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Remarks on the Solution Map for Yudovich Solutions of the Euler Equations

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Abstract. Consider Yudovich solutions to the incompressible Euler equations with bounded initial vorticity in bounded planar domains. We present a purely Lagrangian proof that the solution map is strongly continuous in L^p for all $p \in [1, \infty)$ and is weakly-$*$ continuous in L^∞.

Keywords. Solution map, Weak continuity, Yudovich solution, Euler equations.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^2$ with C^2 boundary $\partial\Omega$. Let Δ_D denote the Dirichlet Laplacian associated to Ω. The vorticity formulation of the 2D incompressible Euler equations is

$$\partial_t \omega + u \cdot \nabla \omega = 0, \quad (x, t) \in \Omega \times (0, \infty), \tag{1.1}$$

where the velocity u is recovered from the vorticity ω through the Biot-Savart law

$$u = \nabla^\perp \Delta_D^{-1} \omega = K * \omega. \tag{1.2}$$

Note that u given by (1.2) is parallel to the boundary $\partial\Omega$. In the celebrated work [6], Yudovich proved the existence and uniqueness of global solutions to (1.1) with bounded initial vorticity. This theory includes the important class of vortex patches [3].

Theorem 1.1 [4,6]. *Let $\omega_0 \in L^\infty(\Omega)$. There exists a unique triple (ω, u, X_t) solution to (1.1) such that $\omega \in L^\infty(\mathbb{R}; L^\infty(\Omega))$, $u(t) = K * \omega(t)$, $X_t : \Omega \to \Omega$ measure-preserving, invertible and*

$$\frac{d}{dt} X_t(x) = u(X_t(x), t), \quad X_0(x) = x \quad \forall x \in \overline{\Omega}, \tag{1.3}$$

$$\omega(x, t) = \omega_0(X_t^{-1}(x)). \tag{1.4}$$

Moreover, the flow $X_t : \overline{\Omega} \to \overline{\Omega}$ is Hölder continuous on $\overline{\Omega}$ with exponent $\exp(-C|t|\|\omega_0\|_{L^\infty(\Omega)})$ for some $C = C(\Omega)$.

The preceding version of Yudovich theory is taken from [4] and is elegant in that the notion of solution is naturally defined in terms of the Lagrangian flow and does not involve test functions. The purpose of this note is to present proofs of folklore about the continuity of the solution map for Yudovich solutions in this purely Lagrangian framework.

To define the inverse of the flow X_t, we let $X_{s,t}(x)$ be the solution of

$$\frac{d}{dt} X_{s,t}(x) = u(X_{s,t}(x), t), \quad X_{s,s}(x) = x. \tag{1.5}$$

In view of (1.3), we denote $X_{0,t} \equiv X_t$. Then we have

$$X_t^{-1} = X_{t,0},$$

and (1.4) becomes

$$\omega(x,t) = \omega_0(X_{t,0}(x)). \tag{1.6}$$

We first state the continuity in time of Yudovich solutions.

Lemma 1.2. *For all initial data $\omega_0 \in L^\infty(\Omega)$, the unique solution ω given by Theorem 1.1 belongs to $C(\mathbb{R}; L^p(\Omega)) \cap C_w(\mathbb{R}; L^\infty(\Omega))$ for all $p \in [1,\infty)$. Here $C_w(\mathbb{R}; L^\infty(\Omega))$ denotes the space of functions that are continuous in time with values in the weak-$*$ topology of $L^\infty(\Omega)$.*

Proof. We first note that since the velocity field u is Log-Lipschitz (see (2.5)), $X_t(x) \in C(\overline{\Omega} \times \mathbb{R})$ (see [4]). Therefore, if $\omega_0 \in C(\overline{\Omega})$ then it is clear that $\omega \in C(\mathbb{R}; L^p(\Omega))$ for all $p \in [1,\infty]$. For $p \in [1,\infty)$ and $\omega_0 \in L^\infty(\Omega) \subset L^p(\Omega)$, using the fact that $C(\overline{\Omega})$ is dense in $L^p(\Omega)$ and X_t is measure-preserving, we obtain $\omega \in C(\mathbb{R}; L^p(\Omega))$.

For any $f \in C(\overline{\Omega})$, (1.4) yields

$$g(t) := \int_\Omega \omega(x,t) f(x) dx = \int_\Omega \omega_0(X_t^{-1}(x)) f(x) dx = \int_\Omega \omega_0(x) f(X_t(x)) dx.$$

Thus $g \in C(\mathbb{R})$ since $X_t(x) \in C(\overline{\Omega} \times \mathbb{R})$. Since $C(\overline{\Omega})$ is dense in $L^1(\Omega)$ and X_t is measure-preserving, it follows that $t \mapsto \int_\Omega \omega(x,t) f(x) dx$ is continuous for all $f \in L^1(\Omega)$. Therefore, $\omega \in C_w(\mathbb{R}; L^\infty(\Omega))$. $\qquad\square$

By virtue of Lemma 1.2, for every $t > 0$, the solution map

$$S_t : L^\infty(\Omega) \ni \omega_0 \mapsto \omega(t) \in L^\infty(\Omega) \tag{1.7}$$

is well defined. We prove that S_t is strongly continuous in $L^p(\Omega)$ for all $p \in [1,\infty)$.

