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Existence of a Stationary Navier–Stokes Flow Past a Rigid Body, with Application to Starting Problem in Higher Dimensions

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Abstract. We consider the large time behavior of the Navier–Stokes flow past a rigid body in $\mathbb{R}^n$ with $n \geq 3$. We first construct a small stationary solution possessing the optimal summability at spatial infinity, which is the same as that of the Oseen fundamental solution. When the translational velocity of the body gradually increases and is maintained after a certain finite time, we then show that the nonstationary fluid motion converges to the stationary solution corresponding to a small terminal velocity of the body as time $t \to \infty$ in L^q with $q \in [n, \infty]$. This is called Finn's starting problem and the three-dimensional case was affirmatively solved by Galdi et al. (Arch Ration Mech Anal 138: 307–318, 1997). The present paper extends Galdi et al. (1997) to the case of higher dimensions. Even for the three-dimensional case, our theorem provides new convergence rate, that is determined by the summability of the stationary solution at infinity and seems to be sharp.

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1. Introduction and Main Results

We consider a viscous incompressible flow past a rigid body $\mathscr{O} \subset \mathbb{R}^n\,(n \geq 3)$. We suppose that $\mathscr{O}$ is translating with a velocity $-\psi(t)ae_1$, where $a > 0$, $e_1 = (1, 0, \cdots, 0)^\top$ and ψ is a function on $\mathbb{R}$ describing the transition of the translational velocity in such a way that

$$\psi \in C^1(\mathbb{R};\mathbb{R}), \quad |\psi(t)| \leq 1 \quad \text{for } t \in \mathbb{R}, \quad \psi(t) = 0 \quad \text{for } t \leq 0, \quad \psi(t) = 1 \quad \text{for } t \geq 1. \tag{1.1}$$

Here and hereafter, $(\cdot)^\top$ denotes the transpose. We take the frame attached to the body, then the fluid motion which occupies the exterior domain $D = \mathbb{R}^n \backslash \mathscr{O}$ with C^2 boundary ∂D and is started from rest obeys

$$\begin{cases} \dfrac{\partial u}{\partial t} + u \cdot \nabla u = \Delta u - \psi(t)a\dfrac{\partial u}{\partial x_1} - \nabla p, & x \in D,\ t > 0, \\ \nabla \cdot u = 0, & x \in D, t \geq 0, \\ u|_{\partial D} = -\psi(t)ae_1, & t > 0, \\ u \to 0 & \text{as } |x| \to \infty, \\ u(x,0) = 0, & x \in D. \end{cases} \tag{1.2}$$

Here, $u = (u_1(x,t), \cdots, u_n(x,t))^\top$ and $p = p(x,t)$ denote unknown velocity and pressure of the fluid, respectively. Since $\psi(t) = 1$ for $t \geq 1$, the large time behavior of solutions is related to the stationary problem

$$\begin{cases} u_s\cdot\nabla u_s = \Delta u_s - a\,\dfrac{\partial u_s}{\partial x_1} - \nabla p_s, & x\in D,\\ \nabla\cdot u_s = 0, & x\in D,\\ u_s|_{\partial D} = -ae_1, & \\ u_s\to 0 & \text{as } |x|\to\infty. \end{cases} \tag{1.3}$$

When $n=3$, the pioneering work due to Leray [27] provided the existence theorem for weak solution to problem (1.3), what is called D-solution, having finite Dirichlet integral without smallness assumption on data. From the physical point of view, solutions of (1.3) should reflect the anisotropic decay structure caused by the translation, but his solution had little information about the behavior at large distances. To fill this gap, Finn [11–14] introduced the class of solutions with pointwise decay property

$$u_s(x) = O(|x|^{-\frac{1}{2}-\varepsilon}) \qquad \text{as } |x|\to\infty \tag{1.4}$$

for some $\varepsilon>0$ and proved that if a is small enough, (1.3) admits a unique solution satisfying (1.4) and exhibiting paraboloidal wake region behind the body $\mathscr{O}$. He called the Navier–Stokes flows satisfying (1.4) physically reasonable solutions. It is remarkable that D-solutions become physically reasonable solutions no matter how large a would be, see Babenko [2], Galdi [18] and Farwig and Sohr [10]. Galdi developed the L^q-theory of the linearized system, that we call the Oseen system, to prove that every D-solution has the same summability as the one of the Oseen fundamental solution without any smallness assumption, see [19, Theorem X.6.4]. It is not straightforward to generalize his result to the case of higher dimensions and it remains open whether the same result holds true for $n\geq 4$. We also refer to Farwig [9] who gave another outlook on Finn's results by using anisotropically weighted Sobolev spaces, and to Shibata [31] who developed the estimates of physically reasonable solutions and then proved their stability in the L^3 framework when a is small. There is less literature concerning the problem (1.3) for the case $n\geq 4$. When $n\geq 3$, Shibata and Yamazaki [32] constructed a solution u_s, which is uniformly bounded with respect to small $a\geq 0$ in the Lorentz space $L^{n,\infty}$, and investigated the behavior of u_s as $a\to 0$. If, in particular, $n\geq 4$ and if $a\geq 0$ is sufficiently small, they also derived

$$u_s\in L^{\frac{n}{1+\rho_1}}(D)\cap L^{\frac{n}{1-\rho_2}}(D), \quad \nabla u_s\in L^{\frac{n}{2+\rho_1}}(D)\cap L^{\frac{n}{2-\rho_2}}(D) \tag{1.5}$$

for some $0<\rho_1,\rho_2<1$, see [32, Remark 4.2].

Let us turn to the initial value problem. Finn [12] conjectured that (1.2) admits a solution which tends to a physically reasonable solution as $t\to\infty$ if a is small enough. This is called Finn's starting problem. It was first studied by Heywood [21], in which a stationary solution is said to be attainable if the fluid motion converges to this solution as $t\to\infty$. Later on, by using Kato's approach [24] (see also Fujita and Kato [15]) together with the L^q-L^r estimates for the Oseen initial value problem established by Kobayashi and Shibata [26], Finn's starting problem was affirmatively solved by Galdi et al. [20]. After that, Hishida and Maremonti [23] constructed a sort of weak solution u that enjoys

$$\|u(t)-u_s\|_\infty = O(t^{-\frac{1}{2}}) \quad \text{as } t\to\infty \tag{1.6}$$

if a is small, but $u(\cdot,0)\in L^3(D)$ can be large. Here and hereafter, $\|\cdot\|_q$ denotes the norm of $L^q(D)$. Although we concentrate ourselves on attainability in this paper, stability of stationary solutions was also studied by Shibata [31], Enomoto and Shibata [8] and Koba [25] in the L^q framework. Those work except [8] studied the three-dimensional exterior problem, while [8] showed the stability of a stationary solution satisfying (1.5) for some $0<\rho_1,\rho_2<1$ in n-dimensional exterior domains with $n\geq 3$. Stability of physically reasonable solutions in 2D is much more involved for several reasons and it has been recently proved by Maekawa [29].

The aim of this paper is two-fold. The first one is to construct a small stationary solution possessing the optimal summability at spatial infinity, which is the same as that of the Oseen fundamental solution

E:

$$\mathbf{E} \in L^q(\{x \in \mathbb{R}^n \mid |x| > 1\}), \quad q > \frac{n+1}{n-1}, \qquad \nabla \mathbf{E} \in L^r(\{x \in \mathbb{R}^n \mid |x| > 1\}), \quad r > \frac{n+1}{n}, \tag{1.7}$$

see Galdi [19, Section VII]. As already mentioned above, this result is well known in three-dimensional case even for large $a > 0$, but it is not found in the literature for higher dimensional case $n \geq 4$. Our theorem covers the three-dimensional case as well and the proof is considerably shorter than the one given by authors mentioned above since we focus our interest only on summability at infinity rather than anisotropic pointwise estimates. The second aim is to give an affirmative answer to the starting problem as long as a is small enough, that is, to show the attainability of the stationary solution obtained above. The result extends Galdi, Heywood and Shibata [20] to the case of higher dimensions. Even for the three-dimensional case, our theorem not only recovers [20] but also provides better decay properties, for instance,

$$\|u(t) - u_s\|_\infty = O(t^{-\frac{1}{2}-\frac{\rho}{2}}) \quad \text{as } t \to \infty \tag{1.8}$$

for some $\rho > 0$, that should be compared with (1.6). This is because the fluid is initially at rest and because the three-dimensional stationary solution u_s belongs to $L^q(D)$ with $q < 3$; to be precise, since q can be close to 2, one can take ρ close to $1/2$ in (1.8). Due to the L^q-L^r estimates of the Oseen semigroup established by Kobayashi and Shibata [26], Enomoto and Shibata [7,8], see Proposition 3.1, this decay rate is sharp in view of presence of u_s, see (1.19), in forcing terms of the Eq. (1.18) for the perturbation. Our result can be also compared with [34] by the present author on the starting problem in which translation is replaced by rotation of the body $\mathscr{O} \subset \mathbb{R}^3$. Under the circumstance of [34], the optimal spatial decay of stationary solutions observed in general is the scale-critical rate $O(|x|^{-1})$, so that they cannot belong to $L^q(D)$ with $q \leq 3 = n$, and therefore, we have no chance to deduce (1.8). Another remark is that, in comparison with stability theorem due to [8] for $n \geq 3$, more properties of stationary solutions are needed to establish the attainability theorem. Therefore, those properties must be deduced in constructing a solution of (1.3).

Let us state the first main theorem on the existence and summability of stationary solutions.

Theorem 1.1. *Let $n \geq 3$. For every $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ satisfying*

$$\frac{n+1}{n-1} < \alpha_1 \leq n+1 \leq \alpha_2 < \frac{n(n+1)}{2}, \quad \frac{n+1}{n} < \beta_1 \leq \frac{n+1}{2} \leq \beta_2 < \frac{n(n+1)}{n+2}, \tag{1.9}$$

there exists a constant $\delta = \delta(\alpha_1, \alpha_2, \beta_1, \beta_2, n, D) \in (0,1)$ such that if

$$0 < a^{\frac{n-2}{n+1}} < \delta,$$

problem (1.3) admits a unique solution u_s along with

$$\|u_s\|_{\alpha_1} + \|u_s\|_{\alpha_2} \leq Ca^{\frac{n-1}{n+1}}, \quad \|\nabla u_s\|_{\beta_1} + \|\nabla u_s\|_{\beta_2} \leq Ca^{\frac{n}{n+1}}, \tag{1.10}$$

where $C > 0$ is independent of a.

The upper bounds of α_2 and β_2 come from (2.2) with $q < n/2$ in Proposition 2.1 on the L^q-theory of the Oseen system, whereas the lower bounds of α_1 and β_1 are just (1.7).

