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Liouville Theorems for a Stationary and Non-stationary Coupled System of Liquid Crystal Flows in Local Morrey Spaces

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Abstract. We consider here the simplified Ericksen–Leslie system on the whole space $\mathbb{R}^3$. This system deals with the incompressible Navier–Stokes equations strongly coupled with a harmonic map flow which models the dynamical behavior for nematic liquid crystals. For both, the stationary (time independent) case and the non-stationary (time dependent) case, using the fairly general framework of a kind of local Morrey spaces, we obtain some a priori conditions on the unknowns of this coupled system to prove that they vanish identically. This results are known as Liouville-type theorems. As a bi-product, our theorems also improve some well-known results on Liouville-type theorems for the particular case of classical Navier–Stokes equations.

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1. Introduction

In this paper, we consider a coupled system of the incompressible Navier–Stokes equations with a harmonic map flow, which is posed on the whole space $\mathbb{R}^3$. This system, also known as the *simplified Ericksen–Leslie system*, was proposed by Lin in [26] as a simplification of the general *Ericksen–Leslie system* which models the hydrodynamic flow of nematic liquid crystal material [11,25]. The simplified Ericksen–Leslie system, has been successful to model various dynamical behavior for nematic liquid crystals. More precisely, it provides a well macroscopic description of the evolution of the material under the influence of fluid velocity field, and moreover, it provides the macroscopic description of the microscopic orientation of fluid velocity of rod-like liquid crystals. See the book [10] for more details.

From the mathematical point of view, the simplified Ericksen–Leslie system has recently attired a lot of interest in the research community, see, *e.g.*, the articles [19,27–29,35] and the references therein, where the major challenge is due to the *strong* coupled structure of this system and the presence of a super-critical non-linear term.

In the *stationary* setting, the simplified Ericksen–Leslie system is given as follows:

$$\begin{cases} -\Delta \mathbf{u} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} + \operatorname{div}(\nabla\otimes\mathbf{v}\odot\nabla\otimes\mathbf{v}) + \nabla p = 0, \\ -\Delta\mathbf{v} + (\mathbf{u}\cdot\nabla)\,\mathbf{v} - |\nabla\otimes\mathbf{v}|^2\,\mathbf{v} = 0, \\ \operatorname{div}(\mathbf{u}) = 0. \end{cases} \tag{1}$$

Here, the fluid velocity $\mathbf{u}:\mathbb{R}^3\to\mathbb{R}^3$, and the pressure $p:\mathbb{R}^3\to\mathbb{R}$ are the classical unknowns of the fluid mechanics. This system also considers a third unknown $\mathbf{v}:\mathbb{R}^3\to\mathbb{S}^2$ (where $\mathbb{S}^2$ denotes the unitary sphere in $\mathbb{R}^3$) which is a *unit vector field* representing the macroscopic orientation of the nematic liquid crystal molecules. For the vector field $\mathbf{v}=(v_i)_{1\leq i\leq 3}$, we denote $\nabla\otimes\mathbf{v}=(\partial_i v_j)_{1\leq i,j\leq 3}$. In the first equation of this system, the *super-critical* non-linear term: $\operatorname{div}(\nabla\otimes\mathbf{v}\odot\nabla\otimes\mathbf{v})$, is given as the divergence of a symmetric tensor $\nabla\otimes\mathbf{v}\odot\nabla\otimes\mathbf{v}$, where, for $1\leq i,j\leq 3$, its components are defined by the expression

$(\nabla\otimes\mathbf{v}\odot\nabla\otimes\mathbf{v})_{i,j}=\sum_{k=1}^{3}\partial_i v_k\partial_j v_k$, and then, each component of the vector field $\operatorname{div}(\nabla\otimes\mathbf{v}\odot\nabla\otimes\mathbf{v})$ is explicitly given by the following expression: $[\operatorname{div}(\nabla\otimes\mathbf{v}\odot\nabla\otimes\mathbf{v})]_i=\sum_{j=1}^{3}\sum_{k=1}^{3}\partial_j(\partial_i v_k\,\partial_j v_k)$. We may observe that due to the double derivatives in this expression, this super-critical non-linear term is actually more *delicate* to treat than the classical non-linear transport term: $(\mathbf{u}\cdot\nabla)\,\mathbf{u}$, and this fact makes challenging the mathematical study of (1). See, *e.g.*, the works [27] and [29].

Let us introduce the Liouville-type problem for the simplified Ericksen–Leslie system. First, we define a weak solution of the coupled system (1) as the triplet $(\mathbf{u},p,\mathbf{v})$ where: $\mathbf{u}\in L^2_{loc}(\mathbb{R}^3)$, $p\in\mathcal{D}^{'}(\mathbb{R}^3)$ and $\mathbf{v}\in L^\infty(\mathbb{R}^3)$ (since by the *physical model* we assume $|\mathbf{v}|=1$) and $\nabla\otimes\mathbf{v}\in L^2_{loc}(\mathbb{R}^3)$. Under these hypothesis all the terms in (1) are well-defined in the distributional sense. Once we have defined the weak solutions, we may remark that the triplet $\mathbf{u}=0$, $p=0$ and $\nabla\otimes\mathbf{v}=0$ (hence $\mathbf{v}$ is a constant unitary vector) is a trivial weak solution of the system (1) and it is natural to ask if this solution is unique (modulo constants). However, it is interesting to observe that the answer to this question is in general *negative* and we are able to exhibit an explicit counterexample. See Appendix A for the computations.

Due to the non-uniqueness of the trivial solution, we are interesting in finding some additional a priori conditions in order to ensure its uniqueness . This problem is commonly known as the *Liouville-type problem*. To the best of our knowledge, the first Liouville-type result for the coupled system (1) was recently obtained by Y. Hao, X. Liu & X. Zhang in [19]. In this work, the authors consider a solution $(\mathbf{u},p,\mathbf{v})$ which verifies $\nabla\otimes\mathbf{u}\in L^2(\mathbb{R}^3)$ and $\nabla\otimes\mathbf{v}\in L^2(\mathbb{R}^3)$ and moreover, under the important assumption: $\mathbf{u}\in L^{9/2}(\mathbb{R}^3)$ and $\nabla\otimes\mathbf{v}\in L^{9/2}(\mathbb{R}^3)$, they obtained the identities $\mathbf{u}=0$, $p=0$ and $\nabla\otimes\mathbf{v}=0$. These a priori conditions are *decaying properties* on $\mathbf{u}$ and $\nabla\otimes\mathbf{v}$ given by the $L^{9/2}-$ norm; and they are interesting if we compare this result with a well-known result on the Liouville problem for the the classical stationary and incompressible Navier–Stokes equations:

$$-\Delta\mathbf{u}+(\mathbf{u}\cdot\nabla)\,\mathbf{u}+\nabla\,p=0,\quad \operatorname{div}(\mathbf{u})=0. \tag{2}$$

For these equations, a celebrated result obtained in [16] by G. Galdi shows that if $\mathbf{u}\in L^{9/2}(\mathbb{R}^3)$ then we have $\mathbf{u}=0$ and $p=0$, and then, the recent result obtained in [19] can be seen as a generalization of Galdi's result to the more complicated setting of the coupled system (1).

Let us recall that the Liouville problem for the stationary Navier–Stokes equations (2) was extensive studied in different functional settings. Galdi's result [16] was extended to setting of the Lorentz spaces by H. Kozono *et. al.* in [22]. Thereafter, this work was improved to a kind of local Lorentz-type spaces by G. Seregin & W. Wang in [33]. Moreover, the Liouville problem for the equations (2) has also largely studied in the more general setting of the Morrey spaces by D. Chamorro *et. al.* in [9] and G. Seregin in [31] and [32]. For more interesting works on the Liouville problem for the stationary Navier–Stokes equations (2) see the articles [5–7,21] and the references therein.

It is natural to improve the Galdi's-type result for the system (1) obtained in [19] to different functional settings. Thus, the first aim of this paper is to study the Liouville problem for the coupled system (1) in a fairly general functional setting.

A kind of *local Morrey spaces* (see the expression (7) below for a definition) which, roughly speaking, characterize the averaged decaying properties of functions, have recently attired the attention in the study of the existence of global in time weak solutions for the classical the Navier–Stokes [3,14], and also for the coupled system of the Magneto-hydrodynamics equations [12,13]. In this paper we show that the *local Morrey spaces* also give us an interesting and general setting to solve the Liouville problem for the coupled system (1). As a bi-product, since the equations (2) are a particular case of the system (1) (when we set $\mathbf{v}$ an unitary constant vector) we also improve some well-known and recent results on the Liouville problem for (2).

Our methods essentially rely on some L^p- local estimates on the functions $\mathbf{u}$ and $\nabla\otimes\mathbf{v}$, and this approach also allows us to study the Liouville problem for *non-stationary* case of the coupled system (1). Thus, in the second part of this paper, we will focus on the following Cauchy problem for the simplified

Ericksen–Leslie system:

$$\begin{cases} \partial_t \mathbf{u} - \Delta \mathbf{u} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} + \operatorname{div}(\nabla\otimes\mathbf{v}\odot\nabla\otimes\mathbf{v}) + \nabla p = 0, \\ \partial_t \mathbf{v} - \Delta \mathbf{v} + (\mathbf{u}\cdot\nabla)\,\mathbf{v} - |\nabla\otimes\mathbf{v}|^2\,\mathbf{v} = 0, \quad (|\mathbf{v}(t,x)| = 1) \\ \operatorname{div}(\mathbf{u}) = 0, \\ \mathbf{u}(0,\cdot) = \mathbf{u}_0, \quad \mathbf{v}(0,\cdot) = \mathbf{v}_0 \quad \operatorname{div}(\mathbf{u}_0) = 0. \end{cases} \tag{3}$$

For the non-stationary case the Liouville-type problem reads as follows: if we consider the initial data $\mathbf{u}_0 = 0$ and $\nabla\otimes\mathbf{v}_0 = 0$, *i.e.*, $\mathbf{v}_0$ is a constant vector, then we ask if the trivial solution $\mathbf{u} = 0$ and $\nabla\otimes\mathbf{v} = 0$ (hence is $\mathbf{v}$ a constant vector) is the unique one; and thus, we are interested in studying some a priori conditions on $\mathbf{u}$ and $\nabla\otimes\mathbf{v}$ to ensure the uniqueness of the trivial solutions arising from the data $\mathbf{u}_0 = 0$ and $\nabla\otimes\mathbf{v}_0 = 0$.

Our general strategy to study this problem is the following: first we look for some a priori conditions on the data $\mathbf{u}_0$ and $\nabla\otimes\mathbf{v}_0$, and moreover, some a priori conditions on the solutions $\mathbf{u}$ and $\nabla\otimes\mathbf{v}$ to prove that they verify a *global energy inequality* (see (17) for the details). Thereafter, with this global energy inequality at hand, the Liouville-type problem explained above can be easily solved when $\mathbf{u}_0 = 0$ and $\nabla\otimes\mathbf{v}_0 = 0$.

Before to explain this strategy more in details, we need first to overview some previous results obtained in the particular case for the Cauchy problem of the incompressible Navier–Stokes equations:

$$\begin{cases} \partial_t \mathbf{u} - \Delta \mathbf{u} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} + \nabla p = 0, \quad \operatorname{div}(\mathbf{u}) = 0. \\ \mathbf{u}(0,\cdot) = \mathbf{u}_0, \quad \operatorname{div}(\mathbf{u}_0) = 0. \end{cases} \tag{4}$$

For this system, J. Serrin proved in [34] that if $\mathbf{u}_0 \in L^2(\mathbb{R}^3)$ and if a Leray weak solution $\mathbf{u}$ satisfies the condition $\mathbf{u} \in L^p(0,T,L^r(\mathbb{R}^3))$, for $p > 2$ and $r > 3$ such that $2/p + 3/r \leq 1$, then $\mathbf{u}$ verifies the *global energy equality*. Thereafter, this result was generalized by H. Kozono *et. al.* in [22] as follows. Recall first that the notion of *weak suitable solutions* for the equations (4) were introduced in the celebrated Cafarelli, Konh and Niremberg theory [4]. Then, in [22], it is introduced the notion of *generalized weak suitable solution* (see Definition 3.1, page 5 of [22]). This notion of generalized weak suitable solution is a generalization of the well-known weak suitable solutions and the main difference is that it assumes neither finite energy: $\sup\limits_{0\leq t\leq T} \|\mathbf{u}(t,\cdot)\|^2_{L^2} < +\infty$, nor finite dissipation $\displaystyle\int_0^T \|\mathbf{u}(t,\cdot)\|^2_{\dot{H}^1} dt < +\infty$. In the setting of the *generalized weak suitable solution*, H. Kozono *et. al.* gave a *new a priori condition* which ensures that the well-know *global energy inequality* holds. More precisely, assuming that $\mathbf{u}_0 \in L^2(\mathbb{R}^3)$ and moreover, within the general framework of the Lorentz spaces and for the parameters $3 \leq p_1, r_1, p_2, r_2 \leq +\infty$ satisfying some technical conditions related to the well-known scaling properties of the equations (4), the condition $\mathbf{u} \in L^3(0,T,L^{p_1,r_1}(\mathbb{R}^3)) \cap L^2(0,T,L^{p_2,r_2}(\mathbb{R}^3))$ ensures that $\mathbf{u} \in L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x(]0,T[\times\mathbb{R}^3)$ and moreover it verifies the global energy inequality, *i.e.*, $\mathbf{u}$ becomes a Leray weak solution.

Following these ideas, in Definition 2.1 below, we will introduce first a notion of *generalized weak suitable solutions* $(\mathbf{u}, p, \mathbf{v})$ for the coupled system (3). Thereafter, assuming that $\mathbf{u}_0 \in L^2(\mathbb{R}^3)$, $\mathbf{v}_0 \in \dot{H}^1(\mathbb{R}^3)$, and moreover, using a time-space version of the *local Morrey spaces* (see the expression (14) below for a definition), we will give some a priori conditions on $\mathbf{u}$ and $\nabla\otimes\mathbf{v}$ to ensure that, for a time $0 < T < +\infty$ arbitrary large, the *generalized weak suitable solutions* of (3) verify a *global energy inequality* (17). As an interesting application, we obtain some *Liouville-type results* for the non-stationary system (3). More precisely, using the global energy inequality we are able to prove the uniqueness of the trivial solution $\mathbf{u} = 0$, $p = 0$ and $\nabla\otimes\mathbf{v} = 0$ for the initial data $\mathbf{u}_0 = 0$ and $\nabla\otimes\mathbf{v}_0 = 0$.

This paper is organized as follows. In Sect. 2 below we expose all the results obtained. Then, in Sect. 3 we summarize some previous results on the local Morrey spaces we shall use here. Section 4 is devoted to a characterization of the pressure term in the coupled systems (1) and (3) which will be useful for the next sections. Finally, in Sect. 5 we study the Liouville problem for the stationary system (1) and in Sect. 6 we study the Liouville problem for the non-stationary system (3).

2. Framework and Statement of the Results

2.1. The Stationary Case

Recall first that in [19], in order to solve the Liouville problem for (1), the authors need the additional hypothesis on the function $\mathbf{v}$: $\nabla \otimes \mathbf{v} \in L^2(\mathbb{R}^3)$. In our results we will *relax* this hypothesis as follows: for $R \geq 1$ we denote $\mathcal{C}(R/2, R) = \{x \in \mathbb{R}^3 : R/2 < |x| < R\}$; and from now on we will assume

$$\sup_{R\geq 1} \int_{\mathcal{C}(R/2,R)} |\nabla \otimes \mathbf{v}|^2 dx < +\infty. \tag{5}$$

Before to state our results, we recall the definition of the Morrey spaces and local Morrey spaces. For more references about this spaces see, *e.g.*, the Chapter 8 of the book [24] and the Section 7 of the paper [15] respectively. Let $1 < p < r < +\infty$, the homogeneous Morrey space $\dot{M}^{p,r}(\mathbb{R}^3)$ is the set of functions $f \in L^p_{loc}(\mathbb{R}^3)$ such that

$$\|f\|_{\dot{M}^{p,r}} = \sup_{R>0,\ x_0\in\mathbb{R}^3} R^{\frac{3}{r}} \left(\frac{1}{R^3} \int_{B(x_0,R)} |f(x)|^p dx \right)^{\frac{1}{p}} < +\infty, \tag{6}$$

where $B(x_0, R)$ denotes the ball centered at x_0 and with radio R. This is a homogeneous space of degree $-\frac{3}{r}$ and moreover we have the following chain of continuous embedding $L^r(\mathbb{R}^3) \subset L^{r,q}(\mathbb{R}^3) \subset \dot{M}^{p,r}(\mathbb{R}^3)$, where, for $r \leq q \leq +\infty$ the space $L^{r,q}(\mathbb{R}^3)$ is a Lorentz space [8].

We observe that in expression (6) we consider the average in terms of the L^p- norm of the function f on the ball $B(x_0, R)$; and the term $R^{\frac{3}{q}}$ describes the decaying of this averaged quantity when R is large.

