Homogenization and Low Mach Number Limit of Compressible Navier-Stokes Equations in Critically Perforated Domains

Peter Bella and Florian Oschmann

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Abstract. In this note, we consider the homogenization of the compressible Navier-Stokes equations in a periodically perforated domain in $\mathbb{R}^3$. Assuming that the particle size scales like ε^3, where $\varepsilon > 0$ is their mutual distance, and that the Mach number decreases fast enough, we show that in the limit $\varepsilon \to 0$, the velocity and density converge to a solution of the incompressible Navier-Stokes equations with Brinkman term. We strongly follow the methods of Höfer, Kowalczyk and Schwarzacher [https://doi.org/10.1142/S0218202521500391], where they proved convergence to Darcy's law for the particle size scaling like ε^α with $\alpha \in (1,3)$.

1. Introduction

We consider a bounded smooth domain $D \subset \mathbb{R}^3$ which for $\varepsilon > 0$ is perforated by tiny obstacles of size ε^3, and show that solutions to the compressible Navier-Stokes equations in this domain converge as $\varepsilon \to 0$ to a solution of the incompressible Navier-Stokes equations with Brinkman term. To the best of our knowledge, this is the first result of homogenization of compressible fluids for a critically sized perforation.

There is a vast of literature concerning the homogenization of fluid flows in perforated domains. We will just cite a few. For incompressible fluids, Allaire found in [2] and [3] that, concerning the ratios of particle size and distance, there are mainly three regimes of particle sizes ε^α, where $\alpha \geq 1$. Heuristically, if the particles are large, the velocity will slow down and finally stop. This phenomenon occurs if (in three dimensions) $\alpha \in [1,3)$ and gives rise to Darcy's law. When the particles are very small, i.e., $\alpha > 3$, they should not affect the fluid, yielding that in the limit, the fluid motion is still governed by the Stokes or Navier-Stokes equations. The third regime is the so-called critical case $\alpha = 3$, where the particles are large enough to put some friction on the fluid, but not too large to stop the flow. For incompressible fluids, the non-critical cases $\alpha \in (1,3)$ and $\alpha > 3$ were considered in [3], while [2] dealt with the critical case $\alpha = 3$. The case $\alpha = 1$ was treated in [1]. In all the aforementioned literature, the proofs were given by means of suitable oscillating test functions, first introduced by Tartar in [17] and later adopted by Cioranescu and Murat in [5] for the Poisson equation.

In the critical case, the additional friction term is the main part of Brinkman's law. Cioranescu and Murat considered in [5] the Poisson equation in a perforated domain, where they found in the limit "a strange term coming from nowhere". This Brinkman term purely comes from the presence of holes in the domain D_ε. It physically represents the energy of boundary layers around each obstacle, as its columns are proportional to the drag force around a single particle [2, Proposition 2.1.4 and Remark 2.1.5].

The assumptions on the distribution of the holes can also be generalized. For the critical case, Giunti, Höfer, and Velázquez considered in [11] homogenization of the Poisson equation in a randomly perforated

domain. They showed that the "strange term" also occurs in their setting. Hillairet considered in [12] the Stokes equations and random obstacles with a hard sphere condition. This condition was removed by Giunti and Höfer [10], where they showed that for incompressible fluids and randomly distributed holes with random radii, the randomness does not affect the convergence to Brinkman's law. More recently, for large particles, Giunti showed in [9] a similar convergence result to Darcy's law.

Unlike as for incompressible fluids, the homogenization theory for compressible fluids is rather sparse. Masmoudi considered in [15] the case $\alpha = 1$ of large particles, giving rise to Darcy's law. For large particles with $\alpha \in (1,3)$, Darcy's law was just recently treated in [13] for a low Mach number limit. The case of small particles ($\alpha > 3$) was treated in [6,7,14] for different growing conditions on the pressure. Random perforations in the spirit of [10] for small particles were considered by the authors in [4], where in the limit, the equations remain unchanged as in the periodic case.

We want to emphasize that the methods presented here are strongly related to those of [13]. As a matter of fact, their techniques used in the case of large holes also apply in our case for holes having critical size.

Notation: Throughout the whole paper, we denote the Frobenius scalar product of two matrices $A, B \in \mathbb{R}^{3 \times 3}$ by $A : B := \sum_{1 \leq i,j \leq 3} A_{ij} B_{ij}$. Further, we use the standard notation for Lebesgue and Sobolev spaces, where we denote this spaces even for vector valued functions as in scalar case, e.g., $L^p(D)$ instead of $L^p(D; \mathbb{R}^3)$. Moreover, $C > 0$ denotes a constant which is independent of ε and might change its value whenever it occurs.

Organization of the paper: The paper is organized as follows:
In Sect. 2, we give a precise definition of the perforated domain D_ε and state our main results for the steady Navier-Stokes equations. In Sect. 3, we introduce oscillating test functions, which will be crucial to show convergence of the velocity, density, and pressure. Sect. 4 is devoted to invoke the concept of Bogovskiǐ's operator as an inverse of the divergence, which is used to give uniform bounds independent of ε. In Sect. 5, we show how to pass to the limit $\varepsilon \to 0$ and obtain the limiting equations.

2. Setting and Main Results

Consider a bounded domain $D \subset \mathbb{R}^3$ with smooth boundary. Let $\varepsilon > 0$ and cover D with a regular mesh of size 2ε. Set $x_i^\varepsilon \in (2\varepsilon \mathbb{Z})^3$ as the center of the cell with index i and $P_i^\varepsilon := x_i^\varepsilon + (-\varepsilon, \varepsilon)^3$. Further, let $T \Subset B_1(0)$ be a compact and simply connected set with smooth boundary and set $T_i^\varepsilon := x_i^\varepsilon + \varepsilon^3 T$. We now define the perforated domain as

$$D_\varepsilon := D \setminus \bigcup_{i \in K_\varepsilon} T_i^\varepsilon, \quad K_\varepsilon := \{i : \overline{P_i^\varepsilon} \subset D\}. \tag{1}$$

By the periodic distribution of the holes, the number of holes inside D_ε satisfy

$$|K_\varepsilon| \leq C \frac{|D|}{\varepsilon^3} \quad \text{for some } C > 0 \text{ independent of } \varepsilon.$$