For the proof of Theorem 1.1, we define a certain closed ball N and a contraction map $\Psi : N \ni v \mapsto u \in N$ which provides the solution to the problem

$$\begin{cases} \Delta u - a\dfrac{\partial u}{\partial x_1} = \nabla p + v \cdot \nabla v, & x \in D, \\ \nabla \cdot u = 0, & x \in D, \\ u|_{\partial D} = -ae_1, & \\ u \to 0 & \text{as } |x| \to \infty. \end{cases} \tag{1.11}$$

In doing so, we rely on the L^q-theory of the Oseen system developed by Galdi [19, Theorem VII.7.1], see Proposition 2.1, which gives us sharp summability estimates of solutions at infinity together with explicit dependence on $a > 0$. As long as we only use Proposition 2.1, the only space in which estimates of Ψ are closed is

$$\{u \in L^{n+1}(D) \mid \nabla u \in L^{\frac{n+1}{2}}(D)\}.$$

From this, we can capture neither the optimal summability at infinity nor regularity required in the study of the starting problem. We thus use at least two spaces $L^{\alpha_i}(D)$ $(i = 1, 2)$ for u and $L^{\beta_i}(D)$ $(i = 1, 2)$ for ∇u, and intend to find a solution within a closed ball N of

$$\{u \in L^{\alpha_1}(D) \cap L^{\alpha_2}(D) \mid \nabla u \in L^{\beta_1}(D) \cap L^{\beta_2}(D)\}. \tag{1.12}$$

However, it is not possible to apply Proposition 2.1 to $f = v \cdot \nabla v$ with

$$v \in L^{\alpha_1}(D), \qquad \nabla v \in L^{\beta_1}(D) \tag{1.13}$$

or

$$v \in L^{\alpha_2}(D), \qquad \nabla v \in L^{\beta_2}(D) \tag{1.14}$$

if α_1 and β_1 are simultaneously close to $(n+1)/(n-1)$ and $(n+1)/n$, or if α_2 and β_2 are simultaneously close to $n(n+1)/2$ and $n(n+1)/(n+2)$, because the relation

$$\frac{2}{n} < \frac{1}{\alpha_2} + \frac{1}{\beta_2} < \frac{1}{\alpha_1} + \frac{1}{\beta_1} < 1$$

required in the linear theory, see Proposition 2.1, is not satisfied. In order to overcome this difficulty, given $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ satisfying (1.9), we choose auxiliary exponents (q_1, q_2, r_1, r_2) fulfilling

$$\alpha_1 \le q_1 \le q_2 \le \alpha_2, \quad \beta_1 \le r_1 \le r_2 \le \beta_2, \quad \frac{2}{n} < \frac{1}{q_i} + \frac{1}{r_i} < 1, \quad i = 1, 2$$

such that the application of Proposition 2.1 to $f = v \cdot \nabla v$ with $v \in L^{q_1}(D)$ and $\nabla v \in L^{r_1}(D)$ (resp. $v \in L^{q_2}(D)$ and $\nabla v \in L^{r_2}(D)$) recovers (1.13) (resp. (1.14)) with u.

Another possibility to prove Theorem 1.1 is combining Proposition 2.1 with the Sobolev inequality. We then get a solution $(u_s, p_s) \in X_q(n)$ for all $q \in (1, \infty)$ with $n/3 \le q \le (n+1)/3$, where $X_q(n)$ is defined in Proposition 2.1. The restriction $n/3 \le q \le (n+1)/3$ is removed by applying a bootstrap argument to decrease the lower bound to 1 and to increase the upper bound to $n/2$. As compared with this way, in our proof, we do not any use a bootstrap argument and directly construct a solution possessing the optimal summability at infinity as well as regularity required in the study of the starting problem.

Let us proceed to the starting problem. To study the attainability of the stationary solution u_s of class (1.12) with $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ satisfying (1.9), it is convenient to set

$$\alpha_1 = \frac{n}{1 + \rho_1}, \qquad \alpha_2 = \frac{n}{1 - \rho_2}, \qquad \beta_1 = \frac{n}{2 + \rho_3}, \qquad \beta_2 = \frac{n}{2 - \rho_4} \tag{1.15}$$

with $(\rho_1, \rho_2, \rho_3, \rho_4)$ satisfying

$$0 < \rho_1 < \frac{n^2 - 2n - 1}{n + 1}, \quad \frac{1}{n + 1} \le \rho_2 < \frac{n - 1}{n + 1}, \quad 0 < \rho_3 < \frac{n^2 - 2n - 2}{n + 1}, \quad \frac{2}{n + 1} \le \rho_4 < \frac{n}{n + 1} \tag{1.16}$$

and we need the additional condition

$$\rho_2 + \rho_4 > 1. \tag{1.17}$$

We note that the set of those parameters is nonvoid. It is reasonable to look for a solution to (1.2) of the form

$$u(x, t) = v(x, t) + \psi(t) u_s, \quad p(x, t) = \phi(x, t) + \psi(t) p_s.$$

Then the perturbation (v, ϕ) satisfies the following initial boundary value problem

$$\begin{cases} \dfrac{\partial v}{\partial t} = \Delta v - a\dfrac{\partial v}{\partial x_1} - v\cdot\nabla v - \psi(t)v\cdot\nabla u_s - \psi(t)u_s\cdot\nabla v + (1-\psi(t))a\dfrac{\partial v}{\partial x_1} \\ \qquad\qquad +h_1(x,t) + h_2(x,t) - \nabla\phi, \quad x\in D,\, t>0, \\ \nabla\cdot v = 0, \quad x\in D,\, t\geq 0, \\ v|_{\partial D} = 0, \quad t>0, \\ v\to 0 \quad \text{as } |x|\to\infty, \\ v(x,0) = 0, \quad x\in D, \end{cases} \tag{1.18}$$

where

$$h_1(x,t) = -\psi'(t)u_s, \tag{1.19}$$

$$h_2(x,t) = \psi(t)\big(1-\psi(t)\big)\Big(u_s\cdot\nabla u_s + a\frac{\partial u_s}{\partial x_1}\Big). \tag{1.20}$$

In what follows, we study the problem (1.18) instead of (1.2). In fact, if we obtain the solution v of (1.18) which converges to 0 as $t\to\infty$, the solution u of (1.2) converges to u_s as $t\to\infty$. Problem (1.18) is converted into

$$\begin{aligned} v(t) = \int_0^t e^{-(t-\tau)A_a} P\Big[&- v\cdot\nabla v - \psi(\tau)v\cdot\nabla u_s - \psi(\tau)u_s\cdot\nabla v \\ &+ \big(1-\psi(\tau)\big)a\frac{\partial v}{\partial x_1} + h_1(\tau) + h_2(\tau)\Big]d\tau \end{aligned} \tag{1.21}$$

by using the Oseen semigroup e^{-tA_a} (see Section 3) as well as the Fujita–Kato projection P from $L^q(D)$ onto $L^q_\sigma(D)$ associated with the Helmholtz decomposition (see Fujiwara and Morimoto [16], Miyakawa [30] and Simader and Sohr [33]):

$$L^q(D) = L^q_\sigma(D) \oplus \{\nabla p\in L^q(D) \mid p\in L^q_{\mathrm{loc}}(\overline{D})\} \quad (1<q<\infty).$$

Here,

$$L^q_\sigma(D) = \overline{C^\infty_{0,\sigma}(D)}^{\|\cdot\|_q}, \quad C^\infty_{0,\sigma}(D) = \{u\in C^\infty_0(D)^n \mid \nabla\cdot u = 0\}.$$

We are now in a position to give the second main theorem on attainability of stationary solutions.

Theorem 1.2. *Let $n\geq 3$ and let ψ be a function on $\mathbb{R}$ satisfying (1.1). We set $M = \max_{t\in\mathbb{R}}|\psi'(t)|$. Suppose that ρ_1, ρ_2, ρ_3 and ρ_4 satisfy (1.16)–(1.17) and let δ be the constant in Theorem 1.1 with (1.15). Then there exists a constant $\varepsilon = \varepsilon(n, D)\in(0,\delta]$ such that if*

$$0 < (M+1)a^{\frac{n-2}{n+1}} < \varepsilon,$$

Equation (1.21) admits a unique solution v within the class

$$\begin{aligned} Y_0 := \big\{ &v\in BC([0,\infty); L^n_\sigma(D)) \mid t^{\frac12}v\in BC((0,\infty); L^\infty(D)), t^{\frac12}\nabla v\in BC((0,\infty); L^n(D)), \\ &\lim_{t\to 0} t^{\frac12}\big(\|v(t)\|_\infty + \|\nabla v(t)\|_n\big) = 0\big\}. \end{aligned} \tag{1.22}$$

Moreover, we have the following.

1. *(sharp decay) Let $n=3$. Then there exists a constant $\varepsilon_* = \varepsilon_*(D)\in(0,\varepsilon]$ such that if $0 < (M+1)a^{1/4} < \varepsilon_*$, the solution v enjoys decay properties*

$$\|v(t)\|_q = O(t^{-\frac12+\frac{3}{2q}-\frac{\rho_1}{2}}), \qquad 3\leq\forall q\leq\infty, \tag{1.23}$$

$$\|\nabla v(t)\|_3 = O(t^{-\frac12-\frac{\rho_1}{2}}) \tag{1.24}$$

as $t\to\infty$.

Let $n\geq 4$ and suppose that $\rho_3>1$ and $1<\rho_1\leq 1+\rho_3$ in addition to (1.16) (the set of those *parameters is nonvoid when $n \geq 4$). Then there exists a constant $\varepsilon_* = \varepsilon_*(n, D) \in (0, \varepsilon]$ such that if $0 < (M+1)a^{(n-2)/(n+1)} < \varepsilon_*$, the solution v enjoys*

$$\|v(t)\|_q = O(t^{-\frac{1}{2}+\frac{n}{2q}-\frac{\rho_1}{2}}), \qquad n \leq \forall q \leq \infty, \tag{1.25}$$

$$\|\nabla v(t)\|_n = O(t^{-\frac{1}{2}-\frac{\rho_1}{2}}) \tag{1.26}$$

as $t \to \infty$.

