehat{K_2}(t-\tau)\widehat{\partial_1 N_2}(\tau)\, d\tau \\ \widehat{\partial_1 b}(t) = \widehat{K_2}\widehat{\partial_1 u_0} + \widehat{K_3}\widehat{\partial_1 b_0} + \displaystyle\int_0^t \widehat{K_2}(t-\tau)\widehat{\partial_1 N_1}(\tau) + \widehat{K_3}(t-\tau)\widehat{\partial_1 N_2}(\tau)\, d\tau.\end{cases} \tag{3.37}$$

To estimate $\|\partial_1 u\|_{L^2(\mathbb{R}^2)}$, we estimate $\|\widehat{\partial_1 u}\|_{L^2(A_1)}$, $\|\widehat{\partial_1 u}\|_{L^2(A_{21})}$ and $\|\widehat{\partial_1 u}\|_{L^2(A_{22})}$. We start with $\|\widehat{\partial_1 u}\|_{L^2(A_1)}$. By (3.37),

$$\begin{aligned}\|\widehat{\partial_1 u}(t)\|_{L^2(A_1)} &\leq \|\widehat{K_1}(t)\widehat{\partial_1 u_0}\|_{L^2(A_1)} + \|\widehat{K_2}(t)\widehat{\partial_1 b_0}\|_{L^2(A_1)} \\ &\quad + \int_0^t \|\widehat{K_1}(t-\tau)\widehat{\partial_1 N_1}(\tau)\|_{L^2(A_1)}\, d\tau \\ &\quad + \int_0^t \|\widehat{K_2}(t-\tau)\widehat{\partial_1 N_2}(\tau)\|_{L^2(A_1)}\, d\tau \\ &:= H_1 + H_2 + H_3 + H_4.\end{aligned}$$

By Proposition 2.2,

$$H_1 \leq \|e^{-c_0(1+\xi_2^2)t}\widehat{\partial_1 u_0}\|_{L^2(\mathbb{R}^2)} \leq e^{-c_0 t}\|\partial_1 u_0\|_{L^2} \leq C\delta(1+t)^{-\frac{1}{2}},$$

where we have used $(1+t)e^{-c_0 t} \le C$. By the same technique, H_2 obeys the same bound, namely,

$$H_2 \le C\delta(1+t)^{-\frac{1}{2}}.$$

H_3 can be decomposed into two parts,

$$\begin{aligned} H_3 &\le \int_0^t \|\widehat{K_1}(t-\tau)\widehat{\partial_1(u\cdot\nabla u)}(\tau)\|_{L^2(A_1)}\, d\tau \\ &\quad + \int_0^t \|\widehat{K_1}(t-\tau)\widehat{\partial_1(b\cdot\nabla b)}(\tau)\|_{L^2(A_1)}\, d\tau \\ &= H_{3,1} + H_{3,2}. \end{aligned}$$

We further divide $H_{3,1}$ into two parts,

$$\begin{aligned} H_{3,1} &= \int_0^{t/2} \|\widehat{K_1}(t-\tau)\widehat{\partial_1(u\cdot\nabla u)}(\tau)\|_{L^2(A_1)}\, d\tau \\ &\quad + \int_{t/2}^t \|\widehat{K_1}(t-\tau)\widehat{\partial_1(u\cdot\nabla u)}(\tau)\|_{L^2(A_1)}\, d\tau \\ &= H_{3,1,1} + H_{3,1,2}. \end{aligned}$$

By Ladyzhenskaya's inequality, Proposition 2.2 and Lemma 3.3,

$$\begin{aligned} H_{3,1,1} &\le \int_0^{t/2} e^{-c_0(t-\tau)}\|\partial_1(u\cdot\nabla u)\|_{L^2}\, d\tau \\ &\le e^{-\frac{c_0}{2}t}\int_0^{t/2} (\|\partial_1 u\|_{L^4}\|\nabla u\|_{L^4} + \|u\|_{L^\infty}\|\partial_1\nabla u\|_{L^2})\, d\tau \\ &\le Ce^{-\frac{c_0}{2}t}\int_0^{t/2} \|\partial_1 u\|_{L^2}^{\frac{1}{2}}\|\Delta u\|_{L^2}\|\nabla u\|_{L^2}^{\frac{1}{2}} + \|u\|_{L^2}^{\frac{1}{4}}\|\partial_1 u\|_{L^2}^{\frac{1}{4}}\|\partial_2 u\|_{L^2}^{\frac{1}{4}}\|\Delta u\|_{L^2}^{\frac{5}{4}}\, d\tau \\ &\le Ce^{-\frac{c_0}{2}t}\int_0^{t/2} \tilde{c}(1+\tau)^{-\frac{1}{2}}\delta^2 + \tilde{c}^{\frac{3}{4}}(1+\tau)^{-\frac{1}{2}}\delta^2\, d\tau \\ &\le C\tilde{c}\delta^2(1+t)^{-\frac{1}{2}} + C\tilde{c}^{\frac{3}{4}}\delta^2(1+t)^{-\frac{1}{2}}, \end{aligned} \tag{3.38}$$

where we have used $e^{-\frac{c_0}{2}t}(1+t)^\gamma \le C(\gamma) < \infty$ for $\gamma > 0$. We write the norm in $H_{3,1,2}$ from frequency space to physical space, by Hölder's inequality, Lemma 3.1 and Lemma 3.2,

$$\begin{aligned} H_{3,1,2} &= \int_{t/2}^t e^{-c_0(t-\tau))}\|e^{-c_0\xi_2^2(t-\tau)}\widehat{\partial_1(u\cdot\nabla u)}\|_{L^2}\, d\tau \\ &\le \int_{t/2}^t e^{-c_0(t-\tau))}\left\|\|e^{-c_0\Lambda_2^2(t-\tau)}\partial_1(u\cdot\nabla u)\|_{L^2_{x_2}}\right\|_{L^2_{x_1}}\, d\tau \\ &\le C\int_{t/2}^t e^{-c_0(t-\tau)}(t-\tau)^{-\frac{1}{4}}\left\|\|\partial_1(u\cdot\nabla u)\|_{L^1_{x_2}}\right\|_{L^2_{x_1}}\, d\tau \\ &\le C\int_{t/2}^t e^{-c_0(t-\tau)}(t-\tau)^{-\frac{1}{4}}\left(\left\|\|\partial_1 u\cdot\nabla u\|_{L^1_{x_2}} + \|u\cdot\partial_1\nabla u\|_{L^1_{x_2}}\right\|_{L^2_{x_1}}\right) d\tau \\ &\le C\int_{t/2}^t e^{-c_0(t-\tau)}(t-\tau)^{-\frac{1}{4}}\left(\left\|\|\partial_1 u\|_{L^2_{x_2}}\|\nabla u\|_{L^2_{x_2}}\right\|_{L^2_{x_1}}\right. \\ &\qquad\qquad \left. + \left\|\|u\|_{L^2_{x_2}}\|\partial_1\nabla u\|_{L^2_{x_2}}\right\|_{L^2_{x_1}}\right) d\tau \\ &\le C\int_{t/2}^t e^{-c_0(t-\tau)}(t-\tau)^{-\frac{1}{4}}\left(\|\partial_1 u\|_{L^2_{x_2}L^\infty_{x_1}}\|\nabla u\|_{L^2} + \|u\|_{L^2_{x_2}L^\infty_{x_1}}\|\Delta u\|_{L^2}\right) d\tau \end{aligned}$$

