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Existence and Exponential Behavior for the Stochastic 2D Cahn–Hilliard–Oldroyd Model of Order One

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Abstract. In this article, we establish the unique global solvability of a 2D stochastic Cahn–Hilliard–Oldroyd model of order one, for the motion of an incompressible isothermal mixture of two (partially) immiscible non-Newtonian fluids having the same density and perturbed by a multiplicative noise of Gaussian and Lévy type. The model consists of the stochastic Oldroyd model of order one, coupled with a stochastic Cahn–Hilliard model. We prove the existence and uniqueness of a strong solution (in the stochastic sense). The proofs are based on the Galerkin approximation technique. Moreover, we also prove that under some conditions on the forcing terms, the strong solution converges exponentially in the mean square and almost surely exponentially to the stationary solutions.

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1. Introduction

The Oldroyd model of order one arises in the dynamics of non-Newtonian fluids and is well-known as the generalization of the initial-boundary value problem for the Navier–Stokes equations. It is used to model the motion of viscoelastic fluids (see [23,31]). The analysis of this model, both in the deterministic and the stochastic cases has been investigated by several authors (see for instance [23,25–27,31,32]). The well-posedness and the exponential stability of the model in two-dimensional bounded and unbounded (Poincaré domains) domains, both in deterministic and stochastic settings is studied in [27]. The proof of the existence and the uniqueness of weak solution in the deterministic case is obtained via a local monotonicity property of the linear and nonlinear operators and a localized version of the Minty-Browder technique. The global solvability results for the stochastic counterpart are obtained by a stochastic generalization of the Minty-Browder technique. In order to describe the behavior of complex fluids in Fluid Mechanics, diffuse-interface methods are widely used by many researchers (see, e.g., [3,15] and references therein). A typical example is a mixture of two incompressible fluids. The evolution of such a system is described by a sufficiently simple model so-called H model (see [19,20,30] and references therein). This consists in a suitable coupling of the Navier–Stokes equations for the (average) fluid velocity u through a capillarity force proportional to $\mu\nabla\phi$, where μ is the chemical potential, with a local or nonlocal Cahn–Hilliard type equation for the order parameter ϕ through a transport term $u\cdot\nabla\phi$. The mathematical and the numerical analysis of the deterministic and the stochastic local and nonlocal Cahn–Hilliard–Navier–Stokes (CH-NS) model has been considered by several authors such as [9–14,17,18,21,22,24,36–38] and references therein. In [13], the authors considered the stochastic 3D globally modified local CH–NS equations with multiplicative Gaussian noise. They proved the existence and uniqueness of strong solution (in the sense of partial differential equations and stochastic analysis) and derived the existence of a weak martingale solution for the stochastic 3D local CH-NS equations. The third author of the paper has proved the existence and uniqueness of the probabilistic strong solution for the stochastic 2D local CH-NS model with multiplicative Gaussian type of noise in [36] and studies the asymptotic stability of the unique probabilistic strong solution of 2D local CH-NS model in [37]. He also proved the existence of weak solution of 3D local CH-NS model with multiplicative non-Gaussian Lévy noise in [38]. Similar results have been obtained for the 2D and 3D nonlocal CH-NS model in [9,11,12]. Generally, the CH-NS model is used to modelise the flow of a Newtonian binary mixture. However, for viscoelastic binary fluids, the behavior of viscoelastic fluids cannot be predicted by the means of usual Newton's constitutive law since they possess a memory of past deformations which is not the case for Newtonian fluids. Consequently, we have to introduce a more general phenomenological model such as Oldroyd model in addition to the Navier–Stokes equation as a constitutive equation to modelise the viscoelasticity. Taking into account this fact, we derive a model where we will call Cahn–Hilliard–Oldroyd model.

In this article, we study a stochastic generalization model of the CH-NS model. More precisely, we consider in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with an increasing family of sub-sigma fields $\{\mathcal{F}_t\}_{0\leq t\leq T}$ of $\mathcal{F}$ such that $\mathcal{F}_0$ contains all elements $F \in \mathcal{F}$ with $\mathbb{P}(F) = 0$ and $\mathcal{F}_t = \cap_{s>t}\mathcal{F}_s$ for $0 \leq t \leq T$, the following stochastic Cahn–Hilliard–Oldroyd model of order one, for the motion of an incompressible isothermal mixture of two immiscible non-Newtonian fluids

$$\begin{cases} du(t) + [-\nu_1\Delta u + (\beta * \Delta u)(t) + (u.\nabla)u + \nabla p - \mathcal{K}\mu\nabla\phi]\, dt \\ \qquad\qquad = \sigma_1(t, u, \phi)dW_t^1 + \int_Z \gamma(t, u(t^-), \phi(t), z)\tilde{\pi}(dt, dz), \\ d\phi(t) = [\nu_2\Delta\mu - u.\nabla\phi]\, dt + \sigma_2(t, u, \phi)dW_t^2, \\ \mu = -\varepsilon\Delta\phi + \alpha f(\phi), \\ \operatorname{div}(u) = 0, \end{cases} \tag{1.1}$$

in $(0, T) \times \mathcal{M}$ with the conditions

$$\begin{cases} \partial_\eta\phi = \partial_\eta\Delta\phi = 0, \quad \text{on } (0, T) \times \partial\mathcal{M}, \\ u = 0 \quad \text{on } (0, T) \times \partial\mathcal{M}, \\ (u, \phi)(0) = (u_0, \phi_0) \quad \text{in } \mathcal{M}, \end{cases} \tag{1.2}$$

where $T > 0$, $\mathcal{M}$ is a bounded open domain in $\mathbb{R}^2$ with a smooth boundary $\partial\mathcal{M}$ and

$$(\beta * \Delta u)(t) = \int_0^t \beta(t - s)\Delta u(s)ds.$$

In (1.1), W_t^i, $i = 1, 2$, are two cylindrical Wiener processes in a separable Hilbert space U defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Also Z is a measurable subspace of some Hilbert space and $\tilde{\pi}(dt, dz) := \pi(dt, dz) - \lambda(dz)dt$ is a compensated Poisson random measure, where $\lambda(dz)$ is a σ-finite Lévy measure on the Hilbert space with an associated Poisson random measure $\pi(dt, dz)$ such that $\mathbb{E}[\pi(dt, dz)] = \lambda(dz)dt$. The processes W_t^i, $i = 1, 2$, and $\tilde{\pi}$ are mutually independent. Let $\nu > 0$ be the coefficient of kinematic viscosity. In (1.1), for $\varsigma, \kappa > 0$, we take

$$\nu_1 = \frac{\kappa}{\varsigma}, \quad \text{the kernel} \quad \beta(t) = \gamma e^{-\delta t}, \quad \text{where} \quad \gamma = \frac{1}{\varsigma}(\nu - \frac{\kappa}{\varsigma}) > 0 \quad \text{and} \quad \delta = \frac{1}{\varsigma} > 0. \tag{1.3}$$

If we take $\nu_1 = \frac{\kappa}{\varsigma}$ (or $\gamma = 0$) in (1.3), then we obtain a model for the phase separation of an incompressible and isothermal Newtonian binary fluid flow or the well-known Cahn–Hilliard–Navier–Stokes model [13,37, 38]. Note that ς is the relaxation time, and is characterized by the fact that after instantaneous cessation of motion, the stresses in the fluid do not vanish instantaneously, but die out like $e^{-\varsigma^{-t}}$. Moreover, the velocities of the flow, after instantaneous removal of the stresses, die out like $e^{-\kappa^{-t}}$, where κ is the retardation time. For the physical background and the mathematical modeling of viscoelastic fluid flows involving memory effects, we refer the interested readers to [2,23,34], where the topic is studied extensively. The quantity μ is the chemical potential of the binary mixture which is given by the variational derivative of the following free energy functional

$$\mathcal{E}_0(\phi) = \int_D \left(\frac{\varepsilon}{2}|\nabla\phi|^2 + \alpha F(\phi)\right) dx, \tag{1.4}$$

where, $F(r) = \int_0^r f(\zeta)d\zeta$ is the suitable double-well potential. The quantities ν_2 and $\mathcal{K}$ are positive constants that correspond to the mobility constant and capillarity (stress) coefficient, respectively. ε and α are two positive parameters describing the interactions between the two phases. In particular, ε is related to the thickness of the interface separating the two fluids. A typical example of potential F is of logarithmic type. However, this potential is very often replaced by a polynomial approximation of the type $F(r) = \gamma_1 r^4 - \gamma_2 r^2$, with γ_1 and γ_2 are positive constants. Note that in (1.1), the ϕ equation modelises the evolution of the concentration of fluids which can be influenced by the thermal fluctuations which is a random phenomena. In order to take into account this thermal fluctuations, we have introduce a noise in addition to the ϕ equation as a constitutive equation. However, a Levy type noise in the ϕ equation is also possible but, the presence of such noise will involve probably more assumptions and will increase significantly the size of the paper.

To the best of our knowledge, there is no mathematical analysis of the model (1.1) even in its deterministic setting. This article is a contribution in that direction. Moreover, in the literature, there is no work on stochastic two-phase flows models with both Gaussian and non-Gaussian type of noise. But a such type of noise have been considered for instance in [27,29] for the Oldroyd model of order one and the Navier–Stokes equation with hereditary viscosity. In general the presence of a noise on the concentration equation in the two-phase flow model makes the analysis of the model more involved (see [13,16,37]). In [13], an existence result has been obtained under the additional strong condition on the potential f. In order to use a weakened condition on the potential function f for the existence result, we have shown that the energy functional $\mathcal{E}_0$ is twice Fréchet differentiable which makes possible the application of Itô's formula to the process $\mathcal{E}_0(\phi)$. Note that in (1.1) it is possible to add a Lévy type of noise on the relative concentration equation, but the analysis will be tedious. The purpose of the present manuscript is to prove some results related to problem (1.1). Our main results are the following: First, we prove the existence and uniqueness of strong solution (in the stochastic sense) for system (1.1). The method for the proof is based in the Galerkin approximation. Secondly, getting the existence of a unique strong solution in hand, we investigate the stability of this solution. More precisely, we prove that under some conditions on the forcing terms, the strong solution converges exponentially in the mean square and almost surely exponentially to the stationary solutions.

The rest of the paper is organized as follows. In the next section, we describe the mathematical setting required to establish the unique solvability of the system (1.1). The hypothesis satisfied by the potential, the noise coefficient and the external forcing are also discussed in the section. In Sect. 3, we introduce the Galerkin approximation of our problem and we derive a priori estimates for its solution. Then we prove the existence and the pathwise uniqueness of strong solution. In the last section, we analyze the stability of stationary solutions.

2. Functional Setting, Hypothesis and Abstract Formulation

In this section, we fix the hypothesis and describe the functional spaces needed to establish the existence and uniqueness of global strong solution of the system (1.1). We discuss the properties of linear and nonlinear operators, and also of the kernel $\beta(t) = \gamma e^{-\delta t}$.

2.1. Functional Setting

We now introduce the functional setup of Eqs. (1.1)–(1.2). If X is real Hilbert space with inner product $(.,.)_X$, then we denote the induced norm by $|.|_X$, while X^* will indicate its dual. Let us consider the Hilbert spaces

$$H_1 := \overline{\{u \in \mathcal{C}_c^\infty((\mathcal{M}))^2 : \text{div } u = 0 \text{ in } \mathcal{M}\}}^{\mathbb{L}^2},\ V_1 := \overline{\{u \in \mathcal{C}_c^\infty((\mathcal{M}))^2 : \text{div } u = 0 \text{ in } \mathcal{M}\}}^{\mathbb{H}_0^1},$$

where $\mathbb{L}^2(\mathcal{M}) := (L^2(\mathcal{M}))^2$, $\mathbb{H}_0^1(\mathcal{M}) := (H_0^1(\mathcal{M}))^2$. We endow H_1 with the L^2-inner product and norm

$$(u,v) := \int_{\mathcal{M}} u.v\ dx,\ |u|_{L^2} := (u,u)^{1/2}.$$

Moreover, the space V_1 is endowed with the scalar product and norm

$$((u,v)) := \sum_{i=1}^{2} (\partial_{x_i} u, \partial_{x_i} v),\ \|u\| = ((u,u))^{1/2}.$$

The norm in V_1 is equivalent to the $\mathbb{H}^1(\mathcal{M})$-norm (due to Poincaré's inequality). We refer the reader to [35] for more details on these spaces.

We now define the operator A_0 by

$$A_0 u = -\mathcal{P}\Delta u,\ \forall u \in D(A_0) = \mathbb{H}^2(\mathcal{M}) \cap V_1,$$

where $\mathcal{P}$ is the Leray-Helmholtz projector in $\mathbb{L}^2(\mathcal{M})$ onto H_1. Then, A_0 is a self-adjoint positive unbounded in H_1 which is associated with the scalar product defined above. Furthermore, A_0^{-1} is a compact linear operator on H_1 and by the classical spectral theorem, there exists a sequence λ_j with $0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n \leq \lambda_{n+1} \leq \cdots$ and a family $w_j \in D(A_0)$ which is an orthonormal basis in H_1 and such that $A_0 w_j = \lambda_j w_j$.

For $u \in H_1$, we denote $u_j = (u, w_j)$. Given $\alpha > 0$, take

$$D(A_0^\alpha) = \{u \in H_1 : \sum_j \lambda_j^{2\alpha} |u_j|^2 < \infty\}, \tag{2.1}$$

and define $A_0^\alpha u = \sum_j \lambda_j^\alpha u_j w_j$, $u \in D(A_0^\alpha)$. We equip $D(A_0^\alpha)$ with the norm $|u|_\alpha^2 := |A_0^\alpha u|_{L^2}^2 = \sum_j \lambda_j^{2\alpha} |u_j|^2$.

We introduce the linear nonnegative unbounded operator on $L^2(\mathcal{M})$

$$A_1\varphi = -\Delta\varphi,\ \forall \varphi \in D(A_1) = \left\{\varphi \in H^2(\mathcal{M}),\ \partial_\eta \varphi = 0,\ \text{on } \partial\mathcal{M}\right\}, \tag{2.2}$$

and we endow $D(A_1)$ with the norm $|A_1 \cdot| + |\langle\cdot\rangle|$, which is equivalent to the H^2-norm. We also define the linear positive unbounded operator on the Hilbert space $L_0^2(\mathcal{M})$ of the L^2-functions with null mean

$$B_n\varphi = -\Delta\varphi,\ \forall\varphi \in D(B_n) = D(A_1) \cap L_0^2(\mathcal{M}). \tag{2.3}$$

Note that B_n^{-1} is a compact linear operator on $L_0^2(\mathcal{M})$. More generally, we can define B_n^s, for any $s \in \mathbb{R}$, noting that $|B_n^{s/2} \cdot |_{L^2}$, $s > 0$, is an equivalent norm to the canonical H^s-norm on $D(B_n^{s/2}) \subset H^s(\mathcal{M}) \cap L_0^2(\mathcal{M})$. Also note that $A_1 = B_n$ on $D(B_n)$. If φ is such that $\varphi - \langle\varphi\rangle \in D(B_n^{s/2})$, we have that $|B_n^{s/2}(\varphi - \langle\varphi\rangle)|_{L^2} + |\varphi - \langle\varphi\rangle|_{L^2}$ is equivalent to the H^s-norm. Moreover, we set $H^{-s}(\mathcal{M}) = (H^s(\mathcal{M}))'$, whenever $s < 0$.

We note that

$$D(A_1) = \left\{\phi \in H^2 : \frac{\partial\phi}{\partial\eta} = 0 \text{ on } \partial\mathcal{M}\right\},$$

$$A_1\phi = -\sum_{l=1}^{d} \frac{\partial^2\phi}{\partial x_j^2},\ \phi \in D(A_1). \tag{2.4}$$

Classically, there exists a sequence β_j with $0 < \beta_1 < \beta_2 \leq \cdots \leq \beta_n \leq \beta_{n+1} \leq \cdots$ and a family $\psi_j \in D(A_1)$ which is an orthonormal basis in $L^2(\mathcal{M})$ and such that $A_1\psi_j = \beta_j\psi_j$.
Now for $\alpha \geq 0$ we define

$$D(A_1^\alpha) = \{\phi \in H_2 : \sum_j^\infty \beta_j^{2\alpha}|(\phi, \psi_j)|^2 < \infty\}, \tag{2.5}$$

endowed with the Hilbertian norm $|\phi|_\alpha^2 := |A_1^\alpha\phi|_{L^2}^2 = \sum_j \beta_j^{2\alpha}|(\phi, \psi_j)|^2$.
We introduce the bilinear operators B_0, B_1 (and their associated trilinear forms b_0, b_1) as well as the coupling mapping R_0 which are defined, from $D(A_0) \times D(A_0)$ into H_1, $D(A_0) \times D(A_1)$ into $L^2(\mathcal{M})$ and $L^2(\mathcal{M}) \times D(A_1^{3/2})$ into H_1, respectively. More precisely, we set

$$\begin{aligned}
(B_0(u, v), w) &= \int_{\mathcal{M}} [(u \cdot \nabla)v]w dx \\
&= b_0(u, v, w),\ \forall u, v, w \in D(A_0), \\
\\
(B_1(u, \varphi), \psi) &= \int_{\mathcal{M}} [(u \cdot \nabla)\varphi]\psi dx \\
&= b_1(u, \varphi, \psi),\ \forall u \in D(A_0), \varphi, \psi \in D(A_1), \\
\\
(R_0(\mu, \varphi), w) &= \int_{\mathcal{M}} [\mu\nabla\varphi]w dx \\
&= b_1(w, \varphi, \mu),\ \forall w \in D(A_0),\ (\mu, \varphi) \in L^2(\mathcal{M}) \times D(A_1^{3/2}).
\end{aligned} \tag{2.6}$$

Note that

$$R_0(\mu, \varphi) = \mathcal{P}\mu\nabla\varphi.$$

Using an integration by part, we can check that

$$b_0(u, v, v) = 0,\ \forall u, v \in V_1,$$

$$b_1(v, \phi, \phi) = 0,\ \forall v \in V_1,\ \phi \in H^1(\mathcal{M}), \tag{2.7}$$

$$\langle R_0(A_1\phi, \phi), v\rangle = \langle B_1(v, \phi), A_1\phi\rangle = b_1(v, \phi, A_1\phi),\ \forall (v, \phi) \in V_1 \times D(A_1). \tag{2.8}$$

We recall from [18] that, using the integration by parts, a suitable generalized Hölder inequality and a suitable Gagliardo-Nirenberg interpolation inequality, we derive that b_0, and b_1 satisfy the following properties.

$$\begin{aligned}
&|b_0(u, v, w)| \leq c|u|_{L^2}^{1/2}\|u\|^{1/2}\|v\||w|_{L^2}^{1/2}\|w\|^{1/2},\ u, v, w \in V_1, \\
&|b_1(u, \phi, \psi)| \leq c|u|_{L^2}^{1/2}\|u\|^{1/2}\|\phi\|^{1/2}|A_1\phi|_{L^2}^{1/2}|\psi|_{L^2}\ u \in V_1,\ \phi \in D(A_1),\ \psi \in H_2, \\
&|b_1(u, \phi, \psi)| \leq c|u|_{L^2}\|\phi\|^{1/2}|A_1\phi|_{L^2}^{1/2}\|\psi\|\ u \in V_1,\ \phi \in D(A_1),\ \psi \in H_2.
\end{aligned} \tag{2.9}$$

$$\|B_0(u, v)\|_{V_1^*} \leq c|u|_{L^2}^{1/2}\|u\|^{1/2}|v|_{L^2}^{1/2}\|v\|^{1/2}\ u, v, w \in V_1. \tag{2.10}$$

$$|R_0(A_1\phi, \rho)|_{V_1^*} \leq c\begin{cases} \|\rho\|^{1/2}|A_1\rho|_{L^2}^{1/2}|A_1\phi|_{L^2},\ \phi, \rho \in D(A_1) \\ \|\rho\|^{1/2}|A_1\rho|_{L^2}^{1/2}\ \|\phi\|^{1/2}\ |A_1^{3/2}\phi|_{L^2}^{1/2},\ \rho \in D(A_1),\ \phi \in D(A_1^{3/2}). \end{cases} \tag{2.11}$$

$$\|B_1(u, \phi)\|_{V_2^*} \leq c|u|_{L^2}\|\phi\|^{1/2}|A_1\phi|_{L^2}^{1/2},\ u \in H_1,\ \phi \in D(A_1). \tag{2.12}$$

owing to $(1.2)_1$ we derive that

$$\partial_\eta\mu = 0 \text{ on } (0, T) \times \partial\mathcal{M}. \tag{2.13}$$

From (2.13), we deduce the mass conservation in he deterministic case. In fact, for $\sigma_2(\cdot) = 0$, from (2.13), we have the conservation of the following quantity

$$\langle\phi(t)\rangle = |\mathcal{M}|^{-1}\int_{\mathcal{M}} \phi(t, x)dx, \tag{2.14}$$

where $|\mathcal{M}|$ stands for the Lebesgue measure of the domain $\mathcal{M}$. More precisely, we have

$$\langle\phi(t)\rangle = \langle\phi(0)\rangle, \quad \forall t \in [0, T]. \tag{2.15}$$