2. (*Uniqueness*) *There exists a constant $\hat{\varepsilon} = \hat{\varepsilon}(n, D) \in (0, \varepsilon]$ such that if $0 < (M+1)a^{(n-2)/(n+1)} < \hat{\varepsilon}$, the solution v obtained above is unique even within the class*

$$Y := \{v \in BC([0,\infty); L^n_\sigma(D)) \mid t^{\frac{1}{2}} v \in BC((0,\infty); L^\infty(D)), t^{\frac{1}{2}} \nabla v \in BC((0,\infty); L^n(D))\}. \tag{1.27}$$

For the sharp decay properties (1.23)–(1.26), the key step is to prove the L^n-decay of the solution, that is,

$$\|v(t)\|_n = O(t^{-\frac{\rho_1}{2}}) \tag{1.28}$$

as $t \to \infty$. Once we have (1.28), the other decay properties can be derived by the similar argument to [8]. Note that the condition $\rho_1 \leq 1 + \rho_3$ is always fulfilled and thus redundant for $n = 3$ since $\rho_1 < 1/2$ and $\rho_3 < 1/4$. On the other hand, it is enough for $n \geq 4$ to consider the case $\rho_1, \rho_3 > 1$. To prove (1.28), we first derive slower decay

$$\|v(t)\|_n = O(t^{-\frac{\rho}{2}})$$

with some $\rho \in (0,1)$ by making use of $u_s \in L^{n/(1+\rho_1)}(D)$ and $\nabla u_s \in L^{n/(2+\rho_3)}(D)$, see Lemma 3.6 in Section 3. When $n = 3$, one can take $\rho := \min\{\rho_1, \rho_3\}$, yielding better decay properties of the other norms of the solution. With them at hand, we repeat improvement of the estimate of $\|v(t)\|_n$ step by step to find (1.28). However, this procedure does not work for $n \geq 4$ because of $\rho_1 > 1$. In order to get around the difficulty, our idea is to deduce the L^{q_0}-decay of the solution with some $q_0 < n$, that is appropriately chosen, see Lemma 3.8. We are then able to repeat improvement of estimates of several terms to arrive at (1.28), where the argument is more involved than the three-dimensional case above. Finally, to prove the uniqueness within Y, we employ the idea developed by Brezis [5], which shows that the solution $v \in Y$ necessarily satisfies the behavior as $t \to 0$ in (1.22).

In the next section we introduce the L^q-theory of the Oseen system and then prove Theorem 1.1. The final section is devoted to the proof of Theorem 1.2.

2. Proof of Theorem 1.1

In order to prove Theorem 1.1, we first recall the result on the Oseen boundary value problem due to Galdi [19, Theorem VII.7.1], see also Galdi [17] for the first proof of this result.

Proposition 2.1. *Let $n \geq 3$ and let $D \subset \mathbb{R}^n$ be an exterior domain with C^2 boundary. Suppose $a > 0$ and $1 < q < (n+1)/2$. Given $f \in L^q(D)$ and $u_* \in W^{2-1/q,q}(\partial D)$, problem*

$$\begin{cases} \Delta u - a\dfrac{\partial u}{\partial x_1} = \nabla p + f, & x \in D, \\ \nabla \cdot u = 0, & x \in D, \\ u|_{\partial D} = u_*, & \\ u \to 0 & \text{as } |x| \to \infty \end{cases} \tag{2.1}$$

admits a unique (up to an additive constant for p) solution (u, p) within the class

$$X_q(n) := \Big\{(u,p) \in L^1_{\mathrm{loc}}(D) \,\Big|\, u \in L^{s_2}(D),\, \nabla u \in L^{s_1}(D),\, \nabla^2 u \in L^q(D), \frac{\partial u}{\partial x_1} \in L^q(D),\, \nabla p \in L^q(D)\Big\},$$

where

$$\frac{1}{s_1}=\frac{1}{q}-\frac{1}{n+1},\quad \frac{1}{s_2}=\frac{1}{q}-\frac{2}{n+1}. \tag{2.2}$$

Here, by $W^{2-1/q,q}(\partial D)$ we denote the trace space on ∂D from the Sobolev space $W^{2,q}(D)$ (see, for instance, [1,19]).

If, in particular, $a\in(0,1]$ and $q<n/2$, then the solution (u,p) obtained above satisfies

$$a^{\frac{2}{n+1}}\|u\|_{s_2}+a\left\|\frac{\partial u}{\partial x_1}\right\|_q+a^{\frac{1}{n+1}}\|\nabla u\|_{s_1}+\|\nabla^2 u\|_q+\|\nabla p\|_q\le C\big(\|f\|_q+\|u_*\|_{W^{2-\frac{1}{q},q}(\partial D)}\big)$$

with a constant $C>0$ dependent on q,n and D, however, independent of a.

For later use, we prepare the following lemma. The proof is essentially same as the one of Young's inequality for convolution, thus we omit it.

Lemma 2.2. *Let R_0, $d>0$. Assume that $1\le q,s\le\infty$ and $1/q+1/s\ge 1$. Suppose $u\in L^q(\mathbb{R}^n)$ with $\operatorname{supp} u\subset B_d:=\{x\in\mathbb{R}^n \mid |x|<d\}$ and $\rho\in L^s(\mathbb{R}^n\backslash B_{R_0})$. Then for all $R\ge R_0+d$, $\rho * u$ is well-defined as an element of $L^r(\mathbb{R}^n\backslash B_R)$ together with*

$$\|\rho * u\|_{L^r(\mathbb{R}^n\backslash B_R)}\le\|\rho\|_{L^s(\mathbb{R}^n\backslash B_{R_0})}\|u\|_{L^q(B_d)},$$

where $*$ denotes the convolution and $1/r:=1/q+1/s-1$.

When the external force f is taken from $L^{q_1}(D)\cap L^{q_2}(D)$ with $1<q_1,q_2<(n+1)/2$ and $q_1\ne q_2$, we can apply Proposition 2.1 to $f\in L^{q_i}(D)$ $(i=1,2)$. The following tells us that the corresponding solutions coincide with each other.

Lemma 2.3. *Suppose $n\ge 3$, $1<q_1,q_2<(n+1)/2$ and $f\in L^{q_1}(D)\cap L^{q_2}(D)$. Let (u_i,p_i) be a unique solution obtained in Proposition 2.1 with $f\in L^{q_i}(D)$ and $u_*=-ae_1$. Then $u_1=u_2$.*

Proof. We first show that u_1-u_2 behaves like the Oseen fundamental solution $\mathbf{E}$ at large distances. We fix $R_0>0$ satisfying $\mathbb{R}^n\backslash D\subset B_{R_0}$. Let $\zeta\in C^\infty(\mathbb{R}^n)$ be a cut-off function such that $\zeta(x)=0$ for $|x|\le R_0$, $\zeta(x)=1$ for $|x|\ge R_0+1$, and set

$$u(x):=u_1(x)-u_2(x),\qquad p(x):=p_1(x)-p_2(x),$$
$$v(x):=\zeta(x)u(x)-\mathbb{B}[u\cdot\nabla\zeta],\qquad \pi(x):=\zeta(x)p(x).$$

Here, $\mathbb{B}$ is the Bogovskiĭ operator defined on the domain $B_{R_0+1}\backslash B_{R_0}$, see Bogovskiĭ [3], Borchers and Sohr [4] and Galdi [19]. Then we have

$$-\Delta v+a\frac{\partial v}{\partial x_1}+\nabla\pi=g(x),\quad \nabla\cdot v=0\quad\text{in }\mathscr{S}'(\mathbb{R}^n), \tag{2.3}$$

where $\mathscr{S}'(\mathbb{R}^n)$ is the set of tempered distributions on $\mathbb{R}^n$ and

$$g(x)=-(\Delta\zeta)u-2(\nabla\zeta\cdot\nabla)u+a\frac{\partial\zeta}{\partial x_1}u+p\nabla\zeta+\Big(\Delta-a\frac{\partial}{\partial x_1}\Big)\mathbb{B}[u\cdot\nabla\zeta].$$

For (2.3) with $g=0$, we have $\operatorname{supp}\hat v\subset\{0\}$ and $\operatorname{supp}\hat\pi\subset\{0\}$, where $\hat{(\cdot)}$ denotes the Fourier transform. We thus find

$$v(x)=\int_{\mathbb{R}^n}\mathbf{E}(x-y)g(y)\,dy+P(x),\qquad \pi(x)=C(n)\int_{\mathbb{R}^n}\frac{x-y}{|x-y|^n}\cdot g(y)\,dy+Q(x)$$

with some polynomials $P(x)$, $Q(x)$ and some constant $C(n)$. In view of $v\in L^{(1/q_1-2/(n+1))^{-1}}(\mathbb{R}^n)$ $+L^{(1/q_2-2/(n+1))^{-1}}(\mathbb{R}^n)$ and $\nabla\pi\in L^{q_1}(\mathbb{R}^n)+L^{q_2}(\mathbb{R}^n)$, we have $P(x)=0$ and $Q(x)=\overline{p}$. Here, $\overline{p}$ is some constant. Then Lemma 2.2 with

$$\rho=\mathbf{E},\ \nabla\mathbf{E},\ \frac{x-y}{|x-y|^n},$$

$u = g$, $d = R_0 + 1$, $q = 1$ and $r = s$ leads us to

$$u \in L^q(\mathbb{R}^n \backslash B_{2R_0+1}), \qquad \nabla u \in L^r(\mathbb{R}^n \backslash B_{2R_0+1}), \qquad p - \overline{p} \in L^s(\mathbb{R}^n \backslash B_{2R_0+1}) \tag{2.4}$$

for all $q > (n+1)/(n-1)$, $r > (n+1)/n$ and $s > n/(n-1)$, see (1.7).

Let $\varphi \in C^\infty[0,\infty)$ be a cut-off function such that $\varphi(t) = 1$ for $t \leq 1$, $\varphi(t) = 0$ for $t \geq 2$, and set $\varphi_R(x) = \varphi(|x|/R)$ for $R \geq 2R_0 + 1$, $x \in \mathbb{R}^n$. We note that there exists a constant $C > 0$ independent of R such that

$$\|\nabla \varphi_R\|_n \leq C. \tag{2.5}$$

It follows from

$$-\Delta u + a\frac{\partial u}{\partial x_1} + \nabla p = 0, \quad \nabla \cdot u = 0 \quad \text{in } D, \quad u|_{\partial D} = 0$$

that

$$\begin{aligned} 0 &= \int_D \Big\{ -\Delta u + a\frac{\partial u}{\partial x_1} + \nabla(p - \overline{p})\Big\} \cdot (\varphi_R u)\, dx \\ &= \int_D |\nabla u|^2 \varphi_R \, dx + \int_{R \leq |x| \leq 2R} \Big\{ (\nabla u \cdot \nabla \varphi_R) u - \frac{a}{2}\frac{\partial \varphi_R}{\partial x_1}|u|^2 - (p - \overline{p})\nabla \varphi_R \cdot u \Big\}\, dx. \end{aligned} \tag{2.6}$$

Since we can see

$$|\nabla u||u|,\ |u|^2,\ (p - \overline{p})|u| \in L^{n/(n-1)}\big(\mathbb{R}^n \backslash B_{2R_0+1}\big)$$

from (2.4), letting $R \to \infty$ in (2.6) yields $\|\nabla u\|_2^2 = 0$ because of (2.5). From this together with $u|_{\partial D} = 0$, we conclude $u_1 = u_2$. □

