



Euler System with a Polytropic Equation of State as a Vanishing Viscosity Limit

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Abstract. We consider the Euler system of gas dynamics endowed with the incomplete $(e - \varrho - p)$ equation of state relating the internal energy e to the mass density ϱ and the pressure p . We show that any sufficiently smooth solution can be recovered as a vanishing viscosity-heat conductivity limit of the Navier–Stokes–Fourier system with a properly defined temperature. The result is unconditional in the case of the Navier type (slip) boundary conditions and extends to the no-slip condition for the velocity under some extra hypotheses of Kato’s type concerning the behavior of the fluid in the boundary layer.

Keywords. Polytropic equation of state, Compressible Euler system, Navier–Stokes–Fourier system, Vanishing dissipation limit.

1. Introduction

The *Euler system* describing the evolution of the density $\varrho = \varrho(t, x)$, the velocity $\mathbf{u} = \mathbf{u}(t, x)$, and the internal energy $e = e(t, x)$ of a compressible inviscid fluid reads

$$\begin{aligned} \partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) &= 0, \\ \partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p &= 0, \\ \partial_t \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e \right] + \operatorname{div}_x \left(\left(\left[\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e \right] + p \right) \mathbf{u} \right) &= 0. \end{aligned} \tag{1.1}$$

The fluid is confined to a bounded domain $\Omega \subset \mathbb{R}^3$, with *impermeable boundary*,

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0. \tag{1.2}$$

The system (1.1) rewritten in terms of the phase variables (ϱ, \mathbf{u}, e) is symmetric hyperbolic, see e.g. Benzoni-Gavage and Serre [4, Chapter 13, Section 13.2.2]. The problem is formally closed by prescribing a suitable *equation of state* (EOS). We consider a polytropic EOS

$$p = (\gamma - 1)\varrho e \text{ with the adiabatic exponent } \gamma > 1. \tag{1.3}$$

The equation of state (1.3) is *incomplete*, in particular, the (absolute) temperature ϑ is not uniquely determined. Indeed Gibbs’ law asserts

$$\vartheta Ds = De + pD \left(\frac{1}{\varrho} \right), \tag{1.4}$$

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where s is a new thermodynamic variable called *entropy*. Here $D = \left(\frac{\partial}{\partial \varrho}, \frac{\partial}{\partial \vartheta}\right)$. Plugging (1.3) in (1.4) we obtain a first order system that can be integrated yielding

$$p(\varrho, \vartheta) = \vartheta^{\frac{\gamma}{\gamma-1}} P\left(\frac{\varrho}{\vartheta^{\frac{1}{\gamma-1}}}\right), \tag{1.5}$$

and, in accordance with (1.3), (1.4),

$$\begin{aligned} e(\varrho, \vartheta) &= \frac{\vartheta}{\gamma-1} \frac{\vartheta^{\frac{1}{\gamma-1}}}{\varrho} P\left(\frac{\varrho}{\vartheta^{\frac{1}{\gamma-1}}}\right), \\ s(\varrho, \vartheta) &= S\left(\frac{\varrho}{\vartheta^{\frac{1}{\gamma-1}}}\right), \quad S'(Z) = -\frac{1}{\gamma-1} \frac{\gamma P(Z) - P'(Z)Z}{Z^2}, \end{aligned} \tag{1.6}$$

for an arbitrary function P . Thus the absolute temperature ϑ is determined by ϱ and e modulo the function P , see Cowperthwaite [6], Müller and Ruggeri [17], or [11, Chapters 2,3].

The *Navier–Stokes–Fourier system* describing the motion of a real viscous and heat conductive gas can be viewed as a viscous regularization of (1.1):

$$\begin{aligned} \partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) &= 0, \\ \partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p &= \operatorname{div}_x \mathbb{S}, \\ \partial_t(\varrho s) + \operatorname{div}_x(\varrho s \mathbf{u}) + \operatorname{div}_x\left(\frac{\mathbf{q}}{\vartheta}\right) &= \frac{1}{\vartheta} \left(\mathbb{S} : \mathbb{D}_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta}\right), \quad \mathbb{D}_x \mathbf{u} \equiv \frac{\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u}}{2}, \end{aligned} \tag{1.7}$$

with the viscous stress \mathbb{S} given by *Newton’s rheological law*

$$\mathbb{S} = 2\tilde{\mu} \left(\mathbb{D}_x \mathbf{u} - \frac{1}{d} \operatorname{div}_x \mathbf{u} \mathbb{I}\right) + \tilde{\eta} \operatorname{div}_x \mathbf{u} \mathbb{I}, \tag{1.8}$$

and the heat flux given by *Fourier’s law*

$$\mathbf{q} = -\tilde{\kappa} \nabla_x \vartheta. \tag{1.9}$$

The second law of thermodynamics requires the entropy production rate

$$\frac{1}{\vartheta} \left(\mathbb{S} : \mathbb{D}_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta}\right)$$

to be non-negative; whence the diffusion *transport coefficients* $\tilde{\mu}$, $\tilde{\eta}$, and $\tilde{\kappa}$ must be non-negative. Note that, unlike in the Euler system (1.1), the knowledge of the temperature ϑ is necessary to determine the entropy as well as the heat flux in (1.7). The internal energy e can be evaluated in terms of ϱ , ϑ through (1.6). Thus solutions of the associated Navier–Stokes–Fourier system (1.7), that may be seen as a viscous regularization of the Euler system (1.1), depend on the choice of P in (1.5).

We consider the vanishing dissipation limit of the Navier–Stokes–Fourier system, specifically, we rescale

$$\mathbb{S}_n \approx \mu_n \mathbb{S}, \quad \mathbf{q}_n \approx \kappa_n \mathbf{q}, \quad \mu_n \searrow 0, \quad \kappa_n \searrow 0. \tag{1.10}$$

Moreover, the existing mathematical theory of the Navier–Stokes–Fourier system (see [11]) is based on the augmentation of the pressure, and, accordingly, the internal energy and entropy, by the radiation component

$$p_R = \frac{a}{3} \vartheta^4, \quad e_R = \frac{a}{\varrho} \vartheta^4, \quad s_R = \frac{4a}{3\varrho} \vartheta^3, \quad a > 0. \tag{1.11}$$

The parameter a is very small and usually neglected in the real world applications. Consistently with (1.10), we therefore consider

$$a = a_n, \quad a_n \searrow 0. \tag{1.12}$$

Suppose that $\gamma > 1$ is given and that the Euler system (1.1)–(1.3) admits a smooth (C^1) solution on a time interval $[0, T]$. Our goal is to identify the function P in (1.5) in such a way that any sequence of weak

solutions to the Navier–Stokes–Fourier system (1.7)–(1.9) converges in the vanishing viscosity/radiation limit (1.10)–(1.12) to the solution of the Euler system in $(0, T) \times \Omega$. Moreover, we show that the convergence is unconditional, if the boundary layer is eliminated by the choice of the complete slip boundary conditions

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad (\mathbb{S} \cdot \mathbf{n}) \times \mathbf{n}|_{\partial\Omega} = 0, \tag{1.13}$$

where \mathbf{n} denotes the outer normal vector to $\partial\Omega$. In the case of the no-slip boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0, \tag{1.14}$$

the convergence is conditioned by extra hypotheses of Kato’s type [15], [16] identified in the compressible setting by Sueur [21] and Wang and Zhu [22].

In comparison with the existing literature, notably [22], our result covers all admissible values of the adiabatic coefficient γ in (1.3) as well as general dependence of the transport coefficients on the temperature in the spirit of the existence theory developed in [11].

The paper is organized as follows. In Sect. 2, we recall the necessary preliminary material concerning the weak solutions to the Navier–Stokes–Fourier system including the relative energy inequality that represents a crucial tool in the analysis. Section 3 contains the main results. In Sect. 4 we show consistency of the vanishing viscosity approximation. Specifically, the viscous stress, the heat flux as well as the radiation components of the pressure, internal energy, and entropy along with the associated fluxes disappear in the regime specified in (1.10), (1.12). This process is “path dependent”, specifically certain relations concerning the asymptotic behaviour of (μ_n, κ_n, a_n) must be imposed in the spirit of [8]. The convergence towards the strong solution of the Euler system is shown in Sect. 5.

2. Preliminary Material

We recall the existing theory of weak solutions to the Navier–Stokes–Fourier system.

2.1. Mathematical Theory of the Closed System

We suppose the fluid is mechanically insulated as we stipulate either the complete slip (1.13) or the no-slip boundary condition (1.14). In view of our final objective, we require the fluid to be energetically isolated, specifically

$$\mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = 0. \tag{2.1}$$

The mathematical theory for closed systems relevant for future analysis was developed in [11]. Note that the extension to *open* systems is also available in the recent works [5, 12], see also the forthcoming monograph [9].

A suitable *weak formulation* of the Navier–Stokes–Fourier system augmented by the radiative terms proposed in [11] reads

$$\begin{aligned} \partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) &= 0, \\ \partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x(p + p_R) &= \mu \operatorname{div}_x \mathbb{S}, \\ \partial_t(\varrho(s + s_R)) + \operatorname{div}_x(\varrho(s + s_R)\mathbf{u}) + \kappa \nabla_x \left(\frac{\mathbf{q}}{\vartheta} \right) &\geq \frac{1}{\vartheta} \left(\mu \mathbb{S} : \mathbb{D}_x \mathbf{u} - \kappa \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right), \\ \frac{d}{dt} \int_{\Omega} \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho(e + e_R) \right) dx &= 0, \end{aligned} \tag{2.2}$$

see [11, Chapter 3]. Note that we anticipate the influence of thermal radiation represented by the extra terms p_R , e_R , and s_R in (2.2). In accordance with (1.12), these terms will vanish in the asymptotic limit. The energy balance appearing in the Euler system in (1.1) is replaced by the entropy inequality supplemented with the total energy balance in (2.2).

2.2. Transport Coefficients

In accordance with the molecular theory of gases (see e.g. Becker [1]), the transport coefficients depend on the temperature. Specifically, we assume that $\tilde{\mu}, \tilde{\eta}, \tilde{\kappa}$ are continuously differentiable functions of ϑ satisfying

$$\begin{aligned} 0 < \underline{\mu}(1 + \vartheta^\alpha) \leq \tilde{\mu}(\vartheta) \leq \bar{\mu}(1 + \vartheta^\alpha), \quad \alpha \geq 0, \\ \sup_{\vartheta \in [0, \infty)} |\tilde{\mu}'(\vartheta)| < \infty, \\ 0 \leq \tilde{\eta}(\vartheta) \leq \bar{\eta}(1 + \vartheta^\alpha), \quad 0 < \underline{\kappa}(1 + \vartheta^3) \leq \tilde{\kappa}(\vartheta) \leq \bar{\kappa}(1 + \vartheta^3) \end{aligned} \tag{2.3}$$

for any $\vartheta \geq 0$. Note that the cubic growth of κ is motivated by the presence of the radiation terms, see Oxenius [18].

2.3. Equation of State

A proper choice of the equation of state for the Navier–Stokes–Fourier system plays of course a crucial role in the present paper. Given $\gamma > 1$, we have to identify the function P in (1.5). For $p = p(\varrho, \vartheta)$, $e = e(\varrho, \vartheta)$, we recall the *hypothesis of thermodynamic stability*

$$\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0, \quad \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0. \tag{2.4}$$

This imposes the following restrictions on P :

$$\begin{aligned} P'(Z) > 0 \text{ for all } Z > 0, \\ \gamma P(Z) - P'(Z)Z > 0 \text{ for all } Z > 0. \end{aligned} \tag{2.5}$$

The following lemma shows existence of a suitable P .

Lemma 2.1. *For all $\underline{Z} > 0$ there exist functions $P, S \in C^1[0, \infty)$ with properties (1.6), (2.5) and such that*

$$P(Z) = Z \quad \text{for all } Z \in [0, \underline{Z}]. \tag{2.6}$$

Moreover P, S satisfy

$$P(0) = 0, \tag{2.7}$$

$$\frac{\gamma P(Z) - P'(Z)Z}{Z} \leq C \quad \text{for all } Z > 0, \tag{2.8}$$

$$\lim_{Z \rightarrow \infty} \frac{P(Z)}{Z^\gamma} > 0, \tag{2.9}$$

$$\lim_{Z \rightarrow \infty} S(Z) = 0. \tag{2.10}$$

Note that according to (2.10), S from Lemma 2.1 is in accordance with the Third law of thermodynamics, namely

$$s(\varrho, \vartheta) \rightarrow 0 \text{ as } \vartheta \rightarrow 0+ \text{ for any fixed } \varrho > 0,$$

cf. Belgiorno [2, 3].