$$\leq C\int_{t/2}^{t} e^{-c_0(t-\tau)}(t-\tau)^{-\frac{1}{4}}\|\partial_1 u\|_{L^2}^{\frac{1}{2}}\|\Delta u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}d\tau$$
$$+C\int_{t/2}^{t} e^{-c_0(t-\tau)}(t-\tau)^{-\frac{1}{4}}\|u\|_{L^2}^{\frac{1}{2}}\|\partial_1 u\|_{L^2}^{\frac{1}{2}}\|\Delta u\|_{L^2}d\tau$$
$$\leq C\tilde{c}^{\frac{3}{4}}\delta^2(1+t)^{-\frac{1}{2}}+C\tilde{c}\delta^2(1+t)^{-\frac{1}{2}}, \tag{3.39}$$

where we used $\int_{t/2}^{t} e^{-c_0(t-\tau)}(t-\tau)^{-\frac{1}{4}}\ d\tau<\infty$. Since $H_{3,2}$ admits the same bound as $H_{3,1}$,

$$H_3 \leq C\tilde{c}\delta^2(1+t)^{-\frac{1}{2}}+C\tilde{c}^{\frac{3}{4}}\delta^2(1+t)^{-\frac{1}{2}}.$$

H_4 obeys the same bound as H_3, hence,

$$\|\widehat{\partial_1 u}(t)\|_{L^2(A_1)}\leq C\delta(1+t)^{-\frac{1}{2}}+C\tilde{c}\delta^2(1+t)^{-\frac{1}{2}}+C\tilde{c}^{\frac{3}{4}}\delta^2(1+t)^{-\frac{1}{2}}. \tag{3.40}$$

Now we estimate $\|\widehat{\partial_1 u}\|_{L^2(A_{21})}$. By (3.37),

$$\begin{aligned}\|\widehat{\partial_1 u}(t)\|_{L^2(A_{21})}&\leq \|\widehat{K_1}(t)\widehat{\partial_1 u_0}\|_{L^2(A_{21})}+\|\widehat{K_2}(t)\widehat{\partial_1 b_0}\|_{L^2(A_{21})}\\&\quad+\int_0^t\|\widehat{K_1}(t-\tau)\widehat{\partial_1 N_1}(\tau)\|_{L^2(A_{21})}\ d\tau\\&\quad+\int_0^t\|\widehat{K_2}(t-\tau)\widehat{\partial_1 N_2}(\tau)\|_{L^2(A_{21})}\ d\tau\\&:=L_1+L_2+L_3+L_4.\end{aligned}$$

By Part (2) in Proposition 2.2 and Lemma 3.1,

$$\begin{aligned}L_1&=\|\widehat{K_1}(t)\widehat{\partial_1 u_0}\|_{L^2(A_{21})}\\&\leq C\|e^{-c_0(1+\xi_2^2)t}\widehat{\partial_1 u_0}\|_{L^2(A_{21})}+C\|e^{-c_0|\xi|^2t}\widehat{\partial_1 u_0}\|_{L^2(A_{21})}\\&\leq C\|e^{-c_0(1+\xi_2^2)t}\widehat{\partial_1 u_0}\|_{L^2(A_{21})}+C\|\xi e^{-c_0|\xi|^2t}\widehat{u_0}\|_{L^2(A_{21})}\\&\leq Ce^{-c_0t}\|\widehat{\partial_1 u_0}\|_{L^2(\mathbb{R}^2)}+C\|e^{-c_0\Lambda^2t}\Lambda u_0\|_{L^2(\mathbb{R}^2)}\\&\leq C(1+t)^{-1}\|\partial_1 u_0\|_{L^2(\mathbb{R}^2)}+Ct^{-1}\|u_0\|_{L^1(\mathbb{R}^2)}\\&\leq C(1+t)^{-1}\|u_0\|_{H^1\cap L^1}\leq C\delta(1+t)^{-\frac{1}{2}}.\end{aligned}$$

where we have used $e^{-c_o t}\leq C(1+t)^{-1}$ for $t\geq 0$. Similarly,

$$L_2\leq C(1+t)^{-1}\|b_0\|_{H^1\cap L^1}\leq C\delta(1+t)^{-\frac{1}{2}}.$$

We divide L_3 into four parts,

$$\begin{aligned}L_3&=\int_0^t\|\widehat{K_1}(t-\tau)\widehat{\partial_1 N_1}(\tau)\|_{L^2(A_{21})}\ d\tau\\&\leq C\int_0^t\|e^{-c_0(1+\xi_2^2)(t-\tau)}\widehat{\partial_1(u\cdot\nabla u)}\|_{L^2(\mathbb{R}^2)}\ d\tau\\&\quad+C\int_0^t\|e^{-c_0(1+\xi_2^2)(t-\tau)}\widehat{\partial_1(b\cdot\nabla b)}\|_{L^2(\mathbb{R}^2)}\ d\tau\\&\quad+C\int_0^t\|e^{-c_0|\xi|^2(t-\tau)}\widehat{\partial_1(u\cdot\nabla u)}\|_{L^2(\mathbb{R}^2)}\ d\tau\\&\quad+C\int_0^t\|e^{-c_0|\xi|^2(t-\tau)}\widehat{\partial_1(b\cdot\nabla b)}\|_{L^2(\mathbb{R}^2)}\ d\tau\\&=L_{3,1}+L_{3,2}+L_{3,3}+L_{3,4}.\end{aligned}$$

Clearly, $L_{3,1}$ can be bounded similarly as H_3, namely,

$$L_{3,1} \leq C\tilde{c}\delta^2(1+t)^{-\frac{1}{2}} + C\tilde{c}^{\frac{3}{4}}\delta^2(1+t)^{-\frac{1}{2}}. \tag{3.41}$$

$L_{3,2}$ admits the same bound as $L_{3,1}$. $L_{3,3}$ is decomposed into two parts,

$$\begin{aligned} L_{3,3} &= C\int_0^{t/2} \|e^{-c_0|\xi|^2(t-\tau)}\widehat{\partial_1(u\cdot\nabla u)}\|_{L^2(\mathbb{R}^2)}\,d\tau \\ &\quad + C\int_{t/2}^{t} \|e^{-c_0|\xi|^2(t-\tau)}\widehat{\partial_1(u\cdot\nabla u)}\|_{L^2(\mathbb{R}^2)}\,d\tau \\ &= L_{3,3,1} + L_{3,3,2}. \end{aligned}$$

