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Global Classical Solutions and Stabilization in a Two-Dimensional Parabolic-Elliptic Keller–Segel–Stokes System

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Abstract. A class of parabolic-elliptic Keller–Segel–Stokes systems generalizing the prototype

$$\begin{cases} n_t + u\cdot\nabla n = \Delta n - C_S\nabla\cdot(n(1+n)^{-\alpha}\nabla c), \quad x\in\Omega, t>0,\\ u\cdot\nabla c = \Delta c - c + n, \quad x\in\Omega, t>0,\\ u_t+\nabla P = \Delta u + n\nabla\phi, \quad x\in\Omega, t>0,\\ \nabla\cdot u = 0, \quad x\in\Omega, t>0 \end{cases} \tag{KSF}$$

is considered under boundary conditions of homogeneous Neumann type for n (the density of the cell population) and c (the chemical concentration), and Dirichlet type for u (the velocity field), in a bounded domain $\Omega\subseteq\mathbb{R}^2$ with smooth boundary, where $C_S>0$ and ϕ is a given sufficiently smooth function. The model is proposed to describe chemotaxis-fluid interaction in cases when the evolution of the chemoattractant is essentially dominated by production through cells. Moreover, the chemical diffuses much faster than the cells move. It is shown that under the condition that

$$\alpha > 0,$$

for any sufficiently smooth initial data (n_0, u_0) satisfying some compatibility conditions, the associated initial-boundary-value problem (KSF) possesses a global bounded classical solution. In comparison to the result for the corresponding fluid-free system, it is easy to see that the restriction on α here is optimal. Building on this boundedness property, it can finally even be proved that the corresponding solution of the system decays to $(\bar{n}_0, \bar{n}_0, 0)$ exponentially if C_S is smaller, where $\bar{n}_0 = \frac{1}{|\Omega|}\int_\Omega n_0$. Our main tool is consideration of the energy functional

$$\int_\Omega n^{1+\alpha},$$

which is a new energy-like functional.

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1. Introduction

We consider the following parabolic-elliptic Keller–Segel–Stokes system

$$\begin{cases} n_t + u\cdot\nabla n = \Delta n - \nabla\cdot(nS(n)\nabla c), \quad x\in\Omega, t>0,\\ \tau c_t + u\cdot\nabla c = \Delta c - c + n, \quad x\in\Omega, t>0,\\ u_t+\nabla P = \Delta u + n\nabla\phi, \quad x\in\Omega, t>0,\\ \nabla\cdot u = 0, \quad x\in\Omega, t>0,\\ (\nabla n - nS(n)\nabla c)\cdot\nu = \nabla c\cdot\nu = 0, u=0, \quad x\in\partial\Omega, t>0,\\ n(x,0)=n_0(x), u(x,0)=u_0(x), \quad x\in\Omega \end{cases} \tag{1.1}$$

in a bounded domain $\Omega\subseteq\mathbb{R}^2$ with smooth boundary, where $\tau=0$ reflects the assumption that the considered chemical diffuses much faster than cells and fluid particles, as having formed technically essential fundamentals already in numerous precedents from the chemotaxis(-fluid) literature (cf., e.g., [18,22] for two representative examples). Herein, the unknown function n denotes the population density of cells, c is the chemical concentration, and u, P and ϕ represent the fluid velocity, the pressure of the fluid, and the potential of the gravitational field, respectively. Moreover, the chemotactic sensitivity S here is assumed to be a given scalar function satisfying

$$S \in C^2(\bar{\Omega}) \text{ and } S(n) \geq 0 \text{ for all } n \geq 0 \tag{1.2}$$

and

$$|S(n)| \leq C_S(1+n)^{-\alpha} \text{ for all } n \geq 0 \tag{1.3}$$

with some $C_S > 0$ and $\alpha \geq 0$. The model (1.1) is proposed to describe chemotaxis-fluid interaction in cases when the evolution of the chemoattractant is essentially dominated by production through cells (see [1,12,32,46]). This kind of model can also be used to the studies of coral broadcast spawning (see [6,7,18]).

Before going into our mathematical analysis, we recall some important progresses on system (1.1) and its variants. If all effects of fluid flow are ignored by letting $u \equiv 0$, model (1.1) can be reduced to quasilinear chemotaxis model

$$\begin{cases} n_t = \Delta n - \nabla \cdot (nS(n)\nabla c), \quad x \in \Omega,\ t > 0, \\ \tau c_t = \Delta c - c + n, \quad x \in \Omega,\ t > 0, \\ (\nabla n - nS(n)\nabla c) \cdot \nu = \nabla c \cdot \nu = 0, \quad x \in \partial\Omega, t > 0, \\ n(x,0) = n_0(x), \tau c(x,0) = \tau c_0(x), \quad x \in \Omega, \end{cases} \tag{1.4}$$

which as an important variant of the classical chemotaxis Keller–Segel model (see [17] and also [1]). This model was first introduced in 1970 by Keller and Segel [17] to model chemotaxis of cell populations. Since 1970, there have been considerably plentiful results about the behavior (boundedness and blow-up) of the solutions to the Keller–Segel model and its corresponding variants (see [3,4,11,13,23,35] and references therein for detailed results). For instance, if $S(n) \equiv 1$, for the parabolic-elliptic classical Keller–Segel model ($\tau = 0$) [4,11,23], it was shown that in dimension 2, there exists a critical mass M^*, such that if $M < M^*$, the solution exists globally; while if $M > M^*$, the solution may blow up in finite time, where $M^* := \int_\Omega n_0(x)dx = \int_\Omega n(x,t)dx$. Whereas in space dimension $N = 1$, all solutions are global and bounded, blow-up occurs for some initial data if $N \geq 3$ [3]. More generally, when $S = S(n)$ is a non-constant scalar function, then its asymptotic behavior decides whether or not the explosion phenomena may occur. Indeed, when $S(n) := (1+n)^{-\alpha}$, there is a critical exponent

$$\alpha_* = 1 - \frac{2}{N} \tag{1.5}$$

which is related to the presence of a so-called volume-filling effect, where N is the space dimension. In particular, all solutions are global and uniformly bounded if $\alpha > \alpha_*$ [13], while the solution may blow up if $\Omega \subseteq \mathbb{R}^N$ ($N \geq 2$) is a ball and $\alpha < \alpha_*$ under some technical assumptions [13,35]. Besides the above works, recent studies have shown that the solution behavior for system (1.4) can be also impacted by the prevention of overcrowding (see [2,12]), the nonlinear diffusion (see [3,47]) and the logistic damping (see [29,36]).

In various situations, however, the migration of cells may be more complex because it can be effected by the changes in their living environment. In order to describe the dynamics of bacterial swimming and oxygen transport near contact lines, Tuval et al. [30] proposed the following chemotaxis-(Navier–)Stokes system

$$\begin{cases} n_t + u \cdot \nabla n = \Delta n - \nabla \cdot (nS(c)\nabla c), \quad x \in \Omega, t > 0, \\ c_t + u \cdot \nabla c = \Delta c - nf(c), \quad x \in \Omega, t > 0, \\ u_t + \kappa(u \cdot \nabla)u + \nabla P = \Delta u + n\nabla\phi, \quad x \in \Omega, t > 0, \\ \nabla \cdot u = 0, \quad x \in \Omega, t > 0, \end{cases} \tag{1.6}$$

where $\kappa \in \mathbb{R}$ and $f(c)$ denote the strength of nonlinear fluid convection and the per-capita oxygen consumption rate, respectively. When the fixed number $\kappa \in \mathbb{R}$ in (1.6) is nonzero, the fluid motion is governed by the full Navier–Stokes equations involving nonlinear convection, whereas if $\kappa = 0$ we consider the simplified Stokes evolution for u which appears to be justified if the fluid flow remains small [21]. The coupled chemotaxis-fluid model is first introduced by Tuval et al. [30] in 2005. After this, this kind of models have been studied by many researchers. In fact, by making use of energy-type functionals, some local and global solvability of corresponding initial value problem for (1.6) in either bounded or unbounded domains have been obtained in the past years (see Duan and Xiang [5], Di Francesco et al. [8], Ishida [14], Tao and Winkler [27,28], Winkler Winkler61215,Winklerdddsss51215 and the references therein for details). For example, for the case of bounded domain $\Omega \subseteq \mathbb{R}^N$, Winkler [37] proved that the initial-boundary value problem of (1.6) admits a unique global classical solution for $N = 2$ and possesses at least one global weak solution for $N = 3$ under the assumption that $\kappa = 0$. Besides these works focused on the well-posedness theory, Winkler [38] (see also [41]) further investigated the qualitative behavior of such solutions. Indeed, he [38] showed that the global classical solutions obtained in [37] stabilize to the spatially uniform equilibrium $(\bar{n}_0, 0, 0)$ with $\bar{n}_0 := \frac{1}{|\Omega|}\int_\Omega n_0$ as $t \to \infty$. While in three-dimensional case, global weak solutions have been constructed in [40], and in [41] it has recently been shown that any such solution becomes eventually smooth and stabilize to the spatially uniform equilibrium $(\bar{n}_0, 0, 0)$ (see [41]). For more literature related to the variant of model (1.6) with nonlinear diffusion or the rotational flux term, one can refer to [19,27,28,39,42] and the references therein.

If the signal is dominated by production of the signal substance (by cells), the corresponding chemotaxis-fluid should be written as

$$\begin{cases} n_t + u \cdot \nabla n = \Delta n - \nabla \cdot (nS(x,n,c)\nabla c), \quad x \in \Omega, t > 0, \\ \tau c_t + u \cdot \nabla c = \Delta c - c + n, \quad x \in \Omega, t > 0, \\ u_t + \kappa(u \cdot \nabla)u + \nabla P = \Delta u + n\nabla\phi, \quad x \in \Omega, t > 0, \\ \nabla \cdot u = 0, \quad x \in \Omega, t > 0, \end{cases} \tag{1.7}$$

where $\tau \in \{0,1\}$ and the chemotactic sensitivity $S(x,n,c)$ is a tensor-valued function or a scalar function which satisfies

$$S \in C^2(\bar{\Omega} \times [0,\infty)^2; \mathbb{R}^{N\times N}) \tag{1.8}$$

and

$$|S(x,n,c)| \le C_S(1+n)^{-\alpha} \text{ for all } (x,n,c) \in \Omega \times [0,\infty)^2 \tag{1.9}$$

with some $C_S > 0$ and $\alpha \ge 0$. Here $N = \{2,3\}$ denotes the space dimension and $\tau = 0$ denotes the chemical diffuses much faster than the cells move (see e.g. [15]).

Due to the signal production (If the signal is consumed, one can immediately obtain uniform bounds on c from the second equation, which led to it being studied more heavily than the framework with signal production by the cells) and the lower regularity for n and c ($\int_\Omega n = \int_\Omega n_0$ and $\int_\Omega c \le \max\{\int_\Omega n_0, \tau\int_\Omega c_0\}$, as the only apparent a priori information available), the mathematical analysis of (1.7) regarding global as well as bounded solutions and stabilization is far from trivial (see Peng and Xiang [25], Wang and Xiang [33,34], Wang [31], Winkler [46], Ke and Zheng [16] and Zheng [48,49,51,54]). Let us briefly summarize some of the results available for (1.7) with parabolic-parabolic version ($\tau = 1$ in the second equation of (1.7)). In fact, when $S(x,n,c) \equiv 1$ and the initial data satisfy certain smallness condition (e.g. $\int_\Omega n_0 < 2\pi$), Winkler [46] proved that 2-D Keller–Segel–Navier–Stokes system (1.7) admits a global generalized solution which may eventually become smooth and stabilizes toward a spatially homogeneous equilibrium. If $\kappa = 0$ and the tensor-valued chemotaxis sensitivity satisfies (1.8), the classical solution in two-dimensional and three-dimensional bounded domain are proved in [32] (when $\alpha > 0$) and [34] (when $\alpha > \frac{1}{2}$), respectively. Recently, Wang et al. [32] extended the above result [33] to the full Navier–Stokes version ($\kappa \neq 0$ in the first equation of (1.1)) with **convex domain**. More recently, we dropped the hypothesis of **convex domain** by using a new functional (see [50]), which is different from [32]. In the three dimensional case, Wang and Liu [20] proved that the full Navier–Stokes ($\kappa \neq 0$ in the first equation of (1.7)) system (1.7) obtains a global weak solution for tensor-valued sensitivity $S(x, n, c)$ satisfying (1.8) and (1.3) with $\alpha > \frac{3}{7}$. This result was improved in [16] (see also [31]) to the case of $\alpha > \frac{1}{3}$ very recently. In [43], Winkler further proved that the initial-boundary value problem (1.7) possesses a global bounded classical solution under the assumption that $\kappa = 0$ and $S(x, n, c) = (1 + n)^{-\alpha}$ with $\alpha > \frac{1}{3}$. In comparison to the result for the corresponding fluid-free system, it is easy to see that the restriction on α in literature [16,31–33,43] is optimal. We should point that the above results are all about parabolic-parabolic version. Compared with parabolic-parabolic ($\tau = 1$ in the second equation of (1.7)) version, the coupled parabolic-elliptic ($\tau = 0$ in the second equation of (1.7)) version of system (1.7) is much less understood. Moreover, the question of identifying an optimal condition on α ensuring global boundedness in the parabolic-elliptic version of system (1.1) remains an open challenge. To the best of our knowledge, this is the first global existence and boundedness result addressing the parabolic-elliptic version of system (1.7).