Hereafter, we assume that $\sigma_2(\cdot)$ is chosen such that (2.14) is satisfied, which is the case if we assume that

$$\left\langle \int_0^t \sigma_2(s,v,\psi)dW_s^2 \right\rangle = 0, \quad \forall t \geq 0,\ v \in H_1,\ \psi \in H_2, \tag{2.16}$$

where H_2 is defined by (2.19) below.
Due to the mass conservation, we have

$$\langle \phi(t) \rangle = \langle \phi(0) \rangle =: M_0,\ \forall t \geq 0. \tag{2.17}$$

Thus, up to a shift of the order parameter field, we can always assume that the mean of ϕ is zero a the initial time and, therefore it will remain zero for all positive times. Hereafter, we assume that

$$\langle \phi(t) \rangle = \langle \phi(0) \rangle = 0,\ \forall t > 0. \tag{2.18}$$

We set

$$H_2 = D(A_1^{1/2}) \quad \text{and} \quad \mathcal{H} = H_1 \times D(A_1^{1/2}). \tag{2.19}$$

The norm in H_2 is denoted by $\|\cdot\|$, where $\|\psi\|^2 = |A_1^{1/2}\psi|_{L^2}^2$. The space $\mathcal{H}$ is a complete metric space with respect to the metric associated with the norm

$$|(v,\psi)|_{\mathcal{H}}^2 = \mathcal{K}^{-1}|v|^2 + \epsilon\|\psi\|^2. \tag{2.20}$$

We set $V_2 = D(A_1)$ and define the Hilbert spaces $\mathcal{U}$ and $\mathbb{V}$ respectively by

$$\mathcal{U} = V_1 \times D(A_1^{3/2}), \qquad \mathbb{V} = V_1 \times V_2 = V_1 \times D(A_1), \tag{2.21}$$

endowed with the scalar product whose associated norm are respectively

$$\|(v,\psi)\|_{\mathcal{U}}^2 = \|v\|^2 + |A_1^{3/2}\psi|_{L^2}^2, \qquad \|(v,\psi)\|_{\mathbb{V}}^2 = \|v\|^2 + |A_1\psi|_{L^2}^2. \tag{2.22}$$

We will denote by λ_0 a positive constant such that

$$\lambda_0|(v,\psi)|_{\mathcal{H}}^2 \leq \|(v,\psi)\|_{\mathcal{U}}^2 \quad \forall (v,\psi) \in \mathcal{U}. \tag{2.23}$$

For $u_1 = (v_1,\phi_1)$, $u_2 = (v_2,\phi_2)$, $u_3 = (v_3,\phi_3) \in \mathbb{V}$, we define

$$\begin{aligned} \langle B(u_1,u_2),u_3\rangle &= b(u_1,u_2,u_3) = b_0(v_1,v_2,v_3) + b_1(u_1,\phi_2,\phi_3), \\ R(u_1,u_2) &= (R_0(A_1\phi_1,\phi_2),0), \quad E(u_1) = (0, A_1 f(\phi_1)). \end{aligned}$$

Lemma 2.1. *The maps B, R and E are locally Lipschitz continuous i.e. for every $r > 0$, there exists a constant C_r such that*

$$\begin{aligned} &\|B(v_1,v_1) - B(v_2,v_2)\|_{\mathbb{V}^*} \leq C_r \|v_1 - v_2\|_{\mathbb{V}}, \\ &\|R(v_1,v_1) - R(v_2,v_2)\|_{\mathbb{V}^*} \leq C_r \|v_1 - v_2\|_{\mathbb{V}}, \\ &\|E(v_1) - E(v_2)\|_{\mathbb{V}^*} \leq C_r \|v_1 - v_2\|_{\mathbb{V}}, \end{aligned} \tag{2.24}$$

for all $v_1 = (u_1,\phi_1)$, $u_2 = (u_2,\phi_2) \in \mathbb{V}$ with $\|v_1\|_{\mathbb{V}}$ and $\|v_2\|_{\mathbb{V}} \leq r$, where $\mathbb{V}^*$ is the dual space of $\mathbb{V}$.

Proof. Let $v_1 = (u_1,\phi_1)$, $u_2 = (u_2,\phi_2) \in \mathbb{V}$ and $(w,\psi) \in \mathbb{V}$. We assume that $\|v_1\|_{\mathbb{V}} \leq r$ and $\|v_2\|_{\mathbb{V}} \leq r$. To prove $(2.24)_1$, we note that

$$\|B(v_1,v_1) - B(v_2,v_2)\|_{\mathbb{V}^*} = \|B_0(u_1,u_1) - B_0(u_2,u_2)\|_{V_1^*} + \|B_1(u_1,\phi_1) - B(v_2,\phi_2)\|_{D(A_1^{-1})}.$$

Also,

$$\begin{aligned} &|\langle B_0(u_1,u_1) - B_0(u_2,u_2), w\rangle| = |b_0(u_1,u_1,w) - b_0(u_1,u_2,w) + b_0(u_1,u_2,w) - b_0(u_2,u_2,w)| \\ &\quad = |b_0(u_1 - u_2,u_2,w) + b_0(u_1,u_1 - u_2,w)| \leq |b_0(u_2 - u_1,u_2,w)| + |b_0(u_1,u_2 - u_1,w)| \\ &\quad \leq c\|u_1 - u_2\|\,\|u_2\|\,\|w\| + c\|u_1\|\,\|u_1 - u_2\|\,\|w\| = 2c\|u_2 - u_1\|\,\|w\|, \end{aligned}$$

which implies that

$$\|B_0(u_1,u_1) - B_0(u_2,u_2)\|_{V_1^*} \leq 2c\|u_2 - u_1\|. \tag{2.25}$$

Proceeding similarly as in (2.25) and using the fact that $D(A_1)$ is continuously embedded in $L^2(\mathcal{M})$, we obtain

$$\begin{aligned}|\langle B_1(u_1,\phi_1)-B_1(u_2,\phi_2),\psi\rangle| &\le |b_1(u_1-u_2,\phi_1,\psi)|+|b_1(u_2,\phi_1-\phi_2,\psi)|\\ &\le c\,\|u_1-u_2\|\,|A_1\phi_1|_{L^2}\,|A_1\psi|_{L^2}+c\,\|u_2\|\,|A_1\psi|_{L^2}\,|A_1(\phi_1-\phi_2)|_{L^2}\\ &\le cr(\|u_1-u_2\|+|A_1(\phi_1-\phi_2)|_{L^2})\,|A_1\psi|_{L^2}\,.\end{aligned}$$

It then follows that

$$\|B_1(u_1,\phi_1)-B(v_2,\phi_2)\|_{D(A_1^{-1})}\le cr(\|u_1-u_2\|+|A_1(\phi_1-\phi_2)|_{L^2}). \tag{2.26}$$

From (2.25) and (2.26) we derive that

$$\begin{aligned}\|B(v_1,v_1)-B(v_2,v_2)\|_{\mathbb{V}^*} &\le 2c\,\|u_2-u_1\|\,\|w\|+cr(\|u_1-u_2\|+|A_1(\phi_1-\phi_2)|_{L^2})\\ &\le C_r\,\|v_1-v_2\|_{\mathbb{V}}\end{aligned}$$

which prove $(2.24)_1$. Now, remark that

$$\|R(v_1,v_1)-R(v_2,v_2)\|_{\mathbb{V}^*}=\|R_0(A_1\phi_1,\phi_1)-R(A_1\phi_2,\phi_2)\|_{V_1^*}\,.$$

However,

$$\begin{aligned}|\langle R_0(A_1\phi_1,\phi_1)-R(A_1\phi_2,\phi_2),w\rangle| &= |b_1(w,\phi_1,A_1(\phi_1-\phi_2))|+|b_1(w,\phi_1-\phi_2,A_1\phi_2)|\\ &\le c\,\|w\|\,|A_1\phi_1|_{L^2}\,|A_1(\phi_1-\phi_2)|_{L^2}+c\,\|w\|\,|A_1\phi_2|_{L^2}\,|A_1(\phi_1-\phi_2)|_{L^2}\\ &\le cr\,\|w\|\,|A_1(\phi_1-A_1\phi_2)|_{L^2}\,.\end{aligned}$$

It follows that

$$\|R_0(A_1\phi_1,\phi_1)-R(A_1\phi_2,\phi_2)\|_{V_1^*}\le C_r\,|A_1(\phi_1-A_1\phi_2)|_{L^2}\le C_r\,\|v_1-v_2\|_{\mathbb{V}}\,,$$

which prove $(2.24)_2$. Also, we remark that

$$\|E(v_1)-E(v_2)\|_{\mathbb{V}^*}=\|A_1f(\phi_1)-A_1f(\phi_2)\|_{D(A_1^{-1})}\,.$$

Let us recall from [18] that, there exists a monotone non-decreasing function $Q_1(x_1,x_2)$ of x_1 and x_2 such that

$$\begin{aligned}|\langle A_1f(\phi_1)-A_1f(\phi_2),\psi\rangle| &= |\langle f(\phi_1)-f(\phi_2),A_1\psi\rangle|\\ &\le Q_1(\|\phi_1\|\,,\|\phi_2\|)\,|A_1(\phi_1-\phi_2)|_{L^2}\,|A_1\psi|_{L^2}\,.\end{aligned}$$

We recall that, since $\|\phi_1\|\le c\,|A_1\phi_1|_{L^2}\le cr$, $\|\phi_2\|\le c\,|A_1\phi_2|_{L^2}\le cr$ and Q_1 is a monotone non-decreasing function, we have $Q_1(\|\phi_1\|\,,\|\phi_2\|)\le Q_1(cr,cr)$. So, we obtain

$$|\langle A_1f(\phi_1)-A_1f(\phi_2),\psi\rangle|\le Q_1(cr,cr)\,|A_1(\phi_1-\phi_2)|_{L^2}\,|A_1\psi|_{L^2}\,.$$

Hence

$$\begin{aligned}\|E(v_1)-E(v_2)\|_{\mathbb{V}^*} &= \|A_1f(\phi_1)-A_1f(\phi_2)\|_{D(A_1^{-1})}\\ &\le Q_1(cr,cr)\,|A_1(\phi_1-\phi_2)|_{L^2}\le C_r\,\|v_1-v_2\|_{\mathbb{V}}\,,\end{aligned}$$

which proves $(2.24)_3$ and ends the proof of Lemma 2.1. □

Now, we discuss some properties of a general kernel $\beta(.)$ and then in particular $\beta(t)=\gamma e^{-\delta t}$. we define

$$(Lu)(t)=(\beta*u)(t)=\int_0^t\beta(t-s)u(s)ds.$$

A function $\beta(\cdot)$ is called positive kernel if the operator L is positive on $L^2(0,T;H_1)$ for all $T>0$. That is,

$$\int_0^T(Lu(t),u(t))=\int_0^T\int_0^t\beta(t-s)(u(s),u(s))dsdt\ge 0,$$

for all $u\in L^2(0,T;H_1)$ and every $T>0$.

Let $\hat{\beta}(\theta)$ be the Laplace transform of $\beta(t)$, i.e.

$$\hat{\beta}(\theta) = \int_0^\infty e^{-\theta r}\beta(r)dr, \qquad \theta \in \mathbb{C}.$$

Then we have from [5, Lemma 4.1] the following result.

Lemma 2.2. *Let $\beta \in L^\infty(0,\infty)$ be such that $Re\hat{\beta}(\theta) > 0$ if $Re(\theta) > 0$. Then, $\beta(t)$ defines a positive kernel.*

We also recall from [29, Lemma A_1] the following result.

Lemma 2.3. *Let $\beta(t) = \gamma e^{-\delta t}$, $\delta > 0$, $t \in [0,T]$. Then for any right continuous function with left limits, $f : [0,T] \to [0,\infty)$, we have*

$$\int_0^T f(t)((\beta * u)(s), u(s))ds \geq 0, \tag{2.27}$$

for all $u \in L^2(0,T;H_1)$.

Remark 2.1. (1) As proved in [28, Lemma 2.6], with a change of variable and change of integrals, it can be easily seen that, if $\beta \in L^1(0,T)$, $f, g \in L^2(0,T)$, for some $T > 0$, then we have

$$\left(\int_0^T g^2(t)dt \left(\int_0^t \beta(t-s)f(s)ds\right)^2\right)^{1/2} \leq \left(\int_0^T |\beta(t)|\,dt\right)\left(\int_0^T g^2(t)f^2(t)dt\right)^{1/2}. \tag{2.28}$$

(2) If we take $g(t) = 1$, and $f(t) = |u(t)|_{L^2}$ in (2.28) with $u \in L^2(0,T;H_1)$, we obtain

$$\left(\int_0^T \left(\int_0^t \beta(t-s)\,|u(s)|_{L^2}^2\,ds\right)^2\right)^{1/2} \leq \left(\int_0^T |\beta(t)|\,dt\right)\left(\int_0^T |u(t)|_{L^2}^2\,dt\right)^{1/2}. \tag{2.29}$$

(3) For $\beta(t) = \gamma e^{-\delta t}$, we know that

$$\int_0^\infty \beta(t)dt = \frac{\gamma}{\delta} \text{ and } \hat{\beta}(\theta) = \frac{\gamma}{\theta+\delta} > 0, \text{ for } Re\theta > 0,$$

and by Lemma 2.2, $\beta(t)$ is a positive kernel. Hence, we have

$$\int_0^T \langle(\beta * A_0 u)(t), u(t)\rangle dt = \int_0^T ((\beta * \nabla u)(t), \nabla u(t))dt \geq 0. \tag{2.30}$$

Using the fact that $\|A_0 u\|_{V_1^*} \leq \|u\|$, we get

$$\|(\beta * A_0 u)(t)\|_{V_1^*} \leq \int_0^t \beta(t-s)\,\|A_0 u(s)\|_{V_1^*}\,ds \leq \int_0^t \beta(t-s)\,\|u(s)\|\,ds, \tag{2.31}$$

and hence by (2.29), we have

$$\begin{aligned}\int_0^T \|(\beta * A_0 u)(t)\|_{V_1^*}^2\,dt &\leq \int_0^T \left(\int_0^t \beta(t-s)\,\|A_0 u(s)\|_{V_1^*}\,ds\right)^2 dt\\ &\leq \left(\int_0^T \beta(t)dt\right)^2 \int_0^T \|u(t)\|^2\,dt \leq \frac{\gamma^2}{\delta^2}\int_0^T \|u(t)\|^2\,dt,\end{aligned} \tag{2.32}$$

for $u \in L^2(0,T;V_1)$.

Remark 2.2. Using the Cauchy-Schwarz and Hölder inequalities and (2.32), we derive that

$$\begin{aligned}\int_0^T \langle(\beta * A_0 u)(t), u(t)\rangle dt &\leq \left(\int_0^t \|(\beta * A_0 u)(s)\|_{V_1^*}^2\,ds\right)^{1/2}\left(\int_0^t \|u(s)\|^2\,ds\right)^{1/2}\\ &\leq \left(\int_0^t \beta(s)ds\right)\int_0^t \|u(s)\|^2\,ds \leq \frac{\gamma}{\delta}\int_0^t \|u(s)\|^2\,ds.\end{aligned} \tag{2.33}$$

2.2. Hypothesis

We assume that W^i, $i=1,2$ are formally given by the expansion

$$W^i(t)=\sum_{j=1}^{\infty}\beta_j(t)e_j,\quad j=1,2, \tag{2.34}$$

where $\beta_j(t)$, $j\in\mathbb{N}$ are independent one dimensional Brownian motions on $(\Omega,\mathcal{F},\mathbb{P})$, and $\{\beta_j\}_{j=1}^{\infty}$ is an orthonormal basis on U. We also define the auxiliary space U_0 containing U, that is defined by

$$U_0=\left\{\sum_{j=1}^{\infty}\alpha_je_j:\quad \sum_{j=1}^{\infty}\frac{\alpha_j^2}{j^2}<\infty\right\},$$

endowed with the scalar product

$$(u,v)_{U_0}=\sum_{j=1}^{\infty}\frac{\alpha_j\beta_j}{j^2},\quad \text{for } u=\sum_{j=1}^{\infty}\alpha_je_j,\ v=\sum_{j=1}^{\infty}\beta_je_j.$$

The stochastic forcing takes the following form

$$\sigma_i(t,u,\phi)dW^i(t)=\sum_{j=1}^{\infty}\sigma_j^i(t,u,\phi)d\beta_j(t),\quad i=1,2, \tag{2.35}$$

with suitable restrictions on the growth of the diffusion coefficients σ_j^i specified below.
Let us denote by $\mathbf{D}([0,T];H_1)$, the set of all H_1-valued functions defined on $[0,T]$, which are right continuous and have left limits (Càdlàg functions) for every $t\in[0,T]$. Also, let

$$\mathcal{M}_T^{2p}(H_1)=L^{2p}(\Omega\times(0,T]\times Z,\mathcal{B}((0,T]\times\mathcal{F}\times Z),dt\otimes\mathbb{P}\otimes\lambda;H_1), \tag{2.36}$$

be the space of all $\mathcal{B}((0,T]\times\mathcal{F}\times Z)$ measurable functions $\gamma:[0,T]\times\Omega\times Z\to H_1$ such that

$$\mathbb{E}\left[\int_0^T\int_Z|\gamma(t,\cdot,z)|_{L^2}^{2p}\,\lambda(dz)dt\right]<+\infty.$$

For any Hilbert space H, we will denote by $\mathcal{L}^2(U;H)$ the separable Hilbert space of Hilbert-Schmidt operators from U into H.
To simplify the notations, we set (without loss of generality) $\nu_1=\nu_2=\varepsilon=\alpha=\mathcal{K}=1$. Let us assume that the potential function f and the noise coefficients $\sigma_i(\cdot,\cdot)$, $i=1,2$ and $\gamma(\cdot,\cdot,\cdot)$ satisfy the following hypothesis.

(H1) $f\in\mathcal{C}^2(\mathbb{R})$ satisfies

$$\begin{cases}\liminf_{|r|\to+\infty}f'(r)>0,\\ |f^{(i)}(r)|\le c_f(1+|r|^{2-i}),\ \forall r\in\mathbb{R},\ i=0,1,2,\end{cases} \tag{2.37}$$

where c_f is some positive constant.

(H2) For all $t\in[0,T]$, $\langle\int_0^t\sigma_2(s,u,\phi)dW_s^2\rangle=0$ and there exist positive constants K_0 ; K_1 such that

$$\begin{aligned}&\int_Z|\gamma(t,u,\phi,z)|_{L^2}^2\,\lambda(dz)\le K_0(1+|(u,\phi)|_{\mathcal{H}}^2),\\&\int_Z|\gamma(t,u,\phi,z)|_{L^2}^4\,\lambda(dz)\le K_1(1+|(u,\phi)|_{\mathcal{H}}^4),\\&|\sigma_2(t,u,\phi)|_{\mathcal{L}^2(U;H_2)}^2=\sum_{j=1}^{\infty}\left\|\sigma_j^2(t,u,\phi)\right\|^2\le K_0,\end{aligned} \tag{2.38}$$

uniformly in $t\in[0,T]$ for all $(u,\phi)\in\mathcal{H}$.

(H4) For all $t \in [0,T]$, there exists a positive constant L such that

$$\sum_{i=1}^{2} |\sigma_i(t,u_1,\phi_1) - \sigma_i(t,u_2,\phi_2)|^2_{\mathcal{L}^2(U;H_i)} + \int_Z |\gamma(t,u_1,\phi_1,z) - \gamma(t,u_2,\phi_2,z)|^2_{L^2}\,\lambda(dz)$$
$$\leq L\,|(u_1,\phi_1)-(u_2,\phi_2)|^2_{\mathcal{H}}, \tag{2.39}$$

for all $(u_i,\phi_i) \in \mathcal{H}$, $i=1,2$.

Remark 2.3. Condition $(2.38)_3$ on the noise is widely employed in literature (see [16]). This condition implies that $\sigma_2(.,0,0) \in L^p(\Omega,\mathcal{F},\mathbb{P};L^2(0,T;\mathcal{L}^2(U;H_2)))$, for all $p \geq 2$. Indeed,

$$\mathbb{E}\left(\int_0^T \|\sigma_2(.,0,0)\|^2_{\mathcal{L}^2(U;H_2)}\,ds\right)^{p/2} \leq K_0^{p/2}T^{p/2} < \infty.$$

For any $(v,\psi) \in \mathcal{H}$, we set

$$\mathcal{E}(v,\psi) = |v|^2_{L^2} + 2\mathcal{E}_0(\psi) + c_1 = |(v,\psi)|^2_{\mathcal{H}} + 2\alpha\langle F(\psi),1\rangle + c_1, \tag{2.40}$$

where $c_1 > 0$ is a constant large enough and independent on (v,ψ) such that $\mathcal{E}(v,\psi)$ is non-negative (note that F_0 is bounded from below).