By Lemma 3.1 and Hölder's inequality,

$$\begin{aligned} L_{3,3,1} &\leq C\int_0^{t/2} \|e^{-c_0|\xi|^2(t-\tau)}\widehat{\partial_1(u\cdot\nabla u)}\|_{L^2(\mathbb{R}^2)}\,d\tau \\ &\leq C\int_0^{t/2} \||\xi_1\xi|e^{-c_0|\xi|^2(t-\tau)}|\widehat{u\otimes u}|\|_{L^2(\mathbb{R}^2)}\,d\tau \\ &\leq C\int_0^{t/2} \||\xi|^2e^{-c_0|\xi|^2(t-\tau)}|\widehat{u\otimes u}|\|_{L^2(\mathbb{R}^2)}\,d\tau \\ &\leq C\int_0^{t/2} (t-\tau)^{-\frac{3}{2}}\|u\otimes u\|_{L^1}\,d\tau \leq C(t/2)^{-\frac{3}{2}}\int_0^{t/2}\|u\|_{L^2}^2\,d\tau \\ &\leq Ct^{-\frac{3}{2}}\int_0^{t/2}(\tilde{c}\delta(1+\tau)^{-\frac{1}{2}})^2\,d\tau \leq C\tilde{c}^2\delta^2t^{-\frac{3}{2}}\ln(1+t/2) \leq C\tilde{c}^2\delta^2(1+t)^{-\frac{1}{2}}. \end{aligned}$$

where we have used $t^{-\frac{1}{2}}\ln(1+t/2) \leq C$ for all $t \geq 1$. By Lemmas 3.1, 3.4 and Hölder's inequality,

$$\begin{aligned} L_{3,3,2} \leq& C\int_{t/2}^{t} \|e^{-c_0|\xi|^2(t-\tau)}\widehat{\partial_1(u\cdot\nabla u)}\|_{L^2(\mathbb{R}^2)}\,d\tau \\ &\leq C\int_{t/2}^{t} \||\xi|e^{-c_0|\xi|^2(t-\tau)}\widehat{\partial_1(u\otimes u)}\|_{L^2}\,d\tau \\ &\leq C\int_{t/2}^{t} (t-\tau)^{-\frac{1}{2}-\frac{2}{2}(\frac{1}{q}-\frac{1}{2})}\|\partial_1(u\otimes u)\|_{L^q}d\tau \\ &\leq C\int_{t/2}^{t} (t-\tau)^{-\frac{1}{q}}\|\partial_1 u\|_{L^2}\|u\|_{L^r}d\tau \qquad (1<q<2, \quad r>2) \\ &\leq C\int_{t/2}^{t} (t-\tau)^{-\frac{1}{q}}\|\partial_1 u\|_{L^2}\|u\|_{L^2}^{\frac{2}{r}}\|\nabla u\|_{L^2}^{1-\frac{2}{r}}\,d\tau \\ &\leq C\int_{t/2}^{t} (t-\tau)^{-\frac{1}{q}}(\tilde{c}\delta(1+\tau)^{-\frac{1}{2}})(\tilde{c}\delta(1+\tau)^{-\frac{1}{2}})\,d\tau \\ &\leq C\tilde{c}^2\delta^2(1+t)^{-1}\int_{t/2}^{t}(t-\tau)^{-\frac{1}{q}}\,d\tau \\ &\leq C\tilde{c}^2\delta^2(1+t)^{-1}t^{1-\frac{1}{q}} \leq C\tilde{c}^2\delta^2(1+t)^{-\frac{1}{q}} \leq C\tilde{c}^2\delta^2(1+t)^{-\frac{1}{2}}. \end{aligned}$$

Therefore,

$$L_{3,3} \leq C\tilde{c}^2\delta^2(1+t)^{-\frac{1}{2}}. \tag{3.42}$$

Similarly, $L_{3,4}$ obeys the same bound. By (3.41) and (3.42),

$$L_3 \leq C\tilde{c}\delta^2(1+t)^{-\frac{1}{2}} + C\tilde{c}^{\frac{3}{4}}\delta^2(1+t)^{-\frac{1}{2}} + C\tilde{c}^2\delta^2(1+t)^{-\frac{1}{2}}. \tag{3.43}$$

L_4 admits the same bound as L_3 in (3.43). By collecting all the bounds for L_1, L_2, L_3 and L_4, we obtain

$$\|\widehat{\partial_1 u}(t)\|_{L^2(A_{21})} \leq C\delta(1+t)^{-\frac{1}{2}} + C\tilde{c}\delta^2(1+t)^{-\frac{1}{2}} + C\tilde{c}^{\frac{3}{4}}\delta^2(1+t)^{-\frac{1}{2}} + C\tilde{c}^2\delta^2(1+t)^{-\frac{1}{2}}. \tag{3.44}$$

Next we estimate $\|\partial_1 u\|_{L^2(A_{22})}$. By (3.14),

$$\begin{aligned}
\|\widehat{\partial_1 u}(t)\|_{L^2(A_{22})} &\leq \|\widehat{K_1}(t)\widehat{\partial_1 u_0}\|_{L^2(A_{22})} + \|\widehat{K_2}(t)\widehat{\partial_1 b_0}\|_{L^2(A_{22})} \\
&\quad + \int_0^t \|\widehat{K_1}(t-\tau)\widehat{\partial_1 N_1}(\tau)\|_{L^2(A_{22})}\, d\tau \\
&\quad + \int_0^t \|\widehat{K_2}(t-\tau)\widehat{\partial_1 N_2}(\tau)\|_{L^2(A_{22})}\, d\tau \\
&:= S_1 + S_2 + S_3 + S_4.
\end{aligned}$$

By Part (3) in Proposition 2.2,

$$\begin{aligned}
S_1 &= \|\widehat{K_1}(t)\widehat{\partial_1 u_0}\|_{L^2(A_{22})} \\
&\leq C\|e^{-c_0(1+\xi_2^2)t}\widehat{\partial_1 u_0}\|_{L^2(A_{22})} + \|e^{-c_0(1+\frac{\xi_1^2}{\xi_2^2})t}\widehat{\partial_1 u_0}\|_{L^2(A_{22})} \\
&\leq C\|e^{-c_0(1+\xi_2^2)t}\widehat{\partial_1 u_0}\|_{L^2(\mathbb{R}^2)} + \|e^{-c_0(1+\frac{\xi_1^2}{\xi_2^2})t}\widehat{\partial_1 u_0}\|_{L^2(\mathbb{R}^2)} \\
&\leq Ce^{-c_0 t}\|\partial_1 u_0\|_{L^2} \leq C(1+t)^{-\frac{1}{2}}\|\partial_1 u_0\|_{L^2} \leq C\delta(1+t)^{-\frac{1}{2}}.
\end{aligned} \tag{3.45}$$

Similarly, S_2 admits the same bound, namely,

$$S_2 \leq C(1+t)^{-\frac{1}{2}}\|\partial_1 b_0\|_{L^2} \leq C\delta(1+t)^{-\frac{1}{2}}. \tag{3.46}$$

We decompose S_3 into two parts,

$$\begin{aligned}
S_3 &= \int_0^t \|\widehat{K_1}(t-\tau)\widehat{\partial_1 N_1}(\tau)\|_{L^2(A_{22})}\, d\tau \\
&= \int_0^{t/2} \|\widehat{K_1}(t-\tau)\widehat{\partial_1 N_1}(\tau)\|_{L^2(A_{22})}\, d\tau + \int_{t/2}^t \|\widehat{K_1}(t-\tau)\widehat{\partial_1 N_1}(\tau)\|_{L^2(A_{22})}\, d\tau \\
&:= S_{3,1} + S_{3,2}.
\end{aligned}$$

