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On Weak-Strong Uniqueness for Stochastic Equations of Incompressible Fluid Flow

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Abstract. We introduce a concept of *dissipative measure-valued martingale solution* to the stochastic Euler equations describing the motion of an *inviscid incompressible* fluid. These solutions are characterized by a parametrized Young measure and a concentration defect measure in the total energy balance. Moreover, they are weak in the probabilistic sense i.e., the underlying probability space and the driving Wiener process are intrinsic parts of the solution. We first exhibit the relative energy inequality for the incompressible Euler equations driven by a *multiplicative* noise and then demonstrate the pathwise weak-strong uniqueness principle. Finally, we also provide a sufficient condition, á la Prodi (Ann Mat Pura Appl 48:173–182, 1959) and Serrin (in: Nonlinear problems, University of Wisconsin Press, Madison, Wisconsin, pp 69–98, 1963), for the uniqueness of weak martingale solutions to the stochastic Navier–Stokes system in the class of finite energy weak martingale solutions.

Keywords. Euler system, Navier–Stokes system, Incompressible fluids, Stochastic forcing, Measure-valued solution, Dissipative solution, Weak-strong uniqueness.

Contents

1. Introduction

In this paper, we introduce a notion of *dissipative measure-valued solution* for the stochastically forced system of the *incompressible Euler system* describing the velocity vector field $\mathbf{u}$ of a fluid and the scalar pressure field p. The system of equations read

$$\begin{cases} \mathrm{d}\mathbf{u}(t,x) + [\operatorname{div}(\mathbf{u}(t,x)\otimes\mathbf{u}(t,x)) + \nabla p(t,x)]\,\mathrm{d}t = \mathcal{G}(\mathbf{u}(t,x))\,\mathrm{d}W(t), & \text{in} \quad \Pi_T, \\ \operatorname{div}\mathbf{u}(t,x) = 0, & \text{in} \quad \Pi_T, \\ \mathbf{u}(0,x) = \mathbf{u}_0(x), & \text{in} \quad \mathbb{T}^3, \end{cases} \tag{1.1}$$

where $\Pi_T := \mathbb{T}^3 \times (0,T)$ with $T > 0$ fixed, u_0 is the given random initial function with sufficient spatial regularity to be specified later. Let $\big(\Omega, \mathbb{F}, \mathbb{P}, (\mathbb{F}_t)_{t\geq 0}\big)$ be a stochastic basis, where $\big(\Omega, \mathbb{F}, \mathbb{P}\big)$ is a probability space and $(\mathbb{F}_t)_{t\geq 0}$ is a complete filtration with the usual assumptions. We assume that W is a cylindrical Wiener process defined on the probability space $(\Omega, \mathbb{F}, \mathbb{P})$, and the coefficient $\mathcal{G}$ is generally nonlinear and satisfies suitable growth assumptions (see Sect. 2 for the complete list of assumptions). In particular, the map $\mathbf{u} \mapsto \mathcal{G}(\mathbf{u})$ is a Hilbert space valued function signifying the *multiplicative* nature of the noise.

The Euler Eq. (1.1) are the classical model for the motion of an inviscid, incompressible fluid. The presence of stochastic term in the governing equations accounts for numerical, physical, and empirical uncertainties in various real-life applications. The theory for the deterministic counterpart of (1.1) has experienced substantial progress in the past decade and has reached some level of maturity thanks to pioneering work by De Lellis and Szekelyhidi [16,17]. In a nutshell, these results show that for suitable (a large class of) initial data there are *infinitely many* weak solutions, even if the solution satisfies an *entropy condition*. Indeed, it was Scheffer [34] (see also [36]) who first constructed a nontrivial weak solution of the two-dimensional incompressible Euler equations with compact support in time. In other words, recent results by De Lellis and Szekelyhidi, and others [9,10,13] suggest that non-uniqueness of weak solutions to incompressible Euler equations in several space dimensions is a fact of life. However, for *general* initial data, the existence of global-in-time weak solutions is still unknown. In the quest for a global-in-time solution, we recall the framework of *measure-valued* solutions, as introduced by DiPerna and Majda [19] (see also [18]) for the incompressible Euler equations. Despite being a weaker notion of solution, thanks to the work by Brenier et al. [8], it is well-known that these measure-valued solutions of incompressible Euler equations enjoy a remarkable *weak-strong uniqueness* property. Moreover, recent work by Szekelyhidi and Wiedemann [38] confirms that the notions of weak solutions and measure-valued solutions coincide for the incompressible Euler equations.

In the stochastic set-up, the existence of *pathwise* strong solutions (defined up to a stopping time) for incompressible Euler Eq. (1.1) driven by a *multiplicative noise*, in a three-dimensional smooth bounded domain, was established by Glatt-Holtz and Vicol [23]. Moreover, a stochastic variant of deterministic results, as developed by Buckmaster and Vicol [9,10], for the stochastic incompressible Euler system have been recently established by Hofmanova et al. [25]. In fact, by making use of the method of convex integration, they have constructed infinitely many solutions to the incompressible Euler system with a random forcing. Note that although measure-valued solution for the deterministic counterpart of (1.1) has a long and intense history (see Lions [30]), their formulation seems rather intricating in the stochastic setting (specially in the multiplicative noise case). Indeed, there have been several attempts to define a suitable notion of measure-valued solutions for the stochastic incompressible Euler equations driven by *additive* noise, starting from the work of Kim [27], Breit and Moyo [5], and most recently by Hofmanova et al. [25], where the authors introduced a class of dissipative solutions which allowed them to demonstrate weak-strong uniqueness property and non-uniqueness of solutions in law. However, none of the above-mentioned frameworks can be applied directly to (1.1) since the driving noise is *multiplicative* in nature.

In this paper, our first objective is to introduce a novel framework of *dissipative measure-valued* solution for incompressible Euler Eq. (1.1) driven by a *multiplicative* stochastic perturbation of Nemytskii type, which will facilitate to demonstrate weak-strong uniqueness principle. In general, weak-strong uniqueness property can be used to establish convergence of numerical schemes [11,12,21], and convergence of various singular limits [24,30]. Note that dissipative measure-valued solutions are measure-valued solutions of (1.1) augmented with an appropriate form of energy inequality (see Definition 2.15 for the details). Note that a similar type of energy inequality was also considered in [5,25]. However, since we are working with a general multiplicative noise, our energy inequality differs from that of [5,25]. The main advantage of this class of solutions is that, for any finite energy initial data, they can be shown to exist globally in time. The existence of such solutions is addressed through the well-known vanishing viscosity method with the help of a stochastic compactness argument combining Prohorov theorem, and Skorokhod–Jakubowski [26] representation theorem. In other words, measure-valued solutions for (1.1) are obtained through a sequence of solutions of Navier–Stokes system subject to stochastic forcing given by

$$
\begin{aligned}
\mathrm{d}\mathbf{u}_\varepsilon + \left[\operatorname{div}(\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) + \nabla p_\varepsilon\right] dt &= \varepsilon\, \Delta \mathbf{u}_\varepsilon\, dt + \mathcal{G}(\mathbf{u}_\varepsilon)\, \mathrm{d}W,\\
\operatorname{div} \mathbf{u}_\varepsilon &= 0.
\end{aligned} \tag{1.2}
$$

There is a vast literature on the mathematical theory for stochastic perturbations of Navier–Stokes Eq. (1.2), being first initiated by Flandoli and Gatarek [22] (see also [15]), where the global-in-time existence of a *weak martingale solution* is shown. From the PDE standpoint, these solutions are weak, i.e., derivatives only exist in the sense of distributions, and from a probabilistic point of view, these solutions are also weak in the sense that the driving noise and associated filtration are part of the solution.

The primary difficulty, in comparison to existing works [5,27], lies in the successful identification (in the limit) of martingale term involving nonlinear noise coefficient, which in turn plays a pivotal role in the proof of weak (measure-valued)-strong uniqueness principle. Indeed, since the martingale solutions of approximate Eq. (1.2) are not unique, associated filtration depends on the approximation parameter ε, and passing to the limit in martingale term seems delicate. We can only establish that the limit object is a martingale, without knowing its explicit structure. However, a key observation reveals that the information on the cross variation of a martingale solution with a strong solution is sufficient to exhibit the weak-strong uniqueness property. This observation is encoded in the Definition 2.15, by stipulating the correct cross variation between a martingale measure-valued solution and a given smooth process. The cross variation term resembles "noise-noise" interaction term appears in "doubling of variables" argument, à la Kružkov, see [2–4,28,29]. Recently, a concept of dissipative measure-valued solutions to stochastic compressible Euler equations has been introduced by Hofmanova et al. [24] and it is used to prove convergence of numerical schemes for stochastic compressible/incompressible Euler equations by Chaudhary et al. [11,12]. However, the present work differs significantly from [11,24], due to the inherent challenges posed by the divergence-free condition associated with (1.1).

Our second objective is to display a sufficient condition for the uniqueness of weak martingale solutions to stochastic Navier–Stokes Eq. (1.2) in a class of finite energy weak martingale solutions. Note that the uniqueness of (Leray-Hopf) weak solutions for the deterministic counterpart of (1.2) is an outstanding open problem. Therefore, there has been growing interest in searching for sufficient conditions for the uniqueness of weak solutions, starting with the celebrated works by Prodi [33] and Serrin [35]. Due to the limitations of general uniqueness result in the stochastic set-up also, we follow Prodi and Serrin (see also Wiedemann [39]) to provide a sufficient condition for uniqueness by exploiting the weak-strong uniqueness property of solutions to stochastic incompressible Navier–Stokes Eq. (1.2).

A brief description of the organization of the rest of this paper is as follows: Sect. 2 outlines all relevant underlying mathematical/technical frameworks and a list of conditions imposed on noise coefficient. We then describe solution concepts and display the main results. In Sect. 3, we establish a priori estimates and demonstrate the convergence of approximate solutions, using stochastic compactness, to show the existence of dissipative measure-valued martingale solutions to (1.1). In light of a suitable relative energy inequality for incompressible Euler system (1.1), we establish weak (measure-valued)—strong uniqueness principle for (1.1) in Sect. 4, while in Sect. 5, we provide a sufficient condition for uniqueness of weak martingale solutions to stochastic incompressible Navier–Stokes Eq. (1.2), under additional conditions on weak martingale solutions.

2. Technical Framework and Main Results

In this section, we recapitulate some of the relevant mathematical tools to be used in the subsequent analysis and state the main results of this paper. To begin with, we fix an arbitrarily large time horizon $T > 0$. Throughout this paper, we use the letters C, K, etc. to denote various generic constants independent of approximation parameters, which may change line to line along the proofs. Explicit tracking of the constants could be possible but it is cumbersome and avoided for the sake of the reader.

2.1. Analytic Framework

Let us denote the Sobolev space $H^s(\mathbb{T}^3)$, for $s \in \mathbb{R}$, as the set of tempered distributions for which the norm

$$\|\mathbf{u}\|^2_{H^s(\mathbb{T}^3)} := \sum_{n\in\mathbb{Z}^3} \big(1+|n|^{2s}\big)|\hat{\mathbf{u}}(n)|^2,$$

is finite. Here $\hat{\mathbf{u}}$ denotes the Fourier transform of $\mathbf{u}$. Let $C^\infty_{\text{div}}(\mathbb{T}^3)$ and $L^2_{\text{div}}(\mathbb{T}^3)$ be the spaces of infinitely differentiable 3-dimensional vector fields u on $\mathbb{T}^3$ satisfying $\nabla\cdot\mathbf{u} = 0$, and closure of $C^\infty_{\text{div}}(\mathbb{T}^3)$ with respect to L^2-norm respectively. In other words,

$$C^\infty_{\text{div}}(\mathbb{T}^3) := \Big\{\boldsymbol{\varphi} \in C^\infty(\mathbb{T}^3) : \nabla\cdot\boldsymbol{\varphi} = 0\Big\},$$
$$L^2_{\text{div}}(\mathbb{T}^3) := \Big\{\boldsymbol{\varphi} \in L^2(\mathbb{T}^3) : \nabla\cdot\boldsymbol{\varphi} = 0\Big\}.$$

In a similar fashion, we denote by $H^\alpha_{\text{div}}(\mathbb{T}^3)$ the closure of $C^\infty_{\text{div}}(\mathbb{T}^3)$ in $H^\alpha(\mathbb{T}^3;\mathbb{R}^3)$, for $\alpha \geq 0$. Identifying $L^2_{\text{div}}(\mathbb{T}^3)$ with its dual space $(L^2_{\text{div}}(\mathbb{T}^3))'$ and identifying $(L^2_{\text{div}}(\mathbb{T}^3))'$ with a subspace of $H^{-\alpha}(\mathbb{T}^3)$(the dual space of $H^\alpha(\mathbb{T}^3)$), we have $H^\alpha_{\text{div}}(\mathbb{T}^3) \subset L^2_{\text{div}}(\mathbb{T}^3) \subset H^{-\alpha}_{\text{div}}(\mathbb{T}^3)$, and we can denote the dual pairing between H^α_{div} and $H^{-\alpha}_{\text{div}}$ by $\langle\cdot,\cdot\rangle_\alpha$ when no confusion may arise, see [22]. If $\mathbf{f}$ and $\mathbf{g}$ are two measurable functions such that $\mathbf{f}\cdot\mathbf{g} \in L^1(\mathbb{T}^3)$, then we denote $\langle \mathbf{f}, \mathbf{g}\rangle = \int_{\mathbb{T}^3} \mathbf{f}\cdot\mathbf{g}\,\mathrm{d}x$.

Moreover, we set $D(A) := H^2_{\text{div}}(\mathbb{T}^3)$, and define the linear operator $A : D(A) \subset L^2_{\text{div}}(\mathbb{T}^3) \to L^2_{\text{div}}(\mathbb{T}^3)$ by $A\mathbf{u} = -\Delta\mathbf{u}$. We then define the bilinear operator $B(\mathbf{u},\mathbf{v}) : H^1_{\text{div}} \times H^1_{\text{div}} \to H^{-1}_{\text{div}}$ as

$$\langle B(\mathbf{u},\mathbf{v}),\mathbf{z}\rangle_1 := \int_{\mathbb{T}^3} \mathbf{z}(x)\cdot(\mathbf{u}(x)\cdot\nabla)\mathbf{v}(x)dx, \quad \text{for all} \quad \mathbf{z} \in H^1_{\text{div}}(\mathbb{T}^3).$$

Note that the bilinear operator B can be extended to a continuous operator

$$B : L^2_{\text{div}}(\mathbb{T}^3) \times L^2_{\text{div}}(\mathbb{T}^3) \to D(A^{-\alpha}) = H^{-2\alpha}_{\text{div}},$$

for certain $\alpha > 1$, for details consult [22]. A straightforward computation using incompressibility condition reveals that

$$\langle B(\mathbf{u},\mathbf{v}),\mathbf{z}\rangle_{2\alpha} = -\langle B(\mathbf{u},\mathbf{z}),\mathbf{v}\rangle_{2\alpha} = -\langle \mathbf{u}\otimes\mathbf{v}, \nabla\mathbf{z}\rangle_{2\alpha}, \tag{2.1}$$

for all $\mathbf{u},\mathbf{v} \in H^1_{\text{div}}(\mathbb{T}^3)$ and $\mathbf{z} \in C^\infty_{\text{div}}(\mathbb{T}^3)$.

An important consequence of elliptic theory is the existence of the Helmholtz decomposition. It allows to decompose any vector-valued function in $L^2(\mathbb{T}^3;\mathbb{R}^3)$ into a divergence free part and a gradient part. In other words, we have the following decomposition

$$L^2(\mathbb{T}^3) = L^2_{\text{div}}(\mathbb{T}^3) \oplus (L^2_{\text{div}}(\mathbb{T}^3))^\perp,$$

where we denote

$$(L^2_{\text{div}}(\mathbb{T}^3))^\perp := \Big\{\mathbf{u} \in L^2(\mathbb{T}^3;\mathbb{R}^3)\,|\,\mathbf{u} = \nabla\psi,\ \psi \in H^1(\mathbb{T}^3;\mathbb{R})\Big\}.$$

The Helmholtz decomposition is defined by

$$\mathbf{u} = \mathcal{P}_H \mathbf{u} + \mathcal{Q}_H \mathbf{u}, \quad \text{for any} \quad \mathbf{u} \in L^2(\mathbb{T}^3),$$

where $\mathcal{P}_H$ denotes the projection operator from $L^2(\mathbb{T}^3)$ to $L^2_{\mathrm{div}}(\mathbb{T}^3)$, and $\mathcal{Q}_H := Id - \mathcal{P}_H$ denotes the projection operator from $L^2(\mathbb{T}^3)$ to $(L^2_{\mathrm{div}}(\mathbb{T}^3))^{\perp}$. Note that this decomposition is orthogonal with respect to $L^2(\mathbb{T}^3)$-inner product. By property of projection operator $\mathcal{P}_H$, we have for $\mathbf{u} \in L^2(\mathbb{T}^3)$

$$\langle \mathcal{P}_H \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle, \quad \text{for all} \quad \mathbf{v} \in L^2_{\mathrm{div}}(\mathbb{T}^3), \tag{2.2}$$

Finally, we recall a compact embedding result from Flandoli and Gatarek [22, Theorem 2.2]. To state the result, let us first denote K to be a separable Hilbert space. Given $q > 1$, $\gamma \in (0,1)$, let $W^{\gamma,q}(0,T;K)$ denotes a K-valued Sobolev space which is characterized by its norm

$$\|u\|^q_{W^{\gamma,q}(0,T;K)} := \int_0^T \|u(t)\|^q_K \, dt + \int_0^T \int_0^T \frac{\|u(t) - u(s)\|^q_K}{|t-s|^{1+q\gamma}} \, dt \, ds.$$

The following compact embedding result follows from Flandoli and Gatarek [22, Theorem 2.2].

Lemma 2.1. *If $H_1 \subset H_2$ are two Banach spaces with compact embedding, and real numbers $\gamma \in (0,1)$, $q > 1$ satisfy $\gamma q > 1$, then the embedding*

$$W^{\gamma,q}(0,T;H_1) \subset C([0,T];H_2)$$

is compact.

2.1.1. Young Measures, Concentration Defect Measures. In this subsection, we briefly recall the notion of Young measures and related results used in this manuscript. For a detailed overview on this topic, we refer to the monograph by Attouch et al. [1]. To begin with, let us denote by $\mathcal{M}_b(F)$, the space of regular signed Borel measures with finite total variation on locally compact Hausdorff space F equipped with the norm given by the total variation of measures. It is well-known that it is the dual space to the space of continuous functions vanishing at infinity $C_0(F)$ with respect to the supremum norm. Moreover, let us denote by $\mathcal{P}(F)$, the space of probability measures on F.

To define Young measure, let us first fix a sigma finite measure space $(Y, \mathcal{N}, \mu)$. A Young measure from Y into $\mathbb{R}^P$ is a weakly-$*$ measurable function $\mathcal{V} : Y \to \mathcal{P}(\mathbb{R}^P)$. In other words, the map $y \to \mathcal{V}_y(S)$ is $\mathcal{N}$-measurable for every Borel set S in $\mathbb{R}^P$. For our purpose, let us recall the following probabilistic generalization of the well-known classical result on Young measures. For a proof, we refer to the monograph by Breit et al. [7, Sect. 2.8].

Lemma 2.2. *Let $P, Q \in \mathbb{N}$, and $\mathcal{D} \subset \mathbb{R}^Q \times (0,T)$ be a domain. Let $(\mathbf{V}_m)_{m\in\mathbb{N}}$, $\mathbf{V}_m : \Omega \times \mathcal{D} \to \mathbb{R}^P$, be a sequence of random variables such that*

$$\mathbb{E}\Big[\|\mathbf{V}_m\|^p_{L^p(\mathcal{D})}\Big] \le C, \quad \textit{for a certain} \quad p \in (1,\infty).$$

Then, on the complete probability space $\big([0,1], \overline{\mathcal{B}[0,1]}, \widetilde{\mathbb{P}} := \mathcal{L}_{\mathbb{R}}\big)$, there exists a new subsequence $(\tilde{\mathbf{V}}_m)_{m\in\mathbb{N}}$ (not relabeled) and a family $\{\tilde{\mathcal{V}}^\omega_x\}_{x\in\mathcal{D}}$ of parameterized random probability measures on $\mathbb{R}^P$. These random probability measures can be treated as random variables taking values in $\big(L^\infty_{w^*}(\mathcal{D}; \mathcal{P}(\mathbb{R}^P)), w^*\big)$. Moreover, the random variables $\mathbf{V}_m$ has the same law as $\tilde{\mathbf{V}}_m$, i.e. $\mathbf{V}_m \sim_d \tilde{\mathbf{V}}_m$, and the following property holds: given any Carathéodory function $H = H(x,Z), x \in \mathcal{D}, Z \in \mathbb{R}^P$,[1] such that

$$|H(x,Z)| \le C(1 + |Z|^q), \quad 1 \le q < p, \quad \textit{uniformly in} \quad x,$$

implies $\mathcal{L}_{\mathbb{R}}$ almost surely,

$$H(\cdot, \tilde{\mathbf{V}}_m) \rightharpoonup \overline{H} \quad \textit{in} \quad L^{p/q}(\mathcal{D}), \quad \textit{where} \quad \overline{H}(x) = \langle \tilde{\mathcal{V}}^\omega_x; H(x,u) \rangle, \quad \textit{for a.e.} \quad x \in \mathcal{D},$$

[1] i.e. H is measurable with respect to first variable and continuous with respect to second variable.

where we have used the notation $\langle \bar{\mathcal{V}}^\omega_x; H(x,\mathbf{u})\rangle := \int_{\mathbb{R}^P} H(x,\mathbf{u}) \mathrm{d}\bar{\mathcal{V}}^\omega_x(\mathbf{u})$ for almost every $x \in \mathcal{D}$. Usually, Young measure theory plays an important role while extracting limits of bounded continuous functions. However, in our context, we have to deal with a nonlinear function H which may not be continuous, enjoying instead the following bound

$$\mathbb{E}\Big[\|H(\mathbf{V}_m)\|^p_{L^1(\mathcal{D})}\Big] \le C, \quad \text{for a certain} \quad p \in (1,\infty), \quad \text{uniformly in} \quad m.$$

In fact, in this situation, it is customary to embed the function space $L^1(\mathcal{D})$ into the space $\mathcal{M}_b(\mathcal{D})$ to characterize the limit object. Indeed, we can infer that $\widetilde{\mathbb{P}}$-a.s.

$$\text{weak-* limit in} \quad \mathcal{M}_b(\mathcal{D}) \,\text{of}\, H(\mathbf{V}_m) = \langle \tilde{\mathcal{V}}^\omega_x; H(\mathbf{v})\rangle \, dx + H_\infty,$$

where $H_\infty \in \mathcal{M}_b(\mathcal{D})$, and H_∞ is called *concentration defect measure (or concentration Young measure).* It is worth noting that, $\widetilde{\mathbb{P}}$-a.s. $\langle \tilde{\mathcal{V}}^\omega_x; H(\mathbf{v})\rangle$ is finite for a.e. $x \in \mathcal{D}$, thanks to a classical truncation error analysis and Fatou's lemma which yield $\|\langle \tilde{\mathcal{V}}^\omega_x; H(\mathbf{v})\rangle\|_{L^1(\mathcal{D})} \le C$, $\widetilde{\mathbb{P}}$ almost surely. For our purpose, we shall repeatedly use the following crucial lemma concerning the concentration defect measure. A proof of this lemma can be furnished, modulo cosmetic changes, using the same arguments given by Feireisl et al. [20, Lemma 2.1].