Theorem 1.3. *Let $p \in [1,\infty)$. Let $\omega_0, \omega_0^n \in L^\infty(\Omega)$ such that $(\omega_0^n)_n$ converges to ω_0 in $L^p(\Omega)$. Then for all $T > 0$ we have*

$$\lim_{n \to \infty} \sup_{t \in [-T,T]} \|S_t(\omega_0^n) - S_t(\omega_0)\|_{L^p(\Omega)} = 0. \tag{1.8}$$

Moreover, S_t is continuous in the weak-$*$ topology of $L^\infty(\Omega)$.

Theorem 1.4. *If $\omega_0^n \xrightarrow{*} \omega_0$ in $L^\infty(\Omega)$, then $S_\cdot(\omega_0^n) \xrightarrow{*} S_\cdot(\omega_0)$ in $L^\infty(\Omega \times (-T,T))$ for all $T > 0$ and $S_t(\omega_0^n) \xrightarrow{*} S_t(\omega_0)$ in $L^\infty(\Omega)$ for all $t \in \mathbb{R}$.*

It was obtained in [2, Corollary 1] that for the torus $\mathbb{T}^2$, the solution map for Yudovich solutions is continuous in L^p *when restricted to bounded sets of L^∞*. Theorem 1.3 dispenses with the restriction to bounded sets of L^∞ and holds for domains with boundary. The proof in [2] is Eulerian and relies on L^2 energy estimates for the velocity and vorticity differences. On the other hand, our proof of Theorem 1.3 is purely Lagrangian: L^p estimates for the vorticity difference is deduced from an L^1 estimate for the difference of the flow maps. The latter is established by employing an idea in [4] for the uniqueness of Yudovich solutions. We remark that Theorem 1.4 is stated without proof in [5] and is used to deduce properties of the omega-limit set of the 2D Euler equations.

On the whole space $\Omega = \mathbb{R}^2$, the same statement in Theorem 1.1 holds with $L^\infty(\Omega)$ replaced by $L_c^\infty(\mathbb{R}^2)$, the space of $L^\infty(\mathbb{R}^2)$ functions with compact support. Note however that the flow map X_t is then only *locally* Hölder continuous with exponent $\exp(-C|t|\|\omega_0\|_{L^1 \cap L^\infty})$, where C is a universal constant and $L^1 \cap L^\infty \equiv L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ is equipped with the norm

$$\|\cdot\|_{L^1 \cap L^\infty} = \|\cdot\|_{L^1(\mathbb{R}^2)} + \|\cdot\|_{L^\infty(\mathbb{R}^2)}.$$

We have the following version of Theorem 1.4.

Theorem 1.5. *Assume that $\omega_0 \in L_c^\infty(\mathbb{R}^2)$, $(\omega_0^n)_n \subset L_c^\infty(\mathbb{R}^2)$ is bounded in $L^1(\mathbb{R}^2)$ and $\omega_0^n \xrightarrow{*} \omega_0$ in $L^\infty(\mathbb{R}^2)$. Then $S_\cdot(\omega_0^n) \xrightarrow{*} S_\cdot(\omega_0)$ in $L^\infty(\Omega \times (-T,T))$ for all $T > 0$ and $S_t(\omega_0^n) \xrightarrow{*} S_t(\omega_0)$ in $L^\infty(\mathbb{R}^2)$ for all $t \in \mathbb{R}$.*

Under the hypothesis of Theorem 1.5, for all $p \in [1, \infty]$ we have that $\|S_t(\omega_0^n)\|_{L^p} = \|\omega_0^n\|_{L^p}$ is uniformly bounded by interpolation. Therefore, the conclusion in Theorem 1.5 implies $S_t(\omega_0^n) \overset{*}{\rightharpoonup} S_t(\omega_0)$ in $\mathcal{M}(\mathbb{R}^2)$, the space of signed Radon measures on $\mathbb{R}^2$, and $S_t(\omega_0^n) \rightharpoonup S_t(\omega_0)$ in $L^p(\Omega)$ for all $p \in (1, \infty)$.