To bound $S_{3,1}$, we first apply Part (3) in Proposition 2.2 to decompose it into four terms

$$\begin{aligned}
S_{3,1} &= \int_0^{t/2} \|\widehat{K_1}(t-\tau)\widehat{\partial_1 N_1}(\tau)\|_{L^2(A_{22})}\, d\tau \\
&\leq C\int_0^{t/2} \|e^{-c_0(1+\xi_2^2)(t-\tau)}\widehat{\partial_1(u\cdot\nabla u)}\|_{L^2(\mathbb{R}^2)}\, d\tau \\
&\quad + C\int_0^{t/2} \|e^{-c_0(1+\xi_2^2)(t-\tau)}\widehat{\partial_1(b\cdot\nabla b)}\|_{L^2(\mathbb{R}^2)}\, d\tau \\
&\quad + C\int_0^{t/2} \|e^{-c_0(1+\frac{\xi_1^2}{\xi_2^2})(t-\tau)}\widehat{\partial_1(u\cdot\nabla u)}\|_{L^2(\mathbb{R}^2)}\, d\tau \\
&\quad + C\int_0^{t/2} \|e^{-c_0(1+\frac{\xi_1^2}{\xi_2^2})(t-\tau)}\widehat{\partial_1(b\cdot\nabla b)}\|_{L^2(\mathbb{R}^2)}\, d\tau.
\end{aligned}$$

Then we use the same techniques as in the estimates of $H_{3,1,1}$ in (3.38) to obtain

$$\begin{aligned} S_{3,1} &\leq C\int_0^{t/2} e^{-c_0(t-\tau)}(\|\partial_1(u\cdot\nabla u)\|_{L^2}+\|\partial_1(b\cdot\nabla b)\|_{L^2})\, d\tau \\ &\leq Ce^{-\frac{c_0}{2}t}\int_0^{t/2}(\|\partial_1(u\cdot\nabla u)\|_{L^2}+\|\partial_1(b\cdot\nabla b)\|_{L^2})d\tau \\ &\leq C\tilde{c}\delta^2(1+t)^{-\frac{1}{2}}+C\tilde{c}^{\frac{3}{4}}\delta^2(1+t)^{-\frac{1}{2}}. \end{aligned} \tag{3.47}$$

Now we use the new bounds in (3.25) and (3.26) to estimate $S_{3,2}$. By Hölder's inequality and (3.25),

$$\begin{aligned} S_{3,2} \leq{}& C\int_{t/2}^{t}\|e^{-c_0(1+\xi_2^2)(t-\tau)}|\widehat{\partial_1(u\cdot\nabla u)}|\|_{L^2}+\|e^{-c_0(1+\xi_2^2)(t-\tau)}|\widehat{\partial_1(b\cdot\nabla b)}|\|_{L^2}\, d\tau \\ &+C\int_{t/2}^{t}\left\|\left(\frac{\nu\eta\xi_2^2+\xi_1^2}{(\eta+\nu\xi_2^2)^2}+1\right)e^{-c_0(1+\xi_2^2)(t-\tau)}\left(|\widehat{\partial_1(u\cdot\nabla u)}|+|\widehat{\partial_1(b\cdot\nabla b)}|\right)\right\|_{L^2} d\tau \\ &+C\int_{t/2}^{t}\left\|\frac{1}{\eta+\nu\xi_2^2}\left(\frac{\nu\eta\xi_2^2+\xi_1^2}{\eta+\nu\xi_2^2}+\eta\right)e^{-\frac{\nu\eta\xi_2^2+\xi_1^2}{\eta+\nu\xi_2^2}(t-\tau)}\left(|\widehat{\partial_1(u\cdot\nabla u)}|+|\widehat{\partial_1(b\cdot\nabla b)}|\right)\right\|_{L^2} d\tau \\ :={}& S_{3,2,1}+S_{3,2,2}+S_{3,2,3}. \end{aligned}$$

We further rewrite $S_{3,2,1}$ into two parts,

$$\begin{aligned} S_{3,2,1} &= C\int_{t/2}^{t}\|e^{-c_0(1+\xi_2^2)(t-\tau)}|\widehat{\partial_1(u\cdot\nabla u)}|\|_{L^2}\, d\tau \\ &\quad +C\int_{t/2}^{t}\|e^{-c_0(1+\xi_2^2)(t-\tau)}|\widehat{\partial_1(b\cdot\nabla b)}|\|_{L^2}\, d\tau \\ &= S_{3,2,1,1}+S_{3,2,1,2}. \end{aligned}$$

By the same estimates as for $H_{3,1,2}$ in (3.39),

$$S_{3,2,1,1}\leq C\tilde{c}^{\frac{3}{4}}\delta^2(1+t)^{-\frac{1}{2}}+C\tilde{c}\delta^2(1+t)^{-\frac{1}{2}}$$

Clearly, $S_{3,2,1,2}$ admits the same bound, namely,

$$S_{3,2,1}\leq C\tilde{c}^{\frac{3}{4}}\delta^2(1+t)^{-\frac{1}{2}}+C\tilde{c}\delta^2(1+t)^{-\frac{1}{2}}. \tag{3.48}$$

Since $\xi\in A_{22}$, we have $\frac{\nu\eta\xi_2^2+\xi_1^2}{(\eta+\nu\xi_2^2)^2}<\frac{3}{16}$. By (3.48),

$$\begin{aligned} S_{3,2,2} &\leq C\int_{t/2}^{t}\left\|\left(\frac{\nu\eta\xi_2^2+\xi_1^2}{(\eta+\nu\xi_2^2)^2}+1\right)e^{-c_0(1+\xi_2^2)(t-\tau)}|\widehat{\partial_1(u\cdot\nabla u)}|\right\|_{L^2} d\tau \\ &\quad +C\int_{t/2}^{t}\left\|\left(\frac{\nu\eta\xi_2^2+\xi_1^2}{(\eta+\nu\xi_2^2)^2}+1\right)e^{-c_0(1+\xi_2^2)(t-\tau)}|\widehat{\partial_1(b\cdot\nabla b)}|\right\|_{L^2} d\tau \\ &\leq C\int_{t/2}^{t}\|e^{-c_0(1+\xi_2^2)(t-\tau)}\widehat{\partial_1(u\cdot\nabla u)}\|_{L^2}\, d\tau \\ &\quad +C\int_{t/2}^{t}\|e^{-c_0(1+\xi_2^2)(t-\tau)}\widehat{\partial_1(b\cdot\nabla b)}\|_{L^2}\, d\tau \\ &\leq CS_{3,2,1}\leq C\tilde{c}^{\frac{3}{4}}\delta^2(1+t)^{-\frac{1}{2}}+C\tilde{c}\delta^2(1+t)^{-\frac{1}{2}}. \end{aligned}$$

Furthermore, $S_{3,2,3}$ can be rewritten as

$$S_{3,2,3}=C\int_{t/2}^{t}\left\|\frac{1}{\eta+\nu\xi_2^2}\left(\frac{\nu\eta\xi_2^2+\xi_1^2}{\eta+\nu\xi_2^2}+\eta\right)e^{-\frac{\nu\eta\xi_2^2+\xi_1^2}{\eta+\nu\xi_2^2}(t-\tau)}|\widehat{\partial_1(u\cdot\nabla u)}|\right\|_{L^2} d\tau$$

$$
\begin{aligned}
&+C\int_{t/2}^{t}\left\|\frac{1}{\eta+\nu\xi_2^2}\left(\frac{\nu\eta\xi_2^2+\xi_1^2}{\eta+\nu\xi_2^2}+\eta\right)e^{-\frac{\nu\eta\xi_2^2+\xi_1^2}{\eta+\nu\xi_2^2}(t-\tau)}|\widehat{\partial_1(b\cdot\nabla b)}|\right\|_{L^2}\,d\tau\\
&=S_{3,2,3,1}+S_{3,2,3,2}.
\end{aligned}
$$

