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On The Two-Dimensional Stokes Problem in Exterior Domains: The Maximum Modulus Theorem

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Abstract. The two-dimensional Stokes IBVP on $(0,T)\times\Omega$ is investigated under the assumptions that $\Omega\subset\mathbb{R}^2$ is a smooth exterior domain, the initial datum v_0 belongs to $L^\infty(\Omega)$ and $(v_0,\nabla\phi)=0$ for all $\phi\in L^1_{\ell oc}(\Omega)$ with $\nabla\phi\in L^1(\Omega)$. The well-posedeness in $L^\infty((0,T)\times\Omega)$ and the *maximum modulus* theorem are achieved, in particular one deduces that the Stokes semigroup on $L^\infty(\Omega)$ is a bounded analytic semigroup.

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1. Introduction

We consider the following initial boundary value problem for the Stokes system:

$$\begin{aligned}&v_t-\Delta v+\nabla\pi=0\,,\ \nabla\cdot v=0\,,\ \text{in } (0,T)\times\Omega\,,\\&v=0 \text{ on } (0,T)\times\partial\Omega\,,\ v=v_0 \text{ on } \{0\}\times\Omega\,,\end{aligned}\tag{1}$$

where $\Omega\subset\mathbb{R}^2$ is a smooth exterior domain. Following [3,4], we consider $v_0\in L^\infty(\Omega)$ with

$$\int_\Omega v_0\cdot\nabla\varphi dx=0\,,\ \text{for all } \varphi\in L^1_{\ell oc}(\Omega) \text{ with } \nabla\varphi\in L^1(\Omega)\,.\tag{2}$$

We are interested to prove

Theorem 1. (Maximum Modulus Theorem) *For all $v_0\in L^\infty(\Omega)$ enjoying (2), there exists a unique solution (v,π) to problem (1) such that*

$$\|v(t)\|_\infty+t\|v_t(t)\|_\infty\le c\|v_0\|_\infty\,,\ \textit{for all } t>0\,,\tag{3}$$

for all $q>2$,

$$\begin{aligned}&\tfrac{t^{\frac12}}{1+t^{\frac12}}\|\nabla v(t)\|_{L^q_{\ell oc}(\overline\Omega)}+\tfrac{t}{1+t}\|\nabla\nabla v(t)\|_{L^q_{\ell oc}(\overline\Omega)}\le c\|v_0\|_\infty\,,\ \textit{for all } t>0\,,\\&\textit{for all } R_0>0\,,\ \lim_{t\to0}\|v(t)-v_0\|_{L^q(\Omega\cap\mathbb{B}_{R_0})}=0\,,\\&\overline\mu\in(0,\frac12)\,,\ \tfrac{t^{\frac12+\overline\mu}}{1+t^{\frac12+\overline\mu}}\|\pi(t)\|_\infty+\tfrac{t}{t+1}\|\nabla\pi(t)\|_q\le c\|v_0\|_\infty\,,\ \textit{for all } t>0\,.\end{aligned}\tag{4}$$

where c is a constant independent of (v,π).

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We conclude the introduction of the result of the theorem with further possible statements.

If the initial datum belongs to $L^\infty(\Omega) \cap C(\Omega)$, then, for all $x \in \Omega$, $\lim_{t\to 0} v(t,x) = v_0(x)$ holds.

If we assume that $v_0(x) \to 0$ for large $|x|$, then, for all $t > 0$, the solution has the same property and $\lim_{t\to\infty} \|v(t)\|_\infty = 0$ holds too.

The pointwise estimate that one could deduce from (3)-$(4)_1$ for $|\nabla v(t,x)|$ is sharp in the following sense: if $v_0 \in L^\infty(\Omega)$ no asymptotic decay holds, and just an $o(1)$ for large t provided that $v_0(x) \to 0$ for large $|x|$. Of course, this relates to the nature of the exterior domain Ω. In the case of the Cauchy problem or IBVP in Ω bounded, one obtains suitable decay properties.

One can proves that the solution (v, π) is smooth for all $t > 0$.

Via the same approach proposed in this note, the uniqueness can be deduced in a wider set of solutions. Roughly speaking, on the "boundary of the uniqueness set" we find the solutions whose associated pressure field grows as $|x_i|$, $i = 1, 2$, that are not unique.

With the exception of the uniqueness remark, all the statements are considered and proved in [16] for $\Omega \subset \mathbb{R}^n$, $n \geq 3$, exterior domain. The argument lines work with no change also in the two-dimensional case. For the sake of brevity we omit any detail. Concerning the non-uniqueness, there is today a wide literature, we refer the interested reader to the one contained in the recent paper [18].

Last, but not least, Theorem 1 is interesting in order to modify the approach used in the papers [20,21] for the 2D-Navier-Stokes IBVP in exterior domains in L^∞-setting and as a consequence to improve some related results.

Theorem 1 follows a series of papers concerning with the well posedeness in L^∞-setting and developed by some authors in the last decades.

The first results of the kind stated in Theorem 1, as far as we know, are due to Solonnikov in the articles [25,27,28]. Succesively, the Stokes initial boundary value problem with an initial datum in L^∞, jointly with L^∞-estimates of the solutions, has been considered by several authors, both with homogeneous boundary data, see e.g. [3–6,13,16], and with non-homogeneous, see e.g. [7].

In the literature devoted to the question, a distinction is made in connection with the nature of domain Ω. The IBVP in Ω bounded can be considered achieved, whether by means of the methods of the potential theory (essentially [25–28]) or by means of the methods of functional analysis ([3,4]). Instead, in the case of the IBVP in exterior domains, the n-dimensional case, $n > 2$, can be considered achieved (cf. [6,16] too), while in the two-dimensional exterior domain the following results hold.

The contributions given in [27,28] related to the non-homogeneous and homogeneous boundary data respectively, based on the theory of the hydrodynamic potentials, while the quoted literature, based on methods of functional analysis, achieves some results in a sequence of different papers [3,4] and [1,2].

The result in [3,4] is partial, in the sense that the L^∞-estimate *a priori* holds locally in time:

$$\|u(t)\|_\infty \leq c\|u_0\|_\infty\,, \text{ for } t \in [0, T_0)\,, \tag{5}$$

where the constant c and the size T_0, *a priori* $< +\infty$, are independent of u.

Subsequently in [1] estimate (5) is obtained for all $t > 0$, but the result holds losing in terms of generality. Indeed, in [1] the author considers the set of solutions for which the net force satisfies $\int_{\partial\Omega} \nu \cdot T(u, \pi_u) d\mathcal{H}^1 = 0$, where the symbol $T(u, \pi_u)$ denotes the stress tensor and ν is the normal to $\partial\Omega$.

Finally, in [2] the author proves that the Stokes operator is a bounded analytic semigroup of angle $\frac{\pi}{2}$ on the subset $L^\infty_\sigma(\Omega)$ of $L^\infty(\Omega)$, in particular estimate (5) holds for all $t > 0$. The symbol $L^\infty_\sigma(\Omega)$ denotes the following set:

$$L^\infty_\sigma(\Omega) := \{u_0 \in L^\infty(\Omega) : \nabla \cdot u_0 = 0 \text{ in } \Omega\,,\ u_0 \cdot \nu = 0 \text{ on } \partial\Omega\}\,,$$

where ν is the normal to the boundary. However, as remarked in [1], $L^\infty_\sigma(\Omega)$ coincide with the set of initial data considered in Theorem 1. As far as we know, the result in [2] is the most complete in the scope of the 2D-Stokes semigroup results.

We do not know an estimate of the pressure field π_v like the one given in (4). In our approach to the uniqueness, this kind of estimate is crucial. In this regard, we deem it appropriate to make a digression in relation to the problem of the uniqueness of the solutions to the problem (54).

In the L^∞-setting the difficulties to obtain a sharp result are connected essentially with the lack of the Helmholtz decomposition of L^∞, decomposition that in L^q-setting, for all $q \in (1,\infty)$, holds. The lack of the Helmholtz decomposition does not allow to state *a posteriori* the existence of the pressure field as in the case of the L^q-theory or to define the Stokes operator like in the L^q-theory.

In the recent paper [18], the present author proves the existence of solutions (u,π) enjoying (3)-$(4)_{1,2}$ with a pressure field $\pi := u^\infty(t)\cdot x+\overline{\pi}$, $\nabla\overline{\pi} \in L^q(\Omega)$, that is different from $(4)_3$ of Theorem 1. Moreover, the field v enjoys the property: $\lim_{|x|\to\infty} |u(t,x) - u^\infty(t)| = 0$. But the limit $u^\infty(t) \neq 0$ is not a datum of the problem. Actually, following [18], one can construct infinite ones. As a consequence a lack of uniqueness arises.

This critical result is produced considering the two dimensional boundary value problem for steady Stokes system in exterior domains, which admits the same pathologic solutions. That is, a solution admits a value at infinity that is not a datum of the problem.

The result given in [18] is not in contradiction with the ones contained in [3,4] (or with the statement of Theorem 1 of the present paper). Actually, the pressure field π does not verify the condition $d(x,\partial\Omega)|\nabla\pi(t,x)| \le c$ with c independent of x, property exhibits in [3,4] for the solutions (or estimates (4) of the present paper). Since the solution with limit $u^\infty(t)$ obtained in [18] is unique,[1] as consequence we get that no solution established in [3,4] admits $u^\infty(t)$ as limit at infinity (as well as the one of Theorem 1).

The present paper is devoted to the memory of Professor Carlo Miranda, he was an Eminent Mathematician in Napoli, this year is the 40th anniversary of his death.

1.1. Outline of The Proof

Before outlining the proof of Theorem 1, we consider useful to recall what approaches the present author employed in previous papers studying the question in nD, $n \ge 3$, exterior domains and the result of non-uniqueness in 2D.

In n-dimensional case, $n \ge 3$, the results are proved by means of a suitable coupling of the results proved in [3] and in [16], subsequently the same approach is reconsidered in [6]. As already recalled, the first paper is concerned with local in time estimates and the second paper is concerned with the extension of the estimates to large time.

In the two-dimensional case the result of the first paper still works, while the result of the second paper does not work. The result in [16] is based on a technique of duality which does not work in two-dimension, roughly speaking, because the solution $\varphi(t,x)$ of the (local) adjoint problem has the behavior $\|\varphi(t)\|_\infty \le c\|\varphi\|_1 t^{-1}$ where the exponent -1 is sharp. Actually, in [16] one translates the original question into the study of the problem

$$\omega_t - \Delta\omega + \nabla\pi_\omega = -F_t^{(1)} + G\,,\ \nabla\cdot\omega = 0\,,\ \text{in } (0,T)\times\Omega\,,\ \omega = 0 \text{ on } (0,T)\times\partial\Omega\,,\ \omega = 0 \text{ on } \{0\}\times\Omega\,,$$

where $F_t^{(1)}$ and G are suitable functions. In this way the difficulty becomes the fact that G, with compact support in $x \in \Omega$ and belonging to $L^\infty((0,T)\times\Omega)$, has no behavior for large t. By a duality approach,

[1] This sentence could seem in contradiction with the previous result of non-uniqueness. For this reason we explain the meaning of all. The non-uniqueness is with respect the initial datum u_0, in fact we find two solutions corresponding to the same u_0 but one tends to a limit $u_\infty(t)$ and another tends to a limit $\overline{u}_\infty(t) \neq u^\infty(t)$ by letting $|x| \to \infty$, not only this, the pressure fields are different too. Nevertheless, in corresponding to the same u_0 and $u_\infty(t)$ the field u and the related pressure π are unique, but, since $u_\infty(t)$ is not a datum, we arrive at an unsatisfactory result.

in order to obtain an estimate for $\|\omega(t)\|_\infty$, one has to tackle the estimate related to

$$\int_0^t (G(\tau),\varphi(t-\tau))d\tau\,, \tag{6}$$

for which, due to the sharpness behavior of $\|\varphi(s)\|_\infty \le c\|\varphi_0\|_1 s^{-1}$, one at most is able to deduce $O(\|\varphi_0\|_1 \log(t+e))$. Hence, via this approach, no uniform bound in t holds for the L^∞-norm of the solution $\omega(t,x)$.

In the paper [18], roughly speaking, the previous problem of ω becomes

$$\omega_t - \Delta\omega + \nabla\pi_\omega = -F^{(2)}_t\,,\ \nabla\cdot\omega = 0\,,\ \text{in } (0,T)\times\Omega\,,\ \omega = 0 \text{ on } (0,T)\times\partial\Omega\,,\ \omega = 0 \text{ on } \{0\}\times\Omega\,,$$

$F^{(2)}$ plays the same role of $F^{(1)}$ but with different properties. The function $F^{(2)}$ in [18], for all $t>0$, is the extension of $-U(t,x)_{|\partial\Omega} + \widehat{U}(t)$ from $\partial\Omega$ into Ω, where U is the solution of heat equation corresponding to v_0^R and $\widehat{U} = |\partial\Omega|^{-1}\int\limits_{\partial\Omega} U(t,\xi)d\mathcal{H}^1$, and the L^q-norm of $F^{(2)}(t,x)$ has a "good" space-time decay in (t,x). For all $t>0$, the extension $F^{(2)}$ is just the solution to the boundary value problem in Ω of the steady Stokes system. Thanks to this construction, for all $v_0 \in L^\infty$ which enjoys (2), we are able to prove the existence of a solution to problem (1) and estimates (3)-$(4)_{1,2}$ of Theorem 1 for v. But we are not able to furnish estimate $(4)_3$ for the pressure field, which is substituted by $\pi := \widehat{U}(t)\cdot x + \overline{\pi}$, with $\nabla\overline{\pi} \in L^q(\Omega)$. Here, we have $v_0^R := v_0 - v_{0R}$, with v_{0R} equal to v_0 in a neighbourhood of $\partial\Omega$ and with compact support in $\overline{\Omega}$. The disadvantage is that for all $R>0$ we can construct a different solution ($\widehat{U}$ depends on R).

In this note, the chief aim is to avoid the difficulties arises by the sharpness of the estimates for the solutions of the two-dimensional adjoint problem.

As it will be clear by the arguments that we develop in the sequel, we realize the task blending the ideas contained in papers [16] and [18]. It is appropriate to say: *in medio stat virtus*.

We consider an initial datum $v_0 \in L^\infty(\Omega)$ that enjoys (2). We make the decomposition

$$v_0 = v_0^c + v_{0c}\,,\ \text{with } v_0^c := (1-g)v_0 + b_0\,,\ \text{and } v_{0c} := gv_0 - b_0\,,$$

where g is a smooth cutoff function with $g=1$ in neighborhood of $\partial\Omega$ and $g=0$ for large x, and b_0 is a Bogovskiĭ solution to the problem $\nabla\cdot b_0 = \nabla g\cdot v_0$. The peculiarity of the decomposition is in v_{0c} with compact support in $\overline{\Omega}$ and v_0^c with support in Ω but far from the boundary $\partial\Omega$. Of course, this last property plays an important role in the construction of the solution (see the comments after the the following items). We consider $v = v^c + v_c$ and $\pi := \pi_{v^c} + \pi_{v_c}$. The pairs (v^c,π_{v^c}) and (v_c,π_{v_c}) are solutions to problem (1) with initial datum v_0^c and v_{0c}, respectively.

The solution (v_c,π_{v_c}) is already known from [3,4]. In fact, the compact nature of the support of the initial datum v_{0c} allows us to employ the result of the L^q-setting (cf. [8,9]), thus the estimate $\|v_c(t)\|_\infty \le ct^{-\frac{1}{q}}\|v_{0c}\|_q \le ct^{-\frac{1}{q}}\|v_0\|_\infty$, achieving an extension, for all $t>T_0$, of the one established in [3,4] on $(0,T_0)$ enjoying (5) together other properties in L^∞ (cf. Corollary 4).