Lemma 2.3. *Let $\{\mathbf{V}_m\}_{m>0}$, $\mathbf{V}_m : \Omega \times \mathcal{D} \to \mathbb{R}^P$ be a sequence generating a Young measure $\{\mathcal{V}^\omega_y\}_{y\in\mathcal{D}}$, where $\mathcal{D}$ is a measurable set in $\mathbb{R}^Q \times (0,T)$. Let $H^1 : \mathbb{R}^P \to [0,\infty)$ be a continuous function such that*

$$\sup_{m>0} \mathbb{E}\Big[\|H^1(\mathbf{V}_m)\|^p_{L^1(\mathcal{D})}\Big] < +\infty, \quad \textit{for a certain} \quad p \in (1,\infty),$$

and let H^2 be a continuous function such that

$$H^2 : \mathbb{R}^P \to \mathbb{R}, \quad |H^2(\boldsymbol{z})| \le H^1(\boldsymbol{z}), \quad \textit{for all} \quad \boldsymbol{z} \in \mathbb{R}^P.$$

Let us denote $\mathbb{P}$-a.s.

$$H^1_\infty := \widetilde{H^1} - \langle \tilde{\mathcal{V}}^\omega_y; H^1(\boldsymbol{v})\rangle \, dy, \quad H^2_\infty := \widetilde{H^2} - \langle \tilde{\mathcal{V}}^\omega_y; H^2(\boldsymbol{v})\rangle \, dy.$$

Here $\widetilde{H^1}, \widetilde{H^2} \in \mathcal{M}_b(\mathcal{D})$ are weak-$*$ limits of $\{H^1(\mathbf{V}^m)\}_{m>0}$, $\{H^2(\mathbf{V}^m)\}_{m>0}$ respectively in $\mathcal{M}_b(\mathcal{D})$. Then $|H^2_\infty| \le H^1_\infty$ almost surely. Here $|H^2_\infty|$ denotes the total variation measure of H^2_∞.

2.2. Basics of Stochastic Framework

Here we briefly recall some aspects of the theory of stochastic analysis which are pertinent to the present work. We start by fixing a stochastic basis $(\Omega, \mathbb{F}, (\mathbb{F}_t)_{t\ge 0}, \mathbb{P}, W)$ with a complete, right-continuous filtration. Here $(\Omega, \mathbb{F}, \mathbb{P})$ is a complete probability space, and the stochastic process W is a cylindrical $(\mathbb{F}_t)$-Wiener process defined on an auxiliary separable Hilbert space $\mathfrak{U}$. It is formally given by the expansion

$$W(t) = \sum_{k\ge 1} e_k W_k(t),$$

where the elements $\{W_k\}_{k\ge 1}$ are a sequence of mutually independent one dimensional standard Brownian motions relative to $(\mathbb{F}_t)_{t\ge 0}$ and $\{e_k\}_{k\ge 1}$ is a complete orthonormal basis of $\mathfrak{U}$. Let H be a Hilbert space. Recall that $L_2(\mathfrak{U}, H)$ contains all bounded linear operators $A \in L(\mathfrak{U}, H)$ such that

$$\|A\|^2_{L_2(\mathfrak{U},H)} := \sum_{k=1}^{\infty} \|Ae_k\|^2_H < \infty.$$

To define the stochastic integral featured in (1.1), for each $\mathbf{u} \in L^2(\mathbb{T}^3; \mathbb{R}^3)$, we introduce a mapping $\mathcal{G}(\mathbf{u}) : \mathfrak{U} \to L^2(\mathbb{T}^3; \mathbb{R}^3)$ given by

$$\mathcal{G}(\mathbf{u}) e_k = \mathbf{G}_k(\mathbf{u}(\cdot)).$$

In particular, we suppose that the coefficients $\mathbf{G}_k : \mathbb{R}^3 \to \mathbb{R}^3$ are C^1-functions that satisfy the following conditions: for every $\xi, \zeta \in \mathbb{R}^3$

$$\sum_{k\geq 1} |\mathbf{G}_k(\xi)|^2 \leq D_0(1 + |\xi|^2), \tag{2.3}$$

$$\sum_{k\geq 1} |\mathbf{G}_k(\xi) - \mathbf{G}_k(\zeta)|^2 \leq D_1|\xi - \zeta|^2. \tag{2.4}$$

The assumption (2.3) imposed on $\mathcal{G}$ implies that

$$\mathcal{G} : L^2(\mathbb{T}^3; \mathbb{R}^3) \to L_2(\mathfrak{U}; L^2(\mathbb{T}^3; \mathbb{R}^3)),$$

where $L_2(\mathfrak{U}; L^2(\mathbb{T}^3; \mathbb{R}^3))$ denotes the space of Hilbert–Schmidt operators from $\mathfrak{U}$ to $L^2(\mathbb{T}^3; \mathbb{R}^3)$. Thus, given a predictable process $\mathbf{u} \in L^2(\Omega; L^2(0, T; L^2(\mathbb{T}^3; \mathbb{R}^3)))$, the stochastic integral

$$\int_0^t \mathcal{G}(\mathbf{u})\ \mathrm{d}W = \sum_{k\geq 1} \int_0^t \mathbf{G}_k(\mathbf{u})\ \mathrm{d}W_k$$

is a well-defined $(\mathbb{F}_t)$-martingale taking values in $L^2(\mathbb{T}^3; \mathbb{R}^3)$; see [7, Sect. 2.3] for a detailed construction. Finally, since $W(t) = \sum_{k\geq 1} e_k W_k(t)$ does not converge in $\mathfrak{U}$, we define the auxiliary space $\mathfrak{U}_0 \supset \mathfrak{U}$ via

$$\mathfrak{U}_0 := \left\{ v = \sum_{k\geq 1} \beta_k e_k;\ \sum_{k\geq 1} \frac{\beta_k^2}{k^2} < \infty \right\},$$

according to the norm

$$\|v\|^2_{\mathfrak{U}_0} = \sum_{k\geq 1} \frac{\beta_k^2}{k^2}, \quad v = \sum_{k\geq 1} \beta_k e_k.$$

Observe that the embedding $\mathfrak{U} \hookrightarrow \mathfrak{U}_0$ is Hilbert–Schmidt. Moreover, the trajectories of the Brownian motion W are almost surely in $C([0, T]; \mathfrak{U}_0)$, thanks to a standard martingale argument.

In order to state the existence of pathwise strong solution for stochastic incompressible Euler equations, we next describe the conditions imposed on the diffusion coefficient $\mathcal{G}$. For details, we refer to the paper by Glatt-Holtz and Vicol [23]. Although the below-mentioned conditions appear to be rather involved, they cover many realistic stochastic models. In what follows, let us denote by $L_2(\mathfrak{U}; \mathbb{R})$, the usual space of Hilbert–Schmidt operators from $\mathfrak{U}$ to $\mathbb{R}$, and for $p \geq 2, m \geq 0$, define

$$\mathbb{L}^{m,p} = \left\{ \sigma : \mathbb{T}^3 \to\ L_2(\mathfrak{U}; \mathbb{R}) \,\middle|\, \sigma_k(\cdot) = \sigma(\cdot)e_k \in W^{m,p}(\mathbb{T}^3),\ \text{and} \sum_{|\beta|\,\leq\, m} \int_{\mathbb{T}^3} \|\partial^\beta \sigma_k\|^p_{L_2(\mathfrak{U};\mathbb{R})}\,\mathrm{d}x < \infty \right\},$$

which is a Banach space endowed with the norm

$$\|\sigma\|^p_{\mathbb{L}^{m,p}} := \sum_{|\beta|\leq\, m} \int_{\mathbb{T}^3} \|\partial^\beta \sigma\|^p_{L_2(\mathfrak{U};\mathbb{R})}\,\mathrm{d}x = \sum_{|\beta|\leq\, m} \int_{\mathbb{T}^3} \left(\sum_{k\geq 1} |\partial^\beta \sigma_k|^2 \right)^{p/2} \mathrm{d}x.$$

Note that here $W^{m,p}(\mathbb{T}^3)$ denotes the well-known Sobolev space, i.e., the space of functions for which the norm

$$\|w\|^p_{W^{m,p}(\mathbb{T}^3)} := \sum_{|\alpha|\leq m} \|\partial^\alpha w\|^p_{L^p(\mathbb{T}^3)} \text{ is finite.}$$

Consider any pair of Banach spaces X, Y such that $X \subset L^\infty(\mathbb{T}^3)$. In what follows, $L^\infty(\mathbb{T}^3)$ norm of any $x \in X$ is denoted by $\|x\|_\infty$. For an increasing, locally bounded function $\gamma(\cdot) \geq 1$, we denote the space of locally bounded maps

$$\mathrm{Bnd}_{u,\mathrm{loc}}(X, Y) := \left\{ \mathbb{G} \in C(X; Y) : \|\mathbb{G}(x)\|_Y \leq \gamma(\|x\|_\infty)(1 + \|x\|_X),\ \forall\, x \in X \right\}.$$

In addition, we also define the space of locally Lipschitz functions,

$$\mathrm{Lip}_{u,\mathrm{loc}}(X,Y)=\left\{\mathbb{G}\in \mathrm{Bnd}_{u,\mathrm{loc}}(X,Y):\|\mathbb{G}(x)-\mathbb{G}(y)\|_Y\le \gamma\Big(\|x\|_\infty+\|y\|_\infty\Big)\,\|x-y\|_X, \forall\, x,y\in X\right\}$$

For the statement of local pathwise existence result (cf. Theorem 2.9), we shall fix $p\ge 2$ and an integer $m>3/p+1$, and suppose that

$$\mathcal{G}\in \mathrm{Lip}_{u,\mathrm{loc}}(L^p(\mathbb{T}^3),\mathbb{L}^{0,p})\cap \mathrm{Lip}_{u,\mathrm{loc}}(W^{m+1,p}(\mathbb{T}^3),\mathbb{L}^{m+1,p})\cap \mathrm{Lip}_{u,\mathrm{loc}}(W^{m+5,2}(\mathbb{T}^3),\mathbb{L}^{m+5,2}). \tag{2.5}$$

Relating to the convergence of approximate solutions, strong convergence in ω variable plays a pivotal role. To that context, we need the Skorokhod embedding theorem, delivering a new probability space and new random variables, with the same laws as the original ones, converging almost surely. However, for technical reasons, we have to use a modified version of the classical Skorokhod embedding theorem [32, Corollary 2] which is stated below.

Theorem 2.4. *Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space and H_1 be a separable complete metric space. Let H_2 be a quasi-Polish (topological) space (i.e., there is a sequence of continuous functions $h_n: H_2\to[-1,1]$ that separates points of H_2). Assume that $\mathcal{B}(H_1)\otimes\mathcal{H}_2$ is a sigma algebra associated with the product space $H_1\times H_2$, where $\mathcal{H}_2$ is the sigma algebra generated by the sequence of continuous functions $\{h_n\}_{n=1}^\infty$. Let $U_n:\Omega\to H_1\times H_2$, $n\in\mathbb{N}$, be a family of random variables, such that the sequence $\{\mathcal{L}aw(U_n): n\in\mathbb{N}\}$ is weakly convergent on $H_1\times H_2$. For $k=1,2$, let $\pi_i: H_1\times H_2$ be the projection onto H_i, i.e.,*

$$U=(U_1,U_2)\in H_1\times H_2\mapsto \pi_i(U)=U_i\in H_i.$$

Finally, let us assume that there exists a random variable $X:\Omega\to H_1$ such that $\mathcal{L}aw(\pi_1(U_n))=\mathcal{L}aw(X)$, $\forall\, n\in\mathbb{N}$. Then, there is a probability space $(\tilde\Omega,\tilde{\mathcal{F}},\tilde{\mathbb{P}})$, a family of $H_1\times H_2$-valued random variables $\{\tilde U_n: n\in\mathbb{N}\}$, on $(\tilde\Omega,\tilde{\mathcal{F}},\tilde{\mathbb{P}})$ and a random variable $\tilde U:\tilde\Omega\to H_1\times H_2$ such that

- (a) $\mathcal{L}aw(\tilde U_n)=\mathcal{L}aw(U_n)$, $\forall\, n\in\mathbb{N}$;
- (b) $\tilde U_n\to\tilde U$ *in* $H_1\times H_2$, $\mathbb{P}-$ *a.s.*
- (c) $\pi_1(\tilde U_n)(\tilde w)=\pi_1(\tilde U)(\tilde w)$, $\forall\,\tilde w\in\tilde\Omega$.

Finally, we recall the celebrated Kolmogorov continuity thereom related to the existence of continuous modifications of stochastic processes.

Lemma 2.5. *Let $Z=\{Z(t)\}_{t\in[0,T]}$ be a real-valued stochastic process defined on a complete filtered probability space $(\Omega,\mathbb{F},(\mathbb{F}_t)_{t\ge0},\mathbb{P})$. Suppose that there are constants $a>1, b>0$, and $C>0$ such that for all $s,t\in[0,T]$,*

$$\mathbb{E}[|Z(t)-Z(s)|^a]\le C|t-s|^{1+b}.$$

Then there exists a continuous modification of the stochastic process Z and the paths of Z are c-Hölder continuous, for every $c\in[0,\frac{b}{a})$.

2.3. Stochastic Incompressible Euler Equations

As we mentioned before, we are primarily interested in establishing a weak (measure-valued)—strong uniqueness principle for dissipative measure-valued solutions to (1.1). Since such an argument requires the existence of a strong solution, we first recall the notion of a local strong pathwise solution for stochastic incompressible Euler equations. We remark that such a solution can be constructed on any given stochastic basis, that is, solutions are probabilistically strong, and it satisfies the underlying Eq. (1.1) pointwise (not only in the sense of distributions), that is, solutions are also strong from the PDE standpoint. The existence of such a solution was first established by Glatt-Holtz and Vicol [23].

Definition 2.6 (*Local strong pathwise solution*). Let $(\Omega, \mathbb{F}, (\mathbb{F}_t)_{t\geq 0}, \mathbb{P})$ be a stochastic basis with a complete right-continuous filtration, and W be an $(\mathbb{F}_t)$-cylindrical Wiener process. Suppose that $p \geq 2$, and $m > 3/p + 1$. Moreover, let $\mathbf{u}_0$ be a $W^{m,p}_{\text{div}}(\mathbb{T}^3)$-valued $\mathbb{F}_0$-measurable random variable, and let $\mathcal{G}$ satisfies (2.5). Then $(\mathbf{u}, \mathfrak{t})$ is said to be a local strong pathwise solution to the system (1.1) provided

(a) $\mathfrak{t}$ is an a.s. strictly positive $(\mathbb{F}_t)$-stopping time;
(b) the velocity $\mathbf{u}$ is a $W^{m,p}_{\text{div}}(\mathbb{T}^3)$-valued $(\mathbb{F}_t)$-predictable measurable process satisfying

$$\mathbf{u}(\cdot \wedge \mathfrak{t}) \in C([0,T]; W^{m,p}_{\text{div}}(\mathbb{T}^3)) \quad \mathbb{P}\text{-a.s.};$$

(c) for all $t \geq 0$,

$$\mathbf{u}(t \wedge \mathfrak{t}) = \mathbf{u}_0 - \int_0^{t\wedge\mathfrak{t}} \mathcal{P}_H(\mathbf{u}\cdot\nabla\mathbf{u})\mathrm{d}s + \int_0^{t\wedge\mathfrak{t}} \mathcal{P}_H\mathcal{G}(\mathbf{u})\ \mathrm{d}W. \tag{2.6}$$

Remark 2.7 By the property of Projection operator (2.2), we can recast the item (c) in Definition 2.6 as follows:

(c') for all $t \in [0,T]$,

$$\langle \mathbf{u}(t\wedge\mathfrak{t}), \psi\rangle = \langle \mathbf{u}_0, \psi\rangle - \int_0^{t\wedge\mathfrak{t}} \langle \mathbf{u}\cdot\nabla\mathbf{u}, \psi\rangle\mathrm{d}s + \int_0^{t\wedge\mathfrak{t}} \langle \mathcal{G}(\mathbf{u}), \psi\rangle\ \mathrm{d}W, \tag{2.7}$$

for all $\psi \in C^\infty_{\text{div}}(\mathbb{T}^3)$.

It is evident that classical solutions require spatial derivatives of the velocity field $\mathbf{u}$ to be continuous $\mathbb{P}$-a.s. This motivates the following definition.

Definition 2.8 (*Maximal strong pathwise solution*). Fix an initial datum, and a complete stochastic basis with a cylindrical Wiener process as in Definition 2.6. Then a triplet

$$(\mathbf{u}, (\tau_L)_{L\in\mathbb{N}}, \mathfrak{t})$$

is said to be a maximal strong pathwise solution to system (1.1) provided

(a) $\mathfrak{t}$ is an a.s. strictly positive $(\mathbb{F}_t)$-stopping time;
(b) $(\tau_L)_{L\in\mathbb{N}}$ is an increasing sequence of $(\mathbb{F}_t)$-stopping times such that $\tau_L < \mathfrak{t}$ on the set $[\mathfrak{t} < T]$, $\lim_{L\to\infty} \tau_L = \mathfrak{t}$ a.s. and

$$\sup_{t\in[0,\tau_L]} \|\mathbf{u}(t)\|_{W^{1,\infty}(\mathbb{T}^3)} \geq L \quad \text{on} \quad [\mathfrak{t} < T]; \tag{2.8}$$

(c) each pair $(\mathbf{u}, \tau_L)$, $L \in \mathbb{N}$, is a local strong pathwise solution in the sense of Definition 2.6.

In view of the above definitions, we are now in a position to state relevant existence theorems. For a proof, we refer to the work by Glatt-Holtz and Vicol [23].

Theorem 2.9 (Local existence for nonlinear multiplicative noise). *Let $(\Omega, \mathbb{F}, (\mathbb{F}_t)_{t\geq 0}, \mathbb{P})$ be a stochastic basis with a complete right-continuous filtration. Suppose that $p \geq 2$, and $m > 3/p + 1$. Let W be an $(\mathbb{F}_t)$-cylindrical Wiener process and $\mathbf{u}_0$ be a $W^{m,p}_{div}(\mathbb{T}^3)$-valued $\mathbb{F}_0$-measurable random variable, and let $\mathcal{G}$ satisfies (2.5). Then there exists a unique maximal strong pathwise solution $(\mathbf{u}, (\tau_L)_{L\in\mathbb{N}}, \mathfrak{t})$ of (1.1) in the sense of Definition 2.8.*

2.4. Stochastic Incompressible Navier–Stokes Equations

There is a large and intense literature concerning the incompressible Navier–Stokes equations driven by noise, starting with the work by Flandoli and Gatarek [22] where the authors proved the existence of weak martingale solutions to (1.2). As expected, these solutions are weak in both analytical and probabilistic sense. However, to prove the existence of dissipative measure-valued solutions for stochastic Euler equations, we first need to introduce the concept of *finite energy weak martingale* solutions to (1.2). Note that such solutions exist globally in time, and the time evolution of the energy for such solutions can be controlled in terms of their initial state.

Definition 2.10 (*Finite energy weak martingale solution*). Let Λ_ε be a Borel probability measure on $L^2_{\mathrm{div}}(\mathbb{T}^3)$. Then $\big[(\Omega_\varepsilon, \mathbb{F}_\varepsilon, (\mathbb{F}_{\varepsilon,t})_{t\geq 0}, \mathbb{P}_\varepsilon); \mathbf{u}_\varepsilon, W_\varepsilon\big]$ is a weak martingale solution of (1.2) if

(a) $\big(\Omega_\varepsilon, \mathbb{F}_\varepsilon, (\mathbb{F}_{\varepsilon,t})_{t\geq 0}, \mathbb{P}_\varepsilon\big)$ is a stochastic basis with a complete right-continuous filtration,

(b) W_ε is a $(\mathbb{F}_{\varepsilon,t})$-cylindrical Wiener process,

(c) the velocity field $\mathbf{u}_\varepsilon$ is $L^2_{\mathrm{div}}(\mathbb{T}^3)$-valued progressively measurable process and $\mathbb{P}-$a.s.

$$\mathbf{u}(\cdot,\omega) \in C([0,T]; H^{-2\alpha}_{\mathrm{div}}(\mathbb{T}^3)) \cap L^\infty(0,T; L^2_{\mathrm{div}}(\mathbb{T}^3)) \cap L^2(0,T; H^1_{\mathrm{div}}(\mathbb{T}^3))$$

(d) $\Lambda_\varepsilon = \mathbb{P}_\varepsilon \circ \big[\mathbf{u}_\varepsilon(0)\big]^{-1}$,

(e) for all $\boldsymbol{\varphi} \in H^{2\alpha}_{\mathrm{div}}(\mathbb{T}^3)$, we have

$$\begin{aligned} \langle \mathbf{u}_\varepsilon(t), \boldsymbol{\varphi}\rangle_{2\alpha} &= \langle \mathbf{u}_\varepsilon(0), \boldsymbol{\varphi}\rangle_{2\alpha} - \int_0^t \langle B(\mathbf{u}_\varepsilon(s), \mathbf{u}_\varepsilon(s)), \boldsymbol{\varphi}\rangle_{2\alpha}\,\mathrm{d}s + \varepsilon \int_0^t \langle \Delta \mathbf{u}_\varepsilon(s)\,, \boldsymbol{\varphi}\rangle_{2\alpha}\mathrm{d}s \\ &\quad + \int_0^t \langle \mathcal{P}_H \mathcal{G}(\mathbf{u}_\varepsilon), \boldsymbol{\varphi}\rangle_{2\alpha}\,\mathrm{d}W \end{aligned} \tag{2.9}$$

$\mathbb{P}$-a.s. for all $t \in [0,T]$,

(f) the energy inequality

$$\begin{aligned} &-\int_0^T \partial_t \phi \int_{\mathbb{T}^3} \frac{1}{2}|\mathbf{u}_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t + \varepsilon \int_0^T \phi \int_{\mathbb{T}^3} |\nabla_x \mathbf{u}_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t \\ &\quad \leq \phi(0)\int_{\mathbb{T}^3} \frac{1}{2}|\mathbf{u}_\varepsilon(0)|^2 + \sum_{k=1}^\infty \int_0^T \phi\Big(\int_{\mathbb{T}^3} \mathcal{P}_H \mathbf{G}_k(\mathbf{u}_\varepsilon)\cdot \mathbf{u}_\varepsilon\,\mathrm{d}x\Big)\mathrm{d}W_k \\ &\quad + \frac{1}{2}\sum_{k=1}^\infty \int_0^T \phi \int_{\mathbb{T}^3} |\mathcal{P}_H \mathbf{G}_k(\mathbf{u}_\varepsilon)|^2\,\mathrm{d}t \end{aligned} \tag{2.10}$$

holds $\mathbb{P}$-a.s., for all $\phi \in C^\infty_c([0,T))$, $\phi \geq 0$.