Proof. Let us first consider the case $\underline{Z} = 1$. Set $P, S \in C^1[0, \infty)$

$$P(Z) := \begin{cases} Z & \text{if } Z \leq 1, \\ \frac{\gamma-1}{\gamma} + \frac{1}{\gamma}Z^\gamma & \text{if } Z > 1, \end{cases}$$

and

$$S(Z) := \begin{cases} -\log(Z) + 1 & \text{if } Z \leq 1, \\ \frac{1}{Z} & \text{if } Z > 1. \end{cases}$$

It is then straightforward to check (1.6), (2.5), (2.6)–(2.10).

Let us now look at $\underline{Z} \neq 1$. We define the P, S constructed above as P_1, S_1 and set

$$P(Z) := P_1 \left(\frac{Z}{\underline{Z}} \right), \quad S(Z) := \frac{1}{\underline{Z}} S_1 \left(\frac{Z}{\underline{Z}} \right).$$

Again straightforward computations show that the properties (1.6), (2.5), (2.6)–(2.10) follow from the corresponding property of P_1, S_1 . \square

Note that for $Z \in [0, \underline{Z}]$, according to (2.6) and (1.5) we simply obtain the Boyle-Mariotte law

$$p(\varrho, \vartheta) = \varrho \vartheta.$$

Hence the temperature for the Euler system (1.1) endowed with the incomplete EOS (1.3) can be recovered by choosing \underline{Z} in Lemma 2.1 appropriately, see Sect. 4.1 for details.

2.4. Relative Energy

The *relative energy* for the Navier–Stokes–Fourier system may be seen as a counterpart of Dafermos’ relative entropy for the (hyperbolic) Euler system, see [7]. Given a trio of “test functions”

$$r > 0, \quad \Theta > 0, \quad \mathbf{U}, \quad (2.11)$$

the relative energy reads

$$E(\varrho, \vartheta, \mathbf{u} \mid r, \Theta, \mathbf{U}) = \frac{1}{2} \varrho |\mathbf{u} - \mathbf{U}|^2 + H_\Theta(\varrho, \vartheta) - \frac{\partial H_\Theta(r, \Theta)}{\partial \varrho} (\varrho - r) - H_\Theta(r, \Theta), \quad (2.12)$$

where

$$H_\Theta(\varrho, \vartheta) = \varrho(e(\varrho, \vartheta) - \Theta s(\varrho, \vartheta))$$

is the ballistic free energy. In the context of the system (2.2) perturbed by the radiation terms, we have

$$H_\Theta(\varrho, \vartheta) = \varrho((e + e_R)(\varrho, \vartheta) - \Theta(s + s_R)(\varrho, \vartheta)).$$

The relative energy augmented by the radiation component will be denoted E_a . We also introduce the standard energy

$$E(\varrho, \vartheta, \mathbf{u}) = \frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta).$$

The following result was proved in [10]: Suppose that:

- $(\varrho, \vartheta, \mathbf{u})$ is a weak solution to the Navier–Stokes–Fourier system (2.2) in $(0, T) \times \Omega$ with the no-flux boundary conditions (2.1) and either the complete slip boundary conditions (1.13) or the no-slip boundary condition (1.14).
- (r, Θ, \mathbf{U}) is a trio of continuously differentiable test functions,

$$r > 0, \quad \Theta > 0 \text{ in } [0, T] \times \overline{\Omega},$$

where \mathbf{U} satisfies either the impermeability boundary condition

$$\mathbf{U} \cdot \mathbf{n}|_{\partial\Omega} = 0,$$

or the no-slip boundary condition

$$\mathbf{U}|_{\partial\Omega} = 0.$$

Then the *relative energy inequality*

$$\begin{aligned}
 & \left[\int_{\Omega} E_a(\varrho, \vartheta, \mathbf{u} \mid r, \Theta, \mathbf{U}) \, dx \right]_{t=0}^{t=\tau} + \int_0^\tau \int_{\Omega} \frac{\Theta}{\vartheta} \left(\mu \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \kappa \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right) \, dx \, dt \\
 & \leq \int_0^\tau \int_{\Omega} \varrho (\mathbf{u} - \mathbf{U}) \cdot \nabla_x \mathbf{U} \cdot (\mathbf{U} - \mathbf{u}) \, dx \, dt \\
 & \quad + \mu \int_0^\tau \int_{\Omega} \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{U} \, dx \, dt - \kappa \int_0^\tau \int_{\Omega} \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \cdot \nabla_x \Theta \, dx \, dt \\
 & \quad + \int_0^\tau \int_{\Omega} \varrho ((s + s_R)(\varrho, \vartheta) - (s + s_R)(r, \Theta)) (\mathbf{U} - \mathbf{u}) \cdot \nabla_x \Theta \, dx \, dt \\
 & \quad + \int_0^\tau \int_{\Omega} \varrho (\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_x \mathbf{U}) \cdot (\mathbf{U} - \mathbf{u}) \, dx \, dt - \int_0^\tau \int_{\Omega} (p + p_R)(\varrho, \vartheta) \operatorname{div}_x \mathbf{U} \, dx \, dt \\
 & \quad - \int_0^\tau \int_{\Omega} \varrho ((s + s_R)(\varrho, \vartheta) - (s + s_R)(r, \Theta)) (\partial_t \Theta + \mathbf{U} \cdot \nabla_x \Theta) \, dx \, dt \\
 & \quad + \int_0^\tau \int_{\Omega} \left(\left(1 - \frac{\varrho}{r}\right) \partial_t (p + p_R)(r, \Theta) - \frac{\varrho}{r} \mathbf{u} \cdot \nabla_x (p + p_R)(r, \Theta) \right) \, dx \, dt \tag{2.13}
 \end{aligned}$$

holds for a.a. $\tau \in (0, T)$.

Finally, we recall the fundamental properties of the relative energy that follow from the hypothesis of thermodynamic stability (2.4). In accordance with hypothesis (2.11), fix

$$\begin{aligned}
 0 &< \underline{\varrho} < \inf_{[0, T] \times \overline{\Omega}} r \leq \sup_{[0, T] \times \overline{\Omega}} r < \overline{\varrho}, \\
 0 &< \underline{\vartheta} < \inf_{[0, T] \times \overline{\Omega}} \Theta \leq \sup_{[0, T] \times \overline{\Omega}} \Theta < \overline{\vartheta},
 \end{aligned}$$

and define

$$[F]_{\text{ess}} = \Phi(\varrho, \vartheta)F, \quad [F]_{\text{res}} = F - [F]_{\text{ess}},$$

where

$$\Phi \in C_c^1(0, \infty)^2, \quad 0 \leq \Phi \leq 1, \quad \Phi(\varrho, \vartheta) = 1 \text{ whenever } \underline{\varrho} \leq \varrho \leq \overline{\varrho} \text{ and } \underline{\vartheta} \leq \vartheta \leq \overline{\vartheta}.$$

Then

$$\begin{aligned}
 E_a(\varrho, \vartheta, \mathbf{u} \mid r, \Theta, \mathbf{u}) &\geq E(\varrho, \vartheta, \mathbf{u} \mid r, \Theta, \mathbf{u}) \geq c([\varrho - r]_{\text{ess}}^2 + [\vartheta - \Theta]_{\text{ess}}^2 + [\mathbf{u} - \mathbf{U}]_{\text{ess}}^2) \\
 E_a(\varrho, \vartheta, \mathbf{u} \mid r, \Theta, \mathbf{u}) &\geq c(1_{\text{res}} + [\varrho(e + e_R)(\varrho, \vartheta)]_{\text{res}} + [\varrho|(s + s_R)(\varrho, \vartheta)|]_{\text{res}}), \\
 E(\varrho, \vartheta, \mathbf{u} \mid r, \Theta, \mathbf{u}) &\geq c(1_{\text{res}} + [\varrho e(\varrho, \vartheta)]_{\text{res}} + [\varrho |s(\varrho, \vartheta)|]_{\text{res}}), \tag{2.14}
 \end{aligned}$$

where the constants depend on $\underline{\varrho}$, $\overline{\varrho}$, $\underline{\vartheta}$, and $\overline{\vartheta}$, see e.g. [11] for details. As a consequence of the hypothesis of thermodynamic stability (2.4), the relative energy expressed in terms of the conservative entropy variables $(\varrho, \mathbf{m} = \varrho \mathbf{u}, \mathcal{S} = \varrho s)$ is a strictly convex function and represents the so-called Bregman distance between $(\varrho, \mathbf{m}, \mathcal{S})$ and $(r, r\mathbf{U}, rs(r, \Theta))$, see e.g. [12]. Note carefully that the relative energy E_a associated to the Navier–Stokes–Fourier system (2.2) is augmented by the radiation component

$$a(\vartheta^4 - \Theta^4) + \frac{4a}{3}\Theta(\Theta^3 - \vartheta^3) \geq 0.$$

3. Main Results

We state the main results in the physically relevant case $\Omega \subset R^3$. We consider three vanishing parameters in the asymptotic limit: the viscosity coefficient μ_n , the heat conductivity coefficient κ_n , and the radiation parameter a_n , cf. (1.10), (1.12).

3.1. Unconditional Convergence in the Absence of Boundary Layer

We start with the Navier–Stokes–Fourier system (2.2), with the complete slip boundary conditions (1.13), and the no-flux boundary condition (2.1).

Theorem 3.1 (Unconditional convergence). *Let $\Omega \subset R^3$ be a bounded Lipschitz domain. Suppose that the Euler system (1.1)–(1.3), with $\gamma > 1$, admits a strong solution*

$$\varrho_E, e_E \in C^1([0, T] \times \overline{\Omega}), \mathbf{u}_E \in C^1([0, T] \times \Omega; R^3)$$

satisfying

$$\inf_{[0, T] \times \overline{\Omega}} \varrho_E > 0, \quad \inf_{[0, T] \times \overline{\Omega}} e_E > 0.$$

Then there exists a $(p\text{-}\varrho\text{-}\vartheta)$ EOS $p = p(\varrho, \vartheta)$ that complies with Gibbs' relation (1.4) as well as the hypothesis of thermodynamic stability (2.4), with the associated internal energy EOS $e = e(\varrho, \vartheta)$ and entropy $s = s(\varrho, \vartheta)$ determined through (1.6), such that the following holds:

Let $(\varrho_n, \vartheta_n, \mathbf{u}_n)_{n=1}^\infty$ be a sequence of weak solutions to the Navier–Stokes–Fourier system (2.2), with the complete slip boundary condition (1.13), and the no-flux boundary conditions (2.1), in the vanishing dissipation/radiation regime:

$$\mu_n \searrow 0, \quad a_n \approx \mu_n^{\frac{4}{1+\alpha}}, \quad \frac{\kappa_n}{a_n^{\frac{3}{4}}} \rightarrow 0, \quad (3.1)$$

where $\alpha \in [\frac{1}{3}, 1]$ is the exponent in hypothesis (2.3). In addition, suppose that the initial data

$$\varrho_{n,0} = \varrho_n(0, \cdot), \quad \vartheta_{n,0} = \vartheta_n(0, \cdot), \quad \mathbf{u}_{n,0} = \mathbf{u}_n(0, \cdot)$$

converge strongly to those of the Euler system, specifically,

$$\begin{aligned} 0 < \underline{\varrho} < \inf_{(0, T) \times \Omega} \varrho_{n,0} &\leq \sup_{(0, T) \times \Omega} \varrho_{n,0} < \overline{\varrho} \text{ uniformly in } n, \quad \varrho_{n,0} \rightarrow \varrho_E(0, \cdot) \text{ in } L^1(\Omega), \\ 0 < \underline{\vartheta} < \inf_{(0, T) \times \Omega} \vartheta_{n,0} &\leq \sup_{(0, T) \times \Omega} \vartheta_{n,0} < \overline{\vartheta} \text{ uniformly in } n, \quad e(\varrho_{n,0}, \vartheta_{n,0})(0, \cdot) \rightarrow e_E(0, \cdot) \text{ in } L^1(\Omega), \\ |\mathbf{u}_{n,0}| \leq \overline{\mathbf{u}} \text{ uniformly in } n, &\quad \mathbf{u}_{n,0} \rightarrow \mathbf{u}_E(0, \cdot) \text{ in } L^1(\Omega; R^3). \end{aligned} \quad (3.2)$$

Then

$$\varrho_n \rightarrow \varrho_E, \quad \varrho_n e(\varrho_n, \vartheta_n) \rightarrow \varrho_E e_E \text{ in } L^1((0, T) \times \Omega), \quad \varrho_n \mathbf{u}_n \rightarrow \varrho_E \mathbf{u}_E \text{ in } L^1((0, T) \times \Omega; R^3). \quad (3.3)$$

Remark 3.2. The reader may consult [11, Chapter 3] for the exact definition of a weak solution of the Navier–Stokes–Fourier system emanating from the initial data $(\varrho_{n,0}, \vartheta_{n,0}, \mathbf{u}_{n,0})$.