Motivated by the above works and analysis, in the present paper, one purpose is try to establish the global solvability and boundedness of (1.1) under an optimal condition on the key parameter α. Although the L^1 -norm of n and c (see Lemma 3.1) is easy to be obtained, the parabolic-elliptic version still cause essential difficulty due to the deficiency of regularity for c. In fact, for the parabolic-parabolic version ($\tau = 1$ in the first equation of (1.1)), one can establish a L^2-estimate for c and $L^{2\alpha}$-estimate for n, and thus may easily obtain L^p-estimate of c for any large $p > 1$ (see the proof Lemma 5.1 and Lemma 6.1 of [32]). And thus, one can establish the boundedness of the functional $\int_\Omega \ln n(\cdot,t)n(\cdot,t) + a\int_\Omega |\nabla c(\cdot,t)|^2$ (see Lemma 6.2 of [32]) or $\int_\Omega n^{1+\alpha}(x,t)dx + b\int_\Omega |\nabla c(x,t)|^2 dx$ (see Lemma 6.2 of [50]) for some suitable positive constants a and b, where n and c are components of the solutions to (1.1) (see Lemma 6.2 of [32]). And therefore, one can further obtain the upper bound of the functional

$$\frac{1}{p}\int_\Omega n^p(x,t)dx + \frac{2}{q}\int_\Omega |\nabla c(x,t)|^{2q}dx \tag{1.10}$$

(see Lemma 7.1 of [32]). Consequently, one could derive the global existence and boundedness by a straightforward manner. When $\tau = 0$, although, we could find the $L^{2\alpha}$-estimate for n and L^1-estimate for c just like [32]. Unfortunately, the deficiency of regularity for c can not be used to estimate the term $u \cdot \nabla c$ and then conclude the higher estimates for n by using the second equation of (1.1). To overcome this difficulty, we use Lemma 3.4 and 3.1, which makes the regularity of c become less important in the process of energy estimates. Using this, one could conclude that there exists a positive constant ρ_* independent of C_S such that

$$\int_\Omega n^{1+\alpha}(x,t) \le \rho_*\Upsilon(C_S) \text{ for all } t > 0,$$

where

$$\Upsilon(C_S) = [(1 + 2\alpha C_S)^{\frac{1+\alpha}{\alpha}} + (1 + 2\alpha C_S) + (1 + 2\alpha C_S)^4 + 1], \tag{1.11}$$

so that, with the help of the Gagliardo–Nirenberg inequality and a well-known arguments from the elliptic regularity theory enables us to derive

$$\|\nabla c(\cdot,t)\|^2_{L^2(\Omega)} + \|c(\cdot,t)\|^2_{L^2(\Omega)} \le \rho_{**}\Upsilon^2(C_S) \text{ for all } t > 0$$

with some positive constant ρ_{**} independent of C_S. And thus, by virtue of the smoothing estimates of the Neumann heat semigroup and the standard elliptic regularity arguments, we can successively establish the boundeness of the solution to the system. We should point that the upper bounds $\rho_{**}\Upsilon^2(C_S)$ and $\rho_*\Upsilon(C_S)$ play a key role in obtaining the large time behavior of the solutions. Additionally, recent studies have shown that the solution behavior can be also impacted by the nonlinear diffusion (see [52]) and tensor-valued sensitivity (see [53]).

Throughout this paper, we assume that

$$\phi \in W^{2,\infty}(\Omega) \tag{1.12}$$

and the initial data (n_0, u_0) fulfills

$$\begin{cases} n_0 \in W^{1,\infty}(\Omega) \text{ with } n_0 > 0 \text{ in } \Omega, \\ u_0 \in D(A_r^\gamma) \text{ for some } \gamma \in (\frac{1}{2}, 1) \text{ and all } r \in (1, \infty) \end{cases} \tag{1.13}$$

with A_r denoting the Stokes operator with domain $D(A_r) := W^{2,r}(\Omega; \mathbb{R}^2) \cap W_0^{1,r}(\Omega; \mathbb{R}^2) \cap L_\sigma^r(\Omega)$, where $L_\sigma^r(\Omega) := \{\varphi \in L^r(\Omega; \mathbb{R}^2) | \nabla \cdot \varphi = 0\}$ for all $r \in (1, \infty)$ [26].

In the context of these assumptions, the first of our main results asserts global existence of a bounded solution in the following sense.

Theorem 1.1. *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded domain with smooth boundary, (1.12) and (1.13) hold. Moreover, suppose that S satisfies (1.2) and (1.3) with some*

$$\alpha > 0. \tag{1.14}$$

Then for any choice of n_0 and u_0 fulfilling (1.13), the problem (1.1) possesses a global classical solution (n, c, u, P) which satisfies

$$\begin{cases} n \in C^0(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty)), \\ c \in C^0(\bar{\Omega} \times [0, \infty)) \cap C^{2,0}(\bar{\Omega} \times (0, \infty)), \\ u \in C^0(\bar{\Omega} \times [0, \infty); \mathbb{R}^2) \cap C^{2,1}(\bar{\Omega} \times (0, \infty); \mathbb{R}^2), \\ P \in C^{1,0}(\bar{\Omega} \times (0, \infty)) \end{cases} \tag{1.15}$$

as well as n and c are nonnegative in $\Omega \times (0, \infty)$. Moreover, this solution is uniformly bounded in the sense that

$$\|n(\cdot, t)\|_{L^\infty(\Omega)} + \|c(\cdot, t)\|_{W^{1,\infty}(\Omega)} + \|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C \text{ for all } t > 0 \tag{1.16}$$

with some positive constant C independent of t.

Remark 1.1. (i) If $u \equiv 0$, Theorem 1.1 is coincides with Theorem 2.4 of [3], which is **optimal** according to the fact that the 2D fluid-free system admits a global bounded classical solution for $\alpha > 0$ as mentioned before.

(ii) One way to relax the restriction on α is to replace the linear diffusion Δn by the porous medium diffusion Δn^m with m suitably large. Using the idea and method of this paper, one can prove that system (1.1) with nonlinear diffusion (Δn is replaced by Δn^m) admits a global existence and boundedness of weak solutions for any $m > 1$, which is also optimal according to [35].

(iii) As far as we know, this seems to be the first rigorous mathematical result on a small-chemotaxis limit in a chemotaxis-fluid system, which connects the existence of solutions and large time asymptotic properties.

To the best of our knowledge, for parabolic-elliptic Keller–Segel–Stokes system, there is few rigorous mathematical results on large time behavior of the solutions. From this point of view, our results can be referred as an enrichment in this respect. In [44], Winkler proved that in the three-dimensional bounded **convex** domains, Keller–Segel–Navier–Stokes system with **logistic source** $\rho n - \mu n^2$ possesses at least one globally generalized solution, moreover, if $\mu > \frac{\chi\sqrt{\rho_+}}{4}$, then this solution converge to the homogeneous steady state with respect to the topology of $L^1(\Omega) \times L^p(\Omega) \times L^2(\Omega)$ for $p \in [1, 6)$. However, leaving open the question of whether the solution of Keller–Segel–Navier–Stokes system without **logistic source** stabilize or not. Motivated by the arguments in [44], we will also investigate asymptotic stability and convergence rates of model (1.1). In fact, by constructing a Lyapunov functional, we can moreover show that all the above solutions approach the spatially homogeneous equilibrium $(\frac{1}{|\Omega|}\int_\Omega n_0, \frac{1}{|\Omega|}\int_\Omega n_0, 0)$ provided that the condition C_S is appropriately small, where C_S is given by (1.3). Our main results with regard to stabilization can be formulated as follows.

Theorem 1.2. *Assume the hypothesis of Theorem 1.1 holds. Moreover, there exists $\chi_0 > 0$ with the property that if*

$$\frac{C_S}{C_N} < \chi_0,$$

one can find $\gamma > 0$ and $C > 0$ such that the global classical solution (n, c, u) of (1.1) satisfies

$$\|n(\cdot, t) - \bar{n}_0\|_{L^\infty(\Omega)} \leq Ce^{-\gamma t}, \text{ for all } t > 0 \tag{1.17}$$

as well as

$$\|c(\cdot, t) - \bar{n}_0\|_{L^\infty(\Omega)} \leq Ce^{-\gamma t}, \text{ for all } t > 0 \tag{1.18}$$

and

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq Ce^{-\gamma t}, \text{ for all } t > 0, \tag{1.19}$$

where $\bar{n}_0 = \frac{1}{|\Omega|}\int_\Omega n_0$ and C_N is the best Poincaré constant.

We sketch here the main ideas and methods used in this article. Our approach underlying the derivation of Theorem 1.1 will be based on an entropy-like estimate involving the functional

$$\int_\Omega n^{1+\alpha} \tag{1.20}$$

for solutions of (1.1) (see Lemma 3.3), which is a new estimate of parabolic-elliptic Keller–Segel–Stokes system. Once this crucial step has been accomplished, one can finally derive the global boundedness of solution to (1.1) by using the standard regularity theory of partial differential equations and the Stokes system (see Lemmas 3.4–3.6).

2. Preliminaries

Let us recall a result on local solvability of (1.1), which has been established in Lemma 2.2 of [45] (see also Bellomo et al. [1] and [37]) by means of a suitable extensibility criterion and a slight modification of the well-established fixed-point arguments (see Lemma 2.1 of [40], [39] and Lemma 2.1 of [24]).

Lemma 2.1. *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded domain with smooth boundary. Assume that $\phi \in W^{2,\infty}(\Omega)$ and the initial data (n_0, u_0) fulfills (1.13). Moreover, let $S \in C^2(\bar{\Omega} \times [0, \infty))$ satisfy (1.3) for some $C_S \geq 0$ and $\alpha \geq 0$. Then there exist $T_{max} \in (0, \infty]$ and a classical solution (n, c, u, P) of (1.1) in $\Omega \times (0, T_{max})$ such that*

$$\begin{cases} n \in C^0(\bar{\Omega} \times [0, T_{max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max})), \\ c \in C^0(\bar{\Omega} \times [0, T_{max})) \cap C^{2,0}(\bar{\Omega} \times (0, T_{max})), \\ u \in C^0(\bar{\Omega} \times [0, T_{max}); \mathbb{R}^2) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max}); \mathbb{R}^2), \\ P \in C^{1,0}(\bar{\Omega} \times (0, T_{max})) \end{cases} \tag{2.1}$$

classically solving (1.1) in $\Omega \times [0, T_{max})$. Moreover, n and c are nonnegative in $\Omega \times (0, T_{max})$, and

$$\text{if } T_{max} < +\infty, \text{ then } \limsup_{t \nearrow T_{max}} [\|n(\cdot, t)\|_{L^\infty(\Omega)} + \|c(\cdot, t)\|_{W^{1,\infty}(\Omega)} + \|u(\cdot, t)\|_{L^\infty(\Omega)}] = \infty. \tag{2.2}$$

Lemma 2.2. ([13]) *If $m \in \{0, 1\}, p \in [1, \infty]$ and $q \in (1, \infty)$ then with some constant $c_1 > 0$, for all $\varphi \in D((-\Delta + 1)^\theta)$ we have*

$$\|\varphi\|_{W^{m,p}(\Omega)} \leq c_1\|(-\Delta + 1)^\theta \varphi\|_{L^q(\Omega)} \text{ provided that } m - \frac{N}{p} < 2\theta - \frac{N}{q}.$$

Moreover, for $p < \infty$ the associated heat semigroup$(e^{t\Delta})_{t \geq 0}$ maps $L^p(\Omega)$ into $D((-\Delta + 1)^\theta)$ in any of the spaces $L^q(\Omega)$ for $q \geq p$, and there exist $c > 0$ and $\lambda > 0$ such that the L^p-L^q estimates

$$\|(-\Delta + 1)^\theta e^{t(\Delta - 1)}\varphi\|_{L^q(\Omega)} \leq c(1 + t^{-\theta - \frac{N}{2}(\frac{1}{p} - \frac{1}{q})})e^{-\lambda t}\|\varphi\|_{L^p(\Omega)} \text{ for all } \varphi \in L^p(\Omega) \text{ and } t > 0$$

and

$$\|(-\Delta+1)^{\theta}e^{t\Delta}\varphi\|_{L^q(\Omega)}$$
$$\le c(1+t^{-\theta-\frac{N}{2}(\frac{1}{p}-\frac{1}{q})})e^{-\lambda t}\|\varphi\|_{L^p(\Omega)} \text{ for all } t>0 \text{ and } \varphi\in L^p(\Omega) \text{ satisfying } \int_\Omega w=0.$$

In addition, given $p\in(1,\infty)$, for any $\varepsilon>0$, there exists $c(\varepsilon)>0$ such that for all $\mathbb{R}^N$-valued $\varphi\in L^p(\Omega)$,

$$\|(-\Delta+1)^{\theta}e^{t\Delta}\nabla\cdot\varphi\|_{L^p(\Omega)} \le c(\varepsilon)(1+t^{-\theta-\frac{1}{2}-\varepsilon})e^{-\lambda t}\|\varphi\|_{L^p(\Omega)} \text{ for all } t>0.$$

Next, we plan to deduce some L^p bound for Du with p suitably large from the low integrability of n, which plays an important role in the arguments of establishing the desired estimates of (1.1). To this end, we first give some notations, which will be used throughout this paper. For any $r\in(1,\infty)$, the Helmholtz projection acts as a bounded linear operator $\mathcal{P}_r$ from $L^r(\Omega)$ onto its subspace $L^r_\sigma(\Omega):=\{\varphi\in L^r(\Omega)|\nabla\cdot\varphi=0\}$ (see also our hypothesis (1.13) on Sect. 1) of all solenoidal vector fields. Furthermore, the realization A_r of the Stokes operator A in $L^r_\sigma(\Omega)$ with domain $D(A_r):=W^{2,r}(\Omega;\mathbb{R}^2)\cap W^{1,r}_0(\Omega;\mathbb{R}^2)\cap L^r_\sigma(\Omega)$ is sectorial in $L^r_\sigma(\Omega)$. And therefore, for each $\gamma\in\mathbb{R}$, A_r possesses closed fractional powers A_r^γ with dense domains $D(A_r^\gamma)$ (see e.g. [9,10]), and A_r generates an analytic semigroup $(e^{-tA_r})_{t\ge0}$ in $L^r_\sigma(\Omega)$. Due to $\mathcal{P}_r$, A_r^γ and $(e^{-tA_r})_{t\ge0}$ are all actually independent of $r\in(1,\infty)$ whenever applied to smooth functions, so that, in the following, we may omit an explicit index r whenever there is no danger of confusion.