The solution (v^c,π_{v^c}) is instead constructed as it follows. We look for

$$v^c := U - \mathfrak{h}\widehat{U} + F + W + \omega \ \text{ and } \ \pi := \pi_\omega\,.$$

In the previous formula:

- U is the solution to the Cauchy problem with initial datum v_0^c extended to zero on $\mathbb{R}^2$,
- $\mathfrak{h}$ is a cutoff function with support depending on t, and, for all $t>0$, $\mathfrak{h}=1$ in a ball including $\partial\Omega$,
- $\widehat{U}(t) := |\partial\Omega|^{-1}\int\limits_{\partial\Omega} U(t,\xi)d\mathcal{H}^1$,
- the trace on $\partial\Omega$ of $-U+\widehat{U}$ has a suitable extension F from $\partial\Omega$ into Ω with compact support,
- W is a solution to the Bogovskiĭ problem $\nabla\cdot W = \nabla\mathfrak{h}\cdot\widehat{U}$ with compact support in Ω,

- finally, ω is the solution to the Stokes problem with homogeneous boundary and initial datum, but with a right-hand side given by

$$-\frac{\partial}{\partial t}\big[-\mathfrak{h}\widehat{U}+F+W\big]+\Delta\big[-\mathfrak{h}\widehat{U}+F+W\big].$$

The chief properties of U are the behaviors in t for the derivatives of U in $L^\infty(\Omega_L)$, where $\Omega_L\supset\partial\Omega$ is bounded, which are not singular in $t=0$, and U with its derivatives evaluated in $L^\infty(\Omega_L)$-norm go to 0, letting $t\to 0$. All this is a consequence of the special initial datum v_0^c, cf. sect. 2.1.

The extension F is obtained by the same technique employed in [16] for $F^{(1)}$. But the new fact is that the boundary value of F is $-U+\widehat{U}$, for which $\|\widehat{U}-U\|_{L^\infty(\partial\Omega)}=\|v_0\|_\infty O(t^{-\frac12})$ holds, that let us to obtain a decay for $\Delta F=\|v_0\|_\infty O(t^{-\frac12})$ (ΔF has the same meaning of G in (5)) (for the construction of F see sect. 2.2). This property from one side allows us to find the right estimate to discuss (6), from another side leads to discuss the additional term $-\mathfrak{h}\widehat{U}$. The role of this term is to realize the homogeneous boundary value of the solution on $(0,T)\times\partial\Omega$. We recall that at $t=0$ we have $\widehat{U}=0$ pointwise on $\{0\}\times\Omega$, so no correction is due in order to obtain the initial value on $\{0\}\times\Omega$.

The function $\mathfrak{h}$ is defined by means of the function $\overline{h}$ with support in the ball $\mathbb{B}_{\frac74}$ and $\nabla\overline{h}$ has compact support in the shell $\{\frac54<|y|<\frac74\}$ (cf. sect. 2.3). The definition of $\mathfrak{h}$ is given by the scale factor $(\overline{R})^{-1}:=(R+\sqrt{t})^{-1}$, that is $\mathfrak{h}:=\overline{h}(\frac{y}{R+\sqrt{t}})$. Hence, $\mathfrak{h}$ has compact support in the ball $\mathbb{B}_{\frac74\overline{R}}$, and $\nabla\mathfrak{h}$ has compact support in the shell $\{x\in\Omega:\frac54(R+\sqrt{t})<|x|<\frac74(R+\sqrt{t})\}$. This property ensures that $\|\nabla\mathfrak{h}\|_q=c(R+\sqrt{t})^{\frac2q-1}$ and $\|\Delta\mathfrak{h}\|_q=c(R+\sqrt{t})^{\frac2q-2}$, that are decaying in t for $q>2$ and $q>1$, respectively. We take advantage this behavior in t in order to discuss the term $\Delta\mathfrak{h}\widehat{U}$. This is a new fact with respect to the behavior of the term G of the n-dimensional case, that arose the difficulty of the estimate (6) in the two-dimensional case. Instead, in the estimates the time derivative of $\mathfrak{h}\widehat{U}$, as a matter of course, go on without difficulties.

However, the term $\mathfrak{h}\widehat{U}$ is not divergence free. Hence, in order to preserve the divergence free of the solution v^c, we introduce the function W. The function W is a solution of the problem $\nabla\cdot W=\nabla\mathfrak{h}\cdot\widehat{U}$ in the shell $\Omega(\overline{R}):=\{(R+\sqrt{t})<|x|<2(R+\sqrt{t})\}\supset\{\frac54(R+\sqrt{t})<|x|<\frac74(R+\sqrt{t})\}\equiv$supp$\nabla\mathfrak{h}$, with homogenous boundary value. The shell $\Omega(\overline{R})$ is variable in t, but, for all $t>0$, there is the homothety with the shell $S:=\{1<|x|<2\}$. Considering a solution to $\nabla\cdot W^S(t)=\nabla\mathfrak{h}^S\cdot\widehat{U}(t)$ in the shell S, with homogeneous boundary value on ∂S, then a solution W in $\Omega(\overline{R})$ is calculated in the following way: $W(t,x):=W^S(t,\frac{x}{R+\sqrt{t}})$. We find the suitable estimates for W and its derivatives considering the ones related to W^S and employing the homothety property of the domain $\Omega(\overline{R})$. It is important to stress that W_t exists, but W_t does not solve the time derivative of the Bogowskiĭ problem. Since no interest there is for this last fact, and since W_t is a "linear" combination of the spatial derivatives of W and of the time derivative of the solution on the fixed shell S, using the homothety property of the domain, we can deduce all the estimates related to W_t (cf. sect. 2.4).

The plan of the paper follows the items detected for the construction of the auxiliary function U, F, $\mathfrak{h}$, W. They are discussed and proved in Sect. 2. In Sect. 3 we give the statement of the results due to K. Abe and Y. Giga, that furnish Theorem 1 for initial data with compact support. In sect. 4 we solve the Stokes problem related to ω. Finally, in Sect. 5 we give the proof of Theorem 1.

Notations.

We assume that the origin $0\in\mathbb{R}^2-\Omega$.

We set $\mathscr{C}_0(\Omega):=\{\varphi\in C_0^\infty(\Omega)$ with $\nabla\cdot\varphi=0\}$. By the symbol $J^p(\Omega)$ we mean the completion of $\mathscr{C}_0(\Omega)$ in $L^p(\Omega)$, $p\in(1,\infty)$, instead, $J^{1,p}(\Omega)$ denotes the completion in $W^{1,p}(\Omega)$, $p\in(1,\infty)$.

The symbol $\mathbb{B}_\rho(x_0)$ denotes a ball in $\mathbb{R}^2$ with center x_0 and radius ρ, in the case of $x_0=0$, we simply write $\mathbb{B}_\rho$.

In the following we consider $R>3\,diam\,(\mathbb{R}^2-\Omega)$.

We set $\Omega_R:=\Omega\cap\mathbb{B}_R$.

For a Lebesgue's measurable set D, the symbol $|D|$ denotes the measure.

By the symbol $L(q,\sigma)(\Omega)$ we mean the G.G. Lorentz spaces and with $\|\cdot\|_{(q,\sigma)}$ its norm. In particular, we consider $L(q,\infty)(\Omega) \equiv L^q_w(\Omega)$, $q \in (1,\infty)$, endowed with the Lorentz norm

$$\|u\|_{(q,\infty)} := \sup_{\substack{|D|<\infty \\ D\subseteq\Omega}} |D|^{\frac{1-q}{q}} \int_D |u| dx \,. \tag{7}$$

For a function $g(t,x)$ and $t \geq 0$, we denote by $\operatorname{supp}_x g(t,x)$ the support in the variable x.

In the following the symbol c denotes a numerical constant whose value is inessential for our aims.

2. Preliminry Results

2.1. Heat Solution

We denote by $H(t,x)$ the heat fundamental solution and we indicate by $H[v_0](t,x)$ the heat solution (transformation) as convolution of $H(t,x)$ and $v_0 \in L^\infty(\mathbb{R}^2)$. It is well known that, for $k,h \in \mathbb{N}\cup\{0\}$,

$$\|D^k_t \nabla^h H[v_0](t)\|_\infty \leq ct^{-k-\frac{h}{2}} \|v_0\|_\infty \,, \text{ for all } t>0 \,, \tag{8}$$

where c is a constant independent of v_0.

For $\mu \geq 0$, $q \in [1,\infty]$ and $\delta > 0$, we set

$$M^c_q(t,x,L,v_0) := t^{-\frac{1}{q}} \|v_0\|_{L^q(B_L(x))} e^{-\frac{\delta^2}{8t}} + \|v_0\|_\infty \frac{t^{\frac{\mu}{2}}}{(L+t^{\frac{1}{2}})^\mu} \,, \tag{9}$$

$(t,x,L,v_0) \in \mathbb{R}^+ \times \mathbb{B}_R \times (R+\delta,\infty) \times \big\{v_0 \in L^\infty(\mathbb{R}^2) \text{ with } \operatorname{supp} v_0 \subset \mathbb{B}^c_{R+\delta}\big\}$.

Lemma 1. *Let $v_0 \in L^\infty(\mathbb{R}^2)$ with $\operatorname{supp} v_0 \subset \mathbb{B}^c_{R+\delta}$ and M^c_q as given in (9). Then, for all $k,h \in \mathbb{N}\cup\{0\}$, there exists a constant $c(\delta,L)$ such that*

$$\begin{aligned} &|D^k_t \nabla^h H[v_0](t,x)| \leq c(L^2+t)^{-k-\frac{h}{2}} M^c_q(t,x,L,v_0) \,, \\ &\|D^k_t \nabla^h U(t)\|_{L^\infty(\mathbb{B}_R)} \leq c\|v_0\|_\infty t^{\frac{\mu}{2}} \,. \end{aligned} \tag{10}$$

Proof. By the definition of M^c_q, we recall that the left hand side of $(10)_1$ has to be considered for $(t,x) \in \mathbb{R}^+ \times \mathbb{B}_R(O)$. For $k,h \in \mathbb{N}\cup\{0\}$ and $\mu > 0$, we have the well known estimate

$$|D^k_t \nabla^h H(z,t)| \leq ct^{\frac{\mu}{2}} (|z|+t^{\frac{1}{2}})^{-2-h-2k-\mu}. \tag{11}$$

If $L \geq R+\delta$ and $x \in \mathbb{B}_R(0)$, we get that $|x-z| \in [R+\delta, 2L]$ implies $|z| \in [\delta, 3L]$, as well as $|x-z| > 2L$ implies $|z| > L$. Hence, applying Hölder's inequality, by virtue of the definition of the support of v_0, we deduce

$$\begin{aligned} |D^k_t \nabla^h H[v_0](t,x)| &\leq \int\limits_{R+\delta<|x-z|<2L} |D^k_t \nabla^h H(t,z)||v_0(x-z)|dz + \int\limits_{|x-z|>2L} |D^k_t \nabla^h H(t,z)||v_0(x-z)|dz \\ &\leq \|D^k_t \nabla^h H(t)\|_{L^{q'}(\delta<|z|<3L)} \|v_0\|_{L^q(\mathbb{B}_{2L}(x))} + \|v_0\|_\infty \int\limits_{|z|>L} \frac{t^{\frac{\mu}{2}}}{(|z|+t^{\frac{1}{2}})^{\mu+2+h+2k}} \, dz \\ &\leq c(\delta,L)(L^2+t)^{-k-\frac{h}{2}} M^c_q(t,x,L,v_0) \,, \end{aligned}$$

Estimate $(10)_2$ is a consequence of the previous estimate and of definition of $M^c_q(t,x,L,v_0)$. □

Lemma 2. *Let $u_0 \in L^\infty(\mathbb{R}^2)$ and let $u := H[u_0](t,x)$. Then, for all $q \in [1,\infty)$ and $R_0 > 0$, we get*

$$\lim_{t\to 0} \|u(t)-u_0\|_{L^q(\mathbb{B}_{R_0})} = 0 \,. \tag{12}$$

Proof. We set

$$u_{2R_0}(t,x) := H[\chi_{2R_0} u_0](t,x) \text{ and } u^{2R_0}(t,x) := H[(1-\chi_{2R_0})u_0](t,x)\,,$$

where χ_{2R_0} denotes the characteristic function of the ball $\mathbb{B}_{2R_0}$. Of course, we have $u = u_{2R_0} + u^{2R_0}$ too. Hence we get

$$\begin{aligned}\|u(t) - u_0\|_{L^q(\mathbb{B}_{R_0})} &\le \|u_{2R_0}(t) - u_0\|_{L^q(\mathbb{B}_{R_0})} + \|u^{2R_0}(t)\|_{L^q(\mathbb{B}_{R_0})}\\ &< \|u_{2R_0}(t) - u_0\|_{L^q(\mathbb{B}_{2R_0})} + \|u^{2R_0}(t)\|_{L^q(\mathbb{B}_{R_0})}\,.\end{aligned}$$

From the L^q-theory we deduce that $\lim_{t\to 0} \|u_{2R_0}(t) - u_0\|_{L^q(\mathbb{B}_{2R_0})} = 0$. Since for all $(x,y) \in \mathbb{B}_{R_0} \times (\mathbb{R}^2 - \mathbb{B}_{2R_0})$ one has $|x-y| \ge |y| - |x| \ge \frac{|y|}{2} \ge R_0$, by virtue of (11), for $\mu > 0$, we deduce

$$\|u^{2R_0}(t)\|_{L^q(\mathbb{B}_{R_0})} \le cR_0^{\frac{2}{q}} \|u^{2R_0}(t)\|_{L^\infty(\mathbb{B}_{R_0})} t^{\frac{\mu}{2}} (R_0 + t^{\frac{1}{2}})^{-\mu}\,.$$

Hence, letting $t \to 0$, we achieve (12). □

2.2. The Extension *F*

We recall some results concerning the boundary value problem in a smooth bounded domain D of the steady Stokes system:

$$\begin{aligned}&\Delta V = \nabla \pi_V\,,\ \nabla\cdot V = 0,\ \text{in } D,\\ &V = a \text{ on } \partial D,\ \int\limits_{\partial D} a\cdot n d\sigma = 0\,.\end{aligned} \tag{13}$$

Lemma 3. *Let $a \in W^{2-\frac{1}{q},q}(\partial D)$, $q > 2$. Then, problem (13) has a unique solution $(V, \pi_V) \in W^{2,q}(D) \cap C^\infty(D) \times W^{1,q}(\overline{D}) \cap C^\infty(D)$, such that*

$$\|V\|_{2,q} \le M\|a\|_{2-\frac{1}{q},q}\,, \tag{14}$$

with M independent of a. In particular, we deduce $(V, \pi_V) \in C^1(\overline{D}) \times C(\overline{D})$.

Proof. The proof of lemma can be found in [10] Ch.IV Lemma 6.1. □

The following is an *a priori* estimate

Lemma 4. *Let $u \in W^{m+2,q}(\Omega) \cap J^{1,q}(\Omega)$, for some $m \in \mathbb{N}_0$. Then there exists a field π_u such that*

$$\|D^{m+2}u\|_q + \|D^m \nabla\pi\|_q \le c\big[\|D^m P\Delta u\|_q + \|u\|_{L^q(\Omega_R)}\big]\,, \tag{15}$$

where c is a constant independent of u.

Proof. This result is contained in [10,22]. Actually, in our hypotheses, for u we can consider the Helmholtz decomposition of Δu, hence, formally u is a solution to the boundary value problem

$$\Delta u - \nabla\pi_u = P\Delta u\,,\ \nabla\cdot u = 0\,,\ \text{in } \Omega\,,\ u = 0 \text{ on } \partial\Omega\,.$$

Then the estimates and regularity follow from the result in [10,22] for solution to the Stokes problem in exterior domains. □

We recall some results concerning the Bogovskiĭ problem. Let E be a smooth bounded domain and

$$\nabla\cdot v = g\,,\ \text{in } E, v = 0 \text{ on } \partial E, \tag{16}$$

with the compatibility condition $\int\limits_E g dx = 0$.

Lemma 5. *If $g \in C_0^\infty(E)$, then there exists at least a solution $v \in C_0^\infty(E)$ to problem (16) such that, for $m \in \mathbb{N}$ and $r \in (1,\infty)$,*

$$\|v\|_{m,r} \le c\|g\|_{m-1,r}\,. \tag{17}$$

For the proof of the Lemma we refer to [10].