2.3. Abstract Formulation

Using the notations above, we rewrite problem (1.1)–(1.2) as:

$$\begin{cases} du(t) + [\nu_1 A_0 u + (\beta * A_0 u)(t) + B_0(u,u) - \mathcal{K}R_0(A_1\phi,\phi)]dt \\ = \sigma_1(t,u,\phi)dW_t^1 + \int_Z \gamma(t,u(t^-),\phi(t),z)\tilde{\pi}(dt,dz), \text{ in } V_1^*, \\ d\phi(t) + [\nu_2 A_1\mu + B_1(u,\phi)]dt = \sigma_2(t,u,\phi)dW_t^2, \text{ in } H^{-1}(D), \\ \mu = -\varepsilon A_1\phi + \alpha f(\phi), \text{ in } H^{-1}(D), \\ (u,\phi)(0) = (u_0,\phi_0), \end{cases} \tag{2.41}$$

which is equivalent to for all $(v,\psi) \in V_1 \times H^1(D)$,

$$\begin{aligned} (u(t),v) &+ \int_0^t \langle \nu_1 A_0 u + (\beta * A_0 u)(t) + B_0(u,u) - \mathcal{K}R_0(A_1\phi,\phi), v\rangle dt = (u_0,v) \\ &+ \int_0^t (\sigma_1(s,u,\phi)dW_s^1, v) + \int_0^t \int_Z (\gamma(s,u(s^-),\phi(s),z),v)\tilde{\pi}(ds,dz), \\ &(\phi(t),\phi) + \int_0^t \langle \nu_2\nabla\mu, \nabla\psi\rangle ds = (\phi_0,\psi) + \int_0^t (\sigma_2(s,u,\phi)dW_s^2,\psi), \\ \mu &= -\varepsilon A_1\phi + \alpha f(\phi) \end{aligned} \tag{2.42}$$

$\mathbb{P}$-a.s. and for all $t \in [0,T]$, for some fixed point (u_0,ϕ_0) in $\mathcal{H}$.

Remark 2.4. In the weak formulation (2.41), the term $\mu\nabla\phi$ is replaced by $A_1\phi\nabla\phi$. This is justified since $f(\phi)\nabla\phi$ is the gradient of $F(\phi)$ and can be incorporated into the pressure gradient, see [18] for details.

Let us now give the definition of a unique global strong solution in the probabilistic sense to the system (2.41).

Definition 2.1. (*Global strong solution*) Let the $\mathcal{F}_0$-measurable initial data $(u_0,\phi_0) \in L^4(\Omega,\mathcal{F},\mathbb{P};\mathcal{H})$ be given. An $\mathcal{H}$-valued $\mathcal{F}_t$-adapted càdlàg process $(u,\phi)(\cdot)$ is called a strong solution to (2.41) if $(u,\phi) \in L^p(\Omega,\mathcal{F},\mathbb{P};L^\infty(0,T;\mathcal{H})) \cap L^p(\Omega,\mathcal{F},\mathbb{P};L^2(0,T;\mathcal{U}))$, for all $p \geq 2$ and satisfies (2.42).

Definition 2.2. A strong solution $(u,\phi)(\cdot)$ to (2.41) is called a unique strong solution if $(\tilde{u},\tilde{\phi})(\cdot)$ is an another strong solution, then

$$\mathbb{P}\Big\{\omega\in\Omega;\ (u,\phi)(t)=(\tilde{u},\tilde{\phi})(t),\ \text{ for all }\ t\in[0,T]\Big\}=1.$$

3. Existence and Uniqueness

In this section, we establish the global solvability of the system (2.41). To simplify the notations, throughout this section, we will set (without loss of generality) $\nu_1=\nu_2=\alpha=\varepsilon=\mathcal{K}=1$. We first prove the following energy type equality.

Proposition 3.1. *If (u,ϕ) is a variational solution to (2.41), then (u,ϕ) satisfies*

$$\begin{aligned}\mathcal{E}(u,\phi)(t)&+\int_0^t((\beta*\nabla u)(s),\nabla u(s))ds+2\int_0^t\Big(\|u(s)\|^2+|\nabla\mu(s)|_{L^2}^2\Big)ds=\mathcal{E}(u_0,\phi_0)\\&+\int_0^t|\sigma_1(s,u,\phi)|^2_{\mathcal{L}^2(U;H_1)}ds+2\int_0^t(\sigma_1(s,u,\phi),u)dW^1(s)+2\int_0^t(\sigma_2(s,u,\phi),\mu)dW^2(s)\\&\int_0^t\sum_{j=1}^{+\infty}\int_{\mathcal{M}}\Big[\big|\nabla\sigma_j^2(s,u,\phi)(x)\big|^2+|f'(\phi(s,x))|\,\big|\sigma_j^2(s,u,\phi)(x)\big|^2\Big]dxds\\&+2\int_0^t\int_Z(\gamma(s,u(s^-),\phi(s),z),u)\tilde{\pi}(ds,dz)+\int_0^t\int_Z\Upsilon(s,z)\pi(ds,dz),\end{aligned}\tag{3.1}$$

where

$$\Upsilon(s,z)=\big|u(s^-)+\gamma(s,u(s^-),\phi(s),z)\big|^2_{L^2}-\big|u(s^-)\big|^2_{L^2}-2(\gamma(s,u(s^-),\phi(s),z),u(s)).$$

Proof. We apply infinite dimensional Itô's formula (see [33]) to the process $|u|^2_{L^2}$ to find

$$\begin{aligned}|u(t)|^2_{L^2}=|u(0)|^2_{L^2}&-2\int_0^t\langle A_0u+(\beta*A_0u)(s)-R_0(A_1\phi,\phi),u\rangle ds\\&+\int_0^t|\sigma_1(s,u,\phi)|^2_{\mathcal{L}^2(U;H_1)}ds+2\int_0^t(\sigma_1(s,u,\phi),u)dW^1(s)\\&+2\int_0^t\int_Z(\gamma(s,u(s^-),\phi(s),z),u)\tilde{\pi}(ds,dz)+\int_0^t\int_Z\Upsilon(s,z)\pi(ds,dz).\end{aligned}\tag{3.2}$$

We want now to write Itô's formula for the free energy functional $\mathcal{E}_0(\phi)$, $\phi\in D(A_1)$. To this end, we should first prove that $\mathcal{E}_0:D(A_1)\to[0,\infty)$ is twice Fréchet differentiable. Let $\phi,\psi\in D(A_1)$, using the Taylor-Lagrange formula, we derive that

$$\left|\mathcal{E}_0(\phi+\psi)-\mathcal{E}_0(\phi)-(\nabla\phi,\nabla\psi)-\int_{\mathcal{M}}f(\phi)\psi dx\right|=\left|\frac{1}{2}\|\psi\|^2+\int_0^1\int_{\mathcal{M}}(1-s)f'(\phi+s\psi)\psi dxds\right|.$$

Owing to the condition (2.37) and the fact that $D(A_1)$ is continuously embedded in H_2 and in $L^\infty(\mathcal{M})$, we infer that

$$
\begin{aligned}
&\left|\mathcal{E}_0(\phi+\psi)-\mathcal{E}_0(\phi)-(\nabla\phi,\nabla\psi)-\int_{\mathcal{M}} f(\phi)\psi dx\right| \\
&\quad\le c\,|A_1\psi|_{L^2}^2+\|\psi\|_{L^\infty}^2 \sup_{s\in[0,1]}|1-s|\int_0^1 |f'(\phi+s\psi)|_{L^1} \\
&\quad\le c\,|A_1\psi|_{L^2}^2+c\,|A_1\psi|_{L^2}^2\int_0^1 |f'(\phi+s\psi)|_{L^1} \\
&\quad\le c\,|A_1\psi|_{L^2}^2\,(1+cf\int_{\mathcal{M}}\int_0^1(1+|\phi+s\psi|))dxds \\
&\quad\le C(\mathcal{M},f)\,|A_1\psi|_{L^2}^2\,(1+|\phi|_{L^2}+|\psi|_{L^2}).
\end{aligned}
$$

Therefore,

$$
\lim_{\psi\to 0,\psi\neq 0}\frac{\left|\mathcal{E}_0(\phi+\psi)-\mathcal{E}_0(\phi)-(\nabla\phi,\nabla\psi)-\int_{\mathcal{M}} f(\phi)\psi dx\right|}{|A_1\psi|_{L^2}}=0.
$$

This proves that the first Fréchet derivative $\mathcal{D}:D(A_1)\to\mathcal{L}(D(A_1);\mathbb{R})$ of $\mathcal{E}_0$ is given by

$$
\mathcal{D}\mathcal{E}_0(\phi)[\psi]=(\nabla\phi,\nabla\psi)+\int_{\mathcal{M}} f(\phi)\psi dx,\quad \phi,\psi\in D(A_1). \tag{3.3}
$$

Also, it is easy to see that $\mathcal{D}\mathcal{E}_0$ is Fréchet-differentiable with $\mathcal{D}^2\mathcal{E}_0 : D(A_1)\to\mathcal{L}(D(A_1);\mathcal{L}(D(A_1);\mathbb{R}))$ given by

$$
\mathcal{D}^2\mathcal{E}_0(\phi)[\psi,\varphi]=(\nabla\psi,\nabla\varphi)-\int_{\mathcal{M}} f'(\phi)\psi\varphi dx,\quad \phi,\psi,\varphi\in D(A_1). \tag{3.4}
$$

Indeed, by direct computation, we can check that

$$
\begin{aligned}
\left|\mathcal{D}\mathcal{E}_0(\phi+\psi)[\varphi]-\mathcal{D}\mathcal{E}_0(\phi)[\varphi]-\mathcal{D}^2\mathcal{E}_0(\phi)[\psi,\varphi]\right| &= \left|\int_{\mathcal{M}}(f(\phi+\psi)-f(\phi))\varphi dx-\int_{\mathcal{M}} f'(\phi)\psi\varphi dx\right| \\
&= \left|\int_0^1\int_{\mathcal{M}}(f'(\phi+s\psi)-f'(\phi))\psi\varphi dx\right|
\end{aligned}
$$

By the embedding of $D(A_1)$ in $L^\infty(\mathcal{M})$ and in $L^2_0(\mathcal{M})$, the mean value theorem and (2.37), we note that

$$
\begin{aligned}
&\left|\int_0^1\int_{\mathcal{M}}(f'(\phi+s\psi)-f'(\phi))\psi\varphi dx\right| \\
&\quad\le |\psi|_{L^\infty}\,|\varphi|_{L^\infty}\int_0^1 |f'(\phi+s\psi)-f'(\phi)|_{L^1}\,ds \\
&\quad\le c\,|A_1\psi|_{L^2}\,|A_1\varphi|_{L^2}\int_0^1 |f'(\phi+s\psi)-f'(\phi)|_{L^1}\,ds \\
&\quad\le c\,|A_1\psi|_{L^2}\,|A_1\varphi|_{L^2}\int_0^1 \left|s\psi f''(\xi)\right|_{L^1}\,ds \\
&\quad\le \frac{c\,|\mathcal{M}|^{1/2}}{2}\,|A_1\psi|_{L^2}\,|A_1\varphi|_{L^2}\,|\psi|_{L^2} \\
&\quad\le \frac{c\,|\mathcal{M}|^{1/2}}{2}\,|A_1\psi|_{L^2}^2\,|A_1\varphi|_{L^2}\,.
\end{aligned} \tag{3.5}
$$

From (3.5), we arrive at

$$
\sup_{|A_1\varphi|_{L^2}\le 1}\left[\frac{1}{|A_1\psi|_{L^2}}\left|\int_0^1\int_{\mathcal{M}}(f'(\phi+s\psi)-f'(\phi))\psi\varphi dx\right|\right]\le C(\mathcal{M})\,|A_1\psi|_{L^2}\to 0 \text{ as } \psi\to 0 \text{ in } D(A_1),
$$

from which we get (3.4). Furthermore, from the hypothesis (H1), we can easily check that the derivatives $\mathcal{D}\mathcal{E}_0$ and $\mathcal{D}^2\mathcal{E}_0$ are continuous and bounded on bounded subsets of $D(A_1)$. Hence, observing that $\mathcal{D}^2\mathcal{E}_0(\phi) = \mu$, we can apply Itô's formula to $\mathcal{E}_0(\phi)$ in the classical version of [8] to derive that

$$\begin{aligned}\mathcal{E}_0(\phi(t)) + \int_0^t |\nabla\mu|_{L^2}^2\, ds + \int_0^t (u\cdot\nabla\phi, \mu) ds = \mathcal{E}_0(\phi_0) + \int_0^t (\sigma_2(s,u,\phi),\mu) dW^2(s)\\ + \frac{1}{2}\int_0^t \sum_{j=1}^{+\infty}\int_{\mathcal{M}} \left[\left|\nabla\sigma_j^2(s,u,\phi)(x)\right|^2 + |f'(\phi(s,x))|\left|\sigma_j^2(s,u_m,\phi_m)(x)\right|^2\right] dxds. \qquad (3.6)\end{aligned}$$

Adding (3.2) with (3.6) and using (2.8) and the fact that $(u\cdot\nabla\phi, f(\phi)) = 0$, we obtain (3.1) which ends the proof of the Proposition 3.1. □

Theorem 3.1. *We suppose that the Assumptions (H1)–(H4) hold. Moreover, we assume that $\sigma_1(\cdot,0,0) \in L^p(\Omega,\mathcal{F},\mathbb{P}; L^2(0,T;\mathcal{L}^2(U;H_1)))$ and that $(u_0,\phi_0) \in L^p(\Omega,\mathcal{F},\mathbb{P},\mathbb{H})$ satisfies $\mathbb{E}\mathcal{E}^p(u_0,\phi_0) < \infty$, for all $p \geq 2$. Then the system (2.41) has a unique strong solution.*

The rest of this section is devoted to the proof of Theorem 3.1. The method relies on Galerkin approximation and deterministic Gronwall's lemma. For the existence part, instead of the Minty-Browder technique used in [27], we prove the existence and certain uniform estimates for the sequence $(u_m,\phi_m)_m$ of the approximation. Then, as in [37], we use the properties of stopping times and some basic convergence principles from functional analysis to prove the existence of the solution.

3.1. Existence of Strong Solution

Let us consider a finite dimensional Galerkin approximation of the system (2.41). Consider $\{(w_j,\psi_j),\ j = 1,\dots\} \subset \mathbb{V}$ be a orthogonal basis of $\mathcal{H}$, where $\{w_j,\ j = 1,\dots\}$ and $\{\psi_j,\ j = 1,\dots\}$ are eigenvectors of A_0 and A_1 respectively given in the previous section. We set for $m \in \mathbb{N}$, $\mathcal{H}_m = \text{span}\{(w_1,\psi_1),\dots,(w_m,\psi_m)\} = H_{1m}\times H_{2m}$. We look for $(u_m,\phi_m) \in \mathcal{H}_m$ solutions to the ordinary differential equations

$$\begin{cases} du_m(t) + \mathcal{P}_m^1[A_0u_m + (\beta * A_0u_m)(t) + B_0(u_m,u_m) - R_0(A_1\phi_m,\phi_m)]dt \\ = \mathcal{P}_m^1\sigma_1(t,u_m,\phi_m)dW_m^1(t) + \int_Z \mathcal{P}_m^1\gamma(t,u_m(t^-),\phi_m(t),z)\tilde{\pi}(dt,dz), \\ d\phi_m(t) + \mathcal{P}_m^2[A_1\mu_m + B_1(u_m,\phi_m)]dt = \mathcal{P}_m^2\sigma_2(t,u_m,\phi_m)dW_m^2(t), \\ \mu_m = \mathcal{P}_m^2[A_1\phi_m + f(\phi_m)], \\ (u_m,\phi_m)(0) = \mathcal{P}_m(u_0,\phi_0), \end{cases} \qquad (3.7)$$

where $\mathcal{P}_m = (\mathcal{P}_m^1,\mathcal{P}_m^2) : H_1\times L^2(\mathcal{M}) \to \mathcal{H}_m$ is the orthogonal projection, $W_m^i(t) = \mathcal{P}_m^i W_t^i$, for $i = 1,2$. Since the deterministic terms of (3.7) are locally Lipschitz (see Lemma 2.1), and $\mathcal{P}_m^i\sigma_i(.)$, $i = 1,2$ and $\mathcal{P}_m^1\gamma(.)$ is globally Lipschitz, the system (3.7) has a unique $\mathcal{H}_m$-valued càdlàg local strong solution $(u_m,\phi_m) \in L^2(\Omega,\mathcal{F},\mathbb{P}; L^\infty(0,T;\mathcal{H}_m))$ with paths $u \in \mathbf{D}(0,T;H_{1m})$ and $\phi \in C([0;T];H_{2m})$, $\mathbb{P}$-a.s. (see [1,27]). Let us now derive the a-priori energy estimates satisfied by the system (3.7).

Proposition 3.2. *Let (u_m,ϕ_m) be the unique solution of the system (3.7) with $(u_0,\phi_0) \in L^p(\Omega,\mathcal{F},\mathbb{P};\mathcal{H})$, for all $p \geq 2$. If (u_0,ϕ_0) is such that $\mathbb{E}\mathcal{E}^p(u_0,\phi_0) < \infty$ for all $p \geq 2$, then there exists a positive constant C independent of m such that for all $p \geq 2$,*

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\mathcal{E}^p(u_m,\phi_m)(s) + \left(\int_0^T \left(|\nabla\mu_m|_{L^2}^2 + \|u_m\|^2\right) ds\right)^p\right] \leq C(1+\mathbb{E}\mathcal{E}^p(u_0,\phi_0)), \qquad (3.8)$$

$$\mathbb{E}\sup_{s\in[0,T]}\mathcal{E}(u_m,\phi_m)(s) + \mathbb{E}\int_0^T \left(|\nabla\mu_m|_{L^2}^2 + \|u_m\|^2\right) ds \leq C(u_0,\phi_0). \qquad (3.9)$$

Proof. By finite dimensional Itô's formula (see [4, Theorem 4.4.7]) and the fact that $b_0(u_m, u_m, u_m) = 0$, we obtain for all $t \in [0, T]$,

$$\begin{aligned} |u_m(t)|_{L^2}^2 &= |u_m(0)|_{L^2}^2 - 2\int_0^t \langle A_0 u_m + (\beta * A_0 u_m)(s) - R_0(A_1\phi_m, \phi_m), u_m\rangle ds \\ &\quad + \int_0^t \left|\mathcal{P}_m^1 \sigma_1(s, u_m, \phi_m)\right|_{\mathcal{L}^2(U;H_1)}^2 ds + 2\int_0^t (\mathcal{P}_m^1 \sigma_1(s, u_m, \phi_m), u_m) dW_m^1(s) \\ &\quad + 2\int_0^t \int_Z (\mathcal{P}_m^1 \gamma(s, u_m(s^-), \phi_m(s), z), u_m)\tilde{\pi}(ds, dz) + \int_0^t \int_Z \Psi_m(s, z)\pi(ds, dz), \end{aligned} \tag{3.10}$$

where

$$\begin{aligned} \Psi_m(s, z) &= \left|u_m(s^-) + \mathcal{P}_m^1 \gamma(s, u_m(s^-), \phi_m(s), z)\right|_{L^2}^2 - \left|u_m(s^-)\right|_{L^2}^2 \\ &\quad - 2(\mathcal{P}_m^1 \gamma(s, u_m(s^-), \phi_m(s), z), u_m). \end{aligned} \tag{3.11}$$

Note that (2.30) easily gives

$$\int_0^t \langle(\beta * A_0 u_m), u_m\rangle ds = \int_0^t ((\beta * \nabla u_m), \nabla u_m) ds \geq 0. \tag{3.12}$$

Therefore, using the fact that $|x|_{L^2}^2 - |y|_{L^2}^2 + |x-y|_{L^2}^2 = 2(x-y, x)$, $\forall x, y \in H_1$, we infer from (3.10) that

$$\begin{aligned} |u_m(t)|_{L^2}^2 &\leq |u_m(0)|_{L^2}^2 - 2\int_0^t \langle A_0 u_m + (\beta * A_0 u_m)(s) - R_0(A_1\phi_m, \phi_m), u_m\rangle ds \\ &\quad + 2\int_0^t (\mathcal{P}_m^1 \sigma_1(s, u_m, \phi_m), u_m) dW_m^1(s) \\ &\quad + \int_0^t \int_Z \left|\mathcal{P}_m^1 \gamma(s, u_m(s), \phi_m(s), z)\right|_{L^2}^2 \pi(ds, dz) + \int_0^t \left|\mathcal{P}_m^1 \sigma_1(s, u_m, \phi_m)\right|_{\mathcal{L}^2(U;H_1)}^2 ds \\ &\quad + 2\int_0^t \int_Z (\mathcal{P}_m^1 \gamma(s, u_m(s^-), \phi_m(s), z), u_m)\tilde{\pi}(ds, dz). \end{aligned} \tag{3.13}$$

Note that, since spam$\{\psi_1, \ldots. \psi_m\} \subset D(A_1)$, we infer that $D\mathcal{E}_0(\phi_m) = \mu_m$. Therefore, applying Itô's formula to the process $\mathcal{E}_0(\phi_m)$, we get

$$\begin{aligned} &2\mathcal{E}_0(\phi_m(t)) + \int_0^t |\nabla \mu_m|_{L^2}^2 ds + \int_0^t (u_m \cdot \nabla \phi_m, \mu_m) ds \\ &\quad = 2\mathcal{E}_0(\phi_m(0)) + 2\int_0^t (\mathcal{P}_m^2 \sigma_2(s, u_m, \phi_m), \mu_m) dW_m^2(s) \\ &\qquad + \int_0^t \sum_{j=1}^m \int_{\mathcal{M}} \left[\left|\nabla \mathcal{P}_m^2 \sigma_j^2(s, u_m, \phi_m)(x)\right|^2 + |f'(\phi_m(s, x))| \left|\mathcal{P}_m^2 \sigma_j^2(s, u_m, \phi_m)(x)\right|^2\right] dx ds. \end{aligned} \tag{3.14}$$