Remark 2.11 Note that in view of Skorohod [37], it is possible to consider,

$$\Big(\Omega_\varepsilon, \mathbb{F}_\varepsilon, \mathbb{P}_\varepsilon\Big) = \Big([0,1], \mathcal{B}([0,1]), \mathcal{L}_\mathbb{R}\Big),$$

for every ε. Moreover, we may assume the existence of a common Wiener space W for all ε, thanks to a classical compactness argument applied to any chosen subsequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ at once. However, it is worth noticing that, it may not be possible to obtain a filtration that is independent of ε, due to the lack of pathwise uniqueness for the underlying system.

Remark 2.12 By using property (2.1)–(2.2), we can recast the item (e) in Definition 2.10 as

(e') For all $\boldsymbol{\varphi} \in C^\infty_{\mathrm{div}}(\mathbb{T}^3)$,

$$\begin{aligned} \langle \mathbf{u}_\varepsilon(t), \boldsymbol{\varphi}\rangle &= \langle \mathbf{u}_\varepsilon(0), \boldsymbol{\varphi}\rangle + \int_0^t \langle \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon(s), \nabla_x \boldsymbol{\varphi}\rangle\,\mathrm{d}s - \varepsilon \int_0^t \langle \nabla_x \mathbf{u}_\varepsilon(s)\,, \nabla_x \boldsymbol{\varphi}\rangle \mathrm{d}s \\ &\quad + \int_0^t \langle \mathcal{G}(\mathbf{u}_\varepsilon), \boldsymbol{\varphi}\rangle\,\mathrm{d}W \end{aligned} \tag{2.11}$$

holds $\mathbb{P}$-a.s., for all $t \in [0,T]$.

Regarding the existence of finite energy martingale solutions, one may follow the arguments given by Flandoli and Gatarek [22] to obtain the following result.

Theorem 2.13 (Existence of martingale solution for Navier–Stokes system). *Assume that Λ_ε is a Borel probability measure on $L^2_{\mathrm{div}}(\mathbb{T}^3)$ such that the following moment estimate*

$$\int_{L^2_{\mathrm{div}}(\mathbb{T}^3)} \|\mathbf{u}\|^p_{L^2_{\mathrm{div}}(\mathbb{T}^3)}\,d\Lambda_\varepsilon(\mathbf{u}) < \infty,$$

holds for all $1 \le p < \infty$*. Moreover, assume that* (2.3) *and* (2.4) *hold. Then there exists a finite energy weak martingale solution of* (1.2) *in the sense of Definition* 2.10 *with initial law* Λ_ε*.*

Proof Existence proof for a weak martingale solution follows from the work of Flandoli and Gatarek [22]. To prove the energy inequality, one may simply apply the Itô formula to obtain the item (f) of Definition 2.10. Indeed, this is very similar to the recent works on compressible fluids, see Breit et al. [7]. The details are left to the interested reader. □

Remark 2.14 (*A different form of energy inequality*). It is well-known that one can establish the weak-strong uniqueness principle only in the class of dissipative weak solutions, i.e., weak solutions satisfying an appropriate energy inequality. To that context, we make use of a different form of energy inequality for the proof of weak-strong uniqueness related to the incompressible Navier–Stokes equations. First observe that, in view of a standard cut-off argument applied to (2.10), energy inequality holds for a.e. $0 \le s < t \in (0, T)$:

$$
\begin{aligned}
&\int_{\mathbb{T}^3} \frac{1}{2}|\mathbf{u}_\varepsilon(t)|^2 \,\mathrm{d}x + \varepsilon \int_s^t \int_{\mathbb{T}^3} |\nabla_x \mathbf{u}_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}s \\
&\quad \le \int_{\mathbb{T}^3} \frac{1}{2}|\mathbf{u}_\varepsilon(s)|^2 \,\mathrm{d}x + \sum_{k=1}^{\infty} \int_s^t \left(\int_{\mathbb{T}^3} \mathcal{P}_H \mathbf{G}_k(\mathbf{u}_\varepsilon) \cdot \mathbf{u}_\varepsilon \,\mathrm{d}x \right) \mathrm{d}W_k + \frac{1}{2} \sum_{k=1}^{\infty} \int_s^t \int_{\mathbb{T}^3} |\mathcal{P}_H \mathbf{G}_k(\mathbf{u}_\varepsilon)|^2 \,\mathrm{d}s.
\end{aligned} \tag{2.12}
$$

It follows from (2.12) that the limits

$$
\operatorname{ess} \lim_{\tau \to s^+} \int_{\mathbb{T}^3} \frac{1}{2}|\mathbf{u}(\tau)|^2 \,\mathrm{d}x, \quad \operatorname{ess} \lim_{\tau \to t^-} \int_{\mathbb{T}^3} \frac{1}{2}|\mathbf{u}(\tau)|^2 \,\mathrm{d}x
$$

exist $\mathbb{P}$-a.s. for a.a. $0 \le s \le t \le T$ including $s = 0$. Finally, in view to the weak lower-semicontinuity of convex functionals, we have for any $t \in [0, T)$ $\mathbb{P}$-a.s.

$$
\liminf_{\tau \to t^-} \int_{\mathbb{T}^3} \frac{1}{2}|\mathbf{u}(\tau)|^2 \,\mathrm{d}x \ \ge \int_{\mathbb{T}^3} \frac{1}{2}|\mathbf{u}(t)|^2 \,\mathrm{d}x
$$

By making use of the above informations, relative energy inequality (2.12) can be rewritten as

$$
\begin{aligned}
&\int_{\mathbb{T}^3} \frac{1}{2}|\mathbf{u}_\varepsilon(t)|^2 \,\mathrm{d}x + \varepsilon \int_0^t \int_{\mathbb{T}^3} |\nabla_x \mathbf{u}_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}s \\
&\quad \le \int_{\mathbb{T}^3} \frac{1}{2}|\mathbf{u}_\varepsilon(0)|^2 \,\mathrm{d}x + \sum_{k=1}^{\infty} \int_0^t \left(\int_{\mathbb{T}^3} \mathcal{P}_H \mathbf{G}_k(\mathbf{u}_\varepsilon) \cdot \mathbf{u}_\varepsilon \,\mathrm{d}x \right) \mathrm{d}W_k + \frac{1}{2} \sum_{k=1}^{\infty} \int_0^t \int_{\mathbb{T}^3} |\mathcal{P}_H \mathbf{G}_k(\mathbf{u}_\varepsilon)|^2 \,\mathrm{d}s
\end{aligned} \tag{2.13}
$$

holds $\mathbb{P}$-a.s., for all $t \in [0, T]$.

2.5. Measure-Valued Martingale Solutions

In general, in view of the energy inequality (2.13), solutions to the incompressible Navier–Stokes equations have only uniform energy bound, usually in $L^2(\mathbb{T}^3)$. However, such *a priori* bound does not guarantee weak convergence of nonlinear terms $\mathbf{u} \otimes \mathbf{u}, \mathbf{G}^2(\mathbf{u}) \in L^1(\mathbb{T}^3)$, due to the presence of oscillations and concentration effects. In this scenario, one can only identify weak limits (corresponding to nonlinear terms) as a combination of Young measure and concentration measure.

Note that the Young measures, which are probability measures on the phase space, capture oscillations in the solution. On the other hand concentration defect measures, which are measures on physical space–time, account for blow-up type collapse due to possible concentration points. In what follows, we use two different forms of concentration defect measures. To illustrate the difference, we consider the following situation:

- Let $\mathbf{v}_\varepsilon$ converges weakly in $L^2(\mathbb{T}^3)$ to a function $\mathbf{v}$, and we assume that $\int_{\mathbb{T}^3} |\mathbf{v}_\varepsilon|^2 \, dx \le C$.

Inspired by Banach–Alaoglu theorem, one can define a defect measure—which is a non-negative Radon measure, as

$$\mu_1 := \text{Weak -* }\lim_{\varepsilon\to 0}\left(|\mathbf{v}_\varepsilon|^2 - |\mathbf{v}|^2\right) \in \mathcal{M}_b^+(\Pi_T).$$

In a similar fashion, by making use of Young measure theory, one can define another defect measure μ_2 as

$$\mu_2 := \text{Weak -* }\lim_{\varepsilon\to 0}\left(|\mathbf{v}_\varepsilon|^2 - \langle \mathcal{V}_{t,x}(\lambda); |\lambda|^2\rangle\right) \in \mathcal{M}_b^+(\Pi_T).$$

It is well-known that μ_1 is too large to describe concentration effects in a useful way, while the main advantage of the concentration defect measure μ_2 is that it allows to describe weak limits in terms of the Young measure. It is easy to see that, thanks to Hölder inequality, $\mu_2 \le \mu_1$. We shall make use of both forms of concentration defect measures below to define a notion of a measure-valued solution.

2.5.1. Dissipative Measure-Valued Martingale Solutions. Keeping in mind the previous discussion, we are ready to introduce the concept of *dissipative measure-valued martingale solution* to the stochastic compressible Euler system. In what follows, let $\mathcal{S} = \mathbb{R}^3$ be the phase space associated to the incompressible Euler system. We also denote by $A : B$ the scalar product $\sum_{i,j} a_{ij} b_{ij}$ between two matrices $A = (a_{ij})$ and $B = (b_{ij})$ of same size.

Definition 2.15 (*Dissipative measure-valued martingale solution*). Let Λ be a Borel probability measure on $L^2_{\text{div}}(\mathbb{T}^3)$. Then $\left[(\Omega, \mathbb{F}, (\mathbb{F}_t)_{t\ge 0}, \mathbb{P}); \mathcal{V}^\omega_{t,x}, W\right]$ is a dissipative measure-valued martingale solution of (1.1), with initial condition $\mathcal{V}^\omega_{0,x}$, if

(a) $\mathcal{V}^\omega$ is a random variable taking values in the space of Young measures on $L^\infty_{w^*}\big([0,T]\times\mathbb{T}^3; \mathcal{P}(\mathcal{S})\big)$. In other words, $\mathbb{P}$-a.s. $\mathcal{V}^\omega_{t,x} : (t,x) \in [0,T]\times\mathbb{T}^3 \to \mathcal{P}(\mathcal{S})$ is a parametrized family of probability measures on $\mathcal{S}$,

(b) $\big(\Omega, \mathbb{F}, (\mathbb{F}_t)_{t\ge 0}, \mathbb{P}\big)$ is a stochastic basis with a complete right-continuous filtration,

(c) W is a $(\mathbb{F}_t)$-cylindrical Wiener process in $\mathfrak{U}$,

(d) the average velocity $\langle \mathcal{V}^\omega_{t,x}; \mathbf{u}\rangle$[2] satisfies, for any $\boldsymbol{\varphi} \in C^\infty_{\text{div}}(\mathbb{T}^3)$, $t \mapsto \langle\langle \mathcal{V}^\omega_{t,x}; \mathbf{u}\rangle(t,\cdot), \boldsymbol{\varphi}\rangle \in C[0,T]$, $\mathbb{P}$-a.s., the function $t \mapsto \langle\langle \mathcal{V}^\omega_{t,x}; \mathbf{u}\rangle(t,\cdot), \boldsymbol{\phi}\rangle$ is progressively measurable, and for any $\boldsymbol{\varphi} \in C^1(\mathbb{T}^3)$,

$$\int_{\mathbb{T}^3} \langle \mathcal{V}^\omega_{t,x}; \mathbf{u}\rangle \cdot \nabla_x \boldsymbol{\varphi}\, dx = 0$$

for almost $t \in [0,T]$, $\mathbb{P}-$a.s., and

$$\mathbb{E}\left[\sup_{t\in(0,T)} \|\langle \mathcal{V}^\omega_{t,x}; \mathbf{u}\rangle(t,\cdot)\|^p_{L^2_{\text{div}}(\mathbb{T}^3)}\right] < \infty$$

for all $1 \le p < \infty$,

(e) $\Lambda = \mathcal{L}[\langle \mathcal{V}^\omega_{0,x}; \mathbf{u}\rangle]$,

(f) there exists a $H^{-1}_{\text{div}}(\mathbb{T}^3)$-valued square integrable continuous martingale $\mathcal{M}^1_E$, such that the integral identity

$$\begin{aligned}
&\int_{\mathbb{T}^3} \langle \mathcal{V}^\omega_{\tau,x}; \mathbf{u}\rangle \cdot \boldsymbol{\varphi}(x)\,\mathrm{d}x - \int_{\mathbb{T}^3} \langle \mathcal{V}^\omega_{0,x}; \mathbf{u}\rangle \cdot \boldsymbol{\varphi}(x)\,\mathrm{d}x \\
&\quad = \int_0^\tau \int_{\mathbb{T}^3} \langle \mathcal{V}^\omega_{t,x}; \mathbf{u}\otimes\mathbf{u}\rangle : \nabla_x \boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{T}^3} \boldsymbol{\varphi} \int_0^\tau d\mathcal{M}^1_E(t)\,\mathrm{d}x + \int_0^\tau \int_{\mathbb{T}^3} \nabla_x \boldsymbol{\varphi} : d\mu_C,
\end{aligned} \tag{2.14}$$

holds $\mathbb{P}$-a.s., for all $\tau \in [0,T)$, and for all $\boldsymbol{\varphi} \in C^\infty_{\text{div}}(\mathbb{T}^3;\mathbb{R}^3)$, where $\mu_C \in L^\infty_{w*}([0,T]; \mathcal{M}_b(\mathbb{T}^3))$, $\mathbb{P}$-a.s., is a tensor-valued measure; μ_C is called *concentration defect measures*;

[2] Here $\langle \mathcal{V}^\omega_{t,x}; f(\mathbf{u})\rangle := \int_{\mathbb{R}^3} f(\mathbf{u}) d\mathcal{V}^\omega_{t,x}(\mathbf{u})$, for any measurable function f.

(g) there exists a real-valued square integrable continuous martingale $\mathcal{M}_E^2$, such that the following inequality

$$\begin{aligned} \mathrm{E}(t+) \leq \mathrm{E}(s-) &+ \frac{1}{2}\sum_{k\geq 1}\int_s^t \int_{\mathbb{T}^3} \left\langle \mathcal{V}^{\omega}_{\tau,x}; |\mathbf{G}_k(\mathbf{u})|^2 \right\rangle \,\mathrm{d}x\,\mathrm{d}\tau \\ &- \frac{1}{2}\sum_{k\geq 1}\int_s^t \int_{\mathbb{T}^3} \Big(\mathcal{Q}_H \left\langle \mathcal{V}^{\omega}_{\tau,x}; |\mathbf{G}_k(\mathbf{u})| \right\rangle\Big)^2 \mathrm{d}x\,\mathrm{d}\tau + \frac{1}{2}\int_s^t\int_{\mathbb{T}^3} d\mu_D + \int_s^t d\mathcal{M}_E^2, \end{aligned} \tag{2.15}$$

holds $\mathbb{P}$-a.s., for all $0 \leq s < t \in (0,T)$ with

$$\mathrm{E}(t-) := \liminf_{r\to 0^+} \frac{1}{r}\int_{t-r}^t \left(\int_{\mathbb{T}^3} \left\langle \mathcal{V}^{\omega}_{s,x}; \frac{|\mathbf{u}|^2}{2} \right\rangle \mathrm{d}x + \mathcal{D}(s)\right)\mathrm{d}s$$

$$\mathrm{E}(t+) := \liminf_{r\to 0^+} \frac{1}{r}\int_{t}^{t+r} \left(\int_{\mathbb{T}^3} \left\langle \mathcal{V}^{\omega}_{s,x}; \frac{|\mathbf{u}|^2}{2} \right\rangle \mathrm{d}x + \mathcal{D}(s)\right)\mathrm{d}s$$

Here $\mu_D \in L^\infty_{w*}([0,T];\mathcal{M}_b(\mathbb{T}^3))$, $\mathbb{P}$-a.s., $\mathcal{D} \in L^\infty(0,T)$, $\mathcal{D} \geq 0$, $\mathbb{P}$-almost surely, and $\mathbb{E}\big[\operatorname{ess\,sup}_{t\in(0,T)} \mathcal{D}(t)\big] < \infty$, with initial energy

$$\mathrm{E}(0-) = \int_{\mathbb{T}^3} \frac{1}{2}|\mathbf{u}_0|^2 \,\mathrm{d}x.$$

(h) there exists a constant $C > 0$ such that

$$\int_0^\tau \int_{\mathbb{T}^3} d|\mu_C| + \int_0^\tau \int_{\mathbb{T}^3} d|\mu_D| \leq C\int_0^\tau \mathcal{D}(t)dt, \tag{2.16}$$

$\mathbb{P}$-a.s., for every $\tau \in (0,T)$.

(i) For any given stochastic process $h(t)$, adapted to $(\mathbb{F}_t)_{t\geq 0}$, given by

$$\mathrm{d}h = F(h)\,\mathrm{d}t + \mathbb{K}(h)\,\mathrm{d}W,$$

satisfying

$$h \in C([0,T];W^{1,q}\cap C(\mathbb{T}^3)), \quad \mathbb{E}\left[\sup_{t\in[0,T]} \|h\|^2_{W^{1,q}}\right]^q < \infty, \quad \mathbb{P}\text{ -a.s. for all } \ 1 \leq q < \infty,$$

with

$$F(h) \in L^q(\Omega;L^q(0,T;W^{1,q}(\mathbb{T}^3))), \quad \mathbb{K}(h) \in L^2(\Omega;L^2(0,T;L_2(\mathfrak{U};L^2_{\mathrm{div}}(\mathbb{T}^3)))),$$

$$\Big(\sum_{k\geq 1}|\mathbb{K}(h)(e_k)|^q\Big)^{\frac{1}{q}} \in L^q(\Omega;L^q(0,T;L^q(\mathbb{T}^3))),$$

the cross variation between h and the square integrable continuous martingale M_E^1 is given by

$$\left\langle\!\!\left\langle h(t), M_E^1(t) \right\rangle\!\!\right\rangle = \sum_{i,j}\left(\sum_{k=1}^\infty \int_0^t \left\langle \mathcal{P}_H \left\langle \mathcal{V}^{\omega}_{s,x}; \mathbf{G}_k(\mathbf{u})\right\rangle, g_i\right\rangle \left\langle \mathbb{K}(h)(e_k), g_j\right\rangle ds\right) g_i \otimes g_j.$$

Here functions $(g_i)_{i\in\mathbb{N}}$ form a orthonormal basis for $H^{-1}_{\mathrm{div}}(\mathbb{T}^3)$ and bracket $\langle\cdot,\cdot\rangle$ denotes inner product in the same space.

Remark 2.16 Notice that, a standard Lebesgue point argument applied to (2.15) reveals that the energy inequality holds for a.e. $0 \leq s < t$ in $(0,T)$:

$$\begin{aligned} \int_{\mathbb{T}^3} \left\langle \mathcal{V}^{\omega}_{t,x}; \frac{|\mathbf{u}|^2}{2} \right\rangle \mathrm{d}x + \mathcal{D}(t) \leq \int_{\mathbb{T}^3} \left\langle \mathcal{V}^{\omega}_{s,x}; \frac{|\mathbf{u}|^2}{2} \right\rangle \mathrm{d}x + \mathcal{D}(s) + \frac{1}{2}\sum_{k\geq 1}\int_s^t \int_{\mathbb{T}^3} \left\langle \mathcal{V}^{\omega}_{s,x}; |\mathbf{G}_k(\mathbf{u})|^2 \right\rangle \,\mathrm{d}x\,\mathrm{d}\tau \\ - \frac{1}{2}\sum_{k\geq 1}\int_s^t \int_{\mathbb{T}^3} \Big(\mathcal{Q}_H \left\langle \mathcal{V}^{\omega}_{s,x}; |\mathbf{G}_k(\mathbf{u})| \right\rangle\Big)^2 \mathrm{d}x\,\mathrm{d}\tau + \frac{1}{2}\int_s^t\int_{\mathbb{T}^3} d\mu_D + \int_s^t d\mathcal{M}_E^2,\ \mathbb{P}-a.s. \end{aligned} \tag{2.17}$$

However, as it is evident from Sect. 4, we require energy inequality to hold for *all* $s, t \in (0,T)$ to demonstrate weak-strong uniqueness principle.

2.6. Statements of Main Results

We now state main results of this paper. To begin with, regarding the existence of dissipative measure-valued martingale solutions, we have the following theorem.

Theorem 2.17 (Existence of Measure-Valued Solution). *Assume* $\mathbf{G}_k$ *satisfies* (2.3), (2.4), *and* $\mathbf{u}_\varepsilon$ *be a family of finite energy weak martingale solutions to the stochastic incompressible Navier–Stokes system* (1.2). *Let the corresponding initial data* $\mathbf{u}_0$, *the initial law* Λ, *given on the space* $L^2_{\mathrm{div}}(\mathbb{T}^3)$, *be independent of* ε, *satisfying the following moment estimate*

$$\int_{L^2_{\mathrm{div}}} \|\mathbf{q}\|^p_{L^2_{\mathrm{div}}(\mathbb{T}^3)}\, d\Lambda(\mathbf{q}) < \infty, \tag{2.18}$$

for all $1 \le p < \infty$. *Then there exists a sequence* $\{\widetilde{\mathbf{u}}_\varepsilon\}_{\varepsilon>0}$ *of finite energy weak martingale solutions to stochastic incompressible Navier–Stokes system* (1.2) *on a probability space* $(\widetilde{\Omega}, \widetilde{\mathbb{P}}, \widetilde{\mathbb{F}})$ *such that the family* $\{\widetilde{\mathbf{u}}_\varepsilon\}_{\varepsilon>0}$ *generates a dissipative measure-valued martingale solution* $\big[\big(\widetilde{\Omega}, \widetilde{\mathbb{F}}, (\widetilde{\mathbb{F}}_t)_{t\ge 0}, \widetilde{\mathbb{P}}\big); \widetilde{\mathcal{V}}^\omega_{t,x}, \widetilde{W}\big]$ *to the stochastic incompressible Euler system* (1.1), *in the sense of Definition* 2.15, *with initial data* $\mathcal{V}^\omega_{0,x} = \delta_{\mathbf{u}_0(x)}$ *almost surely.*

We then establish the following weak (measure-valued)-strong uniqueness principle:

Theorem 2.18 (Weak-Strong Uniqueness For Nonlinear Noise). *Let* $\big[\big(\Omega, \mathbb{F}, (\mathbb{F}_t)_{t\ge 0}, \mathbb{P}\big); \mathcal{V}^\omega_{t,x}, W\big]$ *be a dissipative measure-valued martingale solution to the system* (1.1). *On the same stochastic basis* $\big(\Omega, \mathbb{F}, (\mathbb{F}_t)_{t\ge 0}, \mathbb{P}, W\big)$, *let the stochastic incompressible Euler Eq.* (1.1) *possess the unique maximal strong pathwise solution* $(\bar{\mathbf{u}}, (\tau_L)_{L\in\mathbb{N}}, \mathfrak{t})$ *with the initial data* $\bar{\mathbf{u}}(0)$. *If the initial states coincide i.e.,*

$$\mathcal{V}^\omega_{0,x} = \delta_{\bar{\mathbf{u}}(0,x)},\ \mathbb{P}-a.s., \text{for a.e. } x \in \mathbb{T}^3,$$

then for $L \in \mathbb{N}$, *a.e.* $t \in [0,T]$, $\mathcal{D}(t \wedge \tau_L) = 0$, $\mathbb{P}$*-a.s., and for a.e.* $t \in [0,T]$, $\mathbb{P}$*-a.s.*

$$\mathcal{V}^\omega_{t\wedge\tau_L,x} = \delta_{\bar{\mathbf{u}}(t\wedge\tau_L,x)}, \text{ for a.e. } x \in \mathbb{T}^3.$$

Next, we move our attention to the stochastic incompressible Navier–Stokes equations given by (1.2) (with $\varepsilon = 1$). For the deterministic counterpart of (1.2), Prodi [33] and Serrin [35] established weak-strong uniqueness principle under additional regularity on the solution $\mathbf{U} \in L^r([0,T], L^s(\mathbb{T}^3))$, where r and s satisfies the relation $2/r + 3/s = 1$, with $s \in (3, \infty)$. In the stochastic setup, the uniqueness of finite energy weak martingale solutions for (1.2) seems to be out of reach. However, one can obtain a conditional uniqueness result like Prodi and Serrin. In fact, our objective is to give a sufficient condition under which martingale solutions of Navier–Stokes Eq. (1.2) are unique in the class of finite energy weak martingale solutions. Note that in comparison to deterministic analysis, we need additional continuity assumption on stochastic solutions to deal with the stopping time.