The local Morrey spaces we shall consider here describes the averaged decaying of functions in a more general setting. For $\gamma \geq 0$ and $1 < p < +\infty$, we define the *local Morrey space* $M^p_\gamma(\mathbb{R}^3)$ as the Banach space of functions $f \in L^p_{loc}(\mathbb{R}^3)$ such that

$$\|f\|_{M^p_\gamma} = \sup_{R\geq 1} \left(\frac{1}{R^\gamma} \int_{B(0,R)} |f(x)|^p dx \right)^{1/p} < +\infty. \tag{7}$$

Here the parameter $\gamma \geq 0$ characterizes the behavior of the quantity $\left(\int_{B(0,R)} |f(x)|^p dx \right)^{1/p}$ when R is large. Moreover, for $\gamma_1 \leq \gamma_2$ we have the continuous embedding $M^p_{\gamma_1}(\mathbb{R}^3) \subset M^p_{\gamma_2}(\mathbb{R}^3)$. Remark also that for $1 < p < r < +\infty$, setting the parameter γ such that $3(1 - p/r) < \gamma$, then we have $\dot{M}^{p,r}(\mathbb{R}^3) = M^p_{3(1-p/r)}(\mathbb{R}^3) \subset M^p_\gamma(\mathbb{R}^3)$, and in this sense the local Morrey space $M^p_\gamma(\mathbb{R}^3)$ is as a generalization of the homogeneous Morrey space $\dot{M}^{p,r}(\mathbb{R}^3)$.

Finally, we define the space $M^p_{\gamma,0}(\mathbb{R}^3)$ as the set of functions $f \in M^p_\gamma(\mathbb{R}^3)$ such that

$$\lim_{R\to+\infty} \left(\frac{1}{R^\gamma} \int_{\mathcal{C}(R/2,R)} |f(x)|^p dx \right)^{1/p} = 0. \tag{8}$$

In the setting of the local Morrey spaces $M^p_{\gamma,0}(\mathbb{R}^3)$ and $M^p_\gamma(\mathbb{R}^3)$ defined above we set, from now on, the parameters $0 < \gamma < 3 \leq p < +\infty$. The condition $0 < \gamma < 3$ is required to use some useful properties of the spaces $M^p_\gamma(\mathbb{R}^3)$ which we summarize in Sect. 3. On the other hand, since all our results are based on a *Cacciopoly* type estimate on the term $\vec{\nabla} \otimes \mathbf{u}$ (for more details see the Proposition 5.1) we also need the condition $3 \leq p < +\infty$.

With the parameters $0 < \gamma < 3 \leq p < +\infty$ above, we introduce now the following quantity:

$$\eta = \eta(\gamma, p) = \frac{\gamma}{p} - \frac{3}{p} + \frac{2}{3}, \tag{9}$$

which relates the decaying parameter γ with the local integrability parameter p in the definition of the spaces $M^p_\gamma(\mathbb{R}^3)$ given in (7). Our results deeply depends on the sign of the function $\eta(\gamma,p)$. More precisely, within the *rectangular region* $(\gamma,p)\in]0,3[\times[3,+\infty[$, we will consider first the region of parameters (γ,p) where $\eta(\gamma,p)\leq 0$, and then, the region of parameters (γ,p) where $\eta(\gamma,p)>0$.

In Fig. 1, we draw these regions. In the horizontal (red) axis we have the parameter γ, while in the vertical (green) axis we have the parameter p. Thus, we have $\eta(\gamma,p)>0$ in the blue sky region, we have $\eta(\gamma,p)\leq 0$ in the dark gray region, while $\eta(\gamma,p)$ is not defined in the light gray region queryPlease check and confirm the inserted citation for Fig. 1 is correct..

Theorem 1. *Let $(\boldsymbol{u},p,\boldsymbol{v})$ be a smooth solution of stationary coupled system (1) such that $\boldsymbol{v}$ verifies (5). For $0<\gamma<3\leq p<+\infty$, we assume $\boldsymbol{u}\in M^p_{\gamma,0}(\mathbb{R}^3)$ and $\nabla\otimes\boldsymbol{v}\in M^p_\gamma(\mathbb{R}^3)$.*

1) If (γ,p) are such that $\eta(\gamma,p)\leq 0$, then we have $\boldsymbol{u}=0$, $\nabla\otimes\boldsymbol{v}=0$ and $p=0$.

2) If (γ,p) are such that $\eta(\gamma,p)>0$, and if the velocity $\boldsymbol{u}$ also verifies the following additional decaying condition:

$$\lim_{R\to+\infty} R^{3\eta(\gamma,p)}\left(\frac{1}{R^\gamma}\int_{\mathcal{C}(R/2,R)}|\boldsymbol{u}(x)|^p dx\right)^{1/p}=0, \tag{10}$$

then we have $\boldsymbol{u}=0$, $p=0$ and $\nabla\otimes\boldsymbol{v}=0$.

We observe first that among the two unknowns of the coupled system (1), the velocity $\mathbf{u}$ must have a faster decaying than the derivatives of the vector field $\mathbf{v}$, since we have $\mathbf{u}\in M^p_{\gamma,0}(\mathbb{R}^3)$ and $\nabla\otimes\mathbf{v}\in M^p_\gamma(\mathbb{R}^3)$.

In point 1), for (γ,p) such that $\eta(\gamma,p)\leq 0$, we solve the Liouville problem for the equations (1) in the local Morrey spaces $M^p_{\gamma,0}(\mathbb{R}^3)$ and $M^p_\gamma(\mathbb{R}^3)$. The main interest of this result bases on the fact that we use a *fairly general framework* to solve this problem. Indeed, we have the following remarks.

- First, if $\eta(\gamma,p)\leq 0$, then for $3<r<9/2$ and $3\leq p<r$ we have the large chain of strict embedding

$$L^r(\mathbb{R}^3)\subset L^{r,\infty}(\mathbb{R}^3)\subset \dot{M}^{p,r}(\mathbb{R}^3)\subset M^p_\delta(\mathbb{R}^3)\subset M^p_{\gamma,0}(\mathbb{R}^3),$$

  involving the Lebesgue, Lorentz, Morrey and local Morrey spaces. Here, the last embedding is due to point 1 of Lemma 3.1 below, where we the parameter δ verifies $3(1-p/r)<\delta<\gamma$.
- Moreover, for the particular values $\gamma=1$ and $p=3$, hence we have $\eta(1,3)=0$, and for $r=9/2$ and $9/2<q<+\infty$, we also have the embedding

$$L^{9/2}(\mathbb{R}^3)\subset L^{9/2,q}(\mathbb{R}^3)\subset M^3_{1,0}(\mathbb{R}^3). \tag{11}$$

  Indeed, if $f\in L^{9/2,q}(\mathbb{R}^3)$ then we have $f\in \dot{M}^{3,9/2}(\mathbb{R}^3)$, but due to the identity $\dot{M}^{3,9/2}(\mathbb{R}^3)=M^3_1(\mathbb{R}^3)$, we get $f\in M^3_1(\mathbb{R}^3)$. Moreover, by the following estimate:

$$\int_{\mathcal{C}(R/2,R)}|f|^3dx=\int_{B(0,R)}\left|\mathbb{1}_{\mathcal{C}(R/2,R)}f\right|^3dx\leq c\,R\left\|\mathbb{1}_{\mathbb{C}(R/2,R)}f\right\|^3_{L^{9/2,\infty}}\leq c\,R\left\|\mathbb{1}_{\mathbb{C}(R/2,R)}f\right\|^3_{L^{9/2,q}},$$

  and using the dominated convergence theorem (which is valid in the space $L^{9/2,q}(\mathbb{R}^3)$ for the values $9/2\leq q<+\infty$, see [8]) we obtain: $\displaystyle\lim_{R\to+\infty}\frac{1}{R}\int_{\mathbb{C}(R/2,R)}|f|^3\,dx=0$, hence we have $f\in M^3_{1,0}(\mathbb{R}^3)$. Due to the embedding given in (11), we may see that the recent result obtained in [19] for the coupled system (1) follows from point 1) in Theorem 1.
- Finally, for the values in the threshold $\eta(\gamma,p)=0$, by the expression (9) we have the identity $\gamma=3-2p/3$, and then, for $0<\gamma<3$ we get $3\leq p<9/2$. Thus, for these values of the parameter p, by the first point in Lemma 3.1 below we have the embedding

$$L^p_{w_\gamma}(\mathbb{R}^3)\subset M^p_{\gamma,0}(\mathbb{R}^3), \tag{12}$$

  where, for $w_\gamma(x)=\dfrac{1}{(1+|x|)^\gamma}$, the weighted Lebesgue space $L^p_{w_\gamma}(\mathbb{R}^3)$ is defined as $L^p_{w_\delta}(\mathbb{R}^3)=L^p(w_\delta\,dx)$.

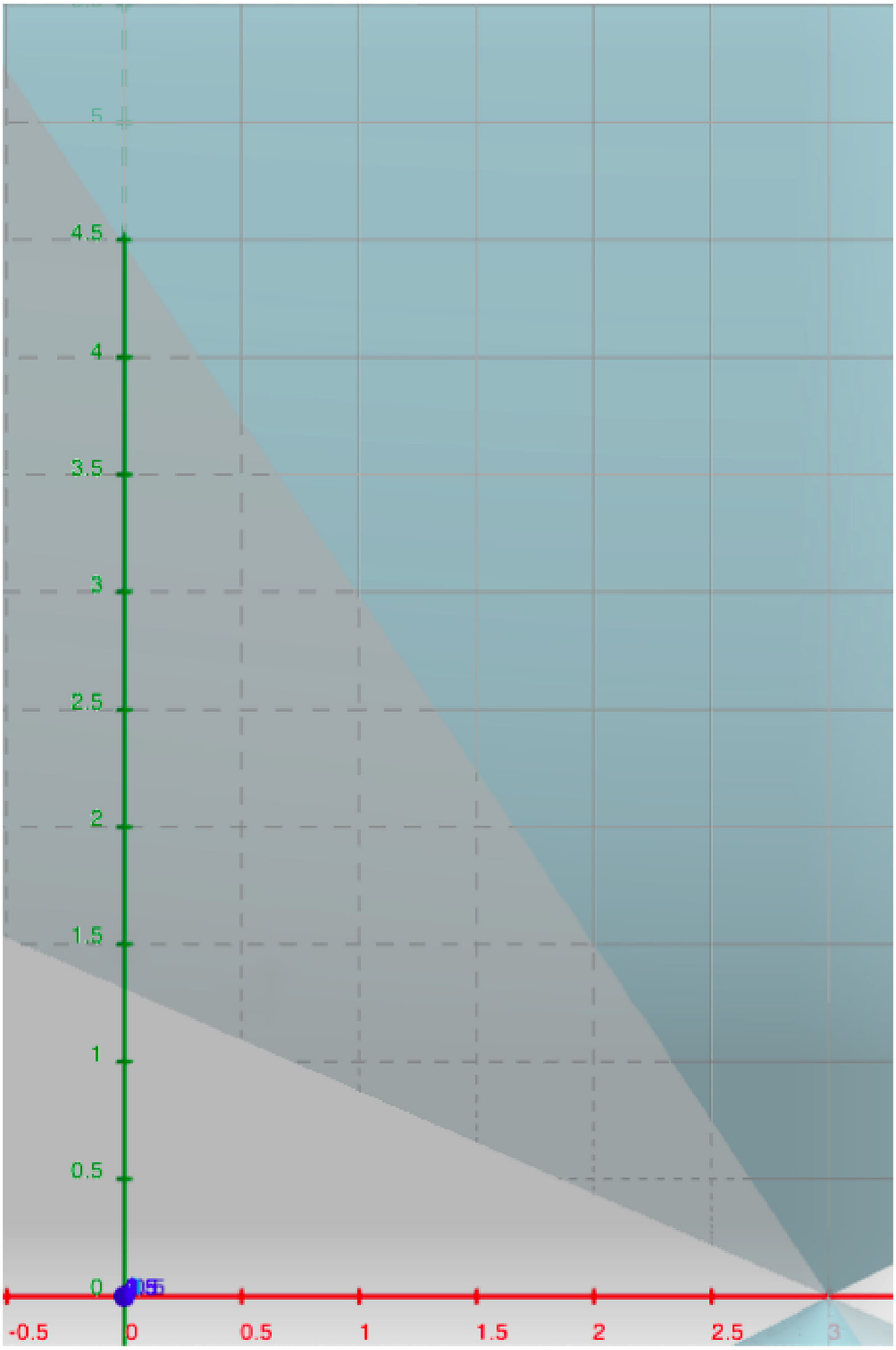

Fig. 1. Signs of $\eta(\gamma, p)$

In point 2) above, we may observe now that for the values (γ, p) where $\eta(\gamma, p) > 0$, the decaying properties given by the space $M^p_{\gamma,0}(\mathbb{R}^3)$ seems not to be sufficient to solve the Liouville problem. More precisely, if we only consider the information $\mathbf{u} \in M^p_{\gamma,0}(\mathbb{R}^3)$, then the velocity $\mathbf{u}$ seems not to decay at infinity fast enough and we need to improve its decaying properties with the expression $R^{3\,\eta(\gamma,p)}$ in (10). Moreover, we remark that this improvement on the decay properties (when $\eta(\gamma, p) > 0$) are only needed for the velocity $\mathbf{u}$ and not for the function $\nabla \otimes \mathbf{v}$.

2.1.1. Some New Results for the Stationary Navier–Stokes Equations. We observe that the stationary coupled Ericksen–Leslies system (1) contains as a particular case the stationary Navier–Stokes equations (2) when setting the unitary vector field $\mathbf{v}$ as a constant vector. Thus, a direct consequence of Theorem 1 is the following new result for the equations (2).

Corollary 1. *Let $(\boldsymbol{u}, p)$ be a smooth solution of the stationary Navier–Stoke equations (2). For $0 < \gamma < 3 \leq p < +\infty$, we assume $\boldsymbol{u} \in M^p_{\gamma,0}(\mathbb{R}^3)$. If (γ, p) are such that $\eta(\gamma, p) \leq 0$, then we have $\boldsymbol{u} = 0$ and $p = 0$.*

It is worth mention how this corollary improves some previous results obtained for the Liouville problem for the stationary Navier–Stoke equations. We observe first that by the embedding (11) our result improves the classical Galdi's result [16] given in the framework of the Lebesgue space $L^{9/2}(\mathbb{R}^3)$ and some recent results [20] obtained in the framework of the Lorentz spaces $L^{9/2,q}(\mathbb{R}^3)$, with $9/2 < q < +\infty$. Moreover, due to the embedding $M^{p,r}(\mathbb{R}^3) \subset M^p_{\gamma,0}(\mathbb{R}^3)$, with $3 < r < 9/2$, our result improves some previous results obtained in the setting of the Morrey spaces in [9] and [20]. Finally, due to the embedding (12), our result also improves some results proven in [30] (see Remark 4.9, page 10) in the setting of weighted spaces.

On the other hand, we are also interested in studying the effects of removing the condition (8) on the velocity $\mathbf{u} \in M^p_{\gamma,0}(\mathbb{R}^3)$; and we consider now $\mathbf{u} \in M^p_{\gamma}(\mathbb{R}^3)$. Within the framework of the larger space $M^p_{\gamma}(\mathbb{R}^3)$, we have the following result.

Proposition 1. *Let $(\boldsymbol{u}, p)$ be a smooth solution of the stationary Navier–Stoke equations (2). For $0 < \gamma < 3 \leq p < +\infty$, we assume $\boldsymbol{u} \in M^p_{\gamma}(\mathbb{R}^3)$, and moreover, we assume that (γ, p) are such that $\eta(\gamma, p) \leq 0$.*

1) *In the case when $\eta(\gamma, p) < 0$, we have $\boldsymbol{u} = 0$ and $p = 0$.*
2) *In the case when $\eta(\gamma, p) = 0$, if there holds $\boldsymbol{u} \in \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)$, then we have $\boldsymbol{u} = 0$ and $p = 0$.*

We observe here that, on the one hand, when $\eta(\gamma, p) < 0$ the condition (8) actually is not required to solve the Liouville problem. On the other hand, when $\eta(\gamma, p) = 0$ we need a supplementary hypothesis on this vector field to ensure its vanishing; and this fact suggests the acuteness of (8).

The supplementary condition is given in the framework of the homogeneous Besov space $\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)$, defined as the set of distributions $f \in \mathcal{S}'(\mathbb{R}^3)$ such that $\|f\|_{\dot{B}^{-1}_{\infty,\infty}} = \sup_{t>0} t^{1/2}\|h_t * f\|_{L^\infty} < +\infty$, where h_t denotes the heat kernel. This space plays a very important role in the analysis on the Navier–Stokes equations (stationary and non stationary) since this is the largest space which is invariant under scaling properties of these equations. See, for instance, the article [2] and the books [23] and [24].

2.2. The Non-stationary Case

From now on, let us fix a time $0 < T < +\infty$. We start by introducing the notion of *generalized weak suitable solution* for the non-stationary Ericksen–Leslie system (3).

Definition 2.1. Let $\mathbf{u}_0 \in L^2(\mathbb{R}^3)$ such that $div(\mathbf{u}_0) = 0$ and let $\mathbf{v}_0 \in \dot{H}^1(\mathbb{R}^3)$. We say that the triplet $(\mathbf{u}, p, \mathbf{v})$ is a generalized weak suitable solution of the coupled system (3) if:

1) $\mathbf{u} \in L^3_{loc}([0, T[\times\mathbb{R}^3)$, $\nabla \otimes \mathbf{u} \in L^3_{loc}([0, T[\times\mathbb{R}^3)$ and $p \in L^{3/2}_{loc}([0, T[\times\mathbb{R}^3)$.
2) $\mathbf{v} \in L^\infty_{loc}([0, T[, L^\infty(\mathbb{R}^3))$, $\nabla \otimes \mathbf{v} \in L^3_{loc}([0, T[\times\mathbb{R}^3)$ and $\Delta\mathbf{v} \in L^3_{loc}([0, T[\times\mathbb{R}^3)$.
3) The triplet $(\mathbf{u}, p, \mathbf{v})$ verifies the first three equations of (3) in $\mathcal{D}'(]0, T[\times\mathbb{R}^3)$.