Proof of Theorem 1.1. Let $n \geq 3$ and let $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ satisfy (1.9). We first choose parameters (q_1, q_2, r_1, r_2) satisfying

$$\frac{n+1}{n-1} < \alpha_1 \leq q_1 \leq n+1 \leq q_2 \leq \alpha_2 < \frac{n(n+1)}{2}, \tag{2.7}$$

$$\frac{n+1}{n} < \beta_1 \leq r_1 \leq \frac{n+1}{2} \leq r_2 \leq \beta_2 < \frac{n(n+1)}{n+2}, \tag{2.8}$$

$$\max\left\{\frac{1}{\alpha_1} + \frac{2}{n+1}, \frac{1}{\beta_1} + \frac{1}{n+1}\right\} \leq \frac{1}{q_1} + \frac{1}{r_1} < 1, \tag{2.9}$$

$$\frac{2}{n} < \frac{1}{q_2} + \frac{1}{r_2} \leq \min\left\{\frac{1}{\alpha_2} + \frac{2}{n+1}, \frac{1}{\beta_2} + \frac{1}{n+1}\right\}. \tag{2.10}$$

It is actually possible to choose those parameters. In fact, we put

$$\alpha_1 = \frac{n+1}{n-1-\gamma_1}, \quad \alpha_2 = \frac{n(n+1)}{2+\gamma_2}, \quad \beta_1 = \frac{n+1}{n-\eta_1}, \quad \beta_2 = \frac{n(n+1)}{n+2+\eta_2}$$

with arbitrarily small $\gamma_i, \eta_i \in (0, n-2]$ and look for (q_1, q_2, r_1, r_2) of the form

$$q_1 = \frac{n+1}{n-1-\tilde{\gamma}_1}, \quad q_2 = \frac{n(n+1)}{2+\tilde{\gamma}_2}, \quad r_1 = \frac{n+1}{n-\tilde{\eta}_1}, \quad r_2 = \frac{n(n+1)}{n+2+\tilde{\eta}_2}.$$

Then the conditions (2.7)–(2.10) are accomplished by

$$\begin{aligned} &n-2 < \tilde{\gamma}_1 + \tilde{\eta}_1 \leq n-2 + \min\{\gamma_1, \eta_1\}, \qquad n-2 < \tilde{\gamma}_2 + \tilde{\eta}_2 \leq n-2 + \min\{\gamma_2, \eta_2\}, \\ &\gamma_i \leq \tilde{\gamma}_i, \qquad \eta_i \leq \tilde{\eta}_i, \quad i = 1, 2. \end{aligned}$$

For each $i = 1, 2$, the set of $(\tilde{\gamma}_i, \tilde{\eta}_i)$ with those conditions is nonvoid for given γ_i and η_i; for instance, we may take $\tilde{\gamma}_i = \gamma_i$, $\tilde{\eta}_i = n-2$ when $\gamma_i \leq \eta_i$ and take $\tilde{\gamma}_i = n-2$, $\tilde{\eta}_i = \eta_i$ when $\gamma_i \geq \eta_i$.

To obtain a small solution, we use the contraction mapping principle. We define

$$B := \{u \in L^{\alpha_1}(D) \cap L^{\alpha_2}(D) \mid \nabla u \in L^{\beta_1}(D) \cap L^{\beta_1}(D)\}$$

which is a Banach space endowed with the norm

$$\|u\|_B := \sum_{i=1}^{2}(a^{\frac{2}{n+1}}\|u\|_{\alpha_i} + a^{\frac{1}{n+1}}\|\nabla u\|_{\beta_i}).$$

Given $v \in B$, which satisfies

$$v\cdot\nabla v \in \bigcap_{i=1}^{2} L^{\kappa_i}(D), \qquad \frac{1}{\kappa_i} = \frac{1}{q_i} + \frac{1}{r_i}, \quad 1 < \kappa_i < \frac{n}{2}$$

for $i = 1, 2$, we can employ Proposition 2.1 with $f = v \cdot \nabla v$, $q = \kappa_i$ $(i = 1, 2)$ and $u_* = -ae_1$. Then, due to Lemma 2.3, the problem (1.11) admits a unique solution (u, p) such that

$$\begin{aligned} &a^{\frac{2}{n+1}}\|u\|_{\mu_i} + a\left\|\frac{\partial u}{\partial x_1}\right\|_{\kappa_i} + a^{\frac{1}{n+1}}\|\nabla u\|_{\lambda_i} + \|\nabla^2 u\|_{\kappa_i} + \|\nabla p\|_{\kappa_i} \\ &\quad \le C'(\|v\cdot\nabla v\|_{\kappa_i} + a) \le C'(\|v\|_{q_i}\|\nabla v\|_{r_i} + a) \le C'(a^{-\frac{3}{n+1}}\|v\|_B^2 + a) \end{aligned}$$

for $i = 1, 2$. Here, $1/\lambda_i = 1/\kappa_i - 1/(n+1)$, $1/\mu_i = 1/\kappa_i - 2/(n+1)$. Furthermore, because the conditions (2.9) and (2.10) ensure $\mu_1 \le \alpha_1 \le \alpha_2 \le \mu_2$ and $\lambda_1 \le \beta_1 \le \beta_2 \le \lambda_2$, we find $u \in B$ with

$$\|u\|_B \le 4C'(a^{-\frac{3}{n+1}}\|v\|_B^2 + a).$$

Hence, we assume

$$a^{\frac{n-2}{n+1}} < \frac{1}{64C'^2} =: \delta \tag{2.11}$$

and set

$$N_a := \{u \in B \mid \|u\|_B \le 8C'a\}$$

to see that the map $\Psi : N_a \ni v \mapsto u \in N_a$ is well-defined. Moreover, for $v_i \in N_a$ $(i = 1, 2)$, set $u_i = \Psi(v_i)$ and let p_i be the pressure associated with u_i. Then we have

$$\begin{cases} \Delta(u_1-u_2) - a\dfrac{\partial}{\partial x_1}(u_1-u_2) = \nabla(p_1-p_2) + (v_1-v_2)\cdot\nabla v_1 + v_2\cdot\nabla(v_2-v_1), & x \in D, \\ \nabla\cdot(u_1-u_2) = 0, \quad x \in D, \\ (u_1-u_2)|_{\partial D} = 0, \\ u_1 - u_2 \to 0 \quad \text{as } |x|\to\infty. \end{cases}$$

By applying Proposition 2.1 again, we find

$$\|u_1-u_2\|_B \le 4C'a^{-\frac{3}{n+1}}(\|v_1\|_B + \|v_2\|_B)\|v_1-v_2\|_B \le 64C'^2 a^{\frac{n-2}{n+1}}\|v_1-v_2\|_B$$

and the map Ψ is contractive on account of (2.11). The proof is complete. □

3. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. We define the operator $A_a : L^q_\sigma(D) \to L^q_\sigma(D)$ $(a > 0, 1 < q < \infty)$ by

$$\mathscr{D}(A_a) = W^{2,q}(D) \cap W^{1,q}_0(D) \cap L^q_\sigma(D), \quad A_a u = -P\left[\Delta u - a\frac{\partial u}{\partial x_1}\right].$$

Here, $W^{1,q}_0(D)$ denotes the completion of $C_0^\infty(D)$ in the Sobolev space $W^{1,q}(D)$. It is well known that $-A_a$ generates an analytic C_0-semigroup e^{-tA_a} called the Oseen semigroup in $L^q_\sigma(D)$, see Miyakawa [30, Theorem 4.2], Enomoto and Shibata [7, Theorem 4.4]. The following L^q-L^r estimates of e^{-tA_a}, which play an important role in the proof of Theorem 1.2, were established by Kobayashi and Shibata [26] in the three-dimensional case and further developed by Enomoto and Shibata [7,8] for $n \ge 3$. We also note that

L^q-L^r estimates in the two-dimensional case were first established by Hishida [22], and recently Maekawa [28] derived those estimates uniformly in small $a > 0$ as a significant improvement of [22].

Proposition 3.1. [7,8,26] *Let $n \ge 3$, $\sigma_0 > 0$ and assume $|a| \le \sigma_0$.*

1. *Let $1 < q \le r \le \infty$ $(q \ne \infty)$. Then we have*
$$\|e^{-tA_a} f\|_r \le Ct^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{r})}\|f\|_q \tag{3.1}$$
for $t > 0$ and $f \in L^q_\sigma(D)$, where $C = C(n, \sigma_0, q, r, D) > 0$ is independent of a.
2. *Let $1 < q \le r \le n$. Then we have*
$$\|\nabla e^{-tA_a} f\|_r \le Ct^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{r})-\frac{1}{2}}\|f\|_q \tag{3.2}$$
for $t > 0$ and $f \in L^q_\sigma(D)$, where $C = C(n, \sigma_0, q, r, D) > 0$ is independent of a.
3. *Let $n/(n-1) \le q \le r \le \infty$ $(q \ne \infty)$. Then we have*
$$\|e^{-tA_a} P\nabla \cdot F\|_r \le Ct^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{r})-\frac{1}{2}}\|F\|_q \tag{3.3}$$
for $t > 0$ and $F \in L^q(D)$, where $C = C(n, \sigma_0, q, r, D) > 0$ is independent of a.

The proof of the assertion 3 is simply based on duality argument together with semigroup property especially for the case $r = \infty$.

We also prepare the following lemma, which plays a role to prove the uniqueness within Y defined by (1.27).

Lemma 3.2. *Let $n \ge 3$ and $a > 0$. For each precompact set $K \subset L^n_\sigma(D)$, we have*
$$\lim_{t\to 0} \sup_{f\in K} t^{\frac{1}{2}}\left(\|e^{-tA_a} f\|_\infty + \|\nabla e^{-tA_a} f\|_n\right) = 0. \tag{3.4}$$

Proof. By applying Proposition 3.1 and approximating $f \in L^n_\sigma(D)$ by a sequence in $C^\infty_{0,\sigma}(D)$, we have
$$\lim_{t\to 0} t^{\frac{1}{2}}\left(\|e^{-tA_a} f\|_\infty + \|\nabla e^{-tA_a} f\|_n\right) = 0 \tag{3.5}$$
for all $f \in L^n_\sigma(D)$. Given $\eta > 0$, let $f_1, \cdots, f_m \in K$ fulfill $K \subset \bigcup_{j=1}^m B(f_j; \eta)$, where $B(f_j; \eta) := \{g \in L^n_\sigma(D) \mid \|g - f_j\|_n < \eta\}$. For each $f \in K$, we choose $f_i \in K$ such that $f \in B(f_i; \eta)$. Then it follows from (3.1) that
$$\begin{aligned} t^{\frac{1}{2}}\|e^{-tA_a} f\|_\infty &\le t^{\frac{1}{2}}\|e^{-tA_a} f_i\|_\infty + t^{\frac{1}{2}}\|e^{-tA_a}(f - f_i)\|_\infty \\ &\le t^{\frac{1}{2}}\|e^{-tA_a} f_i\|_\infty + C\|f - f_i\|_n \le \sum_{j=1}^m t^{\frac{1}{2}}\|e^{-tA_a} f_j\|_\infty + C\eta. \end{aligned}$$
Since the right-hand side is independent of $f \in K$ and since η is arbitrary, (3.5) yields
$$\lim_{t\to 0} \sup_{f\in K} t^{\frac{1}{2}}\|e^{-tA_a} f\|_\infty = 0.$$
We can discuss the L^n norm of the first derivative similarly and thus conclude (3.4). □

We recall a function space Y_0 defined by (1.22), which is a Banach space equipped with norm $\|\cdot\|_Y = \|\cdot\|_{Y,\infty}$, where
$$\begin{aligned} &\|v\|_{Y,t} := [v]_{n,t} + [v]_{\infty,t} + [\nabla v]_{n,t}, \\ &[v]_{q,t} := \sup_{0<\tau<t} \tau^{\frac{1}{2}-\frac{n}{2q}}\|v(\tau)\|_q, \quad q = n, \infty; \qquad [\nabla v]_{n,t} := \sup_{0<\tau<t} \tau^{\frac{1}{2}}\|\nabla v(\tau)\|_n \end{aligned}$$
for $t \in (0, \infty]$. Construction of the solution is based on the following.