Remark 1.6. As we have mentioned earlier, the notion of solution in Theorem 1.1 does not involve test functions. On the other hand, if ω is such a solution, then for any $\phi \in C^1(\overline{\Omega} \times [t_1, t_2])$ we have

$$\int_{t_1}^{t_2} \int_{\Omega} \omega(x,t) \partial_t \phi(x,t) dx dt$$

$$= \int_{t_1}^{t_2} \int_{\Omega} \omega_0(x) (\partial_t \phi)(X_t(x), t) dx dt$$

$$= \int_{t_1}^{t_2} \int_{\Omega} \omega_0(x) \partial_t [\phi(X_t(x), t)] dx dt - \int_{t_1}^{t_2} \int_{\Omega} \omega_0(x) \nabla \phi(X_t(x), t) \cdot \partial_t X_t(x) dx dt$$

$$= \int_{\Omega} \omega_0(x) [\phi(X_{t_2}(x), t_2) - \phi(X_{t_1}(x), t_1)] dx - \int_{t_1}^{t_2} \int_{\Omega} \omega_0(x) \nabla \phi(X_t(x), t) \cdot u(X_t(x), t) dx dt$$

$$= \int_{\Omega} \omega(x, t_2) \phi(x, t_2) - \omega(x, t_1) \phi(x, t_1) dx - \int_{t_1}^{t_2} \int_{\Omega} \omega(x, t) u(x, t) \cdot \nabla \phi(x, t) dx dt.$$

Thus ω obeys the weak form

$$\int_{t_1}^{t_2} \int_{\Omega} \omega(x,t) \left[\partial_t \phi(x,t) + u(x,t) \cdot \nabla \phi(x,t) \right] dx dt = \int_{\Omega} \omega(x, t_2) \phi(x, t_2) - \omega(x, t_1) \phi(x, t_1) dx. \qquad (1.9)$$

2. Proof of Theorem 1.3

We first recall the following estimates for the Biot-Savart kernel K.

Lemma 2.1 [6]. *There exists C depending only on Ω such that for all x, y, a, $b \in \Omega$, we have*

$$|K(x,y)| \le C|x - y|^{-1}, \qquad (2.1)$$

$$\int_{\Omega} |K(x,a) - K(x,b)| dx \le C\varphi(|a - b|), \qquad (2.2)$$

where φ is the Log-Lipschitz modulus of continuity

$$\varphi(r) = r(1 - \ln r) \quad if \ 0 < r \le 1, \quad \varphi(r) = 1 \quad if \ r > 1. \qquad (2.3)$$

As a direct consequence of (2.1) and (2.2), if $\omega \in L^\infty(\Omega)$ then $u = K * \omega$ is bounded and Log-Lipschitz:

$$\|u\|_{L^\infty(\Omega)} \le C\|\omega\|_{L^\infty(\Omega)}, \qquad (2.4)$$

$$|u(x) - u(y)| \le C\|\omega\|_{L^\infty} \varphi(|x - y|) \quad \forall x, \ y \in \Omega. \qquad (2.5)$$

Let (ω^j, u^j, X_t^j), $j = 1, 2$ be two solutions of (1.1) with initial data $\omega_0^j \in L^\infty(\Omega)$. For notational simplicity we shall write $L^p \equiv L^p(\Omega)$. Fix $p \in [1, \infty)$. We have the elementary inequalities

$$(a + b)^p \le 2^{p-1}(a^p + b^p), \quad (a + b + c)^p \le 2^{p-1}a^p + 2^{2p-2}(b^p + c^p) \quad \forall a, b, c \in \mathbb{R}_+. \qquad (2.6)$$

Since the flow maps $X_{s,t}^j$ are measure-preserving, we have

$$\|\omega^1(t) - \omega^2(t)\|_{L^p}^p = \int_\Omega |\omega_0^1(X_{t,0}^1(x)) - \omega_0^2(X_{t,0}^2(x))|^p dx$$

$$\leq 2^{p-1} \int_\Omega |\omega_0^1(X_{t,0}^1(x)) - \omega_0^1(X_{t,0}^2(x))|^p dx$$

$$+ 2^{p-1} \int_\Omega |\omega_0^1(X_{t,0}^2(x)) - \omega_0^2(X_{t,0}^2(x))|^p dx \tag{2.7}$$

$$\leq 2^{p-1} \int_\Omega |\omega_0^1(X_{t,0}^1(x)) - \omega_0^1(X_{t,0}^2(x))|^p dx + 2^{p-1}\|\omega_0^1 - \omega_0^2\|_{L^p}^p.$$

