J. Math. Fluid Mech. (2022) 24:29

1422-6928/22/020001-17
https://doi.org/10.1007/s00021-022-00669-4

Journal of Mathematical Fluid Mechanics

Global Convergence to Compressible Full Navier–Stokes Equations by Approximation with Oldroyd-Type Constitutive Laws

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Communicated by G. P. Galdi

Abstract. We consider smooth solutions to a relaxed Euler system with Oldroyd-type constitutive laws. This system is derived from the one-dimensional compressible full Navier-Stokes equations for a Newtonian fluid by using the Cattaneo–Christov model and the Oldroyd-B model. In a neighborhood of equilibrium states, we construct an explicit symmetrizer and show that the system is symmetrizable hyperbolic with partial dissipation. Moreover, by establishing uniform estimates with respect to the relaxation times, we prove the uniform global existence of smooth solutions and the global-in-time convergence of the system towards the full Navier–Stokes equations.

Mathematics Subject Classification. 35B25, 35L60, 35Q30, 76A05.

Keywords. Full Navier–Stokes equations, Non-Newtonian fluid, Relaxed Euler systems, Oldroyd derivative, Global convergence.

1. Introduction

Euler and Navier-Stokes equations are two important models in fluid dynamics. There are rich literatures on the mathematical analysis around these equations. We refer to [11,23–26] and references therein for mathematical results. There are deep relationships between Euler and Navier-Stokes equations. It is well known that the Euler equations can be derived from Navier-Stokes equations through the vanishing viscosity limit. Meanwhile, the Navier-Stokes equations can also be approximated by first-order partial differential equations using different kinds of constitutive laws for non-Newtonian fluids. These approximate equations are referred to as relaxed Euler systems or hyperbolic Navier-Stokes equations, see for instance [10,16,30,33,34].

In the paper, we study the global-in-time convergence from relaxed Euler-type equations with Oldroyd's constitutive laws to compressible full (non-isentropic) Navier-Stokes equations by letting relaxation times tend to zero. Let $t \geq 0$ be the time variable and $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$ be the space variable. The compressible full Navier-Stokes equations are of the form

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla p = \mathrm{div}\tau, \\ \partial_t(\rho E) + \mathrm{div}(\rho u E + up) + \mathrm{div} q = \mathrm{div}(u\tau), \end{cases} \tag{1.1}$$

in $\mathbb{R}^+ \times \mathbb{R}^d$, where $\rho > 0$ is the density, $u = (u_1, \cdots, u_d)^\top \in \mathbb{R}^d$ is the velocity, θ is the temperature, p is the pressure function, $q \in \mathbb{R}^d$ is the heat flux, τ is the stress tensor and $E = e + \frac{1}{2}|u|^2$ is the total energy per unit mass with e being the specific internal energy. The symbols $\top$ and $\otimes$ represent the transpose and the tensor product, respectively. In (1.1), ρ, u and θ are independent variables, e and p are functions of (ρ, θ). In particular, for the ideal fluid, we have

$$e = c_v \theta, \qquad p = R\rho\theta, \tag{1.2}$$

which satisfy the thermodynamic equation

$$\rho^2 e_\rho = p - \theta p_\theta, \tag{1.3}$$

where R and c_v are positive constants, and $e_\rho = \frac{\partial e}{\partial \rho}$ etc. Generally speaking, the heat flux q satisfies the following Fourier's law

$$q = -\kappa \nabla \theta, \tag{1.4}$$

where $\kappa > 0$ is the heat conduction constant. For Newtonian viscous fluids, the stress tensor τ takes the form

$$\tau = \mu \sigma(u) + \lambda(\mathrm{div} u) I_d, \tag{1.5}$$

with

$$\sigma(u) = \nabla u + (\nabla u)^\top - \frac{2}{d}(\mathrm{div} u) I_d.$$

Here I_d denotes the unit matrix of order d. The parameters $\mu > 0$ and $\lambda > 0$ are the shear and bulk viscosity coefficients, respectively, which are all assumed to be constants.

For the full Navier-Stokes equations (1.1) with constitutive laws (1.4) and (1.5), the construction of the corresponding relaxed Euler systems depends on the way how to decompose the second-order derivatives $\mathrm{div}\tau, \mathrm{div}(u\tau)$ and $\mathrm{div} q$ into first-order derivative terms. Clearly, there are lots of ways to do it, among which the most natural one is to replace (1.4) and (1.5) by the following Maxwell's constitutive laws [28]

$$\varepsilon_1^2 \partial_t q + q = -\kappa \nabla \theta, \tag{1.6}$$

$$\varepsilon_2^2 \partial_t \tau + \tau = \mu \sigma(u) + \lambda(\mathrm{div} u) I_d, \tag{1.7}$$

where $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ are the relaxation times. Equations in (1.1) together with (1.6)-(1.7) form a relaxed Euler system. Formally, letting $(\varepsilon_1, \varepsilon_2) \to 0$ recovers the Navier-Stokes equations (1.1) with (1.4) and (1.5). This idea is not recent. It dates back to 1860s, see for instance [3,4,28]. These approximations have not only the mathematical sense but also physical interpretations. Relation (1.6), also known as the Cattaneo's law, gives rise to heat waves with finite propagation speed. Relation (1.7) describes motions of viscoelastic fluids. The laws are combinations of the Newtonian's law of viscosity and the Hooke's law of elasticity.

The existence of smooth solutions to system (1.1) with Maxwell's constitutive laws (1.6) and (1.7) and their convergence to the classical non-isentropic Navier-Stokes equations (1.1) with (1.4) and (1.5) have been studied in previous works. In the case where $\varepsilon_1 = 0$, the authors of [15] proved the global existence of smooth solutions near constant equilibrium states for fixed $\varepsilon_2 > 0$ and the local-in-time convergence towards the Navier-Stokes equations as $\varepsilon_2 \to 0$. Similar results are obtained in [14] in the case where $\varepsilon_2 = 0$ and $\varepsilon_1 > 0$. In these results, only one of the constitutive laws within (1.6) and (1.7) is used. Hence the systems studied in [14,15] are of mixed hyperbolic-parabolic type in the sense of Shizuta-Kawashima [19,39]. The local-in-time convergence is based on the error estimates between the original system and the limiting system. For the isentropic Navier-Stokes equations with constitutive law (1.7), the author of [42] obtained the local existence and the local convergence to the classical isentropic Navier-Stokes equations under condition $\mathrm{tr}(\tau) = 0$, where $\mathrm{tr}(\tau)$ means the trace of matrix τ. In [30], the first author of the present paper constructed approximate systems with vector variables instead of tensor variables by using Hurwitz-Radon matrices in both compressible and incompressible cases. He proved the uniform (with respect to ε_1 and ε_2) global existence of smooth solutions near constant equilibrium state and the global-in-time convergence of the systems towards classical isentropic Navier-Stokes equations. He also obtained similar results for the isentropic Navier-Stokes equations with constitutive law (1.7) without condition $\mathrm{tr}(\tau) = 0$. For the approximation of incompressible isentropic Navier-Stokes equations with constitutive law (1.7), we also refer to [10,33,34,37,38].

However, these two constitutive laws (1.6) and (1.7) have drawbacks as they do not ensure Galilean invariance. In other words, these laws lead to paradoxical evolution of thermal waves in a moving frame, see [7]. To overcome it, the Oldroyd's upper-convected time derivative (or simply Oldroyd derivative) should be considered. In this paper, we consider the following two constitutive laws. The first one is the Cattaneo-Christov model introduced in [6],

$$\varepsilon_1^2\big(\partial_t q + u\cdot\nabla q - q\cdot\nabla u + (\mathrm{div}u)q\big) = -q - \kappa\nabla\theta, \tag{1.8}$$

in which the terms on the left-hand side are the Oldroyd derivative. It is proved in [6] that the constitutive law (1.8) is Galilean invariant. The second one is the following Oldroyd-B model for the tensor variable τ (see for instance [2,29,35,36] and the references therein)

$$\varepsilon_2^2(\partial_t\tau + u\cdot\nabla\tau + g(\tau,\nabla u)) + \tau = \mu\sigma(u) + \lambda(\mathrm{div}u)I_d, \tag{1.9}$$

where

$$g(\tau,\nabla u) = \tau W(u) - W(u)\tau, \quad \text{with} \quad W(u) = \frac{1}{2}\big(\nabla u - (\nabla u)^\top\big).$$

Hence, the relaxed Euler system for (1.1) with constitutive laws (1.8) and (1.9) is of the form

$$\begin{cases} \partial_t\rho + \mathrm{div}(\rho u) = 0,\\ \partial_t(\rho u) + \mathrm{div}(\rho u\otimes u) + \nabla p = \mathrm{div}\tau,\\ \partial_t(\rho E) + \mathrm{div}(\rho uE + up) + \mathrm{div}q = \mathrm{div}(u\tau),\\ \varepsilon_1^2\big(\partial_t q + u\cdot\nabla q - q\cdot\nabla u + (\mathrm{div}u)q\big) = -q - \kappa\nabla\theta,\\ \varepsilon_2^2(\partial_t\tau + u\cdot\nabla\tau + g(\tau,\nabla u)) + \tau\\ \qquad = \mu\Big(\nabla u + \nabla u^\top - \dfrac{2}{d}(\mathrm{div}u)I_d\Big) + \lambda(\mathrm{div}u)I_d, \end{cases} \tag{1.10}$$

in $\mathbb{R}^+\times\mathbb{R}^d$.