Remark 3.3. We strongly point out that Theorem 3.1 does not contain any claim concerning the existence of weak solutions for the Navier–Stokes–Fourier system. The existence is known only in some particular cases: $\gamma \geq \frac{5}{3}$, $\alpha \in [\frac{2}{5}, 1]$, see [11, Chapter 3, Theorem 3.1], and $\gamma > \frac{3}{2}$, $\alpha = 1$, see Jesslé, Jin, and Novotný [14, Theorem 2.1]. The best known results for the planar flows were obtained recently by Pokorný and Skříšovský [19].

Local in time existence of smooth solutions to the Euler system was established by Schochet [20].

3.2. Conditional Result: Viscous Boundary Layer

The no-slip boundary condition (1.14) imposed on the viscous flow cannot be retained for the limit Euler system and the well known problem of viscous boundary layer appears. We report conditional results à la Kato in the spirit of Sueur [21] and Wang, Zhu [22]. Let

$$\Omega_\delta = \left\{ x \in \Omega \mid \text{dist}[x, \partial\Omega] < \delta \right\}.$$

Any vector field \mathbf{w} can be decomposed into its normal and tangential component with respect to $\partial\Omega$:

$$\begin{aligned} \mathbf{w}(t, x) &= \mathbf{w}_n(t, x) + \mathbf{w}_\tau(t, x), \\ \mathbf{w}_n(t, x) &= (\mathbf{w} \cdot \nabla_x \text{dist}[x, \partial\Omega]) \nabla_x \text{dist}[x, \partial\Omega], \quad \mathbf{w}_\tau(t, x) = \mathbf{w}(t, x) - \mathbf{w}_n(t, x). \end{aligned}$$

Note that $|\nabla_x \text{dist}[x, \partial\Omega]| = 1$, see Sect. 5.1.

We start with a result inspired by Sueur [21].

Theorem 3.4 (Conditional convergence, gradient criterion). *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{2+\nu}$. Suppose that the Euler system (1.1)–(1.3), with $\gamma > 1$, admits a strong solution*

$$\varrho_E, e_E \in C^1([0, T] \times \overline{\Omega}), \quad \mathbf{u}_E \in C^1([0, T] \times \Omega; \mathbb{R}^3)$$

satisfying

$$\inf_{[0, T] \times \overline{\Omega}} \varrho_E > 0, \quad \inf_{[0, T] \times \overline{\Omega}} e_E > 0.$$

Then there exists a $(p\text{-}\varrho\text{-}\vartheta)$ EOS $p = p(\varrho, \vartheta)$ that complies with Gibbs’ relation (1.4) as well as the hypothesis of thermodynamic stability (2.4), with the associated internal energy EOS $e = e(\varrho, \vartheta)$ and entropy $s = s(\varrho, \vartheta)$ determined through (1.6), such that the following holds:

Let $(\varrho_n, \vartheta_n, \mathbf{u}_n)_{n=1}^\infty$ be a sequence of weak solutions to the Navier–Stokes–Fourier system (2.2), with the no-slip boundary condition (1.14), and the no-flux boundary conditions (2.1), in the vanishing dissipation/radiation regime:

$$\mu_n \searrow 0, \quad a_n \approx \mu_n^{\frac{4}{1+\alpha}}, \quad \frac{\kappa_n}{a_n^{\frac{4}{3}}} \rightarrow 0,$$

where $\alpha \in [\frac{1}{3}, 1]$ is the exponent in hypothesis (2.3). In addition, suppose that the initial data

$$\varrho_{n,0} = \varrho_n(0, \cdot), \quad \vartheta_{n,0} = \vartheta_n(0, \cdot), \quad \mathbf{u}_{n,0} = \mathbf{u}_n(0, \cdot)$$

converge strongly to those of the Euler system, specifically,

$$\begin{aligned} 0 < \underline{\varrho} < \inf_{(0, T) \times \Omega} \varrho_{n,0} \leq \sup_{(0, T) \times \Omega} \varrho_{n,0} < \overline{\varrho} \text{ uniformly in } n, \quad \varrho_{n,0} \rightarrow \varrho_E(0, \cdot) \text{ in } L^1(\Omega), \\ 0 < \underline{\vartheta} < \inf_{(0, T) \times \Omega} \vartheta_{n,0} \leq \sup_{(0, T) \times \Omega} \vartheta_{n,0} < \overline{\vartheta} \text{ uniformly in } n, \quad e(\varrho_{n,0}, \vartheta_{n,0})(0, \cdot) \rightarrow e_E(0, \cdot) \text{ in } L^1(\Omega), \\ |\mathbf{u}_{n,0}| \leq \overline{\mathbf{u}} \text{ uniformly in } n, \quad \mathbf{u}_{n,0} \rightarrow \mathbf{u}_E(0, \cdot) \text{ in } L^1(\Omega; \mathbb{R}^3). \end{aligned}$$

Finally, suppose¹

$$\begin{aligned} \mu_n \int_0^T \int_{\Omega_{\mu_n}} |\mathbb{S}(\vartheta_n, \nabla_x \mathbf{u}_n)|^2 dx dt \rightarrow 0, \\ \mu_n \int_0^T \int_{\Omega_{\mu_n}} \left(\frac{\varrho_n |\mathbf{u}_n|^2}{\text{dist}^2[x, \partial\Omega]} + \frac{\varrho_n^2 |(\mathbf{u}_n)_n|^2}{\text{dist}^2[x, \partial\Omega]} \right) dx dt \rightarrow 0 \end{aligned} \tag{3.4}$$

as $n \rightarrow \infty$.

¹Note, that we use the index n both for the sequence and the normal component. Throughout this paper \mathbf{u}_n denotes the n th element of the sequence $(\mathbf{u}_n)_{n=1}^\infty$ and $(\mathbf{u}_n)_n$ its normal component.

Then

$$\varrho_n \rightarrow \varrho_E, \varrho_n e(\varrho_n, \vartheta_n) \rightarrow \varrho_E e_E \text{ in } L^1((0, T) \times \Omega), \varrho_n \mathbf{u}_n \rightarrow \varrho_E \mathbf{u}_E \text{ in } L^1((0, T) \times \Omega; \mathbb{R}^3).$$

Finally, we state a conditional result inspired by Wang and Zhu [22].

Theorem 3.5 (Conditional convergence). *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{2+\nu}$. Suppose that the Euler system (1.1)–(1.3), with $\gamma > 1$, admits a strong solution*

$$\varrho_E, e_E \in C^1([0, T] \times \overline{\Omega}), \mathbf{u}_E \in C^1([0, T] \times \Omega; \mathbb{R}^3)$$

satisfying

$$\inf_{[0, T] \times \overline{\Omega}} \varrho_E > 0, \quad \inf_{[0, T] \times \overline{\Omega}} e_E > 0.$$

Then there exists a $(p\text{-}\varrho\text{-}\vartheta)$ EOS $p = p(\varrho, \vartheta)$ that complies with Gibbs’ relation (1.4) as well as the hypothesis of thermodynamic stability (2.4), with the associated internal energy EOS $e = e(\varrho, \vartheta)$ and entropy $s = s(\varrho, \vartheta)$ determined through (1.6), such that the following holds:

Let $(\varrho_n, \vartheta_n, \mathbf{u}_n)_{n=1}^\infty$ be a sequence of weak solutions to the Navier–Stokes–Fourier system (2.2), with the no-slip boundary condition (1.14), and the no-flux boundary conditions (2.1), in the vanishing dissipation/radiation regime:

$$\mu_n \searrow 0, a_n \approx \mu_n^{\frac{4}{1+\alpha}}, \frac{\kappa_n}{a_n^{\frac{3}{4}}} \rightarrow 0,$$

where $\alpha \in [\frac{1}{3}, 1]$ is the exponent in hypothesis (2.3). In addition, suppose that the initial data

$$\varrho_{n,0} = \varrho_n(0, \cdot), \vartheta_{n,0} = \vartheta_n(0, \cdot), \mathbf{u}_{n,0} = \mathbf{u}_n(0, \cdot)$$

converge strongly to those of the Euler system, specifically,

$$0 < \underline{\varrho} < \inf_{(0, T) \times \Omega} \varrho_{n,0} \leq \sup_{(0, T) \times \Omega} \varrho_{n,0} < \overline{\varrho} \text{ uniformly in } n, \varrho_{n,0} \rightarrow \varrho_E(0, \cdot) \text{ in } L^1(\Omega),$$

$$0 < \underline{\vartheta} < \inf_{(0, T) \times \Omega} \vartheta_{n,0} \leq \sup_{(0, T) \times \Omega} \vartheta_{n,0} < \overline{\vartheta} \text{ uniformly in } n, e(\varrho_{n,0}, \vartheta_{n,0})(0, \cdot) \rightarrow e_E(0, \cdot) \text{ in } L^1(\Omega),$$

$$|\mathbf{u}_{n,0}| \leq \overline{\mathbf{u}} \text{ uniformly in } n, \mathbf{u}_{n,0} \rightarrow \mathbf{u}_E(0, \cdot) \text{ in } L^1(\Omega; \mathbb{R}^3).$$

Finally, suppose there is a sequence $\delta_n \rightarrow 0$ such that

$$\frac{\mu_n}{\delta_n} \rightarrow 0 \text{ as } n \rightarrow \infty,$$

$$\frac{1}{\delta_n} \int_0^T \int_{\Omega_{\delta_n}} \vartheta_n^{1+\alpha} \, dx \, dt \leq c,$$

$$\int_0^T \left(\frac{1}{\delta_n} \|\varrho_n(\mathbf{u}_n)_n\|_{L^{\frac{24}{17+3\alpha}}(\Omega_{\delta_n}; \mathbb{R}^3)} + \frac{1}{\delta_n^2 \mu_n} \|\varrho_n(\mathbf{u}_n)_n\|_{L^{\frac{24}{17+3\alpha}}(\Omega_{\delta_n}; \mathbb{R}^3)}^2 \left\| \vartheta_n^{\frac{1-\alpha}{2}} \right\|_{L^{\frac{8}{1-\alpha}}(\Omega_{\delta_n})}^2 \right) dt \rightarrow 0 \quad (3.5)$$

uniformly for $n \rightarrow \infty$.

Then

$$\varrho_n \rightarrow \varrho_E, \varrho_n e(\varrho_n, \vartheta_n) \rightarrow \varrho_E e_E \text{ in } L^1((0, T) \times \Omega), \varrho_n \mathbf{u}_n \rightarrow \varrho_E \mathbf{u}_E \text{ in } L^1((0, T) \times \Omega; \mathbb{R}^3).$$

Hypothesis (3.5) may seem awkward and much stronger than its counterpart by Wang and Zhu [22], where only the case $\alpha = 1$ is studied. In order to compare our result with [22], we are able to modify Theorem 3.5 for $\alpha = 1$ and obtain the following.