Lemma 2.3. *Let (n,c,u) be the solution of (2.1). Moreover, assume that $p\in[1,\infty)$ and $r\in[1,\infty]$ satisfies that*

$$\begin{cases} r<\frac{2p}{2-p} & \text{if } p\le 2,\\ r\le\infty & \text{if } p>2.\end{cases} \tag{2.3}$$

If there exists $K>0$ such that

$$\|n(\cdot,t)\|_{L^p(\Omega)}\le K \text{ for all } t\in(0,T_{max}), \tag{2.4}$$

then there exists $C:=C(p,r,K)$ such that

$$\|Du(\cdot,t)\|_{L^r(\Omega)}\le C \text{ for all } t\in(0,T_{max}). \tag{2.5}$$

Proof. Without loss of generality, we may assume that $r>p$, since $r\le p$, can be proved similarly and easily. Next, let $\beta>\frac{1}{2}$. Next, we fix $r_0\in(p,r)$ and

$$\beta>\frac{1}{2}+\frac{1}{r_0}-\frac{1}{r}. \tag{2.6}$$

Notice that

$$\left(\frac{1}{2}+\frac{1}{r_0}-\frac{1}{r}\right)-\left\{1-\frac{1}{p}+\frac{1}{r_0}\right\}<0 \tag{2.7}$$

by using (2.3). Hence, we can choose $\beta_0\in(\frac{1}{2},\beta)$ and $\delta\in(0,1)$ satisfying

$$\frac{1}{2}+\frac{1}{r_0}-\frac{1}{r}<\beta_0<\left\{1-\frac{1}{p}+\frac{1}{r_0}\right\} \tag{2.8}$$

and

$$\beta_0+\delta<1-\left(\frac{1}{p}-\frac{1}{r_0}\right). \tag{2.9}$$

Now, we fix some $p_0>p$ sufficiently close to p such that

$$\delta>\frac{1}{p}-\frac{1}{p_0}. \tag{2.10}$$

We apply the fractional power A^γ to the variation-of-constant formula

$$u(\cdot,t)=e^{-tA}u_0+\int_0^t e^{-(t-\tau)A}\mathcal{P}(n(\cdot,\tau)\nabla\phi)d\tau \text{ for all } t\in(0,T_{max}), \tag{2.11}$$

where $A := A_{r_0}$ and $\mathcal{P} = \mathcal{P}_{r_0}$. Now, employing A^{β_0} on both sides of (2.11) and using (1.13) and (1.12), we derive that there exist positive constants $\lambda > 0, \kappa_{0,*}, \kappa_{0,**}$ as well as $\kappa_{0,***}$ and $\kappa_{0,****}$ such that

$$\begin{aligned}
&\|Du(\cdot,t)\|_{L^r(\Omega)}\\
&\quad\le \kappa_{0,*}\|A^{\beta_0}u(\cdot,t)\|_{L^{r_0}(\Omega)}\\
&\quad\le \kappa_{0,**}\|A^{\beta_0}e^{-tA}u_0\|_{L^{r_0}(\Omega)} + \kappa_{0,**}\int_0^t \|A^{\beta_0+\delta}e^{-(t-\tau)A}A^{-\delta}\mathcal{P}[n(\cdot,\tau)\nabla\phi]\|_{L^{r_0}(\Omega)}d\tau\\
&\quad= \kappa_{0,**}\|e^{-tA}A^{\beta_0}u_0\|_{L^{r_0}(\Omega)} + \kappa_{0,**}\int_0^t \|A^{\beta_0+\delta}e^{-(t-\tau)A}A^{-\delta}\mathcal{P}[n(\cdot,\tau)\nabla\phi]\|_{L^{r_0}(\Omega)}d\tau\\
&\quad\le \kappa_{0,**}\|A^{\gamma}e^{-tA}u_0\|_{L^{r_0}(\Omega)} + \kappa_{0,***}\int_0^t (t-\tau)^{-\beta_0-\delta-(\frac{1}{p_0}-\frac{1}{r_0})}e^{-\lambda(t-\tau)}\|A^{-\delta}\mathcal{P}[n(\cdot,\tau)\nabla\phi]\|_{L^{p_0}(\Omega)}d\tau\\
&\quad\le \kappa_{0,****} + \kappa_{0,****}\int_0^t (t-\tau)^{-\beta_0-\delta-(\frac{1}{p_0}-\frac{1}{r_0})}e^{-\lambda(t-\tau)}\|n(\cdot,\tau)\|_{L^p(\Omega)}d\tau
\end{aligned} \tag{2.12}$$

for all $t \in (0, T_{max})$. Noticing that (2.9) entails that

$$\int_0^\infty \tau^{-\beta_0-\delta-(\frac{1}{p_0}-\frac{1}{r_0})}e^{-\lambda\tau}d\tau < +\infty, \tag{2.13}$$

so that, from (2.12) and (2.4) we can deduce the desired result. □

3. A Priori Estimates

In this section, in order to establish the global solvability of system (1.1), we proceed to derive some estimates for the solutions. As the first step, we need to establish some important a priori estimates for n, c and u, where throughout this paper, (n, c, u, P) is the global solution of problem (1.1).

The following two basic properties immediately result from an integration of the first and second equation in (1.1) over Ω.

Lemma 3.1. *The solution of (1.1) satisfies*

$$\int_\Omega n = \int_\Omega n_0 \text{ for all } t > 0 \tag{3.1}$$

as well as

$$\int_\Omega c = \int_\Omega n_0 \text{ for all } t > 0. \tag{3.2}$$

Proof. On integrating the first and the second equation in (1.1) over Ω and using that $\nabla \cdot u = 0$, we obtain the identities

$$\frac{d}{dt}\int_\Omega n = 0 \text{ and } \int_\Omega n = \int_\Omega c \text{ for all } t > 0,$$

which directly imply both (3.1) and (3.2). □

With the L^1 estimate of n in hand (see Lemma 3.1), one can invoke the L^p-L^q estimates for the Neumann heat semigroup to obtain regularity of u in arbitrary L^p spaces, which is presented below for the sake of completeness and easy reference (see also [33]).

Lemma 3.2. *Suppose that (1.2)–(1.3) and (1.12)–(1.13) hold. For any given $p \in (1, \infty)$, there exists a positive constant θ_* which depends only on p, Ω, u_0 as well as n_0 and ϕ such that*

$$\int_\Omega |u(x,t)|^p \le \theta_* \text{ for all } t > 0. \tag{3.3}$$

Proof. First of all, in view of the variation-of-constants representation

$$u(\cdot,t) = e^{-tA}u_0 + \int_0^t e^{-(t-\tau)A}\mathcal{P}(n(\cdot,\tau)\nabla\phi)d\tau \text{ for all } t \in (0, T_{max}),$$

where A and $\mathcal{P}$ are the same as Lemma 2.3. For any $p > 1$, one can fix β such that $\beta \in (1 - \frac{1}{p}, 1)$. Now, relying on (1.12) as well as (1.13) and (3.1), we my employ the L^p-L^q estimates for the Neumann heat semigroup to find $\lambda_* > 0, l_{0,*} > 0, l_{0,**} > 0$ as well as $l_{0,***} > 0$ and $l_{0,****} > 0$ with the property that for any for all $t \in (0, T_{max})$

$$\begin{aligned}
\|u(\cdot,t)\|_{L^p(\Omega)} &\leq l_{0,*}\|e^{-tA}u_0\|_{L^p(\Omega)} + l_{0,*}\int_0^t \|A^\beta e^{-(t-\tau)A}A^{-\beta}\mathcal{P}[n(\cdot,\tau)\nabla\phi]\|_{L^p(\Omega)}d\tau \\
&\leq l_{0,**} + l_{0,**}\int_0^t (t-\tau)^{-\beta}e^{-\lambda_*(t-\tau)}\|A^{-\beta}\mathcal{P}[n(\cdot,\tau)\nabla\phi]\|_{L^p(\Omega)}d\tau \\
&\leq l_{0,**} + l_{0,***}\int_0^t (t-\tau)^{-\beta}e^{-\lambda_*(t-\tau)}\|n(\cdot,\tau)\nabla\phi\|_{L^1(\Omega)}d\tau \\
&\leq l_{0,****},
\end{aligned} \tag{3.4}$$

and thereby precisely arrive at (3.3). □

With the estimates obtained so far, we have already prepared all tools to obtain an $L^p(\Omega)$-estimate for n, for some $p > 1$, which plays a key role in obtaining the $L^\infty(\Omega)$-estimate for n.

Lemma 3.3. *If*

$$\alpha > 0, \tag{3.5}$$

then there exists a positive constant ρ_* independent of C_S such that the solution of (1.1) from Lemma 2.1 satisfies

$$\int_\Omega n^{1+\alpha}(x,t) \leq \rho_*\Upsilon(C_S) \text{ for all } t \in (0, T_{max}), \tag{3.6}$$

where $\Upsilon(C_S)$ is the same as (1.11).

Proof. In the following, we let κ_i, γ_i and $\rho_i (i \in \mathbb{N})$ denote some different constants, which are independent of t and C_S, and if no special explanation, they depend at most on $\Omega, \phi, \alpha, n_0$ and u_0. Taking n^α as the test function for the first equation of (1.1) and using $\nabla \cdot u = 0$, we derive that

$$\begin{aligned}
\frac{1}{1+\alpha}\frac{d}{dt}\|n\|^{1+\alpha}_{L^{1+\alpha}(\Omega)} + \alpha\int_\Omega n^{\alpha-1}|\nabla n|^2 &= -\int_\Omega n^\alpha \nabla\cdot(nS(n)\nabla c) \\
&= \alpha\int_\Omega n^\alpha S(n)\nabla n\cdot\nabla c \\
&= \alpha\int_\Omega \nabla\int_0^n \tau^\alpha S(\tau)d\tau\cdot\nabla c \\
&= -\alpha\int_\Omega\int_0^n \tau^\alpha S(\tau)d\tau\Delta c \\
&= \alpha\int_\Omega\int_0^n \tau^\alpha S(\tau)d\tau(n - c - u\cdot\nabla c) \\
&\leq \alpha C_S\int_\Omega (n^2 + n|u\cdot\nabla c|) \text{ for all } t \in (0, T_{max})
\end{aligned} \tag{3.7}$$

by using (1.3) and the second equation of (1.1). In the following, we will estimate the right-hand side of (3.7). To this end, in view of (3.2), the Gagliardo–Nirenberg inequality entails that there exists $C_1 > 0$ such that

$$\begin{aligned}\int_\Omega |\nabla c|^{\frac{8}{3}} &\le \rho_0 \left(\|\Delta c\|^2_{L^2(\Omega)}\|c\|^{\frac{2}{3}}_{L^1(\Omega)} + \|c\|^{\frac{8}{3}}_{L^1(\Omega)}\right)\\ &\le \rho_1\|\Delta c\|^2_{L^2(\Omega)} + \rho_0\|c\|^{\frac{8}{3}}_{L^1(\Omega)}\\ &= \rho_1\|\Delta c\|^2_{L^2(\Omega)} + \rho_0\|n_0\|^{\frac{8}{3}}_{L^1(\Omega)} \text{ for all } t\in(0,T_{max})\end{aligned} \tag{3.8}$$

with

$$\rho_1 = \rho_0\left(\int_\Omega n_0\right)^{\frac{2}{3}}.$$

Now, by using the Young inequality twice, abbreviating $\kappa_1 = \frac{27\rho_1^3\alpha}{512}$ we thereby deduce that

