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Global Controllability of the Navier–Stokes Equations in the Presence of Curved Boundary with No-Slip Conditions

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Abstract. We consider the issue of the small-time global exact null controllability of the axi-symmetric incompressible Navier–Stokes equation in a 3D finite vertical cylinder with circular section. We assume that we are able to act on the fluid flow on the top and on the bottom of the cylinder while no-slip conditions are prescribed on the boundary of the lateral section. We also make use of a distributed control, which can be chosen arbitrarily small for any Sobolev regularity in space. Our work improves earlier results in Guerrero et al. (C R Math Acad Sci Paris 343:573–577, 2006; J Math Pures Appl (9) 98:689–709, 2012) where the distributed force is small only in a negative Sobolev space and the recent work Coron et al. (Ann PDE 5(2):1–49, 2019) where the case of the 2D incompressible Navier–Stokes equation in a rectangle was considered. Our analysis actually follows quite narrowly the one in Coron et al. (2019) by making use of Coron's return method, of the well-prepared dissipation method and of long-time nonlinear Cauchy–Kovalevskaya estimates. An extra difficulty here is the curvature of the uncontrolled part of the boundary which requires further analysis to apply the well-prepared dissipation method to lower order boundary layer terms.

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Contents

1. Introduction, Statement of the Main Result and Some Open Problems

1.1. Setting

We consider a finite vertical cylinder $\Omega = D_1 \times [0, L] \subset \mathbb{R}^3$ where D_1 is the unit open disc in $\mathbb{R}^2$ and $L > 0$. Let us denote by $\Gamma = (D_1 \times \{0\}) \cup (D_1 \times \{L\})$ the union of the bottom and of the top of the cylinder. This surface Γ has to be thought as the controlled part of the boundary while the complementary part $\partial\Omega\backslash\Gamma = (\partial D_1) \times [0, L]$ of the cylinder's boundary, that is the lateral part of the cylinder's boundary, has to be thought as the uncontrolled part of the boundary. Let us denote by $L^2_{\text{div}}(\Omega)$ the space of the axi-symmetric functions u in $L^2(\Omega)$ such that $\operatorname{div} u = 0$ and $u = 0$ on $\partial\Omega\backslash\Gamma$.

1.2. Main Result

Our main result is the following theorem.

Theorem 1.1. *Let $T > 0$ and $u_0 \in L^2_{\text{div}}(\Omega)$. For any $k \in \mathbb{N}$ and for any $\eta > 0$, there exists a force $f \in L^1((0, T); H^k(\Omega))$ satisfying $\|f\|_{L^1((0,T);H^k(\Omega))} \leq \eta$, and a Leray weak solution $u \in C_{\text{w}}([0, T]; L^2_{\text{div}}(\Omega)) \cap L^2((0, T); H^1(\Omega))$ of*

$$\begin{cases} \partial_t u + u \cdot \nabla u - \Delta u + \nabla p = f \quad \text{and} \quad \operatorname{div} u = 0 \qquad \text{in } \Omega, \\ u = 0 \qquad \text{on } \partial\Omega\backslash\Gamma, \end{cases} \tag{1.1}$$

satisfying $u(0, \cdot) = u_0$ and $u(T, \cdot) = 0$.

1.3. Notion of Controlled Weak Leray Solution

Above, the notion of weak Leray solution corresponds to the following weak formulation:

$$\begin{aligned} - \int_0^T \int_\Omega u \cdot \partial_t \varphi + \int_0^T \int_\Omega (u \cdot \nabla) u \cdot \varphi + 2 \int_0^T \int_\Omega D(u) : D(\varphi) \\ = \int_\Omega u_0 \cdot \varphi(0, \cdot) + \int_0^T \int_\Omega u \cdot f, \end{aligned} \tag{1.2}$$

for any test function $\varphi \in C^\infty([0, T] \times \bar{\Omega})$ which is divergence free, tangent to $\partial\Omega\backslash\Gamma$, vanishes at $t = T$ and vanishes on Γ. This last condition encodes that one controls the part Γ of the boundary, so that no boundary condition is there required. Above the notation $D(\cdot)$ stands for the symmetric part of the gradient.

1.4. Comparison with the Literature

Theorem 1.1 establishes the small-time global exact null controllability of the axi-symmetric incompressible Navier–Stokes equation in a 3D finite cylinder with circular section in the case where the no-slip Dirichlet boundary condition is imposed on the lateral boundary of the cylinder, while we are able to act on the fluid flow on the top and on the bottom of the cylinder, as well as in the interior of the cylinder through a distributed force which can be chosen arbitrarily small for any Sobolev regularity in space. This result improves earlier results in [12,13] where the distributed force is small only in a negative Sobolev space and the recent work [8] where the case of the 2D incompressible Navier–Stokes equation in a rectangle was considered. Let us also mention the paper [6], and its corresponding proceeding [7], on the global null controllability of the Navier–Stokes equation in the case where Navier slip-with-friction boundary conditions are prescribed on the uncontrolled part of the boundary rather than the no-slip conditions. In these references some controlled weak Leray solutions are constructed. This has been improved into smooth solutions (in the case where the initial data is smooth) in the recent paper [15]. The main difference between the case of the Navier conditions and the one of the no-slip conditions, when following the method of proof considered in these references and in the present paper, is the size of the boundary layers which are involved. Indeed in both cases one considers a small-viscosity regime where a controlled auxiliary flow which is built for the inviscid limit of the system has to be corrected by some boundary layer terms on the uncontrolled part of the boundary.

1.5. Difficulty of the Proof

Indeed the proof of Theorem 1.1 below closely follows the analysis developed in [8] to deal with the case of the rectangle by making use of Coron's return method, of the well-prepared dissipation method and of long-time nonlinear Cauchy–Kovalevskaya estimates. One of the main extra difficulty here is the curvature of the uncontrolled part of the cylinder boundary which requires a more delicate analysis, in particular to apply the well-prepared dissipation method, because of lower order boundary layer profiles whose behavior is forced by the higher order boundary layer profiles. This difficulty already appeared in [15] where the case of the Navier boundary conditions, rather than the no-slip condition, was treated.

1.6. Open Problems

It will be interesting to investigate whether or not it is possible to extend the result of Theorem 1.1 to the case of smooth solutions or to the case of non-axisymmetric initial data. We refer to Remark 2.10 below for a few more technical explanations regarding the difficulties of these investigations. An even more challenging issue, as already mentioned in the rectangle case tackled in [8,9], is to get rid of the "phantom" force f.

2. Strategy of the Proof of the Main Result: Theorem 1.1

In this section we explain the general strategy of the proof of Theorem 1.1.

2.1. A Rapid Reminder on Coron's Return Method for the Incompressible Euler Equations

Following Coron's return method, The starting idea is to follow the strategy used by Coron and Glass, see [11], to prove the small-time global exact null controllability of the incompressible Euler equations. In this case, the idea is to introduce an auxiliary flow which is compactly supported in time, and around which the linearized incompressible Euler equations is controllable. To drive the dynamics despite the presence of a non-zero initial data these auxiliary flows have to be chosen with a strong amplitude, which makes them vary quickly. To encode these features we introduce the rescaling

$$u^\varepsilon(t,x) := \varepsilon u(\varepsilon t, x), \quad p^\varepsilon(t,x) := \varepsilon^2 p(\varepsilon t, x), \tag{2.1}$$

where ε is thought as a small positive parameter and we observe that the time interval over which the control is required is then stretched from $(0,T)$ to the large time interval $(0,T/\varepsilon)$.

Some appropriate auxiliary flows can be built for very general geometries of the fluid domain with involved gluing technics, but can be taken in a very simple form in the present case of the finite vertical cylinder Ω as

$$h^0(t)e_3, \tag{2.2}$$

where $\{e_i\}_{i=1}^3$ are the unit vectors of the canonical basis of $\mathbb{R}^3$. Since whatever is the scalar function h^0, the flow given by (2.2) unconditionally satisfies the incompressible Euler equations in Ω, associated with the pressure $-(h^0)'(t)x_3$, while being tangent to the lateral, uncontrolled, part $\partial\Omega\backslash\Gamma$ of the cylinder's boundary. This tangency condition is the natural counterpart of the no-slip condition, actually of any impermeable condition, for the Euler equations (instead of the Navier–Stokes equation).

Moreover the linearization of the incompressible Euler equations around this flow is simply the equation

$$\partial_t v + h^0(t)\partial_3 v + \nabla q = 0 \quad \text{and} \quad \operatorname{div} v = 0, \tag{2.3}$$

where, moreover, the pressure term can be discarded since the divergence free constraint satisfies by the initial data is propagated by transport along the flow (2.2). Together with the tangency condition on the lateral boundary, this leads to the even simpler equation than (2.3), that is the pressure-less equation:

$$\partial_t v + h^0(t)\partial_3 v = 0. \tag{2.4}$$

A controlled solution, in the rescaled variables given by (2.1), can be then constructed as an asymptotic expansion with principal part $h^0(t)e_3 + \varepsilon v$. It is then only a matter to choose the function h^0 in order to flush v outside of Ω and to bound the effect of the remainder term of the expansion to conclude the controllability of the incompressible Euler equations.

2.2. An Almost Returning Solution to the Navier–Stokes Equations Despite the Boundary Layers

Now, the flow (2.2) also satisfies the Navier–Stokes equations inside the cylinder but does not satisfy the no-slip boundary conditions on the boundary of the cross-section. To adapt to the no-slip condition, some boundary layers appear near the boundary. Indeed, under the rescaling (2.1) the (unforced) incompressible Navier–Stokes equations then read as the following small viscosity problem:

$$\partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon - \varepsilon \Delta u^\varepsilon + \nabla p^\varepsilon = 0 \quad \text{and} \quad \operatorname{div} u^\varepsilon = 0.$$

Thanks to the peculiar form of the flow (2.2), the dynamics of this boundary layer is simply given by a heat equation, rather than by Prandtl's equations. Indeed in polar coordinates, for a radial function f, the Laplace operator is given by the formula

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r}, \tag{2.5}$$

where $r := \sqrt{x_1^2 + x_2^2}$ represents the distance to the origin.

The following key intermediate result proves the existence of axisymmetric solutions to the heat equations in the unit disk D_1 with appropriate Dirichlet data which almost return to zero at the final time, while the principal part of the Dirichlet data satisfies a flushing condition at an intermediate time.

Proposition 2.1. *There are h^0, h^1 and h^2 in $C_0^\infty(0,T)$, satisfying the flushing condition*

$$\int_0^{\frac{T}{3}} h^0(t)\,dt = 2L, \tag{2.6}$$

such that the solutions $(v^\varepsilon := v^\varepsilon(t,r))_\varepsilon$ *to*

$$\begin{cases} \partial_t v^\varepsilon - \varepsilon \Delta v^\varepsilon = 0 \qquad t \geq 0, 0 \leq r \leq 1, \\ v^\varepsilon(t,1) = -\sum_{i=0}^{2} \varepsilon^{\frac{i}{2}} h^i(t) \qquad t \geq 0, 0 \leq r \leq 1, \\ v^\varepsilon(0,r) = 0 \qquad 0 \leq r \leq 1, \end{cases} \tag{2.7}$$

verifies

$$\|v^\varepsilon \left(\frac{T}{\varepsilon}, \cdot\right) \|_{L^2(D_1)} \leq C \varepsilon^{\frac{7}{4}}. \tag{2.8}$$

Moreover the function h^0 *has to satisfy the zeroth order moment condition:*

$$\int_0^T h^0(t)\, dt = 0. \tag{2.9}$$

Let us highlight that despite a right hand side $o(\varepsilon)$ in (2.8) would be enough for the sequel, we choose to keep the explicit exponent which comes for free with our method of proof. The zeroth order moment condition (2.9) comes from the construction of V^0 in Sects. 2.5 and 3. Since the source term of the Eq. (2.14) for V^0 below is zero, we find that h^0 satisfies (2.9) from (3.14) in Sect. 3. Indeed the proof of Proposition 2.1 is based on the well-prepared dissipation method, which was initiated in [17], and adapted in [6,8,15]. We give below the core of the proof, postponing to the next sections some intermediate results which are themselves quite intricate.

2.3. A Multi-scale Asymptotic Expansion of v^ε

Let χ be a cut-off function $\chi(r) = 0$ when $r \leq \frac{1}{3}$ and $\chi(r) = 1$ when $r \geq \frac{2}{3}$. Given h^0, h^1 and h^2 in $C_0^\infty(0,T)$, we seek for a multi-scale asymptotic expansion of the solutions $(v^\varepsilon)_\varepsilon$ of the form

$$v^\varepsilon(t,r) :\, = \, -\chi(r) \sum_{i=0}^{2} \varepsilon^{\frac{i}{2}} V^i \left(t, \frac{1-r}{\sqrt{\varepsilon}}\right) + \varepsilon r^\varepsilon(t, x_1, x_2), \tag{2.10}$$

where $r = (x_1^2 + x_2^2)^{\frac{1}{2}}$ and $V^i(t,z) \to 0$ as $z \to 0$ for $0 \leq i \leq 2$ and r^ε is considered as a technical lower-order corrector. To infer which relevant "profiles" $V^i(t,z)$ we should consider, we observe that for a function $g(t,z)$, denoting by $\alpha \in C^\infty([\frac{1}{3}, 1]; \mathbb{R})$ the function such that

$$\frac{1}{r} = 1 + (1-r) + (1-r)^2 \alpha(r), \tag{2.11}$$

we have:

$$\varepsilon \Delta \{g\}_\varepsilon(t,x) = \{\partial_z^2 g\}_\varepsilon(t,x) - \varepsilon^{\frac{1}{2}} \{\partial_z g\}_\varepsilon(t,x) - \varepsilon \{z \partial_z g\}_\varepsilon(t,x) - \varepsilon^{\frac{3}{2}} \alpha(r) \{z^2 \partial_z g\}_\varepsilon(t,x), \tag{2.12}$$

where the notation $\{\cdot\}_\varepsilon$ stands for

$$\{g\}_\varepsilon(t,x) := g\left(t, \frac{1-r}{\sqrt{\varepsilon}}\right), \quad r = (x_1^2 + x_2^2)^{\frac{1}{2}}. \tag{2.13}$$

Therefore, by inserting the ansatz (2.10) in the Eq. (2.7), and then using (2.12) and equalling by powers of ε, we are led to consider the following three problems:

$$\begin{cases} \partial_t V^0 - \partial_z^2 V^0 = 0, \quad t \geq 0, z \geq 0, \\ V^0(t,1) = h^0(t), \qquad t \geq 0, \\ V^0(0,z) = 0, \qquad z \geq 0, \end{cases} \tag{2.14}$$

where we observe that the only non-zero source is on the boundary data,

$$\begin{cases} \partial_t V^1 - \partial_z^2 V^1 = -\partial_z V^0, & t \geq 0, z \geq 0, \\ V^1(t,1) = h^1(t), & t \geq 0, \\ V^1(0,z) = 0, & z \geq 0, \end{cases} \tag{2.15}$$

and

$$\begin{cases} \partial_t V^2 - \partial_z^2 V^2 = -\partial_z V^1 - z\partial_z V^0, & t \geq 0, z \geq 0, \\ V^2(t,1) = h^2(t), & t \geq 0, \\ V^2(0,z) = 0, & z \geq 0. \end{cases} \tag{2.16}$$

2.4. Boundary Forcing and Enhanced Time-Decay of the Free and Forced Heat Equation in a Disk

In this subsection we establish that there exists an appropriate choice of the boundary data, to impose a rapid decay in time of the solutions to the forced heat equation, such as (2.14)–(2.16), for which the choice is respectively the one of the function h^0, h^1 and h^2.

Let us introduce the following weighted Sobolev spaces.

Definition 2.2. For $z \in \mathbb{R}$, we denote $\langle z \rangle := \sqrt{1+z^2}$ and for s and $q \in \mathbb{N}$, we set

$$H^s_q(\mathbb{R}_+) := \Big\{ \ f \in H^s(\mathbb{R}_+) : \ \sum_{j=0}^{s} \int_{\mathbb{R}_+} \langle z \rangle^{2q} |\partial_z^j f(z)|^2 dz < +\infty \ \Big\},$$

endowed with it natural associated norm. In the same way we define $H^s_q(\mathbb{R})$ and the norm

$$\|f\|_{H^s_q(\mathbb{R})} := \left(\sum_{j=0}^{s} \int_{\mathbb{R}} \langle z \rangle^{2q} |\partial_z^j f(z)|^2 dz \right)^{\frac{1}{2}}.$$

Observe that by the Plancherel theorem, we have the following equivalence of norms:

$$\|f\|_{H^s_q(\mathbb{R})} \sim \sum_{j=0}^{q} \left(\int_{\mathbb{R}} \langle \zeta \rangle^{2s} |\partial_\zeta^j \hat{f}(\zeta)|^2 d\zeta \right)^{\frac{1}{2}}, \tag{2.17}$$

where $\hat{f}$ denotes the Fourier transform of f.