Theorem 2.19 (Weak-Strong Uniqueness For Navier–Stokes system). *Let* $\mathbf{u}, \mathbf{U}$ *be two finite energy weak martingale solutions to the system* (1.2) *with same initial data, defined on the same stochastic basis* $\big(\Omega, \mathbb{F}, (\mathbb{F}_t)_{t\ge 0}, W, \mathbb{P}\big)$. *Let the solution* $\mathbf{U}$ *additionally satisfies*

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \|\mathbf{U}\|_{L^s(\mathbb{T}^3)}\Big] < \infty, \quad \text{for some} \quad 3 < s < \infty. \tag{2.19}$$

Then $\mathbb{P}$*-almost surely, for all* $t \in [0,T]$

$$\mathbf{u}(t,x) = \mathbf{U}(t,x), \quad \text{for a.e.} \quad x \in \mathbb{T}^3.$$

3. Proof of Theorem 2.17

The proof of existence is essentially based on the compactness method. For the compactness argument in space and time variables, we make use of Young measure theory, while for the compactness argument in probability variable we rely on the Skorokhod representation theorem. However, as alluded to before, our path spaces are not Polish spaces (which is required for the classical Skorokhod theorem), therefore we rely on the Skorokhod–Jakubowski theorem [32, Corollary 2] which is tailor-made to deal with so-called quasi-Polish spaces. As usual, we obtain the convergence of the approximate sequence on another probability space and the existence of dissipative measure-valued martingale solution follows, thanks to Young measure theory. Observe that the existence of a pathwise solution (typically obtained by Gyöngy–Krylov's characterization of convergence in probability) seems not possible due to the lack of pathwise uniqueness for the underlying system.

3.1. A-Priori Bounds

Recall that, the existence of finite energy weak martingale solution of stochastic incompressible Navier–Stokes system (1.2)

$$\big[\big(\Omega,\mathbb{F},(\mathbb{F}_{\varepsilon,t})_{t\ge 0},\mathbb{P}\big);\mathbf{u}_\varepsilon,W\big]$$

is well established, thanks to the Theorem 2.13. Observe that the filtration $(\mathbb{F}_{\varepsilon,t})_{t\geq 0}$ depends on ε, and lack of pathwise uniqueness for (1.2) does not allow us to choose the filtration independent of ε. Having said this, however, note that the Brownian motion and the probability space can be chosen indepedent of ε.

To obtain a-priori estimate for the approximate solution, we make use of the energy inequality (2.13) to obtain for all $t \in [0,T]$, and $1 \le p < \infty$

$$\mathbb{E}\Big[\|\mathbf{u}_\varepsilon(t)\|^2_{L^2(\mathbb{T}^3)}\Big]^p \le \mathbb{E}\Big[\|\mathbf{u}_0\|^2_{L^2(\mathbb{T}^3)}\Big]^p + C\int_0^t \left(1+\mathbb{E}\Big[\|\mathbf{u}_\varepsilon(s)\|^2_{L^2(\mathbb{T}^3)}\Big]^p\right) ds.$$

Therefore, a simple application Gronwall lemma yield

$$\mathbb{E}\Big[\|\mathbf{u}_\varepsilon\|^2_{L^2(\mathbb{T}^3)}\Big]^p \le C\left(1+\mathbb{E}\Big[\|\mathbf{u}_0\|^2_{L^2(\mathbb{T}^3)}\Big]^p\right)$$

Again by energy inequality (2.13), we have for $1 \le p < \infty$

$$\begin{aligned}
&\mathbb{E}\bigg[\sup_{t\in[0,T]}\|\mathbf{u}_\varepsilon\|^2_{L^2(\mathbb{T}^3)} + \varepsilon\int_0^T \|\nabla_x \mathbf{u}_\varepsilon\|^2_{L^2(\mathbb{T}^3)}\,\mathrm{d}t\bigg]^p \\
&\quad\le C\int_{L^2_{\mathrm{div}}(\mathbb{T}^3)} \|\mathbf{q}\|^{2p}_{L^2_{\mathrm{div}}(\mathbb{T}^3)}\,\mathrm{d}\Lambda(\mathbf{q}) + C\,\mathbb{E}\bigg[\sup_{t\in[0,T]}\int_{\mathbb{T}^3}\int_0^t \mathbf{u}_\varepsilon.\mathcal{P}_H\mathcal{G}(\mathbf{u}_\varepsilon)\,\mathrm{d}W\,\mathrm{d}x\bigg]^p.
\end{aligned}$$

To handle the right most term of the above inequality, we make use of the classical Burkholder–Davis–Gundy (BDG) inequality to obtain

$$\begin{aligned}
&\mathbb{E}\bigg[\sup_{t\in[0,T]}\int_{\mathbb{T}^3}\int_0^t \mathbf{u}_\varepsilon.\mathcal{P}_H\mathcal{G}(\mathbf{u}_\varepsilon)\,\mathrm{d}W\,\mathrm{d}x\bigg]^p \le C\,\mathbb{E}\bigg[\int_0^T \sum_{k\ge 1}\Big(\int_{\mathbb{T}^3}\mathcal{P}_H\mathbf{G}_k(\mathbf{u}_\varepsilon)\mathbf{u}_\varepsilon\,\mathrm{d}x\Big)^2\bigg]^{p/2} \\
&\le C\,\mathbb{E}\bigg[\int_0^T \|\mathbf{u}_\varepsilon(t)\|^2_{L^2(\mathbb{T}^3)}\sum_{k\ge 1}\|\mathbf{G}_k(\mathbf{u}_\varepsilon(t))\|^2_{L^2(\mathbb{T}^3)}\,\mathrm{d}t\bigg]^{p/2} \\
&\le C\,\mathbb{E}\bigg[\int_0^T \|\mathbf{u}_\varepsilon(t)\|^2_{L^2(\mathbb{T}^3)}\,(1+\|\mathbf{u}_\varepsilon(t)\|^2_{L^2(\mathbb{T}^3)})\,\mathrm{d}t\bigg]^{p/2}
\end{aligned}$$

$$\le C\mathbb{E}\bigg[\int_0^T \|\mathbf{u}_\varepsilon(t)\|_{L^2(\mathbb{T}^3)}^2\,\mathrm{d}t\bigg]^p + \mathbb{E}\bigg[\int_0^T (1+\|\mathbf{u}_\varepsilon(t)\|_{L^2(\mathbb{T}^3)}^2)\,\mathrm{d}t\bigg]^p \le C\bigg(1+\int_{L^2_{\mathrm{div}}(\mathbb{T}^3)} \|\mathbf{q}\|_{L^2_{\mathrm{div}}(\mathbb{T}^3)}^{2p}\,\mathrm{d}\Lambda(\mathbf{q})\bigg).$$

This implies that, for any $1 \le p < \infty$, we have

$$\mathbb{E}\bigg[\sup_{0\le t\le T}\int_{\mathbb{T}^3} |\mathbf{u}_\varepsilon|^2\,\mathrm{d}x + \varepsilon\int_0^T\int_{\mathbb{T}^3} |\nabla_x \mathbf{u}_\varepsilon|^2 \mathrm{d}x\,\mathrm{d}s\bigg]^p \le C\bigg(1+\int_{L^2_{\mathrm{div}}(\mathbb{T}^3)} \|\mathbf{q}\|_{L^2_{\mathrm{div}}(\mathbb{T}^3)}^{2p}\,\mathrm{d}\Lambda(\mathbf{q})\bigg) \le C(p,\Lambda,T)$$

Above relation leads to the following uniform bound

$$\mathbf{u}_\varepsilon \in L^p(\Omega; L^\infty(0,T; L^2_{\mathrm{div}}(\mathbb{T}^3))). \tag{3.1}$$

3.2. Tightness and Almost Sure Representations

For our purpose, to secure almost sure convergence in the probability variable (ω-variable) we make use of the Skorokhod–Jakubowski version [26,32] of the classical Skorokhod representation theorem. It is well-known that such a result can be obtained by establishing the tightness of probability measures related to the random variables in quasi-Polish spaces. In what follows, our first aim is to establish the tightness of the probability measures (laws) generated by the approximate solutions. To do so, we first introduce the following path space $\mathcal{Y}$ for these measures:

$$\begin{aligned}
&\mathcal{Y}_{\mathbf{u}} = C_w([0,T]; L^2_{\mathrm{div}}(\mathbb{T}^3)), && \mathcal{Y}_W = C([0,T]; \mathfrak{U}_0),\\
&\mathcal{Y}_C = \big(L^\infty(0,T;\mathcal{M}_b(\mathbb{T}^3)), w^*\big), && \mathcal{Y}_E = \big(L^\infty(0,T;\mathcal{M}_b(\mathbb{T}^3)), w^*\big),\\
&\mathcal{Y}_D = \big(L^\infty(0,T;\mathcal{M}_b(\mathbb{T}^3)), w^*\big) && \mathcal{Y}_{\mathcal{V}} = \big(L^\infty((0,T)\times\mathbb{T}^3;\mathcal{P}(\mathbb{R}^3)), w^*\big),\\
&\mathcal{Y}_X = C([0,T]; H^{-1}_{\mathrm{div}}(\mathbb{T}^3)), && \mathcal{Y}_Y = C([0,T];\mathbb{R}),\\
&\mathcal{Y}_F = \big(L^\infty(0,T;\mathcal{M}_b(\mathbb{T}^3)), w^*\big),
\end{aligned}$$

Let us denote by $\mu_{\mathbf{u}_\varepsilon}$, and μ_{W_ε} respectively, the law of $\mathbf{u}_\varepsilon$, and W_ε on the corresponding path space. Moreover, for the martingale terms, let μ_{X_ε}, and μ_{Y_ε} denote the law $X_\varepsilon := \int_0^t \mathcal{P}_H\mathcal{G}(\mathbf{u}_\varepsilon)\,\mathrm{d}W$, and $Y_\varepsilon := \int_0^t\int_{\mathbb{T}^3} \mathbf{u}_\varepsilon \cdot \mathcal{P}_H\mathcal{G}_k(\mathbf{u}_\varepsilon)\,\mathrm{d}x\,\mathrm{d}W$ on the corresponding path spaces respectively. Furthermore, let μ_{C_ε}, μ_{D_ε}, μ_{E_ε}, and $\mu_{\mathcal{V}_\varepsilon}$ denote the law of

$$C_\varepsilon := \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon, \quad D_\varepsilon := \sum_{k\ge1} |\mathbf{G}_k(\mathbf{u}_\varepsilon)|^2, \quad E_\varepsilon := \frac{1}{2}|\mathbf{u}_\varepsilon|^2, \quad \mathcal{V}_\varepsilon := \delta_{\mathbf{u}_\varepsilon}, \quad F_\varepsilon := \frac{1}{2}\sum_{k\ge1} |\mathcal{Q}_h \mathbf{G}_k(u_\varepsilon)|^2,$$

respectively, on the corresponding path spaces. Finally, we denote by μ^ε, the joint law of all the variables on $\mathcal{Y}$. As stated before, our aim is now to establish tightness of $\{\mu^\varepsilon;\, \varepsilon \in (0,1)\}$. To this end, first note that tightness of μ_{W_ε} is straightforward. Therefore, we focus on proving tightness of other variables.

Proposition 3.1 *The set $\{\mu_{\mathbf{u}_\varepsilon};\, \varepsilon \in (0,1)\}$ is tight on $\mathcal{Y}_{\mathbf{u}}$.*

Proof For convenience, we rewrite the Eq. (2.11) as

$$\int_{\mathbb{T}^3} \mathbf{u}_\varepsilon(t)\cdot\boldsymbol{\varphi}\,\mathrm{d}x = \int_{\mathbb{T}^3} \mathbf{u}_\varepsilon(0)\cdot\boldsymbol{\varphi}\,\mathrm{d}x + \int_0^t\int_{\mathbb{T}^3} I_\varepsilon(s) : \nabla\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_{\mathbb{T}^3} \mathcal{P}_H\mathcal{G}(\mathbf{u}_\varepsilon)\cdot\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}W$$

for all $t \in [0,T]$, for all $\boldsymbol{\varphi} \in C^\infty_{\mathrm{div}}(\mathbb{T}^3)$. where

$$I_\varepsilon := -\varepsilon\nabla\mathbf{u}_\varepsilon + \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon$$

From the a priori estimate in (3.1), we obtain

$$I_\varepsilon \in L^1(\Omega; L^2(0,T; L^1(\mathbb{T}^3))) \subset L^1(\Omega; L^2(0,T; W^{-2,2}(\mathbb{T}^3)))$$

uniformly in ε. Let us consider the functional

$$\langle \mathcal{I}_\varepsilon(t), \boldsymbol{\varphi}\rangle := \int_0^t\int_{\mathbb{T}^3} I_\varepsilon(s) : \nabla\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}s,$$

which is related to the deterministic part of equation. Then we deduce from above the following estimate

$$\mathbb{E}\Big[\|\mathcal{I}_\varepsilon\|_{W^{1,2}(0,T;W^{-3,2}_{\mathrm{div}}(\mathbb{T}^3))}\Big] \le C(T).$$

For the stochastic term, we have, for $a > 2$

$$\begin{aligned}
&\mathbb{E}\Bigg[\Big\|\int_t^s \mathcal{P}_H\mathcal{G}(\mathbf{u}_\varepsilon)dW\Big\|^a_{L^2(\mathbb{T}^3)}\Bigg]\\
&\le C\,\mathbb{E}\Bigg[\Big(\int_t^s \|\mathcal{P}_H\mathcal{G}(\mathbf{u}_\varepsilon)\|^2_{L_2(\mathfrak{U},L^2(\mathbb{T}^3))}d\sigma\Big)^{a/2}\Bigg] \le C\,\mathbb{E}\Bigg[\Big(\int_t^s (1+\|\mathbf{u}_\varepsilon(\sigma)\|^2_{L^2(\mathbb{T}^3)})d\sigma\Big)^{a/2}\Bigg]\\
&\le C\left(|t-s|^{a/2}\Big(1+\mathbb{E}\Big[\sup_{t\in[0,T]}\|\mathbf{u}_\varepsilon(t)\|_{L^2(\mathbb{T}^3)}\Big]^{a/2}\Big)\right) \le C\,|t-s|^{a/2}\Big(1+\mathbb{E}\Big[\|\mathbf{u}_\varepsilon(0)\|^a_{L^2(\mathbb{T}^3)}\Big]\Big).
\end{aligned}$$

As consequence of Kolmogorov continuity theorem (cf. Lemma 2.5), we have

$$\mathbb{E}\Bigg[\Big\|\int_0^\cdot \mathcal{P}_H\mathcal{G}(\mathbf{u}_\varepsilon)dW\Big\|^a_{C^\alpha([0,T];L^2_{\mathrm{div}}(\mathbb{T}^3))}\Bigg] \le C$$

for all $\alpha \in (\frac{1}{a}, \frac{1}{2})$, and $a > 2$. Combining the previous estimates and using the embeddings $W^{1,2}(0,T) \subset C^{1/2}[0,T]$, and $L^2_{\mathrm{div}}(\mathbb{T}^3) \subset W^{-3,2}_{\mathrm{div}}(\mathbb{T}^3)$, we conclude

$$\mathbb{E}\Big[\|\mathbf{u}_\varepsilon\|_{C^\alpha([0,T];W^{-3,2}_{\mathrm{div}}(\mathbb{T}^3))}\Big] \le C(T),$$

for some $\alpha < \frac{1}{2}$. Next, we recall the following compact embedding [6, Chapter 1]

$$C^\alpha([0,T];W^{-3,2}_{\mathrm{div}}) \cap L^\infty(0,T;L^2_{\mathrm{div}}(\mathbb{T}^3)) \subset\subset C_w([0,T];L^2_{\mathrm{div}}(\mathbb{T}^3)),$$

to conclude that $\mu_{\mathbf{u}_\varepsilon}$ is tight. □

Proposition 3.2 *The set* $\{\mu_{C_\varepsilon}, \mu_{D_\varepsilon}, \mu_{E_\varepsilon}, \mu_{F_\varepsilon};\, \varepsilon \in (0,1),\, k \ge 1\}$ *is tight on* $\mathcal{Y}_C \times \mathcal{Y}_D \times \mathcal{Y}_E \times \mathcal{Y}_F$.

Proof By making use of the *a priori* bound (3.1), and the fact that all bounded sets in $L^\infty_{w*}(0,T;\mathcal{M}_b(\mathbb{T}^3))$ are relatively compact with respect to the weak-$*$ topology, we obtain the desired result. □

Proposition 3.3 *The set* $\{\mu_{\mathcal{V}_\varepsilon};\, \varepsilon \in (0,1)\}$ *is tight on* $\mathcal{Y}_\mathcal{V}$.

Proof This follows from the compactness criterion in $\big(L^\infty_{w*}((0,T)\times\mathbb{T}^3;\mathcal{P}(\mathbb{R}^3)), w^*\big)$. To see that, define the set

$$B_M := \Big\{\mathcal{V} \in \big(L^\infty((0,T)\times\mathbb{T}^3;\mathcal{P}(\mathbb{R}^3)), w^*\big);\, \int_0^T\int_{\mathbb{T}^3}\int_{\mathbb{R}^3} |\xi_1|^2\, d\mathcal{V}_{t,x}(\xi)\,\mathrm{d}x\,\mathrm{d}t \le M\Big\},$$

which is relatively compact in $\big(L^\infty((0,T)\times\mathbb{T}^3;\mathcal{P}(\mathbb{R}^3)), w^*\big)$. Notice that

$$\begin{aligned}
\mathcal{L}[\mathcal{V}_\varepsilon](B_M^c) &= \mathbb{P}\left(\int_0^T\int_{\mathbb{T}^3}\int_{\mathbb{R}^3}\Big(|\xi_1|^2\, d\mathcal{V}_{t,x}(\xi)\,\mathrm{d}x\,\mathrm{d}t > M\right)\\
&= \mathbb{P}\left(\int_0^T\int_{\mathbb{T}^3}|\mathbf{u}_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t > M\right) \le \frac{1}{M}\mathbb{E}\Big[\|\mathbf{u}_\varepsilon\|^2_{L^2(\mathbb{T}^3)}\Big] \le \frac{C}{M}.
\end{aligned}$$

The finishes the proof. □

Proposition 3.4 *The set* $\{\mu_{X_\varepsilon};\, \varepsilon \in (0,1)\}$ *is tight on* $\mathcal{Y}_X$.

Proof To prove the result, it is enough to observe that the random variable $X_\varepsilon = \int_0^t \mathcal{P}_H\mathcal{G}(\mathbf{u}_\varepsilon)\,\mathrm{d}W(s) \in L^p\big(\Omega; W^{\alpha,q}(0,T;L^2_{\mathrm{div}}(\mathbb{T}^3))\big)$, for $q \ge 2$ (see [22]). Therefore, a simple application of the compact embedding result given in Lemma 2.1 yields required tightness. □

Proposition 3.5 *The set $\{\mu_{Y_\varepsilon};\, \varepsilon \in (0,1)\}$ is tight on $\mathcal{Y}_Y$.*

Proof It is easy to see that, $Y_\varepsilon(t) = \sum_{k\ge1}\int_0^t\int_{\mathbb{T}^3}\mathbf{u}_\varepsilon\cdot\mathcal{P}_H\mathbf{G}_k(\mathbf{u}_\varepsilon)\,\mathrm{d}x\,\mathrm{d}W$ is a square integrable martingale, for every $\varepsilon\in(0,1)$. Notice that for $a>2$

$$\mathbb{E}\Big[\Big|\sum_{k\ge1}\int_s^t\int_{\mathbb{T}^3}\mathbf{u}_\varepsilon\cdot\mathcal{P}_H\mathbf{G}_k(\mathbf{u}_\varepsilon)\Big|^a\Big] \le \mathbb{E}\Big[\int_s^t\sum_{k=1}^\infty\Big|\int_{\mathbb{T}^3}\mathbf{u}_\varepsilon\cdot\mathbf{G}_k(\mathbf{u}_\varepsilon)\Big|^2\Big]^{a/2}$$
$$\le |t-s|^{a/2}\Big(1+\mathbb{E}\Big[\sup_{0\le t\le T}\|\mathbf{u}_\varepsilon\|^a_{L^2(\mathbb{T}^3)}\Big]\Big)\le C|t-s|^{a/2}.$$

Therefore, we can apply the classical Kolmogorov continuity theorem (cf. Lemma 2.5) to conclude that, for some $\beta>0$

$$\sum_{k\ge1}\int_0^t\int_{\mathbb{T}^3}\mathbf{u}_\varepsilon\cdot\mathcal{P}_H\mathbf{G}_k(\mathbf{u}_\varepsilon)\,\mathrm{d}x\,\mathrm{d}W \in L^a(\Omega; C^\beta(0,T;\mathbb{R})).$$

Therefore, using the well-known compact embedding of C^β into C^0, tightness of law follows. □

Making use of results obtained from Propositions 3.1, 3.2, 3.3, 3.4 and 3.5, we conclude that

Corollary 3.6 *The set $\{\mu^\varepsilon;\, \varepsilon\in(0,1)\}$ is tight on $\mathcal{Y}$.*

Having secured all necessary tightness results, we can now apply Jakubowski–Skorokhod representation theorem (see also Motyl [32]) to extract almost sure convergence on a new probability space. In that context, we infer the following result:

Proposition 3.7 *There exists a subsequence μ^ε (not relabeled), a probability space $(\tilde\Omega,\tilde{\mathbb{F}},\tilde{\mathbb{P}})$ with $\mathcal{Y}$-valued Borel measurable random variables $(\tilde{\mathbf{u}}_\varepsilon,\tilde W_\varepsilon,\tilde C_\varepsilon,\tilde D_\varepsilon,\tilde E_\varepsilon,\tilde X_\varepsilon,\tilde Y_\varepsilon,\tilde F_\varepsilon,\tilde\nu_\varepsilon)$, $\varepsilon\in(0,1)$, and $(\tilde{\mathbf{u}},\tilde W,\tilde C,\tilde D,\tilde E,\tilde X,\tilde Y,\tilde F,\tilde\nu)$ such that*