Using the fact that $H_1 \hookrightarrow L^p(\mathcal{M})$, $p \geq 2$, by assumption (H1), we get

$$\begin{aligned} &\int_0^t \sum_{j=1}^m \int_{\mathcal{M}} \left[\left|\nabla \mathcal{P}_m^2 \sigma_j^2(s, u_m, \phi_m)(x)\right|^2 + |f'(\phi_m(s, x))| \left|\mathcal{P}_m^2 \sigma_j^2(s, u_m, \phi_m)(x)\right|^2\right] dx ds \\ &\quad \leq c\int_0^t \|\sigma_2(s, u_m, \phi_m)\|_{\mathcal{L}^2(U;H_2)}^2 ds + C_f \sum_{j=1}^m \int_0^t \int_{\mathcal{M}} |\phi_m| \left|\mathcal{P}_m^2 \sigma_j^2(s, u_m, \phi_m)(x)\right|^2 dx ds \\ &\quad \leq c\int_0^t \|\sigma_2(s, u_m, \phi_m)\|_{\mathcal{L}^2(U;H_2)}^2 ds + C_f \sum_{j=1}^m \int_0^t \int_{\mathcal{M}} |\phi_m| \left|\mathcal{P}_m^2 \sigma_j^2(s, u_m, \phi_m)\right|^2 (x) dx ds \\ &\quad \leq c\int_0^t \|\sigma_2(s, u_m, \phi_m)\|_{\mathcal{L}^2(U;H_2)}^2 ds + C_f \sum_{j=1}^m \int_0^t \|\phi_m\|_{L^2}^2 \left\|\mathcal{P}_m^2 \sigma_j^2(s, u_m, \phi_m)\right\|_{L^4}^2 ds \end{aligned}$$

$$\leq c\int_0^t \|\sigma_2(s,u_m,\phi_m)\|^2_{\mathcal{L}^2(U;H_2)}\,ds + c\int_0^t \|\phi_m\|^2 \|\sigma_2(s,u_m,\phi_m)\|^2_{\mathcal{L}^2(U;H_2)}\,ds. \tag{3.15}$$

Now, for each $n \geq 1$, we consider the $\mathcal{F}_t$-stopping time τ_n^m defined by:

$$\tau_n^m = \min\left(T, \inf\left\{t\in[0,T] : \mathcal{E}(u_m,\phi_m)(t) + 2\int_0^t \left(\|u_m\|^2 + \|\mu_m\|^2\right) ds \geq n^2\right\}\right). \tag{3.16}$$

For fixed m, the sequence $\{\tau_n^m; n\geq 1\}$ is increasing to T. Adding (3.10) with (3.14) after using (3.12) and (3.15), we get for all $t\in[0,T]$,

$$\begin{aligned}
&\mathcal{E}(u_m,\phi_m)(t\wedge\tau_n^m)\\
&\quad + 2\int_0^{t\wedge\tau_n^m} \left(|\nabla\mu_m|^2_{L^2} + \|u_m\|^2\right) ds \leq \mathcal{E}(u_0,\phi_0)\\
&\quad + \int_0^{t\wedge\tau_n^m} \left|\mathcal{P}_m^1\sigma_1(s,u_m,\phi_m)\right|^2_{\mathcal{L}^2(U;H_1)} ds + 2\int_0^{t\wedge\tau_n^m} (\mathcal{P}_m^2\sigma_2(s,u_m,\phi_m),\mu_m)dW_m^2(s)\\
&\quad + c\int_0^{t\wedge\tau_n^m} \|\sigma_2(s,u_m,\phi_m)\|^2_{\mathcal{L}^2(U;H_2)} ds + c\int_0^t \|\phi_m\|^2 \|\sigma_2(s,u_m,\phi_m)\|^2_{\mathcal{L}^2(U;H_2)} ds\\
&\quad + 2\int_0^{t\wedge\tau_n^m} (\mathcal{P}_m^1\sigma_1(s,u_m,\phi_m),u_m)dW_m^1(s) + \int_0^t\int_Z \Psi_m(s,z)\pi(ds,dz)\\
&\quad + 2\int_0^{t\wedge\tau_n^m}\int_Z (\mathcal{P}_m^1\gamma(s,u_m(s^-),\phi_m(s),z),u_m)\tilde{\pi}(ds,dz).
\end{aligned} \tag{3.17}$$

Now raising both sides to the power $p\geq 2$, taking supremum over $s\in[0,t\wedge\tau_n^m]$ and taking mathematical expectation we have

$$\begin{aligned}
&\mathbb{E}\sup_{r\in[0,t\wedge\tau_n^m]} \mathcal{E}^p(u_m,\phi_m)(r)\\
&\quad +4\mathbb{E}\left(\int_0^{t\wedge\tau_n^m} |\nabla\mu_m|^2_{L^2}\right)^p ds + 4\mathbb{E}\left(\int_0^{t\wedge\tau_n^m} \|u_m\|^2\right)^p ds \leq \mathcal{E}^p(u_0,\phi_0)\\
&\quad +\mathbb{E}\left(\int_0^{t\wedge\tau_n^m} \left|\mathcal{P}_m^1\sigma_1(s,u_m,\phi_m)\right|^2_{\mathcal{L}^2(U;H_1)} ds\right)^p + c\mathbb{E}\left(\int_0^{t\wedge\tau_n^m} (1+\|\phi_m\|^2)ds\right)^p\\
&\quad +\mathbb{E}\sup_{r\in[0,t\wedge\tau_n^m]}\left|\int_0^r (\mathcal{P}_m^2\sigma_2(s,u_m,\phi_m),\mu_m)dW_m^2(r)\right|^p\\
&\quad +2\mathbb{E}\sup_{r\in[0,t\wedge\tau_n^m]}\left|\int_0^r (\mathcal{P}_m^1\sigma_1(s,u_m,\phi_m),u_m)dW_m^1(s)\right|^p\\
&\quad +c\mathbb{E}\sup_{0\leq r\leq t\wedge\tau_n^m} |I_5(r)|^p + c\mathbb{E}\sup_{0\leq r\leq t\wedge\tau_n^m} |I_6(r)|^p.
\end{aligned} \tag{3.18}$$

where

$$\begin{aligned}
I_5(r) &= \int_0^r\int_Z \left\{\left|u_m(\tau^-) + \mathcal{P}_m^1\gamma(\tau,u_m(\tau^-),\phi_m(\tau),z)\right|^2_{L^2} - \left|u_m(\tau^-)\right|^2_{L^2}\right\}\tilde{\pi}(d\tau,dz),\\
I_6(r) &= \int_0^r\int_Z \left\{\left|u_m(s) + \mathcal{P}_m^1\gamma(s,u_m(s),\phi_m(s),z)\right|^2_{L^2} - |u_m(s)|^2_{L^2}\right\}\lambda(dz)ds\\
&\quad - 2\int_0^r\int_Z (\mathcal{P}_m^1\gamma(s,u_m(s),\phi_m(s),z),u_m(s)\lambda(dz)ds
\end{aligned}$$

$$
\begin{aligned}
&\leq c\int_0^r\int_Z \left|\mathcal{P}_m^1\gamma(s,u_m(s^-),\phi_m(s),z)\right|_{L^2}^2\lambda(dz)ds\\
&\leq cK_0\int_0^r(1+|(u_m,\phi_m)|_{\mathcal{H}}^2)ds. \qquad (3.19)
\end{aligned}
$$

As in [6,7], we note that

$$
\begin{aligned}
&\int_Z\left\{\left|u_m(s^-)+\mathcal{P}_m^1\gamma(s,u_m(s-),\phi_m(s),z)\right|_{L^2}^2-\left|u_m(\tau^-)\right|_{L^2}^2\right\}^2\lambda(dz)\\
&\quad\leq \left|u_m(s^-)\right|_{L^2}^2\int_Z\left|\mathcal{P}_m^1\gamma(s,u_m(s^-),\phi_m(s),z)\right|_{L^2}^2\lambda(dz)\\
&\qquad+c\int_Z\left|\mathcal{P}_m^1\gamma(s,u_m(s^-),\phi_m(s),z)\right|_{L^2}^4\lambda(dz)\\
&\quad\leq c_0+c_1\left|(u_m,\phi_m)\right|_{\mathcal{H}}^2+c_2\left|(u_m,\phi_m)\right|_{\mathcal{H}}^4\\
&\quad\leq K_2+K_3\left|(u_m,\phi_m)\right|_{\mathcal{H}}^4. \qquad (3.20)
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\left(\int_0^{t\wedge\tau_n^m}\int_Z\left\{\left|u_m(s^-)+\mathcal{P}_m^1\gamma(s,u_m(s-),\phi_m(s),z)\right|_{L^2}^2-\left|u_m(\tau^-)\right|_{L^2}^2\right\}^2\lambda(dz)ds\right)^{p/2}\\
&\quad\leq c(K_2T)^{p/2}+c(K_3)^{p/2}\left(\int_0^{t\wedge\tau_n^m}\mathcal{E}^2(u_m,\phi_m)ds\right)^{p/2}. \qquad (3.21)
\end{aligned}
$$

Applying Burkholder–Davis–Gundy's inequality (see [33, Theorem 48]) and using (3.20)–(3.21), we derive that

$$
\begin{aligned}
\mathbb{E}\sup_{0\leq r\leq t\wedge\tau_n^m}|I_5(r)|^p&\leq\frac{1}{4}\mathbb{E}\sup_{r\in[0,t\wedge\tau_n^m]}\mathcal{E}^p(u_m,\phi_m)(r)+c(K_2T)^{p/2}+c(K_3)^{p/2}\mathbb{E}\left(\int_0^{t\wedge\tau_n^m}\mathcal{E}(u_m,\phi_m)ds\right)^p\\
&\leq\frac{1}{4}\mathbb{E}\sup_{r\in[0,t\wedge\tau_n^m]}\mathcal{E}^p(u_m,\phi_m)(r)+c(K_2T)^{p/2}+c(K_3)^{p/2}T^{p-1}\mathbb{E}\int_0^{t\wedge\tau_n^m}\mathcal{E}^p(u_m,\phi_m)ds. \qquad (3.22)
\end{aligned}
$$

Using Hölder's inequality, it follows from (3.19) that

$$
\begin{aligned}
\mathbb{E}\left|I_6(t\wedge\tau_n^m)\right|^p&\leq cK_0^p\mathbb{E}\left(\int_0^{t\wedge\tau_n^m}(1+|(u_m,\phi_m)|_{\mathcal{H}}^2)ds\right)^p\\
&\leq C(K_0,T)+cK_0^2\int_0^{t\wedge\tau_n^m}\mathbb{E}\mathcal{E}^p(u_m,\phi_m)(s)ds. \qquad (3.23)
\end{aligned}
$$

By Doob's inequality, we derive will the help of $(2.38)_1$ and Hölder's inequality that

$$
\begin{aligned}
&2\mathbb{E}\sup_{r\in[0,t\wedge\tau_n^m]}\left|\int_0^r(\mathcal{P}_m^1\sigma_1(s,u_m,\phi_m),u_m)dW_m^1(r)\right|^p\\
&\quad\leq c\mathbb{E}\left(\int_0^{t\wedge\tau_n^m}\left|\mathcal{P}_m^1\sigma_1(s,u_m,\phi_m)\right|_{\mathcal{L}^2(U;H_1)}^2|u_m|_{L^2}^2dr\right)^{p/2}\\
&\quad\leq c\mathbb{E}\left[\sup_{0\leq r\leq t\wedge\tau_n^m}\mathcal{E}(u_m,\phi_m)(r)\int_0^t\left\|\mathcal{P}_m^i\sigma_i(s,u_m,\phi_m)\right\|_{\mathcal{L}^2(U;H_1)}^2ds\right]^{p/2}\\
&\quad\leq\frac{1}{4}\mathbb{E}\sup_{0\leq r\leq t\wedge\tau_n^m}\mathcal{E}^p(u_m,\phi_m)(r)+c\mathbb{E}\left(\int_0^{t\wedge\tau_n^m}\left\|\mathcal{P}_m^i\sigma_i(s,u_m,\phi_m)\right\|_{\mathcal{L}^2(U;H_1)}^2dr\right)^p
\end{aligned}
$$

$$\leq \frac{1}{4}\mathbb{E} \sup_{0\leq r\leq t\wedge\tau_n^m} \mathcal{E}^p(u_m,\phi_m)(r) + C(K_0,T) + cK_0^p \int_0^{t\wedge\tau_n^m} \mathbb{E}\mathcal{E}^p(u_m,\phi_m)(s)dr. \tag{3.24}$$

By Hölder's inequality, we derive that

$$\begin{aligned}
&\mathbb{E}\left(\int_0^{t\wedge\tau_n^m} \left|\mathcal{P}_m^1\sigma_1(s,u_m,\phi_m)\right|^2_{\mathcal{L}^2(U;H_1)} ds\right)^p + c\mathbb{E}\left(\int_0^{t\wedge\tau_n^m} (1+\|\phi_m\|^2)ds\right)^p \\
&\quad\leq C(T,L)\mathbb{E}\int_0^{t\wedge\tau_n^m} (1+\mathcal{E}^p(u_m,\phi_m))ds + c\mathbb{E}\left(\int_0^T |\sigma_1(s,0,0)|^2_{\mathcal{L}^2(U;H_1)} ds\right)^p \\
&\quad\leq C(T,L) + C(T,L)\mathbb{E}\int_0^{t\wedge\tau_n^m} \mathcal{E}^2(u_m,\phi_m)(s)ds + c\mathbb{E}\left(\int_0^T |\sigma_1(s,0,0)|^2_{\mathcal{L}^2(U;H_1)} ds\right)^p. \qquad (3.25)
\end{aligned}$$

By (2.37), $(3.7)_4$, and the fact that $H_1 \hookrightarrow L^2(\mathcal{M})$ we infer that

$$|\langle\mu_m\rangle| \leq |\mathcal{M}|^{-1} \|f(\phi_m)\|_{L^1} \leq |\mathcal{M}|^{-1} C_f(1+|\phi_m|^2_{L^2}) \leq c\,|\mathcal{M}|^{-1}(1+\|\phi_m\|^2). \tag{3.26}$$

By Doob's inequality, Hölder's inequality, the condition (2.29), (3.26) and Young's and Poincaré-Wirtinger's inequalities, we infer that

$$\begin{aligned}
&\mathbb{E}\sup_{r\in[0,t\wedge\tau_n^m]}\left|\int_0^r (\mathcal{P}_m^2\sigma_2(s,u_m,\phi_m),\mu_m)dW_m^2(r)\right|^p \\
&\quad\leq c\mathbb{E}\left(\int_0^{t\wedge\tau_n^m} \left|\mathcal{P}_m^1\sigma_1(s,u_m,\phi_m)\right|^2_{\mathcal{L}^2(U;L^2(D))} |\mu_m|^2_{L^2} ds\right)^{p/2} \\
&\quad\leq cK_0^{p/2}\mathbb{E}\left(\int_0^{t\wedge\tau_n^m} |\nabla\mu_m|^2_{L^2} ds + \int_0^{t\wedge\tau_n^m} |\langle\mu_m\rangle|^2 ds\right)^{p/2} \\
&\quad\leq cK_0^{p/2}\mathbb{E}\left(\int_0^{t\wedge\tau_n^m} |\nabla\mu_m|^2_{L^2} ds\right)^{p/2} + cK_0^{p/2}\mathbb{E}\left(\int_0^{t\wedge\tau_n^m} |\langle\mu_m\rangle|^2 ds\right)^{p/2} \\
&\quad\leq cK_0^{p/2}\left[\mathbb{E}\left(\int_0^{t\wedge\tau_n^m} |\nabla\mu_m|^2_{L^2} ds\right)^p\right]^{1/2} + C(T)K_0^{p/2}\mathbb{E}\int_0^{t\wedge\tau_n^m} (1+\|\phi_m\|^{2p})ds \\
&\quad\leq \frac{cK_0^p}{2} + \frac{1}{2}\mathbb{E}\left(\int_0^{t\wedge\tau_n^m} |\nabla\mu_m|^2_{L^2} ds\right)^p + C(T)K_0^{p/2}\mathbb{E}\int_0^{t\wedge\tau_n^m} (1+\|\phi_m\|^{2p})ds \\
&\quad\leq \frac{cK_0^p}{2} + \frac{1}{2}\mathbb{E}\left(\int_0^{t\wedge\tau_n^m} |\nabla\mu_m|^2_{L^2} ds\right)^p + C(T)K_0^{p/2}\mathbb{E}\int_0^{t\wedge\tau_n^m} \mathcal{E}^p(u_m,\phi_m)(s)ds. \qquad (3.27)
\end{aligned}$$

It follows from (3.18)–(3.27) that

$$\begin{aligned}
&\mathbb{E}\sup_{r\in[0,t\wedge\tau_n^m]} \mathcal{E}^p(u_m,\phi_m)(r) \\
&\quad + 3\mathbb{E}\left(\int_0^{t\wedge\tau_n^m} |\nabla\mu_m|^2_{L^2} ds\right)^p + 4\mathbb{E}\left(\int_0^{t\wedge\tau_n^m} \|u_m\|^2 ds\right)^p \leq \mathcal{E}^p(u_0,\phi_0) + C(T,K_0) \\
&\quad + C(T,K_0,L)\int_0^{t\wedge\tau_n^m} \mathbb{E}\mathcal{E}^p(u_m,\phi_m)(s)ds + c\mathbb{E}\left(\int_0^T |\sigma_1(s,0,0)|^2_{\mathcal{L}^2(U;H_1)} ds\right)^p, \qquad (3.28)
\end{aligned}$$

(3.8) follows from Gronwall's lemma and the fact that $\tau_n^m \nearrow T$ as n goes to ∞.

By Young's inequality, and (3.8), we derive that

$$\mathbb{E}\sup_{s\in[0,T]}\mathcal{E}(u_m,\phi_m)(s)\le\frac{1}{2}\mathbb{E}\sup_{s\in[0,T]}\mathcal{E}^2(u_m,\phi_m)(s)+\frac{1}{2}\le C(1+\mathbb{E}\mathcal{E}^2(u_0,\phi_0)),\tag{3.29}$$

$$\begin{aligned}\mathbb{E}\int_0^T\left(|\nabla\mu_m|_{L^2}^2+\|u_m\|^2\right)ds&\le\frac{1}{2}\mathbb{E}\left(\int_0^T\left(|\nabla\mu_m|_{L^2}^2+\|u_m\|^2\right)ds\right)^2+\frac{1}{2}\\&\le C(1+\mathbb{E}\mathcal{E}^2(u_0,\phi_0)).\end{aligned}\tag{3.30}$$

From (3.29) and (3.30) we get (3.9). The Proposition 3.2 is then proved. □

Corollary 3.1. *Under the same hypothesis as in Proposition 3.2, there exists a positive constant C independent of m such that for all $p\ge 2$,*

$$\mathbb{E}\left[\sup_{0\le t\le T}|(u_m,\phi_m)(t)|_{\mathcal{H}}^2+\int_0^T\left(\|(u_m,\phi_m)(s)\|_{\mathbb{V}}^2+\left|A_1^{3/2}\phi_m\right|_{L^2}^2\right)ds\right]\le C,\tag{3.31}$$

$$\mathbb{E}\left[\sup_{0\le t\le T}|(u_m,\phi_m)(t)|_{\mathcal{H}}^p+\left(\int_0^T\left(\|(u_m,\phi_m)(s)\|_{\mathbb{V}}^2+\left|A_1^{3/2}\phi_m\right|_{L^2}^2\right)ds\right)^p\right]\le C,\tag{3.32}$$

$$\mathbb{E}\int_0^T|f(\phi_m)|_{L^2}^2\,ds+\mathbb{E}\int_0^T\|B_1(u_m,\phi_m)(s)\|_{H_2^*}^2\,ds\le C,\tag{3.33}$$

$$\mathbb{E}\int_0^T\|B_0(u_m,u_m)(s)\|_{V_1^*}^2\,ds+\mathbb{E}\int_0^T\|R_0(A_1\phi_m,\phi_m)(s)\|_{V_1^*}^2\,ds\le C,\tag{3.34}$$

$$\begin{aligned}&\sum_{i=1}^2\mathbb{E}\left[\int_0^T\left|\mathcal{P}_m^i\sigma_i(s,u_m,\phi_m)\right|_{\mathcal{L}^2(U;H_i)}^2ds\right]\\&\quad+\mathbb{E}\left[\int_0^T\int_{Z_m}\left|\mathcal{P}_m^1\gamma(s,u_m(s^-),\phi_m(s),z)\right|_{L^2}^2\lambda(dz)ds\right]\le C.\end{aligned}\tag{3.35}$$