We extend ω_0^1 to zero outside Ω and approximate ω_0^1 by $\omega_0^1 * \rho_\varepsilon$, where ρ_ε is the standard mollifier. It follows from (2.7) that

$$\|\omega^1(t) - \omega^2(t)\|_{L^p}^p \leq 2^{p-1}\|\omega_0^1 - \omega_0^2\|_{L^p}^p + 2^{3p-2}\|\omega_0^1 * \rho_\varepsilon - \omega_0^1\|_{L^p}^p$$

$$+ 2^{2p-2} \int_\Omega |(\omega_0^1 * \rho_\varepsilon)(X_{t,0}^1(x)) - (\omega_0^1 * \rho_\varepsilon)(X_{t,0}^2(x))|^p dx$$

$$\leq 2^{p-1}\|\omega_0^1 - \omega_0^2\|_{L^p}^p + 2^{3p-2}\|\omega_0^1 * \rho_\varepsilon - \omega_0^1\|_{L^p}^p \tag{2.8}$$

$$+ 2^{2p-2}\|\omega_0^1 * \rho_\varepsilon\|_{\dot{C}^{1/p}}^p \int_\Omega |X_{t,0}^1(x) - X_{t,0}^2(x)| dx.$$

Set $F(x,t,r) := |X_{t,r}^1(x) - X_{t,r}^2(x)|$. Integrating (1.5) with respect to t we deduce

$$F(x,t,r) \leq \left| \int_t^r |u^1(X_{t,s}^1(x),s) - u^1(X_{t,s}^2(x),s)|ds \right| + \left| \int_t^r |u^1(X_{t,s}^2(x),s) - u^2(X_{t,s}^2(x),s)|ds \right| \tag{2.9}$$

$$:= I_1(x,t,r) + I_2(x,t,r).$$

The Log-Lipschitz bound (2.5) yields

$$|I_1(x,t,r)| \leq C\|\omega_0^1\|_{L^\infty} \left| \int_t^r \varphi(F(x,t,s))ds \right|. \tag{2.10}$$

As for I_2 we use the definition $u^j = K * \omega^j$, (1.6) together the fact that the maps $X_{s,t}^j$ are measure-preserving, giving

$$|I_2(x,t,r)| \leq \left| \int_t^r \left| \int_\Omega K(X_{t,s}^2(x),y)\omega_0^1(X_{s,0}^1(y)) - K(X_{t,s}^2(x),y)\omega_0^2(X_{s,0}^2(y))dy \right| ds \right|$$

$$= \left| \int_t^r \left| \int_\Omega K(X_{t,s}^2(x),X_s^1(y))\omega_0^1(y) - K(X_{t,s}^2(x),X_s^2(y))\omega_0^2(y)dy \right| ds \right|$$

$$\leq \left| \int_t^r \int_\Omega |K(X_{t,s}^2(x),X_s^1(y)) - K(X_{t,s}^2(x),X_s^2(y))||\omega_0^1(y)|dyds \right|$$

$$+ \left| \int_t^r \int_\Omega |K(X_{t,s}^2(x),X_s^2(y))||\omega_0^1(y) - \omega_0^2(y)|dyds \right| := I_2^a + I_2^b.$$

Integrating I_2 in x and using the fact that $X_{t,s}^2$ is measure-preserving, we deduce

$$\int_\Omega |I_2^a(x,t,r)|dx = \left| \int_t^r \int_\Omega \int_\Omega |K(x,X_s^1(y)) - K(x,X_s^2(y))|dx|\omega_0^1(y)|dyds \right|$$

$$\leq C\|\omega_0^1\|_{L^\infty} \left| \int_t^r \int_\Omega \varphi(F(y,0,s))dyds \right|,$$

where we have used (2.2) in the second estimate. Since φ is concave, Jensen's inequality implies

$$\frac{1}{|\Omega|} \int_\Omega |I_2^a(x,t,r)|dx \leq C\|\omega_0^1\|_{L^\infty} \left| \int_t^r \varphi\left(\frac{1}{|\Omega|} \int_\Omega F(y,0,s)dy\right) ds \right|. \tag{2.11}$$

On the other hand, (2.1) gives

$$\int_\Omega |I_2^b(x,t,r)|dx \leq C \left| \int_t^r \int_\Omega \int_\Omega \frac{1}{|X_{t,s}^2(x) - X_s^2(y)|} dx |\omega_0^1(y) - \omega_0^2(y)|dyds \right|$$

$$= C \left| \int_t^r \int_\Omega \int_\Omega \frac{1}{|x - X_s^2(y)|} dx |\omega_0^1(y) - \omega_0^2(y)|dyds \right| \quad (2.12)$$

$$\leq C|t - r|\|\omega_0^1 - \omega_0^2\|_{L^1}.$$

A combination of (2.9), (2.10), (2.11) and (2.12) implies that $\eta(t,r) := |\Omega|^{-1} \int_\Omega F(x,t,r)dx$ satisfies

$$\eta(t,r) \leq C|t-r|\|\omega_0^1 - \omega_0^2\|_{L^1} + C\|\omega_0^1\|_{L^\infty} \left\{ \left| \int_t^r \varphi(\eta(t,s))ds \right| + \left| \int_t^r \varphi(\eta(0,s))ds \right| \right\} \quad (2.13)$$

for all $t, r \in \mathbb{R}$, where $C = C(\Omega)$.