System (1.10) is very complicated. One can observe that it contains d^2+2d+2 equations for fluids in space $\mathbb{R}^d$. So far the symmetrizable hyperbolicity for (1.10) is unknown in cases $d\geq 2$. This makes it hard to establish the existence results. See classical theories [18,21,24]. Moreover, when considering the constitutive laws (1.8) and (1.9) at the same time, there are no apparent dissipative structures for ∇u and $\nabla\theta$ due to the loss of the elliptic structures for u and θ. This is different from the situation for the systems treated in [14,15].

In a recent paper [16], the authors considered (1.10) in one space dimension. In this case, system (1.10) is reduced to the following form

$$\begin{cases} \partial_t\rho + \partial_x(\rho u) = 0,\\ \partial_t(\rho u) + \partial_x(\rho u^2 + p - \tau) = 0,\\ \partial_t(\rho E) + \partial_x(\rho uE + up + q - u\tau) = 0,\\ \varepsilon_1^2(\partial_t q + u\partial_x q) + \kappa\partial_x\theta = -q,\\ \varepsilon_2^2(\partial_t\tau + u\partial_x\tau) - \lambda\partial_x u = -\tau, \end{cases} \tag{1.11}$$

in $\mathbb{R}^+\times\mathbb{R}$, with the initial data

$$(\rho,u,\theta,q,\tau)|_{t=0} = (\rho_0^\varepsilon, u_0^\varepsilon, \theta_0^\varepsilon, q_0^\varepsilon, \tau_0^\varepsilon)(x), \tag{1.12}$$

where $\varepsilon = (\varepsilon_1,\varepsilon_2)$. In the limit as $\varepsilon\to 0$, we have formally

$$q = -\kappa\partial_x\theta, \quad \tau = \lambda\partial_x u.$$

Substituting these relations into (1.11), we recover the following one-dimensional non-isentropic Navier-Stokes equations

$$\begin{cases} \partial_t\rho + \partial_x(\rho u) = 0,\\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x p = \lambda\partial_{xx}u,\\ \partial_t(\rho E) + \partial_x(\rho uE + up) = \kappa\partial_{xx}\theta + \lambda\partial_x(u\partial_x u), \end{cases} \tag{1.13}$$

in $\mathbb{R}^+\times\mathbb{R}$. Equations (1.13) have been widely studied. We refer to [13,17,20,27] for the global existence of smooth solutions. See also [11,23] for related topics and references therein. In (1.11), ρ, u, θ, q and τ

are independent variables, and both the internal energy e and the pressure function p are functions of (ρ, θ, q, τ). In [16], the authors used the following state equations for e and p :

$$e = c_v\theta + \frac{\varepsilon_1^2}{\kappa\rho\theta}q^2 + \frac{\varepsilon_2^2}{2\lambda\rho}\tau^2, \tag{1.14}$$

$$p = R\rho\theta - \frac{\varepsilon_1^2}{2\kappa\theta}q^2 - \frac{\varepsilon_2^2}{2\lambda}\tau^2. \tag{1.15}$$

Equation (1.14) is based on the results in [8], where the authors rigorously proved that the constitutive law (1.8) is consistent with the second law of thermodynamics if and only if the dependence of e on q is quadratic. Similarly, the quadratic dependence of e on τ also implies the compatibility with the second law of thermodynamics. The choice of equation (1.15) makes it consistent with (1.3) and the state equations (1.14)-(1.15) yield formally those for the ideal fluid (1.2) as $\varepsilon \to 0$. For more explanations, see [5,9,16,41].

Let $V_e = (1, 0, 1, 0, 0)^\top$ be an equilibrium state for (1.11). Due to the explicit expressions (1.14) and (1.15), the authors of [16] constructed a strictly convex entropy for system (1.11), see Lemma 3.1 in [16] or Lemma 3.1 below. Based on this, they established the global existence of smooth solutions near V_e for fixed $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, and the local convergence of the system towards the classical full Navier-Stokes equations as $\varepsilon_1 = \varepsilon_2 \to 0$. However, the convergence for large time has not been investigated.

The purpose of this paper is to study the global convergence of system (1.11) with state equations (1.14) and (1.15). The main results of the paper are contained in the following two theorems. The first theorem shows the uniform global existence of smooth solutions to the Cauchy problem (1.11)-(1.12) near V_e. The second one concerns the global convergence of the solution of (1.11) to that of the one-dimensional full Navier-Stokes equations (1.13) for the ideal fluid (1.2) as $\varepsilon \to 0$. Remark that in these results condition $\varepsilon_1 = \varepsilon_2$ is not needed and system (1.11) is not included in the class of systems studied in [22,31,32,43]. The proof of the results is based on uniform estimates with respect to the time and the relaxation parameters. We use the strictly convex entropy given in [16] for the L^2 estimate. A key step is to find an appropriate symmetrizer of system (1.11) for higher-order estimates. It is well-known that the second-order derivative of a strictly convex entropy provides a symmetrizer for a system of conservation laws [1,12]. However, this result cannot be applied to (1.11) because it is a non-conservative system.

It is worth mentioning that the global existence result in the present paper is different from that obtained in [16], which is not uniform with respect to ε_1 and ε_2. More precisely, there are terms $\partial_t q$ and $\partial_t \tau$ in the definition of the energy in [16]. Because of boundary layers in the limit as $\varepsilon \to 0$, such an energy cannot be uniformly bounded with respect to the relaxation parameters. In order to avoid this situation, in the proof of our results, the energy contains only terms of derivative of solutions with respect to x.

Theorem 1.1 (Uniform global existence). *Let $s \geq 2$ be an integer and $(\rho_0^\varepsilon - 1, u_0^\varepsilon, \theta_0^\varepsilon - 1, q_0^\varepsilon, \tau_0^\varepsilon) \in H^s(\mathbb{R})$. There exist two positive constants δ and C, independent of ε_1 and ε_2, such that if*

$$\|\rho_0^\varepsilon - 1\|_s + \|u_0^\varepsilon\|_s + \|\theta_0^\varepsilon - 1\|_s + \varepsilon_1\|q_0^\varepsilon\|_s + \varepsilon_2\|\tau_0^\varepsilon\| \leq \delta,$$

then for all $\varepsilon_1, \varepsilon_2 \in (0, 1]$, the Cauchy problem (1.11)-(1.12) together with (1.14)-(1.15) admits a unique global smooth solution $(\rho^\varepsilon, u^\varepsilon, \theta^\varepsilon, q^\varepsilon, \tau^\varepsilon)$ satisfying

$$\rho^\varepsilon - 1, u^\varepsilon, \theta^\varepsilon - 1, q^\varepsilon, \tau^\varepsilon \in C(\mathbb{R}^+; H^s(\mathbb{R})) \cap C^1(\mathbb{R}^+; H^{s-1}(\mathbb{R})),$$

and

$$\begin{aligned} &\|\rho^\varepsilon(t) - 1\|_s^2 + \|u^\varepsilon(t)\|_s^2 + \|\theta^\varepsilon(t) - 1\|_s^2 + \varepsilon_1^2\|q^\varepsilon(t)\|_s^2 + \varepsilon_2^2\|\tau^\varepsilon(t)\|_s^2 \\ &\quad + \int_0^t \left(\|\partial_x\rho^\varepsilon(t')\|_{s-1}^2 + \|\partial_x u^\varepsilon(t')\|_{s-1}^2 + \|\partial_x\theta^\varepsilon(t')\|_{s-1}^2 + \|q^\varepsilon(t')\|_s^2 + \|\tau^\varepsilon(t')\|_s^2\right)dt' \\ &\leq C\left(\|\rho_0^\varepsilon - 1\|_s^2 + \|u_0^\varepsilon\|_s^2 + \|\theta_0^\varepsilon - 1\|_s^2 + \varepsilon_1^2\|q_0^\varepsilon\|_s^2 + \varepsilon_2^2\|\tau_0^\varepsilon\|_s^2\right), \quad \forall\, t \geq 0, \end{aligned} \tag{1.16}$$

where $\|\cdot\|_k$ denotes the usual norm of $H^k(\mathbb{R})$.

Theorem 1.2 (Global convergence). *Let $\varepsilon = (\varepsilon_1, \varepsilon_2)$ and $(\rho^\varepsilon, u^\varepsilon, \theta^\varepsilon, q^\varepsilon, \tau^\varepsilon)$ be the global solution obtained in Theorem 1.1, then there exist functions $(\bar\rho, \bar u, \bar\theta) \in L^\infty(\mathbb{R}^+; H^s(\mathbb{R}))$ and $(\bar q, \bar\tau) \in L^2(\mathbb{R}^+; H^s(\mathbb{R}))$, such that, as $\varepsilon \to 0$ up to subsequences,*

$$(\rho^\varepsilon, u^\varepsilon, \theta^\varepsilon) \rightharpoonup (\bar\rho, \bar u, \bar\theta) \quad \text{weakly-}* \text{ in } L^\infty(\mathbb{R}^+; H^s(\mathbb{R})), \tag{1.17}$$

$$(q^\varepsilon, \tau^\varepsilon) \rightharpoonup (\bar q, \bar\tau) \quad \text{weakly in } L^2(\mathbb{R}^+; H^s(\mathbb{R})), \tag{1.18}$$

where $(\bar\rho, \bar u, \bar\theta)$ is the solution to the one-dimensional full compressible Navier-Stokes equations (1.13) for the ideal fluid (1.2), with initial value $(\bar\rho_0, \bar u_0, \bar\theta_0)$ which is the weak limit of $(\rho_0^\varepsilon, u_0^\varepsilon, \theta_0^\varepsilon)$ up to subsequences. Moreover,

$$\bar q = -\kappa\partial_x\bar\theta, \quad \bar\tau = \lambda\partial_x\bar u.$$

The rest of the paper is organized as follows. In the next section, we prove results on the hyperbolic structure for system (1.11). These results are crucial in the proof of the above theorems. Section 3 is devoted to uniform global estimates. The proof of the theorems are completed in the last section.