Theorem 3.6 (Conditional convergence, $\alpha = 1$). *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{2+\nu}$. Suppose that the Euler system (1.1)–(1.3), with $\gamma > 1$, admits a strong solution*

$$\varrho_E, e_E \in C^1([0, T] \times \overline{\Omega}), \mathbf{u}_E \in C^1([0, T] \times \Omega; \mathbb{R}^3)$$

satisfying

$$\inf_{[0,T] \times \bar{\Omega}} \varrho_E > 0, \quad \inf_{[0,T] \times \bar{\Omega}} e_E > 0.$$

Then there exists a $(p\text{-}\varrho\text{-}\vartheta)$ EOS $p = p(\varrho, \vartheta)$ that complies with Gibbs' relation (1.4) as well as the hypothesis of thermodynamic stability (2.4), with the associated internal energy EOS $e = e(\varrho, \vartheta)$ and entropy $s = s(\varrho, \vartheta)$ determined through (1.6), such that the following holds:

Let $(\varrho_n, \vartheta_n, \mathbf{u}_n)_{n=1}^\infty$ be a sequence of weak solutions to the Navier–Stokes–Fourier system (2.2), with the no-slip boundary condition (1.14), and the no-flux boundary conditions (2.1), in the vanishing dissipation/radiation regime with $\alpha = 1$:

$$\mu_n \searrow 0, \quad a_n \approx \mu_n^2, \quad \frac{\kappa_n}{a_n^{\frac{3}{4}}} \rightarrow 0.$$

In addition, suppose that the initial data

$$\varrho_{n,0} = \varrho_n(0, \cdot), \quad \vartheta_{n,0} = \vartheta_n(0, \cdot), \quad \mathbf{u}_{n,0} = \mathbf{u}_n(0, \cdot)$$

converge strongly to those of the Euler system, specifically,

$$0 < \underline{\varrho} < \inf_{(0,T) \times \Omega} \varrho_{n,0} \leq \sup_{(0,T) \times \Omega} \varrho_{n,0} < \bar{\varrho} \text{ uniformly in } n, \quad \varrho_{n,0} \rightarrow \varrho_E(0, \cdot) \text{ in } L^1(\Omega),$$

$$0 < \underline{\vartheta} < \inf_{(0,T) \times \Omega} \vartheta_{n,0} \leq \sup_{(0,T) \times \Omega} \vartheta_{n,0} < \bar{\vartheta} \text{ uniformly in } n, \quad e(\varrho_{n,0}, \vartheta_{n,0})(0, \cdot) \rightarrow e_E(0, \cdot) \text{ in } L^1(\Omega),$$

$$|\mathbf{u}_{n,0}| \leq \bar{\mathbf{u}} \text{ uniformly in } n, \quad \mathbf{u}_{n,0} \rightarrow \mathbf{u}_E(0, \cdot) \text{ in } L^1(\Omega; \mathbb{R}^3).$$

Finally, suppose there is a sequence $\delta_n \rightarrow 0$ such that

$$\begin{aligned} \frac{\mu_n}{\delta_n} &\rightarrow 0 \text{ as } n \rightarrow \infty, \\ \frac{1}{\delta_n} \int_0^T \int_{\Omega_{\delta_n}} \vartheta_n^2 \, dx \, dt &\leq c, \\ \frac{1}{\mu_n} \int_0^T \|\varrho_n(\mathbf{u}_n)_n\|_{L^2(\Omega_{\delta_n}; \mathbb{R}^3)}^2 \, dt &\rightarrow 0 \end{aligned} \tag{3.6}$$

as $n \rightarrow \infty$.

Then

$$\varrho_n \rightarrow \varrho_E, \quad \varrho_n e(\varrho_n, \vartheta_n) \rightarrow \varrho_E e_E \text{ in } L^1((0, T) \times \Omega), \quad \varrho_n \mathbf{u}_n \rightarrow \varrho_E \mathbf{u}_E \text{ in } L^1((0, T) \times \Omega; \mathbb{R}^3).$$

Remark 3.7. Indeed hypothesis (3.6) and the assumptions in Wang and Zhu [22] are similar, though not equivalent. Note furthermore, that Wang and Zhu alternatively consider an analogous assumption on the tangential component of \mathbf{u}_n instead of the normal component. In this paper we do not pursue anything of that kind.

The rest of the paper is devoted to the proof of the above results.

4. Consistency of the Vanishing Dissipation/Radiation Approximation

As a preliminary step, we show *consistency* of the vanishing dissipation/radiation approximation.

4.1. Temperature for the Euler System

First we introduce the temperature ϑ_E associated to the limit system. Without loss of generality, we may fix the constants $\underline{\varrho}$, $\bar{\varrho}$ in (3.2) so that

$$0 < \underline{\varrho} < \inf_{[0,T] \times \bar{\Omega}} \varrho_E \leq \sup_{[0,T] \times \bar{\Omega}} \varrho_E < \bar{\varrho}. \quad (4.1)$$

Next, in accordance with the hypotheses of Theorem 3.1,

$$0 < \underline{e} < \inf_{[0,T] \times \bar{\Omega}} e_E \leq \sup_{[0,T] \times \bar{\Omega}} e_E < \bar{e} \quad (4.2)$$

for certain constants \underline{e} , \bar{e} . Let us set

$$\underline{Z} > \frac{\bar{\varrho}}{((\gamma - 1)\underline{e})^{\frac{1}{\gamma-1}}}$$

and apply Lemma 2.1 to obtain suitable functions P, S . Furthermore we define

$$\vartheta_E := (\gamma - 1)e_E.$$

Note that $\vartheta_E > (\gamma - 1)\underline{e}$ and hence

$$\frac{\varrho_E}{(\vartheta_E)^{\frac{1}{\gamma-1}}} < \underline{Z}.$$

By virtue of (2.6), we have

$$e(\varrho_E, \vartheta_E) = e_E \text{ in } [0, T] \times \bar{\Omega}.$$

Moreover, without loss of generality, we may suppose

$$0 < \underline{\vartheta} < \inf_{[0,T] \times \bar{\Omega}} \vartheta_E \leq \sup_{[0,T] \times \bar{\Omega}} \vartheta_E < \bar{\vartheta}, \quad (4.3)$$

with the same constants $\underline{\vartheta}$, $\bar{\vartheta}$ as in (3.2). From this moment on, the pressure law is fixed.

As p , e , and s comply with Gibbs' relation, the smooth solution of the Euler system conserves the entropy:

$$\partial_t(\varrho_E s(\varrho_E, \vartheta_E)) + \operatorname{div}_x(\varrho_E s(\varrho_E, \vartheta_E) \mathbf{u}_E) = 0, \quad (4.4)$$

where s is given by (1.6).

4.2. Consistency

The Navier–Stokes–Fourier system (2.2) may be viewed as a singular perturbation of the Euler system with the extra “error” terms

$$\begin{aligned} E_n^1 &= p_R = \frac{a_n}{3} \vartheta_n^4, \\ E_n^2 &= \mu_n \mathcal{S}(\vartheta_n, \nabla_x \mathbf{u}_n) = \mu_n \left(\tilde{\mu}(\vartheta_n) \left(\nabla_x \mathbf{u}_n + \nabla_x^t \mathbf{u}_n - \frac{2}{3} \operatorname{div}_x \mathbf{u}_n \mathbb{I} \right) + \tilde{\eta}(\vartheta_n) \operatorname{div}_x \mathbf{u}_n \mathbb{I} \right), \\ E_n^3 &= \varrho s_R = \frac{4a_n}{3} \vartheta_n^3, \quad E_n^4 = \varrho s_R \mathbf{u} = \frac{4a_n}{3} \vartheta_n^3 \mathbf{u}_n, \quad E_n^5 \equiv \kappa_n \frac{\mathbf{q}}{\vartheta} = \kappa_n \tilde{\kappa}(\vartheta_n) \frac{\nabla_x \vartheta_n}{\vartheta_n}, \quad E_n^6 = \varrho e_R = a_n \vartheta_n^4. \end{aligned} \quad (4.5)$$

We say that the approximation of the Euler system by the Navier–Stokes–Fourier system is consistent, if the above “error” terms vanish in the asymptotic limit $n \rightarrow 0$. As a matter of fact, we need a milder form of consistency compatible with the relative energy inequality. More specifically, it is sufficient to control the “errors” by the dissipation term

$$\begin{aligned} \mathcal{D}_n &\equiv \mu_n \int_{\Omega} \frac{\tilde{\mu}(\vartheta_n)}{\vartheta_n} \left| \nabla_x \mathbf{u}_n + \nabla_x^t \mathbf{u}_n - \frac{2}{3} \operatorname{div}_x \mathbf{u}_n \mathbb{I} \right|^2 dx + \mu_n \int_{\Omega} \frac{\tilde{\eta}(\vartheta_n)}{\vartheta_n} |\operatorname{div}_x \mathbf{u}|^2 dx \\ &\quad + \kappa_n \int_{\Omega} \frac{\tilde{\kappa}(\vartheta_n)}{\vartheta_n^2} |\nabla_x \vartheta_n|^2 dx, \end{aligned}$$

and the total energy

$$\mathcal{E}_n \equiv \int_{\Omega} \left(\frac{1}{2} \varrho_n |\mathbf{u}_n|^2 + \varrho_n e(\varrho_n, \vartheta_n) + a_n \vartheta_n^4 \right) dx.$$

For each error term E_n^i , $i = 1, \dots, 6$ specified in (4.5) and $\varepsilon > 0$, we have to find $c(\varepsilon)$ such that

$$\|E_n^i\|_{L^1(\Omega)} \leq \varepsilon \mathcal{D}_n + c(\varepsilon) \mathcal{E}_n + c(\varepsilon) \omega_n \text{ uniformly for } n \rightarrow \infty, \omega_n \rightarrow 0. \quad (4.6)$$

Obviously, $E_n^1 = p_R$, $E_n^6 = \varrho e_R$, and $E_n^3 = \varrho s_R$ satisfy (4.6) (with $\varepsilon = 0$), it remains to handle the viscous stress, the heat flux and the entropy convective flux term.

Moreover, we recall some basic estimates that follow directly from the hypotheses (1.6), (2.5), (2.7)–(2.10):

$$\begin{aligned} \varrho^\gamma + \varrho \vartheta &\lesssim \varrho e(\varrho, \vartheta), \\ 0 \leq \varrho s(\varrho, \vartheta) &\lesssim \varrho (1 + |\log(\varrho)| + [\log(\vartheta)]^+). \end{aligned} \quad (4.7)$$

4.2.1. Viscous Stress Consistency. By virtue of hypothesis (2.3),

$$\begin{aligned} &\int_{\Omega} \mu_n \tilde{\mu}(\vartheta_n) \left| \nabla_x \mathbf{u}_n + \nabla_x^t \mathbf{u}_n - \frac{2}{3} \operatorname{div}_x \mathbf{u}_n \mathbb{I} \right| dx \\ &\leq \varepsilon \mu_n \int_{\Omega} \frac{\tilde{\mu}(\vartheta_n)}{\vartheta_n} \left| \nabla_x \mathbf{u}_n + \nabla_x^t \mathbf{u}_n - \frac{2}{3} \operatorname{div}_x \mathbf{u}_n \mathbb{I} \right|^2 dx + c(\varepsilon) \mu_n \int_{\Omega} (1 + \vartheta_n^{1+\alpha}) dx \\ &\leq \varepsilon \mathcal{D}_n + c(\varepsilon) \mu_n + c(\varepsilon) \mu_n \int_{\Omega} [\vartheta_n^{1+\alpha}]_{\text{res}} dx \\ &\leq \varepsilon \mathcal{D}_n + c(\varepsilon) \mu_n + c(\varepsilon) \frac{\mu_n}{a_n^{\frac{1+\alpha}{4}}} \int_{\Omega} \left[a_n^{\frac{1+\alpha}{4}} \vartheta_n^{1+\alpha} \right]_{\text{res}} dx \\ &\leq \varepsilon \mathcal{D}_n + c(\varepsilon) \mu_n + c(\varepsilon) \int_{\Omega} [a_n \vartheta_n^4 + 1]_{\text{res}} dx, \end{aligned}$$

where the last inequality follows from hypothesis (3.1) and the simple fact that $x^{1+\alpha} \leq c(x^4 + 1)$. Thus we obtain the desired estimate (4.6). The bulk viscosity term can be handled in a similar fashion.