$S_{3,2,3,1}$ is naturally divided into two parts,

$$
\begin{aligned}
S_{3,2,3,1}&\le C\int_{t/2}^{t}\left\|\frac{|\xi|^2}{(1+\xi_2^2)^2}e^{-c_0\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)}|\widehat{\partial_1(u\cdot\nabla u)}|\right\|_{L^2}\,d\tau\\
&\quad+C\int_{t/2}^{t}\left\|\frac{|\xi|}{1+\xi_2^2}e^{-c_0\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)}\widehat{u\cdot\nabla u}\right\|_{L^2}\,d\tau\\
&=S_{3,2,3,1,1}+S_{3,2,3,1,2}.
\end{aligned}
$$

The process of estimating $S_{3,2,3,1,1}$ is tedious, so we first estimate $S_{3,2,3,1,2}$. By Lemma 3.3,

$$
\begin{aligned}
S_{3,2,3,1,2}&\le C\int_{t/2}^{t}\left\|\frac{|\xi|}{\sqrt{(1+\xi_2^2)}}(t-\tau)^{\frac12}(t-\tau)^{-\frac12}e^{-c_0\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)}\widehat{u\cdot\nabla u}\right\|_{L^2}\,d\tau\\
&\le C\int_{t/2}^{t}(t-\tau)^{-\frac12}e^{-\frac{c_0}{4}(1+\frac{\xi_1^2}{\xi_2^2})(t-\tau)}\|\widehat{u\cdot\nabla u}\|_{L^2}\,d\tau\\
&\le C\int_{t/2}^{t}(t-\tau)^{-\frac12}e^{-\frac{c_0}{4}(t-\tau)}\|u\cdot\nabla u\|_{L^2}\,d\tau\\
&\le C\int_{t/2}^{t}(t-\tau)^{-\frac12}e^{-\frac{c_0}{4}(t-\tau)}\|u\|_{L^\infty}\|\nabla u\|_{L^2}\,d\tau\\
&\le C\int_{t/2}^{t}(t-\tau)^{-\frac12}e^{-\frac{c_0}{4}(t-\tau)}\|u\|_{L^2}^{\frac14}\|\partial_1u\|_{L^2}^{\frac14}\|\partial_2u\|_{L^2}^{\frac14}\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}^{\frac14}\,d\tau\\
&\le C\int_{t/2}^{t}(t-\tau)^{-\frac12}e^{-\frac{c_0}{4}(t-\tau)}(\tilde c\delta(1+\tau)^{-\frac12})^{\frac32}(\tilde c\delta(1+\tau)^{-1})^{\frac14}(c\delta)^{\frac14}d\tau\\
&\le C\tilde c^2\delta^2(1+t/2)^{-1}\le C\tilde c^2\delta^2(1+t)^{-\frac12},
\end{aligned}
$$

where we have used $\gamma e^{-c_0\gamma^2}\le C$ and $\int_{t/2}^{t}(t-\tau)^{-\frac12}e^{-\frac{c_0}{4}(t-\tau)}\,d\tau<\infty$. To estimate $S_{3,2,3,1,1}$, we first write $\partial_1(u\cdot\nabla u)=\partial_1\partial_2(uu_2)+\partial_1\partial_1(uu_1)$,

$$
\begin{aligned}
S_{3,2,3,1,1}&\le C\int_{t/2}^{t}\left\|\frac{|\xi|^2}{(1+\xi_2^2)^2}e^{-c_0\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)}|\widehat{\partial_1\partial_2(uu_2)+\partial_1\partial_1(uu_1)}|\right\|_{L^2}\,d\tau\\
&\le C\int_{t/2}^{t}\left\|\frac{|\xi|^2}{(1+\xi_2^2)^2}e^{-c_0\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)}|\widehat{\partial_1\partial_2(uu_2)}|\right\|_{L^2}\,d\tau\\
&\quad+C\int_{t/2}^{t}\left\|\frac{|\xi|^2}{(1+\xi_2^2)^2}e^{-c_0\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)}|\widehat{\partial_1\partial_1(uu_1)}|\right\|_{L^2}\,d\tau\\
&=S_{3,2,3,1,1,1}+S_{3,2,3,1,1,2}.
\end{aligned}
$$

Since $\xi\in A_{22}$, $|\xi|^2\le C(1+\xi_2^2)^2$. By Lemma 3.5 and Lemma 3.3,

$$
\begin{aligned}
S_{3,2,3,1,1,1}\le\int_{t/2}^{t}\Bigg\|&\left(\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)\right)^{\sigma}\left(\frac{|\xi|}{(1+\xi_2^2)}\right)^{2-\sigma}\\
&\times|\xi_1|^{1-\sigma}(t-\tau)^{-\sigma}e^{-c_0\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)}|\widehat{\partial_2(uu_2)}|\Bigg\|_{L^2}\,d\tau
\end{aligned}
$$

$$
\begin{aligned}
&\leq C(\sigma)\int_{t/2}^{t}\left\|\left(\frac{|\xi|}{1+\xi_2^2}\right)^{2-\sigma}|\xi_1|^{1-\sigma}(t-\tau)^{-\sigma}e^{-\frac{c_0}{4}(1+\frac{\xi_1^2}{\xi_2^2})(t-\tau)}|\widehat{\partial_2(uu_2)}|\right\|_{L^2}\,d\tau\\
&\leq C(\sigma)\int_{t/2}^{t}\left\|(t-\tau)^{-\sigma}|\xi_1|^{1-\sigma}e^{-\frac{c_0}{4}(1+\frac{\xi_1^2}{\xi_2^2})(t-\tau)}|\widehat{\partial_2(uu_2)}|\right\|_{L^2}\,d\tau\\
&\leq C(\sigma)\int_{t/2}^{t}(t-\tau)^{-\sigma}e^{-\frac{c_0}{4}(t-\tau)}\left\|\widehat{\Lambda_1^{1-\sigma}(\partial_2(uu_2))}\right\|_{L^2}\,d\tau\\
&\leq C(\sigma)\int_{t/2}^{t}(t-\tau)^{-\sigma}e^{-\frac{c_0}{4}(t-\tau)}\|u\|_{L^2}^{\frac{\sigma}{3}}\|\nabla u\|_{L^2}^{\frac{\sigma}{3}}\|\Delta u\|_{L^2}^{1-\frac{2\sigma}{3}}\|\partial_2 u\|_{L^2}\,d\tau\\
&\quad+C(\sigma)\int_{t/2}^{t}(t-\tau)^{-\sigma}e^{-\frac{c_0}{4}(t-\tau)}\|\partial_2 u\|_{L^2}^{\sigma}\|\partial_1\partial_2 u\|_{L^2}^{1-\sigma}\\
&\qquad\qquad\times\|\partial_2 u\|_{L^2}^{\frac14}\|\partial_1 u\|_{L^2}^{\frac14}\|u\|_{L^2}^{\frac14}\|\Delta u\|_{L^2}^{\frac14}\,d\tau\\
&\leq C(\sigma)\int_{t/2}^{t}(t-\tau)^{-\sigma}e^{-\frac{c_0}{4}(t-\tau)}(\tilde{c}\delta(1+\tau)^{-\frac12})^{\frac{2\sigma}{3}}(C\delta)^{1-\frac{2\sigma}{3}}\tilde{c}\delta(1+\tau)^{-1}\,d\tau\\
&\quad+C(\sigma)\int_{t/2}^{t}(t-\tau)^{-\sigma}e^{-\frac{c_0}{4}(t-\tau)}(\tilde{c}\delta(1+\tau)^{-1})^{\sigma+\frac14}(C\delta)^{\frac54-\sigma}(\tilde{c}\delta(1+\tau)^{-\frac12})^{\frac12}\,d\tau\\
&\leq C(\sigma)\tilde{c}^{\frac{2\sigma}{3}+1}\delta^2(1+t)^{-1-\frac{\sigma}{3}}+C(\sigma)\tilde{c}^{\sigma+\frac34}\delta^2(1+t)^{-\sigma-\frac12}\\
&\leq C(\sigma)\tilde{c}^{\frac32}\delta^2(1+t)^{-\frac12},
\end{aligned}
$$