2. Symmetrizable Hyperbolicity

In what follows, $s \geq 2$ denotes an integer. Let $\varepsilon = (\varepsilon_1, \varepsilon_2)$ and C be a generic positive constant independent of ε and any time. We assume that $\varepsilon_1, \varepsilon_2 \in (0, 1]$. For a integer $k \geq 1$, we denote by $\|\cdot\|_k$, $\|\cdot\|$ and $\|\cdot\|_\infty$ the norms of the usual Sobolev spaces $H^k(\mathbb{R})$, $L^2(\mathbb{R})$ and $L^\infty(\mathbb{R})$, respectively. The inner product in $L^2(\mathbb{R})$ is denoted by $\langle\cdot,\cdot\rangle$. In the proof we will frequently use the fact that the embedding from $H^l(\mathbb{R})$ to $L^\infty(\mathbb{R})$ is continuous for all integers $l \geq 1$.

For simplicity, the dependence of solution on the parameters ε_1 and ε_2 is not expressed explicitly. We want to write (1.11) into a first-order quasilinear system of variables $(\rho, u, \theta, q, \tau)$. First, it is clear that by using (1.15) and the first equation in (1.11), the second equation in (1.11) is equivalent to

$$\rho(\partial_t u + u\partial_x u) + p_\rho\partial_x\rho + p_\theta\partial_x\theta + p_q\partial_x q + (p_\tau - 1)\partial_x\tau = 0.$$

Similarly, by using the first two equations in (1.11), the third equation in (1.11) is equivalent to

$$\rho\partial_t e + \rho u\partial_x e + (p - \tau)\partial_x u + \partial_x q = 0. \tag{2.1}$$

This equation can be further treated by using equations (1.3), (1.14) and (1.15). Indeed,

$$\begin{aligned}
\rho\partial_t e &= \rho e_\theta\partial_t\theta + \rho e_\rho\partial_t\rho + \rho e_q\partial_t q + \rho e_\tau\partial_t\tau \\
&= \rho e_\theta\partial_t\theta + \rho e_\rho\partial_t\rho + \rho\frac{2q}{\kappa\theta\rho}(-\varepsilon_1^2 u\partial_x q - q - \kappa\partial_x\theta) + \rho\frac{\tau}{\lambda\rho}(-\varepsilon_2^2 u\partial_x\tau - \tau + \lambda\partial_x u) \\
&= \rho e_\theta\partial_t\theta + \rho e_\rho\partial_t\rho - \frac{2q}{\theta}\partial_x\theta - \frac{2\varepsilon_1^2 uq}{\kappa\theta}\partial_x q - \frac{2q^2}{\kappa\theta} - \frac{\varepsilon_2^2 u\tau}{\lambda}\partial_x\tau - \frac{\tau^2}{\lambda} + \tau\partial_x u,
\end{aligned}$$

and

$$\begin{aligned}
\rho u\partial_x e &= \rho u e_\theta\partial_x\theta + \rho u e_\rho\partial_x\rho + \rho u e_q\partial_x q + \rho u e_\tau\partial_x\tau \\
&= \rho u e_\theta\partial_x\theta + \rho u e_\rho\partial_x\rho + \frac{2\varepsilon_1^2 uq}{\kappa\theta}\partial_x q + \frac{\varepsilon_2^2 u\tau}{\lambda}\partial_x\tau,
\end{aligned}$$

hence,

$$\rho\partial_t e + \rho u\partial_x e = \rho e_\theta\partial_t\theta + \left(\rho u e_\theta - \frac{2q}{\theta}\right)\partial_x\theta + \rho e_\rho(\partial_t\rho + u\partial_x\rho) - \frac{2q^2}{\kappa\theta} - \frac{\tau^2}{\lambda} + \tau\partial_x u.$$

Combining the last equation with (1.3), (2.1) and the first equation in (1.11), we have

$$\rho e_\theta\partial_t\theta + \theta p_\theta\partial_x u + \left(\rho u e_\theta - \frac{2q}{\theta}\right)\partial_x\theta + \partial_x q = \frac{2q^2}{\kappa\theta} + \frac{\tau^2}{\lambda}.$$

Therefore, system (1.11) is equivalent to

$$\begin{cases}\partial_t\rho+\partial_x(\rho u)=0,\\ \rho(\partial_t u+u\partial_x u)+p_\rho\partial_x\rho+p_\theta\partial_x\theta+p_q\partial_x q+(p_\tau-1)\partial_x\tau=0,\\ \rho\theta e_\theta\partial_t\theta+\theta^2p_\theta\partial_x u+(\rho u\theta e_\theta-2q)\partial_x\theta+\theta\partial_x q=\frac{2q^2}{\kappa}+\frac{\theta\tau^2}{\lambda},\\ \varepsilon_1^2(\partial_t q+u\partial_x q)+\kappa\partial_x\theta=-q,\\ \varepsilon_2^2(\partial_t\tau+u\partial_x\tau)-\lambda\partial_x u=-\tau.\end{cases} \tag{2.2}$$

Let $D_0(\varepsilon)$ be a diagonal matrix defined by

$$D_0(\varepsilon)=\mathrm{diag}\left(1,1,1,\varepsilon_1^2,\varepsilon_2^2\right).$$

Then system (2.2) is written as

$$D_0(\varepsilon)V_t+A(V)\partial_x V+BV=F(V), \tag{2.3}$$

where

$$V=(\rho,u,\theta,q,\tau)^\top,\qquad F(V)=\left(0,0,\frac{1}{\rho\theta e_\theta}\Big(\frac{2q^2}{\kappa}+\frac{\theta\tau^2}{\lambda}\Big),0,0\right)^\top,$$

and

$$A(V)=\begin{pmatrix} u & \rho & 0 & 0 & 0\\ \frac{p_\rho}{\rho} & u & \frac{p_\theta}{\rho} & \frac{p_q}{\rho} & \frac{p_\tau-1}{\rho}\\ 0 & \frac{\theta p_\theta}{\rho e_\theta} & u-\frac{2q}{\rho\theta e_\theta} & \frac{1}{\rho e_\theta} & 0\\ 0 & 0 & \kappa & \varepsilon_1^2 u & 0\\ 0 & -\lambda & 0 & 0 & \varepsilon_2^2 u\end{pmatrix},\quad B=\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}.$$

We define

$$A_0(V)=\tilde A_0(V)D_0(\varepsilon),\qquad \tilde A(V)=\tilde A_0(V)A(V),$$

with

$$\tilde A_0(V)=\begin{pmatrix}\theta^2p_\rho p_\theta & 0 & 0 & 0 & 0\\ 0 & \rho^2\theta^2p_\theta & 0 & 0 & 0\\ 0 & 0 & \rho^2\theta e_\theta p_\theta & -\frac{1}{\kappa}(\rho^2qe_\theta) & 0\\ 0 & 0 & -\frac{\varepsilon_1^2}{\kappa}\rho^2qe_\theta & \frac{1}{\kappa}(\rho\theta p_\theta+2\rho qp_q) & 0\\ 0 & 0 & 0 & 0 & \frac{1}{\lambda}\big(\rho\theta^2p_\theta(1-p_\tau)\big)\end{pmatrix}.$$

Then straightforward calculations give

$$A_0(V)=\begin{pmatrix}\theta^2p_\rho p_\theta & 0 & 0 & 0 & 0\\ 0 & \rho^2\theta^2p_\theta & 0 & 0 & 0\\ 0 & 0 & \rho^2\theta e_\theta p_\theta & -\frac{\varepsilon_1^2}{\kappa}(\rho^2qe_\theta) & 0\\ 0 & 0 & -\frac{\varepsilon_1^2}{\kappa}(\rho^2qe_\theta) & \frac{\varepsilon_1^2}{\kappa}\big(\rho\theta p_\theta+2\rho qp_q\big) & 0\\ 0 & 0 & 0 & 0 & \frac{\varepsilon_2^2}{\lambda}\big(\rho\theta^2p_\theta(1-p_\tau)\big)\end{pmatrix},$$

$$\tilde A(V)=\begin{pmatrix}u\theta^2p_\rho p_\theta & \rho\theta^2p_\rho p_\theta & 0 & 0 & 0\\ \rho\theta^2p_\rho p_\theta & \rho^2u\theta^2p_\theta & \rho\theta^2p_\theta^2 & \rho\theta^2p_\theta p_q & \rho\theta^2p_\theta(p_\tau-1)\\ 0 & \rho\theta^2p_\theta^2 & J_1 & \rho\theta p_\theta+\rho^2u\theta e_\theta p_q & 0\\ 0 & \rho\theta^2p_\theta p_q & \rho\theta p_\theta+\rho^2u\theta e_\theta p_q & J_2 & 0\\ 0 & \rho\theta^2p_\theta(p_\tau-1) & 0 & 0 & \frac{\varepsilon_2^2}{\lambda}\big(\rho u\theta^2p_\theta(1-p_\tau)\big)\end{pmatrix},$$

where

$$J_1=\rho^2u\theta e_\theta p_\theta-2\rho qp_\theta-\rho^2qe_\theta,\qquad J_2=\frac{\varepsilon_1^2}{\kappa}\big(\rho u(\theta p_\theta+2qp_q)-\rho q\big).$$

It is clear that $A_0(V)$ and $\tilde{A}(V)$ are symmetric. Let $T > 0$ and

$$N = \rho - 1, \quad \Theta = \theta - 1, \quad W = \big(N, u, \Theta, \varepsilon_1 q, \varepsilon_2 \tau\big)^\top.$$

We denote

$$\mathscr{E}(t) = \|W(t,\cdot)\|_s^2, \qquad \mathscr{E}_T = \sup_{0\le t\le T} \mathscr{E}(t). \tag{2.4}$$

In Theorem 1.1, estimate (1.16) implies that $\mathscr{E}_T$ is uniformly sufficiently small when $\mathscr{E}(0)$ is, although the $L^\infty(0,T;H^s(\mathbb{R}))$ norm of q and τ may not be uniformly small with respect to ε_1 and ε_2. The following result shows that $A_0(V)$ is positive definite when $\mathscr{E}_T$ is sufficiently small.