□

It is known that one solves problem (16) by considering the domain E as a union of domains C_k, $k = 1,\dots,N$, star-shaped with respect to the balls $\mathbb{B}(k)$ of a fixed radius; moreover, using a smooth partition of unity, say $\sum\limits_{k=1}^{N}\psi_k(x) = 1$, with $\mathrm{supp}\psi_k \subset C_k$. Then, a vector field satisfying (16) can be written in the form

$$v(x) = \mathbf{B}[g] = \sum_{k=1}^{N} v_k(x)\,, \tag{18}$$

where

$$v_k(x) = \mathbf{B}^k[\psi_k g] = \int\limits_{C_k} \mathbf{B}^k(x-y,y)\psi_k(y)g(y)dy\,,$$

$$\mathbf{B}^k(z,y) = \frac{z}{|z|^n}\int\limits_{|z|}^{\infty} q^k(y+\xi\frac{z}{|z|})\xi^{n-1}d\xi\,,$$

$$q^k(x) \in C_0^\infty(\mathbb{B}(k)) \text{ and } \int\limits_{\mathbb{B}(k)} q_k(y)dy = 1\,.$$

We also recall that, for each $k = 1,\dots,N$, $\mathbf{B}^k$ is an operator with weakly singular kernel. Actually, $\mathbf{B}^k_j[\,\cdot\,]$ is the integral operator with the kernel

$$\mathbf{B}^k_j(x-y,y) = \frac{x-y}{|x-y|^n}\int\limits_{|x-y|}^{\infty} \frac{\partial}{\partial y_j} q^k(y+\xi\frac{x-y}{|x-y|})\xi^{n-1}d\xi\,, \tag{19}$$

and $\frac{\partial}{\partial x_j}\mathbf{B}^k$ is an operator with singular kernel of Calderon-Zigmund kind. □

Lemma 6. *Let* $A(x) \in W^{2-\frac{1}{q},q}(\partial\Omega)$*,* $q > 2$*, with* $\int\limits_{\partial\Omega} A(x)\cdot n d\sigma = 0$*. Then, the function* A *admits an extension* F *into* Ω*, such that* $F \in W^{2,q}(\Omega)\cap C^\infty(\Omega)$*,* $F(x)$ *has compact support in* $\overline{\Omega}_R$ *and is divergence free in* Ω*, with*

$$\|F\|_{2,q} \le c(R)\|A\|_{2-\frac{1}{q},q}, \tag{20}$$

with $c(R)$ *independent of* A*. In particular, we get* $F(x) \in C^1(\overline{\Omega})$*.*

Proof. Let us consider the boundary value problem (13) for $D \equiv \Omega_R$, with boundary data $a = A$ on $\partial\Omega$ and $a = 0$ on $|x| = R$. By virtue of Lemma 3 there exists a unique solution $(V,\pi_V) \in W^{2,q}(\Omega_R)$, such that

$$\|V\|_{2,q} \le c(R)\|A\|_{2-\frac{1}{q},q}. \tag{21}$$

Moreover, we consider a Bogovskiĭ's solution $\overline{V}$ to the equation (16) assuming $E \equiv \{x : \frac{R}{3} < |x| < \frac{2}{3}R\}$, $g = -\nabla h_R \cdot V$ in E and $\overline{V} = 0$ on ∂E, where h_R is a smooth cut-off function with $h_R = 1$ on $\Omega_{R/3}$ and $h_R = 0$ on $\Omega_R - \Omega_{2R/3}$. By virtue of the estimate of Lemma 5, we get

$$\|\overline{V}\|_{2,q} \le c\|A\|_{2-\frac{1}{q},q}\,. \tag{22}$$

Setting $F = Vh_R + \overline{V}$ we have proved estimate (20). The regularity in Ω is a consequence of the ones doable for V and $\overline{V}$ (see [10]). □

Corollary 1. *Let $A(t,x)$ be a one parameter family of functions with $D^k_tA(t,x) \in W^{2-\frac{1}{q},q}(\partial\Omega)$ and $\int\limits_{\partial\Omega} A(t,x)\cdot n d\sigma = 0$, for all $t \geq 0$. Then, for $t>0$, denoted by $F(t,x)$ the extension obtained in Lemma 6, there exist D^k_tF with*

$$\|D^k_tF(t)\|_{2,q} \leq c\|D^k_tA(t)\|_{2-\frac{1}{q},q}\,, t>0\,, \tag{23}$$

with c independent of t.

Proof. For all $t \geq 0$, we consider the extension $F = Vh_R + \overline{V}$ given in Lemma 6. Hence, recalling the definition of V and $\overline{V}$, there exists $D^k_tF = D^k_tVh_R + D^k_t\overline{V}$. Hence, via estimate (21) for D^k_tV and via representation formula (18) for $\overline{V}_t$, in our hypotheses estimate (23) follows by the same arguments developed for the estimates (20). □

We set

$$U := H[v_0](t,x)\,, \ \text{ and } \ \widehat{U} := |\partial\Omega|^{-1}\int\limits_{\partial\Omega} U(t,\xi)d\mathcal{H}^1\,, \ \text{for all } t>0\,, \tag{24}$$

where U is the solution to the heat equation furnished in Sect. 2.1 and corresponding to v_0 with supp $v_0 \subset \Omega - \mathbb{B}_{R+\delta}$ and enjoying (2).

Lemma 7. *Let $A = -U + \widehat{U}$ in Corollary 1 with U given in (24). For $k \in \mathbb{N}\cup\{0\}$, we get*

$$\|D^k_tF(t)\|_{2,q} \leq c(L^2+t)^{-\frac{1}{2}-k}\|v_0\|_\infty\,, \ \textit{for all } t>0\,, \tag{25}$$

with c independent of v_0.

Proof. By virtue of Corollary 1, estimate (25) easily follows achieving the estimate $||D^k_tA(t)||_{2-\frac{1}{q},q} \leq c(L^2+t)^{-\frac{1}{2}-k}||v_0||_\infty$, for all $t>0$. Then, estimating $\|D^k_tA(t)\|_{2-\frac{1}{q},q}$, the task is to justify the exponent $-\frac{1}{2}$ on the right hand side of estimate (25). The assumption Ω smooth exterior domain leads to assert $\partial\Omega \equiv \bigcup\limits_{m=1}^{p}\partial\Omega_m$. For any continuous function g, the mean value[2] is

$$|\partial\Omega|^{-1}\sum_{h=1}^{p}\int\limits_{\partial\Omega_h} g(\xi)d\mathcal{H}^1 = \Big[\sum_{m=1}^{p}|\partial\Omega_m|\Big]^{-1}\sum_{h=1}^{p}\int\limits_{\partial\Omega_h} g(\xi)d\mathcal{H}^1 = \sum_{h=1}^{p}\frac{|\partial\Omega_h|}{|\partial\Omega|}g(\overline{\xi}_h)\,. \tag{26}$$

In order to estimate $\|D^k_tA_i(t)\|_{L^q(\partial\Omega)} = \|D^k_t(-U_i(t,\xi)+\widehat{U}_i(t))\|_{L^q(\partial\Omega)}$, $i=1,2$, we initially remark that, by virtue of (26), for $i=1,2$ and for all $k \in \mathbb{N}\cup\{0\}$ there exist $\overline{\xi}_h$, $h=1,\dots,p$, such that

$$-D^k_tU_i(t,\xi) + D^k_t\widehat{U}_i(t) = |\partial\Omega|^{-1}\sum_{h=1}^{p}\Big[-D^k_tU_i(t,\xi)+D^k_tU_i(t,\overline{\xi}_h)\Big]|\partial\Omega_h|\,, \tag{27}$$

where $D^k_tU_i(t,\overline{\xi}_h)$ is mean value of the integral on $\partial\Omega_h$. Hence, by virtue of Lagrange's theorem and assumptions on U, from (27) we get

$$\begin{aligned}|D^k_tU_i(t,\xi) - D^k_t\widehat{U}_i(t)| &= |\partial\Omega|^{-1}\Big|\sum_{h=1}^{p}\nabla D^k_tU_i(t,\theta_h)\cdot(\overline{\xi}_h-\xi)\Big| \\ &\leq c\|\nabla D^k_tU(t)\|_\infty \leq c(L^2+t)^{-\frac{1}{2}-k}\|v_0\|_\infty\,, \ \text{for all } \xi\in\partial\Omega\,, \ \text{and } t>0\,,\end{aligned}$$

where in the last step we take (10) into account. This justify the estimate for the $L^q(\partial\Omega)$ norm of D^k_tA. Instead, for the seminorm we have $< \nabla D^k_tA >_{1-\frac{1}{q},q} = < \nabla(D^k_tU - D^k_t\hat{U}) >_{1-\frac{1}{q},q} = < \nabla D^k_tU >_{1-\frac{1}{q},q}$. Hence, considering again estimate (10), *a fortiori* there is for the exponent the increment $-\frac{1}{2}$. □

[2] We recall that in our notations we denoted by $|D|$ the measure of any Lebesgue's measurable set.

2.3. The Function $\mathfrak{h}$

Let $\overline{h}(\sigma)$ be a smooth cut off function such that $\overline{h}(\sigma)=1$ for $\sigma\in[0,\frac54]$, $\overline{h}(\sigma)=0$ for $\sigma\in[\frac74,2]$ and $\overline{h}(\sigma)\in[0,1]$ for $\sigma\in[\frac54,\frac74]$. For all $\tau>0$ and $\rho>0$, we define $h(\tau,\rho):=\overline{h}(\frac{\rho}{R+\sqrt{\tau}})$. One easily verifies the following properties:

$$h(\tau,\rho)\begin{cases}=1\,, & \text{for } \rho\le\frac54(R+\sqrt{\tau})\,,\\ \in[0,1]\,, & \text{for } \rho\in\big[\frac54(R+\sqrt{\tau}),\frac74(R+\sqrt{\tau})\big]\,,\\ =0\,, & \text{for } \rho\ge\frac74(R+\sqrt{\tau})\,.\end{cases}$$

We set $h_\tau:=\frac{\partial}{\partial\tau}h(\tau,\rho)$ and $h_\rho(\tau,\rho):=\frac{\partial}{\partial\rho}h(\tau,\rho)$. A computation gives

$$\begin{aligned}&h_\tau(\tau,\rho)=-\frac12\frac{\rho}{(R+\sqrt{\tau})^2}\frac{1}{\sqrt{\tau}}\overline{h}'(\sigma),\ \text{for } \tau>0 \text{ and } \rho\in\big[\frac{R+\sqrt{\tau}}{2},\frac{2(R+\sqrt{\tau})}{3}\big]\,,\\ &h_\tau(\tau,\rho)=0\,,\ \text{for } \tau>0 \text{ and } \rho\in\mathbb{R}_+-\big[\frac{R+\sqrt{\tau}}{2},\frac{2(R+\sqrt{\tau})}{3}\big]\,,\\ &h_\rho(\tau,\rho)=\frac{1}{R+\sqrt{\tau}}\overline{h}'(\sigma)\,,\ \text{for } \tau>0 \text{ and } \rho\in\big[\frac{R+\sqrt{\tau}}{2},\frac{2(R+\sqrt{\tau})}{3}\big]\,,\\ &h_\rho(\tau,\rho)=0\,,\ \text{for } \tau>0 \text{ and } \rho\in\mathbb{R}_+-\big[\frac{R+\sqrt{\tau}}{2},\frac{2(R+\sqrt{\tau})}{3}\big]\,.\end{aligned}\tag{28}$$

For $\rho:=|x|$, we set

$$\mathfrak{h}(t,x):=h(t,\rho)\,.\tag{29}$$

Recalling that $R>3diam\,\Omega^c$, by our position we have $\mathfrak{h}(t,x)=1$ in $\mathbb{B}_{\frac{R}{2}}\cap\Omega$ for all $t>0$. We have

$$|\nabla\mathfrak{h}(t,x)|\le\ c(R+\sqrt{t})^{-1}\,.\tag{30}$$

and via $(28)_1$

$$|\mathfrak{h}_t(t,x)|\le ct^{-\frac12}(R+\sqrt{t})^{-1}\,.\tag{31}$$

Finally, via (28) we get

$$\begin{aligned}|\nabla\mathfrak{h}_t(t,x)|&\le|D^2_{t,\rho}h(t,\rho)|\le c\frac{1}{\sqrt{t}}\frac{1}{(R+\sqrt{t})^2},\\ |\mathfrak{h}_{tt}(t,x)|&\le c\Big|\frac{d}{dt}\Big[\frac{\rho}{(R+\sqrt{t})^2}\frac{1}{\sqrt{t}}\Big]\Big|+c\Big[\frac{\rho}{(R+\sqrt{t})^2}\frac{1}{\sqrt{t}}\Big]^2|\overline{h}''(\sigma)|,\\ |\Delta\mathfrak{h}(t,x)|&=\big|\overline{h}''+\frac{1}{\rho}\overline{h}'\big|\le c(R+\sqrt{t})^{-2}.\end{aligned}\tag{32}$$

2.4. A Special Bogovskiĭ Problem

For all $t>0$, we consider the Bogovskiĭ problem

$$\nabla\cdot W=\nabla\mathfrak{h}\cdot\widehat{U}(t)\,,\ \text{in } \Omega(\overline{R})\,,\quad W=0 \text{ on } \partial\Omega(\overline{R})\equiv\{|x|=\overline{R}\}\cup\{|x|=2\overline{R}\}\,,\tag{33}$$

where we set $\overline{R}:=R+\sqrt{t}$ and $\Omega(\overline{R}):=\Omega\cap\{R+\sqrt{t}<|x|<2(R+\sqrt{t})\}$. Of course, for all $\overline{R}>0$ the domain $\Omega(\overline{R})$ is homothetic to the shell $S:=1<|z|<2$. For problem (33), since the compatibility condition holds, Lemma 5 holds too. However, here we are interested to state the result employing the following approach, that is more suitable for the special domain $\Omega(\overline{R})$.

Lemma 8. *There exist a constant c and a smooth solution $W(t,x)$ to problem (33) with compact support in $\Omega(\overline{R})$ and such that, for all $t>0$,*

$$\begin{aligned}q\in(1,\infty),\ \|\nabla W(t)\|_q&\le c\|\nabla\mathfrak{h}\cdot\widehat{U}(t)\|_q\,,\\ \|\nabla^2W(t)\|_q&\le c\|\nabla^2\mathfrak{h}\cdot\widehat{U}(t)\|_q\,,\end{aligned}\tag{34}$$

and with

$$
\begin{aligned}
(R+\sqrt{t})\|W_t(t)\|_q+\|\nabla W_t(t)\|_q &\le c\Big[\frac{\|\nabla\mathfrak{h}\cdot\widehat{U}(t)\|_q}{\sqrt{t}(R+\sqrt{t})}+\frac{\|\nabla\nabla\mathfrak{h}\cdot\widehat{U}(t)\|_q}{\sqrt{t}}+\|\nabla\mathfrak{h}\cdot\widehat{U}_t(t)\|_q\Big],\\
\|W_{tt}(t)\|_{(2,\infty)} &\le \frac{c}{t}\Big[\|\nabla\mathfrak{h}\cdot\widehat{U}(t)\|_2\big(\frac{1}{\sqrt{t}}+\frac{1}{R+\sqrt{t}}\big)+\|\nabla\nabla\mathfrak{h}\cdot\widehat{U}(t)\|_2\\
&\quad+\sqrt{t}\|\nabla\mathfrak{h}\cdot\widehat{U}_t(t)\|_2\Big]+c\|\nabla\mathfrak{h}\cdot\widehat{U}_{tt}(t)\|_1.
\end{aligned}
\tag{35}
$$

Proof. For all $z\in S$, we set $\mathfrak{h}^S(z):=\overline{h}(|z|)$. We consider the following problem

$$
\nabla\cdot W^S(t,z)=\nabla\mathfrak{h}^S(z)\cdot\widehat{U}(t)\,,\ \text{in } S\,,\quad W^S=0 \text{ on } \{|z|=1\}\cup\{|z|=2\}\,. \tag{36}
$$

Taking into account that $\mathfrak{h}^S(z)=1$ on $|z|=1$ and $\mathfrak{h}^S(z)=0$ on $|z|=2$, since $\widehat{U}$ is independent of z, for the Bogovskiĭ problem (36) the compatibility condition holds, and so, by virtue of Lemma 5, we establish the existence of a solution $W^S(t,z)$ with compact support in the shell S. Easily one verifies that $W(t,x):=W^S(t,\frac{x}{R+\sqrt{t}})$ is a solution to problem (33) with compact support in $\Omega(\overline{R})$. Being $\Omega(\overline{R})$ homothetic with the shell S, via estimate (17) for $m=1$ and via the following trivial chain, we deduce $(34)_1$:

$$
\begin{aligned}
\int\limits_{\Omega(\overline{R})}|\nabla W(t,x)|^q dx &= \overline{R}^{2-q}\int\limits_S|\nabla_z W^S(t,z)|^q dz\\
&\le c\overline{R}^{2-q}\int\limits_S|\nabla_z\mathfrak{h}^S(z)\cdot\widehat{U}(t)|^q dz = c\int\limits_{\Omega(\overline{R})}|\nabla\mathfrak{h}(\frac{x}{R+\sqrt{t}})\cdot\widehat{U}(t)|^q dx.
\end{aligned}
\tag{37}
$$