where we have set $\sigma=\frac34$, and used $\int_{t/2}^{t}(t-\tau)^{-\sigma}e^{-\frac{c_0}{4}(t-\tau)}\,d\tau<\infty$, (3.29) and the following inequality from (3.30),

$$
\begin{aligned}
\|\Lambda_1^{1-\sigma}(\partial_2(uu_2))\|_{L^2}&\leq C\|u\|_{L^2}^{\frac{\sigma}{3}}\|\nabla u\|_{L^2}^{\frac{\sigma}{3}}\|\Delta u\|_{L^2}^{1-\frac{2\sigma}{3}}\|\partial_2 u\|_{L^2}\\
&\quad+C\|\partial_2 u\|_{L^2}^{\sigma}\|\partial_1\partial_2 u\|_{L^2}^{1-\sigma}\|\partial_2 u\|_{L^2}^{\frac14}\|\partial_1 u\|_{L^2}^{\frac14}\|u\|_{L^2}^{\frac14}\|\Delta u\|_{L^2}^{\frac14}.
\end{aligned}
$$

Similarly, by (3.29),

$$
\begin{aligned}
S_{3,2,3,1,1,2}&\leq\int_{t/2}^{t}\left\|\left(\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)\right)^{\sigma}\left(\frac{|\xi|}{(1+\xi_2^2)}\right)^{2-\sigma}\right.\\
&\qquad\left.\times|\xi_1|^{1-\sigma}(t-\tau)^{-\sigma}e^{-c_0\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)}|\widehat{\partial_1(uu_1)}|\right\|_{L^2}\,d\tau\\
&\leq C(\sigma)\int_{t/2}^{t}\left\|\left(\frac{|\xi|}{1+\xi_2^2}\right)^{2-\sigma}|\xi_1|^{1-\sigma}(t-\tau)^{-\sigma}e^{-\frac{c_0}{4}(1+\frac{\xi_1^2}{\xi_1^2})(t-\tau)}|\widehat{\partial_1(uu_1)}|\right\|_{L^2}\,d\tau\\
&\leq C(\sigma)\int_{t/2}^{t}\left\|(t-\tau)^{-\sigma}|\xi_1|^{1-\sigma}e^{-\frac{c_0}{4}(1+\frac{\xi_1^2}{\xi_2^2})(t-\tau)}|\widehat{\partial_1(uu_1)}|\right\|_{L^2}\,d\tau\\
&\leq C(\sigma)\int_{t/2}^{t}(t-\tau)^{-\sigma}e^{-\frac{c_0}{4}(t-\tau)}\left\|\widehat{\Lambda_1^{1-\sigma}(\partial_1(uu_1))}\right\|_{L^2}\,d\tau\\
&\leq C(\sigma)\int_{t/2}^{t}(t-\tau)^{-\sigma}e^{-\frac{c_0}{4}(t-\tau)}\|u\|_{L^2}^{\frac{\sigma}{3}}\|\nabla u\|_{L^2}^{\frac{\sigma}{3}}\|\Delta u\|_{L^2}^{1-\frac{2\sigma}{3}}\|\partial_1 u\|_{L^2}\,d\tau\\
&\quad+C(\sigma)\int_{t/2}^{t}(t-\tau)^{-\sigma}e^{-\frac{c_0}{4}(t-\tau)}\|\partial_2 u\|_{L^2}^{\sigma}\|\partial_1\partial_1 u\|_{L^2}^{1-\sigma}
\end{aligned}
$$

$$
\begin{aligned}
&\times \|\partial_2 u\|_{L^2}^{\frac14}\|\partial_1 u\|_{L^2}^{\frac14}\|u\|_{L^2}^{\frac14}\|\Delta u\|_{L^2}^{\frac14}\, d\tau\\
&\le C(\sigma)\tilde c^{\frac32}\delta^2(1+t)^{-\frac12},
\end{aligned}
$$

where we have set $\sigma = \frac34$, and used the inequality below following from (3.30),

$$
\begin{aligned}
\|\Lambda_1^{1-\sigma}(\partial_1(uu_1))\|_{L^2} \le{}& C\|u\|_{L^2}^{\frac{\sigma}{3}}\|\nabla u\|_{L^2}^{\frac{\sigma}{3}}\|\Delta u\|_{L^2}^{1-\frac{2\sigma}{3}}\|\partial_1 u\|_{L^2}\\
&+ C\|\partial_1 u\|_{L^2}^{\sigma}\|\partial_1\partial_1 u\|_{L^2}^{1-\sigma}\|\partial_2 u\|_{L^2}^{\frac14}\|\partial_1 u\|_{L^2}^{\frac14}\|u\|_{L^2}^{\frac14}\|\Delta u\|_{L^2}^{\frac14}.
\end{aligned}
$$

Therefore,

$$
S_{3,2,3,1} \le C\tilde c^2\delta^2(1+t)^{-\frac12} + \tilde c^{\frac32}\delta^2(1+t)^{-\frac12}.
$$

Similarly, $S_{3,2,3,2}$ admits the same bound. Collecting the bounds for $S_{3,2,1}, S_{3,2,2}$ and $S_{3,2,3}$ yields

$$
S_{3,2} \le C\tilde c^{\frac34}\delta^2(1+t)^{-\frac12} + C\tilde c^2\delta^2(1+t)^{-\frac12} + C(\sigma)\tilde c^{\frac32}\delta^2(1+t)^{-\frac12}. \tag{3.49}
$$

Combining the estimates for $S_{3,1}$ and $S_{3,2}$ in (3.47) and (3.49) respectively, we have

$$
S_3 \le (\tilde c + \tilde c^{\frac34} + \tilde c^2 + \tilde c^{\frac32})C\delta^2(1+t)^{-\frac12}. \tag{3.50}
$$

Next we estimate S_4. We first divide S_4 into two parts according to (3.26),

$$
\begin{aligned}
S_4 &= \int_0^t \|\widehat{K_2}(t-\tau)\widehat{\partial_1 N_2}(\tau)\|_{L^2(A_{22})}\, d\tau\\
&= \int_0^{t/2} \|\widehat{K_2}(t-\tau)\widehat{\partial_1 N_2}(\tau)\|_{L^2(A_{22})}\, d\tau + \int_{t/2}^t \|\widehat{K_2}(t-\tau)\widehat{\partial_1 N_2}(\tau)\|_{L^2(A_{22})}\, d\tau\\
&:= S_{4,1} + S_{4,2}.
\end{aligned}
$$