Lemma 2.1. *Let $W \in C([0,T];H^s(\mathbb{R}))$ be the smooth solution to (2.2) with (1.14)-(1.15) and (1.12). Then there exist constants $\delta > 0$ and $c_1 > 0$, independent of ε_1 and ε_2, such that if $\mathscr{E}_T^{1/2} \le \delta$, we have*

$$c_1 \le \rho, \theta, p_\rho, p_\theta, e_\theta \le C, \tag{2.5}$$

$$\|p_q\|_s \le C\delta\varepsilon_1, \quad \|p_\tau\|_s \le C\delta\varepsilon_2, \quad \|qp_q\|_s \le C\delta, \tag{2.6}$$

and $A_0(V)$ is positive definite and then system (2.2) is symmetrizable hyperbolic.

Proof. When $\mathscr{E}_T$ is sufficiently small, ρ and θ are sufficiently close to 1, then they have uniform positive upper and lower bounds. From the state equations (1.14) and (1.15), we have

$$p_\rho = R\theta, \quad p_\theta = R\rho + \frac{\varepsilon_1^2}{2\kappa\theta^2} q^2, \quad e_\theta = c_v - \frac{\varepsilon_1^2}{\kappa\rho\theta^2} q^2, \tag{2.7}$$

which imply (2.5) since $\|\varepsilon_1 q\|_\infty \le \mathscr{E}_T$ which is sufficiently small. Besides,

$$p_q = -\frac{\varepsilon_1^2}{\kappa\theta} q, \quad p_\tau = -\frac{\varepsilon_2^2}{\lambda}\tau, \quad qp_q = -\frac{\varepsilon_1^2}{\kappa\theta} q^2, \tag{2.8}$$

which imply (2.6).

Moreover,

$$\begin{vmatrix} \rho^2\theta e_\theta p_\theta & -\frac{\varepsilon_1^2}{\kappa}(\rho^2 q e_\theta) \\ -\frac{\varepsilon_1^2}{\kappa}(\rho^2 q e_\theta) & \frac{\varepsilon_1^2}{\kappa}\big(\rho\theta p_\theta + 2\rho q p_q\big) \end{vmatrix} = \frac{\varepsilon_1^2\rho^3 e_\theta}{\kappa}\left(\theta^2 p_\theta^2 - \frac{\varepsilon_1^2 q^2}{\kappa}(2p_\theta + \rho e_\theta)\right).$$

Then (2.5) implies that the above determinate is positive for sufficiently small $\mathscr{E}_T$. From (2.6) and $\varepsilon_2 \le 1$, we also have $\|p_\tau\|_s \le C\delta$. Therefore, $A_0(V)$ is positive definite from its explicit expression. □

Applying the theory on the symmetrizable hyperbolic system, Lemma 2.1 implies the local existence of smooth solutions to the Cauchy problem (2.2) with (1.14)-(1.15) and (1.12) from the classical iteration technique and fixed point theorems, see for instance [18,21,24].

3. Uniform Global Estimates

Let $T > 0$ and $W = (N, u, \Theta, \varepsilon_1 q, \varepsilon_2\tau)^\top$ be the unique local smooth solution to (2.2) with (1.14)-(1.15) and (1.12), defined on $[0,T]$. We assume that $\mathscr{E}_T$ defined in (2.4) is sufficiently small. This gives rise to the rational assumption that

$$|\rho - 1| \le \frac{1}{2}, \quad |\theta - 1| \le \frac{1}{2}, \quad |p_\tau| \le \frac{1}{2}. \tag{3.1}$$

We want to establish the uniform global estimate for W with respect to the parameters ε_1 and ε_2 and the time T. For this purpose, we introduce the following dissipative energy

$$\mathscr{D}(t) = \|\partial_x N(t)\|_{s-1}^2 + \|\partial_x u(t)\|_{s-1}^2 + \|\partial_x \Theta(t)\|_{s-1}^2 + \|q(t)\|_s^2 + \|\tau(t)\|_s^2.$$

3.1. L^2-estimate

The L^2 estimate relies on the existence of a strictly convex entropy and its corresponding entropy flux.

Lemma 3.1. *For system (2.2), there exists a strictly convex entropy η and its corresponding entropy flux Ψ satisfying*

$$\partial_t \eta(W) + \partial_x \Psi(W) + \frac{q^2}{\kappa\theta^2} + \frac{\tau^2}{\lambda\theta} = 0, \tag{3.2}$$

where

$$\eta(W) = R(\rho\ln\rho - \rho + 1) + \frac{1}{2}\rho u^2 + c_v\rho(\theta - \ln\theta - 1) + \frac{\varepsilon_1^2}{\kappa\theta}\left(1 - \frac{1}{2\theta}\right)q^2 + \frac{\varepsilon_2^2}{2\lambda}\tau^2,$$

and

$$\begin{aligned}\Psi(W) = {} & R\rho(\ln\rho - 1)u + \frac{1}{2}\rho u^3 + c_v\rho u(\theta - \ln\theta - 1) + (p-\tau)u + \Big(1 - \frac{1}{\theta}\Big)q \\ & + \frac{\varepsilon_1^2 u}{\kappa\theta}\Big(1 - \frac{1}{2\theta}\Big)q^2 + \frac{\varepsilon_2^2}{2\lambda}u\tau^2.\end{aligned}$$

In addition, if $\mathscr{E}_T$ is sufficiently small, the following L^2-energy estimate holds

$$\begin{aligned}&\|(\rho-1, u, \theta-1, \varepsilon_1 q, \varepsilon_2\tau)(t)\|^2 + \int_0^t \big(\|q(t')\|^2 + \|\tau(t')\|^2\big)dt' \\ &\quad \le C\left(\|\rho_0 - 1\|^2 + \|u_0\|^2 + \|\theta_0 - 1\|^2 + \varepsilon_1^2\|q_0\|^2 + \varepsilon_2^2\|\tau_0\|^2\right), \quad \forall\, t \in [0,T].\end{aligned} \tag{3.3}$$

Proof. The entropy-entropy flux identity (3.2) was established in [16]. We now prove (3.3). By the Taylor's expansion at $(\rho,\theta) = (1,1)$ and (3.1), we have

$$c_v\rho(\theta - \ln\theta - 1) = \frac{c_v\rho}{2\hat\theta^2}\Theta^2 \ge \frac{1}{9}c_v\Theta^2, \quad R(\rho\ln\rho - \rho + 1) = \frac{R}{2\hat\rho}N^2 \ge \frac{R}{3}N^2,$$

where $\hat\rho$ is between ρ and 1, and $\hat\theta$ is between θ and 1. By Lemma 2.1, this implies that there exists a constant $c_2 > 0$, such that

$$c_2|W|^2 \le \eta(W) \le C|W|^2,$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^5$. Integrating (3.2) over $[0,t]$ for $t \in [0,T]$ implies (3.3). □

3.2. Higher Order Estimates

Let l be an integer with $1 \le l \le s$. Applying ∂_x^l to both sides of the equation (2.3) yields

$$D_0(\varepsilon)\partial_t(\partial_x^l V) + A(V)\partial_x(\partial_x^l V) + \partial_x^l(BV) = \partial_x^l F(V) + K_l,$$

where

$$K_l = A(V)\partial_x^{l+1}V - \partial_x^l(A(V)\partial_x V).$$

Then,

$$A_0(V)\partial_t(\partial_x^l V) + \tilde A(V)\partial_x(\partial_x^l V) + \tilde A_0(V)\partial_x^l(BV) = \tilde A_0(V)\partial_x^l F(V) + \tilde A_0(V)K_l.$$

Taking the inner product of the above system with $\partial_x^l V$ in $L^2(\mathbb{R})$, we have

$$\begin{aligned}&\frac{d}{dt}\big\langle A_0(V)\partial_x^l V, \partial_x^l V\big\rangle + 2\big\langle \tilde A_0(V)\partial_x^l(BV), \partial_x^l V\big\rangle \\ &\quad = \big\langle \partial_t A_0(V)\partial_x^l V, \partial_x^l V\big\rangle + \big\langle \partial_x \tilde A(V)\partial_x^l V, \partial_x^l V\big\rangle + 2\big\langle \tilde A_0(V)\partial_x^l F(V), \partial_x^l V\big\rangle + 2\big\langle \tilde A_0(V)K_l, \partial_x^l V\big\rangle.\end{aligned} \tag{3.4}$$

We deal with (3.4) term by term in a series of lemmas as follows.