Proof. By $(3.7)_4$, (2.37) and the Poincaré–Wirtinger inequality, we note that

$$|f(\phi_m)|_{L^2}^2\le C_f(1+\|\phi_m\|^4)\le C_f(1+\mathcal{E}^2(u_m,\phi_m)),\tag{3.36}$$

$$\begin{aligned}|A_1\phi_m|_{L^2}^2&\le c\,|\mu_m|_{L^2}^2+c\,|f(\phi_m)|_{L^2}^2\\&\le c\,|\nabla\mu_m|_{L^2}^2+c\,|\langle\mu_m\rangle|^2+c\,|f(\phi_m)|_{L^2}^2\\&\le c\,|\nabla\mu_m|_{L^2}^2+c\,|\mathcal{M}|^{-2}\,C_f(1+\|\phi_m\|^2)+c\,|f(\phi_m)|_{L^2}^2\\&\le c\,|\nabla\mu_m|_{L^2}^2+C(\mathcal{M},f)(1+\mathcal{E}^2(u_m,\phi_m)),\end{aligned}\tag{3.37}$$

$$\begin{aligned}\left|A_1^{3/2}\phi_m\right|_{L^2}^2&\le c\,|\nabla\mu_m|_{L^2}^2+c\,|f'(\phi_m)\nabla\phi_m|_{L^2}^2\\&\le c\,|\nabla\mu_m|_{L^2}^2+C_f(\|\phi_m\|^2\,|A_1\phi_m|_{L^2}^2+|A_1\phi_m|_{L^2}^2).\end{aligned}\tag{3.38}$$

The estimates (3.31), (3.32) and the first part of (3.33) follows from (3.36)–(3.38) and the Proposition 3.2. By (2.10), (2.11) and (2.12), we also note that

$$\begin{aligned}&\mathbb{E}\int_0^T\|B_0(u_m,u_m)(s)\|_{V_1^*}^2\,ds+\mathbb{E}\int_0^T\|R_0(A_1\phi_m,\phi_m)(s)\|_{V_1^*}^2\,ds\\&\le c\mathbb{E}\int_0^T|u_m(s)|_{L^2}^2\,\|u_m(s)\|^2\,ds+\mathbb{E}\int_0^T\|\phi_m(s)\|^2\left|A_1^{3/2}\phi_m\right|_{L^2}^2ds\\&\le\left(\mathbb{E}\sup_{0\le s\le T}|u_m(s)|_{L^2}^4\right)^{1/2}\left[\mathbb{E}\left(\int_0^T\|u_m\|^2\,ds\right)^2\right]^{1/2}\end{aligned}$$

$$+ c\left(\mathbb{E}\sup_{0\le s\le T}\|\phi_m(s)\|^4\right)^{1/2}\left[\mathbb{E}\left(\int_0^T\left|A_1^{3/2}\phi_m\right|_{L^2}^2 ds\right)^2\right]^{1/2} \le C(u_0,\phi_0)<\infty, \tag{3.39}$$

$$\begin{aligned}\mathbb{E}\int_0^T\|B_1(u_m,\phi_m)(s)\|_{H_2^*}^2 ds &\le c\mathbb{E}\int_0^T|u_m|_{L^2}^2\|\phi_m\|\,|A_1\phi_m|_{L^2}ds\\ &\le c\mathbb{E}\int_0^T|u_m|_{L^2}^4 ds+\mathbb{E}\int_0^T\|\phi_m\|^2|A_1\phi_m|_{L^2}^2 ds\le C,\end{aligned} \tag{3.40}$$

$$\begin{aligned}&\sum_{i=1}^2\mathbb{E}\left[\int_0^T\left|\mathcal{P}_m^i\sigma_i(s,u_m,\phi_m)\right|_{\mathcal{L}^2(U;H_i)}^2 ds\right]\\ &\quad+\mathbb{E}\left[\int_0^T\int_Z\left|\mathcal{P}_m^1\gamma(s,u_m(s^-),\phi_m(s),z)\right|_{L^2}^2\lambda(dz)ds\right]\\ &\le K_1T+K_1\mathbb{E}\int_0^T|(u_m,\phi_m)(s)|_{\mathcal{H}}^2 ds+\sum_{i=1}^2\mathbb{E}\left[\int_0^T|\sigma_i(s,0,0)|_{\mathcal{L}^2(U;H_i)}^2 ds\right]\\ &\le K_1T+K_1T\mathbb{E}\sup_{0\le s\le T}\mathcal{E}(u_m,\phi_m)(s)\le C(u_0,\phi_0)<\infty.\end{aligned} \tag{3.41}$$

By (3.39)–(3.41) we end the proof of Corollary 3.1. □

From the Corollary 3.1, and along with the Banach-Alaoglu theorem, one can extract a subsequence still denoted by (u_m,ϕ_m) to simplify the notation which converges to the following limits

$$\begin{aligned}&(u_m,\phi_m)\overset{*}{\rightharpoonup}(u,\phi)\quad\text{in}\quad L^p(\Omega,\mathcal{F},\mathbb{P};L^\infty(0,T;\mathcal{H})),\\ &(u_m,\phi_m)\rightharpoonup(u,\phi)\quad\text{in}\quad L^p(\Omega,\mathcal{F},\mathbb{P};L^2(0,T;\mathbb{V})),\\ &(u_m,\phi_m)\rightharpoonup(u,\phi)\quad\text{in}\quad L^2(\Omega,\mathcal{F},\mathbb{P};L^2(0,T;\mathbb{V})),\\ &\phi_m\rightharpoonup\phi\quad\text{in}\quad L^p(\Omega,\mathcal{F},\mathbb{P};L^2(0,T;D(A_1^{3/2}))),\\ &B_0(u_m,\phi_m)\rightharpoonup B_0^0\quad\text{in}\quad L^2(\Omega,\mathcal{F},\mathbb{P};L^2(0,T;V_1^*)),\\ &R_0(A_1\phi_m,\phi_m)\rightharpoonup R_0^0\quad\text{in}\quad L^2(\Omega,\mathcal{F},\mathbb{P};L^2(0,T;V_1^*)),\\ &B_1(u_m,\phi_m)\rightharpoonup B_1^0\quad\text{in}\quad L^2(\Omega,\mathcal{F},\mathbb{P};L^2(0,T;H_2^*)),\\ &f(\phi_m)\rightharpoonup f^0\quad\text{in}\quad L^2(\Omega,\mathcal{F},\mathbb{P};L^2(0,T;L^2(\mathcal{M})),\\ &\mathcal{P}_m^i\sigma_i(\cdot,u_m,\phi_m)\longrightarrow\varPhi_i(\cdot)\quad\text{in}\quad L^2(\Omega,\mathcal{F},\mathbb{P};L^2(0,T;\mathcal{L}^2(U;H_i))),\quad i=1,2,\\ &\mathcal{P}_m^1\gamma(\cdot,u_m(\cdot),\phi_m(\cdot))\rightharpoonup\Psi(\cdot)\quad\text{in}\quad\mathcal{M}_T^2(H_1).\end{aligned} \tag{3.42}$$

With these convergence at hand we see from (3.7) that $(u,\phi)(.)$ satisfies the following Itô stochastic differential: For all $t\in[0,T]$,

$$\begin{aligned}&u(t)+\int_0^t(\beta*A_0u)ds+\int_0^t A_0u ds+\int_0^t B_0^0(s)ds=u_0+\int_0^t R_0^0(s)ds+\int_0^t\varPhi_1(s)dW_s^1\\ &\quad+\int_0^t\int_Z\Psi(s,z)\tilde{\pi}(ds,dz),\\ &\quad\phi(t)+\int_0^t A_1\mu^0 ds+\int_0^t B_1^0(s)ds=\phi_0+\int_0^t\varPhi_2(s)dW_s^2,\\ &\quad\mu^0=A_1\phi+f^0,\end{aligned} \tag{3.43}$$

$\mathbb{P}$-almost surely as equality in $V_1^*\times H_2^*$.

From the energy estimates (see Corollary 3.1), (u_m,ϕ_m) is almost surely uniformly convergent on finite intervals $[0,T]$ to (u,ϕ), from which it follows that (u,ϕ) is $\mathcal{F}_t$-adapted and the $\mathcal{F}_t$-adapted paths of u are càdlàg while the $\mathcal{F}_t$-adapted paths of ϕ are continuous (see [4, Theorem 6.2.3]).

Proposition 3.3. *We have the following identities*

$$\begin{aligned} &B_0(u,u)=B_0^0, \quad R_0(A_1\phi,\phi)=R_0^0, \quad B_1(u,\phi)=B_1^0,\\ &f(\phi_m)=f^0, \quad \gamma(s,u,\phi)=\Psi(s), \quad \sigma_i(s,u,\phi)=\Phi_i(s),\ i=1,2. \end{aligned} \tag{3.44}$$

Proof. Let $(\tilde u_m,\tilde\phi_m,\tilde\mu_m)=\mathcal{P}_m^0(u,\phi,\mu)$, where $\mathcal{P}_m^0=(\mathcal{P}_m^1,\mathcal{P}_m^2,\mathcal{P}_m^2)$. We have

$$\begin{aligned} &\left|(\tilde u_m,\tilde\phi_m)\right|_{\mathcal{H}}\le |(u,\phi)|_{\mathcal{H}},\\ &\left\|(\tilde u_m,\tilde\phi_m)\right\|_{\mathbb{V}}\le c\,\|(u,\phi)\|_{\mathbb{V}},\\ &(\tilde u_m,\tilde\phi_m)\longrightarrow(u,\phi) \text{ in } \mathbb{V} \text{ for almost every } (\omega,t)\in\Omega\times[0,T],\\ &(\tilde u_m,\tilde\phi_m)\longrightarrow(u,\phi) \text{ in } L^2(\Omega,\mathcal{F},\mathbb{P};L^2(0,T;\mathbb{V})). \end{aligned} \tag{3.45}$$

From (3.7) and (3.43), we derive that for $1\le k\le m$,

$$\begin{aligned} &\langle\tilde u_m-u_m(t),w_k\rangle\\ &\quad+\int_0^t\langle A_0(\tilde u_m-u_m),w_k\rangle ds+\int_0^t\langle(\beta * A_0(\tilde u_m-u_m)),w_k\rangle ds\\ &\quad+\int_0^t\langle B_0^0(s)-B_0(u_m,u_m),w_k\rangle=\int_0^t\langle R_0^0-R_0(A_1\phi_m,\phi_m),w_k\rangle ds\\ &\quad+\sum_{j=1}^m\int_0^t\langle\Phi_1(s)e_j-\sigma_1(s,u_m,\phi_m)e_j,w_k\rangle d\beta_s^j+\sum_{j=m+1}^{+\infty}\int_0^t\langle\Phi_1(s)e_j,w_k\rangle d\beta_s^j\\ &\quad+\int_0^t\int_Z(\Psi(s,z)-\gamma(s,u_m(s^-),\phi_m(s),z),w_k)\tilde\pi(ds,dz) \end{aligned} \tag{3.46}$$

$$\begin{aligned} &\langle\tilde\phi_m(t)-\phi(t)_m,\psi_k\rangle\\ &\quad+\int_0^t\langle A_1(\tilde\mu_m-\mu_m),\psi_k\rangle ds+\int_0^t\langle B_1^0(s)-B_1(u_m,\phi_m),\psi_k\rangle ds=0,\\ &\quad+\sum_{j=1}^m\int_0^t\langle\Phi_2(s)e_j-\sigma_2(s,u_m,\phi_m)e_j,w_k\rangle d\beta_s^j+\sum_{j=m+1}^{+\infty}\int_0^t\langle\Phi_2(s)e_j,w_k\rangle d\beta_s^j\\ &\qquad\langle\tilde\mu_m-\mu_m,A_1\psi_k\rangle=\langle A_1(\tilde\phi_m-\phi_m),A_1\psi_k\rangle+\langle f^0-f(\phi_m),A_1\psi_k\rangle. \end{aligned} \tag{3.47}$$

Note that since B_0, R_0 and B_1 are bilinear, we derive that

$$\begin{aligned} B_0^0-B_0(u_m,u_m)&=B_0^0-B_0(\tilde u_m,\tilde u_m)+B_0(\tilde u_m-u_m,\tilde u_m)+B_0(u_m,\tilde u_m-u_m),\\ R_0^0-R_0(A_1\phi_m,\phi_m)&=R_0^0-R_0(A_1\tilde\phi_m,\tilde\phi_m)+R_0(A_1(\tilde\phi_m-\phi_m),\tilde\phi_m)+R_0(A_1\phi_m,\tilde\phi_m-\phi_m)\\ B_1^0-B_1(u_m,\phi_m)&=B_1^0-B_1(\tilde u_m,\tilde\phi_m)+B_1(\tilde u_m-u_m,\tilde\phi_m)+B_1(u_m,\tilde\phi_m-\phi_m),\\ f^0-f(\phi_m)&=f^0-f(\tilde\phi_m)+f(\tilde\phi_m)-f(\phi_m). \end{aligned} \tag{3.48}$$

Let us set $\theta_m=\tilde u_m-u_m$, $\rho_m=\tilde\phi_m-\phi_m$, $\zeta_m=\tilde\mu_m-\mu_m$. From Itô's formula, we have

$$\begin{aligned} d\langle\theta_m,w_k\rangle^2=2\langle\theta_m,w_k\rangle d\langle\theta_m,w_k\rangle&+\sum_{j=1}^m[\langle\Phi_1(t)e_j-\sigma_1(t,u_m,\phi_m)e_j,w_k\rangle]^2dt\\ &+\sum_{j=m+1}^{+\infty}[\langle\Phi_1(t)e_j,w_k\rangle]^2dt+\int_Z\Upsilon(s,z)\pi(dt,dz), \end{aligned} \tag{3.49}$$

where

$$\Upsilon(s,z)=\left|u_m(s^-)+\mathcal{P}_m^1\gamma(s,u_m(s^-),\phi_m(s),z)-\mathcal{P}_m^1\Phi_1(s,z)\right|_{L^2}^2$$

$$
\begin{aligned}
&- \left|u_m(s^-)\right|_{L^2}^2 - 2(\mathcal{P}_m^1\gamma(s,u_m(s^-),\phi_m(s),z),u_m)\\
&= \left|\mathcal{P}_m^1\gamma(s,u_m(s^-),\phi_m(s),z) - \mathcal{P}_m^1\varPhi_1(s,z)\right|_{L^2}^2.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&|\theta_m(t)|_{L^2}^2 + 2\int_0^t (\|\theta_m\|^2 + \langle B_0^0 - B_0(u_m,u_m),\theta_m\rangle)ds = 2\int_0^t \langle R_0^0 - R_0(A_1\phi_m,\phi_m),\theta_m\rangle ds\\
&\quad + 2\sum_{j=1}^m \int_0^t \langle \varPhi_1(s)e_j - \sigma_1(s,u_m,\phi_m)e_j,\theta_m\rangle d\beta_s^j + 2\sum_{j=m+1}^{+\infty}\int_0^t \langle \varPhi_1(s)e_j,\theta_m\rangle d\beta_s^j\\
&\quad + \sum_{j=1}^m \int_0^t \left|\mathcal{P}_m^1[\varPhi_1(s)e_j - \sigma_1(s,u_m,\phi_m)e_j]\right|_{L^2}^2 ds + \sum_{j=m+1}^{+\infty}\int_0^t \left|\mathcal{P}_m^1\varPhi_1(s)e_j\right|_{L^2}^2 ds\\
&\quad + 2\int_0^t\int_Z (\mathcal{P}_m^1(\Psi(s,z) - \gamma(s,u_m(s^-),\phi_m(s),z)),\theta_m)\tilde{\pi}(ds,dz)\\
&\quad + \int_0^t\int_Z \Upsilon(s,z)\pi(ds,dz) - 2\int_0^t ((\beta * \nabla\theta_m),\nabla\theta_m)ds. \qquad (3.50)
\end{aligned}
$$

Also, applying the Itô formula to the process $\|\rho_m\|^2$, and replacing ψ_k in $(3.47)_3$ by $\overline{\zeta}_m - \xi\rho_m$, we obtain

$$
\begin{aligned}
&\|\rho_m(t)\|^2 + 2\int_0^t [\|\overline{\zeta}_m\|^2 + \xi|A_1\rho_m|_{L^2}^2 + \langle B_1^0 - B_1(u_m,\phi_m),A_1\rho_m\rangle]ds\\
&\quad + 2\int_0^t [\xi\langle\zeta_m,A_1\rho_m\rangle + \xi\langle f^b - f(\phi_m),A_1\rho_m\rangle - \langle f^b - f(\phi_m),A_1\zeta_m\rangle]ds\\
&= 2\sum_{j=1}^m\int_0^t (\varPhi_2(s)e_j - \sigma_1(s,u_m,\phi_m)e_j,\rho_m)_{H_2}d\beta_s^j + 2\sum_{j=m+1}^{+\infty}\int_0^t (\varPhi_2(s)e_j,\rho_m)_{H_2}d\beta_s^j\\
&\quad + \sum_{j=1}^m\int_0^t \left\|\mathcal{P}_m^2[\varPhi_2(s)e_j - \sigma_2(s,u_m,\phi_m)e_j]\right\|^2 ds + \sum_{j=m+1}^{+\infty}\int_0^t \left\|\mathcal{P}_m^2\varPhi_1(s)e_j\right\|^2 ds. \qquad (3.51)
\end{aligned}
$$

Note that, owing to $\langle B_0(u_m,\theta_m),\theta_m\rangle = b_0(u_m,\theta_m,\theta_m) = 0$, we have

$$
\begin{aligned}
\langle B_0^0 - B_0(u_m,u_m),\theta_m\rangle &= \langle B_0^0 - B_0(\tilde{u}_m,\tilde{u}_m),\theta_m\rangle + \langle B_0(\theta_m,\widetilde{u}_m),\theta_m\rangle\\
&\le \langle \beta_0^b - B_0(\tilde{u}_m,\tilde{u}_m),\theta_m\rangle + \frac{1}{2}\|\theta_m\|^2 + c\|\tilde{u}_m\|^2|\theta_m|_{L^2}^2. \qquad (3.52)
\end{aligned}
$$

Also we have

$$
\int_0^t ((\beta * \nabla\theta_m),\nabla\theta_m)ds \ge 0, \qquad (3.53)
$$

$$
\begin{aligned}
\langle B_1^0 - B_1(u_m,\phi_m),A_1\rho_m\rangle &= \langle B_1^0 - B_1(\tilde{u}_m,\tilde{\phi}_m),A_1\rho_m\rangle\\
&\quad + \langle B_1(\theta_m,\tilde{\phi}_m),A_1\rho_m\rangle + \langle B_1(u_m,\rho_m),A_1\rho_m\rangle\\
&\le \langle B_1^0 - B_1(\tilde{u}_m,\tilde{\phi}_m),A_1\rho_m\rangle + \frac{1}{4}(\|\theta_m\| + \frac{\xi}{4}|A_1\rho_m|_{L^2}^2)\\
&\quad + c\|\phi_m\|^2|A_1\phi_m|_{L^2}^2|\theta_m|_{L^2}^2 + c|\tilde{u}_m|_{L^2}^2\|\tilde{u}_m\|^2\|\rho_m\|^2, \qquad (3.54)
\end{aligned}
$$

$$
\begin{aligned}
\langle R_0^0 - R_0(A_1\phi_m,\phi_m),\theta_m\rangle &\le \langle R_0^0 - R_0(A_1\tilde{\phi}_m,\tilde{\phi}_m),\theta_m\rangle + \frac{1}{4}(\|\theta_m\| + \frac{\xi}{4}|A_1\rho_m|_{L^2}^2)\\
&\quad + c|A_1\phi_m|_{L^2}^2(|\theta_m|_{L^2}^2 + \|\rho_m\|^2) + c\left\|\tilde{\phi}_m\right\|^2\left|A_1\tilde{\phi}_m\right|_{L^2}^2|\theta_m|_{L^2}^2. \qquad (3.55)
\end{aligned}
$$