Analogously, via (17) for $m=2$, we get

$$
\|\nabla^2 W(t)\|_q=\overline{R}^{\frac{2}{q}-2}\|\nabla_z^2 W^S(t)\|_{L^q(S)}\le c\overline{R}^{\frac{2}{q}-2}\|\nabla_z^2\mathfrak{h}^S\cdot\widehat{U}(t)\|_{L^q(S)}=c\|\nabla^2\mathfrak{h}\cdot\widehat{U}(t)\|_q\,.
$$

Deriving W with respect to t, we get

$$
\begin{aligned}
W_t(t,x) &= -\frac{1}{2\sqrt{t}}\frac{x}{(R+\sqrt{t})^2}\cdot\nabla_\xi W^S(t,\xi)+\frac{\partial}{\partial t}W^S(t,\xi)\\
&= -\frac{1}{2\sqrt{t}}\frac{1}{R+\sqrt{t}}\,\xi\cdot\nabla_\xi W^S(t,\xi)+W_t^S(t,\xi)\,,
\end{aligned}
\tag{38}
$$

we point out that the last term has to be considered as the "Eulerian derivative" which arises via formula (18) written for solution $W^S(t,z)$ where, thanks to the static position, we transport the time derivative on $\widehat{U}(t)$. Now, let us consider ∇W_t. From (38) it follows that

$$
\nabla W_t(t,x)=-\frac{1}{2\sqrt{t}}\frac{1}{(R+\sqrt{t})^2}\nabla W_\xi^S(t,\xi)-\frac{1}{2\sqrt{t}}\frac{x}{(R+\sqrt{t})^3}\cdot\nabla_\xi\nabla_\xi W^S(t,\xi)+\frac{1}{R+\sqrt{t}}\nabla_\xi W_t^S(t,\xi)\,.
$$

Since W and ∇W have compact support in $\Omega(\overline{R})$ and $\big|\frac{x}{R+\sqrt{t}}\big|=|\xi|\le 2$, via (17), we get

$$
\begin{aligned}
\|\nabla W_t(t)\|_q=\|\nabla W_t(t)\|_{L^q(\Omega(\overline{R}))} &\le c\Big[\frac{\|\nabla_\xi W^S(t)\|_{L^q(S)}}{\sqrt{t}(R+\sqrt{t})^{2(1-\frac{1}{q})}}+\frac{\|\nabla_\xi\nabla_\xi W^S(t)\|_{L^q(S)}}{2\sqrt{t}(R+\sqrt{t})^{2(1-\frac{1}{q})}}+\frac{\|\nabla_\xi W_t^S(t)\|_{L^q(S)}}{(R+\sqrt{t})^{1-\frac{2}{q}}}\Big]\\
&\le c\Big[\frac{\|\nabla_\xi\mathfrak{h}^S\cdot\widehat{U}(t)\|_{L^q(S)}}{\sqrt{t}(R+\sqrt{t})^{2(1-\frac{1}{q})}}+\frac{\|\nabla_\xi\nabla_\xi\mathfrak{h}^S\cdot\widehat{U}(t)\|_{L^q(S)}}{\sqrt{t}(R+\sqrt{t})^{2(1-\frac{1}{q})}}+\frac{\|\nabla_\xi\mathfrak{h}^S\cdot\widehat{U}_t(t)\|_{L^q(S)}}{(R+\sqrt{t})^{1-\frac{2}{q}}}\Big]\\
&= c\Big[\frac{\|\nabla\mathfrak{h}\cdot\widehat{U}(t)\|_q}{\sqrt{t}(R+\sqrt{t})}+\frac{\|\nabla\nabla\mathfrak{h}\cdot\widehat{U}(t)\|_q}{\sqrt{t}}+\|\nabla\mathfrak{h}\cdot\widehat{U}_t(t)\|_q\Big],
\end{aligned}
$$

where, taking the homothety between the sets $\Omega(\overline{R})$ and S into account, we argued as made in estimate (37). Hence, we arrive at $(35)_1$ for ∇W, and thanks to the Poincaré inequality we complete the proof of

$(35)_2$. Finally, from (38) it follows that

$$W_{tt}(t,x) = \Big(\frac{1}{4t^{\frac{3}{2}}(R+\sqrt{t})^2} + \frac{1}{4t(R+\sqrt{t})^3}\Big)x\cdot\nabla_\xi W^S(t,\xi) + \frac{1}{4t(R+\sqrt{t})^4}x\otimes x\cdot\nabla_\xi\nabla_\xi W^S(t,\xi) \\ -\frac{1}{\sqrt{t}(R+\sqrt{t})^2}x\cdot\nabla_\xi W^S_t(t,\xi) + W^S_{tt}(t,\xi)\,,$$

where again we stress that the last term is meant as the "Eulerian derivative". We set

$$A(t,\xi) := \Big(\frac{1}{4t^{\frac{3}{2}}(R+\sqrt{t})^2} + \frac{1}{4t(R+\sqrt{t})^3}\Big)x\cdot\nabla_\xi W^S(t,\xi) + \frac{1}{4t(R+\sqrt{t})^4}x\otimes x\cdot\nabla_\xi\nabla_\xi W^S(t,\xi) \\ -\frac{1}{\sqrt{t}(R+\sqrt{t})^2}x\cdot\nabla_\xi W^S_t(t,\xi)\,,$$

and

$$K(t,\xi) := W^S_{tt}(t,\xi)\,.$$

Hence, recalling that $|\xi| = |\frac{x}{R+\sqrt{t}}| \leq 2$, employing the homothetic transformation for the coordinates, via (17), we get

$$\|A(t)\|_2 \leq c\Big[\frac{\|\nabla_\xi W^S(t)\|_{L^2(S)}}{t}\Big(\frac{1}{\sqrt{t}} + \frac{1}{R+\sqrt{t}}\Big) + \frac{\|\nabla_\xi\nabla_\xi W^S(t)\|_{L^2(S)}}{t(R+\sqrt{t})} + \frac{\|\nabla_\xi W^S_t(t)\|_{L^2(S)}}{\sqrt{t}}\Big]\,,$$

then, first via (17), and subsequently applying the homothetic change of variables, we arrive at

$$\|A(t)\|_2 \leq c\Big[\frac{\|\nabla_\xi \mathfrak{h}^S\cdot\widehat{U}(t)\|_{L^2(S)}}{t}\Big(\frac{1}{\sqrt{t}} + \frac{1}{R+\sqrt{t}}\Big) + \frac{\|\nabla_\xi\nabla_\xi \mathfrak{h}^S\cdot\widehat{U}(t)\|_{L^2(S)}}{t(R+\sqrt{t})} + \frac{\|\nabla_\xi \mathfrak{h}^S\cdot\widehat{U}_t(t)\|_{L^2(S)}}{\sqrt{t}}\Big] \\ = c\Big[\frac{\|\nabla\mathfrak{h}\cdot\widehat{U}(t)\|_2}{t}\Big(\frac{1}{\sqrt{t}} + \frac{1}{R+\sqrt{t}}\Big) + \frac{\|\nabla\nabla\mathfrak{h}\cdot\widehat{U}(t)\|_2}{t} + \frac{\|\nabla\mathfrak{h}\cdot\widehat{U}_t(t)\|_2}{\sqrt{t}}\Big].$$

Since W_{tt} has compact support and the sets $\Omega(\overline{R})$ and S are homothetic, we get

$$\|K(t)\|_{(2,\infty)} = \|K(t)\|_{L^2_w(\Omega(\overline{R}))} = (R+\sqrt{t})\|W^S_{tt}\|_{L^2_w(S)} \leq c(R+\sqrt{t})\|\nabla_\xi\mathfrak{h}^S\cdot\widehat{U}_{tt}(t)\|_{L^1(S)} \\ = c\|\nabla\mathfrak{h}\cdot\widehat{U}_{tt}(t)\|_1,$$

where again we consider the "Eulerian derivative", that is, via formula (18) written for solution $W^S(t,z)$, we transported the time derivative on $\widehat{U}(t)$, and then, for the estimate of the Bogovskiĭ solution, we toke into account that the kernel in (18) is weakly singular with exponent $\alpha = 1$. This completes the proof of (35). □

Lemma 9. *Let W be a solution to the Bogovskiĭ problem (33) stated in Lemma 8 and $U \equiv H[v_0]$ as in Lemma 1. Then we get*

$$(R+\sqrt{t})^{2-\frac{2}{q}}\|\nabla^2 W(t)\|_q + (R+\sqrt{t})^{1-\frac{2}{q}}\|\nabla W(t)\|_q + \|W(t)\|_\infty \leq c\|v_0\|_\infty\,, \tag{39}$$

and

$$\begin{aligned}\|W_t(t)\|_q &\leq (R+\sqrt{t})k_1(t)\|v_0\|_\infty\,,\\ \|\nabla W_t(t)\|_q &\leq k_1(t)\|v_0\|_\infty\,,\\ \|W_{tt}(t)\|_{L^2_w} &\leq k_3(t)\|v_0\|_\infty\,,\end{aligned} \tag{40}$$

for all $t > 0$, where we set

$$k_1(t) := c\Big[t^{-\frac{1}{2}}(R+\sqrt{t})^{\frac{2}{q}-2} + (L^2+t)^{-1}(R+\sqrt{t})^{\frac{2}{q}-1}\Big],$$
$$k_3(t) := c\Big[t^{-1}\big(t^{-\frac{1}{2}} + (R+\sqrt{t})^{-1} + \sqrt{t}(L^2+t)^{-1}\big) + (R+\sqrt{t})(L^2+t)^{-2}\Big],$$

and where c is a constant independent of v_0 and t.

Proof. We have

$$|\widehat{U}(t)| \leq \|v_0\|_\infty\,,\ |\widehat{U}_t(t)| \leq c(L^2+t)^{-1}\|v_0\|_\infty\,,\ |\widehat{U}_{tt}(t)| \leq c(L^2+t)^{-2}\|v_0\|_\infty\,,\ \text{for all } t>0\,, \tag{41}$$

where all the estimates are consequence of (10). Moreover, as a consequence of $(10)_2$, for all $\mu>0$, we have

$$|\widehat{U}(t)| + |\widehat{U}_t(t)| \leq ct^{\frac{\mu}{2}}\|v_0\|_\infty\,. \tag{42}$$

For $q \geq 1$, we get

$$\|\nabla\mathfrak{h}\|_{L^q(\Omega(\overline{R}))} = c(R+\sqrt{t})^{\frac{2}{q}-1}\,,\ \|\nabla\nabla\mathfrak{h}\|_{L^q(\Omega(\overline{R}))} = c(R+\sqrt{t})^{\frac{2}{q}-2}\,,\ \text{for all } t>0\,. \tag{43}$$

Hence, as a matter of course, the right hand side of $(34)_1$ is bounded by $c(R+\sqrt{t})^{\frac{2}{q}-1}\|v_0\|_\infty$ and the one of $(34)_2$ is bounded by $c(R+\sqrt{t})^{\frac{2}{q}-2}\|v_0\|_\infty$. Hence we get (39) for ∇W and $\nabla\nabla W$.

Employing Gagliardo-Nirenberg inequality, for $q>2$, we get $\|W(t)\|_\infty \leq c\|\nabla W(t)\|_q^b\|W(t)\|_q^{1-b}$, with exponent $b := \frac{2}{q}$. Since estimate (39) is achieved for $\|\nabla W(t)\|_q$, via the Poincaré inequality, we obtain $\|W(t)\|_\infty \leq (R+\sqrt{t})^{1-b}\|\nabla W(t)\|_q \leq c\|v_0\|_\infty$.

Moreover, by means of the estimates (43) for ∇h and $\nabla\nabla h$, via (41), the right hand side of $(35)_1$ is bounded by $c\Big[t^{-\frac{1}{2}}(R+\sqrt{t})^{\frac{2}{q}-2} + (L^2+t)^{-1}(R+\sqrt{t})^{\frac{2}{q}-1}\Big]\|v_0\|_\infty$, which furnishes $(40)_2$, and then, again via $(35)_1$ we arrive at $(40)_1$.

Estimate given in (43) of $\nabla\mathfrak{h}$ in L^2-norm is a constant c, instead the one for $\nabla\nabla\mathfrak{h}$ in L^2-norm is $c(R+\sqrt{t})^{-1}$. Hence, for the terms involving the L^2-norm on the right hand side of $(35)_2$ we get $ct^{-1}(t^{-\frac{1}{2}} + (R+\sqrt{t})^{-1} + \sqrt{t}(L^2+t)^{-1})\|v_0\|_\infty$. For the estimate of the term involving the L^2_w-norm on the right hand side of $(35)_2$, being $\|\nabla\mathfrak{h}\|_1 = (R+\sqrt{t})$, we get the estimate $c(L^2+t)^{-2}(R+\sqrt{t})\|v_0\|_\infty$. The sum furnishes $(40)_3$. □

Lemma 10. *Let U be the solution of Lemma 1. Then, for $\mu>0$, for all $q\in(1,\infty)$ and $k=0,1,2$, the following holds*

$$\|D_t^kF(t)\|_{W^{2,q}(\Omega)} + \|D_t^k(\mathfrak{h}\widehat{U}(t))\|_\infty + \|W(t)\|_\infty + \|D_tW(t)\|_q + \|D_t^2W(t)\|_{(2,\infty)} \leq c\|v_0\|_\infty t^{\frac{\mu}{2}-\frac{3}{2}}\,, \tag{44}$$

and

$$\|\Delta F(t)\|_q + \|\Delta(\mathfrak{h}\widehat{U}(t))\|_\infty + \|\Delta W(t)\|_q \leq c\|v_0\|_\infty t^{\frac{\mu}{2}}\,, \tag{45}$$

both the estimates evaluated for all $t\in(0,1)$.

Proof. Since U verifies $(10)_2$ and $\widehat{U}$ is defined by (24), then, for all $\mu>0$, we have

$$\|D_t^k\widehat{U}(t)\|_\infty + \|D_t^k(U-\widehat{U}(t))\|_{W^{2-\frac{1}{q},q}(\partial\Omega)} \leq c\|v_0\|_\infty t^{\frac{\mu}{2}}\,,\ t\in(0,1)\,. \tag{46}$$

Function F is the extension furnished by Corollary 1 with $A := -U+\widehat{U}$. As a consequence of (23) and (46) we have (44) and (45) for F.

Function $\mathfrak{h}$ is the cutoff function defined in (29), hence estimates (31) and $(32)_{2,3}$ hold. Then, by virtue of (46) and $R>0$, we deduce (44) for $D_t^k(\mathfrak{h}\widehat{U})$ and (45) for $\Delta\mathfrak{h}\widehat{U}$.

Being W a solution to problem (33), for W we consider (34) and for D_t^kW we consider estimates $(35)_{1,2}$. Since we estimate in neighborhood of $t=0$, the right hand sides of (34) and $(35)_{1,2}$ admit a bound of the kind $t^{-\frac{3}{2}}\|\widehat{U}(t)\|_\infty$, for $t\in(0,1)$. So that, applying (46) to the right hand side, we arrive at (44). Analogously, estimates $(34)_2$ and (46) lead (45) for ΔW. The lemma is proved. □

Of course, since (44)-(45) are stated in a neighborhood of $t=0$ and since $\mu>0$ can be chosen as we want, estimates (44) and (45) are not given in sharpness way, but they are given functional to our aims.

2.5. Some Integral Estimates

The symbols $F, \mathfrak{h}$, and W have the same meaning given in previous section. We recall that we assumed $R > 3diam(\mathbb{R}^2 - \Omega)$ and $L > R$, as well in Lemma 1 we assumed $\text{supp}\, v_0 \subset \mathbb{B}^c_{R+\delta}$, $\delta > 0$.

In this section, for all $\eta > 0$, the function $\varphi \in C(\eta, T; J^{1,q}(\Omega)$, $q \in (1, 2]$, is such that

$$\begin{aligned} & r\in(1,2],\ r\geq q\in(1,2]\,,\ k=0,1,\ \|\nabla D^k_t\varphi(t)\|_r\leq c\begin{cases} t^{-k+\frac{1}{r}-\frac{1}{2}-\frac{1}{q}}\|\varphi_0\|_q, \\ t^{-k+\frac{1}{r}-\frac{3}{2}}\|\varphi_0\|_1, \end{cases} \\ & r>q,\ s\in[1,\infty],\ \text{or}\ r\geq q,\ \text{and}\ s\in[q,\infty]\,,\ k=0,1,\ \|D^k_t\varphi(t)\|_{r,s}\leq c\begin{cases} t^{-k+\frac{1}{r}-\frac{1}{q}}\|\varphi_0\|_q, \\ t^{-k+\frac{1}{r}-1}\|\varphi_0\|_1, \end{cases} \end{aligned} \tag{47}$$

for all $t > 0$, for $r = \infty$ we just consider L^∞, and c is a constant independent of φ.