Lemma 3.3. *Suppose* $0 < a^{(n-2)/(n+1)} < \delta$, *where* δ *is a constant in Theorem 1.1 with (1.15)–(1.17). Let* ψ *be a function on* $\mathbb{R}$ *satisfying (1.1) and set* $M = \max_{t\in\mathbb{R}} |\psi'(t)|$. *Suppose that* u_s *is the stationary solution obtained in Theorem 1.1. For* $u, v \in Y_0$, *we set*

$$G_1(u,v)(t) = \int_0^t e^{-(t-\tau)A_a} P[u\cdot\nabla v](\tau)\,d\tau, \quad G_2(v)(t) = \int_0^t e^{-(t-\tau)A_a} P[\psi(\tau)v\cdot\nabla u_s]\,d\tau,$$
$$G_3(v)(t) = \int_0^t e^{-(t-\tau)A_a} P[\psi(\tau)u_s\cdot\nabla v]\,d\tau,$$
$$G_4(v)(t) = \int_0^t e^{-(t-\tau)A_a} P\left[(1-\psi(\tau))a\frac{\partial v}{\partial x_1}(\tau)\right]d\tau,$$
$$H_1(t) = \int_0^t e^{-(t-\tau)A_a} Ph_1(\tau)\,d\tau, \quad H_2(t) = \int_0^t e^{-(t-\tau)A_a} Ph_2(\tau)\,d\tau,$$

where h_1 *and* h_2 *are defined by (1.19) and (1.20), respectively. Then we have* $G_1(u,v), G_i(v), H_j \in Y_0$ $(i = 2,3,4, j = 1,2)$ *along with*

$$\|G_1(u,v)\|_{Y,t} \le C[u]_{n,t}^{\frac12}[u]_{\infty,t}^{\frac12}[\nabla v]_{n,t}, \tag{3.6}$$
$$\|G_2(v)\|_{Y,t} \le C\big(\|\nabla u_s\|_{\frac{n}{2+\rho_3}} + \|\nabla u_s\|_{\frac{n}{2}} + \|\nabla u_s\|_{\frac{n}{2-\rho_4}}\big)[v]_{\infty,t}, \tag{3.7}$$
$$\|G_3(v)\|_{Y,t} \le C\big(\|u_s\|_{\frac{n}{1+\rho_1}} + \|u_s\|_n + \|u_s\|_{\frac{n}{1-\rho_2}}\big)[\nabla v]_{n,t}, \tag{3.8}$$
$$\|G_4(v)\|_{Y,t} \le Ca[\nabla v]_{n,t}, \tag{3.9}$$
$$\|H_1\|_{Y,t} \le CM\|u_s\|_n, \tag{3.10}$$
$$\|H_2\|_{Y,t} \le C\big(\|u_s\|_{\frac{n}{1-\rho_2}}\|\nabla u_s\|_{\frac{n}{2-\rho_4}} + a\|\nabla u_s\|_{\frac{n}{2-\rho_4}}\big) \tag{3.11}$$

for all $t \in (0,\infty]$ *and*

$$\lim_{t\to 0}\|H_j(t)\|_{Y,t} = 0 \tag{3.12}$$

for $j = 1,2$. *Here,* C *is a positive constant independent of* u, v, ψ, a *and* t.

Proof. The continuity of those functions in t is deduced by use of properties of analytic semigroups together with Proposition 3.1 in the same way as in Fujita and Kato [15]. Since L^∞ estimate is always the same as L^n estimate of the first derivative, the estimate of $[\cdot]_{\infty,t}$ may be omitted. Although (3.6)–(3.8) are discussed in Enomoto and Shibata [8, Lemma 3.1.] we briefly give the proof for completeness. We find that $u \in Y_0$ satisfies $u(t) \in L^{2n}(D)$ and

$$\|u(t)\|_{2n} \le t^{-\frac14}[u]_{n,t}^{\frac12}[u]_{\infty,t}^{\frac12}$$

for all $t > 0$, which together with Proposition 3.1 implies

$$\int_0^t \|e^{-(t-\tau)A_a}P[u\cdot\nabla v](\tau)\|_n\,d\tau \le C\int_0^t (t-\tau)^{-\frac14}\|u(\tau)\|_{2n}\|\nabla v(\tau)\|_n\,d\tau \le C[u]_{n,t}^{\frac12}[u]_{\infty,t}^{\frac12}[\nabla v]_{n,t}$$

and

$$\int_0^t \|\nabla e^{-(t-\tau)A_a}P[u\cdot\nabla v](\tau)\|_n\,d\tau \le C\int_0^t (t-\tau)^{-\frac34}\|u(\tau)\|_{2n}\|\nabla v(\tau)\|_n\,d\tau$$
$$\le Ct^{-\frac12}[u]_{n,t}^{\frac12}[u]_{\infty,t}^{\frac12}[\nabla v]_{n,t}.$$

We thus conclude (3.6). It follows from Proposition 3.1 that

$$\int_0^t \|e^{-(t-\tau)A_a}P[\psi(\tau)v\cdot\nabla u_s]\|_n\,d\tau \le C\int_0^t (t-\tau)^{-\frac12}\|v(\tau)\|_\infty\|\nabla u_s\|_{\frac{n}{2}}\,d\tau \le C[v]_{\infty,t}\|\nabla u_s\|_{\frac{n}{2}} \tag{3.13}$$

and that

$$\begin{aligned}\int_0^t \|\nabla e^{-(t-\tau)A_a} P[\psi(\tau) v\cdot\nabla u_s]\|_n \,d\tau &\le C\int_0^t (t-\tau)^{-1+\frac{\rho_4}{2}} \|v(\tau)\|_\infty \|\nabla u_s\|_{\frac{n}{2-\rho_4}}\,d\tau\\ &\le Ct^{-\frac12+\frac{\rho_4}{2}}[v]_{\infty,t}\|\nabla u_s\|_{\frac{n}{2-\rho_4}} \end{aligned}\tag{3.14}$$

for $t>0$. Furthermore, for $t\ge 2$, we split the integral into

$$\int_0^t \|\nabla e^{-(t-\tau)A_a} P[\psi(\tau) v\cdot\nabla u_s]\|_n \,d\tau = \int_0^{\frac t2} + \int_{\frac t2}^{t-1} + \int_{t-1}^t \tag{3.15}$$

as in [6,8]. By applying (3.2), we have

$$\int_0^{\frac t2} \le C\int_0^{\frac t2} (t-\tau)^{-1}\|v(\tau)\|_\infty \|\nabla u_s\|_{\frac n2}\,d\tau \le Ct^{-\frac12}[v]_{\infty,t}\|\nabla u_s\|_{\frac n2}, \tag{3.16}$$

$$\int_{\frac t2}^{t-1} \le C\int_{\frac t2}^{t-1} (t-\tau)^{-1-\frac{\rho_3}{2}}\|v(\tau)\|_\infty \|\nabla u_s\|_{\frac{n}{2+\rho_3}}\,d\tau \le Ct^{-\frac12}[v]_{\infty,t}\|\nabla u_s\|_{\frac{n}{2+\rho_3}}, \tag{3.17}$$

$$\int_{t-1}^{t} \le C\int_{t-1}^{t} (t-\tau)^{-1+\frac{\rho_4}{2}}\|v(\tau)\|_\infty \|\nabla u_s\|_{\frac{n}{2-\rho_4}}\,d\tau \le Ct^{-\frac12}[v]_{\infty,t}\|\nabla u_s\|_{\frac{n}{2-\rho_4}}. \tag{3.18}$$

Combining (3.13)–(3.18) yields (3.7). By the same manner, we obtain (3.8). We use Proposition 3.1 to find

$$\begin{aligned}\int_0^t \left\|\nabla^k e^{-(t-\tau)A_a} P\left[\big(1-\psi(\tau)\big)a\frac{\partial v}{\partial x_1}\right]\right\|_n d\tau &\le Ca\int_0^{\min\{1,t\}} (t-\tau)^{-\frac k2}\|\nabla v(\tau)\|_n\,d\tau\\ &\le Ca[\nabla v]_{n,t}\int_0^{\min\{1,t\}}(t-\tau)^{-\frac k2}\tau^{-\frac12}\,d\tau\end{aligned}$$

for $k=0,1$, which lead us to (3.9). We see (3.10) from

$$\int_0^t \left\|\nabla^k e^{-(t-\tau)A_a} P[\psi'(\tau)u_s]\right\|_n d\tau \le CM\|u_s\|_n \int_0^{\min\{1,t\}} (t-\tau)^{-\frac k2}\,d\tau \tag{3.19}$$

for $k=0,1$ and (3.11) from

$$\begin{aligned}&\int_0^t \left\|\nabla^k e^{-(t-\tau)A_a} P\left[\psi(\tau)(1-\psi(\tau))\Big(u_s\cdot\nabla u_s + a\frac{\partial u_s}{\partial x_1}\Big)\right]\right\|_n d\tau\\ &\quad\le C\|u_s\|_{\frac{n}{1-\rho_2}}\|\nabla u_s\|_{\frac{n}{2-\rho_4}}\int_0^{\min\{1,t\}}(t-\tau)^{\frac{\rho_2+\rho_4}{2}-1-\frac k2}\,d\tau\\ &\qquad + Ca\|\nabla u_s\|_{\frac{n}{2-\rho_4}}\int_0^{\min\{1,t\}}(t-\tau)^{-\frac12+\frac{\rho_4}{2}-\frac k2}\,d\tau\end{aligned}\tag{3.20}$$

for $k=0,1$, where the condition (1.17) is used. The behavior of $G_1(u,v)(t)$ and $G_i(v)(t)$ as well as the one of $H_j(t)$, see (3.12), as $t\to 0$ follows from (3.6)–(3.9) and (3.19)–(3.20) with $t<1$, so that $G_1(u,v), G_i(v), H_j\in Y_0$ and $\|G_1(u,v)(t)\|_n + \|G_i(v)(t)\|_n + \|H_j(t)\|_n \to 0$ as $t\to 0$. The proof is complete. □

Let us construct a solution of (1.21) by applying Lemma 3.3.