4) For every compact set $K \subset \mathbb{R}^3$, the function $\mathbf{u}(t,\cdot)$ is continuous for $t \in]0,T[$ in the weak topology of $L^2(K)$ and strongly continuous at $t=0$. Moreover, the function $\mathbf{v}(t,\cdot)$ is continuous for $t \in]0,T[$ in the weak topology of $\dot{H}^1(K)$ and strongly continuous at $t=0$.
5) The triplet $(\mathbf{u}, p, \mathbf{v})$ verifies the following *local energy inequality*: there exist a *non-negative*, locally finite measure μ on $]0,T[\times\mathbb{R}^3$ such that:

$$\partial_t\left(\frac{|\mathbf{u}|^2+|\nabla\otimes\mathbf{v}|^2}{2}\right)+|\nabla\otimes\mathbf{u}|^2=-|\Delta\mathbf{v}|^2+\Delta\left(\frac{|\mathbf{u}|^2+|\nabla\otimes\mathbf{v}|^2}{2}\right)$$
$$-\text{div}\left(\left[\frac{|\mathbf{u}|^2+|\nabla\otimes\mathbf{v}|^2}{2}+p\right]\mathbf{u}\right)-\sum_{k=1}^{3}\partial_k([\mathbf{u}\cdot\nabla)\,\mathbf{v}]\cdot\partial_k\mathbf{v})-|\nabla\otimes\mathbf{v}|^2\mathbf{v}\cdot\Delta\mathbf{v}-\mu. \quad (13)$$

Observe that in point 2) we assume $\mathbf{v} \in L^\infty_{loc}([0,T[, L^\infty(\mathbb{R}^3))$ due to the fact that by the *physical model* we have $|\mathbf{v}(t,x)| = 1$. Observe moreover that by the hypothesis given in points 1) and 2) we have that μ is well-defined in the distributional sense. However, the most important fact in this definition is the positivity assumed on μ which is the whole point in the notion of *suitable* solutions.

This notion of generalized weak suitable solution is close to the definition of a weak suitable solution for the coupled system (3) given in [28] (for the case of a bounded and smooth domain $\Omega \subset \mathbb{R}^3$). In comparison with [28], it is worth to remark that here we suppose neither $\mathbf{u} \in L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x$, nor $\mathbf{v} \in L^\infty_t \dot{H}^1_x$, and we consider here only *locally integrable properties*. Moreover, we assume on the pressure term a local $L^{3/2}-$ integrability, while the authors in [28] assume a $L^{5/3}-$ integrability.

We introduce now a time-space version of the *local Morrey spaces*, for more references on these spaces see always the Section 7 of [15]. For $\gamma > 0$ and $1 < p < +\infty$, we define the space $M^p_\gamma L^p(0,T)$ as the Banach space of functions $f \in L^p_{loc}([0,T]\times\mathbb{R}^3)$ such that

$$\|f\|_{M^p_\gamma L^p(0,T)} = \sup_{R\geq 1}\left(\frac{1}{R^\gamma}\int_0^T\int_{B(0,R)}|f(t,x)|^p dx\,dt\right)^{1/p} < +\infty. \quad (14)$$

Moreover, we define the space $M^p_{\gamma,0}L^p(0,T)$ as the set of functions $f \in M^p_\gamma L^p(0,T)$ which verifies

$$\lim_{R\to+\infty}\left(\frac{1}{R^\gamma}\int_0^T\int_{\mathcal{C}(R/2,R)}|f(t,x)|^p dx\,dt\right)^{1/p} = 0. \quad (15)$$

In Definition 2.1 we observe that we need to handle the pressure p and for this, before to state our next result, it is useful to give first the following characterization of the pressure term.

Proposition 2.1. *Let $(\boldsymbol{u}, p, \boldsymbol{v})$ be a solution of the coupled system (3) such that, for $0<\gamma<3$, $2<p<+\infty$, it verifies $\boldsymbol{u} \in M^p_\gamma L^p(0,T)$, $p \in \mathcal{D}'([0,T]\times\mathbb{R}^3)$ and $\nabla\otimes\boldsymbol{v} \in M^p_\gamma L^p(0,T)$. Then, the term ∇p is necessary related to $\boldsymbol{u}$ and $\nabla\otimes\boldsymbol{v}$ through the Riesz transforms $\mathcal{R}_i = \frac{\partial_i}{\sqrt{-\Delta}}$ by the formula*

$$\nabla p = \nabla\left(\sum_{i,j=1}^{3}\mathcal{R}_i\mathcal{R}_j(u_i\,u_j)+\sum_{i,j,k=1}^{3}\mathcal{R}_i\mathcal{R}_j\left(\partial_i v_k\,\partial_j v_k\right)\right). \quad (16)$$

Here, in the general setting of the time-space local Morrey spaces, we show that the pressure p is always related to the velocity $\mathbf{u}$ and the derivatives of the vector field $\mathbf{v}$. This results has also an independent interest when seeking for very general frameworks in which the pressure is related to the other unknowns in the equations (3). See, for instance, [1] and [15], for related works in the case of the Navier–Stokes equations (4).

As mentioned in the introduction, in our next result we give some a priori conditions on the generalized weak suitable solutions defined above to ensure that these solutions verify a global energy inequality.

Theorem 2. *Let $\boldsymbol{u}_0 \in L^2(\mathbb{R}^3)$, with $div(\boldsymbol{u}_0)=0$, and let $\boldsymbol{v}_0 \in \dot{H}^1(\mathbb{R}^3)$ be the initial data. Let $0<T<+\infty$, and let $(\boldsymbol{u}, p, \boldsymbol{v})$ be a* generalized weak suitable solution *of the non-stationary coupled system (3) given in Definition 2.1.*

For $0<\gamma<3\leq p<+\infty$, we assume $\boldsymbol{u}\in M^p_{\gamma,0}L^p(0,T)$ and $\nabla\otimes\boldsymbol{v}\in M^p_\gamma L^p(0,T)$. If (γ,p) are such that the quantity $\eta(\gamma,p)$ in (9) verifies $\eta(\gamma,p)\leq 0$, then we have $\boldsymbol{u}\in L^\infty_t L^2_x\cap L^2_t\dot{H}^1_x([0,T]\times\mathbb{R}^3)$, $\boldsymbol{v}\in L^\infty_t\dot{H}^1_x([0,T]\times\mathbb{R}^3)$, and moreover, for all $t\in[0,T]$ the global energy inequality is verified:

$$\|\boldsymbol{u}(t,\cdot)\|^2_{L^2}+2\int_0^t\|\boldsymbol{u}(s,\cdot)\|^2_{\dot{H}^1}ds+\|\boldsymbol{v}(t,\cdot)\|^2_{\dot{H}^1}\leq\|\boldsymbol{u}_0\|^2_{L^2}+\|\boldsymbol{v}_0\|^2_{\dot{H}^1}. \tag{17}$$

As a direct application of the global energy inequality above we have the following Liouville-type result.

Corollary 2. *Within the framework of Theorem 2, let $(\boldsymbol{u},p,\boldsymbol{v})$ be a* generalized weak suitable solution *of the non-stationary coupled system (3) given by Definition 2.1. Moreover, let $\boldsymbol{u}_0$ and $\boldsymbol{v}_0$ be the initial data. If $\boldsymbol{u}_0=0$ and $\boldsymbol{v}_0$ is a constant vector field, then we have $\boldsymbol{u}=0$, $p=0$ and $\nabla\otimes\boldsymbol{v}=0$ on $[0,T]\times\mathbb{R}^3$.*

To close this section, let us make the following comments. As for the stationary case, we may observe that if we set $\mathbf{v}_0$ and $\mathbf{v}$ two constant unitary vectors, then Theorem 2 and Corollary 2 hold true for the classical Navier–Stokes equations (4) provided that $(\mathbf{u},p)$ is a generalized weak suitable solution in the sense of Definition 2.1 (with $\nabla\otimes\mathbf{v}=0$) and $\mathbf{u}\in M^p_{\gamma,0}L^p(0,T)$, with (γ,p) such that $\eta(\gamma,p)\leq 0$. In this setting, it is interesting to observe that the space $M^p_{\gamma,0}L^p(0,T)$ generalizes some spaces in which these kind of results have been obtained in previous works [34]. More precisely, for $3<p,r\leq 9/2$ such that $2/p+3/r\leq 1$, we have the following chain of embedding

$$L^p\Big(0,T,L^r(\mathbb{R}^3)\Big)\subset L^p\Big(0,T,L^{r,q}(\mathbb{R}^3)\Big)\subset M^p_{\gamma,0}L^p(0,T), \tag{18}$$

with $r<q<+\infty$, which is proven in the Appendix B.

3. The Local Morrey Spaces

In this section, for the completeness of the paper, we summarize some previous results on the local Morrey spaces $M^p_\gamma(\mathbb{R}^3)$ and $M^p_{\gamma,0}(\mathbb{R}^3)$ given in (7) and (8) respectively, and its time-space version $M^p_\gamma L^p(0,T)$ and $M^p_{\gamma,0}L^p(0,T)$ defined in (14) and (15) respectively.

These kind of local Morrey spaces are strongly lied with the weighted Lebesgue spaces $L^p_{w_\gamma}(\mathbb{R}^3)$ which are defined as follows: for $\gamma\geq 0$ we consider the weight

$$w_\gamma(x)=\frac{1}{(1+|x|)^\gamma} \tag{19}$$

and then for $1<p<+\infty$ we define the space $L^p_{w_\gamma}(\mathbb{R}^3)=L^p(w_\gamma\,dx)$. Thus, we have the following useful result.

Lemma 3.1. *[Lemma 2.1 of [13]] Let $0\leq\gamma<\delta$ and $1<p<+\infty$.*

1) *We have the continuous embedding: $L^p_{w_\gamma}(\mathbb{R}^3)\subset M^p_{\gamma,0}(\mathbb{R}^3)\subset M^p_\gamma(\mathbb{R}^3)\subset L^p_{w_\delta}(\mathbb{R}^3)$.*
2) *Moreover, for $0<T<+\infty$ we also have the continuous embedding:*

$$L^p\left([0,T],L^p_{w_\gamma}(\mathbb{R}^3)\right)\subset M^p_{\gamma,0}L^p(0,T)\subset M^p_\gamma L^p(0,T)\subset L^p\left([0,T],L^p_{w_\delta}(\mathbb{R}^3)\right).$$

Thereafter, a second useful result is the following one.

Lemma 3.2. *[Lemma 2.1 of [12] and Corollary 2.1 of [13]] Let $0<\gamma<3$ and $1<p<+\infty$.*

1) *The Riesz transform $\mathcal{R}_i=\dfrac{\partial_i}{\sqrt{-\Delta}}$ is bounded on $L^p_{w_\gamma}(\mathbb{R}^3)$ and we have $\|\mathcal{R}_i f\|_{L^p_{w_\gamma}}\leq c_{p,\gamma}\|f\|_{L^p_{w_\gamma}}$.*
2) *The Hardy-Littlewood maximal function operator $\mathcal{M}$ is also bounded on the space $L^p_{w_\gamma}(\mathbb{R}^3)$ and we have $\|\mathcal{M}_f\|_{L^p_{w_\gamma}}\leq c_{p,\gamma}\|f\|_{L^p_{w_\gamma}}$.*
3) *The points 1) and 2) also hold for the local Morrey space $M^p_\gamma(\mathbb{R}^3)$.*

Proof. We start by proving the points 1 and 2 above. The main idea to prove these points is to verify that the weight $w_\gamma(x)$, defined in (19), belongs to the Muckenhoupt class $\mathcal{A}_p(\mathbb{R}^3)$. For this, we recall that $w_\gamma(x)$ belongs to $\mathcal{A}_p(\mathbb{R}^3)$ if and only if it satisfies the following reserve Hölder inequality:

$$\sup_{x_0\in\mathbb{R}^3,\, R>0}\left(\frac{1}{|B(x_0,R)|}\int_{B(x_0,R)} w_\gamma(x)dx\right)^{\frac{1}{p}}\left(\frac{1}{|B(x_0,R)|}\int_{B(x_0,R)} \frac{1}{w_\gamma^{\frac{1}{p-1}}(x)}dx\right)^{1-\frac{1}{p}} < +\infty. \tag{20}$$

For more details on the Muckenhoupt class $\mathcal{A}_p(\mathbb{R}^3)$ see the Section 9.2.1, page 293 of the book [18].

In order to verify (20) we write

$$\begin{aligned}
&\sup_{x_0\in\mathbb{R}^3,\, R>0}\left(\frac{1}{|B(x_0,R)|}\int_{B(x_0,R)} w_\gamma(x)dx\right)^{\frac{1}{p}}\left(\frac{1}{|B(x_0,R)|}\int_{B(x_0,R)} \frac{1}{w_\gamma^{\frac{1}{p-1}}(x)}dx\right)^{1-\frac{1}{p}}\\
&\leq \sup_{x_0\in\mathbb{R}^3,\, 0<R\leq 1}\left(\frac{1}{|B(x_0,R)|}\int_{B(x_0,R)} w_\gamma(x)dx\right)^{\frac{1}{p}}\left(\frac{1}{|B(x_0,R)|}\int_{B(x_0,R)} \frac{1}{w_\gamma^{\frac{1}{p-1}}(x)}dx\right)^{1-\frac{1}{p}}\\
&\quad+\sup_{x_0\in\mathbb{R}^3,\, R>1}\left(\frac{1}{|B(x_0,R)|}\int_{B(x_0,R)} w_\gamma(x)dx\right)^{\frac{1}{p}}\left(\frac{1}{|B(x_0,R)|}\int_{B(x_0,R)} \frac{1}{w_\gamma^{\frac{1}{p-1}}(x)}dx\right)^{1-\frac{1}{p}}\\
&= I_1 + I_2,
\end{aligned}$$

where, we must find un upper bound for the term I_1 and I_2. For the term I_1, as we have $0 < R \leq 1$ then, for $|x-x_0| < R$, we can write $\frac{1}{2}(1+|x_0|) \leq 1+|x| \leq 2(1+|x_0|)$. Hence, by definition of the weight $w_\gamma(x)$, we get the upper bound $I_1 \leq 4^{\frac{\gamma}{p}}$. For the term I_2 we write

$$\begin{aligned}
I_2 \leq &\sup_{|x_0|\leq 10R,\, R>1}\left(\frac{1}{|B(x_0,R)|}\int_{B(x_0,R)} w_\gamma(x)dx\right)^{\frac{1}{p}}\left(\frac{1}{|B(x_0,R)|}\int_{B(x_0,R)} \frac{1}{w_\gamma^{\frac{1}{p-1}}(x)}dx\right)^{1-\frac{1}{p}}\\
&+\sup_{|x_0|>10R,\, R>1}\left(\frac{1}{|B(x_0,R)|}\int_{B(x_0,R)} w_\gamma(x)dx\right)^{\frac{1}{p}}\left(\frac{1}{|B(x_0,R)|}\int_{B(x_0,R)} \frac{1}{w_\gamma^{\frac{1}{p-1}}(x)}dx\right)^{1-\frac{1}{p}}\\
=& I_{2,1}+I_{2,2},
\end{aligned}$$

where we study now the terms $I_{2,1}$ and $I_{2,2}$. To treat the term $I_{2,1}$, using the polar coordinates ($\rho = |x|$) and the definition of the function $w_\gamma(x)$, where $0<\gamma<3$, we write:

$$\begin{aligned}
I_{2,1} &\leq \sup_{R>1}\left(\frac{1}{|B(0,11R)|}\int_{B(0,11R)} w_\gamma(x)dx\right)^{\frac{1}{p}}\left(\frac{1}{|B(0,11R)|}\int_{B(0,11R)} \frac{1}{w_\gamma^{\frac{1}{p-1}}(x)}dx\right)^{1-\frac{1}{p}}\\
&\leq \sup_{R>1}\left(\frac{1}{R^3}\int_0^{11R}\frac{\rho^2\, d\rho}{(1+\rho)^\gamma}\right)\left(\frac{1}{R^3}\int_0^{11R}\rho^2(1+\rho)^{\frac{\gamma}{p-1}}\, d\rho\right)^{1-\frac{1}{p}}\\
&\leq c_{\gamma,p}\sup_{R>1}\left(\frac{1}{R^3}\int_0^{11R}\rho^{2-\gamma}\, d\rho\right)\left[\left(\frac{1}{R^3}\int_0^{11R}\rho^2\, d\rho\right)^{1-\frac{1}{p}}+\left(\frac{1}{R^3}\int_0^{11R}\rho^{2+\frac{\gamma}{p-1}}\, d\rho\right)^{1-\frac{1}{p}}\right]\\
&\leq C_{\gamma,p} < +\infty.
\end{aligned}$$

Finally, to treat the term $I_{2,2}$, we remark that for all $R>1$ and $|x_0|10R$ we have

$$\frac{9}{10}(1+|x_0|) \leq 1+|x| \leq \frac{11}{10}(1+|x_0|),$$

and then, always by definition of the weight $w_\gamma(x)$ we get $I_{2,2} \leq \left(\frac{11}{9}\right)^{\frac{\gamma}{p}}$.

Gathering all these estimates we finally verify (20). Then, for $1 < \gamma < 3$ the weight $w_\gamma(x)$ belongs to $\mathcal{A}_p(\mathbb{R}^3)$ with $1 < p < +\infty$.

Thus, the boundness of the Riesz transforms and the Hardy-Littlewood maximal function operator directly follow by the basic properties of the Muckenhoupt class $\mathbb{A}_p$. See the Theorem 9.1.9, page 287 of the book [18]. The points 1 and 2 are now verified.