4.2.2. Heat Flux Consistency. Similarly to the preceding part,

$$\begin{aligned} \int_{\Omega} \kappa_n \frac{\tilde{\kappa}(\vartheta_n)}{\vartheta_n} |\nabla_x \vartheta_n| dx &\leq \varepsilon \kappa_n \int_{\Omega} \frac{\tilde{\kappa}(\vartheta_n)}{\vartheta_n^2} |\nabla_x \vartheta_n|^2 dx + c(\varepsilon) \kappa_n \int_{\Omega} \tilde{\kappa}(\vartheta_n) dx \\ &\leq \varepsilon \mathcal{D}_n + c(\varepsilon) \kappa_n + c(\varepsilon) \kappa_n \int_{\Omega} \vartheta_n^3 dx \leq \varepsilon \mathcal{D}_n + c(\varepsilon) \kappa_n + c(\varepsilon) \frac{\kappa_n}{a_n^{\frac{3}{4}}} \left(\int_{\Omega} a_n \vartheta_n^4 dx \right)^{\frac{3}{4}} \\ &\leq \varepsilon \mathcal{D}_n + c(\varepsilon) \kappa_n + c(\varepsilon) \frac{\kappa_n}{a_n^{\frac{3}{4}}} + c(\varepsilon) \mathcal{E}_n, \end{aligned} \quad (4.8)$$

whence (4.6) follows from hypothesis (3.1).

4.2.3. Radiation Entropy Convective Flux Consistency. To close the circle of consistence estimates, we have to handle the integral

$$a_n \int_{\Omega} \vartheta_n^3 |\mathbf{u}_n| \, dx \leq a_n \|\vartheta_n^3\|_{L^{\frac{4}{3}}(\Omega)} \|\mathbf{u}_n\|_{L^4(\Omega; R^3)},$$

that corresponds to the radiation entropy convective flux.

By virtue of Sobolev embedding theorem,

$$\|\mathbf{u}_n\|_{L^4(\Omega; R^3)} \lesssim \|\mathbf{u}_n\|_{W^{1, \frac{8}{5-\alpha}}(\Omega; R^3)} \text{ as long as } \alpha \geq \frac{1}{3},$$

and, by a generalized Korn–Poincaré inequality [11, Theorem 11.23],

$$\|\mathbf{u}_n\|_{W^{1, \frac{8}{5-\alpha}}(\Omega; R^3)} \lesssim \left(\left\| \nabla_x \mathbf{u}_n + \nabla_x^t \mathbf{u}_n - \frac{2}{3} \operatorname{div}_x \mathbf{u}_n \mathbb{I} \right\|_{L^{\frac{8}{5-\alpha}}(\Omega; R^9)} + \int_{\Omega} \varrho_n |\mathbf{u}_n| \, dx \right).$$

Another application of Hölder’s inequality yields

$$\begin{aligned} & \left\| \nabla_x \mathbf{u}_n + \nabla_x^t \mathbf{u}_n - \frac{2}{3} \operatorname{div}_x \mathbf{u}_n \mathbb{I} \right\|_{L^{\frac{8}{5-\alpha}}(\Omega; R^9)} \\ & \leq \|\vartheta_n^{\frac{1-\alpha}{2}}\|_{L^{\frac{8}{1-\alpha}}(\Omega)} \left\| \vartheta_n^{\frac{\alpha-1}{2}} \left(\nabla_x \mathbf{u}_n + \nabla_x^t \mathbf{u}_n - \frac{2}{3} \operatorname{div}_x \mathbf{u}_n \mathbb{I} \right) \right\|_{L^2(\Omega; R^9)}. \end{aligned}$$

Consequently,

$$\begin{aligned} a_n \int_{\Omega} \vartheta_n^3 |\mathbf{u}_n| \, dx & \leq a_n \|\vartheta_n^3\|_{L^{\frac{4}{3}}(\Omega)} \|\mathbf{u}_n\|_{L^4(\Omega; R^3)} \lesssim a_n \|\vartheta_n^3\|_{L^{\frac{4}{3}}(\Omega)} \int_{\Omega} \varrho_n |\mathbf{u}_n| \, dx \\ & \quad + a_n \|\vartheta_n^3\|_{L^{\frac{4}{3}}(\Omega)} \|\vartheta_n^{\frac{1-\alpha}{2}}\|_{L^{\frac{8}{1-\alpha}}(\Omega)} \left\| \vartheta_n^{\frac{\alpha-1}{2}} \left(\nabla_x \mathbf{u}_n + \nabla_x^t \mathbf{u}_n - \frac{2}{3} \operatorname{div}_x \mathbf{u}_n \mathbb{I} \right) \right\|_{L^2(\Omega; R^9)} \\ & \leq \varepsilon \mathcal{D}_n + c(\varepsilon) \frac{a_n^2}{\mu_n} \|\vartheta_n^3\|_{L^{\frac{4}{3}}(\Omega)}^2 \|\vartheta_n^{\frac{1-\alpha}{2}}\|_{L^{\frac{8}{1-\alpha}}(\Omega)}^2 + ca_n \|\vartheta_n^3\|_{L^{\frac{4}{3}}(\Omega)} \int_{\Omega} \varrho_n |\mathbf{u}_n| \, dx. \end{aligned} \quad (4.9)$$

Finally, by virtue of hypothesis (3.1)

$$\frac{a_n^2}{\mu_n} \|\vartheta_n^3\|_{L^{\frac{4}{3}}(\Omega)}^2 \|\vartheta_n^{\frac{1-\alpha}{2}}\|_{L^{\frac{8}{1-\alpha}}(\Omega)}^2 = \frac{a_n^2}{\mu_n} \left(\int_{\Omega} \vartheta_n^4 \, dx \right)^{\frac{7-\alpha}{4}} = \frac{a_n^{\frac{1+\alpha}{4}}}{\mu_n} \left(\int_{\Omega} a_n \vartheta_n^4 \, dx \right)^{\frac{7-\alpha}{4}} \lesssim \mathcal{E}_n^{\frac{7-\alpha}{4}}.$$

The rightmost integral in (4.9) can be handled in a similar fashion. Since $\alpha \in [\frac{1}{3}, 1]$, we have in particular $0 \leq \alpha \leq 3$ and the desired conclusion (4.6) follows from the boundedness of the total energy.

5. Convergence

The proof of convergence consists of using the strong solution $(\varrho_E, \vartheta_E, \mathbf{u}_E)$ of the Euler system as the test functions $r = \varrho_E$, $\Theta = \vartheta_E$, $\mathbf{U} = \mathbf{u}_E$ in the relative energy inequality (2.13). This can be done in a direct manner in the case of the complete slip boundary conditions (1.13), whereas the velocity \mathbf{u}_E must be modified to comply with the homogeneous Dirichlet boundary conditions in the case of no-slip (1.14). We focus on the latter case as the proof in the case of the complete slip boundary conditions can be performed in a way similar to [8].

5.1. Velocity Regularization

If the solutions of the Navier–Stokes–Fourier system satisfy the no-slip boundary conditions, the velocity \mathbf{u}_E is not eligible for the relative energy inequality (2.13) as its tangential component may not vanish on $\partial\Omega$. Instead we consider

$$\mathbf{U} = \mathbf{u}_E - \mathbf{v}_\delta, \tag{5.1}$$

where the perturbation \mathbf{v}_δ is given as

$$\mathbf{v}_\delta(t, x) = \xi \left(\frac{\text{dist}[x, \partial\Omega]}{\delta} \right) \mathbf{u}_E(t, \Pi(x)), \quad \delta > 0, \tag{5.2}$$

where

$$\xi \in C^\infty(\mathbb{R}), \quad \xi' \leq 0, \quad \xi(d) = 1 \text{ if } d \leq 0, \quad \xi(d) = 0 \text{ if } \xi \geq 1,$$

and

$$\Pi(x) \in \partial\Omega \text{ is the nearest point to } x \text{ in } \partial\Omega.$$

If $\partial\Omega$ is of class C^k , $k \geq 2$, then $\text{dist}[x, \partial\Omega] \in C^k(\Omega_\delta)$ for any $0 < \delta < \delta_0$, and

$$\nabla_x \text{dist}[x, \partial\Omega] = \frac{x - \Pi(x)}{|x - \Pi(x)|} = -\mathbf{n}(\Pi(x)) \text{ for any } x \in \Omega_\delta,$$

see Foote [13].

5.2. Application of the Relative Energy Inequality

As $\mathbf{U} = \mathbf{u}_E - \mathbf{v}_\delta$ vanishes on $\partial\Omega$, the trio $(r = \varrho_E, \mathbf{U} = \mathbf{u}_E - \mathbf{v}_\delta, \Theta = \vartheta_E)$ can be used as test functions in the relative energy inequality (2.13). Recall that at this stage we have the following vanishing parameters: μ_n, κ_n, a_n , and $\delta = \delta_n$.

We have

$$\left| E \left(\varrho_n, \vartheta_n, \mathbf{u}_n \mid \varrho_E, \vartheta_E, \mathbf{u}_E \right) - E \left(\varrho, \vartheta, \mathbf{u} \mid \varrho_E, \vartheta_E, \mathbf{u}_E - \mathbf{v}_\delta \right) \right| \lesssim |\varrho_n(\mathbf{u}_n - \mathbf{u}_E) \cdot \mathbf{v}_\delta| + \varrho_n |\mathbf{v}_\delta|^2. \tag{5.3}$$

Seeing that

$$\text{ess sup}_{t \in (0, T)} \|\varrho_n\|_{L^\gamma(\Omega)} + \text{ess sup}_{t \in (0, T)} \|\varrho_n \mathbf{u}_n\|_{L^{\frac{2\gamma}{\gamma+1}}(\Omega; \mathbb{R}^3)} \lesssim 1, \tag{5.4}$$

we may infer that

$$\int_\Omega \left| E \left(\varrho_n, \vartheta_n, \mathbf{u}_n \mid \varrho_E, \vartheta_E, \mathbf{u}_E \right) - E \left(\varrho, \vartheta, \mathbf{u} \mid \varrho_E, \vartheta_E, \mathbf{u}_E - \mathbf{v}_\delta \right) \right| dx \rightarrow 0 \text{ as } \delta \rightarrow 0. \tag{5.5}$$

The first rather straightforward observation is that, under hypothesis (3.2) concerning the initial data,

$$\int_\Omega E_{a_n} \left(\varrho_{0,n}, \vartheta_{0,n}, \mathbf{u}_{0,n} \mid \varrho_E(0, \cdot), \vartheta_E(0, \cdot), \mathbf{u}_E(0, \cdot) - \mathbf{v}_\delta(0, \cdot) \right) dx \rightarrow 0 \text{ for } n \rightarrow \infty, \delta \rightarrow 0.$$

Consequently, we can write (2.13) in the form

$$\begin{aligned} & \int_\Omega E_{a_n} \left(\varrho_n, \vartheta_n, \mathbf{u}_n \mid \varrho_E, \vartheta_E, \mathbf{u}_E - \mathbf{v}_\delta \right) (\tau, \cdot) dx \\ & + \frac{\vartheta}{\vartheta_n} \int_0^\tau \int_\Omega \frac{1}{\vartheta_n} \left(\mu_n \mathbb{S}(\vartheta_n, \nabla_x \mathbf{u}_n) : \nabla_x \mathbf{u}_n - \kappa_n \frac{\mathbf{q}(\vartheta_n, \nabla_x \vartheta_n) \cdot \nabla_x \vartheta_n}{\vartheta_n} \right) dx dt \\ & \leq \int_0^\tau \int_\Omega \varrho_n (\mathbf{u}_n - (\mathbf{u}_E - \mathbf{v}_\delta)) \cdot \nabla_x (\mathbf{u}_E - \mathbf{v}_\delta) \cdot (\mathbf{u}_n - (\mathbf{u}_E - \mathbf{v}_\delta)) dx dt \\ & + \mu_n \int_0^\tau \int_\Omega \mathbb{S}(\vartheta_n, \nabla_x \mathbf{u}_n) : \nabla_x (\mathbf{u}_E - \mathbf{v}_\delta) dx dt - \kappa_n \int_0^\tau \int_\Omega \frac{\mathbf{q}(\vartheta_n, \nabla_x \vartheta_n)}{\vartheta_n} \cdot \nabla_x \vartheta_E dx dt \end{aligned}$$

$$\begin{aligned}
& + \int_0^\tau \int_\Omega \varrho_n ((s + s_R)(\varrho_n, \vartheta_n) - (s + s_R)(\varrho_E, \vartheta_E)) ((\mathbf{u}_E - \mathbf{v}_\delta) - \mathbf{u}_n) \cdot \nabla_x \vartheta_E \, dx \, dt \\
& + \int_0^\tau \int_\Omega \varrho_n \left(\partial_t (\mathbf{u}_E - \mathbf{v}_\delta) + (\mathbf{u}_E - \mathbf{v}_\delta) \cdot \nabla_x (\mathbf{u}_E - \mathbf{v}_\delta) \right) \cdot ((\mathbf{u}_E - \mathbf{v}_\delta) - \mathbf{u}_n) \, dx \, dt \\
& - \int_0^\tau \int_\Omega (p + p_R)(\varrho_n, \vartheta_n) \operatorname{div}_x (\mathbf{u}_E - \mathbf{v}_\delta) \, dx \, dt \\
& - \int_0^\tau \int_\Omega \varrho_n ((s + s_R)(\varrho_n, \vartheta_n) - (s + s_R)(\varrho_E, \vartheta_E)) \left(\partial_t \vartheta_E + (\mathbf{u}_E - \mathbf{v}_\delta) \cdot \nabla_x \vartheta_E \right) \, dx \, dt \\
& + \int_0^\tau \int_\Omega \left(\left(1 - \frac{\varrho_n}{\varrho_E} \right) \partial_t (p + p_R)(\varrho_E, \vartheta_E) - \frac{\varrho_n}{\varrho_E} \mathbf{u}_n \cdot \nabla_x (p + p_R)(\varrho_E, \vartheta_E) \right) \, dx \, dt + h(n, \delta), \quad (5.6)
\end{aligned}$$

which holds for a.a. $\tau \in (0, T)$, where h denotes a generic sequence,

$$h(n, \delta) \rightarrow 0 \text{ as } n \rightarrow \infty, \delta \rightarrow 0.$$