(1) *the law of $(\tilde{\mathbf{u}}_\varepsilon,\tilde W_\varepsilon,\tilde C_\varepsilon,\tilde D_\varepsilon,\tilde E_\varepsilon,\tilde X_\varepsilon,\tilde Y_\varepsilon,\tilde F_\varepsilon,\tilde\nu_\varepsilon)$ is given by μ^ε, $\varepsilon\in(0,1)$,*
(2) *the law of $(\tilde{\mathbf{u}},\tilde W,\tilde C,\tilde D,\tilde E,\tilde X,\tilde Y,\tilde F,\tilde\nu)$, denoted by μ, is a Radon measure,*
(3) *$(\tilde{\mathbf{u}}_\varepsilon,\tilde W_\varepsilon,\tilde C_\varepsilon,\tilde D_\varepsilon,\tilde E_\varepsilon,\tilde X_\varepsilon,\tilde Y_\varepsilon,\tilde F_\varepsilon,\,\tilde\nu_\varepsilon)$ converges $\tilde{\mathbb{P}}$-almost surely to $(\tilde{\mathbf{u}},\tilde W,\tilde C,\tilde D,\tilde E,\tilde X,\tilde Y,\tilde F,\tilde\nu)$ in the topology of $\mathcal{Y}$, i.e.,*

$$\begin{aligned}
&\tilde{\mathbf{u}}_\varepsilon\to\tilde{\mathbf{u}} \text{ in } C_w([0,T];L^2_{\mathrm{div}}(\mathbb{T}^3)), && \tilde W_\varepsilon\to\tilde W \text{ in } C([0,T];\mathcal{U}_0)),\\
&\tilde C_\varepsilon\to\tilde C \text{ weak-}* \text{ in } L^\infty_{w*}(0,T;\mathcal{M}_b(\mathbb{T}^3)), && \tilde D_\varepsilon\to\tilde D \text{ weak-}* \text{ in } L^\infty_{w*}(0,T;\mathcal{M}_b(\mathbb{T}^3)),\\
&\tilde Y_\varepsilon\to\tilde Y \text{ in } C([0,T];\mathbb{R}), && \tilde E_\varepsilon\to\tilde E \text{ weak-}* \text{ in } L^\infty_{w*}(0,T;\mathcal{M}_b(\mathbb{T}^3)),\\
&\tilde\nu_\varepsilon\to\tilde\nu \text{ weak-}* \text{ in } L^\infty_{w*}((0,T)\times\mathbb{T}^3;\mathcal{P}(\mathbb{R}^3)), && \tilde X_\varepsilon\to\tilde X \text{ in } C([0,T];W^{-1,2}_{\mathrm{div}}(\mathbb{T}^3)),\\
&\tilde F_\varepsilon\to\tilde F \text{ weak-}* \text{ in } L^\infty_{w*}(0,T;\mathcal{M}_b(\mathbb{T}^3)),
\end{aligned}$$

(4) *For any ε, $\tilde W_\varepsilon=\tilde W$, $\tilde{\mathbb{P}}$-a.s.*
(5) *For any Carathéodory function $J=J(t,x,\mathbf{u})$, where $(t,x)\in(0,T)\times\mathbb{T}^3$ and $\mathbf{u}\in\mathbb{R}^3$, satisfying for some p the growth condition $|J(t,x,\mathbf{u})|\le1+|\mathbf{u}|^p$, uniformly in (t,x). Then we have $\tilde{\mathbb{P}}$-a.s.*

$$J(\tilde{\mathbf{u}}_\varepsilon)\to\overline{J(\tilde{\mathbf{u}})} \quad \text{in} \quad L^r((0,T)\times\mathbb{T}^3), \quad \text{for all} \quad 1<r\le\frac{2}{p}.$$

Proof Proof of the items (1), (2) and (3) directly follow from Jakubowski–Skorokhod representation theorem. For the proof of the item (4), we refer to Theorem 2.4, and [32]. For the proof of the item (5), we refer to the Lemma 2.2. □

3.2.1. Passing to the Limit. Note that in view of the equality of joint laws, the energy inequality (2.10) and the a priori estimate (3.1) for the new random variables hold on the new probability space. Making use of convergence results given by Proposition 3.7, we can now pass to the limit in approximate Eq. (1.2), and the energy inequality (2.10). First we show that the approximations $\tilde{\mathbf{u}}_\varepsilon$ solve the equation given by (1.2) on the new probability space $(\tilde{\Omega}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}})$. For that purpose, let us denote by $(\tilde{\mathbb{F}}_t^\varepsilon)$ and $(\tilde{\mathbb{F}}_t)$, $\tilde{\mathbb{P}}$-augmented canonical filtrations of the process $(\tilde{\mathbf{u}}_\varepsilon, \tilde{W}_\varepsilon)$ and $(\langle \tilde{\mathcal{V}}^\omega_{t,x}; \tilde{\mathbf{u}}\rangle, \tilde{W}, \tilde{X}, \tilde{Y})$, respectively. This means

$$\tilde{\mathbb{F}}_t^\varepsilon = \sigma\big(\sigma\big(\mathbf{s}_t\tilde{\mathbf{u}}_\varepsilon,\, \mathbf{s}_t\tilde{W}_\varepsilon\big) \cup \big\{N \in \tilde{\mathbb{F}};\ \tilde{\mathbb{P}}(N) = 0\big\}\big), \quad t \in [0,T],$$
$$\tilde{\mathbb{F}}_t = \sigma\big(\sigma\big(\mathbf{s}_t\langle \tilde{\mathcal{V}}^\omega_{t,x}; \tilde{\mathbf{u}}\rangle,\, \mathbf{s}_t\tilde{W},\, \mathbf{s}_t\tilde{X}, \mathbf{s}_t\tilde{Y}\big) \cup \big\{N \in \tilde{\mathbb{F}};\ \tilde{\mathbb{P}}(N) = 0\big\}\big), \quad t \in [0,T],$$

where $\mathbf{s}_t$ is the restriction operator to the interval $[0,t]$ acting on various path spaces.

Proposition 3.8 *For every $\varepsilon \in (0,1)$, $\big((\tilde{\Omega}, \tilde{\mathbb{F}}, (\tilde{\mathbb{F}}_{\varepsilon,t})_{t\geq 0}, \tilde{\mathbb{P}}), \tilde{\mathbf{u}}_\varepsilon, \tilde{W}\big)$ is a finite energy weak martingale solution to (1.2) with the initial law Λ_ε.*

Proof Proof of the above proposition is standard, and one can furnish the proof following the same line of argument, as in the monograph by Breit et al. [7, Theorem 2.9.1]. For brevity, we skip all the details. □

We remark that, in light of the above proposition, the new random variables satisfy the following equations and the energy inequality on the new probability space

- for all $\boldsymbol{\varphi} \in C^\infty_{\mathrm{div}}(\mathbb{T}^3)$ we have

$$\langle \tilde{\mathbf{u}}_\varepsilon(t), \boldsymbol{\varphi}\rangle = \langle \tilde{\mathbf{u}}_\varepsilon(0), \boldsymbol{\varphi}\rangle - \int_0^t \langle \tilde{\mathbf{u}}_\varepsilon \otimes \tilde{\mathbf{u}}_\varepsilon, \nabla_x \boldsymbol{\varphi}\rangle\,\mathrm{d}s + \varepsilon \int_0^t \langle \nabla_x \tilde{\mathbf{u}}_\varepsilon\,, \nabla_x \boldsymbol{\varphi}\rangle\,\mathrm{d}s + \int_0^t \langle \mathcal{G}(\tilde{\mathbf{u}}_\varepsilon), \boldsymbol{\varphi}\rangle\,\mathrm{d}W \tag{3.2}$$

  $\mathbb{P}$-a.s. for all $t \in [0,T]$,
- the energy inequality

$$\begin{aligned}
&- \int_0^T \partial_t \phi \int_{\mathbb{T}^3} \frac{1}{2}|\tilde{\mathbf{u}}_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t + \varepsilon \int_0^T \phi \int_{\mathbb{T}^3} |\nabla_x \tilde{\mathbf{u}}_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\leq \phi(0) \int_{\mathbb{T}^3} \frac{1}{2}|\mathbf{u}_\varepsilon(0)|^2\,\mathrm{d}x + \sum_{k=1}^\infty \int_0^T \phi\Big(\int_{\mathbb{T}^3} \mathbf{G}_k(\tilde{\mathbf{u}}_\varepsilon)\cdot \tilde{\mathbf{u}}_\varepsilon\,\mathrm{d}x\Big)\mathrm{d}W_k \\
&+ \frac{1}{2}\sum_{k=1}^\infty \int_0^T \phi \int_{\mathbb{T}^3} |\mathcal{P}_H \mathbf{G}_k(\tilde{\mathbf{u}}_\varepsilon)|^2\,\mathrm{d}t
\end{aligned} \tag{3.3}$$

  holds $\mathbb{P}$-a.s., for all $\phi \in C_c^\infty([0,T))$, $\phi \geq 0$.

Now we are in a position to pass to the limit in ε in (3.2) and (3.3). To see this, note that we have a-priori estimate (3.1) for the new random variable. Therefore, an application of Lemma 2.2 helps us to conclude that $\tilde{\mathbb{P}}$-a.s.,

$$\tilde{\mathbf{u}}_\varepsilon \rightharpoonup \tilde{\mathbf{u}} = \langle \tilde{\mathcal{V}}^\omega_{t,x}; \tilde{\mathbf{u}}\rangle, \quad \text{weakly in} \quad L^2((0,T); L^2_{\mathrm{div}}(\mathbb{T}^3)).$$

Moreover, making use of item (5) of Lemma 3.7, we conclude that $\mathbb{P}$-a.s.

$$\mathbf{G}_k(\tilde{\mathbf{u}}_\varepsilon) \rightharpoonup \big\langle \tilde{\mathcal{V}}^\omega_{t,x}; \mathbf{G}_k(\tilde{\mathbf{u}})\big\rangle \quad \text{weakly in} \quad L^2((0,T); L^2(\mathbb{T}^3)).$$

This, in particular, implies that $\mathbb{P}$-a.s.

$$\mathcal{Q}_H \mathbf{G}_k(\tilde{\mathbf{u}}_\varepsilon) \rightharpoonup \mathcal{Q}_H\big\langle \tilde{\mathcal{V}}^\omega_{t,x}; \mathbf{G}_k(\tilde{\mathbf{u}})\big\rangle \quad \text{weakly in} \quad L^2((0,T); (L^2(\mathbb{T}^3))^\perp).$$

Indeed, Let $\mathbf{v} \in L^2((0,T); (L^2(\mathbb{T}^3))^\perp)$ then orthogonal property of projection $\mathcal{Q}_H$ implies that

$$\lim_{\varepsilon\to 0}\langle \mathcal{Q}_H \mathbf{G}_k(\tilde{\mathbf{u}}_\varepsilon), \mathbf{v}\rangle = \lim_{\varepsilon\to 0}\langle \mathbf{G}_k(\tilde{\mathbf{u}}_\varepsilon), \mathbf{v}\rangle = \langle \langle \tilde{\mathcal{V}}^\omega_{t,x}; \mathbf{G}_k(\tilde{\mathbf{u}})\rangle, \mathbf{v}\rangle = \langle \mathcal{Q}_H \langle \tilde{\mathcal{V}}^\omega_{t,x}; \mathbf{G}_k(\tilde{\mathbf{u}})\rangle, \mathbf{v}\rangle$$

As usual, to identify the weak limits related to the nonlinear terms present in the equations, we first need to introduce corresponding concentration defect measures

$$\tilde{\mu}_C = \tilde{C} - \left\langle \tilde{\mathcal{V}}^{\omega}_{(\cdot,\cdot)}; \tilde{\mathbf{u}} \otimes \tilde{\mathbf{u}} \right\rangle \mathrm{d}x\,\mathrm{d}t,\ \tilde{\mu}_E = \tilde{E} - \left\langle \tilde{\mathcal{V}}^{\omega}_{(\cdot,\cdot)}; \frac{1}{2}|\tilde{\mathbf{u}}|^2 \right\rangle \mathrm{d}x\,\mathrm{d}t,$$
$$\tilde{\mu}_D = \tilde{D} - \left\langle \tilde{\mathcal{V}}^{\omega}_{(\cdot,\cdot)}; \sum_{k\geq 1} |\mathbf{G}_k(\tilde{\mathbf{u}})|^2 \right\rangle \mathrm{d}x\,\mathrm{d}t,\ \tilde{\mu}_F = \tilde{F} - \left| \mathcal{Q}_H \left\langle \tilde{\mathcal{V}}^{\omega}_{(\cdot,\cdot)}; \sum_{k\geq 1} |\mathbf{G}_k(\tilde{\mathbf{u}})|^2 \right\rangle \right|^2 \mathrm{d}x\,\mathrm{d}t.$$

In view of the discussion in Sect. 2.1.1, and making use of above concentration defect measures, we can conclude that $\tilde{\mathbb{P}}$ almost surely

$$\tilde{C}_\varepsilon \rightharpoonup \left\langle \tilde{\mathcal{V}}^{\omega}_{(\cdot,\cdot)}; \tilde{\mathbf{u}} \otimes \tilde{\mathbf{u}} \right\rangle \mathrm{d}x\,\mathrm{d}t + \tilde{\mu}_C, \quad \text{weak-}* \text{ in } L^\infty_{w*}(0,T;\mathcal{M}_b(\mathbb{T}^3)),$$
$$\tilde{D}_\varepsilon \rightharpoonup \left\langle \tilde{\mathcal{V}}^{\omega}_{(\cdot,\cdot)}; \sum_{k\geq 1} |\mathbf{G}_k(\tilde{\mathbf{u}})|^2 \right\rangle \mathrm{d}x\,\mathrm{d}t + \tilde{\mu}_D,\ \text{weak-}* \text{ in } L^\infty_{w*}(0,T;\mathcal{M}_b(\mathbb{T}^3)),$$
$$\tilde{E}_\varepsilon \rightharpoonup \left\langle \tilde{\mathcal{V}}^{\omega}_{(\cdot,\cdot)}; \frac{1}{2}|\tilde{\mathbf{u}}|^2 \right\rangle \mathrm{d}x\,\mathrm{d}t + \tilde{\mu}_E,\ \text{weak-}* \text{ in } L^\infty_{w*}(0,T;\mathcal{M}^+_b(\mathbb{T}^3)),$$
$$\tilde{F}_\varepsilon \rightharpoonup \left| \mathcal{Q}_H \left\langle \tilde{\mathcal{V}}^{\omega}_{(\cdot,\cdot)}; \sum_{k\geq 1} |\mathbf{G}_k(\tilde{\mathbf{u}})|^2 \right\rangle \right|^2 \mathrm{d}x\,\mathrm{d}t + \tilde{\mu}_F,\ \text{weak-}* \text{ in } L^\infty_{w*}(0,T;\mathcal{M}^+_b(\mathbb{T}^3)).$$

Note that both defect measures $\tilde{\mu}_E$, and $\tilde{\mu}_F$ are positive, thanks to the lower semi-continuity property of norms. Next, we move on to the martingale terms $\tilde{X}_\varepsilon$, appearing in the momentum equation, and $\tilde{Y}_\varepsilon$, appearing in the energy inequality. Regarding convergence of these terms, we state the following propositions.

Proposition 3.9 *For every time $t \in [0,T]$, $\tilde{\mathbb{P}}$-almost surely $\tilde{Y}_\varepsilon(t) \to \tilde{Y}(t)$ in $\mathbb{R}$, where $\tilde{Y}(t)$ is a real valued square-integrable martingale with respect to the filtration $(\tilde{\mathbb{F}}_t)$.*

Proof First of all, in view of the Proposition 3.7, we conclude that $\tilde{Y}_\varepsilon \to \tilde{Y}$ $\tilde{\mathbb{P}}$-a.s. in $C([0,T];\mathbb{R})$. To claim that $\tilde{Y}(t)$ is a martingale, as usual, it is sufficient to show that

$$\tilde{\mathbb{E}}[\tilde{Y}(t)|\tilde{\mathcal{F}}_s] = \tilde{Y}(s),$$

for all $t,s \in [0,T]$ with $s \leq t$. In other words, it is enough to prove that

$$\tilde{\mathbb{E}}\Big[\mathfrak{L}_s(\tilde{\Phi})\big(\tilde{Y}(t) - \tilde{Y}(s)\big)\Big] = 0,$$

where we denote $\tilde{\Phi} := (\langle \tilde{\mathcal{V}}^{\omega}_{t,x}; \tilde{\mathbf{u}} \rangle, \tilde{W}, \tilde{X}, \tilde{Y})$, and on the path space $\underline{\mathcal{Y}} := \mathcal{Y}_{\mathbf{u}} \times \mathcal{Y}_W \times \mathcal{Y}_X \times \mathcal{Y}_Y$, we denote $\mathfrak{L}_s$ by any bounded continuous functional which depends on on the values of $\tilde{\Phi}$ restricted to $[0,s]$. The idea is to use the fact that $\tilde{Y}_\varepsilon(t)$ is a martingale, i.e.,

$$\tilde{\mathbb{E}}\Big[\mathfrak{L}_s(\tilde{\Phi}_\varepsilon)\big(\tilde{Y}_\varepsilon(t) - \tilde{Y}_\varepsilon(s)\big)\Big] = 0,$$

for all bounded continuous functional $\mathfrak{L}_s$ on the same path space, and $\tilde{\Phi}_\varepsilon = (\tilde{\mathbf{u}}_\varepsilon, \tilde{W}, \tilde{X}_\varepsilon, \tilde{Y}_\varepsilon)$. At this point, we recall Proposition 3.7 to conclude that $\tilde{\Phi}_\varepsilon \to \tilde{\Phi}$, $\tilde{\mathbb{P}}$-a.s. in the (weak) topology of $\underline{\mathcal{Y}}$. This, in particular, implies that $\mathfrak{L}_s(\tilde{\Phi}_\varepsilon) \to \mathfrak{L}_s(\tilde{\Phi})$ $\tilde{\mathbb{P}}$-a.s. Now given this property, along with the fact that $\tilde{Y}_\varepsilon(t) \in L^2(\tilde{\Omega})$, we may apply classical Vitali's convergence theorem to pass to the limit in ε to conclude that $\tilde{Y}(t)$ is a martingale. □

Proposition 3.10 *For each time $t \in [0,T]$, $\tilde{X}_\varepsilon(t) \to \tilde{X}(t)$, $\tilde{\mathbb{P}}$-almost surely in the topology of $H^{-1}_{\mathrm{div}}(\mathbb{T}^3)$. Moreover, $\tilde{X}(t)$ is also a $H^{-1}_{\mathrm{div}}(\mathbb{T}^3)$-valued square integrable martingale with respect to the filtration $(\tilde{\mathbb{F}}_t)$.*

Proof Note that as before, we would not be able to identify the structure of the martingale $\tilde{X}(t)$, instead we just prove that $\tilde{X}(t)$ is a martingale. In what follows, with the help of the Proposition 3.7, we can conclude that for each $t \in [0,T]$, $\tilde{X}_\varepsilon(t) \to \tilde{X}(t)$, $\mathbb{P}$-a.s. in the topology of $H^{-1}_{\mathrm{div}}(\mathbb{T}^3)$. Finally to show that $\tilde{X}(t)$ is a martingale, it is enough to demonstrate that for all $i \geq 1$

$$\tilde{\mathbb{E}}\Big[\mathfrak{L}_s(\tilde{\Phi})\big\langle \tilde{X}(t) - \tilde{X}(s), g_i\big\rangle\Big] = 0,$$

where g_i's are given orthonormal basis for the space $H^{-1}_{\mathrm{div}}(\mathbb{T}^3)$. We follow the usual argument to establish the result. To that context, we first use the information that

$$\tilde{\mathbb{E}}\Big[\mathfrak{L}_s(\tilde{\Phi}_\varepsilon)\big\langle \tilde{X}_\varepsilon(t) - \tilde{X}_\varepsilon(s), g_i\big\rangle\Big] = 0,$$

for all $i \geq 1$. Then, like before, we can pass to the limit in the parameter ε to show that $\tilde{M}(t)$ is a martingale. Indeed, this argument requires uniform integrability in ω variable, and can be achieved using BDG inequality:

$$\begin{aligned}\tilde{\mathbb{E}}\Big[\big|\big\langle \tilde{X}_\varepsilon(t), g_i\big\rangle\big|^p\Big] &= \tilde{\mathbb{E}}\Bigg[\bigg|\Big\langle \int_0^t \mathcal{P}_H \mathcal{G}(\tilde{\mathbf{u}}_\varepsilon)\,\mathrm{d}\tilde{W}, g_i\Big\rangle\bigg|^p\Bigg] \leq C\,\tilde{\mathbb{E}}\Bigg[\sup_{0\leq t\leq T}\bigg\|\int_0^t \mathcal{P}_H\mathcal{G}(\tilde{\mathbf{u}}_\varepsilon)\,\mathrm{d}\tilde{W}\bigg\|^p_{H^{-1}_{\mathrm{div}}(\mathbb{T}^3)}\Bigg] \\ &\leq C\,\tilde{\mathbb{E}}\Bigg[\bigg(\int_0^T \|\mathcal{G}(\tilde{\mathbf{u}}_\varepsilon)\|^2_{L_2(\mathcal{U},L^2(\mathbb{T}^3))}\,\mathrm{d}s\bigg)^{p/2}\Bigg] \leq C.\end{aligned}$$