By Part (3) in Proposition 2.2 and by (3.47), $S_{4,1}$ obeys the same bound as $S_{3,1}$, namely,

$$
S_{4,1} \le C\tilde c\delta^2(1+t)^{-\frac12} + C\tilde c^{\frac34}\delta^2(1+t)^{-\frac12}. \tag{3.51}
$$

Since the bound for $\widehat{K_2}$ in (3.26) is not the same as the bound for $\widehat{K_1}$ in (3.25), we need to estimate $S_{4,2}$ differently from $S_{3,2}$.

$$
\begin{aligned}
S_{4,2} \le{}& C\int_{t/2}^t \left\| \frac{|\xi_1|}{1+\xi_2^2} e^{-c_0(1+\xi_2^2)(t-\tau)}\left(|\widehat{\partial_1(u\cdot\nabla b)}| + |\widehat{\partial_1(b\cdot\nabla u)}|\right)\right\|_{L^2} d\tau\\
&+ C\int_{t/2}^t \left\| \frac{|\xi_1|}{1+\xi_2^2} e^{-\frac{\nu\eta\xi_2^2+\xi_1^2}{\eta+\nu\xi_2^2}(t-\tau)}\left(|\widehat{\partial_1(u\cdot\nabla b)}| + |\widehat{\partial_1(b\cdot\nabla u)}|\right)\right\|_{L^2} d\tau\\
:={}& S_{4,2,1} + S_{4,2,2}.
\end{aligned}
$$

Since $\xi\in A_{22}$, $\frac{\xi^2}{(1+\xi_2^2)^2} \le C$. By the same process as in (3.39), we write

$$
\begin{aligned}
S_{4,2,1} \le{}& C\int_{t/2}^t \|e^{-c_0(1+\xi_2^2)(t-\tau)}\widehat{\partial_1(u\cdot\nabla b)}\|_{L^2}\, d\tau\\
&+ C\int_{t/2}^t \|e^{-c_0(1+\xi_2^2)(t-\tau)}\widehat{\partial_1(b\cdot\nabla u)}\|_{L^2}\, d\tau\\
\le{}& C\tilde c^{\frac34}\delta^2(1+t)^{-\frac12} + C\tilde c\delta^2(1+t)^{-\frac12}.
\end{aligned}
$$

We rewrite $S_{4,2,2}$ as

$$
S_{4,2,2} \le C\int_{t/2}^t \left\| \frac{|\xi_1|}{1+\xi_2^2} e^{-\frac{\nu\eta\xi_2^2+\xi_1^2}{\eta+\nu\xi_2^2}(t-\tau)}|\widehat{\partial_1(u\cdot\nabla b)}|\right\|_{L^2} d\tau
$$

$$
\begin{aligned}
&+ C\int_{t/2}^{t}\left\|\frac{|\xi_1|}{1+\xi_2^2}e^{-\frac{\nu\eta\xi_2^2+\xi_1^2}{\eta+\nu\xi_2^2}(t-\tau)}|\widehat{\partial_1(b\cdot\nabla u)}|\right\|_{L^2}\,d\tau\\
&:= S_{4,2,2,1}+S_{4,2,2,2}.
\end{aligned}
$$

To bound $S_{4,2,2,1}$, we write $\partial_1(u\cdot\nabla b)=\partial_1\partial_2(bu_2)+\partial_1\partial_1(bu_1)$,

$$
\begin{aligned}
S_{4,2,2,1} &\le C\int_{t/2}^{t}\left\|\frac{|\xi_1|}{1+\xi_2^2}e^{-c_0\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)}|\widehat{\partial_1\partial_2(bu_2)+\partial_1\partial_1(bu_1)}|\right\|_{L^2}\,d\tau\\
&\le C\int_{t/2}^{t}\left\|\frac{|\xi_1|}{1+\xi_2^2}e^{-c_0\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)}|\widehat{\partial_1\partial_2(bu_2)}|\right\|_{L^2}\,d\tau\\
&\quad + C\int_{t/2}^{t}\left\|\frac{|\xi_1|}{1+\xi_2^2}e^{-c_0\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)}|\widehat{\partial_1\partial_1(bu_1)}|\right\|_{L^2}\,d\tau\\
&= S_{4,2,2,1,1}+S_{4,2,2,1,2}.
\end{aligned}
$$

The first piece is bounded by

$$
\begin{aligned}
S_{4,2,2,1,1} &\le \int_{t/2}^{t}\left\|\left(\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)\right)^{\frac{\sigma+1}{2}}|\xi_1|^{1-\sigma}(t-\tau)^{-\frac{\sigma+1}{2}}e^{-c_0\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)}|\widehat{\partial_2(bu_2)}|\right\|_{L^2}\,d\tau\\
&\le C(\sigma)\int_{t/2}^{t}\left\||\xi_1|^{1-\sigma}(t-\tau)^{-\frac{\sigma+1}{2}}e^{-\frac{c_0}{4}(1+\frac{\xi_1^2}{\xi_2^2})(t-\tau)}|\widehat{\partial_2(bu_2)}|\right\|_{L^2}\,d\tau\\
&\le C(\sigma)\int_{t/2}^{t}(t-\tau)^{-\frac{\sigma+1}{2}}e^{-\frac{c_0}{4}(t-\tau)}\left\|\widehat{\Lambda_1^{1-\sigma}(\partial_2(bu_2))}\right\|_{L^2}\,d\tau\\
&\le C(\sigma)\int_{t/2}^{t}(t-\tau)^{-\frac{\sigma+1}{2}}e^{-\frac{c_0}{4}(t-\tau)}\|u\|_{L^2}^{\frac{\sigma}{3}}\|\nabla u\|_{L^2}^{\frac{\sigma}{3}}\|\Delta u\|_{L^2}^{1-\frac{2\sigma}{3}}\|\partial_2 b\|_{L^2}\,d\tau\\
&\quad + C(\sigma)\int_{t/2}^{t}(t-\tau)^{-\frac{\sigma+1}{2}}e^{-\frac{c_0}{4}(t-\tau)}\|\partial_2 b\|_{L^2}^{\sigma}\|\partial_1\partial_2 b\|_{L^2}^{1-\sigma}\\
&\quad \times\|\partial_2 u\|_{L^2}^{\frac14}\|\partial_1 u\|_{L^2}^{\frac14}\|u\|_{L^2}^{\frac14}\|\Delta u\|_{L^2}^{\frac14}\,d\tau\\
&\quad + C(\sigma)\int_{t/2}^{t}(t-\tau)^{-\frac{\sigma+1}{2}}e^{-\frac{c_0}{4}(t-\tau)}\|b\|_{L^2}^{\frac{\sigma}{3}}\|\nabla b\|_{L^2}^{\frac{\sigma}{3}}\|\Delta b\|_{L^2}^{1-\frac{2\sigma}{3}}\|\partial_2 u_2\|_{L^2}\,d\tau\\
&\quad + C(\sigma)\int_{t/2}^{t}(t-\tau)^{-\frac{\sigma+1}{2}}e^{-\frac{c_0}{4}(t-\tau)}\|\partial_2 u_2\|_{L^2}^{\sigma}\|\partial_1\partial_2 u_2\|_{L^2}^{1-\sigma}\\
&\quad \times\|\partial_2 b\|_{L^2}^{\frac14}\|\partial_1 b\|_{L^2}^{\frac14}\|b\|_{L^2}^{\frac14}\|\Delta b\|_{L^2}^{\frac14}\,d\tau\\
&\le C(\sigma)\int_{t/2}^{t}(t-\tau)^{-\frac{\sigma+1}{2}}e^{-\frac{c_0}{4}(t-\tau)}(\tilde c\delta(1+\tau)^{-\frac12})^{\frac{2\sigma}{3}}(C\delta)^{1-\frac{2\sigma}{3}}\tilde c\delta(1+\tau)^{-1}\,d\tau\\
&\quad + C(\sigma)\int_{t/2}^{t}(t-\tau)^{-\frac{\sigma+1}{2}}e^{-\frac{c_0}{4}(t-\tau)}(\tilde c\delta(1+\tau)^{-1})^{\sigma+\frac14}(C\delta)^{\frac54-\sigma}(\tilde c\delta(1+\tau)^{-\frac12})^{\frac12}\,d\tau\\
&\le C(\sigma)\tilde c^{\frac{2\sigma}{3}+1}\delta^2(1+t)^{-1-\frac{\sigma}{3}}+C(\sigma)\tilde c^{\sigma+\frac34}\delta^2(1+t)^{-\sigma-\frac12}\\
&\le C\tilde c^{\frac32}\delta^2(1+t)^{-\frac12},
\end{aligned}
$$