Definition 2.3. Let $k \in \mathbb{N}, \gamma > 0$ and X a Banach space with norm $\| \cdot \|_X$. We define the space $C^k_\gamma(\mathbb{R}_+; X)$ of the functions $f \in C^k(\mathbb{R}_+; X)$ such that

$$\|f\|_{C^k_\gamma(\mathbb{R}_+;X)} := \sup_{t \geq 0, 0 \leq j \leq k} \left(\|\partial_t^j f(t)\|_X \langle t \rangle^\gamma \right) < +\infty,$$

where

$$C^k(\mathbb{R}_+; X) := \big\{ f : \partial_t^j f \in C(\mathbb{R}_+; X), \ 0 \leq j \leq k \big\}.$$

The proof of the next proposition is given in Sect. 3.

Proposition 2.4. *Let $\gamma > 0$, $k, s, q, n \in \mathbb{N}$ satisfy*

$$n \geq \frac{q}{2} + \gamma - 1. \tag{2.18}$$

and we define

$$\tilde{\gamma} := 2n + 3, \quad \tilde{s} := s + 2k + 2n, \quad \tilde{q} := 2n + 3. \tag{2.19}$$

Given $f \in C^0_{\tilde\gamma}(\mathbb{R}_+; H^{\tilde s}_{\tilde q}(\mathbb{R}_+))$ when $k = 0$ and $f \in C^{k-1}_{\tilde\gamma}(\mathbb{R}_+; H^{\tilde s}_{\tilde q}(\mathbb{R}_+))$ when $k \geq 1$, we can find a nonzero function $h \in C^\infty_0(0, T)$, supported in $(0, \frac{T}{3}] \cup [\frac{2T}{3}, T)$, such that the following system

$$\begin{cases} \partial_t v - \partial_z^2 v = f, & t \geq 0, z \geq 0, \\ v(t, 0) = h(t), & t \geq 0, \\ v(0, z) = 0, & z \geq 0, \end{cases} \tag{2.20}$$

has a unique solution $v \in C^k_\gamma(\mathbb{R}_+; H^s_q(\mathbb{R}_+))$. Moreover, if f is supported away from $t = 0$ as a function of time t, so does v.

2.5. Proof of Proposition 2.1

Let us now see how to conclude the proof of Proposition 2.1 by applying Proposition 2.4 to the three problems (2.14)–(2.16). Let $k_0 = 0, \gamma_0 = 23, s_0 = 28, q_0 = 23$.

- By Proposition 2.4, there exists $h^0 \in C^\infty_0(0, T)$ supported in $(0, \frac{T}{3}] \cup [\frac{2T}{3}, T)$ such that system (2.14) has a unique solution $V^0 \in C^0_{23}(\mathbb{R}_+; H^{28}_{23}(\mathbb{R}_+))$. Moreover V^0 is supported away from $t = 0$ as a function of time t.
- Using Proposition 2.4 again, there exists $h^1 \in C^\infty_0(0, T)$ such that system (2.15) has a unique solution $V^1 \in C^0_7(\mathbb{R}_+; H^7_8(\mathbb{R}_+))$. Moreover V^1 is supported away from $t = 0$ as a function of time t.
- By a final use of Proposition 2.4, there exists $h^2 \in C^\infty_0(0, T)$ and $V^2 \in C^0_2(\mathbb{R}_+; H^2_2(\mathbb{R}_+))$ satisfying (2.16). Moreover V^2 is supported away from $t = 0$ as a function of time t.

By the definition of $\{\cdot\}_\varepsilon$, for a profile $\mathcal{V}$ defined in $\mathbb{R}^2_+$ with $\mathcal{V} \in C(\mathbb{R}_+; L^2(\mathbb{R}_+))$,

$$\|\{\mathcal{V}(t, \cdot)\}_\varepsilon\|_{L^2(D_1)} \leq C\varepsilon^{\frac14}\|\mathcal{V}(t, \cdot)\|_{L^2(\mathbb{R}_+)} \tag{2.21}$$

for a constant $C > 0$. Hence, in view of (2.10),

$$\begin{aligned} \|v^\varepsilon(\frac{T}{\varepsilon}, \cdot)\|_{L^2(D_1)} \leq\ & C\varepsilon^{\frac14}(\|V^0(\frac{T}{\varepsilon}, \cdot)\|_{L^2(\mathbb{R}_+)} + \sqrt{\varepsilon}\|V^1(\frac{T}{\varepsilon}, \cdot)\|_{L^2(\mathbb{R}_+)} + \varepsilon\|V^2(\frac{T}{\varepsilon}, \cdot)\|_{L^2(\mathbb{R}_+)}) \\ & + \varepsilon\|r^\varepsilon\|_{L^2(D_1)} \\ \leq\ & C\varepsilon^{\frac94} + \varepsilon\|r^\varepsilon\|_{L^2(D_1)}, \end{aligned} \tag{2.22}$$

where we use $V^i \in C^0_2(\mathbb{R}_+; L^2(\mathbb{R}_+))$ by construction. Then the proof of Proposition 2.1 is complete up to the following result on the remainder which is postponed to Sect. 4.

Lemma 2.5. *There exists a constant C such that, for $s = 0, 1$,*

$$\|r^\varepsilon\|_{L^\infty(\mathbb{R}_+; L^2(D_1))} \leq C\varepsilon^{\frac34}, \tag{2.23}$$

$$\|r^\varepsilon\|_{L^2(\mathbb{R}_+; H^{2s}(D_1))} \leq C\varepsilon^{-\frac14}. \tag{2.24}$$

2.6. An Auxiliary Solution of the Navier–Stokes Equations in the Infinite Cylinder

We then consider as an auxiliary solution of the Navier–Stokes equations:

$$u^{\text{aux}} = \left(\sum_{i=0}^{2} \varepsilon^{\frac{i}{2}} h^i(t) + v^\varepsilon\right) e_3, \tag{2.25}$$

where $r := \sqrt{x_1^2 + x_2^2}$ and h^0, h^1, h^2 and v^ε is the family of functions defined in the proof of Proposition 2.1.

Let us extend the finite vertical cylinder Ω into a cylinder with infinite height:

$$\mathcal{C} := D_1 \times \mathbb{R}. \tag{2.26}$$

We also needs an extension u_b of the initial data u_0. Actually, we will need in the sequel to work with analytic data, due to the difficulty related to a loss of derivatives due to the boundary layers in the equation satisfied by the remainder. More precisely one will use analyticity with respect to the vertical variable x_3. Therefore we cannot simply extend the initial data u_0 by 0 outside of Ω. The reason why we use the index b is that we will need a first regularization step to get a large enough radius of analyticity to completely overrule the effect of the loss of derivative.

Then, by setting

$$p^{\text{aux}} = -\sum_{i=0}^{2} \varepsilon^{\frac{i}{2}} (h^i)'(t) x_3, \tag{2.27}$$

we obtain that

$$\begin{cases} \partial_t u^{\text{aux}} + u^{\text{aux}} \cdot \nabla u^{\text{aux}} - \varepsilon \Delta u^{\text{aux}} + \nabla p^{\text{aux}} = 0 & \text{in } (0, \frac{T}{\varepsilon}) \times \mathcal{C}, \\ \operatorname{div} u^{\text{aux}} = 0 & \text{in } (0, \frac{T}{\varepsilon}) \times \mathcal{C}, \\ u^{\text{aux}} = 0 & \text{on } (0, \frac{T}{\varepsilon}) \times \partial\mathcal{C}, \\ u^{\text{aux}}|_{t=0} = 0 & \text{in } \mathcal{C}. \end{cases} \tag{2.28}$$

and

$$\|u^{\text{aux}} \left(\frac{T}{\varepsilon}, \cdot \right) \|_{L^2(\Omega)} \leq C \varepsilon^{\frac{7}{4}}. \tag{2.29}$$

We also define (the index "fl" stands for "flushed"):

$$u^{\text{fl}}(t,x) := \mu(t) u_b \left(x - \left(\int_0^t h^0(s) ds \right) e_3 \right), \tag{2.30}$$

where $\mu(t) \in C^\infty(\mathbb{R})$ is a cut-off function which satisfies $\mu(t) = 1$ when $t \leq \frac{T}{3}$ and $\mu(t) = 0$ when $t \geq \frac{2T}{3}$.

We observe that u^{fl} satisfies

$$\begin{cases} \partial_t u^{\text{fl}} + h^0 \partial_3 u^{\text{fl}} = \xi^{\text{fl}} + f^{\text{fl}} & \text{in } (0,T) \times \mathcal{C}, \\ \operatorname{div} u^{\text{fl}} = 0 & \text{in } (0,T) \times \mathcal{C}, \\ u^{\text{fl}} = 0 & \text{on } (0,T) \times \partial\mathcal{C}, \\ u^{\text{fl}} = u_b & \text{on } \{0\} \times \mathcal{C}, \end{cases} \tag{2.31}$$

where $\partial_3 := \partial_{x_3}$ and by virtue of (2.6),

$$\xi^{\text{fl}}(t,x) : = \dot{\mu}(t) u_b(x - 2Le_3)|_{\mathcal{C} \backslash \Omega}, \tag{2.32}$$

$$f^{\text{fl}}(t,x) : = \dot{\mu}(t) u_b(x - 2Le_3)|_{\Omega}. \tag{2.33}$$

We also introduce a domain

$$G := D_1 \times [-2L, -L], \tag{2.34}$$

and a space

$$L^2_{\text{div}}(\mathcal{C}) := \{ v \in L^2(\mathcal{C}) : v \text{ is axi-symmetric, } \operatorname{div} v = 0, v = 0 \text{ on } \partial\mathcal{C} \}. \tag{2.35}$$

It is easy to observe that the control profile ξ^{fl} is supported in $\mathcal{C} \backslash \Omega$, the phantom profile f^{fl} is supported in Ω, and for any $k \in \mathbb{N}$,

$$\|f^{\text{fl}}\|_{L^1(0,T);H^k(\mathcal{C}))} \leq T \|u_b\|_{H^k(G)}. \tag{2.36}$$

Now consider any Leray weak solution

$$u^\varepsilon \in C_{\text{w}}([0, \frac{T}{\varepsilon}]; L^2_{\text{div,loc}}(\mathcal{C})) \cap L^2 \left((0, \frac{T}{\varepsilon}); H^1(\mathcal{C}) \right),$$

of the rescaled system

$$\begin{cases} \partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon - \varepsilon \Delta u^\varepsilon + \nabla p^\varepsilon = \xi^\varepsilon + f^\varepsilon & \text{in } \left(0, \frac{T}{\varepsilon}\right) \times \mathcal{C}, \\ \operatorname{div} u^\varepsilon = 0 \quad \text{in } \left(0, \frac{T}{\varepsilon}\right) \times \mathcal{C}, \\ u^\varepsilon = 0 \quad \text{on } \left(0, \frac{T}{\varepsilon}\right) \times \partial\mathcal{C}, \\ u^\varepsilon|_{t=0} = \varepsilon u_b \quad \text{in } \mathcal{C}. \end{cases} \tag{2.37}$$

We decompose u^ε and p^ε into

$$u^\varepsilon = u^{\text{aux}} + \varepsilon u^{\text{fl}} + \varepsilon R^\varepsilon, \tag{2.38}$$

$$p^\varepsilon = p^{\text{aux}} + \varepsilon \pi^\varepsilon, \tag{2.39}$$

then we take

$$\xi^\varepsilon = \varepsilon \xi^{\text{fl}} \qquad \text{and} \qquad f^\varepsilon = \varepsilon f^{\text{fl}}, \tag{2.40}$$

and we observe that the remainder R^ε satisfies

$$\begin{cases} \partial_t R^\varepsilon - \varepsilon \Delta R^\varepsilon + u^\varepsilon \cdot \nabla R^\varepsilon + R^\varepsilon \cdot \nabla (u^{\text{aux}} + \varepsilon u^{\text{fl}}) + \nabla \pi^\varepsilon = F^\varepsilon, \\ \operatorname{div} R^\varepsilon = 0, \\ R^\varepsilon|_{\partial\mathcal{C}} = 0, \\ R^\varepsilon|_{t=0} = 0, \end{cases} \tag{2.41}$$

where

$$\begin{aligned} F^\varepsilon := & -(-\chi\{V^0\}_\varepsilon + \sqrt{\varepsilon}(h^1 - \chi\{V^1\}_\varepsilon) + \varepsilon(h^2 - \chi\{V^2\}_\varepsilon) + \varepsilon r^\varepsilon)\partial_3 u^{\text{fl}} \\ & + u_r^{\text{fl}} \chi' \{V^0 + \sqrt{\varepsilon} V^1 + \varepsilon V^2\}_\varepsilon e_3 - u_r^{\text{fl}} \frac{\chi}{1-r} \{z \partial_z V^0 + \sqrt{\varepsilon} z \partial_z V^1 + \varepsilon z \partial_z V^2\}_\varepsilon e_3 \\ & - \varepsilon u_r^{\text{fl}} \partial_r r^\varepsilon e_3 - \varepsilon u^{\text{fl}} \cdot \nabla u^{\text{fl}} + \varepsilon \Delta u^{\text{fl}}. \end{aligned} \tag{2.42}$$

where u_r^{fl} is the e_r component of u^{fl} given by (2.55). By construction, u^{fl} is supported in $[0, T]$, so does F^ε.

Indeed, because of the fast variation due to the boundary layer terms in u^{aux}, the term $R^\varepsilon \cdot \nabla u^{\text{aux}}$ in (2.41) is singular with respect to ε. A classical trick here is to treat this singularity in ε against a loss of derivative, see for example [18,19], and (5.18)–(5.20) below, which leads to consider analytic regularity in order to bootstrap some estimates.

Proposition 2.6. *Let $T > 0$. There exists $\rho_b > 0$ such that, for every $u_b \in L^2_{\text{div}}(\Omega)$ for which there exists C_b such that*

$$\forall m \geq 0, \quad \|\partial_{x_3}^m u_b\|_{H^3(\mathcal{C})} \leq \tfrac{m!}{\rho_b^m} C_b, \tag{2.43}$$

$$\|u_b\|_{L^1_{x_3}(H^2(D_1))} \leq C_b. \tag{2.44}$$

for every $k \in \mathbb{N}$, we have the following estimate for R^ε, there exists a constant C such that

$$\sup_{t \in [0, T/\varepsilon]} \|R^\varepsilon(t)\|_{L^2(\mathcal{C})} + \left(\int_0^{T/\varepsilon} \varepsilon \|\nabla R^\varepsilon\|^2_{L^2(\mathcal{C})} \right)^{\frac{1}{2}} \leq C \varepsilon^{\frac{1}{4}}. \tag{2.45}$$

The proof of Proposition 2.6 is postponed to Sect. 5.

Let us now explain how to deduce Theorem 1.1 by following the strategy of [8]. It is a matter to glue together three steps, each of which corresponds to a subpart of the time interval which is imparted to realize the control. To explain this, we go back to the original scaling and first recast the result obtained so far, which will be used as a second step (explaining the notation b as an index for the "initial" data below). Setting, for $\varepsilon \in (0,1)$,

$$u^\varepsilon(t,x) := \varepsilon u(\varepsilon t, x), \quad p^\varepsilon(t,x) := \varepsilon^2 p(\varepsilon t, x), \tag{2.46}$$

$$\xi^\varepsilon(t,x) := \varepsilon^2 \xi(\varepsilon t, x) \quad \text{and} \quad f^\varepsilon(t,x) := \varepsilon^2 f(\varepsilon t, x), \tag{2.47}$$

we deduce from Lemma 2.5, the definition of u^{fl} and Proposition 2.6 the following result about the possibility to drive a large analytic initial data with a sufficiently large analyticity radius can be driven approximately into the null equilibrium.

Proposition 2.7. *Let $T > 0$. There exists $\rho_b > 0$ such that, for every $\sigma > 0$ and each $u_b \in L^2_{\text{div}}(\mathcal{C})$ for which there exists C_b such that (2.43) and (2.44) hold, for every $k \in \mathbb{N}$, there exist two forces $\xi \in C^\infty([0,T] \times \overline{\mathcal{C}\backslash\Omega})$ and $f \in C^\infty([0,T] \times \overline{\Omega})$ satisfying*

$$\|f\|_{L^1((0,T);H^k(\Omega))} \leq C_k \|u_b|_G\|_{H^k(G)}, \tag{2.48}$$

$$\textit{supp}\ \xi \subset (0,T) \times \overline{\mathcal{C}\backslash\Omega}, \tag{2.49}$$

where the constant C_k depends only on k, and a Leray weak solution $u \in C_{\text{w}}([0,T]; L^2_{\text{div}}(\mathcal{C})) \cap L^2((0,T); H^1(\mathcal{C}))$ to (2.52) associated with the initial data u_b, such that there exists $0 < T_c \leq T$ such that $u_c := u(T_c, \cdot)$ satisfies

$$\|u_c|_\Omega\|_{L^2(\Omega)} \leq \sigma. \tag{2.50}$$

Since this result requires some analytic initial data, we use as a prequel the following result regarding the regularization of any finite energy initial data into an analytic function with arbitrary analyticity radius.