This finishes the proof. □

In view of the above discussions, we can pass to the limit in (3.2) to conclude that

$$\begin{aligned}&\int_{\mathbb{T}^3}\langle \tilde{\mathcal{V}}^\omega_{\tau,x};\tilde{\mathbf{u}}\rangle\cdot\boldsymbol{\varphi}(\tau,\cdot)\,\mathrm{d}x - \int_{\mathbb{T}^3}\langle \tilde{\mathcal{V}}^\omega_{0,x};\tilde{\mathbf{u}}\rangle\cdot\boldsymbol{\varphi}(0,\cdot)\,\mathrm{d}x \\ &\quad = \int_0^\tau\int_{\mathbb{T}^3}\langle \tilde{\mathcal{V}}^\omega_{t,x};\tilde{\mathbf{u}}\otimes\tilde{\mathbf{u}}\rangle : \nabla_x\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{T}^3}\boldsymbol{\varphi}\int_0^\tau d\tilde{X}(t)\,\mathrm{d}x + \int_0^\tau\int_{\mathbb{T}^3}\nabla_x\boldsymbol{\varphi} : d\tilde{\mu}_C,\end{aligned}$$

holds $\tilde{\mathbb{P}}$-a.s., for all $\tau \in [0,T)$, and for all $\boldsymbol{\varphi} \in C^\infty_{\mathrm{div}}(\mathbb{T}^3;\mathbb{R}^3)$. This implies that (2.14) holds. Next, we focus on proving the energy inequality (2.15). In that context, making use of identifications of weak limits of various terms involved in the energy inequality, we can pass to the limit in ε in (3.3). This yield

$$\begin{aligned}&-\int_0^T \partial_t\phi\bigg(\int_{\mathbb{T}^3}\Big\langle \tilde{\mathcal{V}}^\omega_{\tau,x};\frac{|\tilde{\mathbf{u}}|^2}{2}\Big\rangle\,\mathrm{d}x + \tilde{\mathcal{D}}(\tau)\bigg)\mathrm{d}\tau \leq \phi(0)\int_{\mathbb{T}^3}\Big\langle \tilde{\mathcal{V}}^\omega_{0,x};\frac{|\tilde{\mathbf{u}}|^2}{2}\Big\rangle\,\mathrm{d}x \\ &\quad + \frac{1}{2}\sum_{k\geq 1}\int_0^T\phi(\tau)\int_{\mathbb{T}^3}\big\langle \mathcal{V}^\omega_{\tau,x};|\mathbf{G}_k(\tilde{\mathbf{u}})|^2\big\rangle\,\mathrm{d}x\,\mathrm{d}\tau - \frac{1}{2}\sum_{k\geq 1}\int_s^t\phi(\tau)\int_{\mathbb{T}^3}\Big(\mathcal{Q}_H\Big\langle \tilde{\mathcal{V}}^\omega_{\tau,x};|\mathbf{G}_k(\tilde{\mathbf{u}})|\Big\rangle\Big)^2\,\mathrm{d}x\,\mathrm{d}\tau \\ &\quad + \frac{1}{2}\int_0^T\phi(\tau)\int_{\mathbb{T}^3}\mathrm{d}\tilde{\mu}_D\,\mathrm{d}\tau - \frac{1}{2}\int_0^T\phi(\tau)\int_{\mathbb{T}^3}\mathrm{d}\tilde{\mu}_F\,\mathrm{d}\tau + \int_0^T\phi(\tau)\,\mathrm{d}\tilde{Y}(\tau)\end{aligned} \tag{3.4}$$

holds $\tilde{\mathbb{P}}$-a.s., for all $\phi \in C^\infty_c([0,T))$, $\phi \geq 0$. Note that here $\tilde{\mathcal{D}}(\tau) := \tilde{\mu}_E(\tau)(\mathbb{T}^3)$. To proceed, we first fix any s and t such that $0 < s < t < T$. For any $r > 0$ with $0 < s - r < t + r < T$, let us denote by ϕ_r, a Lipschitz function which is linear on $[s-r,s]$ and $[t,t+r]$ such that

$$\phi_r(\tau) = \begin{cases} 0, & \text{if} \quad \tau \in [0,s-r] \quad \text{or} \quad \tau \in [t+r,T] \\ 1, & \text{if} \quad \tau \in [s,t]. \end{cases}$$

Then, a standard regularization argument reveals that ϕ_r can be used as an admissible test function in (3.3). Therefore, replacing the test function ϕ by ϕ_r in (3.4), we get $\mathbb{P}$-a.s., for all $0 < s < t < T$

$$\frac{1}{r}\int_t^{t+r}\left(\int_{\mathbb{T}^3}\left\langle \tilde{\mathcal{V}}^{\omega}_{\tau,x};\frac{|\tilde{\mathbf{u}}|^2}{2}\right\rangle \mathrm{d}x+\tilde{\mathcal{D}}(\tau)\right)\mathrm{d}\tau \le \frac{1}{r}\int_{s-r}^{s}\left(\int_{\mathbb{T}^3}\left\langle \tilde{\mathcal{V}}^{\omega}_{\tau,x};\frac{|\tilde{\mathbf{u}}|^2}{2}\right\rangle \mathrm{d}x+\tilde{\mathcal{D}}(\tau)\right)\mathrm{d}\tau$$
$$+\frac{1}{2}\sum_{k\geq 1}\int_{s-r}^{t+r}\phi_r(\tau)\int_{\mathbb{T}^3}\left\langle \mathcal{V}^{\omega}_{\tau,x};|\mathbf{G}_k(\tilde{\mathbf{u}})|^2\right\rangle \mathrm{d}x\,\mathrm{d}\tau-\frac{1}{2}\sum_{k\geq 1}\int_{s-r}^{t+r}\phi_r(\tau)\int_{\mathbb{T}^3}\left(\mathcal{Q}_H\left\langle \tilde{\mathcal{V}}^{\omega}_{\tau,x};|\mathbf{G}_k(\tilde{\mathbf{u}})|\right\rangle\right)^2 \mathrm{d}x\,\mathrm{d}\tau$$
$$+\frac{1}{2}\int_{s-r}^{t+r}\phi_r(\tau)\int_{\mathbb{T}^3}\mathrm{d}\tilde{\mu}_D\,\mathrm{d}\tau-\frac{1}{2}\int_{s-r}^{t+r}\phi_r(\tau)\int_{\mathbb{T}^3}\mathrm{d}\tilde{\mu}_F\,\mathrm{d}\tau+\int_{s-r}^{t+r}\phi_r(\tau)\,\mathrm{d}\tilde{Y}(\tau). \tag{3.5}$$

Now using the non-negativity of the defect measure $\tilde{\mu}_F$, and letting liminf $r\to 0^+$ in (3.5), we obtain $\mathbb{P}$-a.s., for all $0<s<t<T$

$$\liminf_{r\to 0^+}\frac{1}{r}\int_t^{t+r}\left(\int_{\mathbb{T}^3}\left\langle \tilde{\mathcal{V}}^{\omega}_{\tau,x};\frac{|\tilde{\mathbf{u}}|^2}{2}\right\rangle \mathrm{d}x+\tilde{\mathcal{D}}(\tau)\right)\mathrm{d}\tau$$
$$\le \liminf_{r\to 0^+}\frac{1}{r}\int_{s-r}^{s}\left(\int_{\mathbb{T}^3}\left\langle \tilde{\mathcal{V}}^{\omega}_{\tau,x};\frac{|\tilde{\mathbf{u}}|^2}{2}\right\rangle \mathrm{d}x+\tilde{\mathcal{D}}(\tau)\right)\mathrm{d}\tau+\frac{1}{2}\sum_{k\geq 1}\int_s^t\int_{\mathbb{T}^3}\left\langle \mathcal{V}^{\omega}_{\tau,x};|\mathbf{G}_k(\tilde{\mathbf{u}})|^2\right\rangle \mathrm{d}x\,\mathrm{d}\tau$$
$$-\frac{1}{2}\sum_{k\geq 1}\int_s^t\int_{\mathbb{T}^3}\left(\mathcal{Q}_H\left\langle \tilde{\mathcal{V}}^{\omega}_{\tau,x};|\mathbf{G}_k(\tilde{\mathbf{u}})|\right\rangle\right)^2 \mathrm{d}x\,\mathrm{d}\tau+\frac{1}{2}\int_s^t\int_{\mathbb{T}^3}\mathrm{d}\tilde{\mu}_D\,\mathrm{d}\tau+\int_s^t \mathrm{d}\tilde{Y}(\tau) \tag{3.6}$$

We remark that for $s=0$ we need to use a slightly different test function to conclude the result. In this case we take

$$\phi_r(\tau)=\begin{cases}1, & \text{if} \quad \tau\in[0,t]\\ \text{linear}, & \text{if} \quad \tau\in[t,t+r]\\ 0, & \text{otherwise.}\end{cases}$$

and apply the same argument as before to establish that the energy inequality (2.15) holds.

Now we are only left with the verifications of (2.16), and item (i) of Definition 2.15. To proceed, we start with the following lemma.

Lemma 3.11 *Given a stochastic process h, as in item (i) of Definition 2.15*

$$dh = F(h)\,dt+\mathbb{K}(h)\,d\tilde{W},$$

the cross variation with $\tilde{X}$ is given by

$$\left\langle\!\!\left\langle h(t),\tilde{X}(t)\right\rangle\!\!\right\rangle=\sum_{i,j}\left(\sum_{k=1}^{\infty}\int_0^t\left\langle \mathcal{P}_H\left\langle \tilde{\mathcal{V}}^{\omega}_{s,x};\mathbf{G}_k(\tilde{\mathbf{u}})\right\rangle,g_i\right\rangle\left\langle \mathbb{K}(h)(e_k),g_j\right\rangle ds\right)g_i\otimes g_j.$$

where g_i's are orthonormal basis for $H^{-1}_{\mathrm{div}}(\mathbb{T}^3)$ and bracket $\langle\cdot,\cdot\rangle$ denotes inner product in the same space.

Proof Following the definition of cross variation between two Hilbert space valued martingales, given by Da Prato and Zabczyk [14, Sect. 3.4], we have

$$\left\langle\!\!\left\langle h(t),\tilde{X}_\varepsilon(t)\right\rangle\!\!\right\rangle=\sum_{i,j}\left\langle\!\!\left\langle \left\langle h(t),g_i\right\rangle,\left\langle \tilde{X}_\varepsilon(t),g_j\right\rangle\right\rangle\!\!\right\rangle g_i\otimes g_j,$$

where using the properties of the processes $h(t)$ and $\tilde{X}_\varepsilon(t)$, we have

$$\left\langle\!\!\left\langle \left\langle h(t),g_i\right\rangle,\left\langle \tilde{X}_\varepsilon(t),g_j\right\rangle\right\rangle\!\!\right\rangle=\left\langle\!\!\left\langle \sum_{k\geq 1}\int_0^t\langle \mathbb{K}(h)(e_k),g_i\rangle \mathrm{d}\tilde{W}_k,\sum_{k\geq 1}\int_0^t\langle \mathcal{P}_H\mathbf{G}_k(\tilde{\mathbf{u}}_\varepsilon),g_j\rangle \mathrm{d}\tilde{W}_k\right\rangle\!\!\right\rangle$$
$$=\sum_{k\geq 1}\int_0^t\left\langle \mathbb{K}(h)(e_k),g_i\right\rangle\left\langle \mathcal{P}_H\mathbf{G}_k(\tilde{\mathbf{u}}_\varepsilon),g_j\right\rangle \mathrm{d}s$$

Therefore we get

$$\left\langle\!\!\left\langle h(t), \tilde{X}_\varepsilon(t) \right\rangle\!\!\right\rangle = \sum_{i,j} \left(\sum_{k=1}^{\infty} \int_0^t \left\langle \mathbb{K}(h)(e_k), h_i \right\rangle \left\langle \mathcal{P}_H \mathbf{G}_k(\tilde{\mathbf{u}}_\varepsilon), h_j \right\rangle \mathrm{d}s \right) h_i \otimes h_j.$$

This equivalently implies that

$$\tilde{\mathbb{E}}\Big[\mathfrak{L}_s(\tilde{\Phi}_\varepsilon)\Big(\big\langle h(t), g_i\big\rangle\big\langle \tilde{X}_\varepsilon(t), g_j\big\rangle - \sum_{k=1}^{\infty} \int_0^t \left\langle \mathbb{K}(h)(e_k), g_i \right\rangle \left\langle \mathcal{P}_H \mathbf{G}_k(\tilde{\mathbf{u}}_\varepsilon), g_j \right\rangle \mathrm{d}s\Big)\Big] = 0.$$

Since we have $\mathbb{P}$-a.s.

$$\mathbf{G}_k(\tilde{\mathbf{u}}_\varepsilon) \rightharpoonup \big\langle \tilde{\mathcal{V}}^{\omega}_{t,x}; \mathbf{G}_k(\tilde{\mathbf{u}})\big\rangle, \text{ weakly in } L^2((0,T); L^2(\mathbb{T}^3)),$$

This gives $\mathbb{P}$-a.s

$$\mathcal{P}_H\mathbf{G}_k(\tilde{\mathbf{u}}_\varepsilon) \rightharpoonup \mathcal{P}_H\big\langle \tilde{\mathcal{V}}^{\omega}_{t,x}; \mathbf{G}_k(\tilde{\mathbf{u}})\big\rangle, \text{ weakly in } L^2((0,T); L^2_{\mathrm{div}}(\mathbb{T}^3)).$$

Moreover, making use of the a priori estimate (3.1), we have

$$\tilde{\mathbb{E}}\Big[\int_0^T \|\mathcal{P}_H \mathcal{G}(\tilde{\mathbf{u}}_\varepsilon)\|^2_{L_2(\mathcal{U}; H^{-1}_{\mathrm{div}}(\mathbb{T}^3))}\, dt\Big] \le \tilde{\mathbb{E}}\Big[\int_0^T \int_{\mathbb{T}^3} (\tilde{1} + |\tilde{\mathbf{u}}_\varepsilon|^2)\, dx\, dt\Big] \le C.$$

This implies that $\mathbb{P}$-a.s.

$$\mathcal{P}_H\mathbf{G}_k(\tilde{\mathbf{u}}) \rightharpoonup \mathcal{P}_H\big\langle \tilde{\mathcal{V}}^{\omega}_{t,x,}; \mathbf{G}_k(\tilde{\mathbf{u}})\big\rangle, \quad \text{weakly in} \quad L^2((0,T); H^{-1}_{\mathrm{div}}(\mathbb{T}^3)).$$

Therefore we can pass to the limit in $\varepsilon \to 0$, thanks to uniform integrability, to conclude

$$\tilde{\mathbb{E}}\Big[\mathfrak{L}_s(\tilde{\Phi})\Big(\big\langle h(t), g_i\big\rangle\big\langle \tilde{X}(t), g_j\big\rangle - \sum_{k=1}^{\infty} \int_0^t \left\langle \mathbb{K}(h)(e_k), g_i \right\rangle \left\langle \mathcal{P}_H \left\langle \tilde{\mathcal{V}}^{\omega}_{s,x}; \mathbf{G}_k(\tilde{\mathbf{u}}) \right\rangle, g_j \right\rangle \mathrm{d}s\Big)\Big] = 0.$$

This finishes the proof the lemma. □

Finally, regarding the proof of (2.16), we have the following lemma:

Lemma 3.12 *Concentration defect measures $\tilde{\mu}_C$, and $\tilde{\mu}_D$ are dominated by the nonnegative concentration defect measures $\tilde{\mathcal{D}}(\tau) := \tilde{\mu}_E(\tau)(\mathbb{T}^3)$, in the sense of Lemma 2.3. More precisely, there exists a constant $K > 0$ such that*

$$\int_0^\tau \int_{\mathbb{T}^3} d|\tilde{\mu}_C| + \int_0^\tau \int_{\mathbb{T}^3} d|\tilde{\mu}_D| \le K \int_0^\tau \tilde{\mathcal{D}}(\tau)\, dt,$$

$\mathbb{P}$-a.s., for all $\tau \in (0,T)$.

Proof Clearly, by Lemma 2.3, we can conclude that $\tilde{\mu}_E$ dominates the defect measure $\tilde{\mu}_C$. On the other hand, making use of hypotheses (2.3), (2.4), we can show the required dominance of $\tilde{\mu}_E$ over $\tilde{\mu}_D$. Indeed, note that the function

$$\mathbf{u} \mapsto \sum_{k\ge 1} |\mathbf{G}_k(\mathbf{u})|^2 \text{ is continuous,}$$

and clearly dominated by the total energy

$$\sum_{k\ge 1} |\mathbf{G}_k(\mathbf{u})|^2 \le C\left(1 + |\mathbf{u}|^2\right)$$

This finishes the proof of the lemma. □

4. Weak-Strong Uniqueness Principle for Euler System

In this section, we prove Theorem 2.18 through auxiliary results. Essentially the proof relies upon successful identification of the cross variation between two processes, given by a measure-valued Euler solution and a local strong Euler solution. In what follows, we begin with the following lemma.

Lemma 4.1 (Weak Itô Product Formula). *Let* $\mathbf{q}$ *be a stochastic process on* $\big(\Omega,\mathbb{F},(\mathbb{F}_t)_{t\ge0},\mathbb{P}\big)$ *such that*

$$\mathbf{q}\in C_w([0,T];L^2_{\rm div}(\mathbb{T}^3))\cap L^\infty((0,T);L^2_{\rm div}(\mathbb{T}^3)),\ \mathbb{P}-a.s.$$
$$\mathbb{E}\Big[\sup_{t\in[0,T]}\|\mathbf{q}\|^2_{L^2_{\rm div}(\mathbb{T}^3)}\Big]<+\infty.$$

Moreover, it satisfies $\mathbb{P}$*-a.s.*

$$\int_{\mathbb{T}^3}\mathbf{q}(t)\cdot\boldsymbol{\varphi}\,dx=\int_{\mathbb{T}^3}\boldsymbol{q}(0)\cdot\boldsymbol{\varphi}\,dx+\int_0^t\int_{\mathbb{T}^3}\mathbf{q}_1:\nabla\boldsymbol{\varphi}dxds+\int_0^t\int_{\mathbb{T}^3}\nabla\boldsymbol{\varphi}:\,d\mu(x,s)\,ds$$
$$+\int_{\mathbb{T}^3}\boldsymbol{\varphi}\cdot\int_0^t dM\,dx \tag{4.1}$$

for all $t\in[0,T]$*, and test function* $\varphi\in C^\infty_{div}(\mathbb{T}^3)$*. Here* M *is a continuous square integrable* $H^{-1}_{\rm div}(\mathbb{T}^3)$*-valued martingale and* $\mathbf{q}_1,\ \mu$ *are progressively measurable with*

$$\mathbf{q}_1\in L^2(\Omega;L^1(0,T;L^2_{\rm div}(\mathbb{T}^3))),\quad \mu\in L^1(\Omega;L^\infty_{w*}(0,T;\mathcal{M}_b(\mathbb{T}^3))).$$

Let $\boldsymbol{Q}$ *be a stochastic process on* $\big(\Omega,\mathbb{F},(\mathbb{F}_t)_{t\ge0},\mathbb{P}\big)$ *satisfying*

$$\boldsymbol{Q}\in C([0,T];C^1(\mathbb{T}^3)),\,\mathbb{P}-a.s.\ and\ \ \mathbb{E}\big[\sup_{t\in[0,T]}\|\mathbf{Q}\|^2_{L^2_{\rm div}(\mathbb{T}^3)\cap C(\mathbb{T}^3)}\big]<\infty,$$

be such that

$$\mathrm{d}\mathbf{Q}=\mathbf{Q}_1\mathrm{d}t+\mathbf{Q}_2\,\mathrm{d}W.$$

Here $\mathbf{Q}_1,\mathbf{Q}_2$ *are progressively measurable with*

$$\mathbf{Q}_1\in L^2(\Omega;L^1((0,T);L^2_{\rm div}(\mathbb{T}^3))),\quad \boldsymbol{Q}_2\in L^2(\Omega;L^2((0,T);L_2(\mathfrak{U};L^2_{\rm div}(\mathbb{T}^3)))),$$
$$\sum_{k=1}^\infty\int_0^T\|\mathbf{Q}_2(e_k)\|^2_{L^2(\mathbb{T}^3)}\in L^1(\Omega).$$

Then $\mathbb{P}$*-a.s., for all* $t\in[0,T]$*,*

$$\int_{\mathbb{T}^3}\mathbf{q}(t)\cdot\mathbf{Q}(t)\,\mathrm{d}x=\int_{\mathbb{T}^3}\mathbf{q}(0)\cdot\mathbf{Q}(0)\,\mathrm{d}x+\int_0^t\int_{\mathbb{T}^3}\mathbf{q}_1:\nabla\mathbf{Q}\,\mathrm{d}x\,\mathrm{d}s+\int_0^t\int_{\mathbb{T}^3}\nabla\mathbf{Q}:\,\mathrm{d}\mu\,\mathrm{d}s$$
$$+\int_{\mathbb{T}^3}\int_0^t\mathbf{Q}\cdot\mathrm{d}M\,\mathrm{d}x+\int_0^t\int_{\mathbb{T}^3}\mathbf{Q}_1\cdot\mathbf{q}\,\mathrm{d}x\,\mathrm{d}s+\int_{\mathbb{T}^3}\int_0^t\mathbf{q}\cdot\mathbf{Q}_2\,\mathrm{d}W\,\mathrm{d}x$$
$$+\int_{\mathbb{T}^3}\Big\langle\!\Big\langle M(t)\,,\,\mathbf{Q}(t)\Big\rangle\!\Big\rangle\,\mathrm{d}x. \tag{4.2}$$

Proof The proof of this lemma is straightforward. For the sake of completeness, we briefly outline the proof. Note that in order to prove the claim, we need to compute $\mathrm{d}\int_{\mathbb{T}^3}\mathbf{q}\cdot\mathbf{Q}\,\mathrm{d}x$. Due to lack of regularity, it is customary to use regularization by convolutions. To that context, let us denote by $(\boldsymbol{\rho}_\alpha)$, an approximation to the identity on $\mathbb{T}^3$. Let $\boldsymbol{\varphi}\in L^2_{\rm div}(\mathbb{T}^3)$, then $\boldsymbol{\varphi}_\alpha=\boldsymbol{\varphi}*\boldsymbol{\rho}_\alpha\in C^\infty_{\rm div}(\mathbb{T}^3)$. Then using $\boldsymbol{\varphi}_\alpha$ as a test function in (4.1), we obtain $\mathbb{P}$-almost surely, for all $t\in[0,T]$

$$\int_{\mathbb{T}^3}\mathbf{q}_\alpha(t)\cdot\boldsymbol{\varphi}\,\mathrm{d}x=\int_{\mathbb{T}^3}\mathbf{q}_\alpha(0)\cdot\boldsymbol{\varphi}\,\mathrm{d}x+\int_0^t\int_{\mathbb{T}^3}(\mathbf{q}_1)_\alpha:\nabla\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}s+\int_0^t\int_{\mathbb{T}^3}\nabla\boldsymbol{\varphi}:\,d\mu_\alpha(x,s)\,\mathrm{d}s$$
$$+\int_{\mathbb{T}^3}\boldsymbol{\varphi}\cdot\int_0^t dM_\alpha\,\mathrm{d}x.$$

It implies that $\mathbb{P}$-a.s., for all $t \in [0,T]$

$$\mathbf{q}_\alpha(t) = \mathbf{q}_\alpha(0) - \int_0^t \mathcal{P}_H\big(\mathrm{div}(\mathbf{q}_1)_\alpha\big)\mathrm{d}s - \int_0^t \mathcal{P}_H\big(\mathrm{div}\mu_\alpha\big)\,\mathrm{d}x\,\mathrm{d}s + \int_0^t dM_\alpha\,\mathrm{d}x.$$

Now, we can apply Itô's formula to the process $t \to \int_{\mathbb{T}^3} \mathbf{q}_r \cdot \mathbf{Q}\,\mathrm{d}x$, then we obtain for all $t \in [0,T]$, $\mathbb{P}$-a.s.

$$\begin{aligned}\int_{\mathbb{T}^3} \mathbf{q}_\alpha(t)\cdot\mathbf{Q}(t)\,\mathrm{d}x &= \int_{\mathbb{T}^3} \mathbf{q}_\alpha(0)\cdot\mathbf{Q}(0)\,\mathrm{d}x + \int_0^t\int_{\mathbb{T}^3} (\mathbf{q}_1)_\alpha : \nabla\mathbf{Q}\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_{\mathbb{T}^3} \nabla\mathbf{Q} :\, \mathrm{d}\mu_\alpha\,\mathrm{d}s\\ &\quad + \int_0^t\int_{\mathbb{T}^3} \mathbf{Q}\cdot\mathrm{d}M_\alpha\,\mathrm{d}x + \int_0^t\int_{\mathbb{T}^3} \mathbf{Q}_1\cdot\mathbf{q}_\alpha\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_{\mathbb{T}^3} \mathbf{q}_\alpha\cdot\mathbf{Q}_2\,\mathrm{d}W\,\mathrm{d}x\\ &\quad + \int_{\mathbb{T}^3} \Big\langle\!\Big\langle M_\alpha(t)\,,\,\mathbf{Q}(\mathbf{t})\Big\rangle\!\Big\rangle\,\mathrm{d}x\,.\end{aligned}$$

By using the given hypotheses, we can perform the limit $\alpha \to 0$ in above relation to conclude the proof. □

4.1. Relative Energy Inequality (Euler System)

It is well-known that the relative energy inequality is very useful for the comparison of a measure-valued solution and a smooth given function. To see this, let us first introduce the *relative energy (entropy)* functional in the context of measure-valued solutions to the stochastic incompressible Euler system as

$$\mathrm{F}^1_{\mathrm{mv}}\Big(\mathbf{u}\,\Big|\mathbf{U}\Big)(t) := \int_{\mathbb{T}^3} \langle\mathcal{V}^\omega_{t,x};\frac{1}{2}|\mathbf{u}|^2\rangle\,\mathrm{d}x + \mathcal{D}(t) - \int_{\mathbb{T}^3}\big\langle\mathcal{V}^\omega_{t,x};\mathbf{u}\big\rangle\cdot\mathbf{U}\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{T}^3}|\mathbf{U}|^2\,\mathrm{d}x.$$

In view of the energy inequality (2.17), it is clear that the above relative energy functional is defined for all $t \in [0,T]\setminus\mathcal{N}$, where the null (i.e., Lebesgue measure zero) set $\mathcal{N}$ may depends on $\omega \in \Omega$. We also define relative energy functional for all $t \in \mathcal{N}$ as follows:

$$\mathrm{F}^2_{\mathrm{mv}}\Big(\mathbf{u}\,\Big|\mathbf{U}\Big)(t) := \liminf_{r\to 0^+}\frac{1}{r}\int_t^{t+r}\Big[\int_{\mathbb{T}^3}\langle\mathcal{V}^\omega_{s,x};\frac{1}{2}|\mathbf{u}|^2\rangle\,\mathrm{d}x + \mathcal{D}(s)\Big]ds - \int_{\mathbb{T}^3}\big\langle\mathcal{V}^\omega_{t,x};\mathbf{u}\big\rangle\cdot\mathbf{U}\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{T}^3}|\mathbf{U}|^2\,\mathrm{d}x.$$

Using above, we define relative energy functional, which is well-defined defined for all $t \in [0,T]$, as follows:

$$\mathrm{F}_{\mathrm{mv}}\Big(\mathbf{u}\,\Big|\mathbf{U}\Big)(t) := \begin{cases}\mathrm{F}^1_{\mathrm{mv}}(\mathbf{u}\,|\mathbf{U})(t), & \text{if} \quad t\in[0,T]\setminus\mathcal{N};\\ \mathrm{F}^2_{\mathrm{mv}}(\mathbf{u}\,|\mathbf{U})(t), & \text{if} \quad t\in\mathcal{N}.\end{cases} \tag{4.3}$$

To proceed further, we make use of the relative energy (4.3) to derive the relative energy inequality given by (4.6).