In D_ε, we consider the steady compressible Navier-Stokes equations

$$\begin{cases} \operatorname{div}(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) - \operatorname{div} \mathbb{S}(\nabla \mathbf{u}_\varepsilon) + \frac{1}{\varepsilon^\beta} \nabla \varrho_\varepsilon^\gamma = \varrho_\varepsilon \mathbf{f} + \mathbf{g} & \text{in } D_\varepsilon, \\ \operatorname{div}(\varrho_\varepsilon \mathbf{u}_\varepsilon) = 0 & \text{in } D_\varepsilon, \\ \mathbf{u}_\varepsilon = 0 & \text{on } \partial D_\varepsilon, \end{cases} \tag{2}$$

where $\varrho_\varepsilon, \mathbf{u}_\varepsilon$ are the fluids density and velocity, respectively, and $\mathbb{S}(\nabla \mathbf{u}_\varepsilon)$ is the Newtonian viscous stress tensor of the form

$$\mathbb{S}(\nabla \mathbf{u}) = \mu \left(\nabla \mathbf{u} + \nabla^T \mathbf{u} - \frac{2}{3} \operatorname{div}(\mathbf{u}) \mathbb{I} \right) + \eta \operatorname{div}(\mathbf{u}) \mathbb{I} \tag{3}$$

with viscosity coefficients $\mu > 0$, $\eta \geq 0$. Further, we assume that $\gamma \geq 3$, $\beta > 3(\gamma + 1)$, and $\mathbf{f}, \mathbf{g} \in L^\infty(D)$ are given. Since the equations (2) are invariant under adding a constant to the pressure term $\varepsilon^{-\beta}\varrho_\varepsilon^\gamma$, we define

$$p_\varepsilon := \varepsilon^{-\beta}(\varrho_\varepsilon^\gamma - \langle\varrho_\varepsilon^\gamma\rangle_\varepsilon), \tag{4}$$

where $\langle\cdot\rangle_\varepsilon$ denotes the mean value over D_ε, given by

$$\langle f\rangle_\varepsilon = \frac{1}{|D_\varepsilon|}\int_{D_\varepsilon} f\,\mathrm{d}x.$$

We will show convergence of the velocity $\mathbf{u}_\varepsilon$ and the pressure p_ε to limiting functions $\mathbf{u}$ and p, respectively, such that the couple $(\mathbf{u}, p)$ solves the incompressible steady Navier-Stokes-Brinkman equations

$$\begin{cases} \mathrm{div}(\varrho_0\mathbf{u} \otimes \mathbf{u}) - \mu\Delta\mathbf{u} + \nabla p + \mu M\mathbf{u} = \varrho_0\mathbf{f} + \mathbf{g} & \text{in } D, \\ \mathrm{div}(\mathbf{u}) = 0 & \text{in } D, \\ \mathbf{u} = 0 & \text{on } \partial D, \end{cases}$$

where the resistance matrix M is introduced in the next section, and the constant ϱ_0 is the strong limit of ϱ_ε in $L^{2\gamma}(D)$, which is determined by the mass constraint on ϱ_ε as formulated in Definition 2.1 below.

Before stating our main result, we introduce the standard concept of finite energy weak solutions to (2).

Definition 2.1. Let D_ε be as in (1) and $\gamma \geq 3$, $\mathfrak{m} > 0$ be fixed. We say a couple $(\varrho_\varepsilon, \mathbf{u}_\varepsilon)$ is a *finite energy weak solution* to system (2) if

$$\varrho_\varepsilon \in L^{2\gamma}(D_\varepsilon), \quad \mathbf{u}_\varepsilon \in W_0^{1,2}(D_\varepsilon),$$

$$\varrho_\varepsilon \geq 0 \text{ a.e. in } D_\varepsilon, \quad \int_{D_\varepsilon} \varrho_\varepsilon\,\mathrm{d}x = \mathfrak{m},$$

$$\int_{D_\varepsilon} \varrho_\varepsilon\mathbf{u}_\varepsilon \cdot \nabla\psi\,\mathrm{d}x = 0,$$

$$\int_{D_\varepsilon} p_\varepsilon\,\mathrm{div}\,\varphi + (\varrho_\varepsilon\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) : \nabla\varphi - \mathbb{S}(\nabla\mathbf{u}_\varepsilon) : \nabla\varphi + (\varrho_\varepsilon\mathbf{f} + \mathbf{g}) \cdot \varphi\,\mathrm{d}x = 0$$

for all test functions $\psi \in C_c^\infty(D_\varepsilon)$ and all test functions $\varphi \in C_c^\infty(D_\varepsilon; \mathbb{R}^3)$, where p_ε is given in (4), and the energy inequality

$$\int_{D_\varepsilon} \mathbb{S}(\nabla\mathbf{u}_\varepsilon) : \nabla\mathbf{u}_\varepsilon\,\mathrm{d}x \leq \int_{D_\varepsilon} (\varrho_\varepsilon\mathbf{f} + \mathbf{g}) \cdot \mathbf{u}_\varepsilon\,\mathrm{d}x \tag{5}$$

holds.

Remark 2.2. Existence of finite energy weak solutions to system (2) is known for all values $\gamma > 3/2$; see, for instance, [16, Theorem 4.3]. However, we need the assumption $\gamma \geq 3$ to bound the convective term $\mathrm{div}(\varrho_\varepsilon\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon)$ in a proper way, see Sect. 4.

Let us denote the zero extension of a function f with D_ε as its domain of definition by $\tilde{f}$, that is,

$$\tilde{f} = f \text{ in } D_\varepsilon, \quad \tilde{f} = 0 \text{ in } \mathbb{R}^3 \setminus D_\varepsilon.$$

Our main result for the stationary Navier-Stokes equations now reads as follows:

Theorem 2.3. *Let $D \subset \mathbb{R}^3$ be a bounded domain with smooth boundary, $0 < \varepsilon < 1$, D_ε be as in (1), $\gamma \geq 3$, $\mathfrak{m} > 0$ and $\mathbf{f}, \mathbf{g} \in L^\infty(D)$. Let $\beta > 3(\gamma + 1)$ and $(\varrho_\varepsilon, \mathbf{u}_\varepsilon)$ be a sequence of finite energy weak solutions to problem (2). Then, with p_ε defined in (4), we can extract subsequences (not relabeled) such that*

$$\tilde{\varrho}_\varepsilon \to \varrho_0 \quad \textit{strongly in } L^{2\gamma}(D),$$

$$\tilde{p}_\varepsilon \rightharpoonup p \quad \textit{weakly in } L^2(D),$$

$$\tilde{\mathbf{u}}_\varepsilon \rightharpoonup \mathbf{u} \quad \textit{weakly in } W_0^{1,2}(D),$$
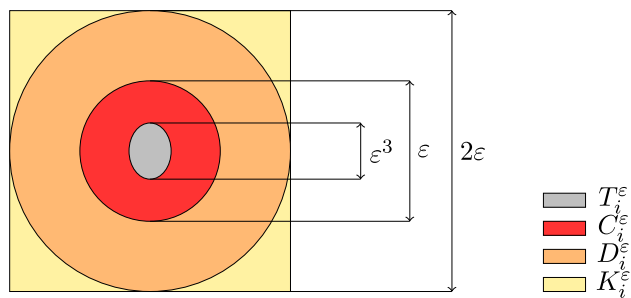
FIG. 1. Splitting of the cell P_i^ε