Proposition 2.8. *Let $T > 0$, $\rho_b > 0$ and $u_0 \in L^2_{\text{div}}(\Omega)$ with $u_0 \cdot \mathbf{n} = 0$ on Γ. For any $k \in \mathbb{N}$ and $\eta_b > 0$, there exists an extension $u_a \in L^2_{\text{div}}(\mathcal{C})$ of u_0 to the domain $\mathcal{C}$, a control force $\xi \in C^\infty([0,T] \times \overline{\mathcal{C}\backslash\Omega})$, a phantom force $f \in C^\infty([0,T] \times \overline{\Omega})$ satisfying*

$$\|f\|_{L^1(0,T);H^k(\Omega)} \leq \eta_b, \tag{2.51}$$

a Leray weak solution $u \in C_{\text{w}}([0,T]; L^2_{\text{div}}(\mathcal{C})) \cap L^2((0,T); H^1(\mathcal{C}))$ to

$$\begin{cases} \partial_t u + u \cdot \nabla u - \Delta u + \nabla p = \xi + f & \textit{in } (0,T) \times \mathcal{C}, \\ \operatorname{div} u = 0 & \textit{in } (0,T) \times \mathcal{C}, \\ u = 0 & \textit{on } (0,T) \times \partial\mathcal{C}, \end{cases} \tag{2.52}$$

associated with initial data u_a, $C_b > 0$ and $0 < T_b \leq T$ such that $u_b := u(T_b, \cdot) \in L^2_{\text{div}}(\mathcal{C})$ satisfies

$$\|u_b|_G\|_{H^k(G)} \leq \eta_b, \tag{2.53}$$

(2.43) and (2.44).

Let us emphasize that for $L^2_{\text{div}}(\Omega)$ initial data, we can find $T_1 \in (0,T)$ such that $u(T_1,\cdot) \in H^1(\Omega)$. Now for this new H^1 "initial" data, we can find a small time $T_2 \in (T_1, T]$ and a solution u to (2.52) with zero force, such that $u \in C^\infty((T_1,T_2) \times \mathcal{C})$, the rest of the proof is almost the same as the proof of Proposition 1.7 of [8] and is therefore left to the reader.

Last, we use that small enough states can be driven exactly to the rest state.

Proposition 2.9. *Let $T > 0$. There exists $\sigma > 0$ such that, for any $u_c \in L^2_{\text{div}}(\Omega)$ which satisfies*

$$\|u_c\|_{L^2(\Omega)} \leq \sigma, \tag{2.54}$$

there exists a Leray weak solution $u \in C_{\text{w}}([0,T]; L^2_{\text{div}}(\Omega)) \cap L^2((0,T); H^1((\Omega))$ to (1.1) with the associated initial data u_c and $f = 0$, which satisfies $u(T,\cdot) = 0$.

We refer to Theorem 2 of [10]. Actually, the original theorem of [10] requires the initial date to be small in $L^4(\Omega) \cap L^2_{\text{div}}(\Omega)$. While if we only have the smallness of the initial data, by using the energy inequality, we can find a $T_1 \in (0, \frac{T}{2})$ such that $u(T_1)$ is small in $L^2(\Omega) \cap H^1(\Omega)$ hence small in $L^4(\Omega) \cap L^2_{\text{div}}(\Omega)$. Therefore we only require the initial data small in $L^2_{\text{div}}(\Omega)$.

Theorem 1.1 is then implied by combining the three propositions above.

Remark 2.10. We observe that in the strategy above we only make use of analyticity in the vertical direction. One can wonder if a use of analyticity in the orthoradial direction could be helpful to extend the the result of Theorem 1.1 to the case of smooth solutions or to the case of non-axisymmetric initial data. Unfortunately this does not seem to be the case. One difficulty is that, to deal with the nonlinear feature in the remainder estimates, see Sect. 5, one typically needs to replace the space L^2 by the Sobolev space $H^{\frac{1}{2}}$ to bootstrap some estimates. This requires to construct yet more accurate asymptotic expansion. Then one observes that the lower order boundary layer profiles which would be natural to consider depend on the vertical variable and satisfy some equations which couples the terms of a heat equation with respect to the fast variable z, like the ones considered above, and some transport terms in the x_3 direction with a prefactor which corresponds to the principal term of u^{aux}. Unfortunately the well-prepared dissipation method seems to be delicate to adapt to such problems.

2.7. Organization of the Rest of the Paper

The rest of the paper is devoted to the proof of the intermediate results which were admitted above in the course of the proof of Theorem 1.1. The two next sections are devoted to the proofs of the intermediate results used to prove Proposition 2.1, that is Proposition 2.4 in Sect. 3, and Lemma 2.5 in Sect. 4. Next, in Sect. 5 we give the proof of Proposition 2.6. Once again, the proof is quite technical, and we will use two technical intermediate results: Proposition 5.4 and Lemma 5.3 whose proofs are postponed respectively to Sects. 6 and 7.

Notations: For any $x \in \mathcal{C}$ defined by (2.26), let $x_1 = r\cos\theta, x_2 = r\cos\theta$, $n = \frac{1}{r}(x_1, x_2, 0)$. Let $\{e_i\}_{i=1}^3$ be the unit vectors of the canonical basis of $\mathbb{R}^3$. Let $e_r = (\cos\theta, \sin\theta, 0), e_\theta = (-\sin\theta, \sin\theta, 0)$. For a vector $a \in \mathbb{R}^3$, we write

$$a = a_1 e_1 + a_2 e_2 + a_3 e_3 = a_r e_r + a_\theta e_\theta + a_3 e_3. \tag{2.55}$$

We always denote $\nabla_{\text{h}} := (\partial_{x_1}, \partial_{x_2})$, $\partial_i := \partial_{x_i}$ for $1 \le i \le 3$, $\partial_r := e_r \cdot \nabla$ and $(a, b) = \int_{\mathcal{C}} a(x)b(x)\,dx$ the $L^2(\mathcal{C})$ inner product of a and b.

3. Proof of Proposition 2.4

This section is devoted to the proof of Proposition 2.4. We will rely on the following result from [15, Lemma 3.4] where, for $n \in \mathbb{N}$ and $x \in \mathbb{R}$, we set

$$s_n(x) := \sum_{k=0}^{n} \frac{x^k}{k!}. \tag{3.1}$$

Lemma 3.1. *Let $\gamma > 0$ and $k, s, q, n \in \mathbb{N}$ satisfying (2.18) and $\tilde{\gamma}$, $\tilde{s}$, and $\tilde{q}$ as in (2.19). Let $v_0 \in H^{s+2k}_{\tilde{q}}(\mathbb{R})$ and $f \in C^0_{\tilde{\gamma}}(\mathbb{R}_+; H^{\tilde{s}}_{\tilde{q}}(\mathbb{R}))$ when $k = 0$ and $f \in C^{k-1}_{\tilde{\gamma}}(\mathbb{R}_+; H^{\tilde{s}}_{\tilde{q}}(\mathbb{R}))$ when $k \ge 1$, such that*

$$\Big(\partial_\zeta^j\big(\hat{v}_0(\zeta) + \int_0^\infty s_n(t\zeta^2)\hat{f}(t,\zeta)dt\big)\Big)\Big|_{\zeta=0} = 0, \quad \textit{for } 0 \le j \le 2n+1. \tag{3.2}$$

Then the following Cauchy problem

$$\begin{cases} \partial_t v - \partial_z^2 v = f, & t \ge 0, z \in \mathbb{R}, \\ v|_{t=0} = v_0, & z \in \mathbb{R}, \end{cases} \tag{3.3}$$

has a unique solution $v \in C^k_\gamma(\mathbb{R}_+; H^s_q(\mathbb{R}))$.

We are now ready to start the proof of Proposition 2.4.

Proof of Proposition 2.4. We first observe that it is sufficient to deal with the case where $k = 0$, since the general case follows by using that for $0 \le i \le k$, for z in $\mathbb{R}$ and $t \ge 0$, $\partial_t^i v = \partial_z^2 \partial_t^{i-1} v + \partial_t^{i-1} f$.

Assume that $h \in C_0^\infty(0,T)$. Let $j_0 = [\frac{\tilde{s}+1}{2}]$,

$$A_0(t) := h(t), \quad A_j(t) := \partial_t A_{j-1}(t) - \partial_z^{2j-2} f|_{z=0^+} \text{ for } 1 \le j \le j_0. \tag{3.4}$$

For $z \ge 0$, we denote

$$A(t,z) := \sum_{j=0}^{j_0} A_j(t) \frac{z^{2j}}{(2j)!} \chi_1(z), \tag{3.5}$$

where $\chi_1(z) \in C_0^\infty(\overline{\mathbb{R}_+})$ is a cut-off function which satisfies $\chi_1(z) = 1$ for $z \in [0, \frac{1}{2}]$ and $\chi_1(z) = 0$ for $z \ge 1$. One can check that

$$A \in C^0_\gamma(\mathbb{R}_+; C_0^\infty(\overline{\mathbb{R}_+})). \tag{3.6}$$

Then the function $V := v - A$ satisfies, for $t \ge 0, z \ge 0$,

$$\begin{cases} \partial_t V - \partial_z^2 V = F, \\ V(t,0) = 0, \\ V(0,z) = 0 \end{cases} \tag{3.7}$$

where

$$F := f - \partial_t A + \partial_z^2 A. \tag{3.8}$$

It follows from the construction of $A_j(t)$, that $\partial_z^{2j} F|_{z=0^+} = 0$ for $0 \le j \le j_0 - 1$. Thus, extending F by $F(t,z) = -F(t,-z)$ to the whole line $z \in \mathbb{R}$, we have

$$F \in C^0_{\tilde{\gamma}}(\mathbb{R}_+; H^{\tilde{s}}_{\tilde{q}}(\mathbb{R})). \tag{3.9}$$

Moreover, if f is supported away from $t = 0$ as a function of time t, so does F, and by using the energy method it is easy to find that V is also supported away from $t = 0$.

Now, let us observe that, to prove Proposition 2.4, it is sufficient to find a function $h \in C_0^\infty(0,T)$, supported in $(0, \frac{T}{3}] \cup [\frac{2T}{3}, T)$, such that the associated function F satisfies the condition:

$$\Big(\partial_\zeta^j \big(\int_0^\infty s_n(t\zeta^2) \hat{F}(t,\zeta) dt \big)\Big)\Big|_{\zeta=0} = 0, \qquad \text{for } 0 \le j \le 2n+1, \tag{3.10}$$

where s_n is defined in (3.1) and $\hat{F}$ denotes the Fourier transform of F with respect to z. Indeed, then by the result of Lemma 3.1, we obtain that $V \in C^0_\gamma(\mathbb{R}_+; H^s_q(\mathbb{R}))$. Moreover, by (3.6), it follows that $v \in C^0_\gamma(\mathbb{R}_+; H^s_1(\mathbb{R}_+))$. Actually, since F is an odd function with respect to z, it suffices to guarantee (3.10) for odd integer j with $1 \le j \le 2n+1$. □

By the definition of $A_j(t)$ in (3.4), for $0 \le j \le j_0$,

$$A_j(t) = \partial_t^j h(t) - \sum_{i=0}^{j-1} \partial_t^{j-1-i} \partial_z^{2i} f|_{z=0^+}. \tag{3.11}$$

Then, by combining (3.5), (3.8) and (3.11), we arrive at

$$\begin{aligned} F &= f - \sum_{j=0}^{j_0} \partial_t A_j(t) \frac{z^{2j}}{(2j)!}\chi_1(z) + \sum_{j=0}^{j_0} A_j(t)\partial_z^2 (\frac{z^{2j}}{(2j)!}\chi_1(z)) \\ &= f - \sum_{j=0}^{j_0-1} \partial_z^{2j} f|_{z=0^+} \frac{z^{2j}}{(2j)!}\chi_1(z) - \partial_t A_{j_0}(t) \frac{z^{2j_0}}{(2j_0)!}\chi_1(z) + A_0(t)\chi_1''(z) \\ &\quad + \sum_{j=1}^{j_0} A_j(t) \frac{z^{2j-1}}{(2j-1)!}(2\chi_1'(z) + \frac{z}{2j}\chi_1''(z)) \\ &= \tilde{f} + \sum_{j=0}^{j_0+1} \partial_t^j h(t)\alpha_j(z), \end{aligned}$$

where

$$\begin{aligned} \tilde{f} :&= f - \sum_{j=0}^{j_0-1} \partial_z^{2j} f|_{z=0^+} \frac{z^{2j}}{(2j)!}\chi_1(z) + \sum_{i=0}^{j_0-1} \partial_t^{j_0-i}\partial_z^{2i} f|_{z=0^+} \frac{z^{2j_0}}{(2j_0)!}\chi_1(z) \\ &\quad - \sum_{j=1}^{j_0}\sum_{i=0}^{j-1} \partial_t^{j-1-i}\partial_z^{2i} f|_{z=0^+} \frac{z^{2j-1}}{(2j-1)!}(2\chi_1'(z) + \frac{z}{2j}\chi_1''(z)), \\ \alpha_0(z) :&= \chi_1''(z), \\ \alpha_j(z) :&= \frac{z^{2j-1}}{(2j-1)!}(2\chi_1'(z) + \frac{z}{2j}\chi_1''(z)), \quad \text{for } 1 \le j \le j_0, \\ \alpha_{j_0+1}(z) :&= -\frac{z^{2j_0}}{(2j_0)!}\chi_1(z). \end{aligned}$$

Next we odd extend $\tilde{f}$, $\chi_1(z)$ and $\chi_1''(z)$, and even extend $\chi_1'(z)$ with respect to z to get a odd extension of F.

When $j = 1$, the condition (3.10) becomes

$$\begin{aligned} 0 &= \partial_\zeta \left(\int_0^\infty s_n(t\zeta^2)\hat{F}(t,\zeta)\,dt \right)|_{\zeta=0} \\ &= \int_0^\infty \partial_\zeta \hat{F}(t,0)\,dt \\ &= \int_0^\infty \partial_\zeta \hat{\tilde{f}}(t,0)\,dt + \left(\sum_{i=0}^{j_0+1} \int_0^\infty \partial_t^i h(t)\,dt \right) \partial_\zeta \widehat{\alpha}_i(0). \end{aligned} \tag{3.12}$$

Since $h \in C_0^\infty((0,T))$, we have that, for $i \ge 1$,

$$\int_0^\infty \partial_t^i h(t)dt = 0.$$

On the other hand,

$$\partial_\zeta \widehat{\alpha}_0(0) = -\mathrm{i}\int_{-\infty}^\infty z\alpha_0(z)\,dz = -2\mathrm{i}\int_0^\infty z\chi_1''(z)\,dz = -2\mathrm{i}, \tag{3.13}$$

where we denote $\mathrm{i} = \sqrt{-1}$. Thus (3.12) is equivalent to

$$\int_0^\infty h(t)\,dt = c_0 := \frac{1}{2\mathrm{i}}\int_0^\infty \partial_\zeta \hat{\tilde{f}}(t,0)dt. \tag{3.14}$$

When $j=3$, the condition (3.10) becomes

$$
\begin{aligned}
0 \;&=\; \partial_\zeta^3\left(\int_0^\infty s_n(t\zeta^2)\hat{F}(t,\zeta)\,dt\right)|_{\zeta=0}\\
&= \int_0^\infty (6t\partial_\zeta \hat{\hat{f}}(t,0)+\partial_\zeta^3\hat{\hat{f}}(t,0))dt\\
&\quad+\int_0^\infty \sum_{i=0}^{j_0+1}(6t\partial_t^i h(t)\partial_\zeta\widehat{\alpha}_i(0)+\partial_t^i h(t)\partial_\zeta^3\widehat{\alpha}_i(0))dt.
\end{aligned}\tag{3.15}
$$

Observe that

$$
\int_0^\infty t\partial_t h(t)dt=-\int_0^\infty h(t)dt=-c_0,
$$

and for $i\geq 2$,

$$
\int_0^\infty t\partial_t^i h(t)dt=-\int_0^\infty \partial_t^{i-1}h(t)dt=0.
$$

We find that (3.15) is equivalent to

$$
\int_0^\infty th(t)dt=c_1,\tag{3.16}
$$

where

$$
c_1:=\frac{1}{12\mathrm{i}}\int_0^\infty(6t\partial_\zeta\hat{\hat{f}}(t,0)+\partial_\zeta^3\hat{\hat{f}}(t,0))dt-\frac{1}{2\mathrm{i}}c_0\partial_\zeta\hat{\alpha_1}(0)+\frac{1}{12\mathrm{i}}c_0\partial_\zeta^3\widehat{\alpha}_0(0).\tag{3.17}
$$