Let $T > 0$ be arbitrary. We first consider $t = 0$ in (2.13). We choose ω_0^j such that $CT\|\omega_0^1 - \omega_0^2\|_{L^1} < 1$. Since $\eta(0,0) = 0$ and $\eta(0,\cdot)$ is continuous, there exists a maximal time $T_1 \in (0,T]$ such that $\eta(0,s) < 1$ for all $s \in [0,T_1)$. Consequently, in (2.13) we have $\varphi(\eta(0,s)) = \eta(0,s)[1 - \ln(\eta(0,s))]$ provided that $s \in [0,T_1)$. An application of Osgood's lemma [1, Lemma 3.4] yields

$$\eta(0,r) \leq e^{1-\exp(-C|r|\|\omega_0^1\|_{L^\infty})} \left(CT\|\omega_0^1 - \omega_0^2\|_{L^1} \right)^{\exp(-C|r|\|\omega_0^1\|_{L^\infty})} \quad \forall r \in [0,T_1]. \quad (2.14)$$

Using (2.14) with $r = T_1$ we find that if

$$CT\|\omega_0^1 - \omega_0^2\|_{L^1} < e^{1-\exp(CT\|\omega_0^1\|_{L^\infty})}, \quad (2.15)$$

then $\eta(0,T_1) < 1$, and hence $T_1 = T$ and (2.14) holds for all $r \in [0,T]$. By the same argument, we obtain (2.14) for all $r \in [-T,T]$. Since φ is increasing, inserting (2.14) into the right-hand side of (2.13), we deduce

$$\eta(t,r) \leq \Phi_{T,\|\omega_0^1\|_{L^\infty}}(\|\omega_0^1 - \omega_0^2\|_{L^1}) + C\|\omega_0^1\|_{L^\infty} \left| \int_t^r \varphi(\eta(t,s))ds \right| \quad (2.16)$$

for all $t, r \in [-T,T]$, where

$$\Phi_{T,\|\omega_0^1\|_{L^\infty}}(z) = CTz + CT\|\omega_0^1\|_{L^\infty} \varphi \left(e \left(CTz \right)^{\exp(-CT\|\omega_0^1\|_{L^\infty})} \right)$$

with C depending only on Ω. Clearly $\Phi(z) \to 0$ as $z \to 0$. Similarly to (2.14), we can apply Osgood's lemma to (2.16) and obtain

$$\eta(t,r) \leq e^{1-\exp(-C|t-r|\|\omega_0^1\|_{L^\infty})} \Phi_{T,\|\omega_0^1\|_{L^\infty}}(\|\omega_0^1 - \omega_0^2\|_{L^1})^{\exp(-C|t-r|\|\omega_0^1\|_{L^\infty})} \quad (2.17)$$

for all $t, r \in [-T,T]$ provided that

$$\Phi_{T,\|\omega_0^1\|_{L^\infty}}(\|\omega_0^1 - \omega_0^2\|_{L^1}) < e^{1-\exp(2CT\|\omega_0^1\|_{L^\infty})}. \quad (2.18)$$

By virtue of (2.17), (2.8) yields

$$\|\omega^1(t) - \omega^2(t)\|_{L^p}^p \leq 2^{p-1}\|\omega_0^1 - \omega_0^2\|_{L^p}^p + 2^{3p-2}\|\omega_0^1 * \rho_\varepsilon - \omega_0^1\|_{L^p}^p$$

$$+ 2^{2p-2}\|\omega_0^1 * \rho_\varepsilon\|_{\dot{C}^{1/p}}^p |\Omega| e^{1-\exp(-CT\|\omega_0^1\|_{L^\infty})} \Phi_{T,\|\omega_0^1\|_{L^\infty}}(\|\omega_0^1 - \omega_0^2\|_{L^1})^{\exp(-CT\|\omega_0^1\|_{L^\infty})} \quad (2.19)$$

for all $t \in [-T,T]$.

To obtain Theorem 1.3, let $\omega_0, \omega_0^n \in L^\infty(\Omega)$ such that $(\omega_0^n)_n$ converges to ω_0 in $L^p(\Omega)$. For $n \geq N$ sufficiently large, the smallness conditions (2.15) and (2.18) hold for $\omega_0 - \omega_0^n$, so that (2.19) holds for $S_t(\omega_0) - S_t(\omega_0^n)$. In (2.19), taking the supremum over $t \in [0,T]$, then letting $n \to \infty$ followed by $\varepsilon \to 0$, we conclude that

$$\lim_{n\to\infty} \sup_{t\in[-T,T]} \|S_t(\omega_0^n) - S_t(\omega_0)\|_{L^p(\Omega)} = 0. \quad (2.20)$$

The proof of Theorem 1.3 is complete.