We set

$$\mathscr{I}_1(t) := -\int_0^t (F_\tau(\tau), \varphi(t-\tau))d\tau + \int_0^t (\mathfrak{h}_\tau(\tau)\widehat{U}(\tau) - \int_0^t (W_\tau, \varphi(t-\tau))d\tau, \varphi(t-\tau))d\tau\,,$$

$$\mathscr{I}_2(t) := \int_0^t (\Delta F(\tau), \varphi(t-\tau))d\tau - \int_0^t (\Delta\mathfrak{h}(\tau)\widehat{U}(\tau), \varphi(t-\tau))d\tau + \int_0^t (\Delta W(\tau), \varphi(t-\tau))d\tau\,.$$

Lemma 11. *For all $q \in [1, 2)$, there exists a constant $C := C(R)$ such that*

$$|\mathscr{I}_1(t)| \leq C t^{1-\frac{1}{q}} \|v_0\|_\infty \|\varphi_0\|_q\,,\ \textit{for all}\ t > 0\,. \tag{48}$$

Proof. We separately look for the estimate of each integral term, called $I_i(t), i = 1, 2, 3$, of the sum. Applying Hölder's inequality, we get

$$\begin{aligned} |I_1(t)| &\leq \int_0^t ||F_\tau(\tau)||_2||\varphi(t-\tau)||_2 d\tau \leq c||v_0||_\infty||\varphi_0||_q \int_0^t (L^2+\tau)^{-\frac{3}{2}}(t-\tau)^{\frac{1}{2}-\frac{1}{q}}d\tau \\ &\leq cR^{-2}t^{1-\frac{1}{q}}||v_0||_\infty||\varphi_0||_q\,, \end{aligned}$$

where increasing we employed (25) for F, $R < L$ and $(47)_2$ for φ. Applying Hölder's inequality and employing (31), we get

$$|I_2(t)| \leq c\int_0^t \|\widehat{U}(\tau)\|_\infty |\operatorname*{supp}_x \mathfrak{h}(\tau,x)|^{\frac{1}{2}}\tau^{-\frac{1}{2}}(R+\sqrt{\tau})^{-1}\|\varphi(t-\tau)\|_2 d\tau\,.$$

Since $|\operatorname*{supp}_x \mathfrak{h}| = c(R+\sqrt{\tau})^2$, via $(47)_2$, we get

$$|I_2(t)| \leq c\|v_0\|_\infty\|\varphi_0\|_q \int_0^t \tau^{-\frac{1}{2}}(t-\tau)^{-\frac{1}{q}+\frac{1}{2}}d\tau\,.$$

Applying Holder's inequality, we get

$$|I_3(t)| \leq \int_0^t ||W_\tau(\tau)||_2||\varphi(t-\tau)||_2 d\tau\,.$$

By virtue of estimate $(40)_2$ for W_t and estimate $(47)_2$ for φ, we obtain

$$|I_3(t)| \leq c\|v_0\|_\infty \|\varphi_0\|_q \int\limits_0^t \tau^{-\frac{1}{2}}(t-\tau)^{-\frac{1}{q}+\frac{1}{2}} d\tau \leq ct^{1-\frac{1}{q}}\|v_0\|_\infty\|\varphi_0\|_q\,.$$

The above estimates furnish (48). □

Lemma 12. *For all $q \in [1,2)$ there exists a constant c such that*

$$|\mathscr{I}_2(t)| \leq ct^{1-\frac{1}{q}}\|v_0\|_\infty\|\varphi_0\|_q\,,\ \textit{for all}\ t>0\,, \tag{49}$$

where constant c is independent of v_0 and φ_0.

Proof. We look for separately the estimate of each integral, called $I_i(t)$, $i = 1,2,3$, of the sum. Via Hölder's inequality, we get

$$\begin{aligned}|I_1(t)| &\leq \int\limits_0^t |(\Delta F(\tau), \varphi(t-\tau))| d\tau \leq \int\limits_0^t \|\Delta F(\tau)\|_2 \|\varphi(t-\tau)\|_2 d\tau \\ &\leq c\|v_0\|_\infty\|\varphi_0\|_q \int\limits_0^t (L^2+\tau)^{-\frac{1}{2}}(t-\tau)^{\frac{1}{2}-\frac{1}{q}} d\tau \leq ct^{1-\frac{1}{q}}\|v_0\|_\infty\|\varphi_0\|_q\,,\end{aligned}$$

where increasing we employed (25) for F and $(47)_1$ for φ. Via Hölder's inequality, and employing (43) for $\Delta\mathfrak{h}$ and $(47)_1$ for φ, we obtain

$$|I_2(t)| \leq \int\limits_0^t |\widehat{U}(\tau)|\|\Delta\mathfrak{h}(\tau)\|_2\|\varphi(t-\tau)\|_2 \leq c\|v_0\|_\infty\|\varphi_0\|_q \int\limits_0^t (R+\sqrt{t})^{-1}(t-\tau)^{\frac{1}{2}-\frac{1}{q}} d\tau \leq ct^{1-\frac{1}{q}}\|v_0\|_\infty\|\varphi_0\|_q.$$

After applying Hölder's inequality, we get

$$\begin{aligned}|I_3(t)| &\leq \int\limits_0^t \|\Delta W(\tau)\|_2\|\varphi(t-\tau)\|_2 d\tau \\ &\leq c\|v_0\|_\infty\|\varphi_0\|_q \int\limits_0^t (R+\sqrt{t})^{-1}(t-\tau)^{\frac{1}{2}-\frac{1}{q}} d\tau = ct^{1-\frac{1}{q}}\|v_0\|_\infty\|\varphi_0\|_q\,,\end{aligned}$$

where in the last step of the estimate we taken into account $(47)_2$ for φ and (39) for ΔW. Collecting the estimates for $I_i(t)$, we arrive at the wanted one for $\mathscr{I}_2(t)$. □

Lemma 13. *The following estimate holds:*

$$\mu > 1\,,\ \ |\mathscr{I}_1(t)| + |\mathscr{I}_2(t)| \leq c\|v_0\|_\infty\|\varphi_0\|_1 t^{\frac{\mu}{2}-1}\,,\ \textit{for}\ t \in (0,1)\,. \tag{50}$$

Proof. Applying Hölder's inequality, for all $t \in (0,1)$, we get

$$\begin{aligned}|\mathscr{I}_1(t)| &\leq \int\limits_0^t \left[\|F_\tau(\tau)\|_2 + \|D_\tau(\mathfrak{h}\widehat{U}(\tau))\|_2 + \|W_\tau(\tau)\|_2\right]\|\varphi(t-\tau)\|_2 d\tau \\ &\leq c\|v_0\|_\infty\|\varphi_0\|_1 \int\limits_0^t \tau^{\frac{\mu}{2}-\frac{3}{2}}(t-\tau)^{-\frac{1}{2}} d\tau = c\|v_0\|_\infty\|\varphi_0\|_1 t^{\frac{\mu}{2}-1}\,,\end{aligned}$$

where increasing we applied (44) for the terms in $\|\cdot\|_2$-norm and (47) for φ. Analogously, for all $\mu > 0$, we obtain the estimate

$$|\mathscr{I}_2(t)| \le \int_0^t \big[\|\Delta F(\tau)\|_2 + \|\Delta(\mathfrak{h}\widehat{U}(\tau))\|_2 + \|\Delta W(\tau)\|_2\big]\|\varphi(t-\tau)\|_2 d\tau$$
$$\le c\|v_0\|_2\|\varphi_0\|_1 \int_0^t \tau^{\frac{\mu}{2}}(t-\tau)^{-\frac12} d\tau = c\|v_0\|_\infty\|\varphi_0\|_1 t^{\frac{\mu}{2}+\frac12}\,,$$

where increasing we employed (45) for the terms in $\|\cdot\|_2$-norm and $(47)_2$ for φ. □

We set

$$\mathscr{I}_3(t) := -\int_0^t (D^2_\tau(F(\tau) - \mathfrak{h}(\tau)\widehat{U}(\tau) + W(\tau)), \varphi(t-\tau) d\tau\,,$$
$$\mathscr{I}_4(t) := \int_0^t (\Delta D_\tau(F(\tau) - \mathfrak{h}(\tau)\widehat{U}(\tau) + W(\tau)), \varphi(t-\tau)) d\tau\,.$$

Lemma 14. *For all $q \in [1,2]$, there exists a constant $C := C(R)$ such that*

$$|\mathscr{I}_3(t)| \le C t^{-\frac1q}\|v_0\|_\infty\|\varphi_0\|_q\,,\ \textit{for all } t > 0\,. \tag{51}$$

Proof. We initially point out that $\mu > 0$ can be chosen in Lemma 10 in such a way that the integral is well posed. Moreover, we recall that $L > R$. In order to deduce (51) we look for separately the estimate of each integral, called $J_i(t)$, $i = 1,2,3$, of the sum. Integrating by parts we get

$$J_1(t) = -(F_t(\frac t2), \varphi(\frac t2)) + \int_0^{\frac t2} (F_\tau(\tau), \varphi_\tau(t-\tau)) d\tau - \int_{\frac t2}^t (F_{\tau\tau}(\tau), \varphi(t-\tau)) d\tau\,,$$

where in $t = 0$ we used the bound (44) for $\mu > 3$. Applying Hölder's inequality, we get

$$|J_1(t)| \le \|F_t(\frac t2)\|_2\|\varphi(\frac t2)\|_2 + \int_0^{\frac t2} \|F_\tau(\tau)\|_1\|\varphi_\tau(t-\tau)\|_\infty d\tau$$
$$+ \int_{\frac t2}^t \|F_{\tau\tau}(\tau)\|_2\|\varphi(t-\tau)\|_2 d\tau$$
$$\le c\|v_0\|_\infty\|\varphi_0\|_q\Big[(L^2+t)^{-\frac32} t^{\frac12-\frac1q} + \int_0^{\frac t2} (L^2+\tau)^{-\frac32}(t-\tau)^{-\frac1q} d\tau$$
$$+ \int_{\frac t2}^t (L+\sqrt{t})^{-\frac52}\tau^{\frac12-\frac1q} d\tau\Big]\,,$$

where for F we toke estimate (25) into account, as well estimates $(47)_2$ for φ. Since $R < L$ we arrive at (51) for J_1.

After integrating by parts, applying Hölder's inequality, we obtain

$$|J_2(t)| \le \|D_t(\mathfrak{h}(\frac{t}{2})\widehat{U}(\frac{t}{2}))\|_2\|\varphi(\frac{t}{2})\|_2+\int\limits_0^{\frac{t}{2}}\|D_\tau(\mathfrak{h}(\tau)\widehat{U}(\tau))\|_2\|\varphi_\tau(t-\tau)\|_2 d\tau$$

$$+\int\limits_{\frac{t}{2}}^{t}\|D^2_\tau(\mathfrak{h}(\tau)\widehat{U}(\tau))\|_2\|\varphi(t-\tau)\|_2 d\tau = J_2^1+J_2^2+J_2^3\,,$$

where in $t=0$, we toke the bound $(44)_2$ for $\mu>3$ into account. Recalling that ≤ 1, employing (31) for $\mathfrak{h}$, $(10)_1$ for U and U_t, and $(47)_2$ for φ, since $|\operatorname{supp}_x \mathfrak{h}| = c(R+\sqrt{\tau})^2$, being $R<L$, we have

$$J_2^1 \le ct^{-\frac{1}{q}}\|v_0\|_\infty\|\varphi_0\|_q\,.$$

Recalling that $|\operatorname{supp}_x \mathfrak{h}| = c(R+\sqrt{\tau})^2$, since, ≤ 1 the estimate (31) for $\mathfrak{h}$ and $(47)_3$ for φ_τ furnish

$$J_2^2 \le c\|\varphi_0\|_q\int\limits_0^{\frac{t}{2}}\big[(R+\sqrt{\tau})\|\widehat{U}_\tau(\tau)\|_\infty + \|\widehat{U}(\tau)\|_\infty\tau^{-\frac{1}{2}}\big](t-\tau)^{-\frac{1}{2}-\frac{1}{q}}d\tau\,.$$

Now, applying $(10)_1$ for the term $\|\widehat{U}_t(t)\|_\infty$ and for the term $\|\widehat{U}\|_\infty$, being $R<L$, we realize

$$J_2^2(t) \le c\|v_0\|_\infty\|\varphi_0\|_q\int\limits_0^{\frac{t}{2}}\tau^{-\frac{1}{2}}(t-\tau)^{-\frac{1}{2}-\frac{1}{q}}d\tau \le ct^{-\frac{1}{q}}\|v_0\|_\infty\|\varphi_0\|_q\,.$$

For the last term we have $D_t^2(\mathfrak{h}\widehat{U}) = \mathfrak{h}_{\tau\tau}\widehat{U}+2h_\tau\widehat{U}_\tau+\mathfrak{h}\widehat{U}_{\tau\tau}$. Recalling that $|\operatorname{supp}_x \mathfrak{h}| = c(R+\sqrt{\tau})^2$, via estimates (10) for $\widehat{U}$ and developing $(32)_2$ for $\mathfrak{h}_{\tau\tau}$, we get

$$J_2^3 \le c\|\varphi_0\|_q\int\limits_{\frac{t}{2}}^{t}(t-\tau)^{\frac{1}{2}-\frac{1}{q}}\tau^{-\frac{3}{2}}\|\widehat{U}(\frac{\tau}{2})\|_\infty d\tau \le ct^{-\frac{1}{q}}\|\varphi_0\|_q\|v_0\|_\infty\,,$$

where increasing we used the semigroup property of U. Collecting the estimates related to J_2^i, we arrive at

$$|J_2(t)| \le ct^{-\frac{1}{q}}\|v_0\|_\infty\|\varphi_0\|_q,\ \text{for all } t>0\,. \tag{52}$$

Integrating by parts, applying Hölder's inequality, we get

$$|J_3(t)| \le \|W_t(\frac{t}{2})\|_2\|\varphi(\frac{t}{2})\|_2+\int\limits_0^{\frac{t}{2}}\|W_\tau(\tau)\|_2\|\varphi_\tau(t-\tau)\|_2 d\tau$$

$$+\int\limits_{\frac{t}{2}}^{t}\|W_{\tau\tau}(\tau)\|_{(2,\infty)}\|\varphi(t-\tau)\|_{(2,1)}d\tau\,,$$

where in $t=0$ we toke the bound (44). Employing $(40)_{1,3}$ for W, we get

$$|J_3(t)| \le c\|v_0\|_\infty\|\varphi_0\|_q\Big[t^{-\frac{1}{q}}+\int\limits_0^{\frac{t}{2}}\tau^{-\frac{1}{2}}(t-\tau)^{-\frac{1}{2}-\frac{1}{q}}d\tau+\int\limits_{\frac{t}{2}}^{t}\tau^{-\frac{3}{2}}(t-\tau)^{\frac{1}{2}-\frac{1}{q}}d\tau\Big],$$

where we considered $(47)_2$ for φ. □

Lemma 15. *For $q \in [1,2]$, there exists a constant $C := C(R)$ such that*

$$|\mathscr{I}_4(t)| \leq Ct^{-\frac{1}{q}}\|v_0\|_\infty\|\varphi_0\|_q \text{ for all } t > 0\,. \tag{53}$$

Proof. Integrating by parts, we get

$$\mathscr{I}_4(t) = -\int_0^t (\nabla F_\tau(\tau) - \nabla\mathfrak{h}_\tau(\tau)\otimes\widehat{U}(\tau) - \nabla\mathfrak{h}\otimes\widehat{U}_\tau(\tau) + \nabla W_\tau(\tau), \nabla\varphi(t-\tau))d\tau\,.$$