Generally, we can find some constants $c_i, 0\leq i\leq n$, such that when $j=2i+1$, (3.10) is equivalent to

$$
\int_0^\infty t^i h(t)dt=c_i.\tag{3.18}
$$

We can prove it by induction. We already know it holds for $i=0,1$, assume $2\leq i\leq n$ and assume that (3.18) holds for $0\leq l<i$, then when $j=2i+1$, the condition (3.10) becomes

$$
\begin{aligned}
0 \;&=\; \partial_\zeta^{2i+1}\left(\int_0^\infty s_n(t\zeta^2)\hat{F}(t,\zeta)\,dt\right)|_{\zeta=0}\\
&= \sum_{l=0}^{i}\int_0^\infty C_{2i+1}^{2l}\frac{(2l)!}{l!}t^l\partial_\zeta^{2i+1-2l}\hat{\hat{f}}(t,0)dt\\
&\quad+\sum_{l=0}^{i}\int_0^\infty C_{2i+1}^{2l}\frac{(2l)!}{l!}t^l\sum_{\nu=0}^{j_0+1}\partial_t^\nu h(t)\partial_\zeta^{2i+1-2l}\hat{\alpha_\nu}(0)dt.
\end{aligned}\tag{3.19}
$$

Note that, for $l,\nu\in\mathbb{N}$,

$$
\int_0^\infty t^l\partial_t^\nu h(t)dt=\begin{cases}0, & \text{if } l<\nu,\\ (-1)^\nu\frac{l!}{(l-\nu)!}\int_0^\infty t^{l-\nu}h(t)dt, & \text{if } l\geq\nu.\end{cases}\tag{3.20}
$$

Using (3.20) and the induction assumption that (3.18) holds for $0 \le l < i$, (3.19) is equivalent to (3.18) with

$$\begin{aligned} c_i :&= \frac{1}{2\mathrm{i}} \cdot \frac{i!}{(2i+1)!} \partial_\zeta^{2i+1} \left(\int_0^\infty s_n(t\zeta^2) \hat{\hat{f}}(t,\zeta)\, dt \right) |_{\zeta=0} \\ &+ \frac{1}{2\mathrm{i}} \cdot \frac{i!}{(2i+1)!} \sum_{l=0}^{i-1} \sum_{\nu=0}^{l} (-1)^\nu C_{2i+1}^{2l} \frac{(2l)!}{(l-\nu)!} c_{l-\nu} \partial_\zeta^{2i+1-2l} \hat{\alpha_\nu}(0) \\ &+ \frac{1}{2\mathrm{i}} \sum_{\nu=1}^{i} (-1)^\nu \frac{i!}{(i-\nu)!} c_{i-\nu} \partial_\zeta \hat{\alpha_\nu}(0). \end{aligned} \tag{3.21}$$

Thus the condition (3.10) is equivalent to (3.18) holds for all $0 \le i \le n$. Given function f, we can define constants c_i as above, the rest of our task is therefore to find a nonzero function $h \in C_0^\infty(0,T)$, supported in $(0, \frac{T}{3}] \cup [\frac{2T}{3}, T)$, such that (3.18) holds for all $0 \le i \le n$.

We introduce a nonnegative cut-off function $\chi_2 \in C_0^\infty(0,T)$ such that $\chi_2(t) = 0$ for $t \in [\frac{T}{3}, \frac{2T}{3}]$ and $\chi_2(t) = 1$ for $t \in [\frac{T}{9}, \frac{2T}{9}] \cup [\frac{7T}{9}, \frac{8T}{9}]$, we seek a function h of form

$$h(t) = \sum_{i=0}^{n+1} \beta_i t^i \chi_2(t), \tag{3.22}$$

where β_i, $0 \le i \le n+1$, are constants to be determined. Then (3.18) holds for $0 \le i \le n$ is equivalent to

$$\mathbf{B}\beta = \mathbf{c}, \tag{3.23}$$

where $\beta = (\beta_0, \ldots, \beta_{n+1})^{Tr}, \mathbf{c} = (c_0, \ldots, c_n)^{Tr}$ and

$$\mathbf{B} = \begin{pmatrix} \int_0^\infty \chi_2(t)dt & \int_0^\infty t\chi_2(t)dt & \cdots & \int_0^\infty t^{n+1}\chi_2(t)dt \\ \int_0^\infty t\chi_2(t)dt & \int_0^\infty t^2\chi_2(t)dt & \cdots & \int_0^\infty t^{n+2}\chi_2(t)dt \\ \cdots & \cdots & \cdots & \cdots \\ \int_0^\infty t^n\chi_2(t)dt & \int_0^\infty t^{n+1}\chi_2(t)dt & \cdots & \int_0^\infty t^{2n+1}\chi_2(t)dt \end{pmatrix}. \tag{3.24}$$

We denote $\mathbf{B}'$ the sub-matrix of $\mathbf{B}$ made of the first n row, $\mathbf{b}'$ the vector of $(n+1)-th$ row of $\mathbf{B}$. It is obvious that $\mathbf{b}'$ is a nonzero vector. In order to find a nonzero vector β such that (3.23), we first show that $\mathbf{B}'$ is a nondegenerate matrix. Otherwise, there exists a vector $\mathbf{y} = (y_0, \ldots, y_n)^{Tr} \ne 0$ such that

$$\mathbf{y}^{Tr}\mathbf{B}'\mathbf{y} = \int_0^\infty |y_0 + y_1 t + \cdots + y_n t^n|^2 \chi_2(t)dt = 0. \tag{3.25}$$

Hence $y_0 + y_1 t + \cdots + y_n t^n \equiv 0$ for $t \in [\frac{T}{9}, \frac{2T}{9}] \cup [\frac{7T}{9}, \frac{8T}{9}]$, this contradict to $\mathbf{y} \ne 0$. So $\mathbf{B}'$ is a nondegenerate matrix. If $\mathbf{c} \ne 0$, we can take $(\beta_0, \ldots, \beta_n)^{Tr} = \mathbf{B}'^{-1}\mathbf{c}$, $\beta_{n+1} = 0$. If $\mathbf{c} = 0$, we can take $\beta_{n+1} = 1$, $(\beta_0, \ldots, \beta_n)^{Tr} = -\mathbf{B}'^{-1}\mathbf{b}'$. Either way, we have found a nonzero vector β such that (3.23) holds true. This means we have hound a nonzero function $h \in C_0^\infty(0,T)$ supported in $(0, \frac{T}{3}] \cup [\frac{2T}{3}, T)$ such that (3.18) holds for $0 \le i \le n$. This concludes our proof.

□

4. Proof of Lemma 2.5

This section is devoted to the proof of Lemma 2.5.

4.1. Equation Satisfied by the Remainder

By using the equation for V^i, for $0 \le i \le 2$, see the systems (2.14)–(2.16), the Eq. (2.7) and the composition rule (2.12), we obtain that the remainder $r^\varepsilon := r^\varepsilon(t, x_1, x_2)$ satisfies

$$\begin{cases} \partial_t r^\varepsilon - \varepsilon \Delta_{\rm h} r^\varepsilon = q^\varepsilon & t \geq 0, (x_1, x_2) \in D_1, \\ r^\varepsilon(t, x_1, x_2) = 0 & t \geq 0, (x_1, x_2) \in \partial D_1, \\ r^\varepsilon(0, x_1, x_2) = 0 & (x_1, x_2) \in D_1, \end{cases} \tag{4.1}$$

where

$$\Delta_{\rm h} := \partial_{x_1}^2 + \partial_{x_2}^2, \tag{4.2}$$

and

$$\begin{aligned} q^\varepsilon \, : \, = & -(\chi'' + \frac{\chi'}{r})\{V^0 + \sqrt{\varepsilon} V^1 + \varepsilon V^2\}_\varepsilon + \frac{2\chi'}{1-r}\{z\partial_z V^0 + \sqrt{\varepsilon} z \partial_z V^1 + \varepsilon z \partial_z V^2\}_\varepsilon \\ & + \sqrt{\varepsilon}\chi\alpha(r)\{z^2 \partial_z V^0\}_\varepsilon + \sqrt{\varepsilon}\chi\{z\partial_z V^1\}_\varepsilon + \varepsilon\chi\alpha(r)\{z^2\partial_z V^1\}_\varepsilon + \frac{\varepsilon\chi}{r}\{\partial_z V^2\}_\varepsilon. \end{aligned} \tag{4.3}$$

Moreover, since V^i, for $0 \leq i \leq 2$, are supported way from $t = 0$ as a function of time t, so is the forcing term q^ε.

4.2. Estimate Satisfied by the Forcing Term of the Remainder's Equation

In this subsection we prove that q^ε satisfies, for a constant C,

$$\|q^\varepsilon\|_{(L^1\cap L^2)(\mathbb{R}_+\times D_1)} \leq C\varepsilon^{\frac{3}{4}}. \tag{4.4}$$

By definition of the cut-off function χ, see Sect. 2.3, there exists a constant C such that

$$|(\chi'' + \frac{\chi'}{r})\frac{1}{1-r}| \leq C, \qquad |\frac{\chi'}{(1-r)^2}| \leq C. \tag{4.5}$$

Thus by virtue of (2.21), we find

$$\begin{aligned} \|(\chi'' + \frac{\chi'}{r})\{V^0\}_\varepsilon\|_{(L^1\cap L^2)(\mathbb{R}_+;L^2(D_1))} &= \|(\chi'' + \frac{\chi'}{r})\frac{\sqrt{\varepsilon}}{1-r}\{zV^0\}_\varepsilon\|_{(L^1\cap L^2)(\mathbb{R}_+;L^2(D_1))} \\ &\leq C\varepsilon^{\frac{3}{4}}\|zV^0\|_{(L^1\cap L^2)(\mathbb{R}_+;L^2(\mathbb{R}_+))} \\ &\leq C\varepsilon^{\frac{3}{4}}\|V^0\|_{C_2^0(\mathbb{R}_+;H_1^0(\mathbb{R}_+))}. \end{aligned} \tag{4.6}$$

Similarly, one has

$$\begin{aligned} & \|(\chi'' + \frac{\chi'}{r})\{\sqrt{\varepsilon}V^1 + \varepsilon V^2\}_\varepsilon\|_{(L^1\cap L^2)(\mathbb{R}_+;L^2(D_1))} \\ & \leq C\varepsilon^{\frac{3}{4}}\big(\|V^1\|_{C_2^0(\mathbb{R}_+;H_0^0(\mathbb{R}_+))} + \sqrt{\varepsilon}\|V^2\|_{C_2^0(\mathbb{R}_+;H_0^0(\mathbb{R}_+))}\big), \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} & \|\frac{2\chi'}{1-r}\{z\partial_z V^0 + \sqrt{\varepsilon}z\partial_z V^1 + \varepsilon z\partial_z V^2\}_\varepsilon\|_{(L^1\cap L^2)(\mathbb{R}_+;L^2(D_1))} \\ & \leq C\varepsilon^{\frac{3}{4}}\big(\|V^0\|_{C_2^0(\mathbb{R}_+;H_2^1(\mathbb{R}_+))} + \|V^1\|_{C_2^0(\mathbb{R}_+;H_1^1(\mathbb{R}_+))} + \sqrt{\varepsilon}\|V^2\|_{C_2^0(\mathbb{R}_+;H_1^1(\mathbb{R}_+))}\big), \end{aligned} \tag{4.8}$$

and

$$\|\sqrt{\varepsilon}\chi\{z\partial_z V^1\}_\varepsilon\|_{(L^1\cap L^2)(\mathbb{R}_+;L^2(D_1))} \leq C\varepsilon^{\frac{3}{4}}\|V^1\|_{C_2^0(\mathbb{R}_+;H_1^1(\mathbb{R}_+))}, \tag{4.9}$$

and

$$\|\frac{\varepsilon\chi}{r}\{\partial_z V^2\}_\varepsilon\|_{(L^1\cap L^2)(\mathbb{R}_+;L^2(D_1))} \leq C\varepsilon^{\frac{5}{4}}\|V^2\|_{C_2^0(\mathbb{R}_+;H_0^1(\mathbb{R}_+))}. \tag{4.10}$$

Since $\alpha(r) \in C^\infty([\frac{1}{3}, 1])$, recalling that the definition is in (2.11),

$$\|\sqrt{\varepsilon}\chi\alpha(r)\{z^2\partial_z V^0\}_\varepsilon\|_{(L^1\cap L^2)(\mathbb{R}_+;L^2(D_1))} \leq C\varepsilon^{\frac{3}{4}}\|V^0\|_{C_2^0(\mathbb{R}_+;H_2^1(\mathbb{R}_+))}, \tag{4.11}$$

$$\|\varepsilon\chi\alpha(r)\{z^2\partial_z V^1\}_\varepsilon\|_{(L^1\cap L^2)(\mathbb{R}_+;L^2(D_1))} \leq C\varepsilon^{\frac{5}{4}}\|V^1\|_{C_2^0(\mathbb{R}_+;H_2^1(\mathbb{R}_+))}. \tag{4.12}$$

By summarizing the above inequalities, we conclude the proof of (4.4).

4.3. Parabolic Estimates

Now we are ready to prove (2.23) and (2.24). By using energy method, we find that

$$\frac{1}{2}\partial_t\|r^\varepsilon\|^2_{L^2(D_1)}+\varepsilon\|\nabla_{\rm h} r^\varepsilon\|^2_{L^2(D_1)}\le\|q^\varepsilon\|_{L^2(D_1)}\|r^\varepsilon\|_{L^2(D_1)},\tag{4.13}$$

where $\nabla_{\rm h}=(\partial_{x_1},\partial_{x_2})$. Thanks to (4.4),

$$\|r^\varepsilon\|_{L^\infty(\mathbb{R}_+;L^2(D_1))}\le\|q^\varepsilon\|_{L^1(\mathbb{R}_+;L^2(D_1))}\le C\varepsilon^{\frac{3}{4}}.\tag{4.14}$$

While by taking $L^2(D_1)$ inner product of (4.1) with $\partial_t r^\varepsilon$, we obtain

$$\begin{aligned}\|\partial_t r^\varepsilon(t)\|^2_{L^2(D_1)}+\frac{\varepsilon}{2}\frac{d}{dt}\|\nabla_h r^\varepsilon(t)\|^2_{L^2(D_1)}\le&\|q^\varepsilon\|_{L^2(D_1)}\|\partial_t r^\varepsilon\|_{L^2(D_1)}\\ \le&\frac{1}{2}\|q^\varepsilon\|^2_{L^2(D_1)}+\frac{1}{2}\|\partial_t r^\varepsilon\|^2_{L^2(D_1)},\end{aligned}$$

from which and (4.4), we infer

$$\varepsilon\|\nabla_{\rm h} r^\varepsilon\|^2_{L^\infty(\mathbb{R}_+;L^2(D_1))}+\|\partial_t r^\varepsilon\|^2_{L^2(\mathbb{R}_+\times D_1)}\le\|q^\varepsilon\|^2_{L^2(\mathbb{R}_+\times D_1)}\le C\varepsilon^{\frac{3}{2}}.\tag{4.15}$$

Thanks to (4.15), we deduce from the r^ε equation of (4.1) that

$$\varepsilon\|\Delta_{\rm h} r^\varepsilon\|_{L^2(\mathbb{R}_+\times D_1)}\le\|\partial_t r^\varepsilon\|_{L^2(\mathbb{R}_+\times D_1)}+\|q^\varepsilon\|_{L^2(\mathbb{R}_+\times D_1)}\le C\varepsilon^{\frac{3}{4}},$$

which together with the homogeneous boundary condition of r^ε on ∂D_1 ensures that

$$\|\nabla^2_{\rm h} r^\varepsilon\|_{L^2(\mathbb{R}_+\times D_1)}\le C\varepsilon^{-\frac{1}{4}}\tag{4.16}$$

Since r^ε vanishes on the boundary ∂D_1 and thanks to Poincaré inequality and interpolation inequality,

$$\|r^\varepsilon\|_{L^2(\mathbb{R}_+\times D_1)}\le C\|\nabla_{\rm h} r^\varepsilon\|_{L^2(\mathbb{R}_+\times D_1)}\le C\|r^\varepsilon\|^{\frac{1}{2}}_{L^2(\mathbb{R}_+\times D_1)}\|\nabla^2_{\rm h} r^\varepsilon\|^{\frac{1}{2}}_{L^2(\mathbb{R}_+\times D_1)}.\tag{4.17}$$

Therefore,

$$\|r^\varepsilon\|_{L^2(\mathbb{R}_+\times D_1)}\le C\|\nabla^2_{\rm h} r^\varepsilon\|_{L^2(\mathbb{R}_+\times D_1)}\le C\varepsilon^{-\frac{1}{4}}.\tag{4.18}$$

By summarizing the estimates (4.14), (4.16), and (4.17) and (4.18), we finish the proof of Lemma 2.5.