where $\varrho_0 = \mathfrak{m}/|D|$ *is constant and* $(p, \mathbf{u}) \in L^2(D) \times W_0^{1,2}(D)$ *with* $\int_D p = 0$ *is a weak solution to the steady incompressible Navier-Stokes-Brinkman equations*

$$\begin{cases} \mathrm{div}(\varrho_0 \mathbf{u} \otimes \mathbf{u}) + \nabla p - \mu \Delta \mathbf{u} + \mu M \mathbf{u} = \varrho_0 \mathbf{f} + \mathbf{g} & in\ D, \\ \mathrm{div}(\mathbf{u}) = 0 & in\ D, \\ \mathbf{u} = 0 & on\ \partial D, \end{cases} \qquad (6)$$

where M will be defined in (11).

Remark 2.4. It it well known that the solution to system (6) is unique if $\mathbf{f}$ and $\mathbf{g}$ are "sufficiently small", see, e.g., [18, Chapter II, Theorem 1.3]. This smallness assumption can be dropped in the case of Stokes equations, i.e., without the convective term $\mathrm{div}(\varrho_0 \mathbf{u} \otimes \mathbf{u})$.

3. The Cell Problem and Oscillating Test Functions

In this section, we introduce oscillating test functions and define the resistance matrix M, following the original work of Allaire [2]. We repeat here the definition of these functions as well as the estimates given in [13].

Consider for a single particle T the solution $(q_k, \mathbf{w}_k)$ to the cell problem

$$\begin{cases} \nabla q_k - \Delta \mathbf{w}_k = 0 & in\ \mathbb{R}^3 \setminus T, \\ \mathrm{div}(\mathbf{w}_k) = 0 & in\ \mathbb{R}^3 \setminus T, \\ \mathbf{w}_k = 0 & on\ \partial T, \\ \mathbf{w}_k = \mathbf{e}_k & at\ infinity, \end{cases} \qquad (7)$$

where $\mathbf{e}_k$ is the k-th unit basis vector of the canonical basis of $\mathbb{R}^3$. Note that the solution exists and is unique, see, e.g., [8, Chapter V]. Let us further recall the definition of oscillating test functions as made in [2] (see also [13]):

We set

$$\mathbf{w}_k^\varepsilon = \mathbf{e}_k, \quad q_k^\varepsilon = 0 \text{ in } P_i^\varepsilon \cap D$$

for each P_i^ε with $P_i^\varepsilon \cap \partial D \neq \emptyset$. Now, we denote $B_i^r := B_r(x_i^\varepsilon)$ and split each cell P_i^ε entirely included in D into the following four parts:

$$\overline{P_i^\varepsilon} = T_i^\varepsilon \cup \overline{C_i^\varepsilon} \cup \overline{D_i^\varepsilon} \cup \overline{K_i^\varepsilon},$$

where C_i^ε is the open ball centered at x_i^ε with radius $\varepsilon/2$ and perforated by the hole T_i^ε, $D_i^\varepsilon = B_i^\varepsilon \setminus B_i^{\varepsilon/2}$ is the ball with radius ε perforated by the ball with radius $\varepsilon/2$, and $K_i^\varepsilon = P_i^\varepsilon \setminus B_i^\varepsilon$ are the remaining corners, see Fig. 1.

In these parts, we define

$$\begin{cases} \mathbf{w}_k^\varepsilon = \mathbf{e}_k \\ q_k^\varepsilon = 0 \end{cases} \text{in } K_i^\varepsilon, \qquad\qquad \begin{cases} \nabla q_k^\varepsilon - \Delta \mathbf{w}_k^\varepsilon = 0 \\ \text{div}(\mathbf{w}_k^\varepsilon) = 0 \end{cases} \text{in } D_i^\varepsilon,$$

$$\begin{cases} \mathbf{w}_k^\varepsilon(x) = \mathbf{w}_k\left(\frac{x}{\varepsilon^3}\right) \\ q_k^\varepsilon(x) = \frac{1}{\varepsilon^3} q_k\left(\frac{x}{\varepsilon^3}\right) \end{cases} \text{in } C_i^\varepsilon, \qquad\qquad \begin{cases} \mathbf{w}_k^\varepsilon = 0 \\ q_k^\varepsilon = 0 \end{cases} \text{in } T_i^\varepsilon,$$

where we impose matching Dirichlet boundary conditions and $(q_k, \mathbf{w}_k)$ is the solution to the cell problem (7). As shown in [13, Lemma 3.5], we have for the functions $(q_k^\varepsilon, \mathbf{w}_k^\varepsilon)$ and all $p > \frac{3}{2}$ the estimates

$$\|\nabla \mathbf{w}_k^\varepsilon\|_{L^p(D)} + \|q_k^\varepsilon\|_{L^p(D)} \leq C\varepsilon^{3\left(\frac{2}{p}-1\right)}, \tag{8}$$

$$\|\nabla q_k^\varepsilon\|_{L^p(\cup_i C_i^\varepsilon)} \leq C\varepsilon^{6\left(\frac{1}{p}-1\right)}, \tag{9}$$

$$\|\nabla \mathbf{w}_k^\varepsilon\|_{L^2(\cup_i B_i^\varepsilon \setminus B_i^{\varepsilon/4})} + \|q_k^\varepsilon\|_{L^2(\cup_i B_i^\varepsilon \setminus B_i^{\varepsilon/4})} \leq C\varepsilon, \tag{10}$$

where the constant $C > 0$ does not depend on ε. Moreover, we have the following Theorem due to Allaire:

Theorem 3.1 ([2, page 214, Proposition 1.1.2 and Lemma 2.3.6]). *The functions* $(q_k^\varepsilon, \mathbf{w}_k^\varepsilon)$ *fulfill:*

(H1) $q_k^\varepsilon \in L^2(D), \quad \mathbf{w}_k^\varepsilon \in W^{1,2}(D)$;

(H2) $\text{div } \mathbf{w}_k^\varepsilon = 0$ *in* D *and* $\mathbf{w}_k^\varepsilon = 0$ *on the holes* T_i^ε;

(H3) $\mathbf{w}_k^\varepsilon \rightharpoonup \mathbf{e}_k$ *in* $W^{1,2}(D), \quad q_k^\varepsilon \rightharpoonup 0$ *in* $L^2(D)/\mathbb{R}$;

(H4) *For any* $\nu_\varepsilon, \nu \in W^{1,2}(D)$ *with* $\nu_\varepsilon = 0$ *on the holes* T_i^ε *and* $\nu_\varepsilon \rightharpoonup \nu$, *and any* $\varphi \in \mathcal{D}(D)$, *we have*

$$\langle \nabla q_k^\varepsilon - \Delta \mathbf{w}_k^\varepsilon, \varphi \nu_\varepsilon \rangle_{W^{-1,2}(D), W_0^{1,2}(D)} \to \langle M\mathbf{e}_k, \varphi \nu \rangle_{W^{-1,2}(D), W_0^{1,2}(D)},$$

where the resistance matrix $M \in W^{-1,\infty}(D)$ *is defined by its entries* M_{ik} *via*

$$\langle M_{ik}, \varphi \rangle_{\mathcal{D}'(D), \mathcal{D}(D)} = \lim_{\varepsilon \to 0} \int_D \varphi \nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_k^\varepsilon \, dx \tag{11}$$

for any test function $\varphi \in \mathcal{D}(D)$.