We separately look for the estimate of each integral, called $J_i(t)$, $i = 1,2,3,4$, of the sum. Applying Hölder's inequality, we get

$$|J_1(t)| \leq \int_0^t ||\nabla F_\tau(\tau)||_2||\nabla\varphi(t-\tau)||_2 d\tau \leq c||v_0||_\infty||\varphi_0||_q\int_0^t (L^2+\tau)^{-\frac{3}{2}}(t-\tau)^{-\frac{1}{q}}d\tau$$
$$\leq ct^{-\frac{1}{q}}||v_0||_\infty||\varphi_0||_q\,,$$

where we toke estimate (25) for F, and $(47)_1$ for φ into account. Being $R < L$, we arrive at (53) for J_1. An integration by parts on $(0,\frac{t}{2})\times\Omega$ and Hölder's inequality furnish

$$|J_2(t)+J_3(t)| \leq \|\Delta\mathfrak{h}(\frac{t}{2})\widehat{U}(\frac{t}{2})\|_2\|\varphi(t-\tau)\|_2 + \int_0^{\frac{t}{2}}\|\Delta\mathfrak{h}(\tau)\|_2\|\widehat{U}(\tau)\|_2\|\varphi_\tau(t-\tau)\|_2 d\tau$$
$$+\int_{\frac{t}{2}}^t \big[\|\nabla\mathfrak{h}_\tau(\tau)\widehat{U}(\tau)\|_2 + \|\nabla\mathfrak{h}\widehat{U}_\tau(\tau)\|_2\big]\|\nabla\varphi(t-\tau)\|_2 d\tau\,,$$

where we employed $(10)_2$ in $t=0$. From $(10)_1$ $|\widehat{U}(t)| + (L^2+t)|\widehat{U}_t(t)| \leq c\|v_0\|_\infty$, via (43) for $\nabla\mathfrak{h}$, $\Delta\mathfrak{h}$ and $(32)_1$ for $\nabla\mathfrak{h}_t$, employing (47) for φ, we get

$$|J_2(t)+J_3(t)| \leq c\|v_0\|_\infty\|\varphi_0\|_q\Big[t^{-\frac{1}{q}} + \int_0^{\frac{t}{2}}\big[\tau^{-\frac{1}{2}}(t-\tau)^{-\frac{1}{2}-\frac{1}{q}}d\tau + \int_{\frac{t}{2}}^t (\tau^{-1} + (L^2+\tau)^{-1})(t-\tau)^{-\frac{1}{q}}d\tau\Big]$$
$$\leq ct^{-\frac{1}{q}}\|v_0\|_\infty\|\varphi_0\|_q\,.$$

Inequality (34) for ∇W and $(10)_2$ for U ensure that $\lim_{t\to 0}\|\nabla W(t)\|_2 = 0$. Integrating by parts on $(0,\frac{t}{2})$ and applying Hölder's inequality, we obtain

$$|J_4(t)| \leq \|\nabla W(\frac{t}{2})\|_2\|\nabla\varphi(\frac{t}{2})\|_2 + \int_0^{\frac{t}{2}}\|\nabla W(\tau)\|_2\|\nabla\varphi_\tau(\tau)\|_2 d\tau$$
$$+\int_{\frac{t}{2}}^t \|\nabla W_\tau(\tau)\|_2\|\nabla\varphi(t-\tau)\|_2 d\tau\,.$$

Recalling estimate (39) for ∇W and $(40)_2$ for ∇W_t, employing estimate $(47)_1$ for φ, we get

$$|J_4(t)| \leq c\|v_0\|_\infty\|\varphi_0\|_q\Big[t^{-\frac{1}{q}} + \int_0^{\frac{t}{2}}(t-\tau)^{-1-\frac{1}{q}}d\tau + \int_{\frac{t}{2}}^t\big[\tau^{-\frac{1}{2}}(R+\sqrt{\tau})^{-1} + (L^2+\tau)^{-1}\big](t-\tau)^{-\frac{1}{q}}d\tau\Big]$$
$$\leq ct^{-\frac{1}{q}}\|v_0\|_\infty\|\varphi_0\|_q.$$

Collecting the above estimates J_i, $i = 1,2,3,4$, we arrive at (51). □

2.6. Some Results of The L^p-Theory for The IBVP

Now, we recall some results concerning the Stokes initial boundary value problem:

$$\begin{aligned} &\vartheta_t - \Delta\vartheta = -\nabla\pi_\vartheta + f,\ \nabla\cdot\vartheta = 0,\ \text{in } (0,T)\times\Omega,\\ &\vartheta(t,x) = 0 \text{ on } (0,T)\times\partial\Omega,\\ &(\vartheta(0),\varphi) = (w_0,\varphi) \text{ for any } \varphi\in\mathscr{C}_0(\Omega). \end{aligned} \tag{54}$$

In problem (54) the initial condition is given in the weak form $(\vartheta(0),\varphi) = (w_0,\varphi)$, $\varphi\in\mathscr{C}_0(\Omega)$, in order to state the initial boundary value problem with an initial data w_0 belonging to the weaker Lebesgue space $L^p(\Omega)$, $p\geq 1$. With the weak formulation in L^p, $p\in(1,\infty)$, the continuity of the equation of divergence (also in weak form) at $t=0$ is lost, as well as the zero value of the normal component of the solution at the initial instant $t=0$. Of course, if the datum is an element of $J^p(\Omega)\subset L^p(\Omega)$, $p>1$, then, the problem is just the classical one.

For our aims the case $w_0\in C_0^1(\Omega)\subset L^1(\Omega)$ has a special interest. Actually, we look for an estimate in $L^\infty(\Omega)$ by means of the variational formulation $\|a\|_\infty = \sup\limits_{\theta\in C_0^1(\Omega)} \dfrac{(a,\theta)}{\|\theta\|_1}$. In Lemma 22, by means of a duality argument, the quoted weak formulation of the θ solution allows us to give the estimate $\|\omega(t)\|_\infty = \sup\limits_{\theta\in C_0^1(\Omega)} \dfrac{(\omega(t),\theta)}{\|\theta\|_1}$ of the auxiliary solution $\omega(t,x)$ to problem (67).

In Theorem 2 the initial boundary value problem (54) can be considered for Ω bounded or exterior, indifferently.

Theorem 2. *Let be $f=0$ in (54). Let $w_0\in C_0^1(\Omega)$. Then, to the data w_0 it corresponds a unique solution (ψ,π_ψ) of problem (54) such that $\psi\in\bigcap\limits_{q>1} C([0,T);J^q(\Omega))$, for $\eta>0$, $\psi\in\bigcap\limits_{q>1} L^q(\eta,T;W^{2,q}(\Omega)\cap J^{1,q}(\Omega))$ and $\nabla\pi_\psi,\psi_t\in\bigcap\limits_{q>1} L^q(\eta,T;L^q(\Omega))$. Moreover, for $q\in(1,\infty]$ and $r\in[1,q]$,*

$$\begin{aligned} &\|\psi(t)\|_q \leq c\|w_0\|_r t^{-\mu}, \qquad \mu = \tfrac{1}{r}-\tfrac{1}{q},\ t>0;\\ &\|\nabla\psi(t)\|_q \leq c\|w_0\|_r t^{-\mu_1},\ \mu_1 = \begin{cases} \frac12+\mu \text{ if } t\in(0,1],\\ \frac12+\mu \text{ if } t>0 \text{ and } q\in(1,n],\\ \frac{n}{2r} \quad\ \text{ if } t>1 \text{ and } q>n;\end{cases}\\ &\|\psi_t(t)\|_q \leq c\|w_0\|_r t^{-\mu_2},\ \ \mu_2 = 1+\mu,\ t>0; \end{aligned} \tag{55}$$

where the constant c is independent of w_0. Finally, $\lim\limits_{t\to 0}(\psi(t),\varphi) = (w_0,\varphi)$ for any $\varphi\in\mathscr{C}_0(\Omega)$.

Proof. For the proof of the above theorem see [17] Theorem 2.1. Actually, the quoted reference is the two-dimensional version of Theorem 3.2 given in [14]. □

Remark 1. We stress that the property $\psi\in C([0,T);J^q(\Omega))$ is meant in the sense that $\lim_{t\to 0}\|\psi(t)-P_q(w_0)\|_q = 0$. In the case of $w_0\in C_0^1(\Omega)$ in $t=0$ at most the weak limit property holds, the one stated in the theorem. Of course, if we assume $w_0\in J^q(\Omega)$, $q\in(1,\infty)$, the result becomes the classical one, in particular the continuity in norm holds.

Corollary 2. *In the same hypothesis of Theorem 2, for $q\in(r,\infty)$, $\sigma\in[1,\infty]$ we also get*

$$\text{for all } \sigma\in[1,\infty]\,,\ t\|\psi_t(t)\|_{(q,\sigma)} + \|\psi(t)\|_{(q,\sigma)} \leq c\|w_0\|_r t^{-\mu}, \mu = \frac{1}{r}-\frac{1}{q},\ t>0; \tag{56}$$

Proof. We recall that problem (1) can be considered in $L(q,\sigma)$-setting (e.g. cf. [29] or [15]). In particular for all $q>p>1$ one obtains the estimate

$$t\|\psi(t)\|_{(q,\sigma)} + \|\psi(t)\|_{(q,\sigma)} \leq c(t-s)^{-\left(\frac1p-\frac1q\right)}\|\psi(s)\|_{(p,\infty)} \leq c(t-s)^{-\left(\frac1p-\frac1q\right)}\|\psi(s)\|_p\,,\ \text{for all } t>s\geq 0\,.$$

Thus, setting $s=\frac{t}{2}$ and employing $(55)_{1,3}$, for $r\leq p$, one arrives at (56). □

Lemma 16. *In problem* (54)*, assume* $f \in L^q(0,T;L^q(\Omega))$*, and* $w_0 = 0$*. Then, there exists a unique solution* $(\vartheta, \pi_\vartheta)$ *to problem* (54) *with* $\vartheta \in C([0,T);J^q(\Omega))$*,*

$$\int_0^T \big[|\vartheta(t)|_{2,q}^q + |\vartheta_t(t)|_q^q + |\nabla \pi_\vartheta(t)|_q^q\big]dt \le c(T)\int_0^T |f(t)|_q^q dt\,. \tag{57}$$

Proof. For a proof of the above theorem see [12,22,23]. □

Let us consider the equation for the pressure:

$$\Delta \Pi = 0 \text{ in } \Omega\,, \quad \frac{d}{d\nu}\Pi = \nabla \times \nabla \times N \cdot \nu \text{ on } \partial\Omega\,, \quad \Pi \to c \text{ for } |x| \to \infty\,. \tag{58}$$

We set

$$< u >_q^\lambda := \Big[\int_{\partial\Omega}\int_{\partial\Omega} \frac{|u(x)-u(y)|^q}{|x-y|^{1+\lambda q}} d\mathcal{H}^1 d\mathcal{H}^1\Big]^{\frac1q}.$$

For $\lambda = 1 - \frac1q$ we get the seminorm of trace space $W^{1-\frac1q,q}(\partial\Omega)$.

We set $a := \nabla \times N$. The following result holds:

Lemma 17. *Assume that* $\Omega' \subset \Omega$ *bounded with* $\partial(\Omega - \Omega') \cap \partial\Omega = \emptyset$*. Assume that* $a \in W^{1-\frac1q,q}(\partial\Omega)$*. Then a solution of problem* (58) *is such that*

$$\lambda \in (0,1)\,,\ \|\Pi\|_{L^q(\Omega')} \le c < a >_q^\lambda \quad \textit{and} \quad \|\nabla \Pi\|_q \le c < a >_q^{1-\frac1q}\,, \tag{59}$$

where c *is constant independent of* a*.*

Proof. Estimate (59) is due to Solonnikov in [25], recently, it is also reproduced in [19]. □

In the following Lemma 18 and Lemma 19 we assume that $\Omega' \subset \Omega$ is a bounded domain with $\partial(\Omega - \Omega') \cap \partial\Omega = \emptyset$.

By virtue of trace theorems, for any element of $a \in W^{1-\frac1q,q}(\partial\Omega)$ admits an extension from $\partial\Omega$ into Ω' which is an element of $W^{1,q}(\Omega')$. We denote by the same symbol a the element of space trace $W^{1-\frac1q,q}(\partial\Omega)$ and its extension as element of $W^{1,q}(\Omega')$.

Lemma 18. *Assume that* $a \in W^{1-\frac1q,q}(\partial\Omega)$*. For a solution of problem* (58) *the following estimate holds:*

$$\begin{aligned}\|\Pi\|_{C(\overline{\Omega}')} \le c\Big[\|a\|_{L^q(\Omega')}^{(1-\frac1d)(1-\alpha)}\|\nabla a\|_{L^q(\Omega')}^{\frac1d(1-\alpha)+\alpha} + \|a\|_{L^q(\Omega')}^{(1-\frac1q)(1-\frac1d)(1-\alpha)}\|\nabla a\|_{L^q(\Omega')}^{\left(\frac1q(1-\frac1d)+\frac1d\right)(1-\alpha)+\alpha}\Big]\\ + c\big(\|a\|_{L^q(\Omega')} + \|\nabla a\|_{L^q(\Omega')}^{\frac1q}\|a\|_{L^q(\Omega')}^{1-\frac1q}\big)^{1-\frac1d}\|\nabla a\|_{L^q(\Omega')}^{\frac1d}\,,\end{aligned} \tag{60}$$

where $q > 2$, $\alpha := \frac2q$, $d := \frac{q}{1+\lambda q}$, $\lambda \in (0, 1-\frac1q)$*, and* c *is a constant independent of* a*.*

Proof. For $q > 2$, by virtue of Lemma 17, we obtain

$$\begin{aligned}\|\Pi\|_{C(\overline{\Omega}')} &\le c(\|\nabla\Pi\|_{L^q(\Omega')}^\alpha \|\Pi\|_{L^q(\Omega')}^{1-\alpha} + \|\Pi\|_{L^q(\Omega')}) \le c\Big[\big(< a >_q^{1-\frac1q}\big)^\alpha\big(< a >_q^\lambda\big)^{1-\alpha} + < a >_q^\lambda\Big]\\ &\le c\Big[\|\nabla a\|_{L^q(\Omega')}^\alpha\big(< a >_q^\lambda\big)^{1-\alpha} + < a >_q^\lambda\Big]\,, \text{ with } \alpha := \frac2q\,.\end{aligned}$$

We have

$$\begin{aligned}< a >_q^\lambda &\le c\|a\|_{L^q(\partial\Omega)}^{1-\frac1d}\big(< a >_q^{1-\frac1q}\big)^{\frac1d} \le c\|a\|_{L^q(\partial\Omega)}^{1-\frac1d}\|\nabla a\|_{L^q(\Omega')}^{\frac1d}\\ &\le c\big(\|a\|_{L^q(\Omega')} + \|\nabla a\|_{L^q(\Omega')}^{\frac1q}\|a\|_{L^q(\Omega')}^{1-\frac1q}\big)^{1-\frac1d}\|\nabla a\|_{L^q(\Omega')}^{\frac1d}\,, \text{ with } d = \frac{q}{1+\lambda q}\,.\end{aligned}$$

Hence, we get

$$\|\Pi\|_{C(\overline{\Omega}')}\leq c\Big[\|a\|_{L^q(\Omega')}^{(1-\frac{1}{d})(1-\alpha)}\|\nabla a\|_{L^q(\Omega')}^{\frac{1}{d}(1-\alpha)+\alpha}+\|a\|_{L^q(\Omega')}^{(1-\frac{1}{q})(1-\frac{1}{d})(1-\alpha)}\|\nabla a\|_{L^q(\Omega')}^{\big(\frac{1}{q}(1-\frac{1}{d})+\frac{1}{d}\big)(1-\alpha)+\alpha}\Big]$$
$$+c\big(\|a\|_{L^q(\Omega')}+\|\nabla a\|_{L^q(\Omega')}^{\frac{1}{q}}\|a\|_{L^q(\Omega')}^{1-\frac{1}{q}}\big)^{1-\frac{1}{d}}\|\nabla a\|_{L^q(\Omega')}^{\frac{1}{d}}.$$

□

Lemma 19. *Assume that $a(t,\xi)$ in Lemma 17 is a smooth one-parameter family of function in $W^{1-\frac{1}{q},q}(\partial\Omega)$, the time $t>0$ is the parameter. Assume that $t^{\frac{\gamma}{2}}||a(t)||_q+t^{\gamma}||\nabla a(t)||_q\leq A$, for some $\gamma>0$ and for $t\in(0,1)$, and $||a(t)||_q+||\nabla a(t)||_q\leq A$ for $t\geq 1$. Then, we get*

$$||\Pi(t)||_{C(\overline{\Omega}')}\leq cA\begin{cases} t^{-(\frac{1}{2}+\overline{\mu})\gamma}, & \textit{for } t\in(0,1)\,,\\ 1\,, & \textit{for } t\geq 1\,,\end{cases}\tag{61}$$

where c is constant independent of a, exponent $\overline{\mu}\in(0,\frac{1}{2})$, and $\Omega'\subset\Omega$ bounded with $\partial(\Omega-\Omega')\cap\partial\Omega=\emptyset$.