5. Proof of Proposition 2.6

The section is devoted to the proof of Proposition 2.6. As already explained, despite we only desire to obtain a L^2 estimate of the remainder term R^ε, the singular feature of the problem satisfied by R^ε, due to the large variations in the boundary layer, combined with the nonlinearity of the Navier–Stokes system, leads us to consider analytic estimates in the spirit of Cauchy–Kowaleskaya estimates. As we need a nonlinear long-time version of such Cauchy–Kowaleskaya estimates, we follow the method initiated by Chemin in [3], see also [4,19], which makes use of Fourier theory and Besov spaces. We introduce, for $s=0$ and $s=\frac{1}{2}$, the Besov spaces $\dot{B}^s$ respectively endowed with the norm

$$\|a\|_{\dot{B}^s}:=\sum_{k\in\mathbb{Z}}2^{ks}\|\dot{\Delta}_k a\|_{L^2(\mathcal{C})}\tag{5.1}$$

where the dyadic operator $\dot{\Delta}_k$ is defined by

$$\begin{gathered}\dot{\Delta}_k a\stackrel{\rm def}{=}\mathcal{F}^{-1}_{\xi\to x_3}(\varphi(2^{-k}|\xi|)\widehat{a}(x_1,x_2,\xi))\ \text{with}\\ \operatorname{Supp}\varphi\subset\Big\{\tau\in\mathbb{R}\,/\ \ \frac{3}{4}\le|\tau|\le\frac{8}{3}\Big\}\ \text{and}\quad\forall\tau>0\,,\ \sum_{j\in\mathbb{Z}}\varphi(2^{-j}\tau)=1,\end{gathered}$$

where $\mathcal{F}^{-1}_{\xi\to x_3}a$ denotes the inverse Fourier transform of the distribution a with respect to the third variable, and $\widehat{a}(x_1,x_2,\xi)=\mathcal{F}_{x_3\to\xi}(a)(x_1,x_2,\xi)$. One may check more details on Littlewood-Paley theory from [1].

For any locally bounded function Φ on $\mathbb{R}^+\times\mathbb{R}$, we define

$$v_\Phi := \mathcal{F}^{-1}_{\xi\to x_3}(e^{\Phi(t,\xi)}\hat{v}(t,x_1,x_2,\xi)). \tag{5.2}$$

Then it follows from (2.41) that, for any function Φ of the form:

$$\Phi(t,\xi) := \rho(t)|\xi| - \beta(t), \tag{5.3}$$

the vector field R^ε_Φ satisfies the Navier–Stokes type system:

$$\begin{cases} \partial_t R^\varepsilon_\Phi - \dot{\rho}|\partial_3|R^\varepsilon_\Phi + \dot{\beta}R^\varepsilon_\Phi + (u^\varepsilon\cdot\nabla R^\varepsilon + R^\varepsilon\cdot\nabla(u^{\mathrm{aux}}+\varepsilon u^{\mathrm{fl}}))_\Phi \\ \qquad -\varepsilon\Delta R^\varepsilon_\Phi + \nabla\pi^\varepsilon_\Phi = F^\varepsilon_\Phi, \\ \operatorname{div} R^\varepsilon_\Phi = 0, \\ R^\varepsilon_\Phi|_{\partial\mathcal{C}} = 0, \\ R^\varepsilon_\Phi|_{t=0} = 0. \end{cases} \tag{5.4}$$

We recall that the source term F^ε is defined in (2.42). Such a form of the exponential Fourier multiplier make appear two possible gains through the second and third terms in the first equation above. The purpose of these gains is to help to deal with the singular part of the convective term, that is the fourth term of the same equation. The choices of $\beta(t)$ and $\rho(t)$ are therefore crucial. We first define $\beta(t)$, by

$$\begin{cases} \dot{\beta}(t) = & C_*\chi_{[0,T]}(t) + C_*(\varepsilon\|\nabla_{\mathrm{h}}e^{\rho_0|\partial_3|}u^{\mathrm{fl}}\|^2_{\dot{B}^{\frac12}} + \|h^1-\chi\{V^1\}_\varepsilon\|^2_{L^\infty(D_1)} \\ & +\varepsilon\|h^2-\chi\{V^2\}_\varepsilon\|^2_{L^\infty(D_1)} + \|V^0\|_{L^\infty_z} + \|z\partial_z V^0\|_{L^\infty_z} + \varepsilon\|r^\varepsilon\|_{L^\infty(D_1)}), \\ \beta(0) = & 0, \end{cases} \tag{5.5}$$

where C_* is a constant which will be determined later.

Proposition 5.1. *If u_b satisfies (2.43) for a constant $C_0>0$ and $\rho_b>\rho_0$, there exists $\beta_*>0$ such that*

$$\sup_{t\in[0,T/\varepsilon]}\beta(t) = \beta(T/\varepsilon) \le \beta_*. \tag{5.6}$$

Proof. Since h^1 is compactly supported in $(0,T)$, χ is a cut-off function which satisfies $0\le\chi\le1$,

$$\int_0^{\frac{T}{\varepsilon}}\|h^1-\chi\{V^1\}_\varepsilon\|^2_{L^\infty(D_1)}\,dt \le 2T\|h^1\|^2_{L^\infty} + 2\int_0^{\frac{T}{\varepsilon}}\|V^1\|^2_{L^\infty_z}\,dt.$$

By construction, $V^1\in C^0_7(\mathbb{R}_+;H^7_8(\mathbb{R}_+))$, and recall Definition 2.3, we find that

$$\int_0^{\frac{T}{\varepsilon}}\|h^1-\chi\{V^1\}_\varepsilon\|^2_{L^\infty(D_1)}\,dt \le 2T\|h^1\|^2_{L^\infty} + C\|V^1\|^2_{C^0_1(\mathbb{R}_+;H^1(\mathbb{R}_+))},$$

for a constant C. Similarly

$$\int_0^{\frac{T}{\varepsilon}}\|h^2-\chi\{V^2\}_\varepsilon\|^2_{L^\infty(D_1)}\,dt \le 2T\|h^2\|^2_{L^\infty} + C\|V^2\|^2_{C^0_1(\mathbb{R}_+;H^1(\mathbb{R}_+))}.$$

and

$$\int_0^{\frac{T}{\varepsilon}}(\|V^0\|_{L^\infty_z} + \|z\partial_z V^0\|_{L^\infty_z})\,dt \le C\|V^0\|_{C^0_2(\mathbb{R}_+;H^2_1(\mathbb{R}_+))}.$$

By Proposition 2.5 and Sobolev imbedding inequality,

$$\begin{aligned}\varepsilon\int_0^{\frac{T}{\varepsilon}}\|r^\varepsilon\|_{L^\infty(D_1)}\,dt &\le C\varepsilon\int_0^{\frac{T}{\varepsilon}}\|r^\varepsilon\|_{H^2(D_1)}\,dt\\ &\le C\varepsilon\|r^\varepsilon\|^{\frac12}_{L^2(\mathbb{R}_+;H^2(D_1))}\varepsilon^{-\frac12}T^{\frac12}\\ &\le C\varepsilon^{\frac38}.\end{aligned}$$

It remains to estimate $\int_0^{\frac{T}{\varepsilon}}\varepsilon\|\nabla_{\rm h}e^{\rho_0|\partial_3|}u^{\rm fl}\|^2_{\dot B^{\frac12}}\,dt$. Since $u^{\rm fl}$ is supported in $[0,T]$ as a function of t we have that

$$\begin{aligned}\int_0^{\frac{T}{\varepsilon}}\varepsilon\|\nabla_{\rm h}e^{\rho_0|\partial_3|}u^{\rm fl}\|^2_{\dot B^{\frac12}}\,dt &\le T\|\nabla_{\rm h}e^{\rho_0|\partial_3|}u_b\|^2_{\dot B^{\frac12}}\\ &\le T\left(\sum_{k\in\mathbb{Z}}2^{\frac k2}\|\dot\Delta_k\nabla_{\rm h}e^{\rho_0|\partial_3|}u_b\|_{L^2(\mathcal{C})}\right)^2\\ &\le CT\big(\|\nabla_{\rm h}e^{\rho_0|\partial_3|}u_b\|_{L^2(\mathcal{C})}+\|\partial_3\nabla_{\rm h}e^{\rho_0|\partial_3|}u_b\|_{L^2(\mathcal{C})}\big)^2\\ &\le CT\|e^{\rho_0|\partial_3|}u_b\|^2_{H^2(\mathcal{C})}.\end{aligned}$$

Since u_b satisfies (2.43) for $\rho_b>\rho$,

$$\int_0^{\frac{T}{\varepsilon}}\varepsilon\|\nabla_{\rm h}e^{\rho_0|\partial_3|}u^{\rm fl}\|^2_{\dot B^{\frac12}}\,dt\le CTC_b^2\left(\frac{\rho_b}{\rho_b-\rho_0}\right)^2.$$

Eventually, gathering the above inequality concludes the proof of Proposition 5.1. □

Remark 5.2. From now on, for simplification, we shall denote the norm $\|\cdot\|_{L^2(\mathcal{C})}$ by $\|\cdot\|$ if there is no ambiguity.

Now we are in a position to complete the proof of Proposition 2.6.

Proof of Proposition 2.6. We define $\rho(t)$ as the solution of the nonlinear ODE:

$$\begin{cases}\dot\rho(t)=-C_*(\varepsilon\|\nabla_{\rm h}R^\varepsilon_\Phi\|^2_{\dot B^0}+\|z\partial_zV^0\|_{L^\infty_z}), & t>0,\\ \rho(0)=\rho_0,\end{cases}\tag{5.7}$$

where

$$\rho_0:=2+C_*\int_0^\infty\|z\partial_zV^0(t)\|_{L^\infty_z}\,dt.\tag{5.8}$$

and C_* is a constant which will be determined later. We set

$$T^*:=\sup\{t\in[0,\frac{T}{\varepsilon}]:\rho(t)\ge1\}.\tag{5.9}$$

We apply the operator $\dot\Delta_k$ to (5.4) and we use energy estimates to find that

$$\begin{aligned}&\tfrac12\tfrac{d}{dt}\|\dot\Delta_kR^\varepsilon_\Phi\|^2+|\dot\rho|(|\partial_3|\dot\Delta_kR^\varepsilon_\Phi,\dot\Delta_kR^\varepsilon_\Phi)+\dot\beta\|\dot\Delta_kR^\varepsilon_\Phi\|^2+\varepsilon\|\nabla\dot\Delta_kR^\varepsilon_\Phi\|^2\\ &\quad+(\dot\Delta_k(u^\varepsilon\cdot\nabla R^\varepsilon+R^\varepsilon\cdot\nabla(u^{\rm aux}+\varepsilon u^{\rm fl}))_\Phi,\dot\Delta_kR^\varepsilon_\Phi)=(\dot\Delta_kF^\varepsilon_\Phi,\dot\Delta_kR^\varepsilon_\Phi),\end{aligned}\tag{5.10}$$

which implies

$$\begin{aligned}&\sum_{k\in\mathbb{Z}}\|\dot\Delta_k R^\varepsilon_\Phi(t)\|+\sum_{k\in\mathbb{Z}}\Big(\int_0^t 2^k|\dot\rho|\|\dot\Delta_k R^\varepsilon_\Phi\|^2\,ds\Big)^{\frac12}\\&\qquad+\sum_{k\in\mathbb{Z}}\Big(\int_0^t \dot\beta\|\dot\Delta_k R^\varepsilon_\Phi\|^2\,ds\Big)^{\frac12}+\sum_{k\in\mathbb{Z}}\Big(\int_0^t \varepsilon\|\nabla\dot\Delta_k R^\varepsilon_\Phi\|^2\,ds\Big)^{\frac12}\\&\le C\sum_{k\in\mathbb{Z}}\Big(\int_0^t |(\dot\Delta_k F^\varepsilon_\Phi,\dot\Delta_k R^\varepsilon_\Phi)|\,ds\Big)^{\frac12}\\&\qquad+C\sum_{k\in\mathbb{Z}}\Big(\int_0^t |(\dot\Delta_k(u^\varepsilon\cdot\nabla R^\varepsilon+R^\varepsilon\cdot\nabla(u^{\rm aux}+\varepsilon u^{\rm fl}))_\Phi,\dot\Delta_k R^\varepsilon_\Phi)|\,ds\Big)^{\frac12}.\end{aligned}\tag{5.11}$$

Since F^ε is supported in $[0,T]$, the first term on the right hand side can be bounded by

$$\sum_{k\in\mathbb{Z}}\left(\int_0^t \chi_{[0,T]}(s)\|\dot\Delta_k R^\varepsilon_\Phi(s)\|^2 ds\right)^{\frac12}+\sum_{k\in\mathbb{Z}}\left(\int_0^T \|\dot\Delta_k F^\varepsilon_\Phi(s)\|^2 ds\right)^{\frac12}.\tag{5.12}$$

It remains to estimate the last term in (5.11). It is easy to observe from (2.10) and (2.25) that $u^{\rm aux}$ is independent of x_3 variable. By using integration by parts and $\operatorname{div} u^{\rm aux}=0$, we find that

$$(\dot\Delta_k(u^{\rm aux}\cdot\nabla R^\varepsilon)_\Phi,\dot\Delta_k R^\varepsilon_\Phi)=(u^{\rm aux}\cdot\nabla\dot\Delta_k R^\varepsilon_\Phi,\dot\Delta_k R^\varepsilon_\Phi)=0.\tag{5.13}$$

While due to $\operatorname{div} u^{\rm fl}=\operatorname{div} R^\varepsilon=0$, one has

$$(\dot\Delta_k((\varepsilon u^{\rm fl}+\varepsilon R^\varepsilon)\cdot\nabla R^\varepsilon)_\Phi,\dot\Delta_k R^\varepsilon_\Phi)=-(\dot\Delta_k((\varepsilon u^{\rm fl}+\varepsilon R^\varepsilon)\otimes R^\varepsilon)_\Phi,\nabla\dot\Delta_k R^\varepsilon_\Phi).\tag{5.14}$$

Next we use the following lemma: □

Lemma 5.3. *For any axi-symmetric functions a,b and c in $\mathcal{C}$, assume that function a vanishes on the boundary $\partial\mathcal{C}$, then for any constant $c_0>0$, there exists $C>0$, such that*

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\left(\int_0^t |(\dot\Delta_k(ab)_\Phi,\dot\Delta_k c_\Phi))|\,ds\right)^{\frac12}&\le c_0\sum_{k\in\mathbb{Z}}\left(\int_0^t \|\dot\Delta_k c_\Phi\|^2\,ds\right)^{\frac12}\\&\quad+C\sum_{k\in\mathbb{Z}}\left(\int_0^t \|\nabla_{\rm h} a_\Phi\|^2_{\dot B^{\frac12}}\|\dot\Delta_k b_\Phi\|^2\,ds\right)^{\frac12}.\end{aligned}$$

When $a=b$ and both vanish on the boundary, we also have

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\left(\int_0^t |(\dot\Delta_k(a^2)_\Phi,\dot\Delta_k c_\Phi))|\,ds\right)^{\frac12}&\le c_0\sum_{k\in\mathbb{Z}}\left(\int_0^t \|\dot\Delta_k c_\Phi\|^2\,ds\right)^{\frac12}\\&\quad+C\sum_{k\in\mathbb{Z}}\left(\int_0^t 2^k\|\nabla_{\rm h} a_\Phi\|^2_{\dot B^0}\|\dot\Delta_k a_\Phi\|^2\,ds\right)^{\frac12}.\end{aligned}$$

The proof of Lemma 5.3 is postponed to Sect. 7. Then by Lemma 5.3, for any $c_0>0$, there exists a constant $C>0$, such that

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\left(\int_0^t |(\dot\Delta_k((\varepsilon u^{\rm fl}+\varepsilon R^\varepsilon)\cdot\nabla R^\varepsilon)_\Phi,\dot\Delta_k R^\varepsilon_\Phi)|\,ds\right)^{\frac12}&\le c_0\sum_{k\in\mathbb{Z}}\left(\int_0^t \varepsilon\|\nabla\dot\Delta_k R^\varepsilon_\Phi\|^2\,ds\right)^{\frac12}\\&\quad+C\sum_{k\in\mathbb{Z}}\left(\int_0^t \varepsilon(\|\nabla_{\rm h} u^{\rm fl}_\Phi\|^2_{\dot B^{\frac12}}+2^k\|\nabla_{\rm h} R^\varepsilon_\Phi\|^2_{\dot B^0})\|\dot\Delta_k R^\varepsilon_\Phi\|^2\,ds\right)^{\frac12}\end{aligned}\tag{5.15}$$

Again, by integration by parts, we find

$$(\dot\Delta_k(R^\varepsilon\cdot\nabla(u^{\rm aux}+\varepsilon u^{\rm fl}))_\Phi,\dot\Delta_k R^\varepsilon_\Phi)=-(\dot\Delta_k(R^\varepsilon\otimes(u^{\rm aux}+\varepsilon u^{\rm fl}))_\Phi,\nabla\dot\Delta_k R^\varepsilon_\Phi).$$

By Lemma 5.3, for any $c_0>0$, there exists a constant $C>0$, such that

$$\begin{aligned}&\sum_{k\in\mathbb{Z}}\left(\int_0^t\varepsilon|(\dot\Delta_k(R^\varepsilon\otimes u^{\rm fl})_\Phi,\nabla\dot\Delta_k R^\varepsilon)|\,ds\right)^{\frac12}\\&\le c_0\sum_{k\in\mathbb{Z}}\left(\int_0^t\varepsilon\|\nabla\dot\Delta_k R^\varepsilon_\Phi\|^2\,ds\right)^{\frac12}+C\sum_{k\in\mathbb{Z}}\left(\int_0^t\varepsilon\|\nabla_{\rm h}u^{\rm fl}_\Phi\|^2_{\dot B^{\frac12}}\|\dot\Delta_k R^\varepsilon_\Phi\|^2\,ds\right)^{\frac12}.\end{aligned}\tag{5.16}$$

and

$$\begin{aligned}&\sum_{k\in\mathbb{Z}}\left(\int_0^t|(\dot\Delta_k(R^\varepsilon\otimes(u^{\rm aux}-(h^0-\chi\{V^0\}_\varepsilon)e_3))_\Phi,\nabla\dot\Delta_k R^\varepsilon_\Phi)|\,ds\right)^{\frac12}\\&\le c_0\sum_{k\in\mathbb{Z}}\left(\int_0^t\varepsilon\|\nabla\dot\Delta_k R^\varepsilon_\Phi\|^2\,ds\right)^{\frac12}\\&\quad+C\sum_{k\in\mathbb{Z}}\left(\int_0^t\|\dot\Delta_k R^\varepsilon_\Phi\|^2(\|h^1-\chi\{V^1\}_\varepsilon\|^2_{L^\infty(D_1)}+\varepsilon\|h^2-\chi\{V^2\}_\varepsilon\|^2_{L^\infty(D_1)}+\varepsilon\|r^\varepsilon\|^2_{L^\infty})\,ds\right)^{\frac12},\end{aligned}\tag{5.17}$$

where in the last step, we used (2.10) and (2.25).