Further, for any $p \geq 1$,

$$\|\mathbf{w}_k^\varepsilon - \mathbf{e}_k\|_{L^p(D)} \to 0.$$

Remark 3.2. This definition of M yields that the matrix is symmetric and positive definite in the sense that for all test functions $\varphi_i \in \mathcal{D}(D)$ and $\Phi = (\varphi_i)_{1 \leq i \leq 3}$,

$$\langle M\Phi, \Phi \rangle_{\mathcal{D}'(D), \mathcal{D}(D)} = \lim_{\varepsilon \to 0} \int_D \left| \sum_{i=1}^3 \varphi_i \nabla \mathbf{w}_i^\varepsilon \right|^2 dx \geq 0,$$

thus implying that there exists at least one solution to system (6).

4. Bogovskiĭ's Operator and Uniform Bounds for the Navier-Stokes Equations

As in [6], we have the following result for the inverse of the divergence operator:

Theorem 4.1 ([6, Theorem 2.3]). *Let* $1 < q < \infty$ *and* D_ε *be defined as in (1). There exists a bounded linear operator*

$$\mathcal{B}_\varepsilon : \left\{ f \in L^q(D_\varepsilon) : \int_{D_\varepsilon} f \, dx = 0 \right\} \to W_0^{1,q}(D_\varepsilon)$$

such that for any $f \in L^q(D_\varepsilon)$ *with* $\int_{D_\varepsilon} f \, dx = 0$,

$$\text{div } \mathcal{B}_\varepsilon(f) = f \text{ in } D_\varepsilon, \quad \|\mathcal{B}_\varepsilon(f)\|_{W_0^{1,q}(D_\varepsilon)} \leq C\left(1 + \varepsilon^{3\left(\frac{2}{q}-1\right)}\right) \|f\|_{L^q(D_\varepsilon)},$$

where the constant $C > 0$ does not depend on ε.

We will use this result to bound the pressure p_ε by the density ϱ_ε. Since the main ideas how to get uniform bounds on $\mathbf{u}_\varepsilon$, ϱ_ε, and p_ε are given in [13], we just sketch the proof in our case. First, by Korn's inequality and (5), we find

$$\mu\|\nabla\mathbf{u}_\varepsilon\|_{L^2(D_\varepsilon)}^2 \leq \|\varrho_\varepsilon\|_{L^{\frac{6}{5}}(D_\varepsilon)}\|\mathbf{u}_\varepsilon\|_{L^6(D_\varepsilon)}\|\mathbf{f}\|_{L^\infty(D)} + \|\mathbf{g}\|_{L^\infty(D)}\|\mathbf{u}_\varepsilon\|_{L^1(D_\varepsilon)}.$$

Together with Sobolev embedding, we obtain

$$\|\mathbf{u}_\varepsilon\|_{L^6(D_\varepsilon)} \leq C\|\nabla\mathbf{u}_\varepsilon\|_{L^2(D_\varepsilon)},$$

which yields

$$\|\mathbf{u}_\varepsilon\|_{L^6(D_\varepsilon)} + \|\nabla\mathbf{u}_\varepsilon\|_{L^2(D_\varepsilon)} \leq C(\|\varrho_\varepsilon\|_{L^{\frac{6}{5}}(D_\varepsilon)} + 1). \tag{12}$$

To get uniform bounds on the velocity, we first have to estimate the density. To this end, let $\mathcal{B}_\varepsilon$ be as in Theorem 4.1. Testing the first equation in (2) with $\mathcal{B}_\varepsilon(p_\varepsilon) \in W_0^{1,2}(D_\varepsilon)$ yields

$$\|p_\varepsilon\|_{L^2(D_\varepsilon)}^2 = \int_{D_\varepsilon} p_\varepsilon \operatorname{div}\mathcal{B}_\varepsilon(p_\varepsilon)\,\mathrm{d}x$$

$$= \int_{D_\varepsilon} \mathbb{S}(\nabla\mathbf{u}_\varepsilon):\nabla\mathcal{B}_\varepsilon(p_\varepsilon) - (\varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon):\nabla\mathcal{B}_\varepsilon(p_\varepsilon) - (\varrho_\varepsilon\mathbf{f} + \mathbf{g})\cdot\mathcal{B}_\varepsilon(p_\varepsilon)\,\mathrm{d}x.$$

Recalling $\varrho_\varepsilon \in L^{2\gamma}(D_\varepsilon)$ and $\gamma \geq 3$, this leads to

$$\|p_\varepsilon\|_{L^2(D_\varepsilon)}^2 \leq C(\|\nabla\mathbf{u}_\varepsilon\|_{L^2(D_\varepsilon)} + \|\varrho_\varepsilon\|_{L^6(D_\varepsilon)}\|\mathbf{u}_\varepsilon\|_{L^6(D_\varepsilon)}^2)\|\nabla\mathcal{B}_\varepsilon(p_\varepsilon)\|_{L^2(D_\varepsilon)}$$

$$+ C\big(\|\mathbf{f}\|_{L^\infty(D_\varepsilon)}\|\varrho_\varepsilon\|_{L^{2\gamma}(D_\varepsilon)} + \|\mathbf{g}\|_{L^\infty(D_\varepsilon)}\big)\|\mathcal{B}_\varepsilon(p_\varepsilon)\|_{L^2(D_\varepsilon)}$$

$$\overset{(12)}{\leq} C(\|\varrho_\varepsilon\|_{L^{\frac{6}{5}}(D_\varepsilon)} + 1 + \|\varrho_\varepsilon\|_{L^6(D_\varepsilon)}(\|\varrho_\varepsilon\|_{L^{\frac{6}{5}}(D_\varepsilon)}^2 + 1))\|\nabla\mathcal{B}_\varepsilon(p_\varepsilon)\|_{L^2(D_\varepsilon)}$$