Our goal is to show

$$\int_\Omega E_{a_n} \left(\varrho_n, \vartheta_n, \mathbf{u}_n \middle| \varrho_E, \vartheta_E, \mathbf{u}_E - \mathbf{v}_\delta \right) (\tau, \cdot) \, dx = h(n, \delta) \text{ uniformly for a.a. } \tau \in (0, T),$$

by means of a Gronwall type argument.

5.3. Integrals Controlled by the Consistency Estimates

Evoking the bounds obtained in Sect. 4.2 we get

$$\begin{aligned}
& \kappa_n \left| \int_\Omega \frac{\mathbf{q}(\vartheta_n, \nabla_x \vartheta_n)}{\vartheta_n} \cdot \nabla_x \vartheta_E \, dx \right| = \kappa_n \left| \int_\Omega \frac{\tilde{\kappa}(\vartheta_n)}{\vartheta_n} \nabla_x \vartheta_n \cdot \nabla_x \vartheta_E \, dx \right| \\
& \leq \kappa_n \left| \int_\Omega \left[\frac{\tilde{\kappa}(\vartheta_n)}{\vartheta_n} \right]_{\text{ess}} \nabla_x \vartheta_n \cdot \nabla_x \vartheta_E \, dx \right| + \kappa_n \left| \int_\Omega \left[\frac{\tilde{\kappa}(\vartheta_n)}{\vartheta_n} \right]_{\text{res}} \nabla_x \vartheta_n \cdot \nabla_x \vartheta_E \, dx \right| \\
& \leq \varepsilon \mathcal{D}_n + c(\varepsilon, \|\nabla_x \vartheta_E\|_{L^\infty}) \kappa_n + \|\nabla_x \vartheta_E\|_{L^\infty} \kappa_n \int_\Omega \left[\frac{\tilde{\kappa}(\vartheta_n)}{\vartheta_n} \right]_{\text{res}} |\nabla_x \vartheta_n| \, dx,
\end{aligned}$$

where, by virtue of (4.8),

$$\begin{aligned}
\kappa_n \int_\Omega \left[\frac{\tilde{\kappa}(\vartheta_n)}{\vartheta_n} \right]_{\text{res}} |\nabla_x \vartheta_n| \, dx & \leq \varepsilon \mathcal{D}_n + c(\varepsilon) \int_\Omega a_n [\vartheta_n^4]_{\text{res}} \, dx \\
& \leq \varepsilon \mathcal{D}_n + c(\varepsilon) \int_\Omega E_{a_n} \left(\varrho_n, \vartheta_n, \mathbf{u}_n \middle| \varrho_E, \vartheta_E, \mathbf{u}_E - \mathbf{v}_\delta \right) \, dx.
\end{aligned}$$

Using the consistency estimates of Sect. 4.2, we can handle other integrals containing vanishing parameters. Accordingly, the inequality (5.6) simplifies to

$$\begin{aligned}
& \int_\Omega E_{a_n} \left(\varrho_n, \vartheta_n, \mathbf{u}_n \middle| \varrho_E, \vartheta_E, \mathbf{u}_E - \mathbf{v}_\delta \right) (\tau, \cdot) \, dx \\
& + \vartheta \int_0^\tau \int_\Omega \frac{1}{\vartheta_n} \left(\mu_n \mathbb{S}(\vartheta_n, \nabla_x \mathbf{u}_n) : \nabla_x \mathbf{u}_n - \kappa_n \frac{\mathbf{q}(\vartheta_n, \nabla_x \vartheta_n) \cdot \nabla_x \vartheta_n}{\vartheta_n} \right) \, dx \, dt \\
& \leq - \int_0^\tau \int_\Omega \varrho_n (\mathbf{u}_n - (\mathbf{u}_E - \mathbf{v}_\delta)) \cdot \nabla_x \mathbf{v}_\delta \cdot (\mathbf{u}_n - (\mathbf{u}_E - \mathbf{v}_\delta)) \, dx \, dt \\
& - \mu_n \int_0^\tau \int_\Omega \mathbb{S}(\vartheta_n, \nabla_x \mathbf{u}_n) : \nabla_x \mathbf{v}_\delta \, dx \, dt \\
& + \int_0^\tau \int_\Omega \varrho_n ((s + s_R)(\varrho_n, \vartheta_n) - (s + s_R)(\varrho_E, \vartheta_E)) ((\mathbf{u}_E - \mathbf{v}_\delta) - \mathbf{u}_n) \cdot \nabla_x \vartheta_E \, dx \, dt
\end{aligned}$$

$$\begin{aligned}
& + \int_0^\tau \int_\Omega \varrho_n \left(\partial_t (\mathbf{u}_E - \mathbf{v}_\delta) + (\mathbf{u}_E - \mathbf{v}_\delta) \cdot \nabla_x (\mathbf{u}_E - \mathbf{v}_\delta) \right) \cdot ((\mathbf{u}_E - \mathbf{v}_\delta) - \mathbf{u}_n) \, dx \, dt \\
& - \int_0^\tau \int_\Omega (p + p_R)(\varrho_n, \vartheta_n) \operatorname{div}_x (\mathbf{u}_E - \mathbf{v}_\delta) \, dx \, dt \\
& - \int_0^\tau \int_\Omega \varrho_n (s(\varrho_n, \vartheta_n) - s(\varrho_E, \vartheta_E)) \left(\partial_t \vartheta_E + (\mathbf{u}_E - \mathbf{v}_\delta) \cdot \nabla_x \vartheta_E \right) \, dx \, dt \\
& + \int_0^\tau \int_\Omega \left(\left(1 - \frac{\varrho_n}{\varrho_E} \right) \partial_t p(\varrho_E, \vartheta_E) - \frac{\varrho_n}{\varrho_E} \mathbf{u}_n \cdot \nabla_x p(\varrho_E, \vartheta_E) \right) \, dx \, dt \\
& + c \int_0^\tau \int_\Omega E_{a_n} \left(\varrho_n, \vartheta_n, \mathbf{u}_n \Big|_{\varrho_E, \vartheta_E, \mathbf{u}_E - \mathbf{v}_\delta} \right) \, dx \, dt + h(n, \delta). \tag{5.7}
\end{aligned}$$

Moreover, as $\mathbf{u}_E \cdot \mathbf{n}|_{\partial\Omega} = 0$,

$$\|\operatorname{div}_x \mathbf{v}_\delta\|_{L^\infty} \lesssim 1 \text{ independently of } \delta, \tag{5.8}$$

and, consequently,

$$\begin{aligned}
& \int_\Omega E_{a_n} \left(\varrho_n, \vartheta_n, \mathbf{u}_n \Big|_{\varrho_E, \vartheta_E, \mathbf{u}_E - \mathbf{v}_\delta} \right) (\tau, \cdot) \, dx \\
& + \vartheta \int_0^\tau \int_\Omega \frac{1}{\vartheta_n} \left(\mu_n \mathbb{S}(\vartheta_n, \nabla_x \mathbf{u}_n) : \nabla_x \mathbf{u}_n - \kappa_n \frac{\mathbf{q}(\vartheta_n, \nabla_x \vartheta_n) \cdot \nabla_x \vartheta_n}{\vartheta_n} \right) \, dx \, dt \\
& \leq - \int_0^\tau \int_\Omega \varrho_n \mathbf{u}_n \cdot \nabla_x \mathbf{v}_\delta \cdot (\mathbf{u}_n - (\mathbf{u}_E - \mathbf{v}_\delta)) \, dx \, dt \\
& - \mu_n \int_0^\tau \int_\Omega \mathbb{S}(\vartheta_n, \nabla_x \mathbf{u}_n) : \nabla_x \mathbf{v}_\delta \, dx \, dt \\
& + \int_0^\tau \int_\Omega \varrho_n [s(\varrho_n, \vartheta_n) + 1]_{\text{res}} |\mathbf{u}_n| |\nabla_x \vartheta_E| \, dx \, dt \\
& + \int_0^\tau \int_\Omega \varrho_n \left(\partial_t \mathbf{u}_E + \mathbf{u}_E \cdot \nabla_x \mathbf{u}_E + \frac{1}{\varrho_E} \nabla_x p(\varrho_E, \vartheta_E) \right) \cdot (\mathbf{u}_E - \mathbf{u}_n) \, dx \, dt \\
& - \int_0^\tau \int_\Omega \varrho_n \left(\partial_t \mathbf{v}_\delta + \mathbf{v}_\delta \cdot \nabla_x \mathbf{u}_E \right) \cdot ((\mathbf{u}_E - \mathbf{v}_\delta) - \mathbf{u}_n) \, dx \, dt \\
& + \int_0^\tau \int_\Omega \left(p(\varrho_E, \vartheta_E) - p(\varrho_n, \vartheta_n) \right) \operatorname{div}_x \mathbf{u}_E \, dx \, dt \\
& - \int_0^\tau \int_\Omega \varrho_n (s(\varrho_n, \vartheta_n) - s(\varrho_E, \vartheta_E)) \left(\partial_t \vartheta_E + \mathbf{u}_E \cdot \nabla_x \vartheta_E \right) \, dx \, dt \\
& + \int_0^\tau \int_\Omega \left(1 - \frac{\varrho_n}{\varrho_E} \right) \left(\partial_t p(\varrho_E, \vartheta_E) + \mathbf{u}_E \cdot \nabla_x p(\varrho_E, \vartheta_E) \right) \, dx \, dt \\
& + c \int_0^\tau \int_\Omega E_{a_n} \left(\varrho_n, \vartheta_n, \mathbf{u}_n \Big|_{\varrho_E, \vartheta_E, \mathbf{u}_E - \mathbf{v}_\delta} \right) \, dx \, dt + h(n, \delta). \tag{5.9}
\end{aligned}$$

Finally, as $(\varrho_E, \vartheta_E, \mathbf{u}_E)$ solves the Euler system,

$$\partial_t \mathbf{u}_E + \mathbf{u}_E \cdot \nabla_x \mathbf{u}_E + \frac{1}{\varrho_E} \nabla_x p(\varrho_E, \vartheta_E) = 0.$$

In addition, it is easy to check that

$$\|\partial_t \mathbf{v}_\delta\|_{L^\infty} + \|\mathbf{v}_\delta\|_{L^\infty} \lesssim 1 \text{ independently of } \delta. \tag{5.10}$$