We prove now the point 3. For this, we will use the real interpolation theory. More precisely, for $\gamma < \delta < +\infty$, by the Proposition 7.1 of [15] we have that the local Morrey space $M^p_\gamma(\mathbb{R}^3)$ can be obtained by interpolation between the Lebesgue space $L^p(\mathbb{R}^3)$ and the weighted Lebesgue space $L^p_{w_\delta}(\mathbb{R}^3)$:

$$M^p_\gamma(\mathbb{R}^3) = [L^p(\mathbb{R}^3), L^p_{w_\delta}(\mathbb{R}^3)]_{\frac{\gamma}{\delta},\infty}.$$

Moreover, the norms $\|\cdot\|_{M^p_\gamma}$ and $\|\cdot\|_{[L^p, L^p_{w_\delta}]_{\frac{\gamma}{\delta},\infty}}$ are equivalents. Here, we recall that the interpolation space $[L^p(\mathbb{R}^3), L^p_{w_\delta}(\mathbb{R}^3)]_{\frac{\gamma}{\delta},\infty}$ is defined as the space of measurable functions f that can be written as $f = \sum_{j\in\mathbb{Z}} f_i$, where this series converges in $L^p(\mathbb{R}^3) + L^p_\gamma(\mathbb{R}^3)$. In addition, we have $f_i \in L^p(\mathbb{R}^3) \cap L^p_{w_\delta}(\mathbb{R}^3)$ and $\left(2^{-j\frac{\gamma}{\delta}} \max\left(\|f_j\|_{L^p},\, 2^j \|f_j\|_{L^p_{w_\gamma}}\right)\right)_{j\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z})$. Moreover, the interpolation norm $\|\cdot\|_{[L^p, L^p_{w_\delta}]_{\frac{\gamma}{\delta},\infty}}$ is defined by the expression:

$$\|f\|_{[L^p, L^p_{w_\delta}]_{\frac{\gamma}{\delta},\infty}} = \min_{(f_j)_{j\in\mathbb{Z}} \in L^p + L^p_{w_\delta}} \left(\sup_{j\in\mathbb{Z}} 2^{-j\frac{\gamma}{\delta}} \|f_j\|_{L^p} + \sup_{j\in\mathbb{Z}} 2^{j\left(1-\frac{\gamma}{\delta}\right)} \|f_j\|_{L^p_{w_\delta}} \right).$$

Thus, the point 3 follows directly from this fact and the points 1 and 2 verified above.

4. Characterization of the Pressure Term

4.1. Proof of Proposition 2.1

First, we define q given by the expression

$$q = \sum_{i,j=1}^{3} \mathcal{R}_i \mathcal{R}_j (u_i\, u_j) + \sum_{i,j,k=1}^{3} \mathcal{R}_i \mathcal{R}_j \left(\partial_i v_k\, \partial_j v_k\right), \tag{21}$$

where, for $9/4 < \delta < 3$ we have $q \in L^{p/2}([0,T], L^{p/2}_{w_\delta}(\mathbb{R}^3))$. Indeed, since we have assumed $\mathbf{u} \in M^p_\gamma L^p(0,T)$ and $\nabla \otimes \mathbf{v} \in M^p_\gamma L^p(0,T)$, with $0 < \gamma < 3/2$, then by point 2) of Lemma 3.1 we get $\mathbf{u} \in L^p([0,T], L^p_{w_\delta}(\mathbb{R}^3))$ and $\nabla \otimes \mathbf{v} \in L^p([0,T], L^p_{w_\delta}(\mathbb{R}^3))$. With this information we are able to write $\mathbf{u} \otimes \mathbf{u} \in L^{p/2}([0,T], L^{p/2}_{w_\delta}(\mathbb{R}^3))$ and $\nabla \otimes \mathbf{v} \odot \nabla \otimes \mathbf{v} \in L^{p/2}([0,T], L^{p/2}_{w_\delta}(\mathbb{R}^3))$, and moreover, as by point 1) of Lemma 3.2 the operator $\mathcal{R}_i\mathcal{R}_i$ is bounded in $L^{p/2}([0,T], L^{p/2}_{w_\delta}(\mathbb{R}^3))$, then we obtain $q \in L^{p/2}([0,T], L^{p/2}_{w_\delta}(\mathbb{R}^3))$.

Now, we will prove the identity $\nabla p = \nabla q$. For this, let $\varepsilon > 0$ (small enough) and let $\alpha \in \mathcal{C}^\infty_0(\mathbb{R})$ be a function such that $\alpha(t) = 0$ for $|t| > \varepsilon$. Moreover, let $\varphi \in \mathcal{C}^\infty_0(\mathbb{R}^3)$. We may observe that we have $(\alpha\varphi) * \nabla p \in \mathcal{D}'(]\varepsilon, T-\varepsilon[\times\mathbb{R}^3)$ and $(\alpha\varphi) * \nabla q \in \mathcal{D}'(]\varepsilon, T-\varepsilon[\times\mathbb{R}^3)$ and then, for $t \in]\varepsilon, T-\varepsilon[$ fix, we define the expression $A_\varepsilon(t) = (\alpha\varphi) * \nabla p(t,\cdot) - (\alpha\varphi) * \nabla q(t,\cdot) \in \mathcal{D}'(\mathbb{R}^3)$, where we must verify that we have $A_\varepsilon(t) \in \mathcal{S}'(\mathbb{R}^3)$. We write $A_\varepsilon(t) = (\alpha\varphi) * \nabla p(t,\cdot) - (\alpha\nabla\varphi) * q(t,\cdot)$. Moreover, since $(\mathbf{u}, p, v)$ verify the coupled system (3) then we have

$$\nabla p = -\partial_t \mathbf{u} + \Delta \mathbf{u} - div(\mathbf{u} \otimes \mathbf{u}) - div(\nabla \otimes \mathbf{v} \odot \nabla \otimes \mathbf{v}),$$

and thus we obtain

$$A_\varepsilon(t) = \left[(-(\partial_t\alpha)\varphi + \alpha\Delta\varphi) * \mathbf{u}\right](t,\cdot) - \left[(\alpha * \nabla\varphi) * (\mathbf{u}\otimes\mathbf{u})\right](t,\cdot)$$
$$- \left[(\alpha * \nabla\varphi) * (\nabla\otimes\mathbf{v}\odot\nabla\otimes\mathbf{v})\right](t,\cdot) - \left[(\alpha\nabla\varphi) * q\right](t,\cdot). \quad (22)$$

In this identity, we will prove that each term in the right side belong to the space $L^{p/2}_{w_\delta}(\mathbb{R}^3)$ (where $9/4 < \delta < 3$). For the first term to the right in (22), recall that we have $\mathbf{u} \in L^p([0,T], L^p_{w_\delta}(\mathbb{R}^3))$. Moreover, since for $\varphi \in \mathcal{C}^\infty_0(\mathbb{R}^3)$ and a function f we have the pointwise estimate $|(\varphi * f)(x)| \leq c_\varphi \mathcal{M}_f(x)$ (where $\mathcal{M}$ always denote the Hardy-Littlewood maximal function operator) then by point 2) of Lemma 3.2 we obtain that convolution with test functions is a bounded operator on $L^p_{w_\delta}(\mathbb{R}^3)$. Thus, we have $\left[(-(\partial_t\alpha)\varphi + \alpha\Delta\varphi) * \mathbf{u}\right](t,\cdot) \in L^p_{w_\delta}(\mathbb{R}^3)$. On the other hand, for $9/4 < \delta < 3$ we have the continuous embedding $L^p_{w_\delta}(\mathbb{R}^3) \subset L^{p/2}_{w_\delta}(\mathbb{R}^3)$. Indeed, by definition of the weight $w_\delta(x)$ given by (19) and using the Cauchy-Schwarz inequalities we write

$$\int_{\mathbb{R}^3} |f|^{p/2} w_\delta dx = \int_{\mathbb{R}^3} |f|^{p/2} w_{3/4} w_{\delta-3/4} \leq \left(\int_{\mathbb{R}^3} |f|^p w_{3/2} dx\right)^{1/2} \left(\int_{\mathbb{R}^3} w_{2\delta-3/2} dx\right)^{1/2},$$

where, as we have $9/4 < \delta < 3$ then the last integral in the right side convergences. Thus we obtain $\left[(-(\partial_t\alpha)\varphi + \alpha\Delta\varphi) * \mathbf{u}\right](t,\cdot) \in L^{p/2}_{w_\delta}(\mathbb{R}^3)$.

For the second and third terms to the right in (22), recall that we have $\mathbf{u}\otimes\mathbf{u} \in L^{p/2}([0,T], L^{p/2}_{w_\delta}(\mathbb{R}^3))$ and $\nabla\otimes\mathbf{v}\odot\nabla\otimes\mathbf{v} \in L^{p/2}([0,T], L^{p/2}_{w_\delta}(\mathbb{R}^3))$, hence, always by the fact that convolution with test functions is a bounded operator on the space $L^{p/2}_{w_\delta}(\mathbb{R}^3)$, we obtain $\left[(\alpha * \nabla\varphi) * (\mathbf{u}\otimes\mathbf{u})\right](t,\cdot) \in L^{p/2}_{w_\delta}(\mathbb{R}^3)$ and $\left[(\alpha * \nabla\varphi) * (\nabla\otimes\mathbf{v}\odot\nabla\otimes\mathbf{v})\right](t,\cdot) \in L^{p/2}_{w_\delta}(\mathbb{R}^3)$ respectively.

Finally, for the fourth term to the right in identity (22), as we have $q \in L^{p/2}([0,T], L^{p/2}_{w_\delta}(\mathbb{R}^3))$ then we obtain $\left[(\alpha\nabla\varphi) * q\right](t,\cdot) \in L^{p/2}_{w_\delta}(\mathbb{R}^3)$.

Getting back to the identity (22) we get $A_\varepsilon(t) \in L^{p/2}_{w_\delta}(\mathbb{R}^3)$ and then we have $A_\varepsilon(t) \in \mathcal{S}'(\mathbb{R}^3)$. On the other hand, since we have $div(\mathbf{u}) = 0$, taking the divergence operator in the first equation of (3) we obtain $\Delta(p-q) = 0$. Then we have $\Delta A_\varepsilon(t) = 0$ and since $A_\varepsilon(t) \in \mathcal{S}'(\mathbb{R}^3)$ we get that $A_\varepsilon(t)$ is a polynomial. But, recalling that $A_\varepsilon(t) \in L^{p/2}_{w_\delta}(\mathbb{R}^3)$, we necessary have $A_\varepsilon(t) = 0$. Finally, we use the approximation of the identity $\frac{1}{\varepsilon^4}\alpha\left(\frac{t}{\varepsilon}\right)\varphi\left(\frac{x}{\varepsilon}\right)$ to write $\nabla(p-q)(t,\cdot) = \lim_{\varepsilon\to 0} A_\varepsilon(t) = 0$.

5. The Stationary Case

5.1. Proof of Theorem 1

Let $(\mathbf{u}, p, \mathbf{v})$ be a smooth solutions of the coupled system (1). We assume now that for $0 < \gamma < 3 \leq p < +\infty$ we have $\mathbf{u} \in M^p_{\gamma,0}(\mathbb{R}^3)$ and $\nabla\otimes\mathbf{v} \in M^p_\gamma(\mathbb{R}^3)$.

Our starting point is to use the Proposition 2.1 to characterize the term ∇p in the first equation of (1). For this, we observe first that as $\mathbf{u}$, p and $\mathbf{v}$ are time-independent functions then they are also a smooth solution of the coupled system (3), since all the terms concerning the time derivatives are equal to zero. Moreover, as we have $\mathbf{u} \in M^p_{\gamma,0}(\mathbb{R}^3)$ and $\nabla\otimes\mathbf{v} \in M^p_\gamma(\mathbb{R}^3)$ then, for a time $0 < T < +\infty$ fix, we have $\mathbf{u} \in M^p_\gamma L^p(0,T)$ and $\nabla\otimes\mathbf{v} \in M^p_\gamma L^p(0,T)$. Thus, by the Proposition 2.1 we have the identity $\nabla p = \nabla q$, where q is given in formula (21). From now on we will consider the equation (1) with the term ∇q instead of the term ∇p.

We study now the following local estimate, also know as a Caccioppoli-type estimate.

Proposition 5.1. *Within the framework of Theorem 1, there exists a constant $c > 0$ such that for all $R \geq 1$ we have:*

$$\int_{B_{R/2}} |\nabla \otimes \boldsymbol{u}|^2 dx \leq c \left[\left(\int_{\mathcal{C}(R/2,R)} |\boldsymbol{u}|^p dx \right)^{2/p} + \left(\int_{\mathcal{C}(R/2,R)} |\nabla \otimes \boldsymbol{v}|^p dx \right)^{2/p} \right.$$
$$\left. + \left(\int_{\mathcal{C}(R/2,R)} |q|^{p/2} dx \right)^{2/p} \right] \times R^{2-9/p} \left(\int_{\mathcal{C}(R/2,R)} |\boldsymbol{u}|^p dx \right)^{1/p} + \frac{c}{R^2} \int_{\mathcal{C}(R/2,R)} |\boldsymbol{u}|^2 dx. \tag{23}$$

Proof. We start by introducing the following cut-off function. Let $\theta \in \mathcal{C}_0^\infty(\mathbb{R}^3)$ be a positive and radial function such that $\theta(x) = 1$ for $|x| < 1/2$ and $\theta(x) = 0$ for $|x| \geq 1$. Then, for $R \geq 1$ we define the function

$$\theta_R(x) = \theta(x/R). \tag{24}$$

Remark that this function verifies the following properties: we have $\theta_R(x) = 1$ for $|x| < R/2$, $\theta_R(x) = 0$ for $|x| > R$, and moreover we have $\|\nabla \theta_R\|_{L^\infty} \leq \dfrac{c}{R}$ and $\|\Delta \theta_R\|_{L^\infty} \leq \dfrac{c}{R^2}$.

For $R \geq 1$, we multiply the first equation of the system (1) by $\theta_R \mathbf{u}$ and integrating on the ball B_R (since we have $supp(\theta_R) \subset B_R$) we obtain:

$$-\int_{B_R} \Delta \mathbf{u} \cdot \theta_R \mathbf{u} dx + \int_{B_R} \operatorname{div}(\mathbf{u} \otimes \mathbf{u}) \cdot \theta_R \mathbf{u} dx + \int_{B_R} \operatorname{div}(\nabla \otimes \mathbf{v} \odot \nabla \otimes \mathbf{v}) \cdot \theta_R \mathbf{u} dx$$
$$+ \int_{B_R} \nabla q \cdot \theta_R \mathbf{u} dx = 0. \tag{25}$$

Moreover, we multiply the second equation of the system (1) by $-\theta_R \Delta \mathbf{v}$, then we integrate on the ball B_R to get:

$$\int_{B_R} \Delta \mathbf{v} \cdot \theta_R \Delta \mathbf{v} dx - \int_{B_R} div(\mathbf{v} \otimes \mathbf{u}) \cdot \theta_R \Delta \mathbf{v} dx + \int_{B_R} |\nabla \otimes \mathbf{v}|^2 \mathbf{v} \cdot \theta_R \Delta \mathbf{v} dx = 0. \tag{26}$$

At this point remark that as $\mathbf{u}, q$ and $\mathbf{v}$ are smooth and locally integrable functions then all the terms in equations (25) and (26) are well-defined.

Now, we need to study each term in these equations. We start by equation (25). For the first term in the left-hand side, by integration by parts we have

$$\begin{aligned}
&-\int_{B_R} \Delta \mathbf{u} \cdot \theta_R \mathbf{u} dx = -\sum_{i,j=1}^{3} \int_{B_R} (\partial_j^2 u_i)(\theta_R u_i) d = \sum_{i,j=1}^{3} \int_{B_R} \partial_j u_i \partial_j (\theta_R u_i) dx \\
&= \sum_{i,j=1}^{3} \int_{B_R} (\partial_j u_i)(\partial_j \theta_R) u_i dx + \sum_{i,j=1}^{3} \int_{B_R} (\partial_j u_i) \theta_R (\partial_j u_i) dx \\
&= \frac{1}{2} \sum_{i,j=1}^{3} \int_{B_R} (\partial_j \theta_R) \partial_j (u_i^2) dx + \int_{B_R} |\nabla \otimes \mathbf{u}|^2 \theta_R dx \\
&= -\frac{1}{2} \int_{B_R} |\mathbf{u}|^2 \Delta \theta_R dx + \int_{B_R} |\nabla \otimes \mathbf{u}|^2 \theta_R dx.
\end{aligned}$$

For the second term in the left-hand side of (25), by integration by parts and moreover, as we have $div(\mathbf{u})=0$, we can write

$$\int_{B_R} div(\mathbf{u}\otimes\mathbf{u})\cdot\theta_R\mathbf{u}dx = \sum_{i,j=1}^{3}\int_{B_R}\partial_j(u_iu_j)\theta_R u_i dx$$
$$= -\sum_{i,j=1}^{3}\int_{B_R}u_iu_j(\partial_j\theta_R)u_idx - \sum_{i,j=1}^{3}\int_{B_R}u_iu_j\theta_R(\partial_j u_i)dx$$
$$= -\int_{B_R}|\mathbf{u}|^2(\mathbf{u}\cdot\nabla\theta_R)dx - \frac{1}{2}\sum_{i,j=1}^{3}\int_{B_R}u_j\theta_R\partial_j(u_i^2)dx$$
$$= -\int_{B_R}|\mathbf{u}|^2(\mathbf{u}\cdot\nabla\theta_R)dx + \frac{1}{2}\sum_{i,j=1}^{3}\int_{B_R}\partial_j(u_j\theta_R)u_i^2dx$$
$$= -\int_{B_R}|\mathbf{u}|^2(\mathbf{u}\cdot\nabla\theta_R)dx + \frac{1}{2}\int_{B_R}(\mathbf{u}\cdot\nabla\theta_R)|\mathbf{u}|^2dx$$
$$= -\frac{1}{2}\int_{B_R}|\mathbf{u}|^2(\mathbf{u}\cdot\nabla\theta_R)dx.$$