$$+ C(\|\varrho_\varepsilon\|_{L^{2\gamma}(D_\varepsilon)} + 1)\|\mathcal{B}_\varepsilon(p_\varepsilon)\|_{L^2(D_\varepsilon)}$$

$$\leq C(\|\varrho_\varepsilon\|_{L^{2\gamma}(D_\varepsilon)} + \|\varrho_\varepsilon\|_{L^6(D_\varepsilon)}\|\varrho_\varepsilon\|_{L^{\frac{6}{5}}(D_\varepsilon)}^2 + 1)\|\mathcal{B}_\varepsilon(p_\varepsilon)\|_{W_0^{1,2}(D_\varepsilon)}$$

$$\leq C(\|\varrho_\varepsilon\|_{L^{2\gamma}(D_\varepsilon)} + \|\varrho_\varepsilon\|_{L^{2\gamma}(D_\varepsilon)}^3 + 1)\|\mathcal{B}_\varepsilon(p_\varepsilon)\|_{W_0^{1,2}(D_\varepsilon)}$$

$$\leq C(\|\varrho_\varepsilon\|_{L^{2\gamma}(D_\varepsilon)} + \|\varrho_\varepsilon\|_{L^{2\gamma}(D_\varepsilon)}^3 + 1)\|p_\varepsilon\|_{L^2(D_\varepsilon)},$$

that is,

$$\|p_\varepsilon\|_{L^2(D_\varepsilon)} \leq C(\|\varrho_\varepsilon\|_{L^{2\gamma}(D_\varepsilon)} + \|\varrho_\varepsilon\|_{L^{2\gamma}(D_\varepsilon)}^3 + 1). \tag{13}$$

Further, we have

$$\langle\varrho_\varepsilon\rangle_\varepsilon = \frac{1}{|D_\varepsilon|}\int_{D_\varepsilon}\varrho_\varepsilon\,\mathrm{d}x = \frac{\mathfrak{m}}{|D_\varepsilon|}$$

and

$$\frac{1}{\varepsilon^\beta}\|\varrho_\varepsilon^\gamma - \langle\varrho_\varepsilon\rangle_\varepsilon^\gamma\|_{L^2(D_\varepsilon)} \leq \frac{C}{\varepsilon^\beta}\|\varrho_\varepsilon^\gamma - \langle\varrho_\varepsilon^\gamma\rangle_\varepsilon\|_{L^2(D_\varepsilon)} \overset{(4)}{=} C\|p_\varepsilon\|_{L^2(D_\varepsilon)},$$

see [13, Section 3.3 and inequality (4.7)]. This yields

$$\frac{1}{\varepsilon^\beta}\|\varrho_\varepsilon^\gamma - \langle\varrho_\varepsilon\rangle_\varepsilon^\gamma\|_{L^2(D_\varepsilon)} \leq C\|p_\varepsilon\|_{L^2(D_\varepsilon)} \leq C(\|\varrho_\varepsilon\|_{L^{2\gamma}(D_\varepsilon)} + \|\varrho_\varepsilon\|_{L^{2\gamma}(D_\varepsilon)}^3 + 1)$$

$$\leq C\left(\|\varrho_\varepsilon^\gamma - \langle\varrho_\varepsilon\rangle_\varepsilon^\gamma\|_{L^2(D_\varepsilon)}^{\frac{1}{\gamma}} + \frac{\mathfrak{m}}{|D_\varepsilon|^{1-1/(2\gamma)}} + \|\varrho_\varepsilon^\gamma - \langle\varrho_\varepsilon\rangle_\varepsilon^\gamma\|_{L^2(D_\varepsilon)}^{\frac{3}{\gamma}} + \frac{\mathfrak{m}^3}{|D_\varepsilon|^{3-3/(2\gamma)}} + 1\right).$$

Together with

$$ab^{\frac{1}{p}} \leq b + a^{p'} \quad \forall a, b > 0,\ \frac{1}{p} + \frac{1}{p'} = 1,$$

which is a consequence of Young's inequality, we obtain, using $\gamma \geq 3$ and the fact that we may assume $\varepsilon \leq 1$ small enough,

$$
\frac{1}{\varepsilon^\beta}\|\varrho_\varepsilon^\gamma - \langle\varrho_\varepsilon\rangle_\varepsilon^\gamma\|_{L^2(D_\varepsilon)} \leq \frac{1}{4\varepsilon^\beta}\|\varrho_\varepsilon^\gamma - \langle\varrho_\varepsilon\rangle_\varepsilon^\gamma\|_{L^2(D_\varepsilon)} + C + \frac{1}{4\varepsilon^\beta}\|\varrho_\varepsilon^\gamma - \langle\varrho_\varepsilon\rangle_\varepsilon^\gamma\|_{L^2(D_\varepsilon)} + C'
$$
$$
= \frac{1}{2\varepsilon^\beta}\|\varrho_\varepsilon^\gamma - \langle\varrho_\varepsilon\rangle_\varepsilon^\gamma\|_{L^2(D_\varepsilon)} + C.
$$

Using that $|\varrho_\varepsilon - \langle\varrho_\varepsilon\rangle_\varepsilon|^\gamma \leq |\varrho_\varepsilon^\gamma - \langle\varrho_\varepsilon\rangle_\varepsilon^\gamma|$, which is a consequence of the triangle inequality for the metric $d(a,b) = |a-b|^{\frac{1}{\gamma}}$ for $\gamma \geq 1$, we conclude

$$
\frac{1}{\varepsilon^\beta}\|\varrho_\varepsilon - \langle\varrho_\varepsilon\rangle_\varepsilon\|^\gamma_{L^{2\gamma}(D_\varepsilon)} \leq \frac{1}{\varepsilon^\beta}\|\varrho_\varepsilon^\gamma - \langle\varrho_\varepsilon\rangle_\varepsilon^\gamma\|_{L^2(D_\varepsilon)} \leq C,
$$

which further gives rise to

$$
\|\varrho_\varepsilon\|_{L^{2\gamma}(D_\varepsilon)} \leq \|\varrho_\varepsilon - \langle\varrho_\varepsilon\rangle_\varepsilon\|_{L^{2\gamma}(D_\varepsilon)} + C\langle\varrho_\varepsilon\rangle_\varepsilon \leq C.
$$

In view of (12) and (13), we finally establish

$$
\begin{aligned}
\|\mathbf{u}_\varepsilon\|_{W_0^{1,2}(D_\varepsilon)} &\leq C, \\
\|\varrho_\varepsilon\|_{L^{2\gamma}(D_\varepsilon)} &\leq C, \\
\|p_\varepsilon\|_{L^2(D_\varepsilon)} &\leq C, \\
\|\varrho_\varepsilon - \langle\varrho_\varepsilon\rangle_\varepsilon\|_{L^{2\gamma}(D_\varepsilon)} &\leq C\varepsilon^{\frac{\beta}{\gamma}}
\end{aligned}
\tag{14}
$$

for some constant $C > 0$ independent of ε.