Proposition 3.4. *Let δ be the constant in Theorem 1.1 with (1.15)–(1.17). Let ψ be a function on $\mathbb{R}$ satisfying (1.1) and set $M=\max_{t\in\mathbb{R}}|\psi'(t)|$. Then there exists a constant $\varepsilon=\varepsilon(n,D)\in(0,\delta]$ such that if $0<(M+1)a^{(n-2)/(n+1)}<\varepsilon$, (1.21) admits a solution $v\in Y_0$ with*

$$\|v\|_Y \le C(M+1)a^{\frac{n-2}{n+1}} \tag{3.21}$$

and

$$\lim_{t\to 0}\|v(t)\|_n = 0. \tag{3.22}$$

Proof. We set

$$\begin{aligned} &v_0(t) = 0,\\ &v_{m+1}(t) = \int_0^t e^{-(t-\tau)A_a} P\Big[-v_m\cdot\nabla v_m - \psi(\tau)v_m\cdot\nabla u_s - \psi(\tau)u_s\cdot\nabla v_m + (1-\psi(\tau))a\frac{\partial v_m}{\partial x_1}\\ &\qquad + h_1(\tau) + h_2(\tau)\Big]d\tau \end{aligned} \tag{3.23}$$

for $m \geq 0$. It follows from Theorem 1.1, Lemma 3.3 and $a \in (0, 1)$ that $v_m \in Y_0$ together with

$$\|v_m\|_{Y,t} \leq \|G_1(v_{m-1}, v_{m-1})\|_{Y,t} + \sum_{i=2}^4 \|G_i(v_{m-1})\|_{Y,t} + \|H_1\|_{Y,t} + \|H_2\|_{Y,t}, \tag{3.24}$$

$$\|v_m\|_Y \leq C_1\|v_{m-1}\|_Y^2 + C_2 a^{\frac{n-2}{n+1}}\|v_{m-1}\|_Y + C_3(M+1)a^{\frac{n-2}{n+1}},$$

$$\|v_{m+1} - v_m\|_Y \leq \{C_1(\|v_m\|_Y + \|v_{m-1}\|_Y) + C_2 a^{\frac{n-2}{n+1}}\}\|v_m - v_{m-1}\|_Y \tag{3.25}$$

for all $m \geq 1$. Hence, if we assume

$$(M+1)a^{\frac{n-2}{n+1}} < \min\left\{\delta, \frac{1}{2C_2}, \frac{1}{16C_1C_3}\right\} =: \varepsilon, \tag{3.26}$$

it holds that

$$\|v_m\|_Y \leq \frac{1 - C_2 a^{\frac{n-2}{n+1}} - \sqrt{(1 - C_2 a^{\frac{n-2}{n+1}})^2 - 4C_1C_3(M+1)a^{\frac{n-2}{n+1}}}}{2C_1} \leq 4C_3(M+1)a^{\frac{n-2}{n+1}},$$

$$\|v_{m+1} - v_m\|_Y \leq \{8C_1C_3(M+1)a^{\frac{n-2}{n+1}} + C_2 a^{\frac{n-2}{n+1}}\}\|v_m - v_{m-1}\|_Y \tag{3.27}$$

for all $m \geq 1$ and that

$$8C_1C_3(M+1)a^{\frac{n-2}{n+1}} + C_2 a^{\frac{n-2}{n+1}} < 1.$$

Therefore, we obtain a solution $v \in Y_0$ satisfying (3.21) with $C = 4C_3$. Moreover, by letting $m \to \infty$ in (3.24) and by using (3.6)–(3.9) and (3.12), we have (3.22), which completes the proof. □

Remark 3.5. Let $b \in L^n_\sigma(D)$. By the same procedure, we can also construct a solution $T(t)b := v(t) \in Y_0$ for the integral equation

$$\begin{aligned} v(t) = e^{-tA_a}b + \int_0^t e^{-(t-\tau)A_a} P\Big[-v\cdot\nabla v - \psi(\tau)v\cdot\nabla u_s - \psi(\tau)u_s\cdot\nabla v\\ + (1-\psi(\tau))a\frac{\partial v}{\partial x_1} + h_1(\tau) + h_2(\tau)\Big]d\tau \end{aligned} \tag{3.28}$$

whenever

$$\|b\|_n + (M+1)a^{\frac{n-2}{n+1}} < \min\left\{\delta, \frac{1}{2C_2}, \frac{1}{16C_1C_0}, \frac{1}{16C_1C_3}\right\}$$

is satisfied. Here, the constant C_0 is determined by the following three estimates:

$$\|e^{-tA_a}b\|_q \leq C_0 t^{-\frac{1}{2}+\frac{3}{2q}}\|b\|_n, \qquad q = n, \infty; \qquad \|\nabla e^{-tA_a}b\|_n \leq C_0 t^{-\frac{1}{2}}\|b\|_n.$$

Moreover, we find that the solution $T(t)b$ is estimated by

$$\|T(\cdot)b\|_Y \leq 4\big(C_0\|b\|_n + C_3(M+1)a^{\frac{n-2}{n+1}}\big).$$

This will be used in the proof of uniqueness of solutions within Y, see (1.27).

We further derive sharp decay properties of the solution $v(t)$ obtained above. To this end, the first step is the following. In what follows, for simplicity of notation, we write

$$G_1(t) = G_1(v,v)(t), \quad G_i(t) = G_i(v)(t)$$

for $i = 2,3,4$, which are defined in Lemma 3.3.

Lemma 3.6. *Let ε be the constant in Proposition 3.4. Given $\rho \in (0,1)$ satisfying $\rho \leq \min\{\rho_1, \rho_3\}$, there exists a constant $\varepsilon' = \varepsilon'(\rho, n, D) \in (0, \varepsilon]$ such that if $0 < (M+1)a^{(n-2)/(n+1)} < \varepsilon'$, then the solution $v(t)$ obtained in Proposition 3.4 satisfies*

$$\|v(t)\|_q = O(t^{-\frac{1}{2}+\frac{n}{2q}-\frac{\rho}{2}}), \qquad n \leq \forall q \leq \infty, \tag{3.29}$$

$$\|\nabla v(t)\|_n = O(t^{-\frac{1}{2}-\frac{\rho}{2}}) \tag{3.30}$$

as $t \to \infty$.

Proof. We start with the case $q = n$, that is,

$$\|v(t)\|_n = O(t^{-\frac{\rho}{2}}) \tag{3.31}$$

as $t \to \infty$. By using (3.1), we have

$$\|G_1(t)\|_n \leq Ct^{-\frac{\rho}{2}}\Big(\sup_{0<\tau<t} \tau^{\frac{1}{2}}\|\nabla v(\tau)\|_n\Big)\Big(\sup_{0<\tau<t} \tau^{\frac{\rho}{2}}\|v(\tau)\|_n\Big) \leq Ct^{-\frac{\rho}{2}}\|v\|_Y \sup_{0<\tau<t} \tau^{\frac{\rho}{2}}\|v(\tau)\|_n, \tag{3.32}$$

$$\|G_2(t)\|_n \leq Ct^{-\frac{\rho_3}{2}}\|\nabla u_s\|_{\frac{n}{2+\rho_3}} \sup_{0<\tau<t} \tau^{\frac{1}{2}}\|v(\tau)\|_\infty \leq Ct^{-\frac{\rho_3}{2}}\|\nabla u_s\|_{\frac{n}{2+\rho_3}}\|v\|_Y \tag{3.33}$$

and

$$\|G_3(t)\|_n \leq Ct^{-\frac{\rho_1}{2}}\|u_s\|_{\frac{n}{1+\rho_1}}\|v\|_Y \tag{3.34}$$

for all $t > 0$. Moreover, we obtain

$$\|G_4(t)\|_n \leq Ca\int_0^{\min\{1,t\}} (t-\tau)^{-\frac{1}{2}}\|v(\tau)\|_n\,d\tau \leq Cat^{-\frac{1}{2}}\|v\|_Y \tag{3.35}$$

for all $t > 0$ by use of (3.3). From (3.1) we see that

$$\|H_1(t)\|_n \leq CMt^{-\frac{\rho_1}{2}}\|u_s\|_{\frac{n}{1+\rho_1}} \tag{3.36}$$

and that

$$\|H_2(t)\|_n \leq Ct^{-\frac{2-\rho_2}{2}}\|u_s\|_{\frac{n}{1-\rho_2}}\|\nabla u_s\|_{\frac{n}{2}} + Cat^{-\frac{1+\rho_3}{2}}\|\nabla u_s\|_{\frac{n}{2+\rho_3}} \tag{3.37}$$

for $t > 0$. Note that $\rho_2 < 1$, see (1.16). Collecting (3.32)–(3.37) for $t > 1$ and (3.21) with $C = 4C_3$ yields

$$\begin{aligned}\sup_{0<\tau<t} \tau^{\frac{\rho}{2}}\|v(\tau)\|_n &\leq C_4\|v\|_Y \sup_{0<\tau<t} \tau^{\frac{\rho}{2}}\|v(\tau)\|_n + C_5\\ &\leq 4C_3C_4(M+1)a^{\frac{n-2}{n+1}} \sup_{0<\tau<t} \tau^{\frac{\rho}{2}}\|v(\tau)\|_n + C_5\end{aligned}$$

with some constants $C_4 = C_4(\rho) > 0$ and $C_5 = C_5(\|v\|_Y, u_s, a, M, \rho_1, \rho_2, \rho_3) > 0$ independent of t, where C_3 comes from estimates of $H_j(t)$ $(j = 1,2)$ in (3.25). Therefore, if we assume

$$(M+1)a^{\frac{n-2}{n+1}} < \min\Big\{\varepsilon, \frac{1}{4C_3C_4}\Big\} =: \varepsilon',$$

we have $\|v(t)\|_n \leq Ct^{-\rho/2}$ for all $t > 0$, which implies (3.31).