$$\begin{aligned}\alpha C_S\int_\Omega (n^2 - n|u\cdot\nabla c|) &\le 2\alpha C_S\int_\Omega\left(n^2 + \frac{1}{4}|u|^2|\nabla c|^2\right)\\ &\le 2\alpha C_S\int_\Omega\left(n^2 + \frac{1}{1024}(\varepsilon_1\times\frac{4}{3})^{-3}|u|^8 + \varepsilon_1|\nabla c|^{\frac{8}{3}}\right)\\ &= 2\alpha C_S\int_\Omega\left(n^2 + \varepsilon_1|\nabla c|^{\frac{8}{3}}\right) + \kappa_1 C_S(1+2\alpha C_S)^3\int_\Omega |u|^8 \text{ for all } t\in(0,T_{max}),\end{aligned} \tag{3.9}$$

where

$$\varepsilon_1 = \frac{1}{4\rho_1(1+2\alpha C_S)}. \tag{3.10}$$

To estimate the term Δc in (3.8), taking $-\Delta c$ as the test function for the second equation of (1.1), we conclude from the Young inequality that for the above $\varepsilon_1 > 0$,

$$\begin{aligned}\int_\Omega |\Delta c|^2 + \int_\Omega |\nabla c|^2 &= -\int_\Omega n\Delta c + \int_\Omega (u\cdot\nabla c)\Delta c\\ &\le \frac{1}{2}\int_\Omega |\Delta c|^2 + \int_\Omega n^2 + \int_\Omega |u|^2|\nabla c|^2\\ &\le \frac{1}{2}\int_\Omega |\Delta c|^2 + \int_\Omega n^2 + \frac{1}{4}\left(\varepsilon_1\times\frac{4}{3}\right)^{-3}\int_\Omega |u|^8 + \varepsilon_1\int_\Omega |\nabla c|^{\frac{8}{3}}\\ &= \frac{1}{2}\int_\Omega |\Delta c|^2 + \int_\Omega n^2 + \frac{27}{4}\rho_1^3(1+2\alpha C_S)^3\int_\Omega |u|^8 + \varepsilon_1\int_\Omega |\nabla c|^{\frac{8}{3}} \text{ for all } t\in(0,T_{max}),\end{aligned}$$

which implies that

$$\frac{1}{2}\int_\Omega |\Delta c|^2 + \int_\Omega |\nabla c|^2 \le \int_\Omega n^2 + \varepsilon_1\int_\Omega |\nabla c|^{\frac{8}{3}} + \kappa_2(1+2\alpha C_S)^3\int_\Omega |u|^8 \text{ for all } t\in(0,T_{max}) \tag{3.11}$$

with

$$\kappa_2 = \frac{27}{4}\rho_1^3.$$

Combining (3.8) with (3.10), we thus infer that

$$\begin{aligned}(1+2\alpha C_S)\varepsilon_1\int_\Omega |\nabla c|^{\frac{8}{3}} =& \frac{1+2\alpha C_S}{4\rho_1(1+2\alpha C_S)}\int_\Omega |\nabla c|^{\frac{8}{3}}\\ \le& \frac{1}{4}\|\Delta c\|^2_{L^2(\Omega)} + \kappa_3\|n_0\|^{\frac{8}{3}}_{L^1(\Omega)} \text{ for all } t\in(0,T_{max}),\end{aligned} \tag{3.12}$$

where

$$\kappa_3 = \frac{\rho_0}{4\rho_1}.$$

Recalling $\alpha > 0$, we can see that $\frac{2}{1+\alpha} < \frac{4}{1+\alpha} < +\infty$, which warrants the embedding $W^{1,2}(\Omega) \hookrightarrow L^{\frac{4}{1+\alpha}}(\Omega) \hookrightarrow L^{\frac{2}{1+\alpha}}(\Omega)$, whence another application of the Gagliardo–Nirenberg inequality together with Lemma 3.2 and the Young inequality that provides positive constant γ_0 such that

$$\begin{aligned}
(1+2\alpha C_S)\int_\Omega n^2 &= (1+2\alpha C_S)\|n^{\frac{1+\alpha}{2}}\|^{\frac{4}{1+\alpha}}_{L^{\frac{4}{1+\alpha}}(\Omega)} \\
&\le (1+2\alpha C_S)\gamma_0(\|\nabla n^{\frac{1+\alpha}{2}}\|^{\frac{2}{1+\alpha}}_{L^2(\Omega)}\|n^{\frac{1+\alpha}{2}}\|^{\frac{2}{1+\alpha}}_{L^{\frac{2}{1+\alpha}}(\Omega)} + \|n^{\frac{1+\alpha}{2}}\|^{\frac{4}{1+\alpha}}_{L^{\frac{2}{1+\alpha}}(\Omega)}) \\
&= \gamma_0(1+2\alpha C_S)(\|\nabla n^{\frac{1+\alpha}{2}}\|^{\frac{2}{1+\alpha}}_{L^2(\Omega)}\|n_0\|_{L^1(\Omega)} + \|n_0\|^2_{L^1(\Omega)}) \\
&\le \varepsilon_2\|\nabla n^{\frac{1+\alpha}{2}}\|^2_{L^2(\Omega)} + \frac{\alpha}{1+\alpha}(\varepsilon_2(1+\alpha))^{-\frac{1}{\alpha}}\gamma_0^{\frac{1+\alpha}{\alpha}}(1+2\alpha C_S)^{\frac{1+\alpha}{\alpha}}\|n_0\|^{\frac{1+\alpha}{\alpha}}_{L^1(\Omega)} \\
&\quad + \gamma_0(1+2\alpha C_S)\|n_0\|^2_{L^1(\Omega)} \\
&= \frac{\alpha}{2}\int_\Omega n^{\alpha-1}|\nabla n|^2 + \kappa_4(1+2\alpha C_S)^{\frac{1+\alpha}{\alpha}}\|n_0\|^{\frac{1+\alpha}{\alpha}}_{L^1(\Omega)} \\
&\quad + \gamma_0(1+2\alpha C_S)\|n_0\|^2_{L^1(\Omega)} \text{ for all } t \in (0, T_{max})
\end{aligned} \tag{3.13}$$

with

$$\varepsilon_2 = \frac{2\alpha}{(1+\alpha)^2}$$

and

$$\kappa_4 = \frac{\alpha}{1+\alpha}(\varepsilon_2(1+\alpha))^{-\frac{1}{\alpha}}\gamma_0^{\frac{1+\alpha}{\alpha}} = \frac{\alpha}{1+\alpha}\left(\frac{2\alpha}{1+\alpha}\right)^{-\frac{1}{\alpha}}\gamma_0^{\frac{1+\alpha}{\alpha}}.$$

Collecting (3.7) and (3.11)–(3.13), we achieve

$$\begin{aligned}
&\frac{1}{1+\alpha}\frac{d}{dt}\|n\|^{1+\alpha}_{L^{1+\alpha}(\Omega)} + \alpha\int_\Omega n^{\alpha-1}|\nabla n|^2 + \frac{1}{2}\int_\Omega|\Delta c|^2 + \int_\Omega|\nabla c|^2 \\
&\le (1+2\alpha C_S)\int_\Omega n^2 + (1+2\alpha C_S)\varepsilon_1\int_\Omega|\nabla c|^{\frac{8}{3}} + [\kappa_1 C_S + \kappa_2](1+2\alpha C_S)^3\int_\Omega|u|^8 \\
&\le \frac{\alpha}{2}\int_\Omega n^{\alpha-1}|\nabla n|^2 + \kappa_4(1+2\alpha C_S)^{\frac{1+\alpha}{\alpha}}\|n_0\|^{\frac{1+\alpha}{\alpha}}_{L^1(\Omega)} + \gamma_0(1+2\alpha C_S)\|n_0\|^2_{L^1(\Omega)} \\
&\quad + \frac{1}{4}\|\Delta c\|^2_{L^2(\Omega)} + \kappa_3\|n_0\|^{\frac{8}{3}}_{L^1(\Omega)} + [\kappa_1 C_S + \kappa_2](1+2\alpha C_S)^3\int_\Omega|u|^8 \text{ for all } t \in (0, T_{max}).
\end{aligned} \tag{3.14}$$

Therefore, namely,

$$\begin{aligned}
&\frac{1}{1+\alpha}\frac{d}{dt}\|n\|^{1+\alpha}_{L^{1+\alpha}(\Omega)} + \frac{\alpha}{2}\int_\Omega n^{\alpha-1}|\nabla n|^2 + \frac{1}{4}\int_\Omega|\Delta c|^2 + \int_\Omega|\nabla c|^2 \\
&\le \kappa_4(1+2\alpha C_S)^{\frac{1+\alpha}{\alpha}}\|n_0\|^{\frac{1+\alpha}{\alpha}}_{L^1(\Omega)} + \gamma_0(1+2\alpha C_S)\|n_0\|^2_{L^1(\Omega)} \\
&\quad + \kappa_3\|n_0\|^{\frac{8}{3}}_{L^1(\Omega)} + [\kappa_1 C_S + \kappa_2](1+2\alpha C_S)^3\int_\Omega|u|^8 \text{ for all } t \in (0, T_{max}).
\end{aligned} \tag{3.15}$$

Next, in view of (3.1), we invoke the Gagliardo–Nirenberg inequality to fix $\gamma_1 > 0$ such that

$$\begin{aligned}
&\|n\|_{L^{1+\alpha}(\Omega)}^{1+\alpha}\\
&= \|n^{\frac{1+\alpha}{2}}\|_{L^2(\Omega)}^2\\
&\le \gamma_1(\|\nabla n^{\frac{1+\alpha}{2}}\|_{L^2(\Omega)}^{\frac{2\alpha}{1+\alpha}}\|n^{\frac{1+\alpha}{2}}\|_{L^{\frac{2}{1+\alpha}}(\Omega)}^{\frac{2}{1+\alpha}} + \|n^{\frac{1+\alpha}{2}}\|_{L^{\frac{2}{1+\alpha}}(\Omega)}^2)\\
&= \gamma_1(\|\nabla n^{\frac{1+\alpha}{2}}\|_{L^2(\Omega)}^{\frac{2\alpha}{1+\alpha}}\|n_0\|_{L^1(\Omega)} + \|n_0\|_{L^1(\Omega)}^2) \text{ for all } t \in (0, T_{max}).
\end{aligned}$$

Upon an application of the Young inequality, the above inequality therefore entails that

$$\begin{aligned}
\|n\|_{L^{1+\alpha}(\Omega)}^{1+\alpha} &\le \|\nabla n^{\frac{1+\alpha}{2}}\|_{L^2(\Omega)}^2 + \frac{1}{1+\alpha}\left(\frac{1+\alpha}{\alpha}\right)^{-\alpha}\|n_0\|_{L^1(\Omega)}^{1+\alpha}\gamma_1^{1+\alpha} + \gamma_1\|n_0\|_{L^1(\Omega)}^2\\
&\le \frac{(1+\alpha)^2}{4}\int_\Omega n^{\alpha-1}|\nabla n|^2\\
&\quad + \frac{1}{1+\alpha}\left(\frac{1+\alpha}{\alpha}\right)^{-\alpha}\|n_0\|_{L^1(\Omega)}^{1+\alpha}\gamma_1^{1+\alpha} + \gamma_1\|n_0\|_{L^1(\Omega)}^2 \text{ for all } t \in (0, T_{max}).
\end{aligned} \tag{3.16}$$

Plugging (3.16) into (3.15) and combining with (3.3) therefore implies that

$$\begin{aligned}
&\frac{1}{1+\alpha}\frac{d}{dt}\|n\|_{L^{1+\alpha}(\Omega)}^{1+\alpha} + \frac{\alpha}{2}\times\frac{4}{(1+\alpha)^2}\int_\Omega n^{1+\alpha} + \frac{1}{4}\int_\Omega|\Delta c|^2 + \int_\Omega|\nabla c|^2\\
&\le \kappa_4(1+2\alpha C_S)^{\frac{1+\alpha}{\alpha}}\|n_0\|_{L^1(\Omega)}^{\frac{1+\alpha}{\alpha}} + \gamma_0(1+2\alpha C_S)\|n_0\|_{L^1(\Omega)}^2\\
&\quad + \kappa_3\|n_0\|_{L^1(\Omega)}^{\frac{8}{3}} + [\kappa_1 C_S + \kappa_2](1+2\alpha C_S)^3\int_\Omega|u|^8\\
&\quad + \frac{2}{\alpha}\times\frac{4}{(1+\alpha)^2}[\frac{1}{1+\alpha}\left(\frac{1+\alpha}{\alpha}\right)^{-\alpha}\|n_0\|_{L^1(\Omega)}^{1+\alpha}\gamma_1^{1+\alpha} + \gamma_1\|n_0\|_{L^1(\Omega)}^2]\\
&\le \kappa_4(1+2\alpha C_S)^{\frac{1+\alpha}{\alpha}}\|n_0\|_{L^1(\Omega)}^{\frac{1+\alpha}{\alpha}} + \gamma_0(1+2\alpha C_S)\|n_0\|_{L^1(\Omega)}^2\\
&\quad + \kappa_3\|n_0\|_{L^1(\Omega)}^{\frac{8}{3}} + (\frac{\kappa_1}{2\alpha} + \kappa_2)(1+2\alpha C_S)^4\sup_{t\in(0,T_{max})}\int_\Omega|u(x,t)|^8\\
&\quad + \frac{2}{\alpha}\times\frac{4}{(1+\alpha)^2}\left[\frac{1}{1+\alpha}\left(\frac{1+\alpha}{\alpha}\right)^{-\alpha}\|n_0\|_{L^1(\Omega)}^{1+\alpha}\gamma_1^{1+\alpha} + \gamma_1\|n_0\|_{L^1(\Omega)}^2\right]\\
&\le \kappa_5[(1+2\alpha C_S)^{\frac{1+\alpha}{\alpha}} + (1+2\alpha C_S) + (1+2\alpha C_S)^4] + \kappa_6 \text{ for all } t \in (0, T_{max}),
\end{aligned} \tag{3.17}$$