Proof. We have to estimate the right hand side of (60). Estimate $(61)_1$ is the behavior of $\|\Pi(t)\|_{C(\overline{\Omega}')}$ in a right neighborhood of $t=0$. Hence, we limit ourselves to consider the terms with the major singularity in $t=0$. This is conditioned by the greater exponent for t^{-1}. Recalling that in estimate (60) we have $a=\nabla\times N(t,x)$ and the domain Ω' is bounded, employing the assumptions $||a(t)||_q\leq cAt^{-\frac{\gamma}{2}}$ and $||\nabla a(t)||_q\leq cAt^{-\gamma}$, then we get

$$\|\Pi(t)\|_{C(\overline{\Omega}')}\leq cAt^{-\gamma\beta},$$

with exponent $\beta:=-\frac{1}{2}(1-\frac{1}{q})(1-\frac{1}{d})(1-\alpha)-\big(\frac{1}{q}(1-\frac{1}{d})+\frac{1}{d}\big)(1-\alpha)-\alpha$. By a computation we obtain

$$\beta=\frac{1}{2}+\frac{\alpha}{2}+(1-\alpha)[\frac{1}{2q}(1-\frac{1}{d})+\frac{1}{2d}]=:\frac{1}{2}+\overline{\mu}$$

where we recall that $\alpha=\frac{2}{q}, q>2$, $d=\frac{q}{1+\lambda q}$ and $\lambda\in(0,1-\frac{1}{q})$. For large q and small λ, we arrive at $\overline{\mu}\in(0,\frac{1}{2})$. Estimate $(61)_2$ is immediate from (60) and assumptions. □

2.7. A Nonlinear Generalization of The Gronwall Inequality

Lemma 20. *Let $y(t)$ be a nonnegative function that satisfies the integral inequality*

$$y(t)\leq A_0+\int_{t_0}^{t}(a(s)y(s)+b(s)y^{\sigma}(s)ds\,,\ A_0\geq 0,\ \sigma\in[0,1)\,,\tag{62}$$

where $a(t)$ and $b(t)$ are continuous nonnegative functions for $t\geq t_0$. Then, the following inequality holds

$$y(t)\leq\Big[A_0^{1-\sigma}\exp\big[(1-\sigma)\int_{t_0}^{t}a(s)ds\big]+(1-\sigma)\int_{t_0}^{t}b(s)\exp\big[(1-\sigma)\int_{s}^{t}a(\tau)d\tau\big]ds\Big]^{\frac{1}{1-\sigma}},\ t>t_0.\tag{63}$$

Proof. This result is a particular case of a more general result due to A.I. Perov. A proof of the result can be found in [24], Theorem 1, p. 360. □

Corollary 3. *Assume that $y(s)$ and $a(s)$ are continuous functions on $[0,T)$, $b(s)=As^{-\frac{1}{2}-\overline{\mu}}$, A a nonnegative constant and $\overline{\mu}\in(0,\frac{1}{2})$. Also, assume that the inequality (62) holds for $t>t_0\geq 0$, $A_0=A(t_0)$, with $A(t_0)$ continuous function. Then, inequality (63), initially achieved for $t>t_0>0$, holds for $t\in(0,T)$ and $t_0=0$.*

Proof. By assumptions, the right hand-side of (63) is convergent for $t_0\to 0$. Hence, letting $t_0\to 0$ we achieve the result. □

3. A Result by K.Abe and Y.Giga

In this section we recall the following fundamental result due to Abe and Giga in [3] and related to the Stokes IBVP in exterior domains.

Theorem 3. *Let us consider the initial boundary value problem (1). Then there exists a* $T_0 > 0$ *such that, for all* $v_0 \in L^\infty(\Omega)$ *enjoying (2), there exists a solution* $(u, \pi_u) \in C^2((0, T_0 \times \overline{\Omega}) \times C^1((0, T_0 \times \overline{\Omega})$ *to the Stokes problem (1), with* $u(t, x)$ $*$*-weakly continuous in* $t = 0$. *Also the following estimates hold*

$$\begin{aligned} &|u(t,x)| \le c\|v_0\|_\infty, \text{ for all } (t,x) \in C([0,T_0) \times \overline{\Omega})\,,\\ &\sum_{|\alpha|=1}^{2} t^{\frac{|\alpha|}{2}} \|D^\alpha u(t)\|_\infty + t\|u_t(t)\|_\infty + t\|\nabla \pi_u(t)\|_\infty \le c\|v_0\|_\infty\,, \text{ for all } t \in [0,T_0] \end{aligned} \tag{64}$$

where c *is independent of* v_0 *and* $|x||\nabla \pi_u(t,x)| \le c$ *for all* $x \in \Omega$.

Lemma 21. *Let* $v_0 \in L^\infty(\Omega) \cap J^p(\Omega)$, $p \in (1,\infty)$. *Denoted by* (u, π_u) *and* (v, π_v) *the solutions corresponding to* v_0 *by virtue of Theorem 3 and Theorem 2, respectively. Then the solutions coincide up to function of* t *for the pressure fields.*

Proof. This result is an immediate consequence of the approach employed in [3,4]. □

Corollary 4. *Let* $v_0 \in L^\infty(\Omega)$ *enjoying (2) and with compact support. Then, for all* $T > 0$, $u \in C([0,T); J^p(\Omega))$,

$$\begin{aligned} &\|u(t)\|_\infty + t\|u_t(t)\|_\infty \le c\|v_0\|_\infty, \text{ for all } t > 0\,,\\ &\sum_{|\alpha|=1}^{2} \frac{t^{\frac{|\alpha|}{2}}}{1+t^{\frac{|\alpha|}{2}}} \|D^\alpha u(t)\|_p + t\|u_t(t)\|_p + \frac{t}{1+t}\|\nabla \pi_u(t)\|_p \le c\|v_0\|_\infty\,, \text{ for all } t > 0\,, \end{aligned} \tag{65}$$

where $p \in (1,\infty)$ *and the constant* c *depends on the support of* v_0. *Moreover, for* $\overline{\mu} \in (0, \frac{1}{2})$, *we get*

$$\begin{aligned} &|\pi_u(t,x)| \le c\|v_0\|_\infty t^{-\frac{1}{2}-\overline{\mu}}\,, \text{ for all } (t,x) \in (0,1) \times \Omega\,,\\ &|\pi_u(t,x)| \le c\|v_0\|_\infty\,, \text{ for all } (t,x) \in [1,\infty) \times \Omega\,. \end{aligned} \tag{66}$$

Proof. The result related to estimate $(65)_1$ is an immediate consequence of Lemma 21. Actually, due to the compact support of the initial datum of the solution, in order to estimate $\|u(t)\|_\infty$ we take advantage of estimate (64) in $(0, T_0)$ and, for $t \ge T_0$, of the ones related to the L^p-setting, see [8,9]. Instead, estimate $(65)_2$ is deduced from the results in L^p-setting.

In order to prove estimates (66), we start remarking that the pressure field π_u is a solution to equation (58) with $N = -u$. Moreover, for the nature of compact support of the initial datum, such a solution is such that $(u, \nabla \pi_u)$ belongs to $L^q(\Omega)$, $q \in (1,2)$, in the sense specified by Theorem 2. This fact ensures that $\int_{\partial\Omega} \nu \cdot \nabla \times \nabla \times u d\mathcal{H}^1 = 0$. Hence, for all $t > 0$, letting $|x| \to \infty$, we get that $\pi_u \to 0$. Thus, by virtue of *maximum principium* for harmonic solutions, we get

$$|\pi_u(t,x)| \le \max_{\Omega'} |\pi_u(t,x)|\,, \text{ for all } (t,x) \in (0,\infty) \times \mathbb{R}^2 - \Omega'\,.$$

On the other hand, via (65), we satisfy the assumptions of Lemma 19 with $A := \|v_0\|_\infty$ and $\gamma = 1$. Hence, in a neighborhood Ω' of $\partial\Omega$ the max value of π_u satisfies estimate (61). In this way we arrive at $(66)_1$ in a neighborhood of $t = 0$, and $(66)_2$ for large t. □

4. An Auxiliary Stokes Problem

In this section we consider the following problem

$$
\begin{aligned}
&\omega_t - \Delta\omega + \nabla\pi_\omega = -\frac{\partial}{\partial t}(F - \mathfrak{h}\widehat{U} + W) + \Delta(F - \mathfrak{h}\widehat{U} + W)\,,\ \text{in } (0,T)\times\Omega\,,\\
&\nabla\cdot\omega = 0\,,\ \text{in } (0,T)\times\Omega\,,\\
&\omega = 0 \text{ on } (0,T)\times\partial\Omega\,,\ \omega = 0 \text{ on } \{0\}\times\Omega\,,
\end{aligned}
\tag{67}
$$

where the functions F, $\mathfrak{h}\widehat{U}$, W are defined in the previous Sects. 2.2, 2.3 and 2.4. In particular, we recall that, for all $t>0$, they have compact support.

We are interested to the following

Lemma 22. *There exists a unique solution (ω,π_ω) to problem (67) such that, for $p\in(1,\infty)$, $\omega\in C([0,T;J^p(\Omega))\cap L^p(0,T;W^{2,p}(\Omega)\cap J^{1,p}(\Omega))$ and $\omega_t, \nabla\pi_\omega\in L^p(0,T;L^p(\Omega))$. Moreover, for $p>2$, we get*

$$
\begin{aligned}
&(1+t^{-\frac{1}{p}})\|\omega(t)\|_p + \|\omega(t)\|_\infty \le c\|v_0\|_\infty\,,\ \textit{for all } t>0\,,\\
&t^{1-\frac{1}{p}}\|\omega_t(t)\|_p + t\|\omega_t(t)\|_\infty \le c\|v_0\|_\infty\,,\ \textit{for all } t>0\,,\\
&\|D^2\omega(t)\|_p + \|\nabla\pi_\omega(t)\|_p \le c(t^{\frac{1}{p}-1}+1)\|v_0\|_\infty\,,\ \textit{for all } t>0\,,\\
&\lim_{t\to 0}\|\omega(t)\|_p = \lim_{t\to 0}\|\omega(t)\|_\infty = 0\,.
\end{aligned}
\tag{68}
$$

Proof. By virtue of Lemma 7 and Lemmas 9-10, and definition of $\mathfrak{h}$, for all $T>0$ and $p\in(1,\infty)$, we get that the right hand side of (67) belongs to $C([0,T;L^p(\Omega))$. Hence, the existence and the uniqueness are a consequence of Lemma 16. We prove that estimates (68) hold. We denote by $\varphi(s,x)$ the solution to the Stokes problem ensured by Theorem 2 with initial datum $w_0\in C_0^1(\Omega)$, and, for all $t>0$, $\varphi(t-\tau,x)$ is the solution $\varphi(s,x)$ written backward in time on interval $(0,t)$. Multiplying equation (67) by $\varphi(t-\tau,x)$, and integrating by parts on $(0,t)\times\Omega$, we get[3]

$$
(\omega(t),w_0) = \mathscr{I}_1(t) + \mathscr{I}_2(t))\,. \tag{69}
$$

Taking into account that we consider $p\in(2,\infty]$, employing Lemmas 11, 12 and 13 with $q=p'$, we arrive at

$$
|(\omega(t),w_0)| \le c(1+t^{\frac{1}{p}})\|v_0\|_\infty\|w_0\|_{p'}\,.
$$

This last, for $p\in(2,\infty]$, furnishes

$$
(1+t^{-\frac{1}{p}})\|\omega(t)\|_p + \|\omega(t)\|_\infty \le c\|v_0\|_\infty\,,\ \text{for all } t>0\,. \tag{70}
$$

[3] Just for the completeness, we justify the limit in end point t for the subsequent formula (69). First of all, in order to achieve (69), we can restrict our considerations to $p\in(2,\infty)$. By Lemma 16 function $\varphi\in J^{p'}(\Omega)$ for all $t>0$ and reaches the initial datum in weak form with test function in $\mathscr{C}_0(\Omega)$. Hence, we argue in the following way:

$$
\begin{aligned}
&\lim_{s\to 0}(\omega(t-s),\varphi(s))\\
&\quad = \lim_{s\to 0}\Big[(\omega(t-s)-\omega(t),\varphi(s)) + (\omega(t)-\omega^m,\varphi(s)) + (\omega^m,\varphi(s))\Big]\\
&\quad =: \lim_{s\to 0}\big[I_1(s)+I_2(s)+I_3(s)\big]\,,
\end{aligned}
$$

where $\{\omega^m\}\subset\mathscr{C}_0(\Omega)$ is convergent to $\omega(t,x)$ in $J^p(\Omega)$. Applying Hölder's inequality, we get

$$
\begin{aligned}
&\lim_{s\to 0}|I_1(s)| \le \lim_{s\to 0}\|\omega(t-s)-\omega(t)\|_p\|\varphi(s)\|_{p'} \le \lim_{s\to 0}\|\omega(t-s)-\omega(t)\|_p\|w_0\|_{p'}\,,\\
&\lim_{s\to 0}|I_2(s)| \le \lim_{s\to 0}\|\omega(t)-\omega^m\|_p\|\varphi(s)\|_{p'} \le \|\omega(t)-\omega^m\|_p\|w_0\|_{p'}\\
&\lim_{s\to 0}(\omega^m,\varphi(s)) = (\omega^m,w_0)\,,
\end{aligned}
$$

where we employed the properties of φ stated in Theorem 2. Hence, letting before $s\to 0$ and then $m\to\infty$, we arrive at (69).

Analogously, deriving with respect to t equation (67) and then multiplying by $\varphi(t-\tau,x)$, after integrating by parts on $(0,t)\times\Omega$, we get

$$(\omega_t(t),w_0)=\mathscr{I}_3(t)+\mathscr{I}_4(t)\,, \tag{71}$$

where for the limit in t we argue as in the case of (69) (actually, for all $\eta>0$, $\omega_t\in C((\eta,T);J^p(\Omega))$ holds, hence we can argue as made in footnote 3). We justify the last formula in $s=0$. Actually, via the equation of ω, we obtain

$$\lim_{s\to 0}(\omega_s(s),\varphi(t-s)) =: \lim_{s\to 0}\big[I_1(s)+I_2(s)+I_3(s)\big]=0\,,$$

with

$$\begin{aligned}I_1(s)&:=\lim_{s\to0}(\Delta\omega(s),\varphi(t-s))=(\omega(s),\Delta\varphi(t-s))\,,\\ I_2(s)&:=\lim_{s\to0}(F_s(s)+D_s(\mathfrak{h}(s)\widehat U(s))-W_s(s),\varphi(t-s))\,,\\ I_3(s)&:=\lim_{s\to0}(\Delta(F(s)+\mathfrak{h}(s)\widehat U(s)-W(s)),\varphi(t-s))\,.\end{aligned}$$

Applying Hölder's inequality, the first limit is zero thanks (70). For the second limit, applying Hölder's inequality, by virtue of (44), we get

$$|\lim_{s\to0}I_2(s)|\le\lim_{s\to0}\big[\|F_s(s)\|_2+\|D_s(\mathfrak{h}(s)\widehat U(s)\|_2+\|W_s(s)\|_2\big]\|\varphi(t-s)\|_2=0\,.$$

Analogously to the limit of the term $I_1(s)$, integrating by parts, we obtain

$$(F(s)+\mathfrak{h}(s)\widehat U(s)-W(s),\Delta\varphi(t-s))\,,$$

that, by virtue of (44), leads to zero limit. Now, from (71), taking into account that for $\mathscr{I}_3(t)$ and $\mathscr{I}_4(t)$ estimates (51) and (53) hold, for $q=p'$, for all $p>2$, we obtain

$$|(\omega_t(t),w_0)|\le ct^{-1+\frac1p}\|v_0\|_\infty\|w_0\|_q\,,\ \text{for all } t>0\,, \tag{72}$$

which furnishes $(68)_2$. In order to complete the proof of (68), we set $G:=D_t(F-\mathfrak{h}\widehat U+W)-\Delta(F-\mathfrak{h}\widehat U+W)$. By virtue of Lemma 4 we deduce

$$\begin{aligned}\|D^2\omega(t)\|_p+\|\nabla\pi_\omega(t)\|_p&\le c\big[\|\omega_t(t)\|_p+\|G(t)\|_p+\|\omega(t)\|_{L^p(\Omega_R)}\big]\\ &\le c\big[t^{-1+\frac1p}\|v_0\|_\infty+\|G(t)\|_p+c(R)\|\omega(t)\|_\infty\big]\\ &\le c\big[(t^{-1+\frac1p}+1)\|v_0\|_\infty+\|G(t)\|_p\big]\,,\ \text{for all } t>0\,.\end{aligned}$$

We estimate $\|G(t)\|_p$ by means of the results of Sect. 2. We get $\|G(t)\|_p\le c\|v_0\|_\infty$, for all $t>0$. Thus, via the above estimate of ω in $L^\infty(\Omega)$ and of ω_t in L^p-norm, we arrive at $(68)_3$. We conclude considering the limit property. The one in L^p-norm follows from $(68)_1$. In order to deduce the limit in L^∞-norm is enough to consider for $\mathscr{I}_1(t)$ and $\mathscr{I}_2(t)$ estimates given in Lemma 13. Actually, from (69) we get

$$|(\omega(t),w_0)|\le c\|v_0\|_\infty\|w_0\|_1t^{\frac\mu2-1}\,,\ \text{for all } t\in(0,1)\,.$$

Hence, it follows that $\|\omega(t)\|_\infty\le ct^{\frac\mu2-1}$, which achieves the zero limit for $t\to0$, provided that $\mu>2$.