In order to study the third term in the left-hand side of (25), we need the following technical identity:

$$div(\nabla\otimes\mathbf{v}\odot\nabla\otimes\mathbf{v}) = \nabla\left(\frac{1}{2}|\nabla\otimes\mathbf{v}|^2\right) + \Delta\mathbf{v}(\nabla\otimes\mathbf{v}).$$

Indeed, recall that for $i=1,2,3$ each component of the vector field $div(\nabla\otimes\mathbf{v}\odot\nabla\otimes\mathbf{v})$ is given by

$$(div(\nabla\otimes\mathbf{v}\odot\nabla\otimes\mathbf{v}))_i = \sum_{j,k=1}^{3}\partial_j(\partial_iv_k\,\partial_jv_k) = \sum_{j,k=1}^{3}\partial_j(\partial_iv_k)\partial_jv_k + \sum_{j,k=1}^{3}\partial_iv_k\partial_j^2v_k$$
$$= \sum_{j,k=1}^{3}\partial_i(\partial_jv_k)\partial_jv_k + \sum_{k=1}^{3}\partial_iv_k\,\Delta v_k = \partial_i\left(\frac{1}{2}\sum_{j,k=1}^{3}(\partial_jv_k)^2\right) + \sum_{k=1}^{3}\Delta v_k\,\partial_iv_k$$
$$= \partial_i\left(\frac{1}{2}|\nabla\otimes\mathbf{v}|^2\right) + (\Delta\mathbf{v}(\nabla\otimes\mathbf{v}))_i.$$

With this identity at hand, we get back to the third term in the left-hand side in (25) and, by integration by parts and the fact that $div(\mathbf{u})=0$, we write

$$\int_{B_R} div(\nabla\otimes\mathbf{v}\odot\nabla\otimes\mathbf{v})\cdot\theta_R\mathbf{u}dx = \sum_{i=1}^{3}\int_{B_R}\partial_i\left(\frac{1}{2}|\nabla\otimes\mathbf{v}|^2\right)\theta_R U_idx$$
$$+\sum_{i,j=1}^{3}\int_{B_R}\Delta V_j(\partial_iV_j)\theta_Ru_idx = -\frac{1}{2}\int_{B_R}|\nabla\otimes\mathbf{v}|^2(\mathbf{u}\cdot\nabla\theta_R)dx$$
$$+\sum_{i,j=1}^{3}\int_{B_R}\Delta v_j(\partial_iv_j)\theta_Ru_idx.$$

Finally, for the fourth term in the left-hand side in (25), always by integration by parts and since $div(\mathbf{u})=0$ we have

$$\int_{B_R}\nabla q\cdot\theta_R\mathbf{u}dx = \sum_{i=1}^{3}\int_{B_R}(\partial_iq)\theta_Ru_idx = -\int_{B_R}q(\mathbf{u}\cdot\nabla\theta_R)dx.$$

Once we dispose these identities, we get back to equation (25) and then we obtain

$$\int_{B_R} |\nabla \otimes \mathbf{u}|^2 \theta_R dx = \int_{B_R} \left(\frac{|\mathbf{u}|^2}{2} + \frac{|\nabla \otimes \mathbf{v}|^2}{2} + q \right) (\mathbf{u} \cdot \nabla \theta_R) dx$$
$$+ \frac{1}{2} \int_{B_R} |\mathbf{u}|^2 \Delta \theta_R dx - \sum_{i,j=1}^{3} \int_{B_R} \Delta v_j (\partial_i v_j) \theta_R u_i dx. \tag{27}$$

We study now the terms in the left-hand side in equation (26). For the first term we write directly

$$\int_{B_R} \Delta \mathbf{v} \cdot \theta_R \Delta \mathbf{v} dx = \int_{B_R} |\Delta \mathbf{v}|^2 \theta_R dx.$$

For the second term, integrating by parts and as $div(\mathbf{u}) = 0$ then we get

$$-\int_{B_R} div(\mathbf{v} \otimes \mathbf{u}) \cdot \theta_R \Delta \mathbf{v} dx = - \sum_{i,j=1}^{3} \int_{B_R} \partial_j (v_i u_j) \theta_R \Delta v_i$$
$$= - \sum_{i,j=1}^{3} \int_{B_R} (\partial_j v_i) u_j \theta_R \Delta v_i = - \sum_{i,j=1}^{3} \int_{B_R} \Delta v_i (\partial_j v_i) \theta_R u_j dx.$$

For the third term we write

$$\int_{B_R} |\nabla \otimes \mathbf{v}|^2 \mathbf{v} \cdot \theta_R \Delta \mathbf{v} dx = \sum_{i=1}^{3} \int_{B_R} |\nabla \otimes \mathbf{v}|^2 v_i \theta_R \Delta v_i dx = \int_{B_R} |\nabla \otimes \mathbf{v}|^2 (\mathbf{v} \cdot \Delta \mathbf{v}) \theta_R dx.$$

Thus, with these identities at hand, from equation (26) we obtain:

$$\int_{B_R} |\Delta \mathbf{v}|^2 \theta_R dx = \sum_{i,j=1}^{3} \int_{B_R} \Delta v_i (\partial_j v_i) \theta_R u_j dx - \int_{B_R} |\nabla \otimes \mathbf{v}|^2 (\mathbf{v} \cdot \Delta \mathbf{v}) \theta_R dx. \tag{28}$$

Now, adding the equations (27) and (28) we get

$$\int_{B_R} |\nabla \otimes \mathbf{u}|^2 \theta_R dx + \int_{B_R} |\Delta \mathbf{v}|^2 \theta_R dx = \int_{B_R} \left(\frac{|\mathbf{u}|^2}{2} + \frac{|\nabla \otimes \mathbf{v}|^2}{2} + q \right) (\mathbf{u} \cdot \nabla \theta_R) dx$$
$$+ \int_{B_R} \frac{|\mathbf{u}|^2}{2} \Delta \theta_R dx \underbrace{- \sum_{i,j=1}^{3} \int_{B_R} \Delta v_j (\partial_i v_j) \theta_R u_i dx + \sum_{i,j=1}^{3} \int_{B_R} \Delta v_i (\partial_j v_i) \theta_R u_j dx}_{(a)}$$
$$- \int_{B_R} |\nabla \otimes \mathbf{v}|^2 (\mathbf{v} \cdot \Delta \mathbf{v}) \theta_R dx,$$

but, we may observe that we have $(a) = 0$ and then we write

$$\int_{B_R} |\nabla \otimes \mathbf{u}|^2 \theta_R dx + \int_{B_R} |\Delta \mathbf{v}|^2 \theta_R dx = \int_{B_R} \left(\frac{|\mathbf{u}|^2}{2} + \frac{|\nabla \otimes \mathbf{v}|^2}{2} + q \right) (\mathbf{u} \cdot \nabla \theta_R) dx$$
$$+ \int_{B_R} \frac{|\mathbf{u}|^2}{2} \Delta \theta_R dx - \int_{B_R} |\nabla \otimes \mathbf{v}|^2 (\mathbf{v} \cdot \Delta \mathbf{v}) \theta_R dx.$$

Moreover, the last term is estimated as follows:

$$- \int_{B_R} |\nabla \otimes \mathbf{v}|^2 (\mathbf{v} \cdot \Delta \mathbf{v}) \theta_R dx \leq \int_{B_R} |\Delta \mathbf{v}|^2 \theta_R dx.$$

Indeed, recall that by hypothesis we have $|\mathbf{v}|^2 = 1$ and then we get $\frac{1}{2}\Delta|\mathbf{v}|^2 = 0$. Thus, we can write

$$\begin{aligned}
-|\nabla \otimes \mathbf{v}|^2 &= -\sum_{i,j=1}^{3} (\partial_i v_j)^2 = -\sum_{i,j=1}^{3} (\partial_i v_i)^2 = -\sum_{i,j=1}^{3} (\partial_i v_j)^2 + \frac{1}{2}\Delta|\mathbf{v}|^2 \\
&= -\sum_{i,j=1}^{3} (\partial_i v_j)^2 + \frac{1}{2}\sum_{i,j=1}^{2} \partial_i^2 (v_i^2) - \sum_{i,j=1}^{3} (\partial_i v_j)^2 + \sum_{i,j=1}^{3} \partial_j \left(\frac{1}{2}\partial_i v_i^2\right) \\
&= -\sum_{i,j=1}^{3} (\partial_i v_j)^2 + \sum_{i,j=1}^{3} \partial_j (v_i \partial_j v_i) - \sum_{i,j=1}^{3} (\partial_i v_j)^2 + \sum_{i,j=1}^{3} (\partial_j v_i)^2 + \sum_{i,j=1}^{3} v_i \partial_j^2 v_i \\
&= \sum_{i,j=1}^{3} v_i \partial_j^2 v_i = \mathbf{v} \cdot \Delta \mathbf{v}. \qquad (29)
\end{aligned}$$

With the identity $-|\nabla \otimes \mathbf{v}|^2 = \mathbf{v} \cdot \Delta \mathbf{v}$ at hand, and moreover, as we have $\theta_R \geq 0$ and as we have $|\mathbf{v}|^2 = 1$, we obtain

$$-\int_{B_R} |\nabla \otimes \mathbf{v}|^2 (\mathbf{v} \cdot \Delta \mathbf{v}) \theta_R dx = \int_{B_R} |\mathbf{v} \cdot \Delta \mathbf{v}|^2 \theta_R dx \leq \int_{B_R} |\mathbf{v}|^2 |\Delta \mathbf{v}|^2 \theta_R dx \leq \int_{B_R} |\Delta \mathbf{v}|^2 \theta_R dx.$$

Once we have this estimate then we can write

$$\begin{aligned}
&\int_{B_R} |\nabla \otimes \mathbf{u}|^2 \theta_R dx + \int_{B_R} |\Delta \mathbf{v}|^2 \theta_R dx \leq \int_{B_R} \left(\frac{|\mathbf{u}|^2}{2} + \frac{|\nabla \otimes \mathbf{v}|^2}{2} + q\right) (\mathbf{u} \cdot \nabla \theta_R) dx \\
&+ \int_{B_R} \frac{|\mathbf{u}|^2}{2} \Delta \theta_R dx + \int_{B_R} |\Delta \mathbf{v}|^2 \theta_R dx,
\end{aligned}$$

hence we get

$$\int_{B_R} |\nabla \otimes \mathbf{u}|^2 \theta_R dx \leq \int_{B_R} \left(\frac{|\mathbf{u}|^2}{2} + \frac{|\nabla \otimes \mathbf{v}|^2}{2} + q\right) (\mathbf{u} \cdot \nabla \theta_R) dx + \int_{B_R} \frac{|\mathbf{u}|^2}{2} \Delta \theta_R dx.$$

Recalling that we have $\theta_R(x) = 1$ for $|x| < R/2$, then we obtain

$$\int_{B_{R/2}} |\nabla \otimes \mathbf{u}|^2 dx \leq \int_{B_R} |\nabla \otimes \mathbf{u}|^2 \theta_R dx,$$

and from the previous inequality we are able to write

$$\int_{B_{R/2}} |\nabla \otimes \mathbf{u}|^2 dx \leq \int_{B_R} \left(\frac{|\mathbf{u}|^2}{2} + \frac{|\nabla \otimes \mathbf{v}|^2}{2} + q\right) (\mathbf{u} \cdot \nabla \theta_R) dx + \int_{B_R} \frac{|\mathbf{u}|^2}{2} \Delta \theta_R dx.$$

Moreover, recalling that we have $supp(\nabla \theta_R) \subset \mathcal{C}(R/2, R)$ and $supp(\Delta \theta_R) \subset \mathcal{C}(R/2, R)$, then we obtain the following estimate

$$\begin{aligned}
\int_{B_{R/2}} |\nabla \otimes \mathbf{u}|^2 dx &\leq \int_{\mathcal{C}(R/2,R)} \left(\frac{|\mathbf{u}|^2}{2} + \frac{|\nabla \otimes \mathbf{v}|^2}{2} + q\right) (\mathbf{u} \cdot \nabla \theta_R) dx + \int_{\mathcal{C}(R/2,R)} \frac{|\mathbf{u}|^2}{2} \Delta \theta_R dx \\
&\leq \int_{\mathcal{C}(R/2,R)} \frac{|\mathbf{u}|^2}{2} (\mathbf{u} \cdot \nabla \theta_R) dx + \int_{\mathcal{C}(R/2,R)} \frac{|\nabla \otimes \mathbf{v}|^2}{2} (\mathbf{u} \cdot \nabla \theta_R) dx \\
&\quad + \int_{\mathcal{C}(R/2,R)} q (\mathbf{u} \cdot \nabla \theta_R) dx + \int_{\mathcal{C}(R/2,R)} \frac{|\mathbf{u}|^2}{2} \Delta \theta_R dx = I_1 + I_2 + I_3 + I_4. \qquad (30)
\end{aligned}$$

From this estimate we will derive the desired inequality (23) and for this we will study each term I_i for $i = 1, \cdots 4$. For the term I_1, by the Hölder inequalities (with $1 = 2/p + 1/q$) and moreover, as we have $\|\nabla\theta_R\|_{L^\infty} \leq c/R$, we get

$$I_1 \leq \int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^2 |\mathbf{u}\cdot\nabla\theta_R| dx \leq \left(\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^p dx\right)^{2/p} \left(\int_{\mathcal{C}(R/2,R)} |\mathbf{u}\cdot\nabla\theta_R|^q dx\right)^{1/q}$$
$$\leq \left(\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^p dx\right)^{2/p} \frac{c}{R}\left(\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^q dx\right)^{1/q}.$$

But, since we have $3 \leq p < +\infty$ and $1 = 2/p + 1/q$ then the parameter q verifies $q \leq 3 \leq p$ and thus, for the last expression we can write

$$\frac{c}{R}\left(\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^q dx\right)^{1/q} \leq \frac{c}{R} R^{3(1/q-1/p)} \left(\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^p dx\right)^{1/p} \leq c\, R^{2-9/p} \left(\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^p dx\right)^{1/p},$$

hence we have

$$I_1 \leq c\left(\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^p dx\right)^{2/p} R^{2-9/p} \left(\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^p dx\right)^{1/p}. \tag{31}$$

Following the same computations, the terms I_2 and I_3 are estimated as follows:

$$I_2 \leq c\left(\int_{\mathcal{C}(R/2,R)} |\nabla\otimes\mathbf{v}|^p dx\right)^{2/p} R^{2-9/p} \left(\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^p dx\right)^{1/p}, \tag{32}$$

and

$$I_3 \leq c\left(\int_{\mathcal{C}(R/2,R)} |q|^{p/2} dx\right)^{2/p} R^{2-9/p} \left(\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^p dx\right)^{1/p}. \tag{33}$$

Finally, for the term I_4, always by the Hölder inequalities, with $1 = 2/p+1/q$, by the fact that $\|\Delta\theta_R\|_{L^\infty} \leq c/R^2$ we obtain

$$I_4 \leq c\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^2 |\Delta\theta_R| dx \leq \frac{c}{R^2}\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^2 dx.$$

With the estimates, we get back to the inequality (30) to obtain the desired estimate (23). Proposition 5.1 is verified.