Consequently,

$$\begin{aligned} & \int_0^\tau \int_\Omega \varrho_n \left(\partial_t \mathbf{v}_\delta + \mathbf{v}_\delta \cdot \nabla_x \mathbf{u}_E \right) \cdot \left((\mathbf{u}_E - \mathbf{v}_\delta) - \mathbf{u}_n \right) dx dt \\ &= \int_0^\tau \int_{\Omega_\delta} \varrho_n \left(\partial_t \mathbf{v}_\delta + \mathbf{v}_\delta \cdot \nabla_x \mathbf{u}_E \right) \cdot \left((\mathbf{u}_E - \mathbf{v}_\delta) - \mathbf{u}_n \right) dx dt \rightarrow 0 \text{ as } \delta \rightarrow 0 \end{aligned}$$

as both $(\varrho_n)_{n \geq 0}$ and $(\varrho_n \mathbf{u}_n)_{n \geq 0}$ are equi-integrable in $(0, T) \times \Omega$. Thus (5.9) reduces to

$$\begin{aligned} & \int_\Omega E_{a_n} \left(\varrho_n, \vartheta_n, \mathbf{u}_n \mid \varrho_E, \vartheta_E, \mathbf{u}_E - \mathbf{v}_\delta \right) (\tau, \cdot) dx \\ &+ \vartheta \int_0^\tau \int_\Omega \frac{1}{\vartheta_n} \left(\mu_n \mathbb{S}(\vartheta_n, \nabla_x \mathbf{u}_n) : \nabla_x \mathbf{u}_n - \kappa_n \frac{\mathbf{q}(\vartheta_n, \nabla_x \vartheta_n) \cdot \nabla_x \vartheta_n}{\vartheta_n} \right) dx dt \\ &\leq - \int_0^\tau \int_\Omega \varrho_n \mathbf{u}_n \cdot \nabla_x \mathbf{v}_\delta \cdot (\mathbf{u}_n - (\mathbf{u}_E - \mathbf{v}_\delta)) dx dt \\ &- \mu_n \int_0^\tau \int_\Omega \mathbb{S}(\vartheta_n, \nabla_x \mathbf{u}_n) : \nabla_x \mathbf{v}_\delta dx dt \\ &+ \int_0^\tau \int_\Omega \varrho_n [s(\varrho_n, \vartheta_n) + 1]_{\text{res}} |\mathbf{u}_n| |\nabla_x \vartheta_E| dx dt \\ &+ \int_0^\tau \int_\Omega \left(p(\varrho_E, \vartheta_E) - p(\varrho_n, \vartheta_n) \right) \text{div}_x \mathbf{u}_E dx dt \\ &- \int_0^\tau \int_\Omega \varrho_n (s(\varrho_n, \vartheta_n) - s(\varrho_E, \vartheta_E)) \left(\partial_t \vartheta_E + \mathbf{u}_E \cdot \nabla_x \vartheta_E \right) dx dt \\ &+ \int_0^\tau \int_\Omega \left(1 - \frac{\varrho_n}{\varrho_E} \right) \left(\partial_t p(\varrho_E, \vartheta_E) + \mathbf{u}_E \cdot \nabla_x p(\varrho_E, \vartheta_E) \right) dx dt \\ &+ c \int_0^\tau \int_\Omega E_{a_n} \left(\varrho_n, \vartheta_n, \mathbf{u}_n \mid \varrho_E, \vartheta_E, \mathbf{u}_E - \mathbf{v}_\delta \right) dx dt + h(n, \delta). \end{aligned} \quad (5.11)$$

5.4. Integrals Independent of the Boundary Layer

Now, we estimate the integrals on the right-hand side of (5.11) that are independent of \mathbf{v}_δ . First, by virtue of (4.7),

$$\begin{aligned} \left| \int_\Omega \varrho_n [s(\varrho_n, \vartheta_n) + 1]_{\text{res}} |\mathbf{u}_n| |\nabla_x \vartheta_E| dx \right| &\lesssim \int_\Omega [\varrho_n]_{\text{res}} |\mathbf{u}_n|^2 dx + \int_\Omega [\varrho_n]_{\text{res}} s^2(\varrho_n, \vartheta_n) dx \\ &\lesssim \int_\Omega [E_{a_n}(\varrho_n, \vartheta_n, \mathbf{u}_n)]_{\text{res}} dx \\ &\lesssim \int_\Omega E_{a_n} \left(\varrho_n, \vartheta_n, \mathbf{u}_n \mid \varrho_E, \vartheta_E, \mathbf{u}_E - \mathbf{v}_\delta \right) dx + h(\delta). \end{aligned} \quad (5.12)$$

We point out that this step depends in an essential way on the fact that s satisfies the Third law of thermodynamics.

Next, we recall two identities that follow from the specific form of EOS (1.3), (1.4), namely

$$\begin{aligned} \partial_t \vartheta_E + \mathbf{u}_E \cdot \nabla_x \vartheta_E &= -(\gamma - 1) \vartheta_E \text{div}_x \mathbf{u}_E, \\ \partial_t p(\varrho_E, \vartheta_E) + \mathbf{u}_E \cdot \nabla_x p(\varrho_E, \vartheta_E) &= -\gamma p(\varrho_E, \vartheta_E) \text{div}_x \mathbf{u}_E. \end{aligned}$$

Consequently, we get

$$\begin{aligned} & \int_\Omega \left(p(\varrho_E, \vartheta_E) - p(\varrho_n, \vartheta_n) \right) \text{div}_x \mathbf{u}_E dx - \int_\Omega \varrho_n (s(\varrho_n, \vartheta_n) - s(\varrho_E, \vartheta_E)) \left(\partial_t \vartheta_E + \mathbf{u}_E \cdot \nabla_x \vartheta_E \right) dx \\ &+ \int_\Omega \left(1 - \frac{\varrho_n}{\varrho_E} \right) \left(\partial_t p(\varrho_E, \vartheta_E) + \mathbf{u}_E \cdot \nabla_x p(\varrho_E, \vartheta_E) \right) dx \end{aligned}$$

$$\begin{aligned}
&= \int_{\Omega} \operatorname{div}_x \mathbf{u}_E \left[p(\varrho_E, \vartheta_E) - p(\varrho_n, \vartheta_n) + (\gamma - 1) \varrho_E \vartheta_E (s(\varrho_n, \vartheta_n) - s(\varrho_E, \vartheta_E)) \right] dx \\
&\quad - \gamma \int_{\Omega} \operatorname{div}_x \mathbf{u}_E \left(1 - \frac{\varrho_n}{\varrho_E} \right) p(\varrho_E, \vartheta_E) dx \\
&\quad + (\gamma - 1) \int_{\Omega} \vartheta_E (\varrho_n - \varrho_E) (s(\varrho_n, \vartheta_n) - s(\varrho_E, \vartheta_E)) \operatorname{div}_x \mathbf{u}_E dx.
\end{aligned} \tag{5.13}$$

Finally, we use the identity

$$\begin{aligned}
&\left(\frac{\varrho_n}{\varrho_E} - 1 \right) \gamma p(\varrho_E, \vartheta_E) + \left(\frac{\partial p(\varrho_E, \vartheta_E)}{\partial \varrho} (\varrho_E - \varrho_n) + \frac{\partial p(\varrho_E, \vartheta_E)}{\partial \vartheta} (\vartheta_E - \vartheta_n) \right) \\
&\quad - (\gamma - 1) \varrho_E \vartheta_E \left(\frac{\partial s(\varrho_E, \vartheta_E)}{\partial \varrho} (\varrho_E - \varrho_n) + \frac{\partial s(\varrho_E, \vartheta_E)}{\partial \vartheta} (\vartheta_E - \vartheta_n) \right) = 0.
\end{aligned} \tag{5.14}$$

Plugging (5.14) into (5.13) yields the desired estimate. Thus (5.11) reduces to

$$\begin{aligned}
&\int_{\Omega} E_{a_n} \left(\varrho_n, \vartheta_n, \mathbf{u}_n \mid \varrho_E, \vartheta_E, \mathbf{u}_E - \mathbf{v}_\delta \right) (\tau, \cdot) dx \\
&\quad + \vartheta \int_0^\tau \int_{\Omega} \frac{1}{\vartheta_n} \left(\mu_n \mathbb{S}(\vartheta_n, \nabla_x \mathbf{u}_n) : \nabla_x \mathbf{u}_n - \kappa_n \frac{\mathbf{q}(\vartheta_n, \nabla_x \vartheta_n) \cdot \nabla_x \vartheta_n}{\vartheta_n} \right) dx dt \\
&\leq - \int_0^\tau \int_{\Omega} \varrho_n \mathbf{u}_n \cdot \nabla_x \mathbf{v}_\delta \cdot (\mathbf{u}_n - (\mathbf{u}_E - \mathbf{v}_\delta)) dx dt \\
&\quad - \mu_n \int_0^\tau \int_{\Omega} \mathbb{S}(\vartheta_n, \nabla_x \mathbf{u}_n) : \nabla_x \mathbf{v}_\delta dx dt \\
&\quad + c \int_0^\tau \int_{\Omega} E_{a_n} \left(\varrho_n, \vartheta_n, \mathbf{u}_n \mid \varrho_E, \vartheta_E, \mathbf{u}_E - \mathbf{v}_\delta \right) dx dt + h(n, \delta).
\end{aligned} \tag{5.15}$$

Note that inequality (5.15) almost completes the proof of Theorem 3.1, where we may take $\mathbf{v}_\delta = 0$. It only remains to show the desired strong convergence claimed in (3.3). This will be done in Sect. 5.6.

However, in order to prove Theorems 3.4–3.6, where $\mathbf{v}_\delta \neq 0$, one has to estimate the first two integrals on the right-hand side of (5.15), which is carried out in the following Sect. 5.5.

5.5. Boundary Layer

It remains to control the first two integrals on the right-hand side of (5.15) that represent the effect of the boundary layer.

5.5.1. Viscous Stress. Similarly to Sect. 4.2, we have

$$\mu_n \left| \int_{\Omega} \mathbb{S}(\vartheta_n, \nabla_x \mathbf{u}_n) : \nabla_x \mathbf{v}_\delta dx \right| \leq \varepsilon \mathcal{D}_n + c(\varepsilon) \mu_n \int_{\Omega} \vartheta_n (1 + \vartheta_n^\alpha) |\nabla_x \mathbf{v}_\delta|^2 dx,$$

where

$$\mu_n \int_{\Omega} \vartheta_n (1 + \vartheta_n^\alpha) |\nabla_x \mathbf{v}_\delta|^2 dx \lesssim \frac{\mu_n}{\delta^2} \int_{\Omega_\delta} (1 + \vartheta_n^{1+\alpha}) dx \lesssim \frac{\mu_n}{\delta} \left(1 + \frac{1}{\delta} \int_{\Omega_\delta} \vartheta_n^{1+\alpha} dx \right).$$

Consequently, when proving Theorems 3.5 and 3.6, the desired estimate follows from hypothesis (3.5) and (3.6), respectively. Note that this type of estimates forces us to consider the thickness δ of the boundary layer asymptotically larger than μ ,

$$\frac{\mu_n}{\delta_n} \rightarrow 0.$$

Alternatively, in order to show Theorem 3.4 and following Sueur [21], we have (3.4), meaning

$$\sqrt{\mu_n} \|\mathbb{S}(\vartheta_n, \nabla_x \mathbf{u}_n)\|_{L^2((0,T) \times \Omega_{\mu_n}; \mathbb{R}^9)} \rightarrow 0.$$

Setting $\mu_n \approx \delta_n$, we get

$$\begin{aligned} & \left| \int_0^T \int_{\Omega} \mathbb{S}(\vartheta_n, \nabla_x \mathbf{u}_n) : \nabla_x \mathbf{v}_\delta \, dx \, dt \right| \\ & \leq \sqrt{\mu_n} \|\mathbb{S}(\vartheta_n, \nabla_x \mathbf{u}_n)\|_{L^2((0,T) \times \Omega_{\mu_n}; R^9)} \|\sqrt{\mu_n} \nabla_x \mathbf{v}_{\mu_n}\|_{L^2((0,T) \times \Omega_{\mu_n}; R^9)} \rightarrow 0. \end{aligned} \quad (5.16)$$

5.5.2. Convective Term. Finally, we consider

$$\int_{\Omega} \varrho_n \mathbf{u}_n \cdot \nabla_x \mathbf{v}_\delta \cdot (\mathbf{u}_n - (\mathbf{u}_E - \mathbf{v}_\delta)) \, dx = \int_{\Omega_\delta} \varrho_n \mathbf{u}_n \cdot \nabla_x \mathbf{v}_\delta \cdot (\mathbf{u}_n - (\mathbf{u}_E - \mathbf{v}_\delta)) \, dx \text{ in } \Omega_\delta.$$