□

Corollary 5. *Let (ω,π_ω) be the solution of Lemma 22. For $\overline\mu\in(0,\frac12)$, we get*

$$\begin{aligned}p\in(2,\infty),\ |\pi_\omega(t,x)|&\le c\|v_0\|_\infty t^{(\frac1p-1)(\frac12+\overline\mu)}\,,\ \textit{for all } (t,x)\in(0,1)\times\Omega\,,\\ |\pi_\omega(t,x)|&\le c\|v_0\|_\infty\,,\ \textit{for all } (t,x)\in[1,\infty)\times\Omega\,.\end{aligned} \tag{73}$$

Proof. The pressure field solves the equation (58) with boundary condition

$$\frac{d\pi_\omega}{d\nu}\ =-\big[\nabla\times\nabla\times\omega-\nabla\times\nabla\times U+\nabla\times\nabla\times F+\big]\cdot\nu\,,\ \text{on } \partial\Omega. \tag{74}$$

We justify (74). The normal component deduced from (67) is the following:

$$\big[\Delta\omega-\frac{\partial}{\partial t}(F-\mathfrak{h}\widehat{U}+W)+\Delta(F-\mathfrak{h}\widehat{U}+W)\big]\cdot\nu\,.$$

Since, on $(0,T)\times\partial\Omega$, we have $F=-U+\widehat{U}$ and $\mathfrak{h}=1$, we deduce

$$F_t-\mathfrak{h}\widehat{U}_t=-U_t=-\Delta U\,,\ \text{on}\ (0,T)\times\partial\Omega\,,$$

where we toke the equation of U into account. For all $t>0$, the field $W=0$ on $\partial\Omega$, hence $W_t=0$ holds. Since $\mathfrak{h}=1$ and $W=0$ hold in a neighborhood of $\partial\Omega$, we get $\Delta(-\mathfrak{h}\widehat{U}+W)=0$. Hence, being the fields ω, F and U divergence free, we arrive at (74).

Since, for $p\in(1,2)$ and $t>0$, $\nabla\pi_\omega\in L^p(\Omega)$, letting $|x|\to\infty$, we get $\pi_\omega(t,x)\to\overline{\pi}(t)$ (cf. [10] § II.5). On the other hand, the right hand side of (74) has integral zero on $\partial\Omega$. Hence we determine a solution $\widetilde{\pi}_\omega$ on Ω that, letting $|x|\to\infty$, tends to zero. Since the difference between π_ω and $\widetilde{\pi}_\omega$ on Ω is at most $\overline{\pi}(t)$, we consider as solution to equation (1) and, as matter of fact to the Neumann problem (74), the one which tends to zero. Since in the sequel there is no confusion, we denote $\widetilde{\pi}_\omega$ by the symbol π_ω. Hence, by virtue of *maximum principium* for harmonic solutions, we get

$$|\pi_\omega(t,x)|\le\max_{\Omega'}|\pi_\omega(t,x)|\,,\ \text{for all}\ (t,x)\in(0,\infty)\times\mathbb{R}^2-\Omega'\,.$$

By virtue estimate of Lemma 17, Lemma 18 and then (61), assuming $\Omega'\supset\Omega\cap\mathbb{B}_R$, for some $\overline{\mu}\in(0,\frac12)$, we get

$$\|\pi_\omega(t)\|_{C(\Omega')}\le c\|v_0\|_\infty(t^{\gamma_1}+t^{\gamma_2}+t^{\gamma_3})^{\frac12+\overline{\mu}}\,,\ \text{for all}\ t\in(0,1)\,.$$

In the last estimate, we consider $\gamma_1:=-1+\frac1p$ for $\Delta\omega$, thanks to $(68)_3$; $\gamma_2=\frac{\mu}{2}$ for U, deduced from $(10)_2$; $\gamma_3=-\frac12$ for F, deduced from (25) setting $k=0$ and $L=0$. Since the estimate with γ_1 holds for $p>2$, we arrive at $(73)_1$. Estimate $(73)_2$ is a consequence of the same previous estimates evaluated for $t>1$, so that, for the sake of brevity, we omit the details. □

5. Proof of Theorem 1

5.1. A First Result

We premise a result. We consider the initial boundary value problem (1) with initial datum $v_0\in L^\infty(\Omega)$ enjoying (2) and with $\operatorname{supp}v_0\subset\Omega-\mathbb{B}_{R+\delta}(0)$. We are going to prove Theorem 1 for this kind of initial data.

Theorem 4. *For all $v_0\in L^\infty(\Omega)$ enjoying (2) and with $\operatorname{supp}v_0\subset\Omega-\mathbb{B}_{R+\delta}(0)$, we get the existence of a solution (v,π) to problem (1) such that, for all $q\in(2,\infty)$,*

$$\begin{aligned}&\|v(t)\|_\infty+t\|v_t(t)\|_\infty\le c\|v_0\|_\infty\,,\ \textit{for all}\ t>0\,,\\&\frac{t^{\frac12}}{1+t^{\frac12}}\|\nabla v(t)\|_{L^q_{\ell oc}(\overline{\Omega})}+\frac{t}{1+t}\|\nabla\nabla v(t)\|_{L^q_{\ell oc}(\overline{\Omega})}\le c\|v_0\|_\infty\,,\ \textit{for all}\ t>0\,,\\&\textit{for all}\ R_0>0\,,\lim_{t\to0}\|v(t)-v_0\|_{L^q(\Omega\cap\mathbb{B}_{R_0})}=0\,.\end{aligned}\tag{75}$$

Finally, for the pressure field we have, for all $q\in(2,\infty)$,

$$\begin{aligned}&\frac{t}{t+1}\|\nabla\pi(t)\|_q\le ct^{\frac1q}\|v_0\|_\infty\,,\ \textit{for all}\ t>0\,,\\&\frac{t^{(1-\frac1q)(\frac12+\overline{\mu})}}{1+t^{(1-\frac1q)(\frac12+\overline{\mu})}}\|\pi(t)\|_\infty\le c\|v_0\|_\infty\,,\ \textit{for all}\ t>0\,.\end{aligned}\tag{76}$$

Proof. We consider the initial datum v_0 extended to 0 in $\mathbb{R}^2$. We define the pair (v, π) as

$$v := U - \mathfrak{h}\widehat{U} + F + W + \omega \ \text{ and } \ \pi := \pi_\omega\,, \tag{77}$$

where we set

- the field $U := H[v_0]$ is the solution to the Cauchy problem enjoying the properties stated in (8) and (10),
- the field $\widehat{U}$, defined in (24), is the mean integral on $\partial\Omega$ of U,
- for all $t > 0$, the field F is the extension from $\partial\Omega$ into Ω with compact support in $\overline{\Omega}$ and with value $-U + \widehat{U}$ on $\partial\Omega$, whose existence is ensured by Lemma 7,
- the field $\mathfrak{h}\widehat{U}$ is the product of $\mathfrak{h}$ defined by (29) and $\widehat{U}$,
- the field W is a solution to the Bogovskiĭ problem (33) given by Lemma 8 and, for $t > 0$, it enjoys the estimates of Lemma 9,
- the pair (ω, π_ω) is the solution to problem (67) furnished by Lemma 22.

By construction the pair (v, π) solves equation $(1)_1$. Moreover, the following estimate holds:

$$\|v(t)\|_\infty \le c\|v_0\|_\infty \ \text{ for all } t > 0, \tag{78}$$

which proves $(75)_1$ for v. For estimate (78) we check the L^∞-norm of each term which appears in (77) for the definition of v. From estimate of Lemma 1 we get $\|U(t) - \mathfrak{h}\widehat{U}(t)\|_\infty \le c\|v_0\|_\infty$, $t > 0$. From estimate (25), recalling that in our hypothesis we have $R < L$, via the embedding Sobolev theorem, we get $\|F(t)\|_\infty \le c\|v_0\|_\infty$, $t > 0$. For the last terms W and ω we recall estimate (39) and estimate $(68)_1$, respectively. Analogously, for $\|v_t(t)\|_\infty$ we employ the estimates related to the time derivative of each term. Hence, we consider (8) to estimate U_t, (31) and (8) to estimate $D_t(\mathfrak{h}\widehat{U})$, (25) to estimate F_t and, via the Gagliardo-Nirenberg inequality $\|W_t\|_\infty \le c\|\nabla W_t\|_q^a \|W_t\|_q^{1-a}$, $a = \frac{2}{q}$, employing $(39)_{2,3}$ we achieve completely the estimate $(75)_1$.

The initial value is assumed in the following sense:

$$\text{for all } q \in (2, \infty) \text{ and } R_0 > 0 \ \lim_{t\to 0} \|v(t) - v_0\|_{L^q(\mathbb{B}_{R_0})} = 0\,.$$

Actually, by virtue of the bounds in L^∞-norm stated in (44), choosing $\mu > 3$, we have zero value limit of F, $\mathfrak{h}\widehat{U}$ and W, and then of ω as in $(68)_4$. Instead, the limit property of U to v_0 is ensured by (12).

Finally, we point out that, for all $q \in (1, \infty)$ and $t > 0$,

$$v \in W^{2,q}_{\ell oc}(\overline{\Omega})\,,\ v_t \in L^q_{\ell oc}(\overline{\Omega}) \text{ and } \nabla\pi \in L^q(\Omega)\,. \tag{79}$$

The property (79) is a consequence of the special construction.

To prove the behavior in t claimed in $(75)_2$, we employ the estimates given in Sect. 2, and we take into account that at $t = 0$ the behavior of U is predominant over that of the other terms of (77). In contrast, for large t the behavior of ω becomes predominant over that of the remaining terms in (77).

The integrability on Ω of $\nabla\pi$ is a consequence of the fact that $\pi \equiv \pi_\omega$ and of Lemma 22. Instead, the estimate of π in L^∞-norm is a consequence of (73).

The theorem is completely proved. □

5.2. Proof of Theorem 1.

For the initial datum $v_0 \in L^\infty(\Omega)$ that enjoys (2), consider the following decomposition:

$$v_0 = v_0^c + v_{0c}\,,\ \text{ where } v_0^c := (1-g)v_0 + b_0 \text{ and } v_{0c} := gv_0 - b_0\,,$$

where, for $R > 3\,diam(\mathbb{R}^2 - \Omega)$, g is a non-negative smooth cutoff function such that $g = 1$ for $|x| \le R + \delta$ and $g = 0$ for $|x| \ge 2(R + \delta)$, $\delta > 0$, also b_0 is a solution to the Bogovskiĭ problem

$$\nabla \cdot b_0 = v_0 \cdot \nabla g \text{ in } \Omega\,,\ g = 0 \text{ on } \{|x| = R + \delta\} \cup \{|x| = 2(R + \delta)\}\,.$$

We tacitly consider v_0^c extended to zero on $\mathbb{R}^2-\operatorname{supp} v_0^c$, as well as v_{0c} extended to zero on $\mathbb{R}^2-\operatorname{supp} v_{0c}$. Being in particular $v_{0c}\in L^\infty(\Omega)\cap J^q(\Omega)$, by virtue of Corollary 4, there exists a solution (v_c,π_{v_c}) to problem (1) with initial datum v_{0c} enjoying estimates (65).

Since $v_0^c\in L^\infty(\Omega)$ verifies the hypotheses of Theorem 4, there exists a solution (v^c,π_{v^c}) to problem (1) with initial datum v_0^c enjoying estimates (75).

Thanks to the linearity of the Stokes system, considering $v:=v^c+v_c$ and $\pi:=\pi_{v^c}+\pi_{v_c}$, by (v,π) we solve the value boundary (1), and (v,π) enjoys estimates (3)-$(4)_1$ as a consequence of (65) and (75). The initial datum is achieved by means of $(4)_2$, which is a consequence of the fact that solution v_c enjoys the limit property from the L^q-theory, and solution v^c enjoys the limit property (75).

The limit property at $t=0$ allows us to claim that the equation of the divergence is satisfied in weak form up $t=0$.

We conclude the existence result claming that the pointwise estimate $(4)_3$ for the pressure field π is a consequence of the estimates (73)-(76), for π_{v_c} and π_{v^c}, respectively.

For the uniqueness of the solution we consider a pair (u,π_u) solution to the homogeneous initial boundary value problem (1), and enjoying properties (3)-$(4)_{1,3}$ and $(4)_2$ substituted by, for all $R_0>0$, $\lim_{t\to0}\|u(t)\|_{L^q(\Omega\cap B_{R_0})}=0$. The goal is to prove that $u=0$. Firstly, we prove that $u\in L^2(\Omega)$ for all $t>0$. Subsequently, we realize the uniqueness. For the first goal we are employing the so called weighted energy method (in this regard cf. [11]), that goes back to the first results of stability and uniqueness of solutions in L^∞-setting for the IBVP in unbounded domains. We multiply the Stokes equation of (u,π_u) by $\exp[-\alpha|x|]u(t,x)$, where $\alpha>0$, integrating on $(s,t)\times\Omega$, we get a weighted energy inequality:

$$\begin{aligned}\|u(t)\exp[-\frac{\alpha}{2}|x|]\|_2^2 &\le \|u(s)\exp[-\frac{\alpha}{2}|x|]\|_2^2+\alpha^2\int_s^t\|u\exp[-\frac{\alpha}{2}|x|]\|_2^2d\tau\\ &\quad+\alpha\int_s^t\|\pi_u\exp[-\frac{\alpha}{2}|x|]\|_2\|u\exp[-\frac{\alpha}{2}|x|]\|_2d\tau.\end{aligned}\tag{80}$$

Since for all $R_0>0$ we have $\lim_{s\to0}\|u(s)\|_{L^2(\Omega\cap\mathbb{B}_{R_0})}=0$, with no difficulty one deduces that the limit for $s\to0$ of the first term on the right hand side is null. Instead, thanks to estimate (3) for u and (4) for the pressure field, which furnishes $\|\pi_u\exp[-\frac{\alpha}{2}|x|]\|_2\le c\alpha^{-1}\|v_0\|_\infty t^{-\frac12-\overline{\mu}}$, both the integral terms admit limit in $s=0$. Hence, we arrive at

$$\|u(t)\exp[-\frac{\alpha}{2}|x|]\|_2^2\le\alpha^2\int_0^t\|u\exp[-\frac{\alpha}{2}|x|]\|_2^2d\tau+\alpha\int_0^t\|\pi_u\exp[-\frac{\alpha}{2}|x|]\|_2\|u\exp[-\frac{\alpha}{2}|x|]\|_2d\tau.\tag{81}$$

The validity of (80)-(81) allows us to apply Corollary 3 related to the Gronwall inequality that furnishes

$$\|u(t)\exp[-\frac{\alpha}{2}|x|]\|_2\le\exp\left[\alpha^2t\right]\alpha\int_0^t\|\pi_u\exp[-\frac{\alpha}{2}|x|]\|_2d\tau\,.$$

Recalling the above estimate for the pressure field, applying the Beppo-Levi theorem, we deduce

$$\|u(t)\|_2\le c\|v_0\|_\infty t^{\frac12-\overline{\mu}}\,,\ \text{for all } t>0\,.$$

This last easily leads to discuss the uniqueness in L^2-setting. The theorem is completely proved. □

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Declarations

Conflict of interest. The author declares that no conflict of interest occurs.

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