where

$$\kappa_5 = \max\{\kappa_4\|n_0\|_{L^1(\Omega)}^{\frac{1+\alpha}{\alpha}}, \gamma_0\|n_0\|_{L^1(\Omega)}^2, \left(\frac{\kappa_1}{2\alpha} + \kappa_2\right)\theta_*\}$$

as well as

$$\kappa_6 = \kappa_3\|n_0\|_{L^1(\Omega)}^{\frac{8}{3}} + \frac{2}{\alpha}\times\frac{4}{(1+\alpha)^2}\left[\frac{1}{1+\alpha}\left(\frac{1+\alpha}{\alpha}\right)^{-\alpha}\|n_0\|_{L^1(\Omega)}^{1+\alpha}\gamma_1^{1+\alpha} + \gamma_1\|n_0\|_{L^1(\Omega)}^2\right]$$

and θ_* is the same as (3.3). Moreover, writing

$$y(t) = \|n(\cdot,t)\|_{L^{1+\alpha}(\Omega)}^{1+\alpha} \text{ for all } t \in (0, T_{max}),$$

whereas (3.17) guarantees that

$$\begin{aligned} & y'(\cdot,t)+\frac{2\alpha}{1+\alpha}y(\cdot,t) \\ & \quad \le (1+\alpha)[\kappa_5[(1+2\alpha C_S)^{\frac{1+\alpha}{\alpha}}+(1+2\alpha C_S)+(1+2\alpha C_S)^4]+\kappa_6] \text{ for all } t\in(0,T_{max}) \end{aligned} \tag{3.18}$$

by dropping the non-negative terms $\frac{1}{4}\int_\Omega|\Delta c|^2$ and $\displaystyle\int_\Omega|\nabla c|^2$. Applying the Gronwall Lemma, (3.18) entails that

$$\begin{aligned} y(\cdot,t)\le{}& e^{-\frac{2\alpha}{1+\alpha}t}y_0+\frac{(1+\alpha)^2}{2\alpha}\left(1-e^{-\frac{2\alpha}{1+\alpha}t}\right)\kappa_5[(1+2\alpha C_S)^{\frac{1+\alpha}{\alpha}}+(1+2\alpha C_S)+(1+2\alpha C_S)^4] \\ & +\frac{(1+\alpha)^2}{2\alpha}\left(1-e^{-\frac{2\alpha}{1+\alpha}t}\right)\kappa_6 \text{ for all } t\in(0,T_{max}), \end{aligned} \tag{3.19}$$

where invoking the fact that $t>0$, we deduce from (3.19) that

$$y(\cdot,t)\le y_0+\frac{(1+\alpha)^2}{2\alpha}\{\kappa_5[(1+2\alpha C_S)^{\frac{1+\alpha}{\alpha}}+(1+2\alpha C_S)+(1+2\alpha C_S)^4]+\kappa_6\} \tag{3.20}$$

for all $t\in(0,T_{max})$. As a consequence of (3.20), (3.6) is valid by a choice of $\rho_*=\max\{y_0+\kappa_6,\frac{(1+\alpha)^2}{2\alpha}\kappa_5\}$.

And the proof of this Lemma is thus completed. □

With the higher regularity for n obtained in Lemma 3.3, one can give some estimates for c which can be proved by using the Gagliardo–Nirenberg inequality and an application of well-known arguments from the elliptic regularity theory.

Lemma 3.4. *If $\alpha>0$, then there exists a positive constant ρ_{**} independent of C_S such that the solution of (1.1) from Lemma 2.1 satisfies*

$$\|\nabla c(\cdot,t)\|^2_{L^2(\Omega)}+\|c(\cdot,t)\|^2_{L^2(\Omega)}\le\rho_{**}\Upsilon^2(C_S) \text{ for all } t\in(0,T_{max}), \tag{3.21}$$

where $\Upsilon(C_S)$ is given by (1.11).

Proof. Testing the second equation of (1.1) by c and combining with the second equation and using $\nabla\cdot u=0$, we have, using the integration by parts, that

$$\begin{aligned} \int_\Omega|\nabla c|^2+\int_\Omega c^2&=\int_\Omega cn \\ &\le\|n\|_{L^{1+\alpha}(\Omega)}\|c\|_{L^{\frac{1+\alpha}{\alpha}}(\Omega)} \\ &\le\rho_*^{\frac{1}{1+\alpha}}\Upsilon^{\frac{1}{1+\alpha}}(C_S)\|c\|_{L^{\frac{1+\alpha}{\alpha}}(\Omega)} \text{ for all } t\in(0,T_{max}) \end{aligned} \tag{3.22}$$

by using the Hölder inequality and Lemma 3.3. Here ρ_* and $\Upsilon(C_S)$ are the same as Lemma 3.3. Now combining the Gagliardo–Nirenberg inequality with the fact that $\int_\Omega c=\int_\Omega n_0$ by (3.2), we can furthermore find $\mu_0>0$ independent of C_S such that

$$\begin{aligned} \|c\|_{L^{\frac{1+\alpha}{\alpha}}(\Omega)}\le{}&\mu_0\left(\|\nabla c\|^{\frac{1}{1+\alpha}}_{L^2(\Omega)}\|c\|^{\frac{\alpha}{1+\alpha}}_{L^1(\Omega)}+\|c\|_{L^1(\Omega)}\right) \\ ={}&\mu_0\left(\|\nabla c\|^{\frac{1}{1+\alpha}}_{L^2(\Omega)}\|n_0\|^{\frac{\alpha}{1+\alpha}}_{L^1(\Omega)}+\|n_0\|_{L^1(\Omega)}\right) \text{ for all } t\in(0,T_{max}), \end{aligned} \tag{3.23}$$

where in the last inequality we have used (3.2). Together with (3.22), an application of the Young inequality yields

$$\begin{aligned} \frac{1}{2}\int_\Omega|\nabla c|^2+\int_\Omega c^2\le{}&\frac{1+2\alpha}{2+2\alpha}(1+\alpha)^{-\frac{1}{1+2\alpha}}\rho_*^{\frac{2}{1+2\alpha}}\Upsilon^{\frac{2}{1+2\alpha}}(C_S)\|n_0\|_{L^1(\Omega)} \\ &+\mu_0\|n_0\|_{L^1(\Omega)}\rho_*^{\frac{1}{1+\alpha}}\Upsilon^{\frac{1}{1+\alpha}}(C_S) \text{ for all } t\in(0,T_{max}), \end{aligned} \tag{3.24}$$

which together with $\frac{2}{1+2\alpha}<2$ implies that (3.21) is valid and thereby completes the proof. □

With Lemma 3.3 at hand, one can improve the estimates of u by virtue of the well-established arguments based on the regularization features of the Stokes semigroup.

Lemma 3.5. *Let $\alpha > 0$ and $\gamma \in (\frac{1}{2}, 1)$. Then one can find a positive constant ρ_{***} independent of C_S such that the solution of* (1.1) *from Lemma* 2.1 *satisfies*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq \rho_{***}\Upsilon(C_S) \text{ for all } t \in (0, T_{max}) \tag{3.25}$$

and

$$\|A^\gamma u(\cdot,t)\|_{L^2(\Omega)} \leq \rho_{***}\Upsilon(C_S) \text{ for all } t \in (0, T_{max}), \tag{3.26}$$

where $\Upsilon(C_S)$ is given by (1.11).

Proof. Firstly, in light of (3.6), there exists a positive constant $\kappa_{*,1}$ independent of C_S such that the solution of (1.1) from Lemma 2.1 satisfies

$$\begin{aligned}\|n(\cdot,t)\|_{L^{1+\alpha}(\Omega)} \leq & \kappa_{*,1}\Upsilon^{\frac{1}{1+\alpha}}(C_S)\\ \leq & \kappa_{*,1}\Upsilon(C_S) \text{ for all } t \in (0, T_{max})\end{aligned} \tag{3.27}$$

by using $\alpha > 0$ and $\Upsilon(C_S) > 1$, where $\Upsilon(C_S)$ is the same as (1.11).

On the basis of the variation-of-constants formula for the projected version of the third equation in (1.1), that is of the identity $u_t + Au = \mathcal{P}[n(\cdot,\tau)\nabla\phi]$, recalling (3.27) and (1.13), according to standard smoothing properties of the Stokes semigroup we see that there exists positive constants $\kappa_{*,2}, \kappa_{*,3}, \kappa_{*,4}$ as well as $\kappa_{*,5}$ and $\lambda > 0$ such that for all $t \in (0, T_{max})$ and $\gamma \in (\frac{1}{2}, 1)$,

$$\begin{aligned}\|A^\gamma u(\cdot,t)\|_{L^2(\Omega)} &\leq \kappa_{*,2}\|A^\gamma e^{-tA}u_0\|_{L^2(\Omega)} + \kappa_{*,2}\int_0^t \|A^\gamma e^{-(t-\tau)A}h(\cdot,\tau)d\tau\|_{L^2(\Omega)}d\tau\\ &\leq \|A^\gamma u_0\|_{L^2(\Omega)} + \kappa_{*,3}\int_0^t (t-\tau)^{-\gamma-\frac{2}{2}(\frac{1}{1+\alpha}-\frac{1}{2})}e^{-\lambda(t-\tau)}\|h(\cdot,\tau)\|_{L^{1+\alpha}(\Omega)}d\tau\\ &\leq \kappa_{*,4} + \kappa_{*,3}\int_0^t (t-\tau)^{-\gamma-\frac{2}{2}(\frac{1}{1+\alpha}-\frac{1}{2})}e^{-\lambda(t-\tau)}\|h(\cdot,\tau)\|_{L^{1+\alpha}(\Omega)}d\tau\\ &\leq \kappa_{*,5}\Upsilon(C_S) \text{ for all } t \in (0, T_{max})\end{aligned} \tag{3.28}$$

with $h = \mathcal{P}[n(\cdot,t)\nabla\phi]$, where we have used the fact that $\Upsilon(C_S) > 1$ and

$$\int_0^t (t-\tau)^{-\gamma-\frac{2}{2}(\frac{1}{1+\alpha}-\frac{1}{2})}e^{-\lambda(t-\tau)}ds \leq \int_0^\infty \sigma^{-\gamma-\frac{2}{2}(\frac{1}{1+\alpha}-\frac{1}{2})}e^{-\lambda\sigma}d\sigma < +\infty$$

by using $1 + \alpha > 1$. As our assumption $\gamma > \frac{1}{2}$ warrants that $D(A^\gamma) \hookrightarrow L^\infty(\Omega)$ (see e.g. [10]), so that, (3.28) also entails

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq \kappa_{*,6}\Upsilon(C_S) \text{ for all } t \in (0, T_{max}) \tag{3.29}$$

with some positive constant $\kappa_{*,6}$ independent of C_S. The proof of Lemma 3.5 is thus completed. □

In conjunction with the estimate for c in $W^{1,2}(\Omega)$ provided by Lemma 3.5 and the estimate for n in $L^{1+\alpha}(\Omega)$ provided by Lemma 3.3, the latter entails boundedness of n and ∇c in $L^\infty(\Omega)$.

Lemma 3.6. *Let $\alpha > 0$ and $\gamma \in (\frac{1}{2}, 1)$. Then one can find $\rho_{****} > 1$ independent of C_S such that the solution of* (1.1) *from Lemma* 2.1 *satisfies*

$$\|n(\cdot,t)\|_{L^\infty(\Omega)} \leq \rho_{****}\left[\Upsilon(C_S) + \Upsilon^2(C_S) + \Upsilon^{1+3\alpha}(C_S))\right]^{\frac{1}{(1+\alpha)(1-b)}} \text{ for all } t \in (0, T_{max}) \tag{3.30}$$

and

$$\|c(\cdot,t)\|_{W^{1,\infty}(\Omega)} \leq \rho_{****}\left[\Upsilon(C_S) + \Upsilon^2(C_S) + \Upsilon^{1+3\alpha}(C_S))\right]^{\frac{1}{(1+\alpha)(1-b)}} \text{ for all } t \in (0, T_{max}), \tag{3.31}$$

where

$$b := \frac{pq_0 - q_0 + p}{pq_0} \in (0, 1) \tag{3.32}$$

as well as

$$q_0 = \min\{\tfrac{2(1+\alpha)}{(1-\alpha)_+}, 4\} \tag{3.33}$$

and

$$p = \min\{\tfrac{2}{(1-\alpha)_+}, 3\}. \tag{3.34}$$

Here $\Upsilon(C_S)$ *is the same as* (1.11).

Proof. In the following, we let $\kappa_{**,i} (i \in \mathbb{N})$ denote some different constants, which are independent of C_S, and if no special explanation, they depend at most on $\Omega, \phi, \alpha, n_0$ and u_0.