Recall that from [18], there exists a monotone non-decreasing function $Q_1(x_1,x_2)$ such that

$$
\begin{aligned}
\langle f^b - f(\phi_m), A_1\zeta_m\rangle &\le \langle f^b - f(\tilde{\phi}_m), A_1\zeta_m\rangle + \frac{1}{2}\|\zeta_m\|^2 + Q_1(\|\tilde{\phi}_m\|, \|\phi_m\|)(|A_1\phi_m|_{L^2}^2 + |A_1\tilde{\phi}_m|_{L^2}^2)\|\rho_m\|^2,\\
\xi\langle f^0 - f(\phi_m), A_1\rho_m\rangle &\le \langle f^b - f(\tilde{\phi}_m), A_1\rho_m\rangle + \frac{\xi}{8}|A_1\rho_m|_{L^2}^2 + Q_1(\|\tilde{\phi}_m\|, \|\phi_m\|)\|\rho_m\|^2,\\
\xi\langle \zeta_m, A_1\rho_m\rangle &\le \frac{c\xi}{2}\|\zeta_m\|^2 + \frac{\xi}{4}|A_1\rho_m|_{L^2}^2,
\end{aligned}
\tag{3.56}
$$

$$
\begin{aligned}
&\sum_{j=1}^m \int_0^t \left|\mathcal{P}_m^1[\varPhi_1(s)e_j - \sigma_1(s,u_m,\phi_m)e_j]\right|_{L^2}^2 ds \le \left\|\mathcal{P}_m^1(\varPhi_1(s) - \sigma_1(s,u_m,\phi_m))\right\|_{\mathcal{L}^2(U;H_1)}^2\\
&\quad \le \left\|\mathcal{P}_m^1(\sigma_1(s,u,\phi) - \sigma_1(s,u_m,\phi_m))\right\|_{\mathcal{L}^2(U;H_1)}\\
&\qquad + 2((\mathcal{P}_m^1(\varPhi_1(s) - \sigma_1(s,u_m,\phi_m)), \mathcal{P}_m^1(\varPhi_1(s) - \sigma_1(s,u,\phi))))_{\mathcal{L}^2(U;H_1)}\\
&\qquad - \left\|\mathcal{P}_m^1(\sigma_1(s,u,\phi) - \varPhi_1(s))\right\|_{\mathcal{L}^2(U;H_1)}^2\\
&\quad \le 2L^2|u - \tilde{u}_m|_{L^2}^2 + 2L^2|(\theta_m,\rho_m)|_{\mathcal{H}}^2\\
&\qquad + 2((\mathcal{P}_m^1(\varPhi_1(s) - \sigma_1(s,u_m,\phi_m)), \mathcal{P}_m^1(\varPhi_1(s) - \sigma_1(s,u,\phi))))_{\mathcal{L}^2(U;H_1)}\\
&\qquad - \left\|\mathcal{P}_m^1(\sigma_1(s,u,\phi) - \varPhi_1(s))\right\|_{\mathcal{L}^2(U;H_1)}^2,
\end{aligned}
\tag{3.57}
$$

$$
\begin{aligned}
&\sum_{j=1}^m \int_0^t \left\|\mathcal{P}_m^2[\varPhi_2(s)e_j - \sigma_2(s,u_m,\phi_m)e_j]\right\|^2 ds \le \left\|\mathcal{P}_m^2(\varPhi_2(s) - \sigma_2(s,u_m,\phi_m))\right\|_{\mathcal{L}^2(U;H_2)}^2\\
&\quad \le 2L^2\left\|\phi - \tilde{\phi}_m\right\|^2 + 2L^2|(\theta_m,\rho_m)|_{\mathcal{H}}^2\\
&\qquad + 2((\mathcal{P}_m^2(\varPhi_2(s) - \sigma_2(s,u_m,\phi_m)), \mathcal{P}_m^2(\varPhi_2(s) - \sigma_2(s,u,\phi))))_{\mathcal{L}^2(U;H_1)}\\
&\qquad - \left\|\mathcal{P}_m^2(\sigma_2(s,u,\phi) - \varPhi_2(s))\right\|_{\mathcal{L}^2(U;H_2)}^2,
\end{aligned}
\tag{3.58}
$$

$$
\begin{aligned}
\Upsilon(s,z) &= \left|\mathcal{P}_m^1\gamma(s,u_m(s^-),\phi_m(s),z) - \mathcal{P}_m^1\varPsi(s,z)\right|_{L^2}^2\\
&= \left|\mathcal{P}_m^1(\gamma(s,u(s^-),\phi(s),z) - \gamma(s,u_m(s^-))), \mathcal{P}_m^1(\phi_m(s),z))\right|_{L^2}^2\\
&\quad + 2(\mathcal{P}_m^1(\varPsi(s,z) - \gamma(s,u_m(s^-),\phi_m(s),z)), \mathcal{P}_m^1(\varPsi(s,z) - \gamma(s,u(s^-),\phi(s),z)))\\
&\quad - \left|\mathcal{P}_m^1(\gamma(s,u(s^-),\phi(s),z) - \varPsi(s,z))\right|_{L^2}\\
&\le 2L^2\left|(u(s^-),\phi(s)) - (\tilde{u}_m(s^-),\tilde{\phi}_m(s))\right|_{\mathcal{H}}^2 + 2L^2\left|(\theta_m(s^-),\rho_m(s))\right|_{\mathcal{H}}^2\\
&\quad - \left|\mathcal{P}_m^1(\gamma(s,u(s^-),\phi(s),z) - \varPsi(s,z))\right|_{L^2} + S_1(s,z),
\end{aligned}
\tag{3.59}
$$

where

$$
S_1(s,z) = 2(\mathcal{P}_m^1(\varPsi(s,z) - \gamma(s,u_m(s^-),\phi_m(s),z)), \mathcal{P}_m^1(\varPsi(s,z) - \gamma(s,u(s^-),\phi(s),z))).
$$

Let us set

$$
\begin{aligned}
Z(t) &= |\theta_m(t)|_{L^2}^2 + \|\rho_m(t)\|^2,\\
Y_1(t) &= c\|\tilde{u}_m\|^2 + c\|\phi_m\|^2 + |A_1\phi_m|_{L^2}^2 + c|\tilde{u}_m|_{L^2}^2\|\tilde{u}_m\|^2 + c|A_1\phi_m|_{L^2}^2\\
&\qquad + c\left|A_1\tilde{\phi}_m\right|_{L^2}^2\left\|\tilde{\phi}_m\right\|^2 + Q_1(\|\tilde{\phi}_m\|, \|\phi_m\|)(1 + |A_1\phi_m|_{L^2}^2 + |A_1\tilde{\phi}_m|_{L^2}^2) + 4L^2,\\
K_2(t) &= \|\theta_m\|^2 + (1 - c\xi)\|\zeta_m\|^2 + c\xi|A_1\rho_m|_{L^2}^2,
\end{aligned}
\tag{3.60}
$$

where ξ is small enough such that $1-c\xi>0$. Also, let us set

$$\sigma(t)=\exp\left(-\int_0^t Y_1(s)ds\right).$$

Adding (3.50) with (3.51), using (3.52)–(3.59), it follows from Itô's formula that

$$\begin{aligned}
&\mathbb{E}\sigma(t)Z(t)+\mathbb{E}\int_0^t \sigma(s)K_2(s)ds+\mathbb{E}\int_0^t\sigma(s)\left\|\mathcal{P}_m^1(\sigma_1(s,u,\phi)-\Phi_1(s))\right\|^2_{\mathcal{L}^2(U;H_1)}ds\\
&\quad+\mathbb{E}\int_0^t\sigma(s)\left\|\mathcal{P}_m^2(\sigma_2(s,u,\phi)-\Phi_2(s))\right\|^2_{\mathcal{L}^2(U;H_2)}ds\\
&\quad+\mathbb{E}\int_0^t\sigma(s)\left|\mathcal{P}_m^1(\gamma(s^-,u(s^-),\phi(s^-),z)-\Psi(s,z))\right|^2_{L^2}ds\\
&\le \mathbb{E}\int_0^t\sigma(s)\langle -B_0^0+B_0(\tilde u_m,\tilde u_m),\theta_m\rangle ds+\mathbb{E}\int_0^t\sigma(s)\langle -B_1^0+B_1(\tilde u_m,\tilde\phi_m),A_1\rho_m\rangle ds\\
&\quad+4L^2\mathbb{E}\int_0^t\sigma(s)\left|(u(s^-),\phi(s))-(\tilde u_m(s^-),\tilde\phi_m(s))\right|^2_{\mathcal{H}}ds\\
&\quad+\sum_{j=m+1}^{+\infty}\mathbb{E}\int_0^t\sigma(s)\left|\mathcal{P}_m^1\Phi_1(s)e_j\right|^2_{L^2}ds+\sum_{j=m+1}^{+\infty}\mathbb{E}\int_0^t\sigma(s)\left\|\mathcal{P}_m^2\Phi_2(s)e_j\right\|^2ds\\
&\quad+2\mathbb{E}\int_0^t\sigma(s)((\Phi_1(s)-\sigma_1(s,u_m,\phi_m),\Phi_1(s)-\sigma_1(s,u,\phi)))_{\mathcal{L}^2(U;H_1)}ds\\
&\quad+2\mathbb{E}\int_0^t\sigma(s)((\Phi_2(s)-\sigma_2(s,u_m,\phi_m),\Phi_2(s)-\sigma_2(s,u,\phi)))_{\mathcal{L}^2(U;H_2)}ds\\
&\quad+\mathbb{E}\int_0^t\int_Z\sigma(s)S_1(s,z)\eta(dz,ds)+\mathbb{E}\int_0^t\sigma(s)\langle R_0^0-R_0(A_1\tilde\phi_m,\tilde\phi_m),\theta_m\rangle ds. \qquad (3.61)
\end{aligned}$$

Now, for each $n\ge 1$, we consider the $\mathcal{F}_t$-stopping time τ_n defined by:

$$\tau_n=\min\left(T,\inf\left\{t\in[0,T];\ \|(u,\phi)\|^2_{\mathcal{H}}+\int_0^t\|(u,\phi)\|^2_{\mathbb{V}}ds\ge n^2\right\}\right).$$

We derive from (3.61) that

$$\begin{aligned}
&\mathbb{E}\sigma(\tau_n)Z(\tau_n)+\mathbb{E}\int_0^{\tau_n}\sigma(s)K_2(s)ds+\mathbb{E}\int_0^{\tau_n}\sigma(s)\left\|\mathcal{P}_m^1(\sigma_1(s,u,\phi)-\Phi_1(s))\right\|^2_{\mathcal{L}^2(U;H_1)}ds\\
&\quad+\mathbb{E}\int_0^{\tau_n}\sigma(s)\left\|\mathcal{P}_m^2(\sigma_2(s,u,\phi)-\Phi_2(s))\right\|^2_{\mathcal{L}^2(U;H_2)}ds\\
&\quad+\mathbb{E}\int_0^{\tau_n}\sigma(s)\left|\mathcal{P}_m^1(\gamma(s,u(s^-),\phi(s),z)-\Psi(s,z))\right|^2_{L^2}ds\\
&\le 4L^2\mathbb{E}\int_0^{\tau_n}\sigma(s)\left|(u(s^-),\phi(s))-(\tilde u_m(s^-),\tilde\phi_m(s))\right|^2_{\mathcal{H}}ds\\
&\quad+\sum_{j=m+1}^{+\infty}\mathbb{E}\int_0^{\tau_n}\sigma(s)\left|\mathcal{P}_m^1\Phi_1(s)e_j\right|^2_{L^2}ds+\sum_{j=m+1}^{+\infty}\mathbb{E}\int_0^{\tau_n}\sigma(s)\left\|\mathcal{P}_m^2\Phi_2(s)e_j\right\|^2ds\\
&\quad+2\mathbb{E}\int_0^{\tau_n}\sigma(s)((\Phi_1(s)-\sigma_1(s,u_m,\phi_m),\Phi_1(s)-\sigma_1(s,u,\phi)))_{\mathcal{L}^2(U;H_1)}ds\\
&\quad+2\mathbb{E}\int_0^{\tau_n}\sigma(s)((\Phi_2(s)-\sigma_2(s,u_m,\phi_m),\Phi_2(s)-\sigma_2(s,u,\phi)))_{\mathcal{L}^2(U;H_2)}ds\\
&\quad+\mathbb{E}\int_0^{\tau_n}\sigma(s)\langle -B_0^0+B_0(\tilde u_m,\tilde u_m),\theta_m\rangle ds+\mathbb{E}\int_0^{\tau_n}\sigma(s)\langle -B_1^0+B_1(\tilde u_m,\tilde\phi_m),A_1\rho_m\rangle ds
\end{aligned}$$

$$+\mathbb{E}\int_0^{\tau_n}\sigma(s)\langle R_0^0-R_0(A_1\tilde{\phi}_m,\tilde{\phi}_m),\theta_m\rangle ds+\mathbb{E}\int_0^{\tau_n}\int_Z\sigma(s)S_1(s,z)\eta(dz,ds). \tag{3.62}$$

Now, we want to prove that the right side of (3.62) goes to 0 as m goes to $+\infty$. We first note that, since $0<\sigma(t)\leq 1$ and $(\tilde{u}_m,\tilde{\phi}_m)\longrightarrow(u,\phi)$ in $L^2(\Omega,\mathcal{F},\mathbb{P};L^2(0,T;\mathbb{V}))$, we have

$$\lim_{m\longrightarrow+\infty}\left(\mathbb{E}\int_0^{\tau_n}\sigma(s)\left|(u(s),\phi(s))-(\tilde{u}_m(s),\tilde{\phi}_m(s))\right|_{\mathcal{H}}^2 ds\right.$$
$$\left.+\sum_{j=m+1}^{+\infty}\mathbb{E}\int_0^{\tau_n}\sigma(s)\left|\mathcal{P}_m^1\Phi_1(s)e_j\right|_{L^2}^2 ds+\sum_{j=m+1}^{+\infty}\mathbb{E}\int_0^{\tau_n}\sigma(s)\left\|\mathcal{P}_m^2\Phi_2(s)e_j\right\|^2 ds\right)=0. \tag{3.63}$$

Following the same way as in [14,36], we derive that

$$\lim_{m\longrightarrow+\infty}\mathbb{E}\int_0^{\tau_n}\sigma(s)\langle -B_0^0+B_0(\tilde{u}_m,\tilde{u}_m),\theta_m\rangle ds=0,$$
$$\lim_{m\longrightarrow+\infty}\mathbb{E}\int_0^{\tau_n}\sigma(s)\langle R_0^0-R_0(A_1\tilde{\phi}_m,\tilde{\phi}_m),\theta_m\rangle ds=0,$$
$$\lim_{m\longrightarrow+\infty}\mathbb{E}\int_0^{\tau_n}\sigma(s)\langle -B_1^0+B_1(\tilde{u}_m,\tilde{\phi}_m),A_1\rho_m\rangle ds=0.$$

Since $\mathcal{P}_m^i\circ\mathcal{P}_m^i=\mathcal{P}_m^i$, and $\left\|\mathcal{P}_m^i\right\|\leq 1$, $i=1,2$, it follows that

$$1_{[0,\tau_n]}\sigma(t)\mathcal{P}_m^i(\Phi_i(s)-\sigma_i(s,u,\phi))\in L^2(\Omega,\mathcal{F},\mathbb{P};L^2(0,T;\mathcal{L}^2(U;H_i)),$$
$$1_{[0,\tau_n]}\sigma(t)\mathcal{P}_m^1(\Psi(s,z)-\gamma(s,u,\phi,z))\in\mathcal{M}_T^2(H_1).$$

Therefore, as $\mathcal{P}_m^1\gamma(\cdot,u_m(\cdot),\phi_m(\cdot))\rightharpoonup\Psi(.)$ in $\mathcal{M}_T^2(H_1)$ and $\mathcal{P}_m^i\sigma_i(\cdot,u_m,\phi_m)\longrightarrow\Phi_i(\cdot)$ in $L^2(\Omega,\mathcal{F},\mathbb{P};L^2(0,T;\mathcal{L}^2(U;H_i)))$, $i=1,2$, we see that

$$\lim_{m\longrightarrow+\infty}\mathbb{E}\int_0^{\tau_n}\int_Z\sigma(s)S_1(s,z)\eta(dz,ds)=0. \tag{3.64}$$

$$\lim_{m\longrightarrow+\infty}\mathbb{E}\int_0^{\tau_n}\sigma(s)((\mathcal{P}_m^i(\Phi_i(s)-\sigma_i(s,u_m,\phi_m)),\mathcal{P}_m^i(\Phi_i(s)-\sigma_i(s,u,\phi))))_{\mathcal{L}^2(U;H_i)}ds=0,\quad i=1,2. \tag{3.65}$$

This concludes that the right side of (3.62) goes to 0 as m goes to $+\infty$.
Now using the fact that $1_{[0,\tau_n]}\sigma(t)\leq 1$, we derive from (3.62) that

$$\lim_{m\longrightarrow+\infty}\mathbb{E}\left|(\theta_m,\rho_m)\right|_{\mathcal{H}}^2=\lim_{m\longrightarrow+\infty}\mathbb{E}\left|(\tilde{u}_m,\tilde{\phi})(\tau_n)-(\theta_m,\psi_m)(\tau_n)\right|_{\mathcal{H}}^2=0$$
$$\lim_{m\longrightarrow+\infty}\mathbb{E}\int_0^{\tau_n}K_2(s)ds=\lim_{m\longrightarrow+\infty}\mathbb{E}\int_0^{\tau_n}(\|\theta_m\|^2+(1-c\xi)\|\zeta_m\|^2+c\xi\left|A_1\rho_m\right|_{L^2}^2)ds$$
$$=\lim_{m\longrightarrow+\infty}\mathbb{E}\int_0^{\tau_n}(\|\tilde{u}_m-u_m\|^2+(1-c\xi)\|\tilde{\mu}_m-\mu_m\|^2+c\xi\left|A_1(\tilde{\phi}_m-\phi_m)\right|_{L^2}^2)ds=0$$
$$\lim_{m\longrightarrow+\infty}\mathbb{E}\int_0^{\tau_n}\left\|\mathcal{P}_m^1(\sigma_1(s,u,\phi)-\Phi_1(s))\right\|_{\mathcal{L}^2(U;H_1)}^2 ds=0$$
$$\lim_{m\longrightarrow+\infty}\mathbb{E}\int_0^{\tau_n}\left\|\mathcal{P}_m^2(\sigma_2(s,u,\phi)-\Phi_2(s))\right\|_{\mathcal{L}^2(U;H_2)}^2 ds=0$$
$$\lim_{m\longrightarrow+\infty}\mathbb{E}\int_0^{\tau_n}\left|\mathcal{P}_m^1(\gamma(s^-,u(s^-),\phi(s^-),z)-\Psi(s,z))\right|_{L^2}^2 ds=0. \tag{3.66}$$

We note that from $(3.66)_{3,4,5}$ and the fact that the sequence $\{\tau_n;\ n\geq 1\}$ is increasing to T, we derive that

$$\sigma_i(s,u,\phi)=\Phi_i(s)\ \text{ in } L^2(\Omega,\mathcal{F},\mathbb{P};L^2(0,T;\mathcal{L}^2(U;H_i))),\ i=1,2,$$

$$\gamma(s,u(s^-),\phi(s),z)=\Psi(s,z),\quad \text{in } \mathcal{M}^2_T(H_1).$$

The end of the proof of the Proposition 3.3 is very similar to [36, Proof of Claim 2]. □

By Proposition 3.3, we infer from (3.43) that (u,ϕ) is a strong solution of problem (1.1) in the sense of Definition 2.1.

3.2. Uniqueness of Strong Solution

Assume that (u_1,ϕ_1) and (u_2,ϕ_2) are two strong solutions to (1.1). We set $(w,\psi,\mu)=(u_1,\phi_1,\mu_1)-(u_2,\phi_2,\mu_2)$, $\tilde{\sigma}_i(\cdot\cdot)=\sigma_i(\cdot,u_1(\cdot),\phi_1(\cdot))-\sigma_i(\cdot,u_2(\cdot),\phi_2(\cdot))$, $i=1,2$ and $\tilde{\gamma}(\cdot,\cdot)=\gamma(\cdot,u_1(\cdot),\phi_1(\cdot),\cdot)-\gamma(\cdot,u_2(\cdot),\phi_2(\cdot),\cdot)$. Then (w,ψ) satisfies the following system

$$\begin{cases} dw+[A_0w+(\beta*A_0w)(t)+B_0(u_2,w)+B_0(w,u_1)]dt \\ =[R_0(A_1\phi_2,\psi)+R_0(A_1\psi,\phi_1)]dt+\tilde{\sigma}_1(t)dW^1_t+\int_Z\tilde{\gamma}(t,z)\tilde{\pi}(dt,dz)\ in\ V_1^*, \\ d\psi+[A_1(\mu-\langle\mu\rangle)+B_1(u_2,\psi)+B_1(w,\phi_1)]dt=+\tilde{\sigma}_2(t)dW^2_t,\ in\ H^{-1}(\mathcal{M}), \\ \mu=A_1\psi+f(\phi_1)-f(\phi_2), \\ (w,\psi)(0)=(0,0). \end{cases} \tag{3.67}$$

We apply infinite dimensional Itô's formula (see [33]) to the process $|w|^2_{L^2}$ and using the fact that $|x|^2_{L^2}-|y|^2_{L^2}+|x-y|^2_{L^2}=2(x-y,x)$, $\forall x,y\in H_1$ to find that

$$\begin{aligned}|w|^2_{L^2}+2\int_0^t\|w(s)\|^2\,ds&=-2\int_0^t((\beta*\nabla w),\nabla w)ds-2\int_0^t b_0(w,u_1,w)ds\\&+2\int_0^t\langle R_0(A_1\phi_2,\psi)+R_0(A_1\psi,\phi_1),w\rangle ds+\int_0^t\int_Z|\tilde{\gamma}(s,z)|^2_{L^2}\,\pi(ds,dz)\\&+\int_0^t\|\tilde{\sigma}_1(s)\|^2_{\mathcal{L}^2(U;H_1)}\,ds+2\int_0^t(\tilde{\sigma}_1(s),w(s))dW^1_s\\&+2\int_0^t\int_Z(\tilde{\gamma}(s,z),w(s^-))\tilde{\pi}(ds,dz).\end{aligned} \tag{3.68}$$

Also, applying the Itô formula to the process $\|\psi\|^2$, we get

$$\begin{aligned}\|\psi\|^2&=-2\int_0^t(A_1(\mu-\langle\mu\rangle),A_1\psi)ds-2\int_0^t b_1(w,\phi_1,A_1\psi)ds\\&-2\int_0^t b_1(u_2,\psi,A_1\psi)ds+\int_0^t\|\tilde{\sigma}_2(s)\|^2_{\mathcal{L}^2(U;H_2)}\,ds+2\int_0^t(\tilde{\sigma}_2(s),\psi(s))_{H_2}dW^2_s.\end{aligned} \tag{3.69}$$