3. Proof of Theorem 1.4

Assume that $\omega_0^n \overset{*}{\rightharpoonup} \omega_0$ in $L^\infty(\Omega)$. Let (ω^n, u^n, X_t^n) (resp. (ω, u, X_t)) be the Yudovich solution of (1.1) with initial data ω_0^n (resp. ω_0). Fix $T > 0$ arbitrary. With $M = \sup_n \|\omega_0^n\|_{L^\infty} < \infty$ we have $\sup_n \|\omega^n\|_{L^\infty(\Omega \times (-T,T))} \leq M$. Thus there exists a subsequence $\omega^{n_k} \overset{*}{\rightharpoonup} \omega^\infty$ in $L^\infty(\Omega \times (-T,T))$. Define

$$u^\infty(t) = K * \omega^\infty(t),$$

$$\frac{d}{dt} X_t^\infty(x) = u^\infty(X_t^\infty(x), t), \quad X_0^\infty(x) = x \quad \forall x \in \overline{\Omega}.$$

Note that u^∞ is divergence-free and Log-Lipschitz, whence X_t^∞ is measure-preserving. We claim that

$$\omega^\infty(x) = \omega_0(X_{t,0}^\infty(x)). \tag{3.1}$$

1. To prove (3.1), we first use the L^∞ bound (2.4) to have

$$\sup_n \left| \frac{d}{dt} X_t^n(x) \right| \leq CM \quad \forall x \in \Omega, \quad C = C(\Omega).$$

Recall in addition from Theorem 1.1 that each X_t^n is Hölder continuous with exponent

$$\exp(-C|t| \|\omega_0\|_{L^\infty(\Omega)}) \geq \exp(-CMT), \quad C = C(\Omega).$$

It follows that the sequence $X_t^n(x)$ is uniformly bounded in $C^\alpha(\overline{\Omega} \times [-T,T])$ for some $\alpha = \alpha(M,T,\Omega)$. By the Arzelà-Ascoli theorem, the subsequence X^{n_k} has a subsequence $X^{n_{k_\ell}} \to Y$ in $C(\overline{\Omega} \times [-T,T])$. Using this strong convergence, we now prove that

$$\omega_0^{n_k}(X_{t,0}^{n_{k_\ell}}(x)) \overset{*}{\rightharpoonup} \omega_0(Y_{t,0}(x)) \quad \text{in } L^\infty(\Omega \times (-T,T)), \quad Y_t(x) \equiv Y(x,t). \tag{3.2}$$

Indeed, for any $f \in C(\overline{\Omega} \times [-T,T])$, we have

$$\int_{-T}^{T} \int_\Omega \omega_0^{n_{k_\ell}}(X_{t,0}^{n_{k_\ell}}(x)) f(x,t) \, dx \, dt = \int_{-T}^{T} \int_\Omega \omega_0^{n_{k_\ell}}(x) f(X_t^{n_{k_\ell}}(x), t) \, dx \, dt$$

$$= \int_{-T}^{T} \int_\Omega \omega_0^{n_{k_\ell}}(x) f(Y_t(x), t) \, dx \, dt$$

$$+ \int_{-T}^{T} \int_\Omega \omega_0^{n_{k_\ell}}(x) [f(X_t^{n_{k_\ell}}(x), t) - f(Y_t(x), t)] \, dx \, dt$$

$$:= I_1 + I_2.$$

Since $\omega_0^n \overset{*}{\rightharpoonup} \omega_0$ in $L^\infty(\Omega)$ and $f(Y_t(\cdot), t) \in L^1(\Omega)$,

$$\lim_{\ell \to \infty} \int_\Omega \omega_0^{n_{k_\ell}}(x) f(Y_t(x), t) \, dx = \int_\Omega \omega_0(x) f(Y_t(x), t) \, dx.$$

In addition, we have

$$\left| \int_\Omega \omega_0^{n_{k_\ell}}(x) f(Y_t(x), t) \, dx \right| \leq M \|f(\cdot, t)\|_{L^1} \in L^1((-T,T)),$$

so that the dominated convergence theorem gives

$$\lim_{\ell \to \infty} I_1 = \int_{-T}^{T} \int_\Omega \omega_0(x) f(Y_t(x), t) \, dx \, dt.$$

Since $X^{n_{k_\ell}} \to Y$ in $C(\overline{\Omega} \times [-T,T])$ and $|\omega_0^{n_{k_\ell}}(x)| \leq M$, I_2 converges to 0 by uniform convergence on the compact set $\overline{\Omega} \times [-T,T]$. Consequently,

$$\lim_{\ell \to \infty} \int_{-T}^{T} \int_{\Omega} \omega_0^{n_{k_\ell}}(X_{t,0}^{n_{k_\ell}}(x)) f(x,t) dx dt = \int_{-T}^{T} \int_{\Omega} \omega_0(x) f(Y_t(x),t) dx \tag{3.3}$$
$$= \int_{-T}^{T} \int_{\Omega} \omega_0(Y_{t,0}(x)) f(x,t) dx.$$