where we have set $\sigma=\frac34$, and used $\int_{t/2}^{t}(t-\tau)^{-\frac{\sigma+1}{2}}e^{-\frac{c_0}{4}(t-\tau)}\,d\tau<\infty$, and

$$
\left(\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)\right)^{\frac{\sigma+1}{2}}e^{-\frac{c_0}{2}\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)}\le C(\sigma).
$$

In addition, we have used the following estimate above, due to Lemma 3.5,

$$\begin{aligned}\|\Lambda_1^{1-\sigma}(\partial_2(bu_2))\|_{L^2} &\le \|\Lambda_1^{1-\sigma}(\partial_2 bu_2)\|_{L^2} + \|\Lambda_1^{1-\sigma}(b\partial_2 u_2)\|_{L^2}\\ &\le C\|\partial_2 b\|_{L^2}\|\Lambda_1^{1-\sigma}u_2\|_{L^\infty} + C\|\Lambda_1^{1-\sigma}\partial_2 b\|_{L^2}\|u_2\|_{L^\infty}\\ &\quad + C\|\partial_2 u_2\|_{L^2}\|\Lambda_1^{1-\sigma}b\|_{L^\infty} + C\|\Lambda_1^{1-\sigma}\partial_2 u_2\|_{L^2}\|b\|_{L^\infty}\\ &\le C\|u\|_{L^2}^{\frac{\sigma}{3}}\|\nabla u\|_{L^2}^{\frac{\sigma}{3}}\|\Delta u\|_{L^2}^{1-\frac{2\sigma}{3}}\|\partial_2 b\|_{L^2}\\ &\quad + C\|\partial_2 b\|_{L^2}^{\sigma}\|\partial_1\partial_2 b\|_{L^2}^{1-\sigma}\|\partial_2 u\|_{L^2}^{\frac14}\|\partial_1 u\|_{L^2}^{\frac14}\|u\|_{L^2}^{\frac14}\|\Delta u\|_{L^2}^{\frac14}\\ &\quad + C\|b\|_{L^2}^{\frac{\sigma}{3}}\|\nabla b\|_{L^2}^{\frac{\sigma}{3}}\|\Delta b\|_{L^2}^{1-\frac{2\sigma}{3}}\|\partial_2 u_2\|_{L^2}\\ &\quad + C\|\partial_2 u_2\|_{L^2}^{\sigma}\|\partial_1\partial_2 u_2\|_{L^2}^{1-\sigma}\|\partial_2 b\|_{L^2}^{\frac14}\|\partial_1 b\|_{L^2}^{\frac14}\|b\|_{L^2}^{\frac14}\|\Delta b\|_{L^2}^{\frac14}.\end{aligned}$$

Similarly,

$$S_{4,2,2,1,2} \le C\tilde{c}^{\frac32}\delta^2(1+t)^{-\frac12}.$$

Since $S_{4,2,2,2}$ obeys the same bound as $S_{4,2,2,1}$, we obtain

$$S_{4,2,2} \le C(\sigma)\tilde{c}^{\frac32}\delta^2(1+t)^{-\frac12}.$$

Therefore,

$$S_{4,2} \le C\tilde{c}^2\delta^2(1+t)^{-\frac12} + C\tilde{c}^{\frac32}\delta^2(1+t)^{-\frac12} + C\tilde{c}^{\frac34}\delta^2(1+t)^{-\frac12}. \tag{3.52}$$

Collecting (3.51) and (3.52) yields

$$S_4 \le (\tilde{c} + \tilde{c}^2 + \tilde{c}^{\frac34} + \tilde{c}^{\frac32})C\delta^2(1+t)^{-\frac12}. \tag{3.53}$$

Combining (3.45), (3.46), (3.50) and (3.53), we obtain

$$\|\widehat{\partial_1 u}(t)\|_{L^2(A_{22})} \le C\delta(1+t)^{-\frac12} + (\tilde{c} + \tilde{c}^{\frac34} + \tilde{c}^2 + \tilde{c}^{\frac32})C\delta^2(1+t)^{-\frac12} \tag{3.54}$$

Putting (3.40), (3.44) and (3.54) together leads to

$$\|\widehat{\partial_1 u}(t)\|_{L^2} \le C_1\delta(1+t)^{-\frac12} + \tilde{c}C_2\delta^2(1+t)^{-\frac12} + (\tilde{c}^{\frac34} + \tilde{c}^2 + \tilde{c}^{\frac32})C_3\delta^2(1+t)^{-\frac12}.$$

If we choose $\tilde{c}$ and δ satisfying

$$C_1 \le \frac{\tilde{c}}{8}, \qquad C_2\delta \le \frac{1}{16}, \qquad (\tilde{c}^{\frac34} + \tilde{c}^2 + \tilde{c}^{\frac32})C_3\delta \le \frac{\tilde{c}}{16},$$

then we obtain

$$\begin{aligned}\|\partial_1 u(t)\|_{L^2} &\le \frac{\tilde{c}}{4}\delta(1+t)^{-\frac12} + \frac{\tilde{c}}{8}\delta(1+t)^{-1} + \frac{\tilde{c}}{8}\delta(1+t)^{-\frac12}\\ &= \frac{\tilde{c}}{2}\delta(1+t)^{-\frac12}.\end{aligned}$$

A similar bound holds for $\|\partial_1 b\|_{L^2}$. Therefore,

$$\|(\partial_1 u(t), \partial_1 b(t))\|_{L^2} \le \frac{\tilde{c}}{2}\delta(1+t)^{-\frac12}.$$

This completes the proof of the second inequality in (3.2). □

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Declarations

Conflict of interest The authors state that there is no conflict of interest.

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