Lemma 3.2. *There exists positive constants c_3 and c_4 such that*

$$2\left|\left\langle \tilde{A}_0(V)\partial_x^l(BV), \partial_x^l V\right\rangle\right| \geq c_3\|\partial_x^l q\|^2 + c_4\|\partial_x^l \tau\|^2 - C\mathscr{E}_T^{1/2}\mathscr{D}(t). \tag{3.5}$$

Proof. By the definition of $\tilde{A}_0(V)$ and B together with (2.8), we have

$$\tilde{A}_0(V)\partial_x^l(BV) = \left(0, 0, 0, -\frac{\rho^2 q e_\theta}{\kappa}\partial_x^l q, \frac{\rho\theta p_\theta + 2\rho q p_q}{\kappa}\partial_x^l q, \frac{\rho\theta^2 p_\theta(1-p_\tau)}{\lambda}\partial_x^l \tau\right)^\top,$$

which implies that

$$\begin{aligned}\left\langle \tilde{A}_0(V)\partial_x^l(BV), \partial_x^l V\right\rangle = &-\left\langle \frac{\rho^2 q e_\theta}{\kappa}\partial_x^l q, \partial_x^l \Theta\right\rangle + \left\langle \frac{\rho\theta p_\theta + 2\rho q p_q}{\kappa}\partial_x^l q, \partial_x^l q\right\rangle \\ &+ \left\langle \frac{\rho\theta^2 p_\theta(1-p_\tau)}{\lambda}\partial_x^l \tau, \partial_x^l \tau\right\rangle.\end{aligned}$$

Obviously,

$$\left|\left\langle \frac{\rho^2 q e_\theta}{\kappa}\partial_x^l q, \partial_x^l \Theta\right\rangle\right| \leq C\|\partial_x^l \Theta\|_\infty \|q\| \|\partial_x^l q\| \leq C\mathscr{E}_T^{1/2}\mathscr{D}(t).$$

For the second term, noticing that $\frac{\rho\theta p_\theta}{\kappa}$ has a uniform lower bound when $\mathscr{E}_T$ is sufficiently small. Besides, the last estimate in (2.6) implies that $\frac{2\rho q p_q}{\kappa}$ is small and consequently, there exists a positive constant c_3 such that

$$2\left\langle \frac{\rho\theta p_\theta + 2\rho q p_q}{\kappa}\partial_x^l q, \partial_x^l q\right\rangle \geq c_3\|\partial_x^l q\|^2.$$

Similarly, (3.1) implies that $1 - p_\tau \geq \frac{1}{2}$. Therefore, there exists a positive constant c_4 such that

$$2\left\langle \frac{\rho\theta^2 p_\theta(1-p_\tau)}{\lambda}\partial_x^l \tau, \partial_x^l \tau\right\rangle \geq c_4\|\partial_x^l \tau\|^2.$$

Combining all these estimates yields (3.5). □

Lemma 3.3. *It holds*

$$\left|\left\langle \partial_t A_0(V)\partial_x^l V, \partial_x^l V\right\rangle\right| \leq C\mathscr{E}_T^{1/2}\mathscr{D}(t). \tag{3.6}$$

Proof. We denote

$$V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}, \quad V_1 = \begin{pmatrix} N \\ u \\ \Theta \end{pmatrix}, \quad V_2 = \begin{pmatrix} q \\ \tau \end{pmatrix},$$

and

$$A_0(V) = \begin{pmatrix} A_0^{11}(V) & A_0^{12}(V) \\ A_0^{21}(V) & A_0^{22}(V) \end{pmatrix}, \qquad \tilde{A}(V) = \begin{pmatrix} \tilde{A}^{11}(V) & \tilde{A}^{12}(V) \\ \tilde{A}^{21}(V) & \tilde{A}^{22}(V) \end{pmatrix},$$

where

$$\begin{cases} A_0^{11}(V) = \begin{pmatrix} \theta^2 p_\rho p_\theta & 0 & 0 \\ 0 & \rho^2\theta^2 p_\theta & 0 \\ 0 & 0 & \rho^2\theta e_\theta p_\theta \end{pmatrix}, \\ A_0^{12}(V) = \left(A_0^{21}(V)\right)^\top = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -\frac{\varepsilon_1^2}{\kappa}\left(\rho^2 q e_\theta\right) & 0 \end{pmatrix}, \\ A_0^{22}(V) = \begin{pmatrix} \frac{\varepsilon_1^2}{\kappa}(\rho\theta p_\theta + 2\rho q p_q) & 0 \\ 0 & \frac{\varepsilon_2^2}{\lambda}\left(\rho\theta^2 p_\theta(1-p_\tau)\right) \end{pmatrix}. \end{cases}$$

Obviously,

$$\left\langle \partial_t A_0(V)\partial_x^l V, \partial_x^l V\right\rangle = \left\langle \partial_t A_0^{11}(V)\partial_x^l V_1, \partial_x^l V_1\right\rangle + 2\left\langle \partial_t A_0^{12}(V)\partial_x^l V_2, \partial_x^l V_1\right\rangle + \left\langle \partial_t A_0^{22}(V)\partial_x^l V_2, \partial_x^l V_2\right\rangle.$$

From (1.14) and (1.15), we see that $A_0^{11}(V)$ is a smooth function of $(N, \Theta, \varepsilon_1^2 q^2)$. Moreover, from (2.2) together with the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\|\partial_t N\|_\infty + \|\partial_t \Theta\|_\infty &\le C\mathscr{E}_T^{1/2} + C\big(\|q\|_s + \|q\|_s^2 + \|\tau\|_s^2\big),\\ \varepsilon_1^2\|\partial_t q\|_\infty &\le C\mathscr{E}_T^{1/2} + C\|q\|_s,\\ \varepsilon_1^2\|\partial_t(q^2)\|_\infty &\le C\mathscr{E}_T^{1/2} + C\|q\|_s + C\|q\|_s^2.\end{aligned}$$

Hence,

$$\|\partial_t A_0^{11}(V)\|_\infty \le C\mathscr{E}_T^{1/2} + C\big(\|q\|_s + \|q\|_s^2 + \|\tau\|_s^2\big).$$

Moreover,

$$2\|q\|_s\|\partial_x^l V_1\|^2 \le \big(\|\partial_x^l V_1\|^2 + \|q\|_s^2\big)\|\partial_x^l V_1\| \le C\mathscr{E}_T^{1/2}\mathscr{D}(t).$$

Since $\mathscr{E}_T$ is uniformly small, we also have

$$\big(\|q\|_s^2 + \|\tau\|_s^2\big)\|\partial_x^l V_1\|^2 \le C\mathscr{E}_T\mathscr{D}(t) \le C\mathscr{E}_T^{1/2}\mathscr{D}(t).$$

Therefore,

$$\big|\big\langle \partial_t A_0^{11}(V)\partial_x^l V_1, \partial_x^l V_1\big\rangle\big| \le C\mathscr{E}_T^{1/2}\mathscr{D}(t).$$

Next, a straightforward calculation yields

$$\big\langle \partial_t A_0^{12}(V)\partial_x^l V_2, \partial_x^l V_1\big\rangle = -\frac{\varepsilon_1^2}{\kappa}\big\langle \partial_t(\rho^2 q e_\theta)\partial_x^l q, \partial_x^l \Theta\big\rangle.$$

From (2.7), we have

$$\rho^2 q e_\theta = c_\nu \rho^2 q - \frac{\varepsilon_1^2 \rho q^3}{\kappa\theta^2}.$$

Then (2.2) together with the bound of $\mathscr{E}_T$ yields

$$\frac{\varepsilon_1^2}{\kappa}\|\partial_t\big(\rho^2 q e_\theta\big)\|_\infty \le C\mathscr{E}_T^{1/2} + C\|q\|_s + C\varepsilon_1\|\tau\|_s^2.$$

Therefore,

$$\begin{aligned}\big|\big\langle \partial_t A_0^{12}(V)\partial_x^l V_2, \partial_x^l V_1\big\rangle\big| &\le \frac{\varepsilon_1^2}{\kappa}\|\partial_t\big(\rho^2 q e_\theta\big)\|_\infty\|\partial_x^l q\|\|\partial_x^l\Theta\|\\ &\le C\mathscr{E}_T^{1/2}\mathscr{D}(t) + C\mathscr{E}_T^{1/2}\|q\|_s^2 + C(\varepsilon_1\|\partial_x^l q\|\|\partial_x^l\Theta\|)\|\tau\|_s^2\\ &\le C\mathscr{E}_T^{1/2}\mathscr{D}(t).\end{aligned}$$

Finally,

$$\big\langle \partial_t A_0^{22}(V)\partial_x^l V_2, \partial_x^l V_2\big\rangle = \frac{\varepsilon_1^2}{\kappa}\big\langle \partial_t(\rho\theta p_\theta + 2\rho q p_q)\partial_x^l q, \partial_x^l q\big\rangle + \frac{\varepsilon_2^2}{\lambda}\big\langle \partial_t\big(\rho\theta^2 p_\theta(1-p_\tau)\big)\partial_x^l\tau, \partial_x^l\tau\big\rangle.$$

Similarly to the estimates above, we have

$$\varepsilon_1^2\big\|\partial_t(\rho\theta p_\theta + 2\rho q p_q)\big\|_\infty \le C\mathscr{E}_T^{1/2} + C\varepsilon_1^2\big(\|q\|_s + \|q\|_s^2 + \|\tau\|_s^2\big),$$

and

$$\varepsilon_2^2\big\|\partial_t\big(\rho\theta^2 p_\theta(1-p_\tau)\big)\big\|_\infty \le C\mathscr{E}_T^{1/2} + C\varepsilon_2^2\big(\|q\|_s + \|q\|_s^2 + \|\tau\|_s + \|\tau\|_s^2\big),$$

which imply that

$$\big|\big\langle \partial_t A_0^{22}(V)\partial_x^l V_2, \partial_x^l V_2\big\rangle\big| \le C\mathscr{E}_T^{1/2}\mathscr{D}(t).$$

This proves the lemma. □

Lemma 3.4. *It holds*

$$\left|\left\langle \partial_x \tilde{A}(V)\partial_x^l V, \partial_x^l V\right\rangle\right| \le C\mathscr{E}_T^{1/2}\mathscr{D}(t). \tag{3.7}$$