5. Convergence Proof

The proof of convergence we give here is essentially the same as in [13]. We thus just sketch the steps done there while highlighting the differences.

Proof of Theorem 2.3. Step 1: Recall that, for a function f defined on D_ε, we denote by $\tilde{f}$ its zero prolongation to $\mathbb{R}^3$. By the uniform estimates (14), we can extract subsequences (not relabeled) such that

$$
\tilde{\mathbf{u}}_\varepsilon \rightharpoonup \mathbf{u} \text{ weakly in } W_0^{1,2}(D),
$$
$$
\tilde{p}_\varepsilon \rightharpoonup p \text{ weakly in } L^2(D),
$$
$$
\tilde{\varrho}_\varepsilon \to \varrho_0 \text{ strongly in } L^{2\gamma}(D),
$$

where $\varrho_0 = \mathfrak{m}/|D| > 0$ is constant. The strong convergence of the density is obtained by

$$
\|\tilde{\varrho}_\varepsilon - \varrho_0\|_{L^{2\gamma}(D)} \leq \|\varrho_0\|_{L^{2\gamma}(D\setminus D_\varepsilon)} + \|\varrho_\varepsilon - \langle\varrho_\varepsilon\rangle_\varepsilon\|_{L^{2\gamma}(D_\varepsilon)} + \|\langle\varrho_\varepsilon\rangle_\varepsilon - \varrho_0\|_{L^{2\gamma}(D_\varepsilon)}
$$
$$
\leq \varrho_0|D\setminus D_\varepsilon|^{\frac{1}{2\gamma}} + C\varepsilon^{\frac{\beta}{\gamma}} + \mathfrak{m}|D_\varepsilon|^{\frac{1}{2\gamma}}\left(\frac{1}{|D_\varepsilon|} - \frac{1}{|D|}\right) \to 0,
$$

since $|D_\varepsilon| \to |D|$. Due to Rellich's theorem, we further have

$$
\tilde{\mathbf{u}}_\varepsilon \to \mathbf{u} \text{ strongly in } L^q(D) \text{ for all } 1 \leq q < 6.
$$

Step 2: We begin by proving that the limiting velocity $\mathbf{u}$ is solenoidal. To this end, let $\varphi \in \mathcal{D}(\mathbb{R}^3)$. By the second equation of (2), we have

$$
0 = \int_{\mathbb{R}^3} \tilde{\varrho}_\varepsilon\tilde{\mathbf{u}}_\varepsilon \cdot \nabla\varphi \, \mathrm{d}x \to \varrho_0 \int_D \mathbf{u} \cdot \nabla\varphi \, \mathrm{d}x.
$$

This together with the compactness of the trace operator yields

$$
\begin{cases}
\operatorname{div} \mathbf{u} = 0 & \text{in } D, \\
\mathbf{u} = 0 & \text{on } \partial D.
\end{cases}
\tag{15}
$$

Step 3: To prove convergence of the momentum equation, let $\varphi \in \mathcal{D}(D)$ and use $\varphi \mathbf{w}_k^\varepsilon$ as test function in the first equation of (2). This yields

$$\int_D \mathbb{S}(\nabla \tilde{\mathbf{u}}_\varepsilon) : \nabla(\varphi \mathbf{w}_k^\varepsilon) \mathrm{d}x = \int_D (\tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon \otimes \tilde{\mathbf{u}}_\varepsilon) : \nabla(\varphi \mathbf{w}_k^\varepsilon) \mathrm{d}x + \int_D \tilde{p}_\varepsilon \operatorname{div}(\varphi \mathbf{w}_k^\varepsilon) \mathrm{d}x + \int_D (\tilde{\varrho}_\varepsilon \mathbf{f} + \mathbf{g}) \cdot (\varphi \mathbf{w}_k^\varepsilon) \mathrm{d}x.$$

Using the definition of $\mathbb{S}$ in (3) and the fact that $\operatorname{div}(\mathbf{w}_k^\varepsilon) = 0$ by (H2) of Theorem 3.1, we rewrite the left hand side as

$$\int_D \mathbb{S}(\nabla \tilde{\mathbf{u}}_\varepsilon) : \nabla(\varphi \mathbf{w}_k^\varepsilon) \,\mathrm{d}x = \mu \int_D \nabla \tilde{\mathbf{u}}_\varepsilon : \nabla(\varphi \mathbf{w}_k^\varepsilon) \,\mathrm{d}x + \left(\frac{\mu}{3} + \eta\right) \int_D \operatorname{div}(\tilde{\mathbf{u}}_\varepsilon) \operatorname{div}(\varphi \mathbf{w}_k^\varepsilon) \,\mathrm{d}x$$

$$= \mu \int_D \nabla \mathbf{w}_k^\varepsilon : \nabla(\varphi \tilde{\mathbf{u}}_\varepsilon) + \nabla \tilde{\mathbf{u}}_\varepsilon : (\mathbf{w}_k^\varepsilon \otimes \nabla \varphi) - \nabla \mathbf{w}_k^\varepsilon : (\tilde{\mathbf{u}}_\varepsilon \otimes \nabla \varphi) \,\mathrm{d}x + \left(\frac{\mu}{3} + \eta\right) \int_D \operatorname{div}(\tilde{\mathbf{u}}_\varepsilon) \mathbf{w}_k^\varepsilon \cdot \nabla \varphi \,\mathrm{d}x$$

and add the term $-\int_D q_k^\varepsilon \operatorname{div}(\varphi \tilde{\mathbf{u}}_\varepsilon) \,\mathrm{d}x$ to both sides to obtain

$$\underbrace{\mu \int_D \nabla \mathbf{w}_k^\varepsilon : \nabla(\varphi \tilde{\mathbf{u}}_\varepsilon) - q_k^\varepsilon \operatorname{div}(\varphi \tilde{\mathbf{u}}_\varepsilon) \,\mathrm{d}x}_{I_1}$$

$$+ \underbrace{\mu \int_D \nabla \tilde{\mathbf{u}}_\varepsilon : (\mathbf{w}_k^\varepsilon \otimes \nabla \varphi) - \nabla \mathbf{w}_k^\varepsilon : (\tilde{\mathbf{u}}_\varepsilon \otimes \nabla \varphi) \,\mathrm{d}x}_{I_2} + \underbrace{\left(\frac{\mu}{3} + \eta\right) \int_D \operatorname{div}(\tilde{\mathbf{u}}_\varepsilon) \mathbf{w}_k^\varepsilon \cdot \nabla \varphi \,\mathrm{d}x}_{I_3}$$

$$= \underbrace{\int_D (\tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon \otimes \tilde{\mathbf{u}}_\varepsilon) : \nabla(\varphi \mathbf{w}_k^\varepsilon) \,\mathrm{d}x}_{I_4} + \underbrace{\int_D \tilde{p}_\varepsilon \mathbf{w}_k^\varepsilon \cdot \nabla \varphi + (\tilde{\varrho}_\varepsilon \mathbf{f} + \mathbf{g}) \cdot (\varphi \mathbf{w}_k^\varepsilon) \,\mathrm{d}x}_{I_5} - \underbrace{\int_D q_k^\varepsilon \operatorname{div}(\varphi \tilde{\mathbf{u}}_\varepsilon) \,\mathrm{d}x}_{I_6}.$$