Once the Caccioppoli type estimate (23) is verified, for $0<\gamma$ we write:

$$\int_{B_{R/2}} |\nabla\otimes\mathbf{u}|^2dx \leq \frac{c}{R^2}\int_{\mathcal{C}(R/2,R)}|\mathbf{u}|^2dx + \frac{c}{R^{\frac{2}{p}\gamma}}\left[\left(\int_{\mathcal{C}(R/2,R)}|\mathbf{u}|^pdx\right)^{2/p}+\left(\int_{\mathcal{C}(R/2,R)}|\nabla\otimes\mathbf{v}|^pdx\right)^{2/p}\right.$$
$$\left.+\left(\int_{\mathcal{C}(R/2,R)}|q|^{p/2}dx\right)^{2/p}\right]\times R^{\frac{2}{p}\gamma+2-\frac{9}{p}}\left(\int_{\mathcal{C}(R/2,R)}|\mathbf{u}|^pdx\right)^{1/p}$$
$$=\frac{c}{R^2}\int_{\mathcal{C}(R/2,R)}|\mathbf{u}|^2dx + c\left[\left(\frac{1}{R^\gamma}\int_{\mathcal{C}(R/2,R)}|\mathbf{u}|^pdx\right)^{2/p}+\left(\frac{1}{R^\gamma}\int_{\mathcal{C}(R/2,R)}|\nabla\otimes\mathbf{v}|^pdx\right)^{2/p}\right.$$
$$\left.+\left(\frac{1}{R^\gamma}\int_{\mathcal{C}(R/2,R)}|q|^{p/2}dx\right)^{2/p}\right]\times R^{\frac{2}{p}\gamma+2-\frac{9}{p}}\left(\int_{\mathcal{C}(R/2,R)}|\mathbf{u}|^pdx\right)^{1/p}.$$
$$=\frac{c}{R^2}\int_{\mathcal{C}(R/2,R)}|\mathbf{u}|^2dx + c\left[\left(\frac{1}{R^\gamma}\int_{\mathcal{C}(R/2,R)}|\mathbf{u}|^pdx\right)^{2/p}+\left(\frac{1}{R^\gamma}\int_{\mathcal{C}(R/2,R)}|\nabla\otimes\mathbf{v}|^pdx\right)^{2/p}\right.$$
$$\left.+\left(\frac{1}{R^\gamma}\int_{\mathcal{C}(R/2,R)}|q|^{p/2}dx\right)^{2/p}\right]\times R^{\frac{3}{p}\gamma+2-\frac{9}{p}}\left(\frac{1}{R^\gamma}\int_{\mathcal{C}(R/2,R)}|\mathbf{u}|^pdx\right)^{1/p}.$$

At this point, recalling that by (9) we define $\eta(\gamma,p)=\frac{\gamma}{p}-\frac{3}{p}+\frac{2}{3}$, then we have $\frac{3}{p}\gamma+2-\frac{9}{p}=3\eta(\gamma,p)$, and we obtain

$$\int_{B_{R/2}} |\nabla\otimes\mathbf{u}|^2dx \leq \frac{c}{R^2}\int_{\mathcal{C}(R/2,R)}|\mathbf{u}|^2dx + c\left[\left(\frac{1}{R^\gamma}\int_{\mathcal{C}(R/2,R)}|\mathbf{u}|^pdx\right)^{2/p}+\left(\frac{1}{R^\gamma}\int_{\mathcal{C}(R/2,R)}|\nabla\otimes\mathbf{v}|^pdx\right)^{2/p}\right.$$
$$\left.+\left(\frac{1}{R^\gamma}\int_{\mathcal{C}(R/2,R)}|q|^{p/2}dx\right)^{2/p}\right]\times R^{3\eta(\gamma,p)}\left(\frac{1}{R^\gamma}\int_{\mathcal{C}(R/2,R)}|\mathbf{u}|^pdx\right)^{1/p}. \tag{34}$$

Here, as we have $\mathbf{u}\in M^p_{\gamma,0}(\mathbb{R}^3)$ and $\nabla\otimes\mathbf{v}\in M^p_\gamma(\mathbb{R}^3)$, for all $R\geq 1$ we have the uniformly bound

$$\left(\frac{1}{R^\gamma}\int_{\mathcal{C}(R/2,R)}|\mathbf{u}|^pdx\right)^{2/p}+\left(\frac{1}{R^\gamma}\int_{\mathcal{C}(R/2,R)}|\nabla\otimes\mathbf{v}|^pdx\right)^{2/p}\leq c\left(\|\mathbf{u}\|^2_{M^p_\gamma}+\|\nabla\otimes\mathbf{v}\|^2_{M^p_\gamma}\right).$$

Moreover, in order to estimate the expression $\left(\frac{1}{R^\gamma}\int_{\mathcal{C}(R/2,R)}|q|^{p/2}dx\right)^{2/p}$, we recall that the term q is defined through $\mathbf{u}$ and $\nabla\otimes\mathbf{v}$ in (21), and then, setting the parameter $0<\gamma$ as $0<\gamma<3$, by the point 3) of Lemma 3.2 we also can write

$$\left(\frac{1}{R^\gamma}\int_{\mathcal{C}(R/2,R)}|q|^{p/2}dx\right)^{2/p}\leq \|q\|_{M^{p/2}_\gamma}\leq c\left(\|\mathbf{u}\|^2_{M^p_\gamma}+\|\nabla\otimes\mathbf{v}\|^2_{M^p_\gamma}\right). \tag{35}$$

Getting back to (34), we have the estimate

$$\int_{B_{R/2}} |\nabla\otimes\mathbf{u}|^2dx \leq \frac{c}{R^2}\int_{\mathcal{C}(R/2,R)}|\mathbf{u}|^2dx + c\left(\|\mathbf{u}\|^2_{M^p_\gamma}+\|\nabla\otimes\mathbf{v}\|^2_{M^p_\gamma}\right)R^{3\eta(\gamma,p)}\left(\frac{1}{R^\gamma}\int_{\mathcal{C}(R/2,R)}|\mathbf{u}|^pdx\right)^{1/p},$$

where, we still must study the first term in the right. Precisely, as $R\geq 1$ we have

$$\frac{c}{R^2}\int_{\mathcal{C}(R/2,R)}|\mathbf{u}|^2dx\leq c\,R^{2\eta(\gamma,p)}\left(\frac{1}{R^\gamma}\int_{\mathcal{C}(R/2,R)}|\mathbf{u}|^p\right)^{2/p}. \tag{36}$$

Indeed, we write

$$\frac{c}{R^2}\int_{\mathcal{C}(R/2,R)}|\mathbf{u}|^2dx\leq cR^{6(1/2-1/p)-2}\left(\int_{\mathcal{C}(R/2,R)}|\mathbf{u}|^p\right)^{2/p}\leq cR^{6(1/2-1/p)-2+2\gamma/p}\left(\frac{1}{R^\gamma}\int_{\mathcal{C}(R/2,R)}|\mathbf{u}|^p\right)^{2/p},$$

where, always by (9) we can write $6(1/2-1/p)-2+2\gamma/p\leq 2(\gamma/p-3/p+1/2)\leq 2\eta(\gamma,p)$, hence the estimate (36) follows.

Thus, we obtain the following estimate

$$\begin{aligned}\int_{B_{R/2}}|\nabla\otimes\mathbf{u}|^2dx\leq\ & c\,R^{2\eta(\gamma,p)}\left(\frac{1}{R^\gamma}\int_{\mathcal{C}(R/2,R)}|\mathbf{u}|^p\right)^{2/p}\\ &+c\left(\|\mathbf{u}\|^2_{M^p_\gamma}+\|\nabla\otimes\mathbf{v}\|^2_{M^p_\gamma}\right)R^{3\eta(\gamma,p)}\left(\frac{1}{R^\gamma}\int_{\mathcal{C}(R/2,R)}|\mathbf{u}|^pdx\right)^{1/p},\end{aligned} \tag{37}$$

and now, we will consider the cases when $\eta(\gamma,p)\leq 0$ and $\eta(\gamma,p)>0$ separately.

1) **The case when** $\eta(\gamma,p)\leq 0$. Here, as $R\geq 1$ then we have $R^{2\eta(\gamma,p)}\leq 1$ and $R^{3\eta(\gamma,p)}\leq 1$. Thus, by the estimate (37) we can write

$$\int_{B_{R/2}}|\nabla\otimes\mathbf{u}|^2dx\leq c\left(\frac{1}{R^\gamma}\int_{\mathcal{C}(R/2,R)}|\mathbf{u}|^p\right)^{2/p}+c\left(\|\mathbf{u}\|^2_{M^p_\gamma}+\|\nabla\otimes\mathbf{v}\|^2_{M^p_\gamma}\right)\left(\frac{1}{R^\gamma}\int_{\mathcal{C}(R/2,R)}|\mathbf{u}|^pdx\right)^{1/p}.$$

Moreover, as $\mathbf{u}\in M^p_{\gamma,0}(\mathbb{R}^3)$, taking the limit when $R\to+\infty$ in each side of the estimate above we obtain $\displaystyle\int_{\mathbb{R}^3}|\nabla\otimes\mathbf{u}|^2dx=0$ and thus $\mathbf{u}$ is a constant vector. But, always by the information $\mathbf{u}\in M^p_{\gamma,0}(\mathbb{R}^3)$ we necessary have the identity $\mathbf{u}=0$.

2) **The case when** $\eta(\gamma,p)>0$. Here, always as $R\geq 1$ then we have $R^{2\eta(\gamma,p)}\leq R^{6\eta(\gamma,p)}$; and thus, by the estimate (37) we write now

$$\begin{aligned}\int_{B_{R/2}}|\nabla\otimes\mathbf{u}|^2dx\leq\ & c\,R^{6\eta(\gamma,p)}\left(\frac{1}{R^\gamma}\int_{\mathcal{C}(R/2,R)}|\mathbf{u}|^p\right)^{2/p}\\ &+c\left(\|\mathbf{u}\|^2_{M^p_\gamma}+\|\nabla\otimes\mathbf{v}\|^2_{M^p_\gamma}\right)R^{3\eta(\gamma,p)}\left(\frac{1}{R^\gamma}\int_{\mathcal{C}(R/2,R)}|\mathbf{u}|^pdx\right)^{1/p}\\ \leq\ & c\left[R^{3\eta(\gamma,p)}\left(\frac{1}{R^\gamma}\int_{\mathcal{C}(R/2,R)}|\mathbf{u}|^p\right)^{1/p}\right]^2\\ &+c\left(\|\mathbf{u}\|^2_{M^p_\gamma}+\|\nabla\otimes\mathbf{v}\|^2_{M^p_\gamma}\right)\left[R^{3\eta(\gamma,p)}\left(\frac{1}{R^\gamma}\int_{\mathcal{C}(R/2,R)}|\mathbf{u}|^pdx\right)^{1/p}\right].\end{aligned}$$

Hence, as $\mathbf{u}\in\mathbf{u}\in M^p_{\gamma,0}(\mathbb{R}^3)$, and moreover, assuming the supplementary decaying condition (10), have the identity $\mathbf{u}=0$.

Until now we have proven that $\mathbf{u}=0$ and then it remains to prove the identities $\nabla\otimes\mathbf{v}=0$ and $q=0$. We start by proving that $\nabla\otimes\mathbf{v}=0$. As $\mathbf{u}=0$ then by (1) we have that $\mathbf{v}$ solves the following elliptic equation

$$-\Delta\mathbf{v}-|\nabla\otimes\mathbf{v}|^2\mathbf{v}=0.$$

In this equation, we multiply by $\theta_R((x\cdot\nabla)\mathbf{v})$, where for $R\geq 1$ the cut-off function $\theta_R(x)$ was defined in (24), and integrating on the ball B_R by [19], page 6, we have the following local estimate:

$$\int_{B_{R/2}}|\nabla\otimes\mathbf{v}|^2dx\leq c\int_{\mathcal{C}(R/2,R)}|\nabla\otimes\mathbf{v}|^2dx. \tag{38}$$

Now, recall that $\mathbf{v}$ verifies (5) and then we have

$$\int_{B_{R/2}}|\nabla\otimes\mathbf{v}|^2dx\leq c\sup_{R\geq 1}\int_{\mathcal{C}(R/2,R)}|\nabla\otimes\mathbf{v}|^2dx<+\infty,$$

hence we obtain $\int_{\mathbb{R}^3}|\nabla\otimes\mathbf{v}|^2dx<+\infty$. With this information, we get back to (38) and taking the limit when $R\to+\infty$ we get $\int_{\mathbb{R}^3}|\nabla\otimes\mathbf{v}|^2dx=0$. Hence we have $\nabla\otimes\mathbf{v}=0$. Once we have the identities $\mathbf{u}=0$ and $\nabla\otimes\mathbf{v}=0$, the identity $q=0$ follows directly from the estimate (35). Finally, always by the identity $\nabla p=\nabla q$ given by Proposition 2.1, we conclude that p is a constant vector.

To finish the proof of Theorem 1, we will verify now that we necessarily have the identity $p=0$. Indeed, in the equation

$$-\Delta\mathbf{u}+(\mathbf{u}\cdot\nabla)\,\mathbf{u}+\operatorname{div}(\nabla\otimes\mathbf{v}\odot\nabla\otimes\mathbf{v})+\nabla p=0,$$

we apply the divergence operator, and moreover, as we have $div(\mathbf{u})=0$ then we get

$$\operatorname{div}\left((\mathbf{u}\cdot\nabla)\,\mathbf{u}\right)+\operatorname{div}\left(\operatorname{div}(\nabla\otimes\mathbf{v}\odot\nabla\otimes\mathbf{v})\right)+\Delta p=0.$$

Thereafter, using the fact that the vector fields $(\mathbf{u}\cdot\nabla)\,\mathbf{u}$ and $\operatorname{div}(\nabla\otimes\mathbf{v}\odot\nabla\otimes\mathbf{v})$ are defined component by component as

$$[(\mathbf{u}\cdot\nabla)\,\mathbf{u}]_i=\sum_{j=1}^{3}\partial_j(u_ju_i),\quad [\operatorname{div}(\nabla\otimes\mathbf{v}\odot\nabla\otimes\mathbf{v})]_i=\sum_{j=1}^{3}\sum_{k=1}^{3}\partial_j(\partial_iv_k\,\partial_jv_k),\quad i=1,2,3,$$

in the last identity we have

$$\sum_{i=1}^{3}\sum_{i=j}^{3}\partial_i\partial_j(u_ju_i)+\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\partial_i\partial_j(\partial_iv_k\,\partial_jv_k)+\Delta p=0.$$

Hence, the pressure p can we written as

$$\begin{aligned}p&=\sum_{i=1}^{3}\sum_{i=j}^{3}\frac{1}{-\Delta}\left(\partial_i\partial_j(u_ju_i)\right)+\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\frac{1}{-\Delta}\left(\partial_i\partial_j(\partial_iv_k\,\partial_jv_k)\right)\\&=\sum_{i=1}^{3}\sum_{i=j}^{3}\left(\frac{\partial_i}{\sqrt{-\Delta}}\frac{\partial_j}{\sqrt{-\Delta}}(u_ju_i)\right)+\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\left(\frac{\partial_i}{\sqrt{-\Delta}}\frac{\partial_j}{\sqrt{-\Delta}}(\partial_iv_k\,\partial_jv_k)\right),\end{aligned}$$

where the operators $\frac{1}{-\Delta}$ and $\frac{1}{\sqrt{-\Delta}}$ can be defined in the Fourier variable by the symbols $|\xi|^{-2}$ and $|\xi|^{-1}$ respectively. Moreover, recalling the definition of the Riesz transforms $\mathcal{R}_i=\frac{\partial_i}{\sqrt{-\Delta}}$, we have

$$p=\sum_{i=1}^{3}\sum_{i=j}^{3}\left(\mathcal{R}_i\mathcal{R}_j(u_ju_i)\right)+\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\left(\mathcal{R}_i\mathcal{R}_j(\partial_iv_k\,\partial_jv_k)\right)=q. \tag{39}$$

At this point we remark that all the terms in the second expression above are well defined by the third point of the Lemma 3.2. Finally, as $q=0$ we have $p=0$. Theorem 1 is proven.

5.2. Proof of Corollary 1

We observe first that if $(\mathbf{u}, p)$ is a smooth solution of the equations (2), then, for a constant vector field $\mathbf{v} \in \mathbb{S}^{n-1}$ the triplet $(\mathbf{u}, p, \mathbf{v})$ is also a smooth solution of the coupled system (1). Thus, assuming the velocity $\mathbf{u}$ verifies $\mathbf{u} \in M^{p}_{\gamma,0}(\mathbb{R}^3)$, where $0 < \gamma < 3 \leq p < +\infty$ are such that $\eta(\gamma, p) \leq 0$, and moreover, as we have $\nabla \otimes \mathbf{v} = 0$ and consequently $\vec{\nabla} \otimes \mathbf{v} \in M^p_\gamma(\mathbb{R}^3)$, the result stated in this corollary directly follows from Theorem 1.

5.3. Proof of Proposition 1

As mentioned above, the stationary Navier–Stokes equations (2) can be observed as a particular of the coupled Ericksen–Leslie system (1) when the unitary vector field $\mathbf{v}$ is a constant vector. Then, the Proposition 2.1 holds true for the equations (2), and we write the term ∇q instead of the term ∇p, where, $\nabla \otimes \mathbf{v} = 0$, the term q is given by the identity $q = \displaystyle\sum_{i,j=1}^{3} \mathcal{R}_i \mathcal{R}_j (u_i\, u_j)$.

We also observe that Proposition 5.1 holds true for the equations (2), and, always as we have $\nabla \otimes \mathbf{v} = 0$, then we are able to write the following estimate:

$$\int_{B_{R/2}} |\nabla \otimes \mathbf{u}|^2 dx \leq c \left[\left(\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^p dx \right)^{2/p} + \left(\int_{\mathcal{C}(R/2,R)} |q|^{p/2} dx \right)^{2/p} \right] \times R^{2-9/p} \left(\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^p dx \right)^{1/p} + \frac{c}{R^2} \int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^2 dx.$$

Hence, following the same computations performed in the estimate (37) we obtain

$$\int_{B_{R/2}} |\nabla \otimes \mathbf{u}|^2 dx \leq \; c\, R^{2\eta(\gamma,p)} \left(\frac{1}{R^\gamma} \int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^p \, dx \right)^{2/p} + c \left(\|\mathbf{u}\|^2_{M^p_\gamma} \right) R^{3\eta(\gamma,p)} \left(\frac{1}{R^\gamma} \int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^p dx \right)^{1/p},$$

and moreover, recalling the definition of the quantity $\|\mathbf{u}\|_{M^p_\gamma}$ given in (7) we finally have the following estimate:

$$\int_{B_{R/2}} |\nabla \otimes \mathbf{u}|^2 dx \leq \; c\, R^{2\eta(\gamma,p)} \|\mathbf{u}\|^2_{M^p_\gamma} + c\, R^{3\eta(\gamma,p)} \|\mathbf{u}\|^3_{M^p_\gamma}. \tag{40}$$

In this estimate, we will distinguish two cases when $\eta(\gamma, p) < 0$ and when $\eta(\gamma, p) = 0$.

1) The case $\eta(\gamma, p) < 0$. Here, in each side of the estimate (40) we take the limit when $R \to +\infty$ to obtain the identity $\mathbf{u} = 0$. Moreover, by the identities $q = \displaystyle\sum_{i,j=1}^{3} \mathcal{R}_i \mathcal{R}_j (u_i\, u_j)$ and $\nabla q = \nabla p$, we conclude that p is a constant.
2) The case $\eta(\gamma, p) = 0$. In this case, by the estimate (40) we have

$$\int_{B_{R/2}} |\nabla \otimes \mathbf{u}|^2 dx \leq \; c\, \|\mathbf{u}\|^2_{M^p_\gamma} + c\, \|\mathbf{u}\|^3_{M^p_\gamma},$$

and taking the limit when $R \to +\infty$ we obtain $\displaystyle\int_{\mathbb{R}^3} |\nabla \otimes \mathbf{u}|^2 dx \leq \; c\, \|\mathbf{u}\|^2_{M^p_\gamma} + c\, \|\mathbf{u}\|^3_{M^p_\gamma}$, hence we can write $\mathbf{u} \in \dot{H}^1(\mathbb{R}^3)$. We will use now the additional hypothesis $\mathbf{u} \in \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)$ to conclude the identity $\mathbf{u} = 0$.