We next show that

$$\|v(t)\|_\infty + \|\nabla v(t)\|_n = O(t^{-\frac{1}{2}-\frac{\rho}{2}})$$

as $t \to \infty$, which together with (3.31) implies (3.29) and (3.30). It suffices to show that

$$t^{\frac{1}{2}}\|v(t)\|_\infty + t^{\frac{1}{2}}\|\nabla v(t)\|_n \leq C\Big\|v\Big(\frac{t}{2}\Big)\Big\|_n \tag{3.38}$$

for all $t \geq 2$. The following argument is similar to Enomoto and Shibata [8]. When $t \geq T > 1$, we have

$$v(t) = e^{-(t-T)A_a}v(T) - \int_T^t e^{-(t-\tau)A_a}P\big[v\cdot\nabla v + v\cdot\nabla u_s + u_s\cdot\nabla v\big]\,d\tau. \tag{3.39}$$

By the same argument as in the proof of Lemma 3.3 and by (1.10), (3.26) as well as (3.21) with $C = 4C_3$, the integral of (3.39) is estimated as

$$\begin{aligned}
&\int_T^t \|e^{-(t-\tau)A_a}P[\cdots]\|_\infty\,d\tau + \int_T^t \|\nabla e^{-(t-\tau)A_a}P[\cdots]\|_n\,d\tau \\
&\le C_1(t-T)^{-\frac12}\big(\sup_{T\le\tau\le t}\|v(\tau)\|_n\big)^{\frac12}\big(\sup_{T\le\tau\le t}(\tau-T)^{\frac12}\|v(\tau)\|_\infty\big)^{\frac12}\big(\sup_{T\le\tau\le t}(\tau-T)^{\frac12}\|\nabla v(\tau)\|_n\big) \\
&\quad + C_2 a^{\frac{n-2}{n+1}}(t-T)^{-\frac12}\big\{\sup_{T\le\tau\le t}(\tau-T)^{\frac12}\|v(\tau)\|_\infty + \sup_{T\le\tau\le t}(\tau-T)^{\frac12}\|\nabla v(\tau)\|_n\big\} \\
&\le C_1(t-T)^{-\frac12}\|v\|_Y\sup_{T\le\tau\le t}(\tau-T)^{\frac12}\|\nabla v(\tau)\|_n \\
&\quad + \frac12(t-T)^{-\frac12}\big\{\sup_{T\le\tau\le t}(\tau-T)^{\frac12}\|v(\tau)\|_\infty + \sup_{T\le\tau\le t}(\tau-T)^{\frac12}\|\nabla v(\tau)\|_n\big\} \\
&\le \frac34(t-T)^{-\frac12}\sup_{T\le\tau\le t}(\tau-T)^{\frac12}\|\nabla v(\tau)\|_n + \frac12(t-T)^{-\frac12}\sup_{T\le\tau\le t}(\tau-T)^{\frac12}\|v(\tau)\|_\infty.
\end{aligned}$$

Therefore, we have

$$\sup_{T\le\tau\le t}(\tau-T)^{\frac12}\|\nabla v(\tau)\|_n + \sup_{T\le\tau\le t}(\tau-T)^{\frac12}\|v(\tau)\|_\infty \le C\|v(T)\|_n$$

for all $t \geq T$. This combined with $t^{1/2} \le \sqrt{2}(t-T)^{1/2}$ for $t \geq 2T$ asserts that

$$t^{\frac12}\|\nabla v(t)\|_n + t^{\frac12}\|v(t)\|_\infty \le C\|v(T)\|_n$$

for all $t \geq 2T$. We then put $T = t/2$ $(t \geq 2)$ to conclude (3.38). □

Sharp decay properties (1.23)–(1.24) for the case $n = 3$ are established in the following proposition.

Proposition 3.7. *Let $n = 3$ and set $\varepsilon_* := \varepsilon'(\rho, 3, D)$ which is the constant in Lemma 3.6 with $\rho := \min\{\rho_1, \rho_3\}$ (recall that $0 < \rho_1 < 1/2$, $0 < \rho_3 < 1/4$ for $n = 3$). If $0 < (M+1)a^{1/4} < \varepsilon_*$, then the solution $v(t)$ obtained in Proposition 3.4 enjoys (1.23) and (1.24).*

Proof. The case $\rho_1 \le \rho_3$ directly follows from Lemma 3.6. To discuss the other case $\rho_3 < \rho_1$, we show by induction that if $0 < (M+1)a^{1/4} < \varepsilon_*$, then

$$\|v(t)\|_3 = O(t^{-\sigma_k}), \qquad \sigma_k := \min\Big\{\frac{k}{2}\rho_3,\ \frac{\rho_1}{2}\Big\} \tag{3.40}$$

as $t \to \infty$ for all $k \geq 1$. We already know (3.40) with $k = 1$ from Lemma 3.6.

Let $k \geq 2$ and suppose (3.40) with $k-1$. By taking (3.21) (near $t = 0$) and (3.38) into account, we have

$$J_{k-1}(v) := \sup_{\tau>0}(1+\tau)^{\sigma_{k-1}}\|v(\tau)\|_3 + \sup_{\tau>0}\tau^{\frac12}(1+\tau)^{\sigma_{k-1}}\big(\|v(\tau)\|_\infty + \|\nabla v(\tau)\|_3\big) < \infty.$$

We use this to see that

$$\begin{aligned}
\|G_1(t)\|_3 &\le C\int_0^t (t-\tau)^{-\frac12}\tau^{-\frac12}(1+\tau)^{-2\sigma_{k-1}}\,d\tau \\
&\qquad \times \big(\sup_{\tau>0}(1+\tau)^{\sigma_{k-1}}\|v(\tau)\|_3\big)\big(\sup_{\tau>0}\tau^{\frac12}(1+\tau)^{\sigma_{k-1}}\|\nabla v(\tau)\|_3\big) \\
&\le Ct^{-2\sigma_{k-1}}J_{k-1}(v)^2,
\end{aligned}$$

and that

$$\|G_2(t)\|_3 \le C\int_0^t (t-\tau)^{-\frac{1+\rho_3}{2}}\tau^{-\frac12}(1+\tau)^{-\sigma_{k-1}}\,d\tau\,\|\nabla u_s\|_{\frac{3}{2+\rho_3}}\sup_{\tau>0}\tau^{\frac12}(1+\tau)^{\sigma_{k-1}}\|v(\tau)\|_\infty$$
$$\le Ct^{-\frac{\rho_3}{2}-\sigma_{k-1}}\|\nabla u_s\|_{\frac{3}{2+\rho_3}}J_{k-1}(v)$$

for $t>0$ due to $\sigma_{k-1}\le \rho_1/2<1/4$. From these and (3.34)–(3.37), we obtain (3.40) with k. We thus conclude (1.23) with $q=3$, which together with (3.38) completes the proof. □

To derive even more rapid decay properties of the solution $v(t)$ for $n\ge 4$, we need the following lemma, which gives the L^{q_0}-decay of $v(t)$ with a specific q_0, see (3.43).

Lemma 3.8. *Let $n\ge 4$. Suppose $1<\rho_1\le 1+\rho_3$ in addition to* (1.16) (*the set of those parameters is nonvoid when $n\ge 4$*). *Let ε be the constant in Proposition 3.4 and $v(t)$ the solution obtained there. Given γ satisfying*

$$\max\left\{0,\ \frac{\rho_1+3-n}{2}\right\}<\gamma<\frac12 \tag{3.41}$$

(*note that* (1.16) *yields $\rho_1<n-2$*), *there exists a constant $\varepsilon''=\varepsilon''(\gamma,n,D)\in(0,\varepsilon]$ such that if $0<(M+1)a^{(n-2)/(n+1)}<\varepsilon''$, then $v(t)\in L^{q_0}(D)$ for all $t>0$ and*

$$\sup_{\tau>0}(1+\tau)^\gamma\|v(\tau)\|_{q_0}<\infty, \tag{3.42}$$

where

$$q_0 := \frac{n}{1+\rho_1-2\gamma}\,(<n). \tag{3.43}$$

Proof. We show that there exists a constant $\varepsilon''(\gamma,n,D)\in(0,\varepsilon]$ such that if $0<(M+1)a^{(n-2)/(n+1)}<\varepsilon''$, then $v_m(t)\in L^{q_0}(D)$ for all $t>0$ along with

$$K_m := \sup_{\tau>0}(1+\tau)^\gamma\|v_m(\tau)\|_{q_0}<\infty,\qquad K_m\le \frac12 K_{m-1}+C(M+1)a^{\frac{n-1}{n+1}} \tag{3.44}$$

for all $m\ge 1$, where $v_m(t)$ is the approximate solution defined by (3.23) and C is a positive constant independent of a and m. We use (3.1) to see that

$$\int_0^t \|e^{-(t-\tau)A_a}Ph_1(\tau)\|_{q_0}\,d\tau \le CM\|u_s\|_{\frac{n}{1+\rho_1}}\int_0^{\min\{1,t\}}(t-\tau)^{-\gamma}\,d\tau \le CM\|u_s\|_{\frac{n}{1+\rho_1}}(1+t)^{-\gamma} \tag{3.45}$$

for $t>0$. Moreover, it holds that

$$\int_0^t \left\|e^{-(t-\tau)A_a}P\Big[\psi(\tau)\big(1-\psi(\tau)\big)a\frac{\partial u_s}{\partial x_1}\Big]\right\|_{q_0}d\tau \le Ca\|\nabla u_s\|_r$$

for $t\le 2$, where $r:=\min\{n/(2-\rho_4),q_0\}$ and that

$$\int_0^t \left\|e^{-(t-\tau)A_a}P\Big[\psi(\tau)\big(1-\psi(\tau)\big)a\frac{\partial u_s}{\partial x_1}\Big]\right\|_{q_0}d\tau \le Ca\|\nabla u_s\|_{\frac{n}{2+\rho_3}}\int_0^1 (t-\tau)^{-\gamma-\frac{1+\rho_3-\rho_1}{2}}\,d\tau$$
$$\le Ca\|\nabla u_s\|_{\frac{n}{2+\rho_3}}t^{-\gamma}$$

for $t>2$ as well as that

$$\int_0^t \|e^{-(t-\tau)A_a}P[\psi(\tau)\big(1-\psi(\tau)\big)u_s\cdot\nabla u_s]\|_{q_0}\,d\tau \le C\|u_s\|_{\frac{n}{1+\kappa}}\|\nabla u_s\|_{\frac n2}\int_0^{\min\{1,t\}}(t-\tau)^{-1-\gamma+\frac{\rho_1-\kappa}{2}}\,d\tau$$
$$\le C\|u_s\|_{\frac{n}{1+\kappa}}\|\nabla u_s\|_{\frac n2}(1+t)^{-\gamma}$$

for $t>0$, where $\max\{0,\rho_1-2\}<\kappa<\min\{n-3,\rho_1-2\gamma\}$ (note that (1.16) yields $\rho_1<n-1$). These estimates imply

$$\int_0^t \|e^{-(t-\tau)A_a}Ph_2(\tau)\|_{q_0}\,d\tau \le C(a\|\nabla u_s\|_r + a\|\nabla u_s\|_{\frac{n}{2+\rho_3}}+\|u_s\|_{\frac{n}{1+\kappa}}\|\nabla u_s\|_{\frac n2})(1+t)^{-\gamma}$$

for $t > 0$, which together with (3.45) and (1.10) leads us to $v_1(t) \in L^{q_0}(D)$ for all $t > 0$ with

$$K_1 \le C(M+1)a^{\frac{n-1}{n+1}}. \tag{3.46}$$

This proves (3.44) with $m = 1$ since $K_0 = 0$.