Now since $W^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$ for any $p > 1$, therefore, this combined with the boundedness of $\|c(\cdot,t)\|_{W^{1,2}(\Omega)}$ (see (3.21)) provides $\kappa_{**,1} > 0$ fulfilling

$$\int_\Omega |c(x,t)|^p dx \le \kappa_{**,1}\Upsilon^2(C_S) \text{ for all } t \in (0, T_{max}) \tag{3.35}$$

by using the Poincaré inequality, where $\Upsilon(C_S)$ is given by (1.11).

Now, applying the L^p estimate for the second equation of (1.1), we easily infer that there exist positive constants $\kappa_{**,2}$ as well as $\kappa_{**,3}$ and $\kappa_{**,4}$ independent of C_S such that

$$\begin{aligned}\|\Delta c(\cdot,t)\|^{1+\alpha}_{L^{1+\alpha}(\Omega)} &\le \kappa_{**,2}\|n(\cdot,t) - c(\cdot,t) - u(\cdot,t)\cdot\nabla c(\cdot,t)\|^{1+\alpha}_{L^{1+\alpha}(\Omega)}\\ &\le \kappa_{**,3}(\|n(\cdot,t)\|^{1+\alpha}_{L^{1+\alpha}(\Omega)} + \|c(\cdot,t)\|^{1+\alpha}_{L^{1+\alpha}(\Omega)} + \|u(\cdot,t)\|_{L^\infty(\Omega)}\|\nabla c(\cdot,t)\|^{1+\alpha}_{L^{1+\alpha}(\Omega)})\\ &\le \kappa_{**,4}[\Upsilon(C_S) + \Upsilon^2(C_S) + \Upsilon(C_S)\|\nabla c(\cdot,t)\|^{1+\alpha}_{L^{1+\alpha}(\Omega)}]\end{aligned} \tag{3.36}$$

for all $t \in (0, T_{max})$. To estimate the third term on the right hand of (3.36), we notice that by Lemma 3.1 and the Gagliardo–Nirenberg inequality, there exist positive constants $\kappa_{**,5}$ and $\kappa_{**,6}$ such that

$$\begin{aligned}\|\nabla c(\cdot,t)\|^{1+\alpha}_{L^{1+\alpha}(\Omega)} &\le \kappa_{**,5}(\|\Delta c(\cdot,t)\|^{\frac{\frac{3}{2}(1+\alpha)-1}{2-\frac{1}{1+\alpha}}}_{L^{1+\alpha}(\Omega)} \|c(\cdot,t)\|^{1+\alpha-\frac{\frac{3}{2}(1+\alpha)-1}{2-\frac{1}{1+\alpha}}}_{L^1(\Omega)} + \|c(\cdot,t)\|^{1+\alpha}_{L^1(\Omega)})\\ &\le \kappa_{**,6}(\|\Delta c(\cdot,t)\|^{\frac{\frac{3}{2}(1+\alpha)-1}{2-\frac{1}{1+\alpha}}}_{L^{1+\alpha}(\Omega)} + 1) \text{ for all } t \in (0, T_{max}).\end{aligned} \tag{3.37}$$

For any $\alpha > 0$, we deduce from an elementary computation that

$$\frac{\frac{3}{2}(1+\alpha) - 1}{2 - \frac{1}{1+\alpha}} < 1 + \alpha,$$

which enables us to make use of the Young inequality to gain

$$\kappa_{**,4}\Upsilon(C_S)\|\nabla c(\cdot,t)\|^{1+\alpha}_{L^{1+\alpha}(\Omega)} \le \frac{1}{2}\|\Delta c(\cdot,t)\|^{1+\alpha}_{L^{1+\alpha}(\Omega)} + \kappa_{**,7}\Upsilon^{1+3\alpha}(C_S) \text{ for all } t \in (0, T_{max})$$

with some $\kappa_{**,7} > 0$ independent of C_S. Inserting the above inequality into (3.36), we moreover find that there exists $\kappa_{**,8} > 0$ such that

$$\|\Delta c(\cdot,t)\|_{L^{1+\alpha}(\Omega)} \le \kappa_{**,8}(\Upsilon(C_S) + \Upsilon^2(C_S) + \Upsilon^{1+3\alpha}(C_S))^{\frac{1}{1+\alpha}} \text{ for all } t \in (0, T_{max}). \tag{3.38}$$

Now, applying the Sobolev embedding theorems, we derive

$$W^{2,1+\alpha}(\Omega) \hookrightarrow W^{1,\frac{2(1+\alpha)}{(1-\alpha)_+}}(\Omega),$$

so that, by virtue of $\frac{2(1+\alpha)}{(1-\alpha)_+} \geq q_0$ (see (3.33)), we further deduce from (3.35) and (3.38) that there is $\kappa_{**,9} > 0$ independent of C_S such that

$$\|c(\cdot,t)\|_{W^{1,q_0}(\Omega)} \leq \kappa_{**,9}\{\Upsilon(C_S) + \Upsilon^2(C_S) + \Upsilon^{1+3\alpha}(C_S)\}^{\frac{1}{1+\alpha}} \text{ for all } t \in (0,T_{max}). \tag{3.39}$$

Given $T\in(0,T_{max})$, in order to prepare an estimation of the finite number $M(T):=\sup_{t\in(0,T)} \|n(\cdot,t)\|_{L^\infty(\Omega)}$ we write $\tilde{h} := S(n)\nabla c+u$ and then obtain from (3.39) as well as (1.3) and (3.35) that there exists $\kappa_{**,10} > 0$ fulfilling

$$\|\tilde{h}(\cdot,t)\|_{L^{q_0}(\Omega)} \leq \kappa_{**,10}\{\Upsilon(C_S) + \Upsilon^2(C_S) + \Upsilon^{1+3\alpha}(C_S)\}^{\frac{1}{1+\alpha}} \text{ for all } t \in (0,T_{max}). \tag{3.40}$$

Since $n_t = \Delta n - \nabla\cdot(n\tilde{h})$ in $\Omega\times(0,T_{max})$ due to the fact that $\nabla\cdot u = 0$, so that, by virtue of an associate variation-of-constants formula we can represent $n(\cdot,t)$ for each $t\in(0,T)$ according to

$$n(\cdot,t) = e^{(t-t_0)\Delta}n(\cdot,t_0) - \int_{t_0}^t e^{(t-s)\Delta}\nabla\cdot(n(\cdot,s)\tilde{h}(\cdot,s))ds,\ t\in(t_0,T) \tag{3.41}$$

with $t_0 := (t-1)_+$. In the case of $t\in(0,1]$, the maximum principle warrants that

$$\|e^{(t-t_0)\Delta}n(\cdot,t_0)\|_{L^\infty(\Omega)} \leq \|n_0\|_{L^\infty(\Omega)}, \tag{3.42}$$

whereas for $t > 1$ it can be derived from Lemma 3.1 and the standard L^p-L^q estimates for Neumann heat semigroup that

$$\|e^{(t-t_0)\Delta}n(\cdot,t_0)\|_{L^\infty(\Omega)} \leq \kappa_{**,11}(t-t_0)^{-\frac{2}{2}}\|n(\cdot,t_0)\|_{L^1(\Omega)} \leq \kappa_{**,12} \text{ for all } t\in(t_0,T) \tag{3.43}$$

with some positive constants $\kappa_{**,11}$ and $\kappa_{**,12}$ indeppendent of C_S. The last term on the right-hand side of (3.41) is estimated as follows. Let p and b be the same as (3.34) and (3.32), respectively. Then $p = \frac{2+q_0}{2} \in (2,q_0)$ by using $\alpha > 0$. And then once more invoking the known smoothing properties of the Stokes semigroup and the Hölder inequality to find $\kappa_{**,13} > 0$ and $\kappa_{**,14} > 0$ independent of C_S such that

$$\begin{aligned}
&\int_{t_0}^t \|e^{(t-s)\Delta}\nabla\cdot(n(\cdot,s)\tilde{h}(\cdot,s)\|_{L^\infty(\Omega)}ds \\
&\leq \kappa_{**,13}\int_{t_0}^t (t-s)^{-\frac{1}{2}-\frac{2}{2p}}\|n(\cdot,s)\tilde{h}(\cdot,s)\|_{L^p(\Omega)}ds \\
&\leq \kappa_{**,13}\int_{t_0}^t (t-s)^{-\frac{1}{2}-\frac{2}{2p}}\|n(\cdot,s)\|_{L^{\frac{pq_0}{q_0-p}}(\Omega)}\|\tilde{h}(\cdot,s)\|_{L^{q_0}(\Omega)}ds \\
&\leq \kappa_{**,14}\int_{t_0}^t (t-s)^{-\frac{1}{2}-\frac{2}{2p}}\|n(\cdot,s)\|^b_{L^\infty(\Omega)}\|n(\cdot,s)\|\|^{1-b}_{L^1(\Omega)}\|\tilde{h}(\cdot,s)\|_{L^{q_0}(\Omega)}ds \\
&\leq \kappa_{**,14}M^b(T)\{\Upsilon(C_S) + \Upsilon^2(C_S) + \Upsilon^{1+3\alpha}(C_S)\}^{\frac{1}{1+\alpha}} \text{ for all } t\in(0,T)
\end{aligned} \tag{3.44}$$

by using (3.1) and (4.28), where b is given by (3.32). Since $p > 2$, we conclude that $-\frac{1}{2} - \frac{2}{2p} > -1$. Collecting (3.41)–(3.44) and using the definition of $M(T)$, one has a $\kappa_{**,15} > \frac{1}{2} > 0$ independet of C_S such that

$$M(T) \leq \kappa_{**,15} + \kappa_{**,15}M^b(T)\{\Upsilon(C_S) + \Upsilon^2(C_S) + \Upsilon^{1+3\alpha}(C_S)\}^{\frac{1}{1+\alpha}} \text{ for all } T\in(0,T_{max}), \tag{3.45}$$

which in light of $b < 1$ particularly entails that

$$\begin{aligned}
\|n(\cdot,t)\|_{L^\infty(\Omega)} &\leq \max\{1, \left[2\kappa_{**,15}(\Upsilon(C_S) + \Upsilon^2(C_S) + \Upsilon^{1+3\alpha}(C_S))\right]^{\frac{1}{(1+\alpha)(1-b)}}\} \\
&\leq \left[2\kappa_{**,15}(\Upsilon(C_S) + \Upsilon^2(C_S) + \Upsilon^3(C_S) + \Upsilon^{1+3\alpha}(C_S))\right]^{\frac{1}{(1+\alpha)(1-b)}} \text{ for all } t\in(0,T_{max})
\end{aligned} \tag{3.46}$$

by using $\kappa_{**,15} > \frac{1}{2}$ and $T\in(0,T_{max})$ was arbitrary.

In the following, we will derive the boundedness of $\|c(\cdot,t)\|_{W^{1,\infty}(\Omega)}$ for all $t \in (0, T_{max})$. In fact, in light of (3.25) and (3.46), we may apply the elliptic regularity theory to the second equation in (1.1) to obtain a positive constant $\kappa_{**,16} > 0$ indepentdent of C_S such that

$$\begin{aligned}&\sup_{t\in(0,T_{max})} \|c(\cdot,t)\|_{W^{2,p}(\Omega)} \\ &\leq \kappa_{**,16} \left[2\kappa_{**,15}(\Upsilon(C_S) + \Upsilon^2(C_S) + \Upsilon^3(C_S) + \Upsilon^{1+3\alpha}(C_S))\right]^{\frac{1}{(1+\alpha)(1-b)}}\end{aligned} \tag{3.47}$$

for all $p \in (1, +\infty)$. Thus, choosing $p > 2$, we infer from the Sobolev embedding theorem that there is $\kappa_{**,17} > 0$ indepentdent of C_S fulfilling

$$\begin{aligned}&\sup_{t\in(0,T_{max})} \|c(\cdot,t)\|_{W^{1,\infty}(\Omega)} \\ &\leq \kappa_{**,17} \left[2\kappa_{**,15}(\Upsilon(C_S) + \Upsilon^2(C_S) + \Upsilon^3(C_S) + \Upsilon^{1+3\alpha}(C_S))\right]^{\frac{1}{(1+\alpha)(1-b)}}\end{aligned} \tag{3.48}$$

and thereby completes the proof.

□

By virtue of (2.2) and Lemma 3.5, the local-in-time solution can be extended to the global-intime solution.