It remains to estimate $(\dot\Delta_k(R^\varepsilon\cdot\nabla(h^0-\chi\{V^0\}_\varepsilon))_\Phi,\dot\Delta_k R^\varepsilon_{3,\Phi})$. Observing that

$$\begin{aligned}&(\dot\Delta_k(R^\varepsilon\cdot\nabla(h^0-\chi\{V^0\}_\varepsilon))_\Phi,\dot\Delta_k R^\varepsilon_{3,\Phi})\\&\quad=-(\dot\Delta_k R^\varepsilon_{r,\Phi}(\chi'\{V^0\}_\varepsilon+\frac{\chi}{1-r}\{z\partial_z V^0\}_\varepsilon),\dot\Delta_k R^\varepsilon_{3,\Phi}),\end{aligned}$$

which implies

$$\begin{aligned}&\left|(\dot\Delta_k(R^\varepsilon\cdot\nabla(h^0-\chi\{V^0\}_\varepsilon))_\Phi,\dot\Delta_k R^\varepsilon_{3,\Phi})\right|\\&\quad\le\|V^0\|_{L^\infty_z}\|\dot\Delta_k R^\varepsilon_\Phi\|^2-(\dot\Delta_k R^\varepsilon_{r,\Phi}\frac{\chi}{1-r}\{z\partial_z V^0\}_\varepsilon,\dot\Delta_k R^\varepsilon_{3,\Phi})\end{aligned}\tag{5.18}$$

Since R^ε vanishes on the boundary of $\mathcal{C}$, we have

$$\frac{\dot\Delta_k R^\varepsilon_{r,\Phi}}{1-r}=-\int_0^1(\partial_r\dot\Delta_k R^\varepsilon_{r,\Phi})(t,1-(1-r)s,x_3)ds.\tag{5.19}$$

By the divergence free condition $\operatorname{div}R^\varepsilon=0$,

$$\partial_r\dot\Delta_k R^\varepsilon_{r,\Phi}=-\frac{\dot\Delta_k R^\varepsilon_{r,\Phi}}{r}-\partial_3\dot\Delta_k R^\varepsilon_{3,\Phi}.\tag{5.20}$$

Note that $\chi(r)=0$ when $r\le\frac13$, we find

$$\begin{aligned}&|(\dot\Delta_k R^\varepsilon_{r,\Phi}\frac{\chi}{1-r}\{z\partial_z V^0\}_\varepsilon,\dot\Delta_k R^\varepsilon_{3,\Phi})|\\&=\left|\int_{\mathcal{C}}\int_0^1(\partial_r\dot\Delta_k R^\varepsilon_{r,\Phi})(t,1-(1-r)s,x_3)\chi(r)\{z\partial_z V^0\}_\varepsilon(t,r)\dot\Delta_k R^\varepsilon_{3,\Phi}(t,x)dsdx\right|\\&\le(3\|\dot\Delta_k R^\varepsilon_{r,\Phi}\|+\|\partial_3\dot\Delta_k R^\varepsilon_{3,\Phi}\|)\|z\partial_z V^0\|_{L^\infty_z}\|\dot\Delta_k R^\varepsilon_{3,\Phi}\|\\&\le C(1+2^k)\|z\partial_z V^0\|_{L^\infty_z}\|\dot\Delta_k R^\varepsilon_\Phi\|^2,\end{aligned}$$

which implies

$$\sum_{k\in\mathbb{Z}}\left(\int_0^t |(\dot\Delta_k(R^\varepsilon\cdot\nabla(h^0-\chi\{V^0\}_\varepsilon))_\Phi,\dot\Delta_k R^\varepsilon_{3,\Phi})|\,ds\right)^{\frac12}$$
$$\le C\sum_{k\in\mathbb{Z}}\left(\int_0^t(\|V^0\|_{L^\infty_z}+(1+2^k)\|z\partial_z V^0\|_{L^\infty_z})\|\dot\Delta_k R^\varepsilon_\Phi\|^2\,ds\right)^{\frac12}. \tag{5.21}$$

Finally, by summing up the estimates (5.13), (5.15)–(5.17) and (5.21), for any $c_0>0$, there exists a constant $C>0$ such that

$$\sum_{k\in\mathbb{Z}}\left(\int_0^t |(\dot\Delta_k(u^\varepsilon\cdot\nabla R^\varepsilon+R^\varepsilon\cdot\nabla(u^{\rm aux}+\varepsilon u^{\rm fl}))_\Phi,\dot\Delta_k R^\varepsilon_\Phi)|\,ds\right)^{\frac12}$$
$$\le c_0\sum_{k\in\mathbb{Z}}\left(\int_0^t \varepsilon\|\nabla\dot\Delta_k R^\varepsilon_\Phi\|^2\,ds\right)^{\frac12}$$
$$+C\sum_{k\in\mathbb{Z}}\left(\int_0^t\left(\varepsilon 2^k\|\nabla_{\rm h}R^\varepsilon_\Phi\|^2_{\dot B^0}+2^k\|z\partial_z V^0\|_{L^\infty_z}+\mathcal{K}_\varepsilon\right)\|\dot\Delta_k R^\varepsilon_\Phi\|^2\,ds\right)^{\frac12}, \tag{5.22}$$

where

$$\mathcal{K}_\varepsilon:=\varepsilon\|\nabla_{\rm h}u^{\rm fl}_\Phi\|^2_{\dot B^{\frac12}}+\|h^1-\chi\{V^1\}_\varepsilon\|^2_{L^\infty(D_1)}+\varepsilon\|h^2-\chi\{V^2\}_\varepsilon\|^2_{L^\infty(D_1)}$$
$$+\|V^0\|_{L^\infty_z}+\|z\partial_z V^0\|_{L^\infty_z}+\varepsilon\|r^\varepsilon\|^2_{L^\infty(D_1)}. \tag{5.23}$$

In view of (5.11) and (5.14), there exists a constant $C_*>0$ such that, for any $t\in[0,T^*]$,

$$\sum_{k\in\mathbb{Z}}\|\dot\Delta_k R^\varepsilon_\Phi(t)\|+\sum_{k\in\mathbb{Z}}\left(\int_0^t 2^k|\dot\rho|\|\dot\Delta_k R^\varepsilon_\Phi\|^2\,ds\right)^{\frac12}$$
$$+\sum_{k\in\mathbb{Z}}\left(\int_0^t\dot\beta\|\dot\Delta_k R^\varepsilon_\Phi\,ds\right)^{\frac12}+\sum_{k\in\mathbb{Z}}\left(\int_0^t\varepsilon\|\nabla\dot\Delta_k R^\varepsilon_\Phi\|^2\,ds\right)^{\frac12}$$
$$\le C_*\sum_{k\in\mathbb{Z}}\left(\int_0^T\|\dot\Delta_k F^\varepsilon_\Phi\|^2\,ds\right)^{\frac12}+\sum_{k\in\mathbb{Z}}\left(\int_0^t C_*(\chi_{[0,T]}+\mathcal{K}_\varepsilon)\|\dot\Delta_k R^\varepsilon_\Phi\|^2\,ds\right)^{\frac12}$$
$$+\sum_{k\in\mathbb{Z}}\left(\int_0^t C_*2^k(\varepsilon\|\nabla_{\rm h}R^\varepsilon_\Phi\|^2_{\dot B^0}+\|z\partial_z V^0\|_{L^\infty_z})\|\dot\Delta_k R^\varepsilon_\Phi\|^2\,ds\right)^{\frac12}. \tag{5.24}$$

Recall the definition of $\beta(t)$ in (5.5) and $\rho(t)$ in (5.7), we find that, for $t\le T^*$,

$$\sum_{k\in\mathbb{Z}}\|\dot\Delta_k R^\varepsilon_\Phi(t)\|+\sum_{k\in\mathbb{Z}}\left(\int_0^t\varepsilon\|\nabla\dot\Delta_k R^\varepsilon_\Phi\|^2\,ds\right)^{\frac12}\le C_*\sum_{k\in\mathbb{Z}}\left(\int_0^T\|\dot\Delta_k F^\varepsilon_\Phi\|^2\,ds\right)^{\frac12}. \tag{5.25}$$

Let us admit the following proposition for the time being.

Proposition 5.4. *If u_b satisfies (2.43) and (2.44) for a constant $C_b>0$, there is a constant C_F such that for any $\varepsilon\in(0,1)$,*

$$\sum_{k\in\mathbb{Z}}\left(\int_0^T\|\dot\Delta_k F^\varepsilon_\Phi\|^2\,dt\right)^{\frac12}\le C_F\varepsilon^{\frac14}. \tag{5.26}$$

The proof of Proposition 5.4 is postponed to Sect. 6.

We then deduce that for $t \leq T^*$,

$$\sum_{k\in\mathbb{Z}} \|\dot{\Delta}_k R^{\varepsilon}_{\Phi}(t)\| + \sum_{k\in\mathbb{Z}} \left(\int_0^t \varepsilon \|\nabla \dot{\Delta}_k R^{\varepsilon}_{\Phi}\|^2 \, ds\right)^{\frac{1}{2}} \leq C_* C_F \varepsilon^{\frac{1}{4}}. \tag{5.27}$$

For $t \leq T^*$, by the Minkowski inequality,

$$\begin{aligned} \int_0^t \varepsilon \|\nabla R^{\varepsilon}_{\Phi}\|^2_{\dot{B}^0} \, ds \; &= \int_0^t \varepsilon \left(\sum_{k\in\mathbb{Z}} \|\dot{\Delta}_k \nabla R^{\varepsilon}_{\Phi}\|\right)^2 ds \\ &\leq \left(\sum_{k\in\mathbb{Z}} \left(\int_0^t \varepsilon \|\dot{\Delta}_k \nabla R^{\varepsilon}_{\Phi}\|^2 \, ds\right)^{\frac{1}{2}}\right)^2 \\ &\leq C_*^2 C_F^2 \varepsilon^{\frac{1}{2}}. \end{aligned} \tag{5.28}$$

So that

$$\rho(T^*) \geq 2 - C_*^3 C_F^2 \varepsilon^{\frac{1}{2}}. \tag{5.29}$$

Thus, for ε small enough, $\rho(T^*) > 1$ and thus $T^* = \frac{T}{\varepsilon}$ and we have, for $t \in [0, \frac{T}{\varepsilon}]$,

$$\|R^{\varepsilon}_{\Phi}(t)\| + \left(\int_0^t \varepsilon \|\nabla R^{\varepsilon}_{\Phi}\|^2\right)^{\frac{1}{2}} \leq C\varepsilon^{\frac{1}{4}}, \tag{5.30}$$

which implies (2.45) since $\beta(t) \leq \beta_*$ by Proposition 5.1 and $\rho(t) \geq 1$ for $t \in [0, \frac{T}{\varepsilon}]$. This completes the proof of Proposition 2.6. □

6. Proof of Proposition 5.4

This section is devoted to the proof of Proposition 5.4. We start with the following observation.

Lemma 6.1. *For any axi-symmetric function f in D_1, if f vanishes on the boundary ∂D_1 and $\nabla_{\rm h} f \in L^2(D_1)$, where $\nabla_{\rm h} = (\partial_{x_1}, \partial_{x_2})$, then there exists a constant C such that*

$$\|f\|_{L^\infty(D_1)} \leq C \|\nabla_{\rm h} f\|_{L^2(D_1)}. \tag{6.1}$$

Proof. For $r \in [0, 1]$, since f vanishes on ∂D_1,

$$|f(r)|^2 = -\int_r^1 2f(\tau)\partial_\tau f(\tau)\, d\tau \leq 2\left(\int_r^1 |\partial_\tau f(\tau)|^2 \tau \, d\tau\right)^{\frac{1}{2}} \left(\int_r^1 \frac{|f(\tau)|^2}{\tau}\, d\tau\right)^{\frac{1}{2}}.$$

On the other hand, one has

$$\|\nabla_{\rm h} f\|^2_{L^2(D_1)} = 2\pi \int_0^1 \left(|\partial_r f(r)|^2 + \frac{|f(r)|^2}{r^2}\right) r\, dr,$$

so that

$$\|f\|_{L^\infty(D_1)} \leq \frac{1}{\sqrt{\pi}} \|\nabla_{\rm h} f\|_{L^2(D_1)},$$

which leads to (6.1). □

First, for any profile $\mathcal{V}\in L^2((0,T)\times D_1)$, any axi-symmetric function v defined in $(0,T)\times\mathcal{C}$, vanishes on the boundary $\mathcal{C}$, and function $g\in L^2((0,T);L^\infty(D_1))$, we deduce from (2.21) and Lemma 6.1 that

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\left(\int_0^T\|\dot\Delta_k(v\{\mathcal{V}\}_\varepsilon)_\Phi\|^2\,dt\right)^{\frac12}&\le\sum_{k\in\mathbb{Z}}\sup_{t\in[0,T]}\|\dot\Delta_k v_\Phi\|_{L^\infty_{x_\mathrm{h}}L^2_{x_3}}\left(\int_0^T\|\{\mathcal{V}\}_\varepsilon\|^2_{L^2_{x_\mathrm{h}}}\,dt\right)^{\frac12}\\&\le C\varepsilon^{\frac14}\|\mathcal{V}\|_{L^2((0,T)\times\mathbb{R}_+)}\sum_{k\in\mathbb{Z}}\|\nabla_\mathrm{h}\dot\Delta_k v_\Phi\|_{L^\infty((0,T);L^2(\mathcal{C}))},\end{aligned}\tag{6.2}$$

and

$$\sum_{k\in\mathbb{Z}}\left(\int_0^T\|\dot\Delta_k(vg)_\Phi\|^2\,dt\right)^{\frac12}\le C\sum_{k\in\mathbb{Z}}\|\dot\Delta_k v_\Phi\|_{L^\infty((0,T);L^2(\mathcal{C}))}\|g\|_{L^2((0,T);L^\infty(D_1))}.\tag{6.3}$$

We recall from (2.42) that

$$\begin{aligned}F^\varepsilon:=\;&\chi(\{V^0\}_\varepsilon+\sqrt{\varepsilon}\{V^1\}_\varepsilon+\varepsilon\{V^2\}_\varepsilon)\partial_3u^{\mathrm{fl}}-(\sqrt{\varepsilon}h^1+\varepsilon h^2+\varepsilon r^\varepsilon)\partial_3u^{\mathrm{fl}}\\&+u_r^{\mathrm{fl}}\chi'\{V^0+\sqrt{\varepsilon}V^1+\varepsilon V^2\}_\varepsilon e_3-u_r^{\mathrm{fl}}\frac{\chi}{1-r}\{z\partial_zV^0+\sqrt{\varepsilon}z\partial_zV^1+\varepsilon z\partial_zV^2\}_\varepsilon e_3\\&-\varepsilon u_r^{\mathrm{fl}}\partial_r r^\varepsilon e_3-\varepsilon u^{\mathrm{fl}}\cdot\nabla u^{\mathrm{fl}}+\varepsilon\Delta u^{\mathrm{fl}}.\end{aligned}\tag{6.4}$$

Now we estimate $\sum_{k\in\mathbb{Z}}\big(\int_0^T\|\dot\Delta_kF^\varepsilon_\Phi\|^2\,dt\big)^{\frac12}$ term by term.