Since $\nu_\varepsilon := \tilde{\mathbf{u}}_\varepsilon$ and $\nu := \mathbf{u}$ fulfill hypothesis (H4) of Theorem 3.1, we have

$$I_1 \to \mu \langle M \mathbf{e}_k, \varphi \mathbf{u}\rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the dual product of $W^{-1,2}(D)$ and $W_0^{1,2}(D)$. Further, by $\tilde{\mathbf{u}}_\varepsilon \to \mathbf{u}$ strongly in $L^2(D)$ and $\nabla \mathbf{w}_k^\varepsilon \rightharpoonup 0$ by hypothesis (H3),

$$I_2 \to \mu \int_D \nabla \mathbf{u} : (\mathbf{e}_k \otimes \nabla \varphi) \,\mathrm{d}x.$$

Because of $\mathbf{w}_k^\varepsilon \to \mathbf{e}_k$ strongly in $L^2(D)$ and (15), we deduce

$$I_3 \to 0, \quad I_5 \to \int_D p \, \mathbf{e}_k \cdot \nabla \varphi + (\varrho_0 \mathbf{f} + \mathbf{g}) \cdot (\varphi \mathbf{e}_k) \,\mathrm{d}x.$$

Step 4: To show convergence of I_4, we proceed as follows. First, since $\mathbf{u}_\varepsilon = 0$ on ∂D_ε and $\tilde{\mathbf{u}}_\varepsilon \rightharpoonup \mathbf{u}$ in $W^{1,2}(D)$, we have $\widetilde{\nabla \mathbf{u}_\varepsilon} = \nabla \tilde{\mathbf{u}}_\varepsilon \rightharpoonup \nabla \mathbf{u}$ in $L^2(D)$. Second, as shown above for $\gamma \geq 3$, $\tilde{\varrho}_\varepsilon \to \varrho_0$ strongly in $L^{2\gamma}(D)$ and $\tilde{\mathbf{u}}_\varepsilon \to \mathbf{u}$ strongly in $L^q(D)$ for any $1 \leq q < 6$, in particular in $L^4(D)$. Together with the strong convergence of $\mathbf{w}_k^\varepsilon$ in any $L^p(D)$ (see Theorem 3.1), in particular in $L^{12}(D)$, we get

$$\tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon \otimes \mathbf{w}_k^\varepsilon \to \varrho_0 \mathbf{u} \otimes \mathbf{e}_k \text{ strongly in } L^2(D).$$

This together with $\operatorname{div}(\varrho_\varepsilon \mathbf{u}_\varepsilon) = 0$ yields

$$I_4 = \int_{D_\varepsilon} (\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) : \nabla(\varphi \mathbf{w}_k^\varepsilon) \,\mathrm{d}x = -\int_{D_\varepsilon} \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla \mathbf{u}_\varepsilon \cdot \varphi \mathbf{w}_k^\varepsilon \,\mathrm{d}x = -\int_{D_\varepsilon} \varphi \nabla \mathbf{u}_\varepsilon : (\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{w}_k^\varepsilon) \,\mathrm{d}x$$

$$= -\int_D \varphi \nabla \tilde{\mathbf{u}}_\varepsilon : (\tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon \otimes \mathbf{w}_k^\varepsilon) \,\mathrm{d}x \to -\int_D \varphi \nabla \mathbf{u} : (\varrho_0 \mathbf{u} \otimes \mathbf{e}_k) \,\mathrm{d}x = \int_D (\varrho_0 \mathbf{u} \otimes \mathbf{u}) : \nabla(\varphi \mathbf{e}_k) \,\mathrm{d}x.$$

In the case $\gamma > 3$, one can also proceed by seeing that

$$\tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon \otimes \tilde{\mathbf{u}}_\varepsilon \to \varrho_0 \mathbf{u} \otimes \mathbf{u} \text{ strongly in } L^2(D),$$

where we used that $\tilde{\mathbf{u}}_\varepsilon \to \mathbf{u}$ strongly in $L^q(D)$ for $q = 4\gamma/(\gamma - 1) < 6$.

Step 5: It remains to show convergence of I_6. First, recall $B_i^r = B_r(x_i^\varepsilon)$. We follow the idea of [13] and introduce a further splitting of the integral:

Let $\psi \in C_c^\infty(B_{1/2}(0))$ be a cut-off function with $\psi = 1$ on $B_{1/4}(0)$, define for $x \in B_i^{\varepsilon/2}$ the function $\psi_\varepsilon^i(x) := \psi((x - x_i^\varepsilon)/\varepsilon)$, and extend ψ_ε^i by zero to the whole of D. Set finally $\psi_\varepsilon(x) := \sum\limits_{i : \overline{P_i^\varepsilon} \subset D} \psi_\varepsilon^i(x)$, where

P_i^ε is the cell of size 2ε with center $x_i^\varepsilon \in (2\varepsilon\,\mathbb{Z})^3$. Then we have $\psi_\varepsilon \in C_c^\infty(\bigcup_i B_i^{\varepsilon/2})$ and

$$\psi_\varepsilon = 1 \text{ in } \bigcup_i B_i^{\varepsilon/4}, \quad |\nabla\psi_\varepsilon| \le C\varepsilon^{-1}. \tag{16}$$