Proof. From (1.14) and (1.15) and the explicit expression of $\tilde{A}(V)$, we see that all elements of $\tilde{A}(V)$ are smooth functions of $(\rho, u, \theta, \varepsilon_1^2 q^2, \varepsilon_2^2 \tau^2)$ except J_1. Noticing that J_1 appears at the position of the third line and the third column of $\tilde{A}(V)$ and

$$\|\partial_x J_1\|_\infty \le C(\|\partial_x W\|_{s-1} + \|q\|_s),$$

we obtain

$$\begin{aligned}
\left|\left\langle \partial_x \tilde{A}(V)\partial_x^l V, \partial_x^l V\right\rangle\right| &\le C\|\partial_x W\|_\infty \|\partial_x^l V\|^2 + \left|\left\langle \partial_x J_1 \partial_x^l \Theta, \partial_x^l \Theta\right\rangle\right| \\
&\le C\mathscr{E}_T^{1/2}\mathscr{D}(t) + C\|\partial_x J_1\|_\infty \|\partial_x^l \Theta\|^2 \\
&\le C\mathscr{E}_T^{1/2}\mathscr{D}(t) + C\|q\|_s \|\partial_x^l \Theta\|^2 \\
&\le C\mathscr{E}_T^{1/2}\mathscr{D}(t).
\end{aligned}$$

This proves (3.7). □

Lemma 3.5. *It holds*

$$\left|\left\langle \tilde{A}_0(V)\partial_x^l F(V), \partial_x^l V\right\rangle\right| \le C\mathscr{E}_T^{1/2}\mathscr{D}(t). \tag{3.8}$$

Proof. By the definition of $\tilde{A}_0(V)$ and F, we have

$$\tilde{A}_0(V)\partial_x^l F(V) = \left(0, 0, \rho^2\theta e_\theta p_\theta \partial_x^l \left(\frac{1}{\rho\theta e_\theta}\left(\frac{2q^2}{\kappa} + \frac{\theta\tau^2}{\lambda}\right)\right), -\frac{\varepsilon_1^2\rho^2 q e_\theta}{\kappa}\partial_x^l \left(\frac{1}{\rho\theta e_\theta}\left(\frac{2q^2}{\kappa} + \frac{\theta\tau^2}{\lambda}\right)\right), 0\right)^\top.$$

It follows that

$$\begin{aligned}
&\left|\left\langle \tilde{A}_0(V)\partial_x^l F(V), \partial_x^l V\right\rangle\right| \\
&\le \left|\left\langle \rho^2\theta e_\theta p_\theta \partial_x^l \left(\frac{1}{\rho\theta e_\theta}\left(\frac{2q^2}{\kappa} + \frac{\theta\tau^2}{\lambda}\right)\right), \partial_x^l \Theta\right\rangle\right| + \left|\left\langle \frac{\varepsilon_1^2\rho^2 q e_\theta}{\kappa}\partial_x^l \left(\frac{1}{\rho\theta e_\theta}\left(\frac{2q^2}{\kappa} + \frac{\theta\tau^2}{\lambda}\right)\right), \partial_x^l q\right\rangle\right| \\
&\le C(\|\partial_x^l \Theta\|_\infty + \|\varepsilon_1 \partial_x^l q\|\|\varepsilon_1 q\|_\infty)(\|q\|_s^2 + \|\tau\|_s^2) \\
&\le C\mathscr{E}_T^{1/2}\mathscr{D}(t),
\end{aligned}$$

which proves (3.8). □

Lemma 3.6. *It holds*

$$\left|\left\langle \tilde{A}_0(V)K_l, \partial_x^l V\right\rangle\right| \le C\mathscr{E}_T^{1/2}\mathscr{D}(t). \tag{3.9}$$

Proof. Recall that

$$K_l = A(V)\partial_x^{l+1} V - \partial_x^l (A(V)\partial_x V).$$

Similarly to the proof of Lemma 3.4, it is easy to see that all elements of $A(V)$ are smooth functions of $(\rho, u, \theta, \varepsilon_1^2 q^2, \varepsilon_2^2 \tau^2)$ except the element at the position of the third line and the third column. We denote by J_3 this element. Then

$$J_3 = u - \frac{2q}{\rho\theta e_\theta},$$

and

$$\|\partial_x J_3\|_{s-1} \le C(\|\partial_x W\|_{s-1} + \|q\|_s).$$

The only two terms that contain J_3 are the following

$$\left\langle \partial_x^l \Theta, \rho^2\theta e_\theta p_\theta \left(\partial_x^l (J_3 \partial_x \Theta) - J_3 \partial_x^{l+1}\Theta\right)\right\rangle \quad \text{and} \quad -\left\langle \partial_x^l q, \frac{\varepsilon_1^2\rho^2 q e_\theta}{\kappa}\left(\partial_x^l (J_3 \partial_x \Theta) - J_3 \partial_x^{l+1}\Theta\right)\right\rangle,$$

which can be treated similarly to the proof of Lemma 3.4 and are obviously bounded by $C\mathscr{E}_T^{1/2}\mathscr{D}(t)$.

On the other hand, each element of $\tilde{A}_0(V)$ is uniformly bounded in $L^\infty([0,T]\times\mathbb{R})$ except for the element at the position of the third line and the fourth column. This element is $-\frac{1}{\kappa}(\rho^2 q e_\theta)$ which only touches the fourth line of K_l in the product $\tilde{A}_0(V)K_l$. Now the only nonzero element on the fourth line of K_l is $\varepsilon_1^2(u\partial_x^{l+1}q-\partial_x^l(u\partial_x q))$. Then the fourth component of $\tilde{A}_0(V)K_l$ is $-\frac{\varepsilon_1^2}{\kappa}(\rho^2 q e_\theta)(u\partial_x^{l+1}q-\partial_x^l(u\partial_x q))$. By the Moser-type calculus inequalities (see [24] for instance), we have

$$\varepsilon_1^2\big|\big\langle \rho^2 q e_\theta(u\partial_x^{l+1}q-\partial_x^l(u\partial_x q)),\partial_x^l\Theta\big\rangle\big|\le C\mathscr{E}_T^{1/2}\mathscr{D}(t).$$

The estimates for the other terms can be easily obtained. This proves (3.9). □

3.3. Dissipative Estimates for $\partial_x N$, $\partial_x u$ and $\partial_x\Theta$

Lemma 3.7 (Dissipative estimates for $\partial_x\Theta$). *It holds*

$$\varepsilon_1^2\sum_{m=0}^{s-1}\frac{d}{dt}\big\langle \partial_x^m q,\partial_x^{m+1}\Theta\big\rangle+\frac{\kappa}{4}\|\partial_x\Theta\|_{s-1}^2\le C\nu\|\partial_x u\|_{s-1}^2+C\|q\|_s^2+C\mathscr{E}_T^{1/2}\mathscr{D}(t), \tag{3.10}$$

where $\nu>0$ is a small positive constant to be determined later.

Proof. Let m be an integer with $0\le m\le s-1$. Applying ∂_x^m to the fourth equation in (2.2) and making the inner product of the resulting equation with $\partial_x^{m+1}\Theta$ in $L^2(\mathbb{R})$, we have

$$\begin{aligned}\kappa\|\partial_x^{m+1}\Theta\|^2=&-\varepsilon_1^2\frac{d}{dt}\big\langle\partial_x^m q,\partial_x^{m+1}\Theta\big\rangle+\varepsilon_1^2\big\langle\partial_t\partial_x^{m+1}\Theta,\partial_x^m q\big\rangle\\&-\varepsilon_1^2\big\langle\partial_x^{m+1}\Theta,\partial_x^m(u\partial_x q)\big\rangle-\big\langle\partial_x^m q,\partial_x^{m+1}\Theta\big\rangle.\end{aligned}$$

Obviously,

$$\big|\varepsilon_1^2\big\langle\partial_x^{m+1}\Theta,\partial_x^m(u\partial_x q)\big\rangle+\big\langle\partial_x^m q,\partial_x^{m+1}\Theta\big\rangle\big|\le\frac{\kappa}{2}\|\partial_x^{m+1}\Theta\|^2+C\|q\|_s^2.$$

Therefore,

$$\varepsilon_1^2\frac{d}{dt}\big\langle\partial_x^m q,\partial_x^{m+1}\Theta\big\rangle+\frac{\kappa}{2}\|\partial_x^{m+1}\Theta\|^2\le\varepsilon_1^2\big\langle\partial_t\partial_x^{m+1}\Theta,\partial_x^m q\big\rangle+C\|q\|_s^2. \tag{3.11}$$

By using the third equation in (2.2) and an integration by parts, we have

$$\begin{aligned}\varepsilon_1^2\left|\big\langle\partial_t\partial_x^{m+1}\Theta,\partial_x^m q\big\rangle\right|\le&\,\varepsilon_1\left|\left\langle\partial_x^m\left(u\partial_x\theta+\frac{\theta p_\theta}{\rho e_\theta}\partial_x u-\frac{2q}{\rho\theta e_\theta}\partial_x\theta+\frac{1}{\rho e_\theta}\partial_x q\right),\partial_x^{m+1}(\varepsilon_1 q)\right\rangle\right|\\&+\varepsilon_1\left|\left\langle\partial_x^m\left(\frac{2q^2}{\kappa\rho\theta e_\theta}+\frac{\tau^2}{\lambda\rho e_\theta}\right),\partial_x^{m+1}(\varepsilon_1 q)\right\rangle\right|.\end{aligned}$$