Now we take the duality of $(3.67)_3$ with $A_1(\mu-\langle\mu\rangle)-\xi A_1\psi$, where $\xi>0$ is small enough and will be selected later. Adding the resulting equality to (3.68) and (3.69), we derive that

$$\begin{aligned}|(w,\psi)|^2_{\mathcal{H}}+2\int_0^t(\|w(s)\|^2+\|\mu-\langle\mu\rangle\|^2+\xi\,|A_1\psi|^2_{L^2})ds&=-2\int_0^t((\beta*\nabla w),\nabla w)ds\\&-2\int_0^t b_0(w,u_1,w)ds+2\int_0^t\langle R_0(A_1\phi_2,\psi),w\rangle ds-2\int_0^t b_1(u_2,\psi,A_1\psi)ds\\&+2\int_0^t[\xi(f(\phi_1)-f(\phi_2),A_1\psi)-(f(\phi_1)-f(\phi_2),A_1(\mu-\langle\mu\rangle))]ds\\&+2\int\xi(\mu-\langle\mu\rangle,A_1\psi)ds+\int_0^t\|\tilde{\sigma}_1(s)\|^2_{\mathcal{L}^2(U;H_1)}\,ds+2\int_0^t(\tilde{\sigma}_1(s),w(s))dW^1_s\end{aligned}$$

$$+ \int_0^t \|\tilde{\sigma}_2(s)\|^2_{\mathcal{L}^2(U;H_2)} ds + 2\int_0^t (\tilde{\sigma}_2(s), \psi(s))_{H_2} dW_s^2$$
$$+ \int_0^t \int_Z |\tilde{\gamma}(s,z)|^2_{L^2} \pi(ds,dz) + 2\int_0^t \int_Z (\tilde{\gamma}(s,z), w(s^-))\tilde{\pi}(ds,dz). \tag{3.70}$$

We note that

$$|b_0(w,u_1,w)| \le \frac{1}{4}\|w\|^2 + c\|u_1\|^2 |w|^2_{L^2}, \tag{3.71}$$

$$|b_1(u_2,\psi,A_1\psi)| \le \frac{\xi}{8}|A_1\psi|^2_{L^2} + c|u_2|^2_{L^2}\|u_2\|^2\|\psi\|^2, \tag{3.72}$$

$$|\langle R_0(A_1\phi_2,\psi),w\rangle| \le \frac{1}{4}(\|w\|^2 + \xi|A_1\psi|^2_{L^2}) + c(|w|^2_{L^2} + |\nabla\psi|^2_{L^2})\|\phi_2\|^2|\phi_2|^2_{H^2}, \tag{3.73}$$

$$\xi|\langle f(\phi_1) - f(\phi_2), A_1\psi\rangle| \le \frac{\xi}{16}|A_1\psi|^2_{L^2} + Q_1(\|\phi_1\|,\|\phi_2\|)\|\psi\|^2, \tag{3.74}$$

$$|\langle f(\phi_1) - f(\phi_2), \mu - \langle\mu\rangle\rangle| \le \frac{\xi}{2}\|\mu - \langle\mu\rangle\|^2$$
$$+ Q_1(\|\phi_1\|,\|\phi_2\|)(|A_1\phi_1|^2_{L^2} + |A_1\phi_2|^2_{L^2})\|\psi\|^2, \tag{3.75}$$

$$\xi|(\mu - \langle\mu\rangle, A_1\psi)_{L^2}| \le \frac{\xi}{16}|A_1\psi|^2_{L^2} + c\xi|\nabla(\mu - \langle\mu\rangle)|^2_{L^2}, \tag{3.76}$$

$$\|\tilde{\sigma}_1(s)\|^2_{\mathcal{L}^2(U;H_1)} + \|\tilde{\sigma}_2(s)\|^2_{\mathcal{L}^2(U;H_2)} + \int_Z |\tilde{\gamma}(s,z)|^2_{L^2}\lambda(dz) \le L|(w,\psi)|^2_{\mathcal{H}}, \tag{3.77}$$

where Q_1 is a suitable monotone non-decreasing function independent on time and the initial condition. Now, let us set $\mathcal{Y}_2(t) = |w(t)|^2_{L^2} + \|\psi\|^2$, and

$$K_1(t) = c(\|u_1\|^2 + |u_2|^2_{L^2}\|u_2\|^2 + \|\phi_2\|^2|A_1\phi_2|^2_{L^2})$$
$$+ Q_1(\|\phi_1\|,\|\phi_2\|)(|A_1\phi_1|^2_{L^2} + |A_1\phi_2|^2_{L^2}), \tag{3.78}$$
$$\sigma(t) = \exp\left(-\int_0^t K_1(s)ds\right).$$

So, applying Itô's formula, to the real-valued process $\sigma(t)\mathcal{Y}_2(t)$, using (3.70) and the inequalities (3.71)–(3.77), we derive that

$$\mathbb{E}\sigma(t)\mathcal{Y}_2(t) + \mathbb{E}\int_0^t \sigma(s)(\|w(s)\|^2 + (1-c\xi)\|(\mu - \langle\mu\rangle)\|^2 + \xi|A_1\psi|^2_{L^2})ds$$
$$+ 2\mathbb{E}\int_0^t \sigma(s)((\beta * \nabla w), \nabla w)ds \le L\mathbb{E}\int_0^t \sigma(s)\mathcal{Y}_2(s)ds, \qquad 0 \le t \le T.$$

Note that the expectation of the stochastic integrals in (3.70) varnishes. Therefore we obtain

$$\mathbb{E}\sigma(t)\mathcal{Y}_2(t) \le L\mathbb{E}\int_0^t \sigma(s)\mathcal{Y}_2(s)ds, \qquad 0 \le t \le T.$$

It follows from the deterministic Gronwall lemma that $\mathcal{Y}_2(t) = 0$ $\mathbb{P}$-a.s., for all $t \in [0,T]$. Hence $(u_1,\phi_1) = (u_2,\phi_2)$, $\mathbb{P}$-a.s., for all $t \in [0,T]$. Note that in (3.70), we choose $\xi > 0$ and small enough such that $1 - c\xi > 0$.

4. Exponential Behavior

In this section, we show some aspects of the effects produced in the long-time behavior of the solution to a two dimensional Cahn–Hilliard–Oldroyd model with order one for the non-Newtonian two phase fluid flows under the presence of stochastic perturbations. More precisely, we discuss the moment exponential

stability and almost sure exponential stability of strong solutions (u,ϕ) of stochastic 2D Cahn–Hilliard–Oldroyd model under some conditions.
We will consider the following system

$$\begin{cases} du(t)+[\nu_1 A_0 u+(\beta * A_0 u)(t)+B_0(u,u)-\mathcal{K}R_0(\varepsilon A_1\phi,\phi)]dt \\ = g_1(t)dt+\sigma_1(t,u,\phi)dW_t^1+\int_Z \gamma(t,u(t^-),\phi(t),z)\tilde{\pi}(dt,dz)dt, \text{ in } V_1^*, \\ d\phi(t)+[\nu_2 A_1\mu+B_1(u,\phi)]dt = g_2(t)dt+\sigma_2(t,u,\phi)dW_t^2, \text{ in } H^{-1}(\mathcal{M}), \\ \mu=\varepsilon A_1\phi+\alpha f(\phi)(u,\phi)(0)=(u_0,\phi_0), \end{cases} \tag{4.1}$$

where $g=(g_1,g_2):[0,T]\to V_1^*\times H_2$ is Borel measurable function such that $g\in L^2(0,T;V_1^*\times H_2)$.

Remark 4.1. From the previous section, it is clear that for $g=(g_1,g_2)\in L^2(0,T;V_1^*\times H_2)$, there exists a unique (pathwise) global strong solution for the system (4.1) under the hypothesis (H1)–(H4).

Hereafter, as in [37], we assume that f satisfies the additional condition. For all $\phi_1,\phi_2\in D(A_1^{3/2})$,

$$\langle \alpha A_1 f(\phi_1)-\alpha A_1 f(\phi_2), A_1\phi_1-A_1\phi_2\rangle \geq -\alpha_0 \left|A_1^{3/2}(\phi_1-\phi_2)\right|_{L^2}^2, \tag{4.2}$$

$$\langle \alpha A_1 f(\phi_1), A_1\phi_1\rangle \geq -\alpha_0 \left|A_1^{3/2}\phi_1\right|_{L^2}^2, \tag{4.3}$$

where $\alpha_0>0$ is a positive constants independent of ϕ_1 and ϕ_2.

Assuming that g is independent of t, we now consider the following stationary equation

$$\begin{cases} \left(\nu_1+\frac{\gamma}{\delta}\right)A_0u^*+B_0(u^*,u^*)-\mathcal{K}R_0(\varepsilon A_1\phi^*,\phi^*)=g_1, \\ \nu_2\varepsilon A_1^2\phi^*+\alpha A_1 f(\phi^*)+B_1(u^*,\phi^*)=g_2. \end{cases} \tag{4.4}$$

Then we recall the following solvability result for the system (4.4) for $\nu=(\nu_1+\frac{\gamma}{\delta})>0$, where the proof is very similar to [37, Section 3.1].

Lemma 4.1. *If $g=(g_1,g_2)\in V_1^*\times V_2^*$, then there exists a stationary solution $(u^*,\phi^*)\in\mathcal{U}$ to system (4.4), Moreover, for $\varepsilon>0$ large enough such that $\alpha_2=\min(\mathcal{K}^{-1}\nu,\varepsilon^2\nu_2-\varepsilon\alpha_0)$ is non negative, if $\alpha_2-2(\|g_1\|_{V_1^*}^2+\|g_2\|_{V_2^*})>0$, then the stationary solution to (4.4) is unique.*

Now, we give the definition of exponential stability.

Definition 4.1. We say that a strong solution $(u,\phi)(t)$ to (4.1) converges to $(u^*,\phi^*)\in\mathcal{H}$ exponentially in the mean square if there exists $a>0$ and $M_0=M_0((u,\phi)(0))>0$ such that

$$\mathbb{E}\left|(u,\phi)(t)-(u^*,\phi^*)\right|_{\mathcal{H}}^2 \leq M_0 e^{-at}, \quad t\geq 0. \tag{4.5}$$

If (u^*,ϕ^*) is a solution to (4.4), we say that (u^*,ϕ^*) is exponentially stable in the mean square provided that every strong solution to (4.1) converges to (u^*,ϕ^*) exponentially in the mean square with the same exponential order $a>0$.

Theorem 4.1. *Let (u^*,ϕ^*) be the unique stationary solution of (4.4) and $\sigma_i(s,u^*,\phi^*)=0$, $i=1,2$, $\gamma(s,u^*,\phi^*,z)$, for all $s>0$ and $z\in Z$. Suppose that the assumption (H1)–(H5) are satisfied, then the strong solution $(u,\phi)(t)$ of system (4.1) converges to the stationary solution (u^*,ϕ^*) of the system (4.4) is exponentially stable in the mean square provided that ε is large enough such that $\alpha_2>0$ and the following inequality holds*

$$\alpha_2 > 2c_1\left\|(u^*,\phi^*)\right\|_{\mathcal{U}}+\frac{\alpha_3 L}{\lambda_0}, \tag{4.6}$$

where $\alpha_2=\min(\mathcal{K}^{-1}\nu_1,\nu_2\varepsilon^2-\alpha_0\varepsilon)$, $\alpha_3=\max(\mathcal{K}^{-1},\varepsilon)$ and $c_1>0$ is given below.

Proof. With the condition (4.6), one can chose a constant $a > 0$ such that

$$0 < a < \min\left\{\delta, \lambda_0\left(\frac{\alpha_2}{2} - 2c_1\|(u^*,\phi^*)\|_{\mathcal{U}} - \frac{\alpha_3 L}{2\lambda_0}\right)\right\}. \tag{4.7}$$

We set $(w,\psi)(t) = (u,\phi)(t) - (u^*,\phi^*)$.

Applying the infinite dimensional Itô formula to the process $\mathcal{K}^{-1}e^{2at}|w(t)|^2_{L^2}$ we get

$$\begin{aligned}
&\mathcal{K}^{-1}e^{2at}|w(t)|^2_{L^2} - \mathcal{K}^{-1}|w(0)|^2_{L^2}\\
&\quad= 2a\int_0^t e^{2as}\mathcal{K}^{-1}|w|^2_{L^2}ds - 2\mathcal{K}^{-1}\nu_1\int_0^t e^{2as}\langle A_0u, w\rangle ds\\
&\qquad - 2\mathcal{K}^{-1}\int_0^t e^{2as}\langle(\beta * A_0u)(s), w\rangle ds - 2\mathcal{K}^{-1}\int_0^t e^{2as}\langle B_0(u,u), w\rangle ds\\
&\qquad + 2\int_0^t e^{2as}\langle R_0(\varepsilon A_1\phi,\phi), w\rangle ds + 2\mathcal{K}^{-1}\int_0^t e^{2as}\langle g_1, w\rangle ds\\
&\qquad + \mathcal{K}^{-1}\int_0^t e^{2as}\|\sigma_1(s,u,\phi)\|^2_{\mathcal{L}^2(U;H_1)}ds + 2\mathcal{K}^{-1}\int_0^t e^{2as}\langle\sigma_1(s,u,\phi), w\rangle dW^1_s\\
&\qquad + \mathcal{K}^{-1}\int_0^t e^{2as}\int_Z|\gamma(s,u(s),\phi(s),z)|^2_{L^2}\pi(dz,ds)ds\\
&\qquad + 2\mathcal{K}^{-1}\int_0^t e^{2as}\int_Z(\gamma(s,u(s^-),\phi(s),z),w)\tilde{\pi}(dz,ds).
\end{aligned} \tag{4.8}$$

Applying the Itô formula to the process $\varepsilon e^{2at}|\nabla\psi(t)|^2_{L^2}$, we derive that

$$\begin{aligned}
&\varepsilon e^{2at}|\nabla\psi(t)|^2_{L^2}\\
&\quad= \varepsilon|\nabla\psi(0)|^2_{L^2} + 2a\varepsilon\int_0^t e^{2as}|\nabla\psi(s)|^2_{L^2}ds - 2\nu_2\varepsilon\int_0^t e^{2as}\langle A_1^2\phi, \varepsilon A_1\psi\rangle ds\\
&\qquad - 2\nu_2\alpha\int_0^t e^{as}\langle A_1 f(\phi), \varepsilon A_1\psi\rangle ds - 2\int_0^t e^{2as}\langle B_1(u,\phi), \varepsilon A_1\psi\rangle ds\\
&\qquad + 2\varepsilon\int_0^t e^{2as}(g_2, A_1\psi)ds + \varepsilon\int_0^t e^{2as}\|\sigma_2(s,u,\phi)\|^2_{\mathcal{L}^2(U;H_2)}ds\\
&\qquad + 2\int_0^t e^{2as}(\sigma_2(s,u,\phi),\psi)_{H_2}dW^2_s.
\end{aligned} \tag{4.9}$$

Summing (4.8) and (4.9), after using (4.3), we derive that

$$\begin{aligned}
&e^{2at}|(w,\psi)(t)|^2_{\mathcal{H}}\\
&\quad= |(w,\psi)(0)|^2_{\mathcal{H}} + 2a\int_0^t e^{2as}|(w,\psi)(s)|^2_{\mathcal{H}}ds - 2\mathcal{K}^{-1}\nu_1\int_0^t e^{2as}\langle A_0u, w\rangle ds\\
&\qquad - 2\nu_2\varepsilon\int_0^t e^{2as}\langle A_1^2\phi, \varepsilon A_1\psi\rangle ds - 2\nu_2\alpha\int_0^t e^{as}\langle A_1 f(\phi), \varepsilon A_1\psi\rangle ds\\
&\qquad - 2\mathcal{K}^{-1}\int_0^t e^{2as}\langle(\beta * A_0u)(s), w\rangle ds - 2\mathcal{K}^{-1}\int_0^t e^{2as}\langle B_0(u,u), w\rangle ds\\
&\qquad + 2\int_0^t e^{2as}\langle R_0(\varepsilon A_1\phi,\phi), w\rangle ds - 2\int_0^t e^{2as}\langle B_1(u,\phi), \varepsilon A_1\psi\rangle ds\\
&\qquad + 2\mathcal{K}^{-1}\int_0^t e^{2as}\langle g_1, w\rangle ds + 2\varepsilon\int_0^t e^{2as}(g_2, A_1\psi)ds + \varepsilon\int_0^t e^{2as}\|\sigma_2(s,u,\phi)\|^2_{\mathcal{L}^2(U;H_2)}ds
\end{aligned}$$

$$
\begin{aligned}
&+ \mathcal{K}^{-1}\int_0^t e^{2as}\left\|\sigma_1(s,u,\phi)\right\|^2_{\mathcal{L}(U;H_1)}ds + \mathcal{K}^{-1}\int_0^t e^{2as}\int_Z |\gamma(s,u(s),\phi(s),z)|^2_{L^2}\,\pi(dz,ds)\\
&+ 2\mathcal{K}^{-1}\int_0^t e^{2as}\int_Z (\gamma(s,u(s^-),\phi(s),z),w)\tilde{\pi}(dz,ds)\\
&+ 2\mathcal{K}^{-1}\int_0^t e^{2as}\langle\sigma_1(s,u,\phi),w\rangle dW^1_s + 2\int_0^t e^{2as}(\sigma_2(s,u,\phi),\psi)_{H_2}dW^2_s. \qquad (4.10)
\end{aligned}
$$

Using the definition of $\beta = \gamma e^{-\delta t}$, we note that

$$
\int_0^t e^{2as}\langle\beta * A_0u^*, w(s)\rangle ds = \frac{\gamma}{\delta}\int_0^t e^{2as}(1-e^{-\delta s})\langle A_0u^*, w(s)\rangle ds. \qquad (4.11)
$$

Using (4.11), we infer from (4.4) that (u^*,ϕ^*) satisfies

$$
\begin{aligned}
&2\mathcal{K}^{-1}\nu_1\int_0^t e^{2as}\langle A_0u^*,w\rangle ds + 2\mathcal{K}^{-1}\int_0^t e^{2as}\langle\beta * A_0u^*, w(s)\rangle ds\\
&\quad+ \frac{2\mathcal{K}^{-1}\gamma}{\delta}\int_0^t e^{(2a-\delta)s}\langle A_0u^*, w(s)\rangle ds\\
&\quad+ 2\nu_2\varepsilon\int_0^t e^{2as}\langle A_1^2\phi^*, \varepsilon A_1\psi\rangle ds + 2\mathcal{K}^{-1}\int_0^t e^{2as}\langle B_0(u^*,u^*),w\rangle ds\\
&\quad- 2\int_0^t e^{2as}\langle R_0(\varepsilon A_1\phi^*,\phi^*),w\rangle ds + 2\int_0^t e^{2as}\langle B_1(u^*,\phi^*),\varepsilon A_1\psi\rangle ds\\
&\quad+ 2\nu_1\alpha\int_0^t e^{2as}\langle A_1 f(\phi^*),\varepsilon A_1\psi\rangle ds = 2\mathcal{K}^{-1}\int_0^t e^{2as}\langle g_1,w\rangle ds + 2\varepsilon\int_0^t e^{2as}(g_2,A_1\psi)ds. \qquad (4.12)
\end{aligned}
$$

Using (4.10) and (4.12), we derive that

$$
\begin{aligned}
&e^{2at}\mathbb{E}\,|(w,\psi)(t)|^2_{\mathcal{H}}\\
&\quad= \mathbb{E}\,|(w,\psi)(0)|^2_{\mathcal{H}} + 2a\int_0^t e^{2as}\mathbb{E}\,|(w,\psi)(s)|^2_{\mathcal{H}}\,ds\\
&\qquad- 2\mathcal{K}^{-1}\nu_1\int_0^t e^{2as}\mathbb{E}\,\|w(s)\|^2\,ds - 2\nu_2\varepsilon^2\int_0^t e^{2as}\mathbb{E}\left|A_1^{3/2}\psi(s)\right|^2_{L^2}ds\\
&\qquad- 2\nu_2\alpha\mathbb{E}\int_0^t e^{as}\langle A_1 f(\phi) - A_1 f(\phi^*),\varepsilon A_1\psi\rangle ds - 2\mathcal{K}^{-1}\mathbb{E}\int_0^t e^{2as}\langle(\beta * A_0 w)(s),w\rangle ds\\
&\qquad+ \frac{2\mathcal{K}^{-1}\gamma}{\delta}\mathbb{E}\int_0^t e^{(2a-\delta)s}\langle A_0u^*,w(s)\rangle ds - 2\mathcal{K}^{-1}\mathbb{E}\int_0^t e^{2as}b_0(w,u^*,w)ds\\
&\qquad- 2\mathbb{E}\int_0^t e^{2as}b_1(u^*,\psi,\varepsilon A_1\psi)ds + 2\mathbb{E}\int_0^t e^{2as}b_1(w,\psi,\varepsilon A_1\phi^*)ds\\
&\qquad+ \varepsilon\mathbb{E}\int_0^t e^{2as}\left\|\sigma_2(s,u,\phi)\right\|^2_{\mathcal{L}^2(U;H_2)}ds + \mathcal{K}^{-1}\mathbb{E}\int_0^t e^{2as}\left\|\sigma_1(s,u,\phi)\right\|^2_{\mathcal{L}^2(U;H_1)}ds\\
&\qquad+ \mathcal{K}^{-1}\mathbb{E}\int_0^t e^{2as}\int_Z|\gamma(s,u(s),\phi(s),z)|^2_{L^2}\lambda(dz)ds. \qquad (4.13)
\end{aligned}
$$