Proposition 4.2 (Relative Energy). *Let* $\big[(\Omega,\mathbb{F},(\mathbb{F}_t)_{t\ge0},\mathbb{P});\mathcal{V}^\omega_{t,x},W\big]$ *be a dissipative measure-valued martingale solution to the system* (1.1)*. Suppose* $\mathbf{U}$ *be stochastic processes which is adapted to the filtration* $(\mathbb{F}_t)_{t\ge0}$ *and satisfies*

$$d\mathbf{U} = \mathbf{U}_1\,dt + \mathcal{P}_H\mathbf{U}_2\,dW,$$

with

$$\mathbf{U}\in C([0,T];C^1_{\mathrm{div}}(\mathbb{T}^3)),\quad \mathbb{P}\text{-}a.s.,\quad \mathbb{E}\Big[\sup_{t\in[0,T]}\|\mathbf{U}\|^2_{L^2_{\mathrm{div}}(\mathbb{T}^3)}\Big]<\infty, \tag{4.4}$$

Moreover, $\mathbf{U}$ *satisfies*

$$\begin{aligned}&\mathbf{U}_1\in L^2(\Omega;L^2(0,T;L^2_{\mathrm{div}}(\mathbb{T}^3))),\quad \mathbf{U}_2\in L^2(\Omega;L^2(0,T;L_2(\mathfrak{U};L^2(\mathbb{T}^3)))),\\ &\int_0^T\sum_{k\ge1}\|\mathcal{P}_H\mathbf{U}_2(e_k)\|^2_{L^2(\mathbb{T}^3)}\in L^1(\Omega).\end{aligned} \tag{4.5}$$

Then the following relative energy inequality *holds:*

$$F_{mv}\left(\mathbf{u}\left|\mathbf{U}\right.\right)(t) \le F_{mv}\left(\mathbf{u}\left|\mathbf{U}\right.\right)(0) + \mathcal{M}_{RE}(t) + \int_0^t \mathfrak{R}_{mv}(\mathbf{u}\,|\,\mathbf{U})(s)\,ds \tag{4.6}$$

$\mathbb{P}$*-almost surely, for all* $t \in [0,T]$ *with*

$$\begin{aligned}\mathfrak{R}_{mv}(\mathbf{u}\,|\,\mathbf{U}) = &\int_{\mathbb{T}^3} \left\langle \mathcal{V}^\omega_{t,x}; \mathbf{u}\otimes\mathbf{u}\right\rangle : \nabla_x \mathbf{u}\,dx + \int_{\mathbb{T}^3}\left\langle \mathcal{V}^\omega_{t,x};\mathbf{u}\right\rangle\cdot \mathbf{U}_1\,\mathrm{d}x\,\mathrm{d}t - \int_{\mathbb{T}^3}\nabla_x\mathbf{U} : d\mu_C + \frac{1}{2}\int_{\mathbb{T}^3} d\mu_D\\ &+ \frac{1}{2}\sum_{k\in\mathbb{N}}\int_{\mathbb{T}^3}\left\langle \mathcal{V}^\omega_{t,x}; \left|\mathbf{G}_k(\mathbf{u}) - \mathbf{U}_2(e_k)\right|^2\right\rangle dx. \end{aligned} \tag{4.7}$$

Here $\mathcal{M}_{RE}(t)$ *is a* $\mathbb{R}$*-valued square integrable martingale, whose norm depends on the norms of smooth function* $\mathbf{U}$ *in the aforementioned spaces.*

Proof We follow the usual strategy and express all the integrals on the right-hand side of (4.3) by making use of the energy inequality (2.15) and the field Eq. (2.14). Therefore, we shall make use of Itô's formula and the energy inequality (2.15) to compute the right-hand side of (4.3).

Step 1: In order to compute $\mathrm{d}\int_{\mathbb{T}^3}\left\langle \mathcal{V}^\omega_{t,x};\mathbf{u}\right\rangle\cdot\mathbf{U}\,\mathrm{d}x$, we first recall that $\mathbf{q} = \left\langle \mathcal{V}^\omega_{t,x};\mathbf{u}\right\rangle$ satisfies hypothesis of Lemma 4.1. Therefore we can apply the Lemma 4.1 to conclude that $\mathbb{P}$-almost surely

$$\begin{aligned}\mathrm{d}\left(\int_{\mathbb{T}^3}\left\langle \mathcal{V}^\omega_{t,x};\mathbf{u}\right\rangle\cdot\mathbf{U}\,\mathrm{d}x\right) = &\int_{\mathbb{T}^3}\left[\left\langle \mathcal{V}^\omega_{t,x};\mathbf{u}\right\rangle\cdot\mathbf{U}_1 + \left\langle \mathcal{V}^\omega_{t,x};\mathbf{u}\otimes\mathbf{u}\right\rangle : \nabla_x\mathbf{u}\right]\,\mathrm{d}x\mathrm{d}t\\ &+\sum_{k\ge1}\int_{\mathbb{T}^3}\mathcal{P}_H\mathbf{U}_2(e_k)\cdot\mathcal{P}_H\left\langle \mathcal{V}^\omega_{t,x};\mathbf{G}_k(\mathbf{u})\right\rangle\,\mathrm{d}x\,\mathrm{d}t\\ &+\int_{\mathbb{T}^3}\nabla_x\mathbf{U} : d\mu_C\,dt + \mathrm{d}\mathcal{M}_1,\end{aligned} \tag{4.8}$$

where the square integrable martingale $\mathcal{M}_1(t)$ is given by

$$\mathcal{M}_1(t) = \int_{\mathbb{T}^3}\int_0^t \mathbf{U}\,dM_E^1\,dx + \int_0^t\int_{\mathbb{T}^3}\left\langle \mathcal{V}^\omega_{t,x};\mathbf{u}\right\rangle\cdot\mathcal{P}_H\mathbf{U}_2\,\mathrm{d}x\,\mathrm{d}W$$

We remark that the item (i) of the Definition 2.15 is used to identify the cross variation in (4.8). Indeed, notice that

$$\begin{aligned}\int_{\mathbb{T}^3}\left\langle\!\left\langle f(t), M_E^1(t)\right\rangle\!\right\rangle dx &= \int_{\mathbb{T}^3}\sum_{i,j}\left(\sum_{k=1}^\infty\int_0^t\left\langle\mathcal{P}_H\left\langle \mathcal{V}^\omega_{s,x};\mathbf{G}_k(\mathbf{u})\right\rangle, h_i\right\rangle\left\langle\mathbb{D}^s f(e_k), h_j\right\rangle ds\right) h_i\otimes h_j\mathrm{d}x.\\ &= \sum_{k\ge1}\int_{\mathbb{T}^3}\int_0^t\mathcal{P}_H\mathbf{U}_2(e_k)\cdot\mathcal{P}_H\left\langle \mathcal{V}^\omega_{t,x};\mathbf{G}_k(\mathbf{u})\right\rangle\,\mathrm{d}x\mathrm{d}t\end{aligned}$$

Step 2: Next, we see that

$$\mathrm{d}\left(\int_{\mathbb{T}^3}\frac{1}{2}|\mathbf{U}|^2\,\mathrm{d}x\right) = \frac{1}{2}\sum_{k\ge1}\int_{\mathbb{T}^3}|\mathcal{P}_H\mathbf{U}_2(e_k)|^2\,\mathrm{d}x\;\mathrm{d}t + \mathrm{d}\mathcal{M}_2, \tag{4.9}$$

where

$$\mathcal{M}_2(t) = \int_0^t\int_{\mathbb{T}^3}\mathbf{U}\cdot\mathcal{P}_H\mathbf{U}_2\,\mathrm{d}x\,\mathrm{d}W.$$

Step 3: We have from energy inequality

$$\begin{aligned}\mathrm{E}(t+) \le \mathrm{E}(s-) &+ \frac{1}{2}\sum_{k\ge1}\int_s^t\int_{\mathbb{T}^3}\left\langle \mathcal{V}^\omega_{s,x};|\mathbf{G}_k(\mathbf{u})|^2\right\rangle\,\mathrm{d}x\mathrm{d}\tau - \frac{1}{2}\sum_{k\ge1}\int_s^t\int_{\mathbb{T}^3}\left(\mathcal{Q}_H\left\langle \mathcal{V}^\omega_{s,x};|\mathbf{G}_k(\mathbf{u})|\right\rangle\right)^2\mathrm{d}x\mathrm{d}\tau\\ &+\frac{1}{2}\int_s^t\int_{\mathbb{T}^3}d\mu_D + \int_s^\tau d\mathcal{M}_E^2.\end{aligned} \tag{4.10}$$

We now manipulate the product term in the equality (4.8) using properties of projections $\mathcal{P}_H$ and $\mathcal{Q}_H$. Indeed, note that

$$\begin{aligned}&\int_{\mathbb{T}^3} \mathcal{P}_H \mathbf{U}_2(e_k)\cdot \mathcal{P}_H\big\langle \mathcal{V}^\omega_{t,x}; \mathbf{G}_k(\mathbf{u})\big\rangle \,\mathrm{d}x\\ &\quad= \int_{\mathbb{T}^3} \mathbf{U}_2(e_k)\cdot \big\langle \mathcal{V}^\omega_{t,x}; \mathbf{G}_k(\mathbf{u})\big\rangle \,\mathrm{d}x - \int_{\mathbb{T}^3} \mathcal{Q}_H \mathbf{U}_2(e_k)\cdot \mathcal{Q}_H\big\langle \mathcal{V}^\omega_{t,x}; \mathbf{G}_k(\mathbf{u})\big\rangle \,\mathrm{d}x,\end{aligned}$$

and

$$\int_{\mathbb{T}^3} |\mathcal{P}_H \mathbf{U}_2(e_k)|^2 \,\mathrm{d}x = \int_{\mathbb{T}^3} |\mathbf{U}_2(e_k)|^2 \,\mathrm{d}x - \int_{\mathbb{T}^3} |\mathcal{Q}_H \mathbf{U}_2(e_k)|^2 \,\mathrm{d}x.$$

These properties of projections imply that

$$\begin{aligned}&\frac{1}{2}\sum_{k\geq 1}\int_{\mathbb{T}^3} |\mathcal{P}_H \mathbf{U}_2(e_k)|^2 \,\mathrm{d}x - \sum_{k\geq 1}\int_{\mathbb{T}^3} \mathcal{P}_H \mathbf{U}_2(e_k)\cdot \mathcal{P}_H\big\langle \mathcal{V}^\omega_{t,x}; \mathbf{G}_k(\mathbf{u})\big\rangle \,\mathrm{d}x\\ &\quad+\frac{1}{2}\sum_{k\geq 1}\int_s^t\int_{\mathbb{T}^3} \big\langle \mathcal{V}^\omega_{s,x}; |\mathbf{G}_k(\mathbf{u})|^2\big\rangle \,\mathrm{d}x\,\mathrm{d}\tau - \frac{1}{2}\sum_{k\geq 1}\int_s^t\int_{\mathbb{T}^3} \Big(\mathcal{Q}_H\big\langle \mathcal{V}^\omega_{s,x}; |\mathbf{G}_k(\mathbf{u})|\big\rangle\Big)^2 \,\mathrm{d}x\,\mathrm{d}\tau\\ &= \frac{1}{2}\sum_{k\geq 1}\int_{\mathbb{T}^3} \Big\langle \mathcal{V}^\omega_{t,x}; \big|\mathbf{G}_k(\mathbf{u}) - \mathbf{U}_2(e_k)\big|^2\Big\rangle \,\mathrm{d}x - \frac{1}{2}\sum_{k\geq 1}\int_{\mathbb{T}^3} \Big|\mathcal{Q}_H \mathbf{U}_2(e_k) - \mathcal{Q}_H\big\langle \mathcal{V}^\omega_{t,x}; \mathbf{G}_k(\mathbf{u})\big\rangle\Big|^2 \,\mathrm{d}x\\ &\leq \frac{1}{2}\sum_{k\geq 1}\int_{\mathbb{T}^3} \Big\langle \mathcal{V}^\omega_{t,x}; \big|\mathbf{G}_k(\mathbf{u}) - \mathbf{U}_2(e_k)\big|^2\Big\rangle \,\mathrm{d}x.\end{aligned} \tag{4.11}$$

Finally, in view of the above observations given by (4.8)–(4.11), we can now add the resulting expressions to establish (4.6). Note that the square integrable martingale $\mathcal{M}_{RE}(t)$ is given by $\mathcal{M}_{RE}(t) := \mathcal{M}_1(t) + \mathcal{M}_2(t) + \mathcal{M}_E^2(t)$. □

4.2. Proof of Theorem 2.18

In this subsection, we aim at establishing the desired weak (measure-valued)—strong uniqueness principle given by Theorem 2.18. To do so, we need to apply the relative energy inequality (2.15) with a specific choice of the smooth given function $\mathbf{U} = \bar{\mathbf{u}}(\cdot \wedge \tau_L)$, where $(\bar{\mathbf{u}}, (\tau_L)_{L\in\mathbb{N}}, \tau)$ is the unique maximal strong pathwise solution to (1.1). For technical reason, note that the stopping time τ_L announces the blow-up and satisfies

$$\sup_{t\in[0,\tau_L]} \|\bar{\mathbf{u}}(t)\|_{1,\infty} \geq L \quad \text{on} \quad [\mathfrak{t} < T];$$

Furthermore, it is evident that $\bar{\mathbf{u}}$ satisfies the equation (4.2), with

$$\mathbf{U}_1 = \mathcal{P}_H(\bar{\mathbf{u}}\cdot\nabla_x\bar{\mathbf{u}}), \quad \mathbf{U}_2 = \mathcal{G}(\bar{\mathbf{u}}).$$

Clearly, in view of Theorem 2.9 and (2.3)–(2.4), the conditions (4.4) and (4.5) are satisfied for $t \leq \mathfrak{t}_L$. Therefore, the inequality (4.6) holds, and we can also deduce from (4.7) that

$$\mathrm{F}_{\mathrm{mv}}\left(\mathbf{u}\middle|\bar{\mathbf{u}}\right)(t\wedge\tau_L) \leq \mathrm{F}_{\mathrm{mv}}\left(\mathbf{u}\middle|\bar{\mathbf{u}}\right)(0) + M_{RE}(t\wedge\tau_L) + \int_0^{t\wedge\tau_L} \mathfrak{R}_{\mathrm{mv}}\big(\mathbf{u}\,|\,\bar{\mathbf{u}}\big)(s)\,\mathrm{d}s, \tag{4.12}$$

holds for each $L\in\mathbb{N}$, for all $t\in[0,T]$, $\mathbb{P}$-almost surely. Here after manipulating terms in (4.7), as in [39], we obtain

$$\begin{aligned}\mathfrak{R}_{\mathrm{mv}}\left(\mathbf{u}\middle|\bar{\mathbf{u}}\right) &= \int_{\mathbb{T}^3} \big\langle \mathcal{V}^\omega_{t,x}; |(\mathbf{u}-\bar{\mathbf{u}})\otimes(\bar{\mathbf{u}}-\mathbf{u})|\big\rangle\,|\nabla_x\bar{\mathbf{u}}|\,dx + \frac{1}{2}\sum_{k\in\mathbb{N}}\int_{\mathbb{T}^3}\Big\langle \mathcal{V}^\omega_{t,x}; \big|\mathbf{G}_k(\mathbf{u}) - \mathbf{G}_k(\bar{\mathbf{u}})\big|^2\Big\rangle \,\mathrm{d}x\\ &\quad+ \int_{\mathbb{T}^3} |\nabla_x\bar{\mathbf{u}}|\cdot d|\mu_C| + \int_{\mathbb{T}^3} d|\mu_D|.\end{aligned}$$

Since $\|\bar{\mathbf{u}}\|_{W^{1,\infty}(\mathbb{T}^3)} \le c(L)$ for $t \le \tau_L$, we can control the terms $|\nabla_x \bar{\mathbf{u}}|$ by some constant. It is also clear that

$$|(\mathbf{u}-\bar{\mathbf{u}}) \otimes (\bar{\mathbf{u}}-\mathbf{u})| \le |\mathbf{u}-\bar{\mathbf{u}}|^2,$$

and

$$\sum_{k\ge 1} \big|\mathbf{G}_k(\mathbf{u}) - \mathbf{G}_k(\bar{\mathbf{u}})\big|^2 \le D_1 |\mathbf{u}-\bar{\mathbf{u}}|^2.$$

Finally, we also see that

$$\frac{1}{2}\sum_{k\in\mathbb{N}} \int_{\mathbb{T}^3} \big\langle \mathcal{V}^\omega_{t,x}; \big|\mathbf{G}_k(\mathbf{u}) - \mathbf{G}_k(\bar{\mathbf{u}})\big|^2 \big\rangle \,\mathrm{d}x \le c(L)\, \mathrm{F}^1_{\mathrm{mv}}\big(\mathbf{u}\,|\,\bar{\mathbf{u}}\big).$$

Collecting all the above estimates and using the item (h) of Definition 2.15, we conclude that for all $t \in [0,T]$, $\mathbb{P}$-a.s.

$$\int_0^{t\wedge\tau_L} \mathfrak{R}_{\mathrm{mv}}\big(\mathbf{u}\,|\,\bar{\mathbf{u}}\big)\,\mathrm{d}s \le c(L) \int_0^{t\wedge\tau_L} \Big(\mathrm{F}_{\mathrm{mv}}\big(\mathbf{u}\,|\,\bar{\mathbf{u}}\big)(s)\Big)\,\mathrm{d}s. \tag{4.13}$$

We now combine (4.13) and (4.12), and apply classical Gronwall's lemma, to obtain for all $t \in [0,T]$

$$\mathbb{E}\Big[\mathrm{F}_{\mathrm{mv}}\big(\mathbf{u}\,|\,\bar{\mathbf{u}}\big)(t\wedge\tau_L)\Big] \le c(L)\,\mathbb{E}\Big[\mathrm{F}_{\mathrm{mv}}\big(\mathbf{u}\,|\,\bar{\mathbf{u}}\big)(0)\Big].$$

We recall that

$$\mathrm{F}_{\mathrm{mv}}\big(\varrho,\mathbf{m}\,|\,\bar{\varrho},\bar{\mathbf{u}}\big)(0) = \int_{\mathbb{T}^3} \Big\langle \mathcal{V}^\omega_{0,x}; \frac{1}{2}\varrho_0\big|\mathbf{u}_0 - \bar{\mathbf{u}}_0\big|^2 \Big\rangle\,\mathrm{d}x,$$

which, by assumption, vanishes in expectation. Therefore, we conclude that

$$\mathbb{E}\Big[\mathrm{F}_{\mathrm{mv}}\big(\mathbf{u}\,|\,\bar{\mathbf{u}}\big)(t\wedge\tau_L)\Big] = 0, \text{ for all } t\in[0,T].$$

This also implies that

$$\lim_{r\to 0+} \frac{1}{r}\int_t^{t+r} \mathbb{E}\Big[\mathrm{F}_{\mathrm{mv}}\big(\mathbf{u}\,|\,\bar{\mathbf{u}}\big)(s\wedge\tau_L)\Big]\,\mathrm{d}s = 0.$$

Keeping in mind a priori estimates, a standard Lebesgue point argument in combination with classical Fubini's theorem reveals that for a.e. $t \in [0,T]$,

$$\mathbb{E}\Big[\mathrm{F}^1_{\mathrm{mv}}\big(\mathbf{u}\,|\,\bar{\mathbf{u}}\big)(t\wedge\tau_L)\Big] = 0.$$

But since the defect measure $\mathcal{D} \ge 0$, above equality implies for a.e. $t\in[0,T]$, $\mathbb{P}$-almost surely

$$\mathcal{D}(t\wedge\tau_L) = 0, \quad \text{and} \quad \delta_{\bar{\mathbf{u}}(x,t\wedge\tau_L)} = \mathcal{V}^\omega_{t\wedge\tau_L,x} \quad \text{for a.e.} \quad x\in\mathbb{T}^3.$$

5. Weak-Strong Uniqueness for Navier–Stokes System

Let $\mathbf{u}$ and $\mathbf{U}$ be two finite energy weak martingale solutions to (1.2), with same initial data u_0, defined on the same stochastic basis. The commonly used form of the *relative energy* functional in the context of weak solutions to the incompressible Navier–Stokes system reads

$$\mathrm{F}^{\mathrm{NS}}_{\mathrm{mv}}\Big(\mathbf{u}\,\Big|\,\mathbf{U}\Big) := \int_{\mathbb{T}^3} \frac{1}{2}|\mathbf{u}|^2\,\mathrm{d}x - \int_{\mathbb{T}^3} \mathbf{u}\cdot\mathbf{U}\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{T}^3} |\mathbf{U}|^2\,\mathrm{d}x. \tag{5.1}$$

The proof of (weak-strong) uniqueness for finite energy weak martingale solutions to (1.2) essentially uses similar arguments, as depicted in Sect. 4. However, the main difficulty lies in the successful identification of cross variation of two martingale solutions. Indeed, the regularity of finite energy weak martingale solutions is not enough to identify the cross variation between them and requires one solution to be more regular. In what follows, we start with the following lemma, whose proof is a simple consequence of the Hölder and Sobolev inequalities, see [31, Lemma 2.4].