Indeed, with the information $\mathbf{u} \in \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)$ we can apply the improved Sobolev inequalities (see the article [17] for a proof of these inequalities) and we write $\|\mathbf{u}\|_{L^4} \leq c\|\mathbf{u}\|^{\frac{1}{2}}_{\dot{H}^1}\|\mathbf{u}\|^{\frac{1}{2}}_{\dot{B}^{-1}_{\infty,\infty}}$. Once we dispose of the information $\mathbf{u} \in L^4(\mathbb{R}^3)$ we can derive now the identity $\mathbf{u} = 0$ as follows: multiplying equation (2) by $\mathbf{u}$ and integrating on the whole space $\mathbb{R}^3$ we have

$$\int_{\mathbb{R}^3} (-\Delta \mathbf{u}) \cdot \mathbf{u} dx = \int_{\mathbb{R}^3} ((\mathbf{u} \cdot \nabla)\mathbf{u}) \cdot \mathbf{u} dx + \int_{\mathbb{R}^3} \nabla p \cdot \mathbf{u} dx,$$

where due to the fact $\mathbf{u} \in \dot{H}^1 \cap L^4(\mathbb{R}^3)$ each term in this identity is well-defined. Indeed, for the term in the left-hand side remark that as $\mathbf{u} \in \dot{H}^1(\mathbb{R}^3)$ then we have $-\Delta \mathbf{u} \in \dot{H}^{-1}(\mathbb{R}^3)$. Then, for the first term in the right-hand side, as $div(\mathbf{u}) = 0$ we write $(\mathbf{u} \cdot \vec{\nabla})\mathbf{u} = div(\mathbf{u} \otimes \mathbf{u})$ where, as $\mathbf{u} \in L^4(\mathbb{R}^3)$ by the Hölder inequalities we have $\mathbf{u} \otimes \mathbf{u} \in L^2(\mathbb{R}^3)$ and then $div(\mathbf{u} \otimes \mathbf{u}) \in \dot{H}^{-1}(\mathbb{R}^3)$. Finally, in order to study the second term in the right-hand side, we write the pressure p as $p = \frac{1}{-\Delta} div(div(\mathbf{u} \otimes \mathbf{u}))$ hence we get $p \in L^2(\mathbb{R}^3)$ (since we have $\mathbf{u} \otimes \mathbf{u} \in L^2(\mathbb{R}^3)$) and then $\nabla p \in \dot{H}^{-1}(\mathbb{R}^3)$.

Now, integrating by parts each term in the identity above we have that $\int_{\mathbb{R}^3} (-\Delta \mathbf{u}) \cdot \mathbf{u} dx = \int_{\mathbb{R}^3} |\nabla \otimes \mathbf{u}|^2 dx$, and moreover $\int_{\mathbb{R}^3} ((\mathbf{u} \cdot \nabla)\mathbf{u}) \cdot \mathbf{u} dx = 0$ and $\int_{\mathbb{R}^3} \nabla p \cdot \mathbf{u} dx = 0$. With these identities we get $\int_{\mathbb{R}^3} |\nabla \otimes \mathbf{u}|^2 dx = 0$ and thus we have $\mathbf{u} = 0$. Moreover, following the same ideas used to prove the identity (39) we have $p = q = 0$. Proposition 1 is now proven.

6. The Non-stationary Case

6.1. Proof of Theorem 2

Recall first that by hypothesis of Theorem 2 we have $\mathbf{u} \in M^p_{\gamma,0} L^p(0,T)$ and $\nabla \otimes \mathbf{v} \in M^p_\gamma L^p(0,T)$, where the parameters $0 < \gamma < 3 \leq p < +\infty$ are such that $\eta(\gamma, p) \leq 0$. Then, by Proposition 2.1 we have the identity $\nabla p = \nabla q$, where the quantity q in defined in expression (21); and from now on we will consider the equations (3) with the term ∇q instead of the term ∇p.

We will apply the local energy balance (13) to a suitable test function and for this we will follow some of the ideas of [14]. Let $0 < t_0 < t_1 < T$. For a parameter $\varepsilon > 0$, we will consider a function $\alpha_{\varepsilon,t_0,t_1}(t)$ which converges *a.e.* to $\mathbb{1}_{[t_0,t_1]}(t)$ and such that $\dfrac{d}{dt}\alpha_{\varepsilon,t_0,t_1}(t)$ is the difference between two identity approximations: the first one in t_0 and the second one in t_1. For this, let $\alpha \in \mathcal{C}^\infty(\mathbb{R})$ be a function such that $\alpha(t) = 0$ for $-\infty < t < 1/2$ and $\alpha(t) = 1$ for $1 < t < +\infty$. Then, for $\varepsilon < \min(t_0/2, T - t_1)$ we set the function $\alpha_{\varepsilon,t_0,t_1}(t) = \alpha\left(\dfrac{t-t_0}{\varepsilon}\right) - \alpha\left(\dfrac{t-t_1}{\varepsilon}\right)$.

On the other hand, for $R \geq 1$ let $\theta_R(x)$ be function test given in (24).Then, we consider the function test $\alpha_{\varepsilon,t_0,t_1}(t)\theta_R(x)$ and by (13) we can write

$$\begin{aligned}
&-\int_{\mathbb{R}}\int_{\mathbb{R}^3} \frac{|\mathbf{u}|^2 + |\nabla \otimes \mathbf{v}|^2}{2} \partial_s \alpha_{\varepsilon,t_0,t_1} \theta_R \, dx\, ds + \int_{\mathbb{R}}\int_{\mathbb{R}^3} |\nabla \otimes \mathbf{u}|^2 \alpha_{\varepsilon,t_0,t_1} \theta_R dx\, ds + \int_{\mathbb{R}}\int_{\mathbb{R}^3} |\Delta \mathbf{v}|^2 \alpha_{\varepsilon,t_0,t_1} \theta_R dx\, ds \\
&\quad \leq \int_{\mathbb{R}}\int_{\mathbb{R}^3} \left(\frac{|\mathbf{u}|^2 + |\nabla \otimes \mathbf{v}|^2}{2}\right) \alpha_{\varepsilon,t_0,t_1} \Delta\theta_R dx\, ds + \int_{\mathbb{R}}\int_{\mathbb{R}^3} \left(\left[\frac{|\mathbf{u}|^2 + |\nabla \otimes \mathbf{v}|^2}{2} + q\right] \mathbf{u}\right) \cdot \alpha_{\varepsilon,t_0,t_1} \nabla\theta_R dx\, ds \\
&\int_{\mathbb{R}}\int_{\mathbb{R}^3} \sum_{k=1}^{3} ([\mathbf{u} \cdot \nabla)\, \mathbf{v}] \cdot \partial_k \mathbf{v}) \alpha_{\varepsilon,t_0,t_1} \partial_k \theta_R dx\, ds - \int_{\mathbb{R}}\int_{\mathbb{R}^3} |\nabla \otimes \mathbf{v}|^2 \mathbf{v} \cdot \Delta \mathbf{v}\, \alpha_{\varepsilon,t_0,t_1} \theta_R dx\, ds.
\end{aligned}$$

Now, taking the limit when $\varepsilon \to 0$, by the dominated convergence theorem we obtain (when the limit in the left side is well-defined)

$$
\begin{aligned}
&-\lim_{\varepsilon\to 0}\int_{\mathbb{R}}\int_{\mathbb{R}^3}\frac{|\mathbf{u}|^2+|\nabla\otimes\mathbf{v}|^2}{2}\partial_s\alpha_{\varepsilon,t_0,t_1}\theta_R\,dx\,ds+\int_{t_0}^{t_1}\int_{\mathbb{R}^3}|\nabla\otimes\mathbf{u}|^2\theta_R dx\,ds+\int_{t_0}^{t_1}\int_{\mathbb{R}^3}|\Delta\mathbf{v}|^2\theta_R dx\,ds\\
&\quad\le\int_{t_0}^{t_1}\int_{\mathbb{R}^3}\left(\frac{|\mathbf{u}|^2+|\nabla\otimes\mathbf{v}|^2}{2}\right)\Delta\theta_R dx\,ds+\int_{t_0}^{t_1}\int_{\mathbb{R}^3}\left(\left[\frac{|\mathbf{u}|^2+|\nabla\otimes\mathbf{v}|^2}{2}+q\right]\mathbf{u}\right)\cdot\nabla\theta_R dx\,ds\\
&\int_{t_0}^{t_1}\int_{\mathbb{R}^3}\sum_{k=1}^{3}([\mathbf{u}\cdot\nabla)\,\mathbf{v}]\cdot\partial_k\mathbf{v})\partial_k\theta_R dx\,ds-\int_{t_0}^{t_1}\int_{\mathbb{R}^3}|\nabla\otimes\mathbf{v}|^2\mathbf{v}\cdot\Delta\mathbf{v}\,\theta_R dx\,ds.
\end{aligned}
$$

At this point, we must study the expression $-\lim_{\varepsilon\to 0}\int_{\mathbb{R}}\int_{\mathbb{R}^3}\frac{|\mathbf{u}|^2+|\nabla\otimes\mathbf{v}|^2}{2}\partial_s\alpha_{\varepsilon,t_0,t_1}\theta_R\,dx\,ds$. To make the writing more simple, let us define the function $A_R(s)=\int_{\mathbb{R}^3}\frac{|\mathbf{u}(s,x)|^2+|\nabla\otimes\mathbf{v}(s,x)|^2}{2}\theta_R\,dx$. Then, assuming that t_0 and t_1 are Lebesgue points of the function $A_R(s)$, and moreover, since

$$
\int_{\mathbb{R}}\int_{\mathbb{R}^3}\frac{|\mathbf{u}|^2+|\nabla\otimes\mathbf{v}|^2}{2}\partial_s\alpha_{\varepsilon,t_0,t_1}\theta_R\,dx\,ds=-\frac{1}{2}\int_{\mathbb{R}}A_R(s)\partial_s\alpha_{\varepsilon,t_0,t_1}ds,
$$

then we have

$$
-\lim_{\varepsilon\to 0}\int_{\mathbb{R}}\int_{\mathbb{R}^3}\frac{|\mathbf{u}|^2+|\nabla\otimes\mathbf{v}|^2}{2}\partial_s\alpha_{\varepsilon,t_0,t_1}\theta_R\,dx\,ds=\frac{1}{2}(A_R(t_1)-A_R(t_0)).
$$

On the other hand, recall that by point 4) in Definition 2.1 we have that the functions $\mathbf{u}(t,\cdot)$ and $\nabla\otimes\mathbf{v}(t,\cdot)$ are strong continuous at $t=0$ and then we can replace t_0 by 0. Moreover, for $0<t<T$, always by point 4) in Definition 2.1 we have that the functions $\mathbf{u}(t,\cdot)$ and $\nabla\otimes\mathbf{v}(t,\cdot)$ are weak continuous at t and then we obtain $A_R(t)\le\liminf_{t_1\to t}A_R(t_1)$. Thus, we can also replace t_1 for t.

With this information, for every $0\le t\le T$ we can write

$$
\begin{aligned}
&\int_{\mathbb{R}^3}\frac{|\mathbf{u}(t,\cdot)|^2+|\nabla\otimes\mathbf{v}(t,\cdot)|^2}{2}\theta_R\,dx+\int_0^t\int_{\mathbb{R}^3}|\nabla\otimes\mathbf{u}|^2\theta_R dx\,ds+\int_0^t\int_{\mathbb{R}^3}|\Delta\mathbf{v}|^2\theta_R dx\,ds\\
&\quad\le\int_{\mathbb{R}^3}\frac{|\mathbf{u}_0|^2+|\nabla\otimes\mathbf{v}_0|^2}{2}\theta_R\,dx+\int_0^t\int_{\mathbb{R}^3}\left(\frac{|\mathbf{u}|^2+|\nabla\otimes\mathbf{v}|^2}{2}\right)\Delta\theta_R dx\,ds\\
&\qquad+\int_0^t\int_{\mathbb{R}^3}\left(\left[\frac{|\mathbf{u}|^2+|\nabla\otimes\mathbf{v}|^2}{2}+q\right]\mathbf{u}\right)\cdot\nabla\theta_R dx\,ds+\int_0^t\int_{\mathbb{R}^3}\sum_{k=1}^{3}([\mathbf{u}\cdot\nabla)\,\mathbf{v}]\cdot\partial_k\mathbf{v})\partial_k\theta_R dx\,ds\\
&\quad-\int_0^t\int_{\mathbb{R}^3}|\nabla\otimes\mathbf{v}|^2\mathbf{v}\cdot\Delta\mathbf{v}\,\theta_R dx\,ds.
\end{aligned}
$$

In this inequality we must study now the term $-\int_0^t\int_{\mathbb{R}^3}|\nabla\otimes\mathbf{v}|^2\mathbf{v}\cdot\Delta\mathbf{v}\theta_R dx\,ds$. Recall that by (29) we have the identity (in the distributional sense) $|\nabla\otimes\mathbf{v}|^2=-\mathbf{v}\cdot\Delta\mathbf{v}$, moreover, as we have $|\mathbf{v}|=1$, then we can write

$$
\begin{aligned}
&-\int_0^t\int_{\mathbb{R}^3}|\nabla\otimes\mathbf{v}|^2\mathbf{v}\cdot\Delta\mathbf{v}\theta_R dx\,ds\\
&=\int_0^t\int_{\mathbb{R}^3}|\mathbf{v}\cdot\Delta\mathbf{v}|^2\theta_R dx\,ds\le\int_0^t\int_{\mathbb{R}^3}|\mathbf{v}|^2|\Delta\mathbf{v}|^2\theta_R dx\,ds\le\int_0^t\int_{\mathbb{R}^3}|\Delta\mathbf{v}|^2\theta_R dx\,ds.
\end{aligned}
$$

By this estimate and the previous inequality we get

$$\int_{\mathbb{R}^3} \frac{|\mathbf{u}|^2 + |\nabla \otimes \mathbf{v}|^2}{2} \theta_R \, dx + \int_0^t \int_{\mathbb{R}^3} |\nabla \otimes \mathbf{u}|^2 \theta_R dx\, ds \leq \int_{\mathbb{R}^3} \frac{|\mathbf{u}_0|^2 + |\nabla \otimes \mathbf{v}_0|^2}{2} \theta_R \, dx$$
$$+ \int_0^t \int_{\mathbb{R}^3} \left(\frac{|\mathbf{u}|^2 + |\nabla \otimes \mathbf{v}|^2}{2} \right) \Delta\theta_R dx\, ds + \int_0^t \int_{\mathbb{R}^3} \left(\left[\frac{|\mathbf{u}|^2 + |\nabla \otimes \mathbf{v}|^2}{2} + q \right] \mathbf{u} \right) \cdot \nabla\theta_R dx\, ds$$
$$- \int_0^s \int_{\mathbb{R}^3} \sum_{k=1}^{3} \partial_k([\mathbf{u} \cdot \nabla)\, \mathbf{v}] \cdot \partial_k \mathbf{v}) \theta_R dx\, ds.$$

Now, as we have $u_0 \in L^2(\mathbb{R}^3)$ and $\mathbf{v}_0 \in \dot{H}^1(\mathbb{R}^3)$, and moreover, recalling that $supp(\theta_R) \subset B_R$, $supp(\nabla\theta_R) \subset \mathcal{C}(R/2, R)$ and $supp(\Delta\theta_R) \subset \mathcal{C}(R/2, R)$, then we write

$$\int_{B_R} \frac{|\mathbf{u}|^2 + |\nabla \otimes \mathbf{v}|^2}{2} \theta_R \, dx + \int_0^t \int_{B_R} |\nabla \otimes \mathbf{u}|^2 \theta_R dx\, ds \leq \|u_0\|^2_{L^2} + \|\mathbf{v}_0\|^2_{\dot{H}^1}$$
$$+ \int_0^t \int_{\mathcal{C}(R/2,R)} \left(\frac{|\mathbf{u}|^2 + |\nabla \otimes \mathbf{v}|^2}{2} \right) \Delta\theta_R dx\, ds + \int_0^t \int_{\mathcal{C}(R/2,R)} \left(\left[\frac{|\mathbf{u}|^2 + |\nabla \otimes \mathbf{v}|^2}{2} + q \right] \mathbf{u} \right) \nabla\theta_R dx\, ds$$
$$+ \int_0^t \int_{\mathcal{C}(R/2,R)} \sum_{k=1}^{3} ([\mathbf{u} \cdot \nabla)\, \mathbf{v}] \cdot \partial_k \mathbf{v}) \partial_k \theta_R dx\, ds$$
$$= \|u_0\|^2_{L^2} + \|\mathbf{v}_0\|^2_{\dot{H}^1} + I_1 + I_2 + I_3, \tag{41}$$

where we will show that we have $\lim_{R \to +\infty} I_i = 0$ for $i = 1, 2, 3$. Indeed, for the term I_1 recall that we have $\|\Delta\theta_R\|_{L^\infty} \leq \frac{c}{R^2}$, and the we get

$$I_1 \leq \frac{c}{R^2} \int_0^t \int_{\mathcal{C}(R/2,R)} (|\mathbf{u}|^2 + |\nabla \otimes \mathbf{v}|^2) dx\, ds \leq c\, R^{1-6/p} \int_0^t \left(\int_{\mathcal{C}(R/2,R)} (|\mathbf{u}|^p + |\nabla \otimes \mathbf{v}|^p) dx \right)^{2/p} ds,$$

thereafter, by the Hölder inequalities in the temporal variable (with $1 = 2/p + (p-2)/p$), and moreover, recalling that we define $\eta(\gamma, p) = \gamma/p - 3/p + 2/3$, we have

$$c\, R^{1-6/p} \int_0^t \left(\int_{\mathcal{C}(R/2,R)} (|\mathbf{u}|^p + |\nabla \otimes \mathbf{v}|^p) dx \right)^{2/p} ds$$
$$\leq c\, R^{1-6/p} \left(\int_0^t \int_{\mathcal{C}(R/2,R)} (|\mathbf{u}|^p + |\nabla \otimes \mathbf{v}|^p) dx\, ds \right)^{2/p} t^{(p-2)/p}$$
$$\leq c\, R^{1-6/p+2\gamma/p} \left(\frac{1}{R^\gamma} \int_0^t \int_{\mathcal{C}(R/2,R)} (|\mathbf{u}|^p + |\nabla \otimes \mathbf{v}|^p) dx\, ds \right)^{2/p} t^{(p-2)/p}$$
$$\leq c\, R^{2(1/2-3/p+\gamma/p)} \left(\frac{1}{R^\gamma} \int_0^t \int_{\mathcal{C}(R/2,R)} (|\mathbf{u}|^p + |\nabla \otimes \mathbf{v}|^p) dx\, ds \right)^{2/p} t^{(p-2)/p}$$
$$\leq c\, R^{2(2/3-3/p+\gamma/p-1/6)} \left(\frac{1}{R^\gamma} \int_0^t \int_{\mathcal{C}(R/2,R)} (|\mathbf{u}|^p + |\nabla \otimes \mathbf{v}|^p) dx\, ds \right)^{2/p} t^{(p-2)/p}$$
$$\leq c\, R^{2\eta(\gamma,p)-1/3} \left(\frac{1}{R^\gamma} \int_0^t \int_{\mathcal{C}(R/2,R)} (|\mathbf{u}|^p + |\nabla \otimes \mathbf{v}|^p) dx\, ds \right)^{2/p} T^{(p-2)/p}.$$