Using (2.30), we have

$$
\mathbb{E}\int_0^t e^{2as}\langle(\beta * A_0 w)(s),w\rangle ds = \mathbb{E}\int_0^t e^{2as}((\beta * \nabla w)(s),\nabla w)ds \geq 0. \qquad (4.14)
$$

Using Cauchy-Schwarz, Hölder's and Young's inequalities, we get

$$
\mathbb{E}\int_0^t e^{(2a-\delta)s}\langle A_0u^*,w(s)\rangle ds \leq \mathbb{E}\int_0^t e^{(2a-\delta)s}\|u^*\|\,\|w(s)\|\,ds
$$

$$\leq \left(\int_0^t e^{2(a-\delta)s} \|u^*\|^2 ds\right)^{1/2} \left(\mathbb{E}\int_0^t e^{2as} \|w(s)\|^2 ds\right)^{1/2}$$
$$\leq \frac{\alpha_2\delta}{2\mathcal{K}^{-1}\gamma}\mathbb{E}\int_0^t e^{2as} \|w(s)\|^2 ds + \frac{\mathcal{K}^{-1}\gamma}{4\alpha_2\delta(\delta-a)} \|u^*\|^2, \tag{4.15}$$

for $0 < a < \delta$.
Note that

$$\mathcal{K}^{-1} |b_0(w,u^*,w)| \leq c_1 \|u^*\| \|w\|^2 \tag{4.16}$$
$$|b_1(w,\psi,\varepsilon A_1\phi^*)| \leq \varepsilon c |A_1\psi|_{L^2} |A_1\phi^*|_{L^2} \|w\|$$
$$\leq c_1 \left|A_1^{3/2}\phi^*\right|_{L^2} (\|w\|^2 + \left|A_1^{3/2}\psi\right|^2_{L^2}) \tag{4.17}$$
$$|b_1(u^*,\psi,\varepsilon A_1\psi)| \leq c_1 \|u^*\| \left|A_1^{3/2}\psi\right|^2_{L^2} \tag{4.18}$$
$$\nu_2\langle\alpha A_1 f(\phi) - \alpha A_1 f(\phi^*), \varepsilon A_1\psi\rangle \geq -\alpha_0\nu_2\varepsilon \left|A_1^{3/2}\psi\right|^2_{L^2}. \tag{4.19}$$

Using (2.39), (4.16)–(4.19) and (2.23) in (4.13) we get

$$e^{2at}\mathbb{E}\,|(w,\psi)(t)|^2_{\mathcal{H}} + 2\left(\frac{\alpha_2}{2} - \frac{(2a+\alpha_3 L)}{2\lambda_0} - c_1 \|(u^*,\phi^*)\|_{\mathcal{U}}\right)\int_0^t e^{2as}\mathbb{E}\,\|(w,\psi)(s)\|^2_{\mathcal{U}} ds$$
$$\leq \mathbb{E}\,|(w,\psi)(0)|^2_{\mathcal{H}} + \frac{\mathcal{K}^{-2}\gamma^2}{2\alpha_2\delta^2(\delta-a)} \|(u^*,\phi^*)\|^2_{\mathcal{U}}. \tag{4.20}$$

Since a satisfies (4.7), we finally have

$$\mathbb{E}\,|(w,\psi)(t)|^2_{\mathcal{H}} \leq e^{-2at}\left[\mathbb{E}\,|(w,\psi)(0)|^2_{\mathcal{H}} + \frac{\mathcal{K}^{-2}\gamma^2}{2\alpha_2\delta^2(\delta-a)} \|(u^*,\phi^*)\|^2_{\mathcal{U}}\right], \tag{4.21}$$

and hence $(u,\phi)(t)$ converges to (u^*,ϕ^*) exponentially in the mean square. □

Theorem 4.2. *Suppose that all conditions given in Theorem 4.1 are satisfied, then the strong solution $(u,\phi)(t)$ of (4.1) converges to the stationary solution (u^*,ϕ^*) of (4.4) almost surely exponentially.*

Proof. Let $n = 1, 2, \ldots,$ and $h > 0$. By the Itô formula, for any $t \geq N$ we have

$$|(w,\psi)(t)|^2_{\mathcal{H}}$$
$$= |(w,\psi)(nh)|^2_{\mathcal{H}} - 2\mathcal{K}^{-1}\nu_1\int_{nh}^t \|w(s)\|^2 ds - 2\nu_2\varepsilon^2\int_{nh}^t \left|A_1^{3/2}\psi(s)\right|^2_{L^2} ds$$
$$- 2\nu_2\alpha\int_{nh}^t \langle A_1 f(\phi) - A_1 f(\phi^*), \varepsilon A_1\psi\rangle ds - 2\mathcal{K}^{-1}\int_{nh}^t \langle(\beta * A_0 w)(s), w\rangle ds$$
$$+ \frac{2\mathcal{K}^{-1}\gamma}{\delta}\int_{nh}^t e^{-\delta s}\langle A_0 u^*, w(s)\rangle ds - 2\mathcal{K}^{-1}\int_{nh}^t b_0(w,u^*,w) ds$$
$$- 2\int_{nh}^t b_1(u^*,\psi,\varepsilon A_1\psi) ds + 2\int_{nh}^t b_1(w,\psi,\varepsilon A_1\phi^*) ds$$
$$+ \varepsilon\int_{nh}^t \|\sigma_2(s,u,\phi)\|^2_{\mathcal{L}(U;H_2)} ds + \mathcal{K}^{-1}\int_{nh}^t \|\sigma_1(s,u,\phi)\|^2_{\mathcal{L}^2(U;H_1)} ds$$
$$+ \mathcal{K}^{-1}\int_{nh}^t\int_Z |\gamma(s,u(s),\phi(s),z)|^2_{L^2} \lambda(dz) ds$$
$$+ 2\mathcal{K}^{-1}\int_{nh}^t \langle\sigma_1(s,u,\phi), w\rangle dW_s^1 + 2\varepsilon\int_{nh}^t (\sigma_2(s,u,\phi),\psi)_{H_2} dW_s^2. \tag{4.22}$$

Taking supremum from nh to $(n+1)h$ and then taking expectation in (4.22) after using (4.14)–(4.19) with $a = 1$, we find

$$\begin{aligned}
&\mathbb{E}\sup_{nh\le t\le (n+1)h}|(w,\psi)(t)|_{\mathcal{H}}^2 + \alpha_2\mathbb{E}\int_{nh}^{(n+1)h}\|(w,\psi)(s)\|_{\mathcal{U}}^2\,ds \le \mathbb{E}\,|(w,\psi)(nh)|_{\mathcal{H}}^2\\
&\quad + \frac{\mathcal{K}^{-2}\gamma^2}{2\alpha_2\delta^3}\|(u^*,\phi^*)\|_{\mathcal{U}}^2\,e^{-2\delta nh} + 2c_1\|(u^*,\phi^*)\|_{\mathcal{U}}\,\mathbb{E}\int_{nh}^{(n+1)h}\|(w,\psi)(s)\|_{\mathcal{U}}^2\,ds\\
&\quad + \varepsilon\mathbb{E}\int_{nh}^{(n+1)h}\|\sigma_2(s,u,\phi)\|_{\mathcal{L}(U;H_2)}^2\,ds + \mathcal{K}^{-1}\mathbb{E}\int_{nh}^{(n+1)h}\|\sigma_1(s,u,\phi)\|_{\mathcal{L}(U;H_1)}^2\,ds\\
&\quad + \mathcal{K}^{-1}\mathbb{E}\int_{nh}^{(n+1)h}\int_Z|\gamma(s,u(s),\phi(s),z)|_{L^2}^2\,\lambda(dz)ds\\
&\quad + 2\mathcal{K}^{-1}\mathbb{E}\sup_{nh\le t\le (n+1)h}\left|\int_{nh}^{t}\int_Z(\gamma(s,u(s^-),\phi(s),z),w)\tilde{\pi}(dz,ds)\right|\\
&\quad + 2\mathcal{K}^{-1}\mathbb{E}\sup_{nh\le t\le (n+1)h}\left|\int_{nh}^{t}\langle\sigma_1(s,u,\phi),w\rangle dW_s^1\right|\\
&\quad + 2\varepsilon\mathbb{E}\sup_{nh\le t\le (n+1)h}\left|\int_{nh}^{t}(\sigma_2(s,u,\phi),\psi)_{H_2}dW_s^2\right|. \qquad (4.23)
\end{aligned}$$

By, Davis', Hölder's, and Young's inequalities we derive

$$\begin{aligned}
&\mathcal{K}^{-1}\mathbb{E}\sup_{nh\le t\le (n+1)h}\left|\int_{nh}^{t}\langle\sigma_1(s,u,\phi),w\rangle dW_s^1\right| + \varepsilon\mathbb{E}\sup_{nh\le t\le (n+1)h}\left|\int_{nh}^{t}(\sigma_2(s,u,\phi),\psi)_{H_2}dW_s^2\right|\\
&\le c\mathcal{K}^{-1}\mathbb{E}\left(\int_{nh}^{(n+1)h}\|\sigma_1(s,u,\phi)\|_{\mathcal{L}(U;H_1)}^2|w|_{L^2}^2\,ds\right)^{1/2}\\
&\quad + c\varepsilon\mathbb{E}\left(\int_{nh}^{(n+1)h}\|\sigma_2(s,u,\phi)\|_{\mathcal{L}(U;H_2)}^2\|\phi\|^2\,ds\right)^{1/2}\\
&\le \frac{1}{8}\mathbb{E}\sup_{nh\le t\le (n+1)h}|(w,\psi)(t)|_{\mathcal{H}}^2 + c_2\sum_{i=0}^{2}\mathbb{E}\int_{nh}^{(n+1)h}\|\sigma_i(s,u,\phi)\|_{\mathcal{L}(U;H_i)}^2\,ds \qquad (4.24)
\end{aligned}$$

An application of the Burkholder–Davis–Gundy inequality (see [33, Theorem 48]), Hölder's, and Young's inequalities yield

$$\begin{aligned}
&\mathcal{K}^{-1}\mathbb{E}\sup_{nh\le t\le (n+1)h}\left|\int_{nh}^{t}\int_Z(\gamma(s,u(s^-),\phi(s),z),w)\tilde{\pi}(dz,ds)ds\right|\\
&\quad\le c2\mathcal{K}^{-1}\mathbb{E}\left(\int_{nh}^{(n+1)h}\int_Z|\gamma(s,u(s),\phi(s),z)|_{L^2}^2|w|_{L^2}^2\,\pi(dz,ds)\right)^{1/2}\\
&\quad\le c2\mathcal{K}^{-1}\mathbb{E}\sup_{nh\le t\le (n+1)h}|w|_{L^2}\left(\int_{nh}^{(n+1)h}\int_Z|\gamma(s,u(s),\phi(s),z)|_{L^2}^2\,\pi(dz,ds)\right)^{1/2}\\
&\quad\le \frac{1}{8}\mathbb{E}\sup_{nh\le t\le (n+1)h}|(w,\psi)(t)|_{\mathcal{H}}^2 + c_2\mathbb{E}\int_{nh}^{(n+1)h}\int_Z|\gamma(s,u(s),\phi(s),z)|_{L^2}^2\,\lambda(dz)ds. \qquad (4.25)
\end{aligned}$$

Combining (4.24) and (4.25), substituting it in (4.23), and then using (2.39), we get

$$\mathbb{E}\sup_{nh\le t\le (n+1)h}|(w,\psi)(t)|_{\mathcal{H}}^2 + \alpha_4\mathbb{E}\int_{nh}^{(n+1)h}\|(w,\psi)(s)\|_{\mathcal{U}}^2\,ds$$

$$\leq 2\mathbb{E}\left|(w,\psi)(nh)\right|_{\mathcal{H}}^2 + \frac{\mathcal{K}^{-2}\gamma^2}{\alpha_2\delta^3}\left\|(u^*,\phi^*)\right\|_{\mathcal{U}}^2 e^{-2\delta nh}, \tag{4.26}$$

where

$$\alpha_3 = 2\lambda_0\left(\alpha_2 - 2c_1\left\|(u^*,\phi^*)\right\|_{\mathcal{U}} + \frac{c_2L}{\lambda_0}\right) > 0.$$

Using (4.21) in (4.26), we arrive at

$$\mathbb{E}\sup_{nh\leq t\leq(n+1)h}\left|(w,\psi)(t)\right|_{\mathcal{H}}^2 \leq \left(2\mathbb{E}\left|(w,\psi)(0)\right|_{\mathcal{H}}^2 + \frac{\mathcal{K}^{-2}\gamma^2(2\delta-a)}{\alpha_2\delta^3(\delta-a)}\left\|(u^*,\phi^*)\right\|_{\mathcal{U}}^2\right)e^{-2anh}, \tag{4.27}$$

For $\theta\in(0,a)$, we set

$$\Omega_{n,h}^a = \left\{\omega\in\Omega : \sup_{nh\leq t\leq(n+1)h}\left|(w,\psi)(t)\right|_{\mathcal{H}}^2 > e^{-(a-\theta)nh}\right\}.$$

By Chebychev's inequality, we also have

$$\begin{aligned}\mathbb{P}(\Omega_{n,h}^a) &\leq e^{2(a-\theta)nh}\mathbb{E}\sup_{nh\leq t\leq(n+1)h}\left|(w,\psi)(t)\right|_{\mathcal{H}}^2\\ &\leq \left(2\mathbb{E}\left|(w,\psi)(0)\right|_{\mathcal{H}}^2 + \frac{\mathcal{K}^{-2}\gamma^2(2\delta-a)}{\alpha_2\delta^3(\delta-a)}\left\|(u^*,\phi^*)\right\|_{\mathcal{U}}^2\right)e^{-2\theta nh},\end{aligned} \tag{4.28}$$

which implies that

$$\sum_{n=1}^{\infty}\mathbb{P}(\Omega_{n,h}^a) \leq \left(2\mathbb{E}\left|(w,\psi)(0)\right|_{\mathcal{H}}^2 + \frac{\mathcal{K}^{-2}\gamma^2(2\delta-a)}{\alpha_2\delta^3(\delta-a)}\left\|(u^*,\phi^*)\right\|_{\mathcal{U}}^2\right)\frac{1}{e^{2\theta h}-1} < +\infty.$$

Therefore, by the Borel-Cantelli lemma, there is a finite integer $n_0(\omega)$ such that

$$\sup_{nh\leq t\leq(n+1)h}\left|(w,\psi)(t)\right|_{\mathcal{H}}^2 \leq e^{-(a-\theta)nh}, \quad \mathbb{P}-a.s., \tag{4.29}$$

for all $n\geq n_0$, and the Theorem 4.2 is then proved. □

For the next theorem we assume that g_1 and g_2 depend on $u(\cdot)$, $\phi(\cdot)$ and satisfy the following Lipschitz condition: For all $(u_1,\phi_1),(u_2,\phi_2)\in\mathcal{U}$,

$$\begin{aligned}\left\|g_1(u_1,\phi_2) - g_1(u_1,\phi_2)\right\|_{V_1^*} &\leq L_1\left\|(u_1,\phi_1)-(u_2,\phi_2)\right\|_{\mathcal{U}},\\ \left\|g_2(u_1,\phi_1) - g_2(u_2,\phi_2)\right\|_{H_2} &\leq L_2\left|(u_1,\phi_1)-(u_2,\phi_2)\right|_{\mathcal{H}}.\end{aligned} \tag{4.30}$$

Theorem 4.3. *If $g_i(0,0)=0$, $\sigma_i(t,0,0)=0$, $i=1,2$ and $\gamma(t,0,0,z)=0$, for all $t>0$ and $z\in Z$, then any strong solution $(u,\phi)(t)$ to (4.1) converges to zero almost surely exponentially if*

$$\alpha_2 > \mathcal{K}^{-1}L_1 - \frac{(L_2+\alpha_3L)}{2\lambda_0}. \tag{4.31}$$

Proof. Owing to (4.31), one can chose a constant $a>0$ such that

$$0 < a < \lambda_0\left(\alpha_2 - \mathcal{K}^{-1}L_1 - \frac{(L_2+\alpha_3L)}{2\lambda_0}\right). \tag{4.32}$$

Applying the infinite dimensional Itô formula to the process $\mathcal{K}^{-1}e^{2at}\left|u(t)\right|_{L^2}^2$ and $\varepsilon e^{2at}\left|\nabla\psi(t)\right|_{L^2}^2$ respectively, summing the results and using (2.7)–(2.8), we get

$$\begin{aligned}&e^{2at}\left|(u,\phi)(t)\right|_{\mathcal{H}}^2\\ &= \left|(u,\phi)(0)\right|_{\mathcal{H}}^2 + 2a\int_0^t e^{2as}\left|(u,\phi)(s)\right|_{\mathcal{H}}^2 ds - 2\mathcal{K}^{-1}\nu_1\int_0^t e^{2as}\langle A_0u,u\rangle ds\\ &\quad - 2\nu_2\varepsilon\int_0^t e^{2as}\langle A_1^2\phi,\varepsilon A_1\phi\rangle ds - 2\nu_2\alpha\int_0^t e^{as}\langle A_1f(\phi),\varepsilon A_1\phi\rangle ds\end{aligned}$$

$$
\begin{aligned}
&- 2\mathcal{K}^{-1}\int_0^t e^{2as}\langle(\beta * A_0 u)(s), u\rangle ds + 2\mathcal{K}^{-1}\int_0^t e^{2as}\langle g_1(u,\phi), u\rangle ds \\
&+ 2\varepsilon\int_0^t e^{2as}(\nabla g_2(u,\phi), \nabla\phi)ds + \varepsilon\int_0^t e^{2as}\left\|\sigma_2(s,u,\phi)\right\|^2_{\mathcal{L}(U;H_2)} ds \\
&+ \mathcal{K}^{-1}\int_0^t e^{2as}\left\|\sigma_1(s,u,\phi)\right\|^2_{\mathcal{L}(U;H_1)} ds \\
&+ \mathcal{K}^{-1}\int_0^t e^{2as}\int_Z |\gamma(s,u(s),\phi(s),z)|^2_{L^2}\,\pi(dz,ds) \\
&+ 2\mathcal{K}^{-1}\int_0^t e^{2as}\int_Z (\gamma(s,u(s^-),\phi(s),z),u)\tilde{\pi}(dz,ds) \\
&+ 2\mathcal{K}^{-1}\int_0^t e^{2as}\langle\sigma_1(s,u,\phi),u\rangle dW^1_s + 2\int_0^t e^{2as}(\sigma_2(s,u,\phi),\psi)_{H_2} dW^2_s. \qquad (4.33)
\end{aligned}
$$

Using (2.30), (4.3), (2.39) and (4.30), we infer from (4.33) that

$$
\begin{aligned}
&e^{2at}\mathbb{E}\,|(u,\phi)(t)|^2_{\mathcal{H}} + 2\alpha_2\mathbb{E}\int_0^t e^{2as}\left\|(u,\phi)(s)\right\|^2_{\mathcal{U}} ds \le \mathbb{E}\,|(u,\phi)(0)|^2_{\mathcal{H}} + 2a\mathbb{E}\int_0^t e^{2as}\left|(u,\phi)(s)\right|^2_{\mathcal{H}} ds \\
&\quad + 2\mathcal{K}^{-1}\mathbb{E}\int_0^t e^{2as}\langle g_1(u,\phi),u\rangle ds + 2\varepsilon\mathbb{E}\int_0^t e^{2as}(\nabla g_2(u,\phi),\nabla\phi)ds + \varepsilon\mathbb{E}\int_0^t e^{2as}\left\|\sigma_2(s,u,\phi)\right\|^2_{\mathcal{L}(U;H_2)} ds \\
&\quad + \mathcal{K}^{-1}\mathbb{E}\int_0^t e^{2as}\left\|\sigma_1(s,u,\phi)\right\|^2_{\mathcal{L}(U;H_1)} ds + \mathcal{K}^{-1}\mathbb{E}\int_0^t e^{2as}\int_Z \left|\gamma(s,u(s^-),\phi(s),z)\right|^2_{L^2}\lambda(dz)ds \\
&\quad \le \mathbb{E}\,|(u,\phi)(0)|^2_{\mathcal{H}} + 2\mathcal{K}^{-1}L_1\mathbb{E}\int_0^t e^{2as}\left\|(u,\phi)(s)\right\|^2_{\mathcal{U}} ds + (2a + L_2 + \alpha_3 L)\mathbb{E}\int_0^t e^{2as}\left|(u,\phi)(s)\right|^2_{\mathcal{H}} ds.
\end{aligned}
$$

This implies that

$$
e^{2at}\mathbb{E}\,|(u,\phi)(t)|^2_{\mathcal{H}} + 2\left(\alpha_2 - \mathcal{K}^{-1}L_1 - \frac{(2a + L_2 + \alpha_3 L)}{2\lambda_0}\right)\mathbb{E}\int_0^t e^{2as}\left\|(u,\phi)(s)\right\|^2_{\mathcal{U}} ds \le \mathbb{E}\,|(u,\phi)(0)|^2_{\mathcal{H}}.
$$

under the condition (4.31), it is immediate that

$$
\mathbb{E}\,|(u,\phi)(t)|^2_{\mathcal{H}} \le \mathbb{E}\,|(u,\phi)(0)|^2_{\mathcal{H}}\, e^{-2at}.
$$

This implies that the strong solution of (4.1) converges to zero exponentially in the mean square. We can then finish the proof using the same method as in the proof of Theorem 4.2. □

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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