Since $C(\overline{\Omega} \times [-T,T])$ is dense in $L^1(\Omega \times (-T,T))$ and $\omega_0^n(X_{t,0}^n(x))$ is uniformly bounded in $L^\infty(\Omega \times (-T,T))$, (3.3) implies (3.2). On the other hand, $\omega_0^{n_k}(X_{t,0}^{n_{k_\ell}}(x)) = \omega^{n_{k_\ell}}(x,t) \overset{*}{\rightharpoonup} \omega^\infty(x,t)$, so that (3.2) implies

$$\omega^\infty(x,t) = \omega_0(Y_{t,0}(x)). \tag{3.4}$$

Thus (3.1) would follow from (3.4) provided that

$$Y_t(x) = X_t^\infty(x). \tag{3.5}$$

To prove (3.5) we start with

$$\frac{d}{dt} X_t^n(x) = u^n(X_t^n(x),t), \quad u^n(t) = K * \omega^n(t). \tag{3.6}$$

Using that $\omega^{n_k} \overset{*}{\rightharpoonup} \omega^\infty$ in $L^\infty(\Omega \times (-T,T))$ and $\int_\Omega |K(x,y)| dy \leq C(\Omega)$, we deduce $u^{n_k} \overset{*}{\rightharpoonup} u^\infty$ in $L^\infty(\Omega \times (-T,T))$. Arguing as in the proof of (3.2) we obtain

$$u^{n_k}(X_t^{n_k}(x),t) \overset{*}{\rightharpoonup} u^\infty(Y_t(x),t) \quad \text{in } L^\infty(\Omega \times (-T,T)),$$

whence (3.6) gives $\frac{d}{dt} Y_t(x) = u^\infty(Y_t(x),t)$. Therefore, $Y_t(x) = X_t^\infty(x)$ by the uniqueness of trajectories generated by Log-Lipschitz velocity fields. This finishes the proof of (3.5) and hence of (3.1).

2. With (3.1) established, the triple $(\omega^\infty, u^\infty, X_t^\infty)$ is a Yudovich solution of (1.1) with initial data $\omega^\infty|_{t=0} = \omega_0$. By the uniqueness part in Theorem 1.1, $(\omega^\infty, u^\infty, X_t^\infty) \equiv (\omega, u, X_t)$. In fact, the above argument shows that every subsequence of ω^n has a subsequence converging weakly-* to ω in $L^\infty(\Omega \times (-T,T))$. It follows that the entire sequence $\omega^n \overset{*}{\rightharpoonup} \omega$ in $L^\infty(\Omega \times (-T,T))$.

Now for each $t \in \mathbb{R}$, $\omega^n(t)$ is well-defined in $L^\infty(\Omega)$ by virtue of Lemma 1.2. Moreover, $\|\omega^n(t)\|_{L^\infty} \leq M$, whence $\omega^{n_k} \overset{*}{\rightharpoonup} h(t)$ in $L^\infty(\Omega)$ for some subsequence n_k which a priori depends on t. For any $f \in C(\overline{\Omega})$ we have

$$\int_\Omega \omega^n(x,t) f(x) dx = \int_\Omega \omega_0^n(x) f(X_t^n(x)) dx \to \int_\Omega \omega_0(x) f(X_t(x)) dx$$

in views of the facts that $\omega_0^n \overset{*}{\rightharpoonup} \omega_0$ in $L^\infty(\Omega)$ and $X_t^n \to X_t^\infty \equiv X_t$ in $C(\overline{\Omega})$. It follows that

$$\int_\Omega h(x,t) f(x) dx = \int_\Omega \omega_0(x) f(X_t(x)) dx = \int_\Omega \omega_0(X_{t,0}(x)) f(x) dx,$$

and thus $h(x,t) = \omega_0(X_{t,0}(x)) = \omega(x,t)$ a.e. $x \in \Omega$. In fact, we have proved that every subsequence of $\omega^n(\cdot,t)$ has a subsequence converging weakly-* to $\omega(\cdot,t)$ in $L^\infty(\Omega)$. Therefore, the entire sequence $\omega^n(\cdot,t) \overset{*}{\rightharpoonup} \omega(\cdot,t)$ in $L^\infty(\Omega)$. This concludes the proof of Theorem 1.4.