Lemma 5.1 *Let r, s satisfy*

$$\frac{3}{s}+\frac{2}{r}=1, \quad s \in (3,\infty)$$

and let $\mathbf{u}, \mathbf{w} \in L^2((0,T);H^1_{\text{div}}(\mathbb{T}^3))$, *and* $\mathbf{u} \in L^r(0,T;L^s(\mathbb{T}^3))$. *Then*

$$\left|\int_0^T \langle \boldsymbol{v}\cdot\nabla \boldsymbol{w}, \boldsymbol{u}\rangle\right| \leq C\left(\int_0^T \|\nabla \boldsymbol{w}\|^2_{L^2(\mathbb{T}^3)}dt\right)^{\frac{1}{2}}\left(\int_0^T \|\nabla \boldsymbol{v}\|^2_{L^2(\mathbb{T}^3)}\right)^{\frac{3}{2s}}\left(\int_0^T \|\mathbf{u}\|^r_{L^s(\mathbb{T}^3)}\|\boldsymbol{v}\|^2_{L^2(\mathbb{T}^3)}dt\right)^{\frac{1}{r}}.$$

To make use of the above inequality, let us assume that two finite energy weak martingale solutions $\mathbf{u}, \mathbf{U} \in L^\infty((0,T);L^2(\mathbb{T}^3)) \cap L^2((0,T);H^1(\mathbb{T}^3))$ be diveregence free. Assume, in addition, that $\mathbf{U} \in L^r((0,T);L^s(\mathbb{T}^3))$. Then, in view of the above Lemma 5.1, for every $\tau \in (0,T)$

$$\begin{aligned}
&\left|\int_0^\tau \int_{\mathbb{T}^3} \nabla(\mathbf{u}-\mathbf{U}) : ((\mathbf{u}-\mathbf{U})\otimes\mathbf{U})\,\mathrm{d}x\,\mathrm{d}t\right| \\
&\quad \leq C\left(\int_0^\tau \int_{\mathbb{T}^3} |\nabla(\mathbf{u}-\mathbf{U})|^2\,\mathrm{d}x\right)^{1-1/r}\left(\int_0^\tau \|\mathbf{U}\|^r_{L^s(\mathbb{T}^3))}\int_{\mathbb{T}^3}|\mathbf{u}-\mathbf{U}|^{\mathbf{2}}\,\mathrm{d}x\,\mathrm{d}t\right)^{1/r}. \qquad (5.2)
\end{aligned}$$

In order to calculate the evolution equation satisfies by the second term of the relative energy (5.1), we need to apply Itô product rule. To do so, we first regularize (2.11) (with $\varepsilon = 1$), for both solutions $\mathbf{u}$ and $\mathbf{U}$, by taking a spatial convolution with a suitable family of regularizing kernels. We denote by $\mathbf{v}_r$, the regularization of $\mathbf{v}$. For a test function $\boldsymbol{\varphi} \in L^2_{\text{div}}(\mathbb{T}^3)$, we have $\boldsymbol{\varphi_r} \in C^\infty_{\text{div}}(\mathbb{T}^3)$. For both weak solutions $\mathbf{u}$ and $\mathbf{U}$, we may write equations

$$\langle \mathbf{u}(t), \boldsymbol{\varphi}_r\rangle = \langle \mathbf{u}(0), \boldsymbol{\varphi_r}\rangle + \int_0^t \langle \mathbf{u}\otimes\mathbf{u}(s), \nabla_x\boldsymbol{\varphi_r}\rangle\,\mathrm{d}s - \int_0^t \langle \nabla_x \mathbf{u}_(s)\,\nabla_x\boldsymbol{\varphi_r}\rangle \mathrm{d}s + \int_0^t \langle \mathcal{G}(\mathbf{u}), \boldsymbol{\varphi_r}\rangle\,\mathrm{d}W,$$

$$\langle \mathbf{U}(t), \boldsymbol{\varphi}_r\rangle = \langle \mathbf{U}(0), \boldsymbol{\varphi_r}\rangle + \int_0^t \langle \mathbf{U}\otimes\mathbf{U}(s), \nabla_x\boldsymbol{\varphi_r}\rangle\,\mathrm{d}s - \int_0^t \langle \nabla_x \mathbf{U}(s)\,, \nabla_x\boldsymbol{\varphi_r}\rangle \mathrm{d}s + \int_0^t \langle \mathcal{G}(\mathbf{U}), \boldsymbol{\varphi_r}\rangle\,\mathrm{d}W.$$

After shifting regularizing kernel from test to solutions term, we obtain

$$\mathbf{u}_r(t) = \mathbf{u}_r(0) - \int_0^t \mathcal{P}_H(\text{div}(\mathbf{u}\otimes\mathbf{u})_r)\mathrm{d}s + \int_0^t \Delta\mathbf{u}_r(s)\mathrm{d}s + \int_0^t \mathcal{P}_H(\mathcal{G}(\mathbf{u})_r)\mathrm{d}W,$$

$$\mathbf{U}_r(t) = \mathbf{U}_r(0) - \int_0^t \mathcal{P}_H(\text{div}(\mathbf{U}\otimes\mathbf{U})_r)\mathrm{d}s + \int_0^t \Delta\mathbf{U}_r(s)\mathrm{d}s + \int_0^t \mathcal{P}_H(\mathcal{G}(\mathbf{U})_r)\mathrm{d}W,$$

in $(L^2_{\text{div}}(\mathbb{T}^3))'$. We can now apply classical Itô's formula to the process $t \to \int_{\mathbb{T}^3} \mathbf{u}_r \cdot \mathbf{U_r}\,\mathrm{d}\mathbf{x}$, to obtain $\mathbb{P}$-almost surely

$$\begin{aligned}
\mathrm{d}\left(\int_{\mathbb{T}^3} \mathbf{u}_r\cdot\mathbf{U}_r\,\mathrm{d}x\right) &= \int_{\mathbb{T}^3}\Big[-\mathbf{u}_r\cdot\text{div}(\mathbf{U}\otimes\mathbf{U})_r + (\mathbf{u}\otimes\mathbf{u})_r : \nabla_x\mathbf{u}_r\Big]\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \sum_{k\geq 1}\int_{\mathbb{T}^3} \mathcal{P}_H(\mathbf{G}_k(\mathbf{U})_r)\cdot\mathcal{P}_H(\mathbf{G}_k(\mathbf{u})_r)\,\mathrm{d}x\,\mathrm{d}t - 2\int_{\mathbb{T}^3}\nabla_x\mathbf{U}_r : \nabla_x\mathbf{u}_r\,\mathrm{d}x\,\mathrm{d}t \qquad (5.3) \\
&\quad + \int_{\mathbb{T}^3}(\mathbf{u}_r\cdot\mathcal{G}(\mathbf{U})_r + \mathbf{U}_r\cdot\mathcal{G}(\mathbf{u}_r))\,\mathrm{d}x\,\mathrm{d}W,
\end{aligned}$$

We now wish to let $r \to 0$ in the above relation (5.3). The main difficulty in passing to the limits in the parameter r stems from the nonlinear terms, the treatments of other terms are classical. Indeed, we may apply the classical BDG inequality, with the help of a priori estimate and given conditions for noise coefficients, to handle stochastic terms that appeared in (5.3). In what follows, we show, with the help of Lemma (5.1) and extra regularity of $\mathbf{U}$, we can pass to the limit in the nonlinear terms. Observe that

$$\begin{aligned}
&\int_0^t \int_{\mathbb{T}^3} \mathbf{u}_r \cdot \operatorname{div}(\mathbf{U}\otimes\mathbf{U})_r \,\mathrm{d}s\,\mathrm{d}x - \int_0^t \int_{\mathbb{T}^3} \mathbf{u} \cdot \operatorname{div}(\mathbf{U}\otimes\mathbf{U}) \,\mathrm{d}s\,\mathrm{d}x \\
&\quad = \left(\int_0^t \int_{\mathbb{T}^3} (\mathbf{u}_r)_r \cdot \operatorname{div}(\mathbf{U}\otimes\mathbf{U}) \,\mathrm{d}s\,\mathrm{d}x - \int_0^t \int_{\mathbb{T}^3} \mathbf{u}_r \cdot \operatorname{div}(\mathbf{U}\otimes\mathbf{U}) \,\mathrm{d}s\,\mathrm{d}x \right) \\
&\qquad + \left(\int_0^t \int_{\mathbb{T}^3} \mathbf{u}_r \cdot \operatorname{div}(\mathbf{U}\otimes\mathbf{U}) \,\mathrm{d}s\,\mathrm{d}x - \int_0^t \int_{\mathbb{T}^3} \mathbf{u} \cdot \operatorname{div}(\mathbf{U}\otimes\mathbf{U}) \,\mathrm{d}s\,\mathrm{d}x \right) =: I_1^r + I_2^r,
\end{aligned}$$

For the second nonlinear term,

$$\begin{aligned}
&\int_0^t \int_{\mathbb{T}^3} \mathbf{U}_r \cdot \operatorname{div}(\mathbf{u}\otimes\mathbf{u})_r \,\mathrm{d}s\,\mathrm{d}x - \int_0^t \int_{\mathbb{T}^3} \mathbf{U} \cdot \operatorname{div}(\mathbf{u}\otimes\mathbf{u}) \,\mathrm{d}s\,\mathrm{d}x \\
&\quad = \left(\int_0^t \int_{\mathbb{T}^3} (\mathbf{U}_r)_r \cdot \operatorname{div}(\mathbf{u}\otimes\mathbf{u}) \,\mathrm{d}s\,\mathrm{d}x - \int_0^t \int_{\mathbb{T}^3} \mathbf{U}_r \cdot \operatorname{div}(\mathbf{u}\otimes\mathbf{u}) \,\mathrm{d}s\,\mathrm{d}x \right) \\
&\qquad + \left(\int_0^t \int_{\mathbb{T}^3} \mathbf{U}_r \cdot \operatorname{div}(\mathbf{u}\otimes\mathbf{u}) \,\mathrm{d}s\,\mathrm{d}x - \int_0^t \int_{\mathbb{T}^3} \mathbf{U} \cdot \operatorname{div}(\mathbf{u}\otimes\mathbf{u}) \,\mathrm{d}s\,\mathrm{d}x \right) =: J_1^r + J_2^r,
\end{aligned}$$

Moreover, thanks to Lemma 5.1, we conclude

$$|I_1^r|, |I_2^r| \le C\left(\int_0^t \|\nabla(\mathbf{u}_r - \mathbf{u})\|_{L^2(\mathbb{T}^3)}^2 d\tau \right)^{1/2}, \quad |J_1^r|, |J_2^r| \le C\|\mathbf{U}_r - \mathbf{U}\|_{L^r([0,T];L^s(\mathbb{T}^3))},$$

where the constant C depends on $\mathbf{U}$ and $\mathbf{u}$ only. Therefore, a simple property of convolution reveals that

$$\lim_{r\to 0} I_1^r = \lim_{r\to 0} I_2^r = \lim_{r\to 0} J_1^r = \lim_{r\to 0} J_2^r = 0. \tag{5.4}$$

Finally, letting $r \to 0$ in (5.3), and using (5.4), we obtain

$$\begin{aligned}
\mathrm{d}\left(\int_{\mathbb{T}^3} \mathbf{u}\cdot\mathbf{U}\,\mathrm{d}x \right) &= \int_{\mathbb{T}^3} \left[-\mathbf{u}\cdot\operatorname{div}(\mathbf{U}\otimes\mathbf{U}) + \mathbf{u}\otimes\mathbf{u} : \nabla_x \mathbf{u} \right] \mathrm{d}x\,\mathrm{d}t \\
&\quad + \sum_{k\ge 1} \int_{\mathbb{T}^3} \mathbf{G}_k(\mathbf{U})\cdot\mathbf{G}_k(\mathbf{u})\,\mathrm{d}x\,\mathrm{d}t - \int_{\mathbb{T}^3} \nabla_x\mathbf{U} : \nabla_x\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t + \mathrm{d}\mathcal{Q}_1,
\end{aligned} \tag{5.5}$$

where $\mathcal{Q}_1(t)$ is a square integrable martingale given by

$$\mathcal{Q}_1(t) := \int_0^t \int_{\mathbb{T}^3} (\mathbf{u}\cdot\mathcal{G}(\mathbf{U}) + \mathbf{U}\cdot\mathcal{G}(\mathbf{u})\,\mathrm{d}x\,\mathrm{d}W.$$

5.1. Proof of Theorem 2.19

We closely follow the strategy depicted in Sect. 4.2. To proceed, we first introduce a stopping time

$$\kappa_L := \inf\left\{ t\in(0,T) \,\Big|\, \|\mathbf{U}(t)\|_{L^s(\mathbb{T}^3)} \ge L \right\}$$

Since $\mathbb{E}\Big[\sup_{t\in[0,T]}\|\mathbf{U}\|_{L^s(\mathbb{T}^3)}\Big] < \infty$ by assumption, we have

$$\begin{aligned}
\mathbb{P}[\kappa_L < T] \le \mathbb{P}\Big[\sup_{t\in[0,T]} \|\mathbf{U}\|_{L^s(\mathbb{T}^3)} \ge L \Big] &\le \frac{1}{L}\mathbb{E}\Big[\sup_{t\in[0,T]} \|\mathbf{U}\|_{L^s(\mathbb{T}^3)} \Big] \to 0, \\
\mathbb{P}\Big[\lim_{L\to\infty} \kappa_L = T \Big] &= 1.
\end{aligned}$$

Therefore, it is enough to show the result for a fixed L. We now make use of the relative energy (5.1), the energy inequality (2.13), and (5.5), for both solutions, to conclude that for all $t \in [0,T]$, $\mathbb{P}$-a.s.

$$
\begin{aligned}
&\mathrm{F}^{\mathrm{NS}}_{\mathrm{mv}}\left(\mathbf{u}\middle|\mathbf{U}\right)(t\wedge\kappa_L) + \int_0^{t\wedge\kappa_L}\int_{\mathbb{T}^3}|\nabla_x(\mathbf{u}-\mathbf{U})|^2\,\mathrm{d}x\\
&= \frac{1}{2}\int_{\mathbb{T}^3}|\mathbf{u}(t\wedge\kappa_L)|^2\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{T}^3}|\mathbf{U}(t\wedge\kappa_L)|^2\,\mathrm{d}x + \int_0^{t\wedge\kappa_L}\int_{\mathbb{T}^3}|\nabla_x\mathbf{u}|^2\,\mathrm{d}x dt\\
&\quad + \int_0^{t\wedge\kappa_L}\int_{\mathbb{T}^3}|\nabla_x\mathbf{U}|^2\,\mathrm{d}x dt - \int_{\mathbb{T}^3}\mathbf{u}(t\wedge\kappa_L)\cdot\mathbf{U}(t\wedge\kappa_L)\,\mathrm{d}x - 2\int_0^{t\wedge\kappa_L}\int_{\mathbb{T}^3}\nabla_x\mathbf{u}:\nabla_x\mathbf{U}\,\mathrm{d}x dt\\
&\le \mathrm{F}^{\mathrm{NS}}_{\mathrm{mv}}\left(\mathbf{u}(0)\middle|\mathbf{U}(0)\right) + \int_0^{t\wedge\kappa_L}\int_{\mathbb{T}^3}\nabla_x\mathbf{u}:((\mathbf{u}-\mathbf{U})\otimes\mathbf{U})\,\mathrm{d}x dt + \mathcal{Q}_{RE}(t\wedge\kappa_L)\\
&\qquad + \frac{1}{2}\int_0^{t\wedge\kappa_L}\int_{\mathbb{T}^3}|\mathcal{P}_H(\mathcal{G}(\mathbf{u})-\mathcal{G}(\mathbf{U}))|^2\,\mathrm{d}x dt\\
&= \mathrm{F}^{\mathrm{NS}}_{\mathrm{mv}}\left(\mathbf{u}(\mathbf{0})\middle|\mathbf{U}(0)\right) + \int_0^{t\wedge\kappa_L}\int_{\mathbb{T}^3}\nabla_x(\mathbf{u}-\mathbf{U}):((\mathbf{u}-\mathbf{U})\otimes\mathbf{U})\,\mathrm{d}x + \mathcal{Q}_{RE}(t\wedge\kappa_L)\\
&\quad + \frac{1}{2}\int_0^{t\wedge\kappa_L}\int_{\mathbb{T}^3}|\mathcal{P}_H(\mathcal{G}(\mathbf{u})-\mathcal{G}(\mathbf{U}))|^2\,\mathrm{d}x dt\\
&\le \mathrm{F}^{\mathrm{NS}}_{\mathrm{mv}}\left(\mathbf{u}(0)\middle|\mathbf{U}(0)\right) + C\left(\int_0^{t\wedge\kappa_L}\int_{\mathbb{T}^3}|\nabla(\mathbf{u}-\mathbf{U})|^2\right)^{1-1/r}\left(\int_0^{t\wedge\kappa_L}\|\mathbf{U}\|^r_{L^s}\int_{\mathbb{T}^3}|\mathbf{u}-\mathbf{U}|^2\,dxdt\right)^{1/r}\\
&\qquad + \mathcal{Q}_{RE}(t\wedge\kappa_L) + \frac{1}{2}\int_0^{t\wedge\kappa_L}\int_{\mathbb{T}^3}|\mathcal{G}(\mathbf{u})-\mathcal{G}(\mathbf{U})|^2\,\mathrm{d}x dt\\
&\le \mathrm{F}^{\mathrm{NS}}_{\mathrm{mv}}\left(\mathbf{u}(0)\middle|\mathbf{U}(0)\right) + \int_0^{t\wedge\kappa_L}\int_{\mathbb{T}^3}|\nabla_x(\mathbf{u}-\mathbf{U})|^2\,\mathrm{d}x + C\left(\int_0^{t\wedge\kappa_L}\|\mathbf{U}\|^r_{L^s}\int_{\mathbb{T}^3}|\mathbf{u}-\mathbf{U}|^2\,dxdt\right)\\
&\qquad + \mathcal{Q}_{RE}(t\wedge\kappa_L) + \frac{D_1}{2}\int_0^{t\wedge\kappa_L}\int_{\mathbb{T}^3}|\mathbf{u}-\mathbf{U}|^2\,\mathrm{d}x dt\\
&\le \mathrm{F}^{\mathrm{NS}}_{\mathrm{mv}}\left(\mathbf{u}(0)\middle|\mathbf{U}(0)\right) + \int_0^{t\wedge\kappa_L}\int_{\mathbb{T}^3}|\nabla_x(\mathbf{u}-\mathbf{U})|^2\,\mathrm{d}x + C(L)\left(\int_0^{t\wedge\kappa_L}\int_{\mathbb{T}^3}|\mathbf{u}-\mathbf{U}|^2\,dxdt\right)\\
&\quad + \mathcal{Q}_{RE}(t\wedge\kappa_L) + \frac{D_1}{2}\int_0^{t\wedge\kappa_L}\int_{\mathbb{T}^3}|\mathbf{u}-\mathbf{U}|^2\,\mathrm{d}x dt,
\end{aligned}
$$

where

$$
\mathcal{Q}_{RE}(t) = \mathcal{Q}_1(t) + \int_0^t\int_{\mathbb{T}^3}\Big(\mathbf{u}\,\mathcal{P}_H\mathcal{G}(\mathbf{u}) + \mathbf{U}\,\mathcal{P}_H\mathcal{G}(\mathbf{U})\Big)\,\mathrm{d}x\,\mathrm{d}W.
$$

Note that in the above calculations, we have used the estimate (5.2), and classical Young's inequality

$$
xy \le \frac{\gamma x^a}{a} + \frac{y^b}{b\gamma^{b/a}}
$$

for any $\gamma>0$, $\frac{1}{a}+\frac{1}{b}=1$, and $x,y>0$. Therefore, we obtain for all $t\in[0,T]$, $\mathbb{P}$-a.s.

$$
\mathrm{F}^{\mathrm{NS}}_{\mathrm{mv}}\left(\mathbf{u}\,\middle|\mathbf{U}\right)(t\wedge\kappa_L) \le \mathrm{F}^{\mathrm{NS}}_{\mathrm{mv}}\left(\mathbf{u}(0)\middle|\mathbf{U}(0)\right) + C(L)\int_0^{t\wedge\kappa_L}\mathrm{F}^{\mathrm{NS}}_{\mathrm{mv}}\left(\mathbf{u}\,\middle|\mathbf{U}\right)(s)ds + \mathcal{Q}_{RE}(t\wedge\kappa_L).
$$

After taking expectation both side, we have

$$
\mathbb{E}\left[\mathrm{F}^{\mathrm{NS}}_{\mathrm{mv}}\left(\mathbf{u}\,\middle|\mathbf{U}\right)(t\wedge\kappa_L)\right] \le \mathbb{E}\left[\mathrm{F}^{\mathrm{NS}}_{\mathrm{mv}}\left(\mathbf{u}(0)\middle|\mathbf{U}(0)\right) + C(L)\int_0^{t\wedge\kappa_L}\mathrm{F}^{\mathrm{NS}}_{\mathrm{mv}}\left(\mathbf{u}\,\middle|\mathbf{U}\right)(s)ds\right].
$$

Since $\mathbb{E}\left[\mathrm{F}^{\mathrm{NS}}_{\mathrm{mv}}\left(\mathbf{u}(\mathbf{0})\middle|\mathbf{U}(\mathbf{0})\right)\right]=0$, we may use Gronwall's inequality to conclude that

$$
\mathbb{E}\left[\mathrm{F}^{\mathrm{NS}}_{\mathrm{mv}}\left(\mathbf{u}\,\middle|\mathbf{U}\right)(t\wedge\kappa_L)\right] = 0.
$$

We can now pass to the limit as $L \to \infty$ to conclude that

$$\mathbb{E}\left[\mathrm{F}^{\mathrm{NS}}_{\mathrm{mv}}\left(\mathbf{u}\,\middle|\,\mathbf{U}\right)(t)\right] = 0, \quad \text{for all} \quad t \in [0,T].$$

This implies that $\mathbb{P}$-almost surely, for all $t \in [0,T]$,

$$\mathbf{u}(x,t) = \mathbf{U}(x,t), \quad \text{for a.e.} \quad x \in \mathbb{T}^3.$$

This finishes the proof of the theorem.

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Declarations

Conflict of interest All authors declare that they have no conflict of interest.

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