Obviously,

$$\begin{aligned}&\varepsilon_1\left|\left\langle\partial_x^m\left(u\partial_x\theta+\frac{\theta p_\theta}{\rho e_\theta}\partial_x u-\frac{2q}{\rho\theta e_\theta}\partial_x\theta+\frac{1}{\rho e_\theta}\partial_x q\right),\partial_x^{m+1}(\varepsilon_1 q)\right\rangle\right|\\&\le\frac{\kappa}{4}\|\partial_x\Theta\|_{s-1}^2+\nu\|\partial_x u\|_{s-1}^2+C\|q\|_s^2+C\mathscr{E}_T^{1/2}\mathscr{D}(t),\end{aligned}$$

and

$$\begin{aligned}\varepsilon_1\left|\left\langle\partial_x^m\left(\frac{2q^2}{\kappa\rho\theta e_\theta}+\frac{\tau^2}{\lambda\rho e_\theta}\right),\partial_x^{m+1}(\varepsilon_1 q)\right\rangle\right|&\le C\varepsilon_1\|\varepsilon_1 q\|_s\|q\|_s^2+C\varepsilon_1\|\varepsilon_1 q\|_s\|\tau\|_s^2\\&\le C\mathscr{E}_T^{1/2}\mathscr{D}(t).\end{aligned}$$

Combining these estimates, we arrive at

$$\varepsilon_1^2\left|\big\langle\partial_t\partial_x^{m+1}\Theta,\partial_x^m q\big\rangle\right|\le\frac{\kappa}{4}\|\partial_x\Theta\|_{s-1}^2+\nu\|\partial_x u\|_{s-1}^2+C\|q\|_s^2+C\mathscr{E}_T^{1/2}\mathscr{D}(t).$$

This together with (3.11) yields

$$\varepsilon_1^2 \frac{d}{dt}\langle \partial_x^m q, \partial_x^{m+1}\Theta\rangle + \frac{\kappa}{4}\|\partial_x^{m+1}\Theta\|^2 \leq \nu\|\partial_x u\|_{s-1}^2 + C\|q\|_s^2 + C\mathscr{E}_T^{1/2}\mathscr{D}(t).$$

Adding this inequality for all $0 \leq m \leq s-1$ yields (3.10). □

Lemma 3.8 (Dissipative estimates for $\partial_x u$). *It holds*

$$-\varepsilon_2^2 \sum_{m=0}^{s-1} \frac{d}{dt}\langle \partial_x^m \tau, \partial_x^{m+1} u\rangle + \frac{\lambda}{2}\|\partial_x u\|_{s-1}^2 \leq C\nu(\|\partial_x N\|_{s-1}^2 + \|\partial_x u\|_{s-1}^2 + \|\partial_x \Theta\|_{s-1}^2) + C\big(\|q\|_s^2 + \|\tau\|_s^2\big) + C\mathscr{E}_T^{1/2}\mathscr{D}(t), \tag{3.12}$$

where $\nu > 0$ is a small positive constant to be determined later.

Proof. For $0 \leq m \leq s-1$, applying ∂_x^m to the fifth equation in (2.2) and making the inner product of the resulting equation with $\partial_x^{m+1} u$ in $L^2(\mathbb{R})$, we have

$$\lambda\|\partial_x^{m+1}u\|^2 = \varepsilon_2^2 \frac{d}{dt}\langle \partial_x^m \tau, \partial_x^{m+1} u\rangle - \varepsilon_2^2\langle \partial_t\partial_x^{m+1}u, \partial_x^m\tau\rangle + \varepsilon_2^2\langle \partial_x^{m+1}u, \partial_x^m(u\partial_x\tau)\rangle + \langle \partial_x^{m+1}u, \partial_x^m\tau\rangle.$$

Obviously,

$$\big|\varepsilon_2^2\langle \partial_x^{m+1}u, \partial_x^m(u\partial_x\tau)\rangle + \langle \partial_x^{m+1}u, \partial_x^m\tau\rangle\big| \leq \frac{\lambda}{2}\|\partial_x^{m+1}u\|^2 + C\|\tau\|_s^2 + C\mathscr{E}_T^{1/2}\mathscr{D}(t).$$

Therefore,

$$-\varepsilon_2^2 \frac{d}{dt}\langle \partial_x^m \tau, \partial_x^{m+1} u\rangle + \frac{\lambda}{2}\|\partial_x^{m+1}u\|^2 \leq -\varepsilon_2^2\langle \partial_t\partial_x^{m+1}u, \partial_x^m\tau\rangle + C\|\tau\|_s^2 + C\mathscr{E}_T^{1/2}\mathscr{D}(t). \tag{3.13}$$

By using the second equation in (2.2) and an integration by parts, we have

$$\begin{aligned}\big|\varepsilon_2^2\langle \partial_t\partial_x^{m+1}u, \partial_x^m\tau\rangle\big| &\leq \varepsilon_2^2\left|\left\langle \partial_x^m\left(\frac{\rho u\partial_x u + p_\rho\partial_x\rho + p_\theta\partial_x\theta}{\rho}\right), \partial_x^{m+1}\tau\right\rangle\right| \\ &\quad + \varepsilon_2^2\left|\left\langle \partial_x^m\left(\frac{p_q\partial_x q + (p_\tau - 1)\partial_x\tau}{\rho}\right), \partial_x^{m+1}\tau\right\rangle\right| \\ &\leq \nu\big(\|\partial_x N\|_{s-1}^2 + \|\partial_x u\|_{s-1}^2 + \|\partial_x\Theta\|_{s-1}^2\big) + C\big(\|q\|_s^2 + \|\tau\|_s^2\big).\end{aligned}$$

Substituting this estimate into (3.13) and adding the resulting equation for all $0 \leq m \leq s-1$ yield (3.12). □

Lemma 3.9 (Dissipative estimates for $\partial_x N$). *It holds*

$$\sum_{m=0}^{s-1} \frac{d}{dt}\langle \partial_x^m u, \partial_x^{m+1} N\rangle + \frac{R}{6}\|\partial_x N\|_{s-1}^2 \leq C\big(\|\partial_x u\|_{s-1}^2 + \|\partial_x\Theta\|_{s-1}^2\big) + C\big(\|q\|_s^2 + \|\tau\|_s^2\big) + C\mathscr{E}_T^{1/2}\mathscr{D}(t). \tag{3.14}$$

Proof. We first write the second equation in (2.2) as

$$\partial_t u + \frac{R\theta}{\rho}\partial_x N + \frac{1}{\rho}\big(\rho u\partial_x u + p_\theta\partial_x\theta + p_q\partial_x q + (p_\tau - 1)\partial_x\tau\big) = 0.$$

For $m \le s-1$, applying ∂_x^m to the above equation and making the inner product of the resulting equation with $\partial_x^{m+1}N$ in $L^2(\mathbb{R})$, we have

$$\left\langle \partial_x^{m+1}N, \frac{R\theta}{\rho}\partial_x^{m+1}N \right\rangle = -\frac{d}{dt}\langle \partial_x^m u, \partial_x^{m+1}N\rangle + \langle \partial_x^m u, \partial_t\partial_x^{m+1}N\rangle - \left\langle \partial_x^m \left(\frac{\rho u\partial_x u + p_\theta\partial_x\theta + p_q\partial_x q + (p_\tau - 1)\partial_x\tau}{\rho} \right), \partial_x^{m+1}N \right\rangle - \left\langle \partial_x^m \left(\frac{R\theta}{\rho}\partial_x N \right) - \frac{R\theta}{\rho}\partial_x^{m+1}N, \partial_x^{m+1}N \right\rangle.$$

Noticing (3.1), we have

$$\left\langle \partial_x^{m+1}N, \frac{R\theta}{\rho}\partial_x^{m+1}N \right\rangle \ge \frac{R}{3}\|\partial_x^{m+1}N\|^2.$$

It is clear that

$$\left| \left\langle \partial_x^m \left(\frac{\rho u\partial_x u + p_\theta\partial_x\theta + p_q\partial_x q + (p_\tau - 1)\partial_x\tau}{\rho} \right), \partial_x^{m+1}N \right\rangle \right| \le \frac{R}{6}\|\partial_x^{m+1}N\|^2 + C\big(\|\partial_x u\|_{s-1}^2 + \|\partial_x\Theta\|_{s-1}^2\big) + C\big(\|q\|_s^2 + \|\tau\|_s^2\big),$$

and by the Moser-type calculus inequalities,

$$\left| \left\langle \partial_x^m \left(\frac{R\theta}{\rho}\partial_x N \right) - \frac{R\theta}{\rho}\partial_x^{m+1}N, \partial_x^{m+1}N \right\rangle \right| \le C\|\partial_x\theta\|_{s-1}\|\partial_x N\|_{s-1}^2 + C\|\partial_x N\|_{s-1}^3 \le C\mathscr{E}_T^{1/2}\mathscr{D}(t).$$

Moreover, by the mass equation in (2.2) and the integration by parts, we obtain

$$\big|\langle \partial_x^m u, \partial_t\partial_x^{m+1}N\rangle\big| = \big|\langle \partial_x^{m+1}u, \partial_x^{m+1}(\rho u)\rangle\big| \le C\|\partial_x u\|_{s-1}(\|\partial_x^m(\partial_x N u + \rho\partial_x u)\|) \le C\|\partial_x u\|_{s-1}^2 + C\mathscr{E}_T^{1/2}\mathscr{D}(t).$$

Combining all these estimates, we arrive at

$$\frac{d}{dt}\langle \partial_x^m u, \partial_x^{m+1}N\rangle + \frac{R}{6}\|\partial_x^{m+1}N\|^2 \le C\big(\|\partial_x u\|_{s-1}^2 + \|\partial_x\Theta\|_{s-1}^2\big) + C\big(\|q\|_s^2 + \|\tau\|_s^2\big) + C\mathscr{E}_T^{1/2}\mathscr{D}(t).$$

Adding the above estimate for all $m \le s-1$ yields (3.14). $\square$

4. Proof of Theorems 1.1-1.2

Lemma 4.1 (Final energy estimate). *If $\mathscr{E}_T$ is sufficiently small, then*

$$\mathscr{E}(t) + \int_0^t \mathscr{D}(t')dt' \le C\mathscr{E}(0), \quad \forall\, t \in [0, T]. \tag{4.1}$$