Hence, as $\eta(\gamma,p)\leq 0$ and $R\geq 1$, we finally write

$$I_1 \leq c\,\frac{T^{(p-2)/p}}{R^{1/3}}\|\mathbf{u}\|^2_{M^p_\gamma L^p(0,T)} + c\frac{T^{(p-2)/p}}{R^{1/3}}\|\nabla\otimes\mathbf{v}\|^2_{M^3_1 L^3(0,T)}.$$

But, as we have the information $\mathbf{u}\in M^p_{\gamma,0}L^p(0,T)$ and $\nabla\otimes\mathbf{v}\in M^p_\gamma L^p(0,T)$, taking the limit when $R\to+\infty$ we obtain $\lim\limits_{R\to+\infty} I_1 = 0$.

For the term I_2, by the estimates (31), (32) and (33), we have

$$\begin{aligned} I_2 \leq & \int_0^t \left(\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^p dx\right)^{2/p} R^{2-9/p}\left(\int_{\mathcal{C}(R/2,R)}|\mathbf{u}|^p dx\right)^{1/p} ds\\ & + \int_0^t \left(\int_{\mathcal{C}(R/2,R)} |\nabla\otimes\mathbf{v}|^p dx\right)^{2/p} R^{2-9/p}\left(\int_{\mathcal{C}(R/2,R)}|\mathbf{u}|^p dx\right)^{1/p} ds\\ & + \int_0^t \left(\int_{\mathcal{C}(R/2,R)} |q|^{p/2} dx\right)^{2/p} R^{2-9/p}\left(\int_{\mathcal{C}(R/2,R)}|\mathbf{u}|^3 dx\right)^{1/p} ds, \end{aligned}$$

hence, since $\eta(\gamma,p)=\gamma/p-3/p+2/3$ then we write

$$\begin{aligned} I_2 \leq & \int_0^t \left(\frac{1}{R^\gamma}\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^p dx\right)^{2/p} R^{2-9/p+3\gamma/p}\left(\frac{1}{R^\gamma}\int_{\mathcal{C}(R/2,R)}|\mathbf{u}|^p dx\right)^{1/p} ds\\ & + \int_0^t \left(\frac{1}{R^\gamma}\int_{\mathcal{C}(R/2,R)} |\nabla\otimes\mathbf{v}|^p dx\right)^{2/p} R^{2-9/p+3\gamma/p}\left(\frac{1}{R^\gamma}\int_{\mathcal{C}(R/2,R)}|\mathbf{u}|^p dx\right)^{1/p} ds\\ & + \int_0^t \left(\frac{1}{R^\gamma}\int_{\mathcal{C}(R/2,R)} |q|^{p/2} dx\right)^{2/p} R^{2-9/p+3\gamma/p}\left(\frac{1}{R^\gamma}\int_{\mathcal{C}(R/2,R)}|\mathbf{u}|^3 dx\right)^{1/p} ds\\ \leq & R^{3\eta(\gamma,p)}\left[\int_0^t \left(\frac{1}{R^\gamma}\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^p dx\right)^{2/p} \left(\frac{1}{R^\gamma}\int_{\mathcal{C}(R/2,R)}|\mathbf{u}|^p dx\right)^{1/p} ds\right.\\ & + \int_0^t \left(\frac{1}{R^\gamma}\int_{\mathcal{C}(R/2,R)} |\nabla\otimes\mathbf{v}|^p dx\right)^{2/p} \left(\frac{1}{R^\gamma}\int_{\mathcal{C}(R/2,R)}|\mathbf{u}|^p dx\right)^{1/p} ds\\ & \left.\int_0^t \left(\frac{1}{R^\gamma}\int_{\mathcal{C}(R/2,R)} |q|^{p/2} dx\right)^{2/p} \left(\frac{1}{R^\gamma}\int_{\mathcal{C}(R/2,R)}|\mathbf{u}|^3 dx\right)^{1/p} ds\right]. \end{aligned}$$

Using first the fact that $\eta(\gamma,p)\leq 0$, and moreover, applying the Hölder inequalities in the temporal variable (with $1=2/p+1/p+(p-3)/p$), we obtain

$$\begin{aligned} I_2 \leq & \left(\frac{1}{R^\gamma}\int_0^t\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^p dx\,ds\right)^{2/p} \left(\frac{1}{R^\gamma}\int_0^t\int_{\mathcal{C}(R/2,R)}|\mathbf{u}|^p dx\,ds\right)^{1/p} t^{(p-3)/p}\\ & + \left(\frac{1}{R^\gamma}\int_0^t\int_{\mathcal{C}(R/2,R)} |\nabla\otimes\mathbf{v}|^p dx\,ds\right)^{2/p} \left(\frac{1}{R^\gamma}\int_0^t\int_{\mathcal{C}(R/2,R)}|\mathbf{u}|^p dx\,ds\right)^{1/p} t^{(p-3)/p}\\ & + \left(\frac{1}{R^\gamma}\int_0^t\int_{\mathcal{C}(R/2,R)} |q|^{p/2} dx\,ds\right)^{2/p} \left(\frac{1}{R^\gamma}\int_0^t\int_{\mathcal{C}(R/2,R)}|\mathbf{u}|^p dx\,ds\right)^{1/p} t^{(p-3)/p} \end{aligned}$$

$$\leq \left(\|\mathbf{u}\|^2_{M^p_\gamma L^p(0,T)} + \|\nabla\otimes\mathbf{v}\|^2_{M^p_\gamma L^p(0,T)} + \|q\|_{M^{p/2}_\gamma L^{p/2}(0,T)} \right) \left(\frac{1}{R^\gamma} \int_0^T \int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^p dx\, ds \right)^{1/p} T^{(p-3)/p}.$$

At this point, as we have $\mathbf{u} \in M^p_{\gamma,0}L^p(0,T)$ and $\nabla\otimes\mathbf{v} \in M^p_\gamma L^p(0,T)$, the by the point 3) of Lemma 3.2 we get

$$\|q\|_{M^{p/2}_\gamma L^{p/2}(0,T)} \leq c \left(\|\mathbf{u}\|^2_{M^p_\gamma L^p(0,T)} + \|\nabla\otimes\mathbf{v}\|^2_{M^p_\gamma L^p(0,T)} \right).$$

Thus, getting back to the previous estimate we can write

$$I_2 \leq c \left(\|\mathbf{u}\|^2_{M^p_\gamma L^p(0,T)} + \|\nabla\otimes\mathbf{v}\|^2_{M^p_\gamma L^p(0,T)} \right) \left(\frac{1}{R^\gamma} \int_0^T \int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^p dx\, ds \right)^{1/p} T^{(p-3)/p},$$

and then, taking the limit when $R \to +\infty$ we have $\lim\limits_{R\to+\infty} I_2 = 0$.

Finally, for the term I_3, applying the Hölder inequalities in the spatial variable (with $1 = 1/p + 2/p + (p-3)/p$), we have

$$\begin{aligned} I_3 &= \sum_{i,j,k=1}^{3} \int_0^t \int_{\mathcal{C}(R/2,R)} (u_j\partial_j v_i)(\partial_k v_i)\partial_k\theta_R dx\, ds \leq c \int_0^t \int_{\mathbb{C}(R/2,R)} |\mathbf{u}||\nabla\otimes\mathbf{v}|^2|\nabla\theta_R| dx\, ds \\ &\leq c \int_0^t \left(\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^p dx \right)^{1/p} \left(\int_{\mathcal{C}(R/2,R)} |\nabla\otimes\mathbf{v}|^p dx \right)^{2/p} \left(\int_{\mathcal{C}(R/2,R)} |\nabla\theta_R|^{p/(p-3)} dx \right)^{(p-3)/p} ds. \end{aligned}$$

Moreover, in the last term, as $\|\nabla\theta_R\|_{L^\infty} \leq \frac{c}{R}$ then we can write

$$\begin{aligned} I_3 &\leq c \left[\int_0^t \left(\int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^p dx \right)^{1/p} \left(\int_{\mathcal{C}(R/2,R)} |\nabla\otimes\mathbf{v}|^p dx \right)^{2/p} ds \right] R^{2-9/p} \\ &\leq c \left[\int_0^t \left(\frac{1}{R^\gamma} \int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^p dx \right)^{1/p} \left(\frac{1}{R^\gamma} \int_{\mathcal{C}(R/2,R)} |\nabla\otimes\mathbf{v}|^p dx \right)^{2/p} ds \right] R^{2-9/p+3\gamma/p}. \end{aligned}$$

Then, recalling that $\eta(\gamma,p) = 2/3 - 3/p + \gamma/p$, and moreover, as we assume $\eta(\gamma,p) \leq 0$, we obtain

$$\begin{aligned} I_3 &\leq c \left[\int_0^t \left(\frac{1}{R^\gamma} \int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^p dx \right)^{1/p} \left(\frac{1}{R^\gamma} \int_{\mathcal{C}(R/2,R)} |\nabla\otimes\mathbf{v}|^p dx \right)^{2/p} ds \right] R^{3\eta(\gamma,p)} \\ &\leq c \int_0^t \left(\frac{1}{R^\gamma} \int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^p dx \right)^{1/p} \left(\frac{1}{R^\gamma} \int_{\mathcal{C}(R/2,R)} |\nabla\otimes\mathbf{v}|^p dx \right)^{2/p} ds. \end{aligned}$$

We apply now the Hölder inequalities in the temporal variable (with $1 = 1/p + 2/p + (p-3)/p$) to write

$$\begin{aligned} I_3 &\leq c \left(\frac{1}{R^\gamma} \int_0^t \int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^p dx\, ds \right)^{1/p} \left(\frac{1}{R^\gamma} \int_0^t \int_{\mathcal{C}(R/2,R)} |\nabla\otimes\mathbf{v}|^p dx\, ds \right)^{2/p} t^{(p-3)/p} \\ &\leq c \left(\frac{1}{R^\gamma} \int_0^t \int_{\mathcal{C}(R/2,R)} |\mathbf{u}|^p dx\, ds \right)^{1/p} \|\nabla\otimes\mathbf{v}\|^2_{M^p_\gamma L^p(0,T)} T^{(p-3)/p}. \end{aligned}$$

Hence, as $\mathbf{u} \in M^p_{\gamma,0}L^p(0,T)$, taking the limit when $R \to +\infty$ we obtain $\lim\limits_{R\to+\infty} I_3 = 0$.

Once we have proven that $\lim\limits_{R\to+\infty} I_i = 0$ for $i = 1,2,3$, we get back to (41) where we take the limit when $R \to +\infty$, and thus for $0 \leq t \leq T$ we get the global energy inequality (17). Theorem 2 is proven. □

6.2. Proof of Corollary 2

This proof is straightforward. We just observe that by the global energy inequality (17) if the initial datum verify $\mathbf{u}_0 = 0$ and $\nabla \otimes \mathbf{v}_0 = 0$ then for all time $0 < t \leq T$ we have $\|\mathbf{u}(t,\cdot)\|^2_{L^2} = 0$ and $\|\mathbf{v}(t,\cdot)\|^2_{\dot{H}^1} = 0$, hence $\mathbf{u} = 0$ and $\nabla \otimes \mathbf{v} = 0$ on $[0,T] \times \mathbb{R}^3$. Moreover, always following the computations done to obtain the identity (39) we have $p = q = 0$.

Declarations

Conflict of interest All authors declare that they have no conflicts of interest.

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A Appendix

Consider the velocity field $\mathbf{u}$, the pressure term p and the vector field $\mathbf{v}$ defined as follows:

$$\mathbf{u}(x_1,x_2,x_3) = (2x_1, 2x_2, -4x_3), \quad p(x_1,x_2,x_3) = -(2x_1^2 + 2x_2^2 + 8x_2^3),$$
$$\text{and} \quad \mathbf{v}(x_1,x_2,x_3) = \begin{cases} (x_1,x_2,0), & \text{if } x_1^2 + x_2^2 = 1, \\ \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), & \text{if } x_1^2 + x_2^2 \neq 1. \end{cases} \tag{42}$$

We have $|\mathbf{v}| = 1$, and using some basic rules of the vector calculus we easily get that the triplet $(\mathbf{u}, p, \mathbf{v})$ defined as above is also a solution of the system (1). Indeed, for the first equation in the system (1), by definition of the vector field $\mathbf{v}(x_1,x_2,x_2)$ above we observe first that we have $\text{div}(\nabla \otimes \mathbf{v} \odot \nabla \otimes \mathbf{v}) = 0$. Thereafter, if we set the scalar field $\psi(x_1,x_2,x_3) = x_1^2 + x_2^2 - 2x_3^2$ we may observe that we have $\mathbf{u} = \nabla \psi$ and moreover we have $\Delta \psi = 0$. With these identities we can write the following computations. First, we have $\Delta \mathbf{u} = \Delta(\nabla \psi) = \nabla(\Delta \psi) = 0$. On the other hand we have $(\mathbf{u} \cdot \nabla)\mathbf{u} = \dfrac{1}{2}|\mathbf{u}| + (\nabla \wedge \mathbf{u}) \wedge \mathbf{u}$, and as $\mathbf{u} = \nabla \psi$ then we get $(\mathbf{u} \cdot \nabla)\mathbf{u} = \dfrac{1}{2}|\mathbf{u}|$. With this identity and the definition of the pressure term p given above we find that $(\mathbf{u}, p, \mathbf{v})$ verify the first equation of (1). Moreover we have $div(\mathbf{u}) = 0$.

For the second equation in (1), we observe first that for the case $x_1^2 + x_2^2 \neq 1$ the vector field $\mathbf{v}(x_1,x_2,x_3)$ defined above is a constant vector and then the second equation in (1) trivially holds. For the other case, when $x_1^2 + x_2^2 = 1$, we have $\mathbf{v}(x_1,x_2,x_3) = (x_1,x_2,0)$ and then we get $\Delta \mathbf{v} = 0$. Moreover, by definition of the vector field $\mathbf{u}(x_1,x_2,x_3)$ is easy to see that we have the identity $(\mathbf{u} \cdot \nabla)\mathbf{v} = |\nabla \otimes \mathbf{v}|^2 \mathbf{v}$. Then, the second equation in (1) also holds true.

B Appendix

Here we give a proof of the embedding (18). It is enough to prove the last inclusion in this embedding, and for this, for all $R \geq 1$ and $t \geq 0$ fixed, we have the estimate

$$\int_{B_R} |f(t,x)|^p dx \leq c R^{3(1-p/r)} \|f(t,\cdot)\|^p_{L^{r,\infty}} \leq c R^{3(1-p/r)} \|f(t,\cdot)\|^p_{L^{r,q}},$$

with $r < q < +\infty$. For a proof of this estimate the Proposition 1.1.10, page 21 of the book [8]. Then, for $0 < \gamma < 3$ we write

$$\frac{1}{R^\gamma} \int_{B_R} |f(t,x)|^p dx \leq c R^{3(1-p/r)-\gamma} \|f(t,\cdot)\|^p_{L^{r,q}}.$$

In the right side of this estimate we impose now the condition $3(1-p/r)-\gamma\leq 0$, which is equivalent to the inequality $2/3-3/r\leq \gamma/p-3/p+2/3=\eta(\gamma,p)$. Moreover, as we assume $\eta(\gamma,p)\leq 0$ we get $2/3-3/r\leq 0$ which give us the restriction on the parameter r: $r\leq 9/2$.

Thus, when $3(1-p/r)-\gamma\leq 0$ holds, we can write

$$\frac{1}{R^\gamma}\int_{B_R}|f(t,x)|^p dx \leq cR^{3(1-p/r)-\gamma}\|f(t,\cdot)\|^p_{L^{r,\infty}}\leq c\|f(t,\cdot)\|^p_{L^{r,q}},$$

hence, integrating in the interval of time $[0,T]$, and moreover, following the same to prove (11), we finally obtain the last inclusion in (18).

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