We first get, by applying (6.2) to $\mathcal{V}:=V_0$ and $v:=\chi\partial_3u^{\mathrm{fl}}$, that

$$\begin{aligned}&\sum_{k\in\mathbb{Z}}\left(\int_0^T\|\dot\Delta_k(\chi\partial_3u^{\mathrm{fl}}\{V^0\}_\varepsilon)_\Phi\|^2\,dt\right)^{\frac12}\\&\quad\le C\varepsilon^{\frac14}\|V^0\|_{L^2((0,T)\times\mathbb{R}_+)}\sum_{k\in\mathbb{Z}}\|\nabla_\mathrm{h}\dot\Delta_k(\chi\partial_3u^{\mathrm{fl}}_\Phi)\|_{L^\infty((0,T);L^2(\mathcal{C}))}.\end{aligned}\tag{6.5}$$

Note that

$$u^{\mathrm{fl}}(t,x)=\mu(t)u_b(x-\int_0^t h^0(s)ds e_3).$$

So that

$$\sum_{k\in\mathbb{Z}}\|\nabla_\mathrm{h}\dot\Delta_k(\chi\partial_3u^{\mathrm{fl}}_\Phi)\|_{L^\infty((0,T);L^2(\mathcal{C})}\le C\sum_{k\in\mathbb{Z}}\|\dot\Delta_ke^{\rho_0|\partial_3|}\partial_3u_b\|_{H^1(\mathcal{C})}.\tag{6.6}$$

Thanks to Cauchy inequality and the properties of operator $\dot\Delta_k$,

$$\begin{aligned}&\sum_{k\in\mathbb{Z}}\|\dot\Delta_ke^{\rho_0|\partial_3|}\partial_3u_b\|_{H^1(\mathcal{C})}\\&\quad\le C\sum_{k\in\mathbb{Z}}2^k\|\dot\Delta_ke^{\rho_0|\partial_3|}u_b\|_{H^1(\mathcal{C})}\\&\quad\le C(\sum_{k\le0}\|\dot\Delta_ke^{\rho_0|\partial_3|}u_b\|^2_{H^1(\mathcal{C})})^{\frac12}+C(\sum_{k>0}2^{4k}\|\dot\Delta_ke^{\rho_0|\partial_3|}u_b\|^2_{H^1(\mathcal{C})})^{\frac12}\\&\quad\le C\|e^{\rho_0|\partial_3|}u_b\|_{H^3(\mathcal{C})}\\&\quad\le CC_b,\end{aligned}\tag{6.7}$$

where we used (2.43) by taking $\rho_b=2\rho_0$.

By inserting the above estimates into (6.5), we obtain

$$\sum_{k\in\mathbb{Z}}\left(\int_0^T\|\dot\Delta_k(\chi\partial_3u^{\mathrm{fl}}\{V^0\}_\varepsilon)_\Phi\|^2\,dt\right)^{\frac12}\le C\varepsilon^{\frac14}.\tag{6.8}$$

Similarly, one has

$$\sum_{k\in\mathbb{Z}}\left(\int_0^T\|\dot{\Delta}_k(\chi\partial_3 u^{\mathrm{fl}}(\sqrt{\varepsilon}\{V^1\}_\varepsilon+\varepsilon\{V^2\}_\varepsilon))_\Phi\|^2\,dt\right)^{\frac{1}{2}}\le C\varepsilon^{\frac{3}{4}}. \tag{6.9}$$

We apply (6.3) to $v=\partial_3 u^{\mathrm{fl}}$ and $g=\sqrt{\varepsilon}h^1+\varepsilon h^2+\varepsilon r^\varepsilon$,

$$\begin{aligned}\sum_{k\in\mathbb{Z}}&\left(\int_0^T\|\dot{\Delta}_k((\sqrt{\varepsilon}h^1+\varepsilon h^2+\varepsilon r^\varepsilon)\partial_3 u^{\mathrm{fl}})_\Phi\|^2\,dt\right)^{\frac{1}{2}}\\&\le C\sum_{k\in\mathbb{Z}}\|\dot{\Delta}_k\partial_3 u^{\mathrm{fl}}_\Phi\|_{L^\infty((0,T);L^2(\mathcal{C}))}\|\sqrt{\varepsilon}h^1+\varepsilon h^2+\varepsilon r^\varepsilon\|_{L^2((0,T);L^\infty(D_1))}.\end{aligned} \tag{6.10}$$

In the same way as (6.6) and (6.7), we obtain that

$$\sum_{k\in\mathbb{Z}}\|\dot{\Delta}_k\partial_3 u^{\mathrm{fl}}_\Phi\|_{L^\infty((0,T);L^2(\mathcal{C}))}\le CC_b. \tag{6.11}$$

While it follows from Sobolev imbedding inequality and Lemma 2.5 that

$$\|r^\varepsilon\|_{L^2((0,T);L^\infty(D_1))}\le C\|r^\varepsilon\|_{L^2((0,T);H^2(D_1))}\le C\varepsilon^{-\frac{1}{4}}. \tag{6.12}$$

We deduce that

$$\sum_{k\in\mathbb{Z}}\left(\int_0^T\|\dot{\Delta}_k((\sqrt{\varepsilon}h^1+\varepsilon h^2+\varepsilon r^\varepsilon)\partial_3 u^{\mathrm{fl}})_\Phi\|^2\,dt\right)^{\frac{1}{2}}\le C\varepsilon^{\frac{1}{2}}. \tag{6.13}$$

We apply (6.2) to $v=u^{\mathrm{fl}}_r\chi'$ and $\mathcal{V}=V^0$ to obtain that

$$\begin{aligned}\sum_{k\in\mathbb{Z}}&\left(\int_0^T\|\dot{\Delta}_k(u^{\mathrm{fl}}_r\chi'\{V^0\}_\varepsilon)_\Phi\|^2\,dt\right)^{\frac{1}{2}}\\&\le C\varepsilon^{\frac{1}{4}}\|V^0\|_{L^2((0,T)\times\mathbb{R}_+)}\sum_{k\in\mathbb{Z}}\|\nabla_{\mathrm{h}}\dot{\Delta}_k(u^{\mathrm{fl}}_r\chi')_\Phi\|_{L^\infty((0,T);L^2(\mathcal{C}))}\\&\le C\varepsilon^{\frac{1}{4}}\|V^0\|_{L^2((0,T)\times\mathbb{R}_+)}\sum_{k\in\mathbb{Z}}\|\dot{\Delta}_k e^{\rho_0|\partial_3|}u_b\|_{L^2_{x_3}(H^1(D_1))}.\end{aligned} \tag{6.14}$$

Thanks to Cauchy inequality and the properties of operator $\dot{\Delta}_k$,

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\|\nabla_{\mathrm{h}}\dot{\Delta}_k e^{\rho_0|\partial_3|}u_b\|_{L^2(\mathcal{C})}&\le\left(\sum_{k\in\mathbb{Z}}2^{-\frac{|k|}{2}}\right)^{\frac{1}{2}}\left(\sum_{k\in\mathbb{Z}}2^{\frac{|k|}{2}}\|e^{\rho_0|\partial_3|}\dot{\Delta}_k u_b\|^2_{L^2_{x_3}(H^1(D_1))}\right)^{\frac{1}{2}}\\&\le C\left(\int_{\mathbb{R}}(|\xi|^{\frac{1}{2}}+|\xi|^{-\frac{1}{2}})e^{2\rho_0|\xi|}\|\mathcal{F}u_b(\xi)\|^2_{H^1(D_1)}d\xi\right)^{\frac{1}{2}},\end{aligned} \tag{6.15}$$

where $\mathcal{F}u_b(\xi)$ is the Fourier transform of u_b in the direction of x_3.

- For low frequencies, by (2.44),

$$\begin{aligned}&\int_{|\xi|\le 1}(|\xi|^{\frac{1}{2}}+|\xi|^{-\frac{1}{2}})e^{2\rho_0|\xi|}\|\mathcal{F}u_b(\xi)\|^2_{H^1(D_1)}d\xi\\&\le e^{2\rho_0}\|\mathcal{F}u_b\|^2_{L^\infty_\xi(H^1(D_1))}\int_{|\xi|\le 1}(|\xi|^{\frac{1}{2}}+|\xi|^{-\frac{1}{2}})\,d\xi\le C\|u_b\|^2_{L^1_{x_3}(H^1(D_1))}\le CC_b^2.\end{aligned} \tag{6.16}$$

- For high frequencies, by (2.43),

$$\begin{aligned}&\int_{|\xi|\geq 1}(|\xi|^{\frac{1}{2}}+|\xi|^{-\frac{1}{2}})e^{2\rho_0|\xi|}\|\mathcal{F}u_b(\xi)\|^2_{H^1(D_1)}d\xi\\&\leq \int_{|\xi|\geq 1}(|\xi|+1)^2e^{2\rho_0|\xi|}\|\mathcal{F}u_b(\xi)\|^2_{H^1(D_1)}d\xi\\&\leq C\|e^{\rho_0|\partial_3|}u_b\|^2_{H^2(\mathcal{C})}\leq CC_b^2.\end{aligned} \tag{6.17}$$

Gathering the estimates (6.14)–(6.17), we get

$$\sum_{k\in\mathbb{Z}}\left(\int_0^T\|\dot\Delta_k(u_r^{\mathrm{fl}}\chi'\{V^0\}_\varepsilon)_\Phi\|^2\,dt\right)^{\frac12}\leq C\varepsilon^{\frac14}. \tag{6.18}$$

Similarly, one has

$$\sum_{k\in\mathbb{Z}}\left(\int_0^T\|\dot\Delta_k(u_r^{\mathrm{fl}}\chi'(\sqrt{\varepsilon}\{V^1\}_\varepsilon+\varepsilon\{V^2\}_\varepsilon))_\Phi\|^2\,dt\right)^{\frac12}\leq C\varepsilon^{\frac34}. \tag{6.19}$$

We apply (6.2) to $v=z\partial_zV^0$ and $v=u_r^{\mathrm{fl}}\frac{\chi}{1-r}$ and use (2.30):

$$\begin{aligned}&\sum_{k\in\mathbb{Z}}\left(\int_0^T\|\dot\Delta_k(u_r^{\mathrm{fl}}\frac{\chi}{1-r}\{z\partial_zV^0\}_\varepsilon)_\Phi\|^2\,dt\right)^{\frac12}\\&\leq C\varepsilon^{\frac14}\|z\partial_zV^0\|_{L^2((0,T)\times\mathbb{R}_+)}\sum_{k\in\mathbb{Z}}\|\nabla_{\mathrm h}\dot\Delta_k(u_r^{\mathrm{fl}}\frac{\chi}{1-r})_\Phi\|_{L^\infty((0,T);L^2(\mathcal{C}))}\\&\leq C\varepsilon^{\frac14}\|V^0\|_{C_0^0(\mathbb{R}_+;H_1^1(\mathbb{R}_+))}\sum_{k\in\mathbb{Z}}\|\dot\Delta_ke^{\rho_0|\partial_3|}\frac{u_b}{1-r}\|_{L^2_{x_3}(H^1(D_1))}.\end{aligned}$$

Since u_b vanishes on the boundary of $\mathcal{C}$,

$$\frac{u_b}{1-r}=-\int_0^1(\partial_ru_b)(1-(1-r)s)ds.$$

Proceeding in the same way as for the treatment of (6.15)–(6.17), and by using (2.43) and (2.44), we find that

$$\begin{aligned}&\sum_{k\in\mathbb{Z}}\|\dot\Delta_ke^{\rho_0|\partial_3|}\frac{u_b}{1-r}\|_{L^2_{x_3}(H^1(D_1))}\\&\leq C\left(\int_{\mathbb{R}}(|\xi|^{\frac12}+|\xi|^{-\frac12})e^{2\rho_0|\xi|}\left(\int_0^1\|\mathcal{F}(\partial_ru_b)(1-(1-r)s)\|_{H^1(D_1)}\,ds\right)^2d\xi\right)^{\frac12}\\&\leq C\left(\int_{\mathbb{R}}(|\xi|^{\frac12}+|\xi|^{-\frac12})e^{2\rho_0|\xi|}\|\mathcal{F}\partial_ru_b\|^2_{H^1(D_1)}d\xi\right)^{\frac12}\\&\leq C\|u_b\|_{L^1_{x_3}(H^2(D_1))}+C\|e^{\rho_0|\partial_3|}u_b\|_{H^3(\mathcal{C})}\\&\leq CC_b.\end{aligned}$$

Thus, we obtain

$$\sum_{k\in\mathbb{Z}}\left(\int_0^T\|\dot\Delta_k(u_r^{\mathrm{fl}}\frac{\chi}{1-r}\{z\partial_zV^0\}_\varepsilon)_\Phi\|^2\,dt\right)^{\frac12}\leq C\varepsilon^{\frac14}. \tag{6.20}$$

Similarly, one has

$$\sum_{k\in\mathbb{Z}}\left(\int_0^T \|\dot\Delta_k(u_r^{\mathrm{fl}}\frac{\chi}{1-r}(\sqrt{\varepsilon}\{z\partial_z V^1\}_\varepsilon+\varepsilon\{z\partial_z V^2\}_\varepsilon))_\Phi\|^2\,dt\right)^{\frac12}\le C\varepsilon^{\frac34}. \tag{6.21}$$

We apply (6.3) to $v=\varepsilon u_r^{\mathrm{fl}}$ and $g=\partial_r r^\varepsilon$,

$$\begin{aligned}&\sum_{k\in\mathbb{Z}}\Big(\int_0^T \|\dot\Delta_k(\varepsilon u_r^{\mathrm{fl}}\partial_r r^\varepsilon)_\Phi\|^2\,dt\Big)^{\frac12}\\ &\le C\varepsilon\sum_{k\in\mathbb{Z}}\|\dot\Delta_k u_{r,\Phi}^{\mathrm{fl}}\|_{L^\infty((0,T);L^2(\mathcal{C}))}\|\partial_r r^\varepsilon\|_{L^2((0,T);L^\infty(D_1))}.\end{aligned}$$

Notice that

$$\sum_{k\in\mathbb{Z}}\|\dot\Delta_k u_{r,\Phi}^{\mathrm{fl}}\|_{L^\infty((0,T);L^2(\mathcal{C}))}\le\sum_{k\in\mathbb{Z}}\|\dot\Delta_k e^{\rho_0|\partial_3|}u_b\|_{L^2(\mathcal{C})}.$$

In the same way as (6.15)–(6.17), we can get

$$\sum_{k\in\mathbb{Z}}\|\dot\Delta_k e^{\rho_0|\partial_3|}u_b\|_{L^2(\mathcal{C})}\le CC_b.$$

By using Lemmas 6.1 and 2.5, we obtain that

$$\|\partial_r r^\varepsilon\|_{L^2((0,T);L^\infty(D_1))}\le C\|\nabla_{\mathrm{h}}\partial_r r^\varepsilon\|_{L^2((0,T)\times D_1)}\le C\varepsilon^{-\frac14}.$$

Therefore, we achieve

$$\sum_{k\in\mathbb{Z}}\left(\int_0^T \|\dot\Delta_k(\varepsilon u_r^{\mathrm{fl}}\partial_r r^\varepsilon)_\Phi\|^2\,dt\right)^{\frac12}\le C\varepsilon^{\frac34}. \tag{6.22}$$

In view of (2.30), one has

$$\sum_{k\in\mathbb{Z}}\left(\int_0^t \|\dot\Delta_k\Delta u_\Phi^{\mathrm{fl}}\|^2\,dt\right)^{\frac12}\le C\sum_{k\in\mathbb{Z}}\|\dot\Delta_k e^{\rho_0|\partial_3|}u^{\mathrm{fl}}\|_{H^2(\mathcal{C})}.$$

In the same way as (6.15)–(6.17), we can get

$$\sum_{k\in\mathbb{Z}}\|\dot\Delta_k e^{\rho_0|\partial_3|}u_b\|_{H^2(\mathcal{C})}\le C\|u_b\|_{L^1_{x_3}(H^2(D_1))}+C\|\langle\partial_3\rangle e^{\rho_0|\partial_3|}u_b\|_{H^2(\mathcal{C})}\le CC_b.$$

Thus we obtain

$$\sum_{k\in\mathbb{Z}}\left(\int_0^t \|\dot\Delta_k(\varepsilon\Delta u^{\mathrm{fl}})_\Phi\|^2\,dt\right)^{\frac12}\le C\varepsilon. \tag{6.23}$$

It remains to estimate the last term $\varepsilon u^{\mathrm{fl}}\cdot\nabla u^{\mathrm{fl}}$ of F^ε. By using (2.30), we find

$$\sum_{k\in\mathbb{Z}}\left(\int_0^T \|\dot\Delta_k(u^{\mathrm{fl}}\cdot\nabla u^{\mathrm{fl}})_\Phi\|^2\,dt\right)^{\frac12}\le C\sum_{k\in\mathbb{Z}}\|\dot\Delta_k(u_b\cdot\nabla u_b)_\Phi\|_{L^2(\mathcal{C})}. \tag{6.24}$$