With this at hand, we write

$$\langle \varrho_\varepsilon \rangle_\varepsilon \cdot I_6 = \langle \varrho_\varepsilon \rangle_\varepsilon \int_{D_\varepsilon} q_k^\varepsilon \psi_\varepsilon \operatorname{div}(\varphi \mathbf{u}_\varepsilon)\, \mathrm{d}x + \langle \varrho_\varepsilon \rangle_\varepsilon \int_{D_\varepsilon} q_k^\varepsilon (1 - \psi_\varepsilon)\varphi \operatorname{div}(\mathbf{u}_\varepsilon)\, \mathrm{d}x$$

$$+ \langle \varrho_\varepsilon \rangle_\varepsilon \int_{D_\varepsilon} q_k^\varepsilon (1 - \psi_\varepsilon)\mathbf{u}_\varepsilon \cdot \nabla\varphi\, \mathrm{d}x$$

$$=: I^1 + I^2 + I^3.$$

Observe that since supp $\psi_\varepsilon \subset \cup_i B_i^{\varepsilon/2}$, the term I^1 covers the behavior of q_k^ε "near" the holes, whereas I^2 and I^3 cover the behavior "far away". Since q_k^ε and ψ_ε are (2ε)-periodic functions and $q_k^\varepsilon \psi_\varepsilon \in L^2(D)$, we have $q_k^\varepsilon \psi_\varepsilon \rightharpoonup 0$ in $L^2(D)/\mathbb{R}$. This together with $\tilde{\mathbf{u}}_\varepsilon \to \mathbf{u}$ strongly in $L^2(D)$ yields

$$|I^3| \to 0.$$

For I^2, we use the definition of q_k^ε and (10) to find

$$|I^2| \le C \int_{D \setminus \cup_i B_i^{\varepsilon/4}} |q_k^\varepsilon|\,|\operatorname{div}(\mathbf{u}_\varepsilon)|\, \mathrm{d}x \overset{(14)}{\le} C\|q_k^\varepsilon\|_{L^2(D \setminus \cup_i B_i^{\varepsilon/4})} = C\|q_k^\varepsilon\|_{L^2(\cup_i B_i^\varepsilon \setminus B_i^{\varepsilon/4})} \le C\varepsilon \to 0.$$

To prove $I^1 \to 0$, we write, using $\operatorname{div}(\varrho_\varepsilon \mathbf{u}_\varepsilon) = 0$,

$$I^1 = \int_{D_\varepsilon} \nabla(q_k^\varepsilon \psi_\varepsilon \varphi) \cdot (\varrho_\varepsilon \mathbf{u}_\varepsilon)\, \mathrm{d}x - \int_{D_\varepsilon} \nabla(q_k^\varepsilon \psi_\varepsilon \varphi) \cdot (\langle \varrho_\varepsilon \rangle_\varepsilon \mathbf{u}_\varepsilon)\, \mathrm{d}x + \langle \varrho_\varepsilon \rangle_\varepsilon \int_{D_\varepsilon} q_k^\varepsilon \psi_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla\varphi\, \mathrm{d}x$$

$$= \int_{D_\varepsilon} \nabla(q_k^\varepsilon \psi_\varepsilon \varphi)(\varrho_\varepsilon - \langle \varrho_\varepsilon \rangle_\varepsilon) \cdot \mathbf{u}_\varepsilon\, \mathrm{d}x + o(1).$$

Here, we used again the periodicity of q_k^ε and ψ_ε to conclude $q_k^\varepsilon \psi_\varepsilon \rightharpoonup 0$ in $L^2(D)/\mathbb{R}$. This and the strong convergence of $\tilde{\mathbf{u}}_\varepsilon$ to $\mathbf{u}$ in $L^2(D)$ shows that the last term vanishes in the limit $\varepsilon \to 0$. For the remaining integral, we find, recalling supp $\psi_\varepsilon \subset \cup_i B_i^{\varepsilon/2}$ and $C_i^\varepsilon = B_i^{\varepsilon/2} \setminus T_i^\varepsilon$,

$$|I^1| \le \|\nabla(q_k^\varepsilon \psi_\varepsilon \varphi)\|_{L^{\frac{2\gamma}{\gamma-1}}(\cup_i C_i^\varepsilon)} \|\varrho_\varepsilon - \langle \varrho_\varepsilon \rangle_\varepsilon\|_{L^{2\gamma}(D_\varepsilon)} \|\mathbf{u}_\varepsilon\|_{L^2(D_\varepsilon)} + o(1)$$

$$\le C\varepsilon^{\frac{\beta}{\gamma}} \|\nabla(q_k^\varepsilon \psi_\varepsilon \varphi)\|_{L^{\frac{2\gamma}{\gamma-1}}(\cup_i C_i^\varepsilon)} + o(1).$$

Since $|\nabla\psi_\varepsilon| \le C\varepsilon^{-1}$, we have

$$|\nabla(q_k^\varepsilon \psi_\varepsilon \varphi)| \le C\left(|\nabla q_k^\varepsilon| + \frac{1}{\varepsilon}|q_k^\varepsilon|\right),$$

thus

$$|I^1| \le C\varepsilon^{\frac{\beta}{\gamma}}\left(\|\nabla q_k^\varepsilon\|_{L^{\frac{2\gamma}{\gamma-1}}(\cup_i C_i^\varepsilon)} + \frac{1}{\varepsilon}\|q_k^\varepsilon\|_{L^{\frac{2\gamma}{\gamma-1}}(\cup_i C_i^\varepsilon)}\right) + o(1).$$

Together with (8) and (9) for $p = 2\gamma/(\gamma - 1) > 3/2$, we establish

$$|I^1| \le C\varepsilon^{\frac{\beta}{\gamma}}\left(\varepsilon^{-3-\frac{3}{\gamma}} + \varepsilon^{-1-\frac{3}{\gamma}}\right) + o(1) \le C\varepsilon^{-3+\frac{\beta-3}{\gamma}} + o(1) \to 0,$$

provided

$$\beta > 3\,(\gamma + 1).$$

To summarize, we have in the limit $\varepsilon \to 0$ for all functions $\varphi \in \mathcal{D}(D)$

$$\mu\,\langle M\mathbf{e}_k, \varphi\mathbf{u}\rangle - \mu\,\langle \Delta\mathbf{u}, \varphi\mathbf{e}_k\rangle = -\langle \mathrm{div}(\varrho_0\mathbf{u} \otimes \mathbf{u}), \varphi\mathbf{e}_k\rangle + \langle \varrho_0\mathbf{f} + \mathbf{g} - \nabla p, \varphi\mathbf{e}_k\rangle.$$

Since M is symmetric, this is

$$\nabla p + \varrho_0\mathbf{u} \cdot \nabla\mathbf{u} - \mu\Delta\mathbf{u} + \mu M\mathbf{u} = \varrho_0\mathbf{f} + \mathbf{g} \text{ in } \mathcal{D}'(D),$$

which is the first equation of (6). This finishes the proof.

$\square$

Declaration
Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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Peter Bella and Florian Oschmann
Technische Universität Dortmund
Dortmund Nordrhein-Westfalen
Germany
e-mail: peter.bella@tu-dortmund.de