4. Proof of Theorem 1.5

We follow closely the proof of Theorem 1.4 and use the same notation whenever possible. We note that $\omega^n(t)$ has compact support for all $t \in \mathbb{R}$ but X_t^n does not in general. Since X^n is uniformly Hölder continuous on any compact set of $\mathbb{R}_x^2 \times \mathbb{R}_t$, any subsequence n_k has a subsequence n_{k_ℓ} such that

$$\forall R > 0, \quad X^{n_{k_\ell}} \to Y \quad \text{in } C(\overline{B_R} \times [-T,T]) \tag{4.1}$$

by the Arzelà-Ascoli theorem and a diagonal procedure. Here B_R denotes the ball of radius R centered at $0 \in \mathbb{R}^2$. To prove (3.2) we take $f \in C_c(\mathbb{R}^2 \times (-T,T))$, a dense subspace of $L^1(\Omega \times (-T,T))$, and only

consider I_2 since the above argument for I_1 does not make use of the boundedness of Ω. If supp $f \subset B_R \times (-T, T)$ then

$$|I_2| \le M \int_{-T}^{T} \int_{X_{t,0}^{n_{k_\ell}}(B_R) \cup Y_{t,0}(B_R)} |f(X_t^{n_{k_\ell}}(x), t) - f(Y_t(x), t)| dx dt,$$

where $M = \sup_n \|\omega_0^n\|_{L^\infty}$. We have

$$\|u^n\|_{L^\infty} \le C\|\omega^n\|_{L^1 \cap L^\infty} = C\|\omega_0^n\|_{L^1 \cap L^\infty} \le CN,$$

where $N := \sup_n \|\omega_0^n\|_{L^1 \cap L^\infty} + \|\omega_0\|_{L^1 \cap L^\infty}$. This implies

$$X_{t,0}^{n_{k_\ell}}(B_R) \cup Y_{t,0}(B_R) \subset B_{R+TCN} \quad \forall |t| \le T,$$

whence

$$|I_2| \le M \int_{-T}^{T} \int_{B_{R+TCN}} |f(X_t^{n_{k_\ell}}(x), t) - f(Y_t(x), t)| dx dt.$$

Therefore, $\lim_{\ell \to \infty} I_2 = 0$ by the uniform convergence (4.1) on the compact set $\overline{B_{R+TCN}} \times [-T, T]$. This yields (3.2).

Regarding (3.6), we prove that $\omega^{n_k} \overset{*}{\rightharpoonup} \omega^\infty$ in $L^\infty(\mathbb{R}^2 \times (-T, T))$ implies $u^{n_k} \overset{*}{\rightharpoonup} u^\infty := K * \omega^\infty$ in $L^\infty(\mathbb{R}^2 \times (-T, T))$. Note that the Biot-Savart kernel for $\mathbb{R}^2$ is

$$K(x, y) \equiv K(x - y), \quad K(x) = \frac{(-x_2, x_1)}{2\pi|x|^2},$$

and K does not belong to $L^1(\mathbb{R}^2)$. For any $f \in C_c(\mathbb{R}^2 \times (-T, T))$ we have

$$\int_{-T}^{T} \int_{\mathbb{R}^2} u^{n_k}(x, t) f(x, t) dx dt = \int_{-T}^{T} \int_{\mathbb{R}^2} \omega^{n_k}(y, t) \int_{\mathbb{R}^2} K(x - y) f(x, t) dx dy dt \tag{4.2}$$
$$:= \int_{-T}^{T} \int \omega^{n_k}(y, t) g(y, t) dy dt.$$

Since $K \notin L^1(\mathbb{R}^2)$, we do not have $g(t) \in L^1(\mathbb{R}^2)$ to use the weak-$*$ convergence of ω^{n_k}. On the other hand, upon splitting the x-integration in g into $|x - y| \le 1$ and $|x - y| > 1$ and applying suitable Young inequalities, we deduce $\|g(t)\|_{L^3} \le C\|f(t)\|_{L^1 \cap L^3}$ and thus $g \in L^\infty((-T, T); L^3)$. Now, ω^n is uniformly bounded in $L^\infty(\mathbb{R}; L^{\frac{3}{2}})$ by interpolation, whence $\omega^{n_k} \rightharpoonup \omega^\infty$ in $L^{\frac{3}{2}}(\mathbb{R}^2 \times (-T, T))$. It then follows from (4.2) that

$$\lim_{k \to \infty} \int_{-T}^{T} \int_{\mathbb{R}^2} u^{n_k}(x, t) f(x, t) dx dt = \int_{-T}^{T} \int_{\mathbb{R}^2} \omega^\infty(y, t) g(y, t) dy dt$$
$$= \int_{-T}^{T} \int_{\mathbb{R}^2} u^\infty(x, t) f(x, t) dx dt$$

for all $f \in C_c(\mathbb{R}^2 \times (-T, T))$. Using this, the fact that u_n is uniformly bounded in $L^\infty(\mathbb{R}^2)$ and a density argument, we conclude $u^{n_k} \overset{*}{\rightharpoonup} u^\infty$ in $L^\infty(\mathbb{R}^2 \times (-T, T))$. The remainder of the proof follows along the same lines of the proof of Theorem 1.4.

Data availability This manuscript has no associated data.

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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