In the same way as (6.15)–(6.17), one has

$$\begin{aligned}&\sum_{k\in\mathbb{Z}}\|\dot\Delta_k(u_b\cdot\nabla u_b)_\Phi\|_{L^2(\mathcal{C})}\\ &\quad\le C\|u_b\cdot\nabla u_b\|_{L^1_{x_3}(L^2(D_1))}+C\|\langle\partial_3\rangle e^{\rho_0|\partial_3|}(u_b\cdot\nabla u_b)\|_{L^2(\mathcal{C})}.\end{aligned} \tag{6.25}$$

It follows from Lemma 6.1 that

$$\begin{aligned}\|u_b\cdot\nabla u_b\|_{L^1_{x_3}(L^2(\mathcal{C}))} &\le \|u_b\|_{L^2_{x_3}(L^\infty(D_1))}\|\nabla u_b\|_{L^2(\mathcal{C})}\\ &\le C\|\nabla_{\rm h}u_b\|_{L^2(\mathcal{C})}\|\nabla u_b\|_{L^2(\mathcal{C})}\\ &\le CC_b^2. \end{aligned}\tag{6.26}$$

By the Plancherel theorem and the inequalities

$$|\xi|\le|\eta|+|\xi-\eta| \text{ and } (1+|\xi|^2)\le 2(1+|\eta|^2)(1+|\xi-\eta|^2)\qquad \forall\,\xi,\eta\in\mathbb{R},$$

we get

$$\begin{aligned}&\|\langle\partial_3\rangle e^{\rho_0|\partial_3|}(u_b\cdot\nabla u_b)\|^2_{L^2(\mathcal{C})}=\int_{\mathbb{R}}\langle\xi\rangle^2 e^{2\rho_0|\xi|}\|\mathcal{F}(u_b\cdot\nabla u_b)\|^2_{L^2(D_1)}d\xi\\ &=\int_{\mathbb{R}}\langle\xi\rangle^2 e^{2\rho_0|\xi|}\|\int_{\mathbb{R}}\mathcal{F}u_b(\eta)\cdot\mathcal{F}(\nabla u_b)(\xi-\eta)d\eta\|^2_{L^2(D_1)}d\xi\\ &\le C\int_{\mathbb{R}}\Big\|\int_{\mathbb{R}}\langle\eta\rangle e^{\rho_0|\eta|}|\mathcal{F}(u_b)(\eta)|\,\langle\xi-\eta\rangle e^{\rho_0|\xi-\eta|}|\mathcal{F}(\nabla u_b)(\xi-\eta)|d\eta\Big\|^2_{L^2(D_1)}d\xi\\ &\le C\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\big\|\langle\eta\rangle e^{\rho_0|\eta|}|\mathcal{F}(u_b)(\eta)|\big\|_{L^\infty(D_1)}\big\|\langle\xi-\eta\rangle e^{\rho_0|\xi-\eta|}|\mathcal{F}(\nabla u_b)(\xi-\eta)|\big\|_{L^2(D_1)}\,d\eta\right)^2 d\xi\\ &\le C\big\|\langle\xi\rangle e^{\rho_0|\xi|}|\mathcal{F}(u_b)(\xi)|\big\|^2_{L^1_\xi(L^\infty(D_1))}\big\|\langle\xi\rangle e^{\rho_0|\xi|}|\mathcal{F}(\nabla u_b)(\xi)|\big\|^2_{L^2_\xi(L^2(D_1))}.\end{aligned}$$

By using again the Plancherel theorem and Lemma 6.1, one has

$$\begin{aligned}\big\|\langle\xi\rangle e^{\rho_0|\xi|}|\mathcal{F}(u_b)(\xi)|\big\|_{L^1_\xi(L^\infty(D_1))} &\le C\int_{\mathbb{R}}\big\|\langle\xi\rangle e^{\rho_0|\xi|}|\mathcal{F}(\nabla_{\rm h}u_b)(\xi)|\big\|_{L^2(D_1)}\,d\xi\\ &\le C\left(\int_{\mathbb{R}}\big\|\langle\xi\rangle^2 e^{\rho_0|\xi|}|\mathcal{F}(\nabla_{\rm h}u_b)(\xi)|\big\|^2_{L^2(D_1)}\,d\xi\right)^{\frac12}\\ &\le C\|e^{\rho_0|\partial_3|}u_b\|_{H^3(\mathcal{C})},\end{aligned}$$

so that we deduce from (2.43) that

$$\begin{aligned}&\|\langle\partial_3\rangle e^{\rho_0|\partial_3|}(u_b\cdot\nabla u_b)\|_{L^2(\mathcal{C})}\\ &\le C\|e^{\rho_0|\partial_3|}u_b\|_{H^3(\mathcal{C})}\|\langle\partial_3\rangle e^{\rho_0|\partial_3|}\nabla u_b\|_{L^2(\mathcal{C})}\\ &\quad\le C\|e^{\rho_0|\partial_3|}u_b\|^2_{H^3(\mathcal{C})}\le CC_b^2.\end{aligned}\tag{6.27}$$

By combining the estimates (6.24)–(6.27), we arrive at

$$\sum_{k\in\mathbb{Z}}\left(\int_0^T\|\dot{\Delta}_k(\varepsilon u^{\rm fl}\cdot\nabla u^{\rm fl})_\Phi\|^2\,dt\right)^{\frac12}\le C\varepsilon.\tag{6.28}$$

Finally, by gathering (6.4) and inequalities (6.8), (6.9), (6.13), (6.18)–(6.22) and (6.28), we obtain that there exist a constant C_F such that

$$\sum_{k\in\mathbb{Z}}\left(\int_0^T\|\dot{\Delta}_kF^\varepsilon_\Phi\|^2\,dt\right)^{\frac12}\le C_F\varepsilon^{\frac14},\tag{6.29}$$

which finishes the proof of Proposition 5.4.

7. Proof of Lemma 5.3

This section is devoted to the proof of Lemma 5.3. As in Remark 5.2, we always denote the norm $\|\cdot\|_{L^2(\mathcal{C})}$ by $\|\cdot\|$ in this section.

Proof of Lemma 5.3. We first get, by using Bony's decomposition from [2] in the x_3 variable, that

$$ab = T^{\rm v}_a b + R^{\rm v}(a,b) + T^{\rm v}_b a,$$

where

$$T^{\rm v}_a b \stackrel{\rm def}{=} \sum_{k\in\mathbb{Z}} \dot{S}_{k-1} a \dot{\Delta}_k b, \text{ and } R^{\rm v}(a,b) \stackrel{\rm def}{=} \sum_{k\in\mathbb{Z}} \dot{\Delta}_k a \widetilde{\dot{\Delta}}_k b, \text{ with } \widetilde{\dot{\Delta}}_k b \stackrel{\rm def}{=} \sum_{|k-k'|\leq 1} \dot{\Delta}_{k'} b.$$

For a function $a \in L^2(\mathcal{C})$, we introduce the notation as in [8],

$$a^+ := \mathcal{F}^{-1}_{\xi\to x_3} |\hat{a}|. \tag{7.1}$$

It is easy to observe from Lemma 6.1 and Bernstein's inequality that

$$\begin{aligned} \|\dot{S}_{k-1} a^+_\Phi\|_{L^\infty(\mathcal{C})} &\leq C \sum_{k'\leq k-2} \|\dot{\Delta}_{k'} a^+_\Phi\|_{L^\infty(\mathcal{C})} \\ &\leq C \sum_{k'\leq k-2} 2^{\frac{k'}{2}} \|\nabla_{\rm h} \dot{\Delta}_{k'} a_\Phi\| \leq C \|\nabla_{\rm h} a_\Phi\|_{\dot{B}^{\frac12}}, \end{aligned} \tag{7.2}$$

so that we get, by a similar proof of Lemma 5.7 of [8] and Proposition 5.1 that

$$\begin{aligned} |(\dot{\Delta}_k (T^{\rm v}_a b)_\Phi, \dot{\Delta}_k c_\Phi)| &\leq C \sum_{|k'-k|\leq 1} \|\dot{S}_{k'-1} a^+_\Phi\|_{L^\infty(\mathcal{C})} \|\dot{\Delta}_{k'} b_\Phi\| \|\dot{\Delta}_k c_\Phi\| \\ &\leq C \|\nabla_{\rm h} a_\Phi\|_{\dot{B}^{\frac12}} \sum_{|k'-k|\leq 1} \|\dot{\Delta}_{k'} b_\Phi\| \|\dot{\Delta}_k c_\Phi\|. \end{aligned} \tag{7.3}$$

Hence for any $c_0 > 0$, there exists $C > 0$ such that

$$\begin{aligned} \sum_{k\in\mathbb{Z}} \left(\int_0^t |(\dot{\Delta}_k (T^{\rm v}_a b)_\Phi, \dot{\Delta}_k c_\Phi)| \, ds \right)^{\frac12} &\leq c_0 \sum_{k\in\mathbb{Z}} \left(\int_0^t \|\dot{\Delta}_k c_\Phi\|^2 \, ds \right)^{\frac12} \\ &\quad + C \sum_{k\in\mathbb{Z}} \left(\int_0^t \|\nabla_{\rm h} a_\Phi\|^2_{\dot{B}^{\frac12}} \|\dot{\Delta}_k b_\Phi\|^2 \, ds \right)^{\frac12}. \end{aligned} \tag{7.4}$$

Similarly, by applying Bernstein's inequality and Lemma 6.1, we find

$$\begin{aligned} |(\dot{\Delta}_k (R^{\rm v}(a,b))_\Phi, \dot{\Delta}_k c_\Phi)| &\leq C 2^{\frac{k}{2}} \sum_{k'\geq k-3} \|\widetilde{\dot{\Delta}}_{k'} a^+_\Phi\|_{L^2_{x_3}(L^\infty(D_1))} \|\dot{\Delta}_{k'} b_\Phi\| \|\dot{\Delta}_k c_\Phi\| \\ &\leq C 2^{\frac{k}{2}} \sum_{k'\geq k-3} \|\nabla_{\rm h} \widetilde{\dot{\Delta}}_{k'} a_\Phi\| \|\dot{\Delta}_{k'} b_\Phi\| \|\dot{\Delta}_k c_\Phi\| \\ &\leq C \sum_{k'\geq k-3} 2^{\frac{k-k'}{2}} \|\nabla_{\rm h} a_\Phi\|_{\dot{B}^{\frac12}} \|\dot{\Delta}_{k'} b_\Phi\| \|\dot{\Delta}_k c_\Phi\|. \end{aligned} \tag{7.5}$$

Then we use the Minkowski inequality and the Hölder inequality to get

$$
\begin{aligned}
&\sum_{k\in\mathbb{Z}}\left(\int_0^t |(\dot\Delta_k(R^{\mathrm{v}}(a,b))_\Phi, \dot\Delta_k c_\Phi)|\,ds\right)^{\frac12}\\
&\le c_0\sum_{k\in\mathbb{Z}}\left(\int_0^t \|\dot\Delta_k c_\Phi\|^2\,ds\right)^{\frac12} + C\sum_{k\in\mathbb{Z}}\left(\int_0^t\left(\sum_{k'\ge k-3} 2^{\frac{k-k'}{2}}\|\nabla_{\mathrm{h}} a_\Phi\|_{\dot B^{\frac12}}\|\dot\Delta_{k'} b_\Phi\|\right)^2\right)^{\frac12}\\
&\le c_0\sum_{k\in\mathbb{Z}}\left(\int_0^t \|\dot\Delta_k c_\Phi\|^2\,ds\right)^{\frac12} + C\sum_{k\in\mathbb{Z}}\left(\int_0^t \|\nabla_{\mathrm{h}} a_\Phi\|_{\dot B^{\frac12}}^2\|\dot\Delta_k b_\Phi\|^2\,ds\right)^{\frac12}. \qquad (7.6)
\end{aligned}
$$

For the last term, we observe that

$$
\begin{aligned}
|(\dot\Delta_k(T_b^{\mathrm{v}} a)_\Phi, \dot\Delta_k c_\Phi)| &\le C\sum_{|k'-k|\le1}\|\dot S_{k'-1} b_\Phi\|_{L^2_{x_{\mathrm{h}}}(L^\infty_{x_3})}\|\dot\Delta_{k'} a_\Phi^+\|_{L^\infty_{x_{\mathrm{h}}}(L^2_{x_3})}\|\dot\Delta_k c_\Phi\|\\
&\le C\sum_{|k'-k|\le1}\sum_{\ell\le k'-2} 2^{\frac{\ell}{2}}\|\dot\Delta_\ell b_\Phi\|\|\nabla_{\mathrm{h}}\dot\Delta_{k'} a_\Phi\|\|\dot\Delta_k c_\Phi\|\\
&\le C\sum_{\ell\le k-1} 2^{\frac{\ell-k}{2}}\|\dot\Delta_\ell b_\Phi\|\|\nabla_{\mathrm{h}} a_\Phi\|_{\dot B^{\frac12}}\|\dot\Delta_k c_\Phi\|.
\end{aligned}
$$

Thus we get

$$
\begin{aligned}
&\sum_{k\in\mathbb{Z}}\left(\int_0^t |(\dot\Delta_k(T_b^{\mathrm{v}} a)_\Phi, \dot\Delta_k c_\Phi)|\,ds\right)^{\frac12}\\
&\le c_0\sum_{k\in\mathbb{Z}}\left(\int_0^t \|\dot\Delta_k c_\Phi\|^2\,ds\right)^{\frac12} + C\sum_{k\in\mathbb{Z}}\left(\int_0^t\left(\sum_{\ell\le k-1} 2^{\frac{\ell-k}{2}}\|\nabla_{\mathrm{h}} a_\Phi\|_{\dot B^{\frac12}}\|\dot\Delta_l b_\Phi\|\right)^2 ds\right)^{\frac12}\\
&\le c_0\sum_{k\in\mathbb{Z}}\left(\int_0^t \|\dot\Delta_k c_\Phi\|^2\,ds\right)^{\frac12} + C\sum_{k\in\mathbb{Z}}\left(\int_0^t \|\nabla_{\mathrm{h}} a_\Phi\|_{\dot B^{\frac12}}^2\|\dot\Delta_k b_\Phi\|^2\,ds\right)^{\frac12}. \qquad (7.7)
\end{aligned}
$$

By combining the estimates (7.4),(7.6) and (7.7), we conclude the proof of the first part of Lemma 5.3. □

Let us now turn to the case where $a = b$. First, observing from (7.2) that

$$
\|S_{k-1} a_\Phi^+\|_{L^\infty(\mathcal{C})} \le C2^{\frac{k}{2}}\|\nabla_{\mathrm{h}} a_\Phi\|_{\dot B^0},
$$

we get, by a similar derivation of (7.3), that

$$
|(\dot\Delta_k(T_a^{\mathrm{v}} a)_\Phi, \dot\Delta_k c_\Phi)| \le C2^{\frac{k}{2}}\|\nabla_{\mathrm{h}} a_\Phi\|_{\dot B^0}\|\dot\Delta_k c_\Phi\|\sum_{|k'-k|\le1}\|\dot\Delta_{k'} a_\Phi\|,
$$

which inplies

$$
\begin{aligned}
&\sum_{k\in\mathbb{Z}}\left(\int_0^t |(\dot\Delta_k(T_a^{\mathrm{v}} a)_\Phi, \dot\Delta_k c_\Phi)|\,ds\right)^{\frac12}\\
&\quad\le c_0\sum_{k\in\mathbb{Z}}\left(\int_0^t \|\dot\Delta_k c_\Phi\|\,ds\right)^{\frac12} + C\sum_{k\in\mathbb{Z}}\left(\int_0^t 2^k\|\nabla_{\mathrm{h}} a_\Phi\|_{\dot B^0}^2\|\dot\Delta_k a_\Phi\|^2\,ds\right)^{\frac12}. \qquad (7.8)
\end{aligned}
$$

We deduce from the third inequality of (7.5) that

$$|(\dot{\Delta}_k(R^{\mathrm{v}}(a,a))_\Phi, \dot{\Delta}_k c_\Phi)| \le C \sum_{k' \ge k-3} 2^{\frac{k}{2}} \|\nabla_{\mathrm{h}} a_\Phi\|_{\dot{B}^0} \|\dot{\Delta}_{k'} a_\Phi\| \|\dot{\Delta}_k c_\Phi\|.$$

Then we can use the Minkowski inequality, again, to find that

$$\sum_{k\in\mathbb{Z}} \Big(\int_0^t |(\dot{\Delta}_k(R^{\mathrm{v}}(a,a))_\Phi, \dot{\Delta}_k c_\Phi)|\,ds\Big)^{\frac12} \le c_0 \sum_{k\in\mathbb{Z}} \Big(\int_0^t \|\dot{\Delta}_k c_\Phi\|\,ds\Big)^{\frac12} + C\sum_{k\in\mathbb{Z}} \Big(\int_0^t 2^k \|\nabla_{\mathrm{h}} a_\Phi\|_{\dot{B}^0}^2 \|\dot{\Delta}_k a_\Phi\|^2\,ds\Big)^{\frac12}, \tag{7.9}$$

which finishes the proof of the second part of Lemma 5.3. □

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