Recall that

$$\begin{aligned} \mathbf{w}(t, x) &= \mathbf{w}_n(t, x) + \mathbf{w}_\tau(t, x), \\ \mathbf{w}_n(t, x) &= (\mathbf{w} \cdot \nabla_x \text{dist}[x, \partial\Omega]) \nabla_x \text{dist}[x, \partial\Omega], \quad \mathbf{w}_\tau(t, x) = \mathbf{w}(t, x) - \mathbf{w}_n(t, x). \end{aligned}$$

Similarly, for a scalar function F , we decompose

$$\nabla_x F = \nabla_n F + \nabla_\tau F, \quad \nabla_n F = (\nabla_x \text{dist}[x, \partial\Omega] \cdot \nabla_x F) \nabla_x \text{dist}[x, \partial\Omega].$$

In accordance with the definition of \mathbf{v}_δ , we get

$$(\mathbf{v}_\delta)_n = 0, \quad \|\nabla_\tau \mathbf{v}_\delta\|_{L^\infty} \lesssim 1, \quad \|\nabla_n \mathbf{v}_\delta\|_{L^\infty} \lesssim \frac{1}{\delta}. \quad (5.17)$$

Now,

$$\begin{aligned} & \int_{\Omega_\delta} \varrho_n \mathbf{u}_n \cdot \nabla_x \mathbf{v}_\delta \cdot (\mathbf{u}_n - (\mathbf{u}_E - \mathbf{v}_\delta)) \, dx \\ &= \int_{\Omega_\delta} \varrho_n (\mathbf{u}_n)_n \cdot \nabla_x \mathbf{v}_\delta \cdot (\mathbf{u}_n - (\mathbf{u}_E - \mathbf{v}_\delta)) \, dx + \int_{\Omega_\delta} \varrho_n (\mathbf{u}_n)_\tau \cdot \nabla_x \mathbf{v}_\delta \cdot (\mathbf{u}_n - (\mathbf{u}_E - \mathbf{v}_\delta)) \, dx \\ &= \int_{\Omega_\delta} \varrho_n (\mathbf{u}_n)_n \cdot \nabla_n (\mathbf{v}_\delta)_\tau \cdot (\mathbf{u}_n - (\mathbf{u}_E - \mathbf{v}_\delta)) \, dx + \int_{\Omega_\delta} \varrho_n (\mathbf{u}_n)_\tau \cdot \nabla_\tau \mathbf{v}_\delta \cdot (\mathbf{u}_n - (\mathbf{u}_E - \mathbf{v}_\delta)) \, dx, \end{aligned}$$

where, by virtue of (5.17),

$$\begin{aligned} & \left| \int_{\Omega_\delta} \varrho_n (\mathbf{u}_n)_\tau \cdot \nabla_\tau \mathbf{v}_\delta \cdot (\mathbf{u}_n - (\mathbf{u}_E - \mathbf{v}_\delta)) \, dx \right| \\ & \lesssim \int_{\Omega} E(\varrho_n, \vartheta_n, \mathbf{u}_n | \varrho_E, \vartheta_E, \mathbf{u}_E - \mathbf{v}_\delta) \, dx + \int_{\Omega_\delta} \varrho_n |\mathbf{u}_E - \mathbf{v}_\delta| |\mathbf{u}_n - (\mathbf{u}_E - \mathbf{v}_\delta)| \, dx. \end{aligned}$$

In view of (5.4), $(\varrho_n)_{n \geq 0}$, $(\varrho_n \mathbf{u}_n)_{n \geq 0}$ are equi-integrable; whence

$$\int_0^T \int_{\Omega_\delta} \varrho_n |\mathbf{u}_E - \mathbf{v}_\delta| |\mathbf{u}_n - (\mathbf{u}_E - \mathbf{v}_\delta)| \, dx \, dt \rightarrow 0 \text{ as } \delta \rightarrow 0$$

uniformly in n .

Thus it remains to handle the integral

$$\int_{\Omega_\delta} \varrho_n (\mathbf{u}_n)_n \cdot \nabla_n (\mathbf{v}_\delta)_\tau \cdot (\mathbf{u}_n - (\mathbf{u}_E - \mathbf{v}_\delta)) \, dx.$$

Let us first look at Theorem 3.5. By Hölder's inequality and (5.17),

$$\begin{aligned} & \left| \int_{\Omega_\delta} \varrho_n (\mathbf{u}_n)_n \cdot \nabla_n (\mathbf{v}_\delta)_\tau \cdot (\mathbf{u}_n - (\mathbf{u}_E - \mathbf{v}_\delta)) \, dx \right| \\ & \leq \frac{1}{\delta} \|\varrho_n (\mathbf{u}_n)_n\|_{L^{\frac{24}{17+3\alpha}}(\Omega_\delta; R^3)} \|\mathbf{u}_n - (\mathbf{u}_E - \mathbf{v}_\delta)\|_{L^{\frac{24}{7-3\alpha}}(\Omega_\delta; R^3)}, \end{aligned} \quad (5.18)$$

where $\frac{24}{7-3\alpha}$ is the critical exponent in the Sobolev–Poincaré inequality

$$\|\mathbf{u}_n\|_{L^{\frac{24}{7-3\alpha}}(\Omega_\delta; R^3)} \lesssim \|\nabla_x \mathbf{u}_n\|_{L^{\frac{8}{5-\alpha}}(\Omega_\delta; R^9)}. \quad (5.19)$$

As $\mathbf{u}_n|_{\partial\Omega} = 0$, Korn's inequality yields

$$\begin{aligned} \|\mathbf{u}_n\|_{L^{\frac{24}{7-3\alpha}}(\Omega_\delta; R^3)} &\lesssim \|\nabla_x \mathbf{u}_n\|_{L^{\frac{8}{5-\alpha}}(\Omega_\delta; R^9)} \lesssim \left\| \nabla_x \mathbf{u}_n + \nabla_x \mathbf{u}_n^t - \frac{2}{3} \operatorname{div}_x \mathbf{u}_n \mathbb{I} \right\|_{L^{\frac{8}{5-\alpha}}(\Omega_\delta; R^9)} \\ &\lesssim \left\| \vartheta_n^{\frac{1-\alpha}{2}} \right\|_{L^{\frac{8}{1-\alpha}}(\Omega_\delta)} \left\| \vartheta_n^{\frac{\alpha-1}{2}} \left(\nabla_x \mathbf{u}_n + \nabla_x \mathbf{u}_n^t - \frac{2}{3} \operatorname{div}_x \mathbf{u}_n \mathbb{I} \right) \right\|_{L^2(\Omega; R^9)}. \end{aligned}$$

Note that the constants are independent of δ as \mathbf{u}_n can be extended to be zero outside Ω .

Thus going back to (5.18) we deduce

$$\begin{aligned} &\left| \int_{\Omega_\delta} \varrho_n(\mathbf{u}_n)_n \cdot \nabla_n(\mathbf{v}_\delta)_\tau \cdot (\mathbf{u}_n - (\mathbf{u}_E - \mathbf{v}_\delta)) \, dx \right| \\ &\leq \frac{c(\varepsilon)}{\delta} \|\varrho_n(\mathbf{u}_n)_n\|_{L^{\frac{24}{17+3\alpha}}(\Omega_\delta; R^3)} + \frac{c(\varepsilon)}{\delta^2 \mu_n} \|\varrho_n(\mathbf{u}_n)_n\|_{L^{\frac{24}{17+3\alpha}}(\Omega_\delta; R^3)}^2 \left\| \vartheta_n^{\frac{1-\alpha}{2}} \right\|_{L^{\frac{8}{1-\alpha}}(\Omega_\delta)}^2 \\ &\quad + \varepsilon \mathcal{D}_n. \end{aligned}$$

in accordance with hypothesis (3.5).

Next, in view of Theorem 3.6 we consider $\alpha = 1$ and replace the critical exponent $\frac{24}{17+3\alpha}$ in (5.18) by the L^2 -norm. Consequently,

$$\begin{aligned} &\left| \int_{\Omega_\delta} \varrho_n(\mathbf{u}_n)_n \cdot \nabla_n(\mathbf{v}_\delta)_\tau \cdot (\mathbf{u}_n - (\mathbf{u}_E - \mathbf{v}_\delta)) \, dx \right| \\ &\leq \frac{1}{\delta} \|\varrho_n(\mathbf{u}_n)_n\|_{L^2(\Omega_\delta; R^3)} \|\mathbf{u}_n - (\mathbf{u}_E - \mathbf{v}_\delta)\|_{L^2(\Omega_\delta; R^3)} \\ &\lesssim \frac{1}{\delta} \|\varrho_n(\mathbf{u}_n)_n\|_{L^2(\Omega_\delta; R^3)} \|\mathbf{u}_n\|_{L^2(\Omega_\delta; R^3)} + \frac{1}{\sqrt{\delta}} \|\varrho_n(\mathbf{u}_n)_n\|_{L^2(\Omega_\delta; R^3)} \\ &\lesssim \frac{1}{\delta} \|\varrho_n(\mathbf{u}_n)_n\|_{L^2(\Omega_\delta; R^3)} \|\mathbf{u}_n\|_{L^2(\Omega_\delta; R^3)} + \sqrt{\frac{\mu_n}{\delta}} \left(1 + \frac{1}{\mu_n} \|\varrho_n(\mathbf{u}_n)_n\|_{L^2(\Omega_\delta; R^3)}^2 \right). \end{aligned} \tag{5.20}$$

Now, replacing (5.19) by Hardy–Sobolev inequality, we gain the multiplicative factor δ ,

$$\|\mathbf{u}_n\|_{L^2(\Omega_\delta; R^3)} \lesssim \delta \|\nabla_x \mathbf{u}_n\|_{L^2(\Omega_\delta; R^9)}. \tag{5.21}$$

Thus the final inequality reads

$$\begin{aligned} &\left| \int_{\Omega_\delta} \varrho_n(\mathbf{u}_n)_n \cdot \nabla_n(\mathbf{v}_\delta)_\tau \cdot (\mathbf{u}_n - (\mathbf{u}_E - \mathbf{v}_\delta)) \, dx \right| \\ &\leq c(\varepsilon) \left(\sqrt{\frac{\mu_n}{\delta}} \left(1 + \frac{1}{\mu_n} \|\varrho_n(\mathbf{u}_n)_n\|_{L^2(\Omega_\delta; R^3)}^2 \right) + \frac{1}{\mu_n} \|\varrho_n(\mathbf{u}_n)_n\|_{L^2(\Omega_\delta; R^3)}^2 \right) \\ &\quad + \varepsilon \mathcal{D}_n \end{aligned}$$

in accordance with (3.6).

Finally, in order to show Theorem 3.4, one has to estimate the left-hand side of (5.18) using hypothesis (3.4). This works exactly as in Sueur [21].

5.6. Strong Convergence

We have established the convergence

$$\int_{\Omega} E_{a_n} \left(\varrho_n, \vartheta_n, \mathbf{u}_n \mid \varrho_E, \vartheta_E, \mathbf{u}_E - \mathbf{v}_{\delta_n} \right) (\tau, \cdot) \, dx \rightarrow 0 \text{ as } n \rightarrow \infty$$

uniformly for a.a. $\tau \in (0, T)$. This obviously yields

$$\int_{\Omega} E \left(\varrho_n, \vartheta_n, \mathbf{u}_n \mid \varrho_E, \vartheta_E, \mathbf{u}_E \right) (\tau, \cdot) \, dx \rightarrow 0 \text{ as } n \rightarrow \infty.$$

In addition, as the energy of the initial data converges and both Euler and the Navier–Stokes–Fourier system conserve energy, we get

$$\int_{\Omega} \left(\frac{1}{2} \varrho_n |\mathbf{u}_n|^2 + \varrho_n e(\varrho_n, \vartheta_n) \right) dx \rightarrow \int_{\Omega} \left(\frac{1}{2} \varrho_E |\mathbf{u}_E|^2 + \varrho_E e(\varrho_E, \vartheta_E) \right) dx \text{ in } L^1(0, T).$$

This yields the desired strong convergence claimed in (3.3).

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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