Proposition 3.1. *Let*

$$\varrho(C_S) = \max\{\Upsilon(C_S), \left[\Upsilon(C_S) + \Upsilon^2(C_S) + \Upsilon^{1+3\alpha}(C_S))\right]^{\frac{1}{(1+\alpha)(1-b)}}\}, \tag{3.49}$$

where q_0, b are the same as Lemma 3.6 and $\Upsilon(C_S)$ is given by (1.11). Then the solution of (1.1) is global on $[0, \infty)$. Moreover, one can find

$\lambda_* > 0$ independent of C_S such that the solution of (1.1) satisfies

$$\|n(\cdot,t)\|_{L^\infty(\Omega)} \leq \lambda_* \varrho(C_S) \text{ for all } t \in (0,\infty) \tag{3.50}$$

as well as

$$\|c(\cdot,t)\|_{W^{1,\infty}(\Omega)} \leq \lambda_* \varrho(C_S) \text{ for all } t \in (0,\infty) \tag{3.51}$$

and

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq \lambda_* \varrho(C_S) \text{ for all } t \in (0,\infty). \tag{3.52}$$

Moreover, for all $\gamma \in (\frac{1}{2}, 1)$, we also have

$$\|A^\gamma u(\cdot,t)\|_{L^2(\Omega)} \leq \lambda_* \varrho(C_S) \text{ for all } t \in (0,\infty). \tag{3.53}$$

Proof. Firstly, relying on Lemmas 3.5–3.6, we my find $\lambda_{1,*} > 0$ independent of C_S with the property that for all $t \in (0, T_{max})$,

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|A^\gamma u(\cdot,t)\|_{L^2(\Omega)} + \|n(\cdot,t)\|_{L^\infty(\Omega)} + \|c(\cdot,t)\|_{W^{1,\infty}(\Omega)} \leq \lambda_{1,*}\varrho(C_S) \tag{3.54}$$

with $\varrho(C_S)$ is the same as (3.49). In view of the extensibility criterion (2.2), we thus infer that $T_{max} = \infty$, i.e., the solution (n, c, u, P) is global in time. Moreover, again based on Lemmas 3.5–3.6, we can deduce that (3.50)–(3.53) hold. This completes the proof of Proposition 3.1. □

The combination of the previous results now immediately establishes Theorem 1.1.

Theorem 1.1. We can merge the results of Proposition 3.1 to immediately arrive at the conclusion. □

4. Large Time Behavior. Proof of Theorem 1.2

Having dealt with issues of boundedness so far, in this section, we next turn our attention to the claimed asymptotic behavior of solutions in (1.1). To this end, we will show that in the large time limit, the classical global solution of (1.1) converges to $(\bar{n}_0, \bar{n}_0, 0)$ exponentially if C_S is smaller.

Lemma 4.1. *Let λ_* and $\varrho(C_S)$ be the same as Proposition 3.1 as well as C_N be the best Poincaré constant and C_S be given by (1.3). Suppose that*

$$C_S \lambda_* \varrho(C_S) < 2\sqrt{C_N}. \tag{4.1}$$

Then there exists $B > 0$ such that

$$\begin{aligned} &\frac{B}{2}\frac{d}{dt}\|n(\cdot,t)-\bar{n}_0\|^2_{L^2(\Omega)} + \left(\frac{BC_N}{2} - \frac{1}{4}\right)\int_\Omega |n-\bar{n}|^2 \\ &+ \left(1 - \frac{BC_S^2\rho^2_{****}\left[\Upsilon(C_S)+\Upsilon^2(C_S)+\Upsilon^{1+3\alpha}(C_S)\right]^{\frac{2}{(1+\alpha)(1-b)}}}{2}\right)\int_\Omega |\nabla c|^2 \\ &\le 0 \text{ for all } t > 0, \end{aligned} \tag{4.2}$$

where

$$\bar{n}_0 = \frac{1}{|\Omega|}\int_\Omega n_0. \tag{4.3}$$

Proof. Testing the first equation in (1.1) by $n(\cdot,t) - \bar{n}_0$ and using the fact that $\nabla \cdot u = 0$, we make use of the Young inequality to have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|n(\cdot,t)-\bar{n}_0\|^2_{L^2(\Omega)} &= \int_\Omega (n-\bar{n}_0)[\Delta n - u\cdot\nabla n - \nabla\cdot(nF(n)S(x,n,c)\nabla c)] \\ &\le -\int_\Omega |\nabla n|^2 + \int_\Omega |n||S(x,n,c)||\nabla n||\nabla c| \\ &\le -\frac{1}{2}\int_\Omega |\nabla n|^2 + \frac{C_S^2}{2}\int_\Omega \frac{n^2}{(1+n)^{2\alpha}}|\nabla c|^2 \\ &\le -\frac{1}{2}\int_\Omega |\nabla n|^2 + \frac{C_S^2 \sup_{t>0}\|n(\cdot,t)\|^2_{L^\infty(\Omega)}}{2}\int_\Omega |\nabla c|^2 \\ &\le -\frac{1}{2}\int_\Omega |\nabla n|^2 + \frac{C_S^2\lambda_*^2\varrho^2(C_S)}{2}\int_\Omega |\nabla c|^2 \text{ for all } t > 0 \end{aligned} \tag{4.4}$$

by using (3.30) as well as (1.3) and $\alpha \ge 0$, where

$$\bar{n}_0 = \frac{1}{|\Omega|}\int_\Omega n_0 \tag{4.5}$$

and C_S is given by (1.3). Here λ_* and $\varrho(C_S)$ are the same as Proposition 3.1. We note from the Poincaré inequality that there is $C_N > 0$ such tha

$$\|\varphi - \frac{1}{|\Omega|}\int_\Omega \varphi\|^2_{L^2(\Omega)} \le C_N \int_\Omega |\nabla\varphi|^2 \text{ for all } \varphi \in W^{1,2}(\Omega). \tag{4.6}$$

This combined with (4.4) yields to for all $t > 0$,

$$\frac{1}{2}\frac{d}{dt}\|n(\cdot,t)-\bar{n}_0\|^2_{L^2(\Omega)} \le -\frac{C_N}{2}\int_\Omega |n-\bar{n}_0|^2 + \frac{C_S^2\lambda_*^2\varrho^2(C_S)}{2}\int_\Omega |\nabla c|^2, \tag{4.7}$$

which in light of (3.1) entails that

$$\frac{1}{2}\frac{d}{dt}\|n(\cdot,t)-\bar{n}_0\|^2_{L^2(\Omega)} \le -\frac{C_N}{2}\int_\Omega |n-\bar{n}_0|^2 + \frac{C_S^2\lambda_*^2\varrho^2(C_S)}{2}\int_\Omega |\nabla c|^2 \tag{4.8}$$

for all $t > 0$. Next, by means of the testing procedure, an application of the Young inequality warrants that

$$\begin{aligned} 0 &= \int_\Omega (c-\bar{n}_0)[\Delta c - u\cdot\nabla c - (c-\bar{n}_0) + (n-\bar{n}_0)] \\ &= \int_\Omega (c-\bar{n}_0)(\Delta c - u\cdot\nabla c) - \int_\Omega (c-\bar{n}_0)^2 + \int_\Omega (c-\bar{n}_0)(n-\bar{n}_0) \\ &\le -\int_\Omega |\nabla c|^2 + \frac{1}{4}\int_\Omega (n-\bar{n}_0)^2 \text{ for all } t>0, \end{aligned} \tag{4.9}$$

where we have used the fact that $\nabla\cdot u = 0$ and $u|_{\partial\Omega} = 0$.

In view of (4.1), we can choose $B > 0$ such that

$$1 - B\frac{C_S^2\lambda_*^2\varrho^2(C_S)}{2} > 0 \tag{4.10}$$

and

$$\frac{B}{2}C_N - \frac{1}{4} > 0. \tag{4.11}$$

Collecting (4.8)–(4.11), we thus infer that

$$\begin{aligned} &\frac{B}{2}\frac{d}{dt}\|n(\cdot,t)-\bar{n}_0\|^2_{L^2(\Omega)} + \left(\frac{BC_N}{2} - \frac{1}{4}\right)\int_\Omega |n-\bar{n}_0|^2 + \left(1 - \frac{BC_S^2\lambda_*^2\varrho^2(C_S)}{2}\right)\int_\Omega |\nabla c|^2 \\ &\quad \le 0 \text{ for all } t>0, \end{aligned} \tag{4.12}$$

which completes the proof. □

With the previous result at hand, we can derive the following stabilization property of n, which will be used in Lemma 4.2 below.

Corollary 4.1. *Under the assumptions of Lemma 4.1, then for any $t > 0$, there exists $\rho_{1,*} > 0$ such that*

$$\|n(\cdot,t)-\bar{n}_0\|^2_{L^2(\Omega)} \le e^{-\rho_{1,*}t}[\|n_0(\cdot,t)-\bar{n}_0\|^2_{L^2(\Omega)}, \tag{4.13}$$

and there exists $C_{1,*} > 0$ such that

$$\int_0^\infty\int_\Omega |\nabla c|^2 + \int_0^\infty\int_\Omega |n-\bar{n}_0|^2 + \int_0^\infty\int_\Omega |\nabla n|^2 \le C_{1,*}, \tag{4.14}$$

where $\bar{n}_0$ is given by (4.3).

Proof. Let $y(t) = \frac{B}{2}\|n(\cdot,t)-\bar{n}_0\|^2_{L^2(\Omega)}$. Consequently, (4.2) can be turned into the inequality

$$y'(t) + (C_N - \frac{1}{2B})y(t) \le 0,$$

and thereby integrate in time to obtain (4.13) by applying the definition of $y(t)$ and (4.1), whereafter (4.14) follows by integrating (4.2) in time and using (4.4). □

Thanks to the decay property of $\|n(\cdot,t)-\bar{n}_0\|^2_{L^2(\Omega)}$ provided by Corollary 4.1, we can now obtain convergence with respect to the norm in $L^2(\Omega)$ also of the crucial quantity c.

Lemma 4.2. *Under the assumptions of Lemma 4.1, then for any $t > 0$, there exists $\rho_{2,*} > 0$ such that*

$$\|c(\cdot,t)-\bar{n}_0\|^2_{L^2(\Omega)} \le e^{-\rho_{2,*}t}\|n_0(\cdot,t)-\bar{n}_0\|^2_{L^2(\Omega)}, \tag{4.15}$$

where $\bar{n}_0$ is given by (4.3).

Proof. First, in view of the testing procedure, and using the fact that $\nabla \cdot u = 0$ and $u|_{\partial\Omega} = 0$ we may derive from the Young inequality that

$$\begin{aligned} 0 &= \int_\Omega (c - \bar{n}_0)[\Delta c - u \cdot \nabla c - (c - \bar{n}_0) + (n - \bar{n}_0)] \\ &= \int_\Omega (c - \bar{n}_0)(\Delta c - u \cdot \nabla c) - \int_\Omega (c - \bar{n}_0)^2 + \int_\Omega (c - \bar{n}_0)(n - \bar{n}_0) \\ &\le -\int_\Omega |\nabla c|^2 - \frac{1}{2}\int_\Omega (c - \bar{n}_0)^2 + \int_\Omega (n - \bar{n}_0)^2 \text{ for all } t > 0. \end{aligned} \tag{4.16}$$

This together with (4.13) warrants

$$\begin{aligned} \int_\Omega |\nabla c|^2 + \frac{1}{2}\int_\Omega (c - \bar{n}_0)^2 \le & \int_\Omega (n - \bar{n}_0)^2 \\ & \le e^{-\rho_{1,*}t}\|n_0(\cdot, t) - \bar{n}_0\|^2_{L^2(\Omega)} \text{ for all } t > 0, \end{aligned} \tag{4.17}$$

where $\rho_{1,*}$ and $\bar{n}_0$ are given by (4.13) and (4.3), respectively. Hence, (4.15) holds. □

In the following, we are now prepared to prove the claimed asymptotic behavior of u. To begin with, some uniform decay properties for the solution of (1.1) are given in the following lemma.

Lemma 4.3. *Under the assumptions of Lemma 4.1, there are positive constants $C_{*,3}$ and $\rho_{*,3}$ such that for any $t > 0$,*

$$\|u(\cdot, t)\|_{L^2(\Omega)} \le C_{*,3}e^{-\rho_{*,3}t}. \tag{4.18}$$

Proof. From straightforward calculations, while relying on (1.12), we derive the third equation in (1.1) that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u(\cdot, t)\|^2_{L^2(\Omega)} &= -\int_\Omega |\nabla u|^2 + \int_\Omega n\nabla\phi \cdot u - \int_\Omega \nabla P \cdot u \\ &= -\int_\Omega |\nabla u|^2 + \int_\Omega (n - \bar{n}_0)\nabla\phi \cdot u \\ &\le -\int_\Omega |\nabla u|^2 + \|\nabla\phi\|_{L^\infty(\Omega)}\left(\int_\Omega |n - \bar{n}_0|^2\right)^{\frac{1}{2}}\left(\int_\Omega |u|^2\right)^{\frac{1}{2}} \text{ for all } t > 0, \end{aligned} \tag{4.19}$$

where we have used the fact that $\nabla \cdot u = 0$ and $u|_{\partial\Omega} = 0$. Recalling the Poincaré inequality, one can find a constant $\eta_N > 0$ fulfilling

$$\eta_N \int_\Omega |u|^2 \le \int_\Omega |\nabla u|^2,$$

therefore, collecting Corollary 4.1 and (4.19), we can find $\kappa_{1,***} > 0$ and $\varrho_{1,***} > 0$ such that

$$\int_\Omega |u(x, t)|^2 dx \le \kappa_{1,***}e^{-\varrho_{1,***}t} \text{ for all } t > 0, \tag{4.20}$$

which directly yields our conclusion. □

Again due to the regularity properties asserted by Proposition 3.1, we can thereby improve our knowledge on spatial regularity of c as follows.