Proof. Combining (3.4) and Lemmas 3.2-3.6, and adding for all $1 \le l \le s$, we arrive at

$$\sum_{l=1}^{s}\frac{d}{dt}\langle A_0(V)\partial_x^l V, \partial_x^l V\rangle + c_3\|\partial_x q\|_{s-1}^2 + c_4\|\partial_x\tau\|_{s-1}^2 \le C\mathscr{E}_T^{1/2}\mathscr{D}(t). \tag{4.2}$$

Since $A_0(V)$ is positive definite with respect to W, there exists a constant $c_5 > 0$, independent of ε_1 and ε_2, such that

$$\langle A_0(V)\partial_x^l V, \partial_x^l V\rangle \ge c_5\|\partial_x^l W\|^2.$$

Integrating (4.2) over $[0,T]$ together with (3.3) yields

$$\|W(t)\|_s^2 + \int_0^t \big(\|q(t')\|_s^2 + \|\tau(t')\|_s^2\big)dt' \le C\mathscr{E}(0) + C\mathscr{E}_T^{1/2}\int_0^t \mathscr{D}(t')dt'. \tag{4.3}$$

On the other hand, combining the estimates in Lemmas 3.7-3.9, we have

$$\begin{aligned}
&\sum_{m=0}^{s-1}\frac{d}{dt}\big(\varepsilon_1^2\langle\partial_x^m q,\partial_x^{m+1}\Theta\rangle - \varepsilon_2^2\langle\partial_x^m\tau,\partial_x^{m+1}u\rangle + \alpha\langle\partial_x^m u,\partial_x^{m+1}N\rangle\big)\\
&\quad + \frac{R\alpha}{6}\|\partial_x N\|_{s-1}^2 + \frac{\lambda}{2}\|\partial_x u\|_{s-1}^2 + \frac{\kappa}{4}\|\partial_x\Theta\|_{s-1}^2\\
&\quad \le C\nu\|\partial_x N\|_{s-1}^2 + C(\nu+\alpha)(\|\partial_x u\|_{s-1}^2 + \|\partial_x\Theta\|_{s-1}^2) + C\big(\|q\|_s^2 + \|\tau\|_s^2\big) + C\mathscr{E}_T^{1/2}\mathscr{D}(t),
\end{aligned}$$

where $\alpha > 0$ is a constant. We choose $\nu > 0$ and $\alpha > 0$ sufficiently small such that

$$2C\nu \le \frac{R\alpha}{6}, \quad 8C(\nu+\alpha) \le \min(\lambda,\kappa).$$

It follows that there is a constant $a_0 > 0$, independent of ε_1 and ε_2, such that

$$\begin{aligned}
&\sum_{m=0}^{s-1}\frac{d}{dt}\big(\varepsilon_1^2\langle\partial_x^m q,\partial_x^{m+1}\Theta\rangle - \varepsilon_2^2\langle\partial_x^m\tau,\partial_x^{m+1}u\rangle + \alpha\langle\partial_x^m u,\partial_x^{m+1}N\rangle\big)\\
&\quad + a_0\big(\|\partial_x N\|_{s-1}^2 + \|\partial_x u\|_{s-1}^2 + \|\partial_x\Theta\|_{s-1}^2\big)\\
&\quad \le C\big(\|q\|_s^2 + \|\tau\|_s^2\big) + C\mathscr{E}_T^{1/2}\mathscr{D}(t).
\end{aligned} \tag{4.4}$$

It is clear that

$$\big|\varepsilon_1^2\langle\partial_x^m q,\partial_x^{m+1}\Theta\rangle - \varepsilon_2^2\langle\partial_x^m\tau,\partial_x^{m+1}u\rangle + \alpha\langle\partial_x^m u,\partial_x^{m+1}N\rangle\big| \le C\|W\|_s^2.$$

Integrating (4.4) over $[0,T]$ yields

$$\begin{aligned}
&-C\|W(t)\|_s^2 + a_0\int_0^t \big(\|\partial_x N(t')\|_{s-1}^2 + \|\partial_x u(t')\|_{s-1}^2 + \|\partial_x\Theta(t')\|_{s-1}^2\big)dt'\\
&\quad \le C\mathscr{E}(0) + C\int_0^t \big(\|q(t')\|_s^2 + \|\tau(t')\|_s^2\big) + C\mathscr{E}_T^{1/2}\int_0^t \mathscr{D}(t')dt'.
\end{aligned}$$

This inequality together with (4.3) yields

$$\mathscr{E}(t) + \int_0^t \mathscr{D}(t')dt' \le C\mathscr{E}(0) + C\mathscr{E}_T^{1/2}\int_0^t \mathscr{D}(t')dt',$$

which implies (4.1) since $\mathscr{E}_T$ is sufficiently small. □

Proof of Theorem 1.1. The estimate in Lemma 4.1 shows that the smooth solution W is uniformly bounded in $L^\infty([0,T];H^s(\mathbb{R}))$ with respect to ε and T. By the bootstrap principle, it yields uniformly global solution. In particular, this estimate gives (1.16). □

Proof of Theorem 1.2. From (1.11), $(\rho^\varepsilon, u^\varepsilon, \theta^\varepsilon, q^\varepsilon, \tau^\varepsilon)$ satisfies the following system

$$\begin{cases}
\partial_t\rho^\varepsilon + \partial_x(\rho^\varepsilon u^\varepsilon) = 0,\\
\partial_t(\rho^\varepsilon u^\varepsilon) + \partial_x(\rho^\varepsilon(u^\varepsilon)^2 + p^\varepsilon - \tau^\varepsilon) = 0,\\
\partial_t(\rho^\varepsilon E^\varepsilon) + \partial_x(\rho^\varepsilon u^\varepsilon E^\varepsilon + u^\varepsilon p^\varepsilon + q^\varepsilon - u^\varepsilon\tau^\varepsilon) = 0,\\
\varepsilon_1^2(\partial_t q^\varepsilon + u^\varepsilon\partial_x q^\varepsilon) + \kappa\partial_x\theta^\varepsilon = -q^\varepsilon,\\
\varepsilon_2^2(\partial_t\tau^\varepsilon + u^\varepsilon\partial_x\tau^\varepsilon) - \lambda\partial_x u^\varepsilon = -\tau^\varepsilon,
\end{cases} \tag{4.5}$$

in $\mathbb{R}^+ \times \mathbb{R}$, where

$$\begin{cases} e^\varepsilon = c_v\theta^\varepsilon + \dfrac{\varepsilon_1^2}{\kappa\rho^\varepsilon\theta^\varepsilon}(q^\varepsilon)^2 + \dfrac{\varepsilon_2^2}{2\lambda\rho^\varepsilon}(\tau^\varepsilon)^2, \\ p^\varepsilon = R\rho^\varepsilon\theta^\varepsilon - \dfrac{\varepsilon_1^2}{2\kappa\theta^\varepsilon}(q^\varepsilon)^2 - \dfrac{\varepsilon_2^2}{2\lambda}(\tau^\varepsilon)^2, \\ E^\varepsilon = e^\varepsilon + \dfrac{1}{2}(u^\varepsilon)^2. \end{cases} \tag{4.6}$$

The uniform estimate (1.16) implies that the sequences $(\rho^\varepsilon - 1)_\varepsilon$, $(u^\varepsilon)_\varepsilon$ and $(\theta^\varepsilon - 1)_\varepsilon$ are bounded in $L^\infty(\mathbb{R}^+; H^s(\mathbb{R}))$ and the sequence $(q^\varepsilon)_\varepsilon$ and $(\tau^\varepsilon)_\varepsilon$ are bounded in $L^2(\mathbb{R}^+; H^s(\mathbb{R}))$. It follows that there exist functions $(\bar\rho, \bar u, \bar\theta) \in L^\infty(\mathbb{R}^+; H^s(\mathbb{R}))$ and $(\bar q, \bar\tau) \in L^2(\mathbb{R}^+; H^s(\mathbb{R}))$, such that (1.17)-(1.18) hold. In addition, as $\varepsilon \to 0$,

$$\varepsilon_1^2(\partial_t q^\varepsilon + u^\varepsilon\partial_x q^\varepsilon) \rightharpoonup 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}),$$

and

$$\varepsilon_2^2(\partial_t \tau^\varepsilon + u^\varepsilon\partial_x \tau^\varepsilon) \rightharpoonup 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}).$$

Moreover, from the first three equations in (4.5), it is easy to see that $(\partial_t\rho^\varepsilon)_\varepsilon$, $(\partial_t u^\varepsilon)_\varepsilon$ and $(\partial_t\theta^\varepsilon)_\varepsilon$ are bounded in $L^2(\mathbb{R}^+; H^{s-1}(\mathbb{R}))$. Hence, by a classical compactness theorem [40], for all $T > 0$, $(\rho^\varepsilon)_\varepsilon, (u^\varepsilon)_\varepsilon$ and $(\theta^\varepsilon)_\varepsilon$ are relatively compact in $C([0,T]; H^{s-1}_{loc}(\mathbb{R}))$. As a consequence, as $\varepsilon \to 0$, up to subsequences,

$$(\rho^\varepsilon, u^\varepsilon, \theta^\varepsilon) \to (\bar\rho, \bar u, \bar\theta) \quad \text{strongly in } C([0,T]; H^{s-1}_{loc}(\mathbb{R})).$$

This is sufficient to pass the limit in (4.5)-(4.6) in the sense of distributions and to obtain the Navier-Stokes equations for the ideal fluid. This ends the proof of Theorem 1.2. □

Declarations

Conflict of interest There is no conflict of interest.

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(accepted: January 29, 2022; published online: March 2, 2022)