Lemma 4.4. *Let $\alpha > 0$. Then for any $p > 2$, there exists a positive constant $C_{*,4} > 0$ such that*

$$\|c(\cdot, t)\|_{W^{2,p}(\Omega)} \le C_{*,4} \text{ for all } t > 0. \tag{4.21}$$

Proof. Applying the L^p estimate for the second equation of (1.1), we derive from Proposition 3.1 that there exist positive constants $\kappa_{***,1}$ as well as $\kappa_{***,2}$ and $\kappa_{***,3}$ independent of C_S such that

$$\begin{aligned}\|\Delta c(\cdot,t)\|^p_{L^p(\Omega)} \leq & \kappa_{***,1}\|n(\cdot,t)-c(\cdot,t)-u(\cdot,t)\cdot\nabla c(\cdot,t)\|^p_{L^p(\Omega)}\\ & \leq \kappa_{***,2}(\|n(\cdot,t)\|^p_{L^p(\Omega)}+\|c(\cdot,t)\|^p_{L^p(\Omega)}+\|u(\cdot,t)\|_{L^\infty(\Omega)}\|\nabla c(\cdot,t)\|^p_{L^p(\Omega)})\\ & \leq \kappa_{***,3}\varrho(C_S) \text{ for all } t>0,\end{aligned} \tag{4.22}$$

which together with (3.51) as well as the Gagliardo–Nirenberg inequality implies (4.21) holds. Here $\varrho(C_S)$ is the same as Proposition 3.1. □

Lemma 4.5. *Assume that the conditions in Theorem 1.1 are satisfied. Let (n,c,u) be a global classical solution of (1.1). Then there exists a positive constant $C_{*,5}$ such that*

$$\|n(\cdot,t)\|_{W^{1,\infty}(\Omega)}+\|u(\cdot,t)\|_{W^{1,\infty}(\Omega)} \leq C_{*,5}. \tag{4.23}$$

Proof. Firstly, based on the regularity of n as well as c and u, one can readily get a constant $\kappa_{****,1}>0$ such that

$$\|c(\cdot,t)\|_{W^{2,p}(\Omega)}+\|n(\cdot,t)\|_{L^\infty(\Omega)}+\|u(\cdot,t)\|_{L^\infty(\Omega)}+\|c(\cdot,t)\|_{W^{1,\infty}(\Omega)} \leq \kappa_{****,1} \text{ for all } t>0. \tag{4.24}$$

Next, we use Lemma 3.4 to obtain

$$\|Du(\cdot,t)\|_{L^r(\Omega)} \leq \kappa_{****,2} \text{ for all } t>0 \text{ and } 1\leq r\leq\infty \text{ and some } \kappa_{****,2}>0, \tag{4.25}$$

which together with Lemma 3.2 and the interpolation inequality yields the existence of $\kappa_{****,3}>0$ satisfying

$$\|u(\cdot,t)\|_{W^{1,\infty}(\Omega)} \leq \kappa_{****,3} \text{ for all } t>0. \tag{4.26}$$

Next, we can rewrite the first equation of (1.1) as

$$n_t-\Delta n+n=a(n,c,u), \tag{4.27}$$

where

$$\begin{aligned}a(x,t)=&a(n(x,t),c(x,t),u(x,t))\\ =&-u\cdot\nabla n-\nabla\cdot(nS(n)\nabla c)+n\\ =&-u\cdot\nabla n-nS'(n)\nabla n\cdot\nabla c-S(n)\nabla n\cdot\nabla c-nS(n)\Delta c+n.\end{aligned}$$

To prove the boundedness of $\|\nabla n(\cdot,t)\|_{L^\infty(\Omega)}$ on $t>0$, by Duhamels principle, we see that the solution of (4.27) can be expressed as follows

$$n(x,t)=e^{-t(\Delta-1)}n_0(x)+\int_0^t e^{-t(\Delta-1)}a(x,\tau)d\tau \text{ for all } t>0.$$

By (4.24) as well as (1.2) and (1.3), then for any $p>1$, there exists $\kappa_{****,4}>0$ such that

$$\begin{aligned}\|a(\cdot,t)\|_{L^p(\Omega)} \leq & \|-u\cdot\nabla n-nS'(n)\nabla n\cdot\nabla c-S(n)\nabla n\cdot\nabla c-nS(n)\Delta c+n\|_{L^p(\Omega)}\\ \leq & \kappa_{****,4}(\|\nabla n(\cdot,t)\|_{L^p(\Omega)}+1) \text{ for all } t>0.\end{aligned} \tag{4.28}$$

On the other hand, multiplying (4.27) by $-\Delta n$, and integrating it over Ω, we derive from the Young inequlity that there is $\kappa_{****,5}>0$ such that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega|\nabla n|^2+\int_\Omega|\Delta n|^2+\int_\Omega|\nabla n|^2 =& -\int_\Omega a(n,c,u)\Delta n\\ & \leq \frac{1}{2}\int_\Omega|\Delta n|^2+\int_\Omega|a(n,c,u)|^2\\ & \leq \frac{1}{2}\int_\Omega|\Delta n|^2+\kappa_{****,5}(\|\nabla n(\cdot,t)\|^2_{L^2(\Omega)}+1) \text{ for all } t>0\end{aligned} \tag{4.29}$$

by using (4.28). Recalling (3.1), so that, the Gagliardo–Nirenberg inequality tells that there exist constants $\kappa_{****,6}$ and $\kappa_{****,7}$ satisfying

$$\begin{aligned}\|\nabla n(\cdot,t)\|^2_{L^2(\Omega)} \leq & \kappa_{****,6}(\|\Delta n(\cdot,t)\|^{\frac{4}{3}}_{L^2(\Omega)}\|n(\cdot,t)\|^{\frac{2}{3}}_{L^1(\Omega)} + \|n(\cdot,t)\|^2_{L^1(\Omega)})\\ \leq & \kappa_{****,7}(\|\Delta n(\cdot,t)\|^{\frac{4}{3}}_{L^2(\Omega)} + 1) \text{ for all } t > 0.\end{aligned}$$

Inserting the above inequality into (4.29) and using the Young inequality, there exists a positive constant $\kappa_{****,8}$ such that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla n|^2 + \frac{1}{4}\int_\Omega |\Delta n|^2 + \int_\Omega |\nabla n|^2 \leq \kappa_{****,8} \text{ for all } t > 0.$$

Thereupon, by means of an ODE comparison argument, we derive

$$\|\nabla n(\cdot,t)\|_{L^2(\Omega)} \leq \kappa_{****,9} \text{ for all } t > 0 \tag{4.30}$$

with some $\kappa_{****,9} > 0$. Now, in view of (4.28), by the L^p–L^q estimate for the Neumann heat semigroup, there exist positive constants $\lambda_1, \kappa_{****,10}$ as well as $\kappa_{****,11}$ and $\kappa_{****,12}$ fulfilling

$$\begin{aligned}\|n(\cdot,t)\|_{W^{1,4}(\Omega)} &\leq \kappa_{****,10}\|\nabla e^{-t(\Delta-1)}n_0(x) + \nabla\int_0^t e^{-t(\Delta-1)}a(x,\tau)d\tau\|_{L^4(\Omega)}\\ &\leq \kappa_{****,11}\|n_0\|_{L^4(\Omega)} + \kappa_{****,11}\int_0^t (t-s)^{-\frac{1}{2}-\frac{2}{2}(\frac{1}{2}-\frac{1}{4})}e^{-\lambda_1(t-s)}\|a(\cdot,s)\|_{L^2(\Omega)}ds\\ &\leq \kappa_{****,12} + \kappa_{****,12}\int_0^t (t-s)^{-\frac{3}{4}}e^{-\lambda_1(t-s)}(\|\nabla n(\cdot,s)\|_{L^2(\Omega)} + 1)ds\\ &\leq \kappa_{****,12} \text{ for all } t > 0\end{aligned} \tag{4.31}$$

by using (4.30). Hence, using the L^p–L^q estimates associated with the heat semigroup as well as (4.28) and (1.13), we derive that for some positive constants λ_2, $\kappa_{****,13}$, $\kappa_{****,14}$ as well as $\kappa_{****,15}$ and $\kappa_{****,16}$,

$$\begin{aligned}\|n(\cdot,t)\|_{W^{1,\infty}(\Omega)} &\leq \kappa_{****,13}\|\nabla e^{-t(\Delta-1)}n_0(x) + \nabla\int_0^t e^{-t(\Delta-1)}a(x,\tau)d\tau\|_{L^\infty(\Omega)}\\ &\leq \kappa_{****,14}e^{-\lambda_2 t}\|n_0\|_{L^\infty(\Omega)} + \kappa_{****,14}\int_0^t (t-s)^{-\frac{1}{2}-\frac{2}{2}(\frac{1}{4}-\frac{1}{\infty})}e^{-\lambda_2(t-s)}\|a(\cdot,s)\|_{L^4(\Omega)}ds\\ &\leq \kappa_{****,15} + \kappa_{****,15}\int_0^t (t-s)^{-\frac{3}{4}}e^{-\lambda(t-s)}(\|\nabla n(\cdot,s)\|_{L^4(\Omega)} + 1)ds\\ &\leq \kappa_{****,16} \text{ for all } t > 0\end{aligned} \tag{4.32}$$

and thereby proves (4.23) by using (4.24) and (4.26). □

With the above preparation (see Corollary 4.1 and Lemmas 4.2–4.5), we can make use of the Gagliardo–Nirenberg inequality as well as Corollary 4.1 and Lemmas 4.2–4.3 to achieve that the solution (n, c, u) exponentially converges to the constant equilibria $(\bar{n}_0, \bar{n}_0, 0)$ in the norm of $L^\infty(\Omega)$ as $t \to \infty$, where $\bar{n}_0 = \frac{1}{|\Omega|}\int_\Omega n_0$.

Lemma 4.6. *Assume the hypothesis of Theorem 1.1 holds. Moreover, there exists $\chi_0 > 0$ with the property that if*

$$\frac{C_S}{C_N} < \chi_0,$$

one can find $\gamma > 0$ and $C > 0$ such that the global classical solution (n, c, u) of (1.1) satisfies

$$\|n(\cdot,t) - \bar{n}_0\|_{L^\infty(\Omega)} \leq Ce^{-\gamma t}, \text{ for all } t > 0 \tag{4.33}$$

as well as

$$\|c(\cdot,t)-\bar{n}_0\|_{L^\infty(\Omega)} \le Ce^{-\gamma t},\ \textit{for all}\ t>0 \tag{4.34}$$

and

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le Ce^{-\gamma t},\ \textit{for all}\ t>0, \tag{4.35}$$

where $\bar{n}_0 = \frac{1}{|\Omega|}\int_\Omega n_0$ *and* C_N *is the best Poincaré constant.*

Proof. Firstly, applying Corollary 4.1 and Lemmas 4.2–4.3, there exist positive constants C_1 and γ_1 such that

$$\|n(\cdot,t)-\bar{n}_0\|_{L^2(\Omega)} \le C_1e^{-\gamma_1 t},\ \text{for all}\ t>0 \tag{4.36}$$

as well as

$$\|c(\cdot,t)-\bar{n}_0\|_{L^2(\Omega)} \le C_1e^{-\gamma_1 t},\ \text{for all}\ t>0 \tag{4.37}$$

and

$$\|u(\cdot,t)\|_{L^2(\Omega)} \le C_1e^{-\gamma_1 t},\ \text{for all}\ t>0. \tag{4.38}$$

Furthermore, we also derive from Proposition 3.1 as well as Lemma 4.5 and (1.13) that there is $C_2>0$ fulfilling

$$\|n(\cdot,t)-\bar{n}_0\|_{W^{1,\infty}(\Omega)} + \|c(\cdot,t)-\bar{n}_0\|_{W^{1,\infty}(\Omega)} + \|u(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le C_2 \text{ for all } t>0, \tag{4.39}$$

where by the Gagliardo–Nirenberg inequality and (4.36) with some $C_3>0$ as well as $C_4>0$ and $C_5>0$ we have

$$\begin{aligned}\|n(\cdot,t)-\bar{n}_0\|_{L^\infty(\Omega)} \le& C_3(\|n(\cdot,t)-\bar{n}_0\|^{\frac12}_{W^{1,\infty}(\Omega)}\|n(\cdot,t)-\bar{n}_0\|^{\frac12}_{L^2(\Omega)} + \|n(\cdot,t)-\bar{n}_0\|_{L^2(\Omega)})\\ \le& C_4\|n(\cdot,t)-\bar{n}_0\|^{\frac12}_{L^2(\Omega)}\\ \le& C_5e^{-\gamma t} \text{ for all } t>0\end{aligned} \tag{4.40}$$

with $\gamma = \frac{\gamma_1}{2}$. Similarly, using (4.36) together with the Gagliardo–Nirenberg inequality we can find positive constants C_6 and C_7 fulfilling

$$\|c(\cdot,t)-\bar{n}_0\|_{L^\infty(\Omega)} \le C_6e^{-\gamma t} \text{ for all } t>0 \tag{4.41}$$

and

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C_7e^{-\gamma t} \text{ for all } t>0. \tag{4.42}$$

Finally, setting

$$C = \max\{C_5, C_6, C_7\},$$

we see that (4.33)–(4.35) follows by applying (4.40)–(4.42). □

The previous lemma at hand, we can conclude Theorem 1.2 in a straightforward manner.

Proof of Theorem 1.2. The statement is evidently implied by Lemma 4.6. □

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Declarations

Conflict of interest The author declares that he has no conflict of interest.

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