Let $m \ge 2$ and suppose that $v_{m-1}(t) \in L^{q_0}(D)$ for all $t > 0$ and (3.44) with $m-1$. Then we have $G_1(v_{m-1}, v_{m-1})(t) \in L^{q_0}(D)$ for $t > 0$ with

$$\sup_{\tau>0}(1+\tau)^\gamma \|G_1(v_{m-1}, v_{m-1})(\tau)\|_{q_0} \le CK_{m-1} \sup_{\tau>0} \tau^{\frac{1}{2}} \|\nabla v_{m-1}(\tau)\|_n. \tag{3.47}$$

Let $t \ge 2$ and split the integral into

$$\int_0^t \|e^{-(t-\tau)A_a} P[\psi(\tau) u_s \cdot \nabla v_{m-1}]\|_{q_0}\, d\tau = \int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^{t-1} + \int_{t-1}^t.$$

Let $\lambda \in (0, \rho_1]$ satisfy $\lambda < n - 3 + 2\gamma - \rho_1$; in fact, we can take such λ due to (3.41). Then (3.3) with $F = v_{m-1} \otimes u_s$ implies

$$\begin{aligned}
\int_0^{\frac{t}{2}} &\le C \int_0^{\frac{t}{2}} (t-\tau)^{-1} \|u_s\|_n \|v_{m-1}(\tau)\|_{q_0}\, d\tau \le Ct^{-\gamma}\|u_s\|_n K_{m-1}, \\
\int_{\frac{t}{2}}^{t-1} &\le C \int_{\frac{t}{2}}^{t-1} (t-\tau)^{-1-\frac{\lambda}{2}} \|u_s\|_{\frac{n}{1+\lambda}} \|v_{m-1}(\tau)\|_{q_0}\, d\tau \le Ct^{-\gamma}\|u_s\|_{\frac{n}{1+\lambda}} K_{m-1}, \\
\int_{t-1}^{t} &\le C \int_{t-1}^{t} (t-\tau)^{-1+\frac{\rho_2}{2}} \|u_s\|_{\frac{n}{1-\rho_2}} \|v_{m-1}(\tau)\|_{q_0}\, d\tau \le Ct^{-\gamma}\|u_s\|_{\frac{n}{1-\rho_2}} K_{m-1}
\end{aligned}$$

for $t \ge 2$. Moreover, we use (3.3) again to see that

$$\begin{aligned}
\int_0^t \|e^{-(t-\tau)A_a} P[\psi(\tau) u_s \cdot \nabla v_{m-1}]\|_{q_0}\, d\tau &\le C \int_0^t (t-\tau)^{-1+\frac{\rho_2}{2}} \|u_s\|_{\frac{n}{1-\rho_2}} \|v_{m-1}(\tau)\|_{q_0}\, d\tau \\
&\le C\|u_s\|_{\frac{n}{1-\rho_2}} K_{m-1}
\end{aligned}$$

for $t \le 2$. We thus conclude $G_3(v_{m-1})(t) \in L^{q_0}(D)$ for $t > 0$ with

$$\sup_{\tau>0}(1+\tau)^\gamma \|G_3(v_{m-1})(\tau)\|_{q_0} \le C(\|u_s\|_n + \|u_s\|_{\frac{n}{1+\lambda}} + \|u_s\|_{\frac{n}{1-\rho_2}})K_{m-1}. \tag{3.48}$$

By the same calculation, we have $G_2(v_{m-1})(t) \in L^{q_0}(D)$ for $t > 0$ with

$$\sup_{\tau>0}(1+\tau)^\gamma \|G_2(v_{m-1})(\tau)\|_{q_0} \le C(\|u_s\|_n + \|u_s\|_{\frac{n}{1+\lambda}} + \|u_s\|_{\frac{n}{1-\rho_2}})K_{m-1}. \tag{3.49}$$

We also have

$$\begin{aligned}
\int_0^t \left\| e^{-(t-\tau)A_a} P\Big[\big(1-\psi(\tau)\big) a \frac{\partial v_{m-1}}{\partial x_1}\Big] \right\|_{q_0} &\le Ca \int_0^{\min\{1,t\}} (t-\tau)^{-\frac{1}{2}} \|v_{m-1}(\tau)\|_{q_0}\, d\tau \\
&\le CaK_{m-1}(1+t)^{-\frac{1}{2}} \le CaK_{m-1}(1+t)^{-\gamma}
\end{aligned}$$

for $t > 0$ by (3.3). This together with (3.46)–(3.49), (1.10) and (3.27) yields $v_m(t) \in L^{q_0}(D)$ for $t > 0$ and

$$\begin{aligned}
K_m &\le C(M+1)a^{\frac{n-1}{n+1}} + \widetilde{C}_1 \Big\{ \big(\sup_{\tau>0} \tau^{\frac{1}{2}} \|\nabla v_{m-1}(\tau)\|_n \big) + \|u_s\|_n + \|u_s\|_{\frac{n}{1+\lambda}} + \|u_s\|_{\frac{n}{1-\rho_2}} + a \Big\} K_{m-1} \\
&\le C(M+1)a^{\frac{n-1}{n+1}} + \widetilde{C}_1(4C_3 + \widetilde{C}_2)(M+1)a^{\frac{n-2}{n+1}} K_{m-1}.
\end{aligned}$$

Suppose

$$(M+1)a^{\frac{n-2}{n+1}} < \min\left\{ \varepsilon,\ \frac{1}{2\widetilde{C}_1(4C_3 + \widetilde{C}_2)} \right\} =: \varepsilon'',$$

then we get (3.44) with m and, thereby, conclude

$$K_m \le 2C(M+1)a^{\frac{n-1}{n+1}}$$

for all $m \geq 1$. Since we know that $\|v_m(t) - v(t)\|_n \to 0$ as $m \to \infty$ for each $t > 0$, we obtain $v(t) \in L^{q_0}(D)$ for $t > 0$ with

$$\sup_{\tau>0}(1+\tau)^\gamma \|v(\tau)\|_{q_0} \leq 2C(M+1)a^{\frac{n-1}{n+1}} < \infty,$$

which completes the proof. □

In view of Lemmas 3.6 and 3.8, we prove sharp decay properties (1.25)–(1.26) for $n \geq 4$.

Proposition 3.9. *Let $n \geq 4$. Suppose $\rho_3 > 1$ and $1 < \rho_1 \leq 1 + \rho_3$ in addition to (1.16) (the set of those parameters is nonvoid when $n \geq 4$). Let ε be the constant in Proposition 3.4. There exists a constant $\varepsilon_* = \varepsilon_*(n, D) \in (0, \varepsilon]$ such that if $0 < (M+1)a^{(n-2)/(n+1)} < \varepsilon_*$, then the solution $v(t)$ obtained in Proposition 3.4 enjoys (1.25) and (1.26).*

Proof. Fix $1/2 < \rho < 1$ and $\gamma > 0$ such that

$$\max\left\{\frac{1}{2} - \frac{\rho}{2}, \frac{\rho_1 + 3 - n}{2}\right\} < \gamma < \frac{1}{2}. \tag{3.50}$$

Let $\varepsilon'(\rho, n, D)$ and $\varepsilon''(\gamma, n, D)$ be the constants in Lemmas 3.6 and 3.8, respectively. We show by induction that if

$$(M+1)a^{\frac{n-2}{n+1}} < \min\{\varepsilon'(\rho, n, D), \varepsilon''(\gamma, n, D)\} =: \varepsilon_*(n, D),$$

then $v(t)$ satisfies

$$\|v(t)\|_n = O(t^{-\sigma_k}), \qquad \sigma_k := \min\left\{\frac{k}{2}\rho, \frac{\rho_1}{2}\right\} \tag{3.51}$$

as $t \to \infty$ for all $k \geq 1$. This implies (1.25) with $q = n$, which together with (3.38) completes the proof. Since $\rho < \rho_1$, (3.51) with $k = 1$ follows from Lemma 3.6. We note that $\sigma_1 < 1/2$ and $\sigma_k > 1/2$ for $k \geq 2$.

Let $k \geq 2$ and suppose (3.51) with $k - 1$. Then

$$L_{k-1}(v) := \sup_{\tau>0}(1+\tau)^{\sigma_{k-1}}\|v(\tau)\|_n + \sup_{\tau>0}\tau^{\frac{1}{2}}(1+\tau)^{\sigma_{k-1}}\big(\|v(\tau)\|_\infty + \|\nabla v(\tau)\|_n\big) < \infty$$

holds due to (3.21) (near $t = 0$) as well as (3.38). In what follows, we always assume $t \geq 2$. From (3.42), it follows that

$$\|G_1(t)\|_n \leq \int_0^{\frac{t}{2}} (t-\tau)^{-\frac{n}{2q_0}} \|v(\tau)\|_{q_0} \|\nabla v(\tau)\|_n \, d\tau + \int_{\frac{t}{2}}^t (t-\tau)^{-\frac{1}{2}} \|v(\tau)\|_n \|\nabla v(\tau)\|_n \, d\tau =: I + II \tag{3.52}$$

with

$$I \leq Ct^{-\frac{n}{2q_0}}\big(\sup_{\tau>0}(1+\tau)^\gamma \|v(\tau)\|_{q_0}\big)L_{k-1}(v) \leq Ct^{-\frac{\rho_1}{2}}\big(\sup_{\tau>0}(1+\tau)^\gamma \|v(\tau)\|_{q_0}\big)L_{k-1}(v), \tag{3.53}$$

where (3.43) and (3.50) are taken into account and

$$II \leq Ct^{-2\sigma_{k-1}} L_{k-1}(v)^2. \tag{3.54}$$

For $G_2(t)$, we split the integral into

$$\int_0^t \|e^{-(t-\tau)A_a} P[\psi(\tau) v \cdot \nabla u_s]\|_n \, d\tau = \int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^{t-1} + \int_{t-1}^t .$$

Then we find

$$\begin{aligned} \int_0^{\frac{t}{2}} &\leq C\int_0^{\frac{t}{2}} (t-\tau)^{-\frac{1+\rho_3}{2}} \tau^{-\frac{1}{2}} (1+\tau)^{-\sigma_{k-1}} \, d\tau \, \|\nabla u_s\|_{\frac{n}{2+\rho_3}} \big(\sup_{\tau>0} \tau^{\frac{1}{2}} (1+\tau)^{\sigma_{k-1}} \|v(\tau)\|_\infty\big) \\ &\leq \begin{cases} Ct^{-\frac{\rho_3}{2} - \sigma_{k-1}} \|\nabla u_s\|_{\frac{n}{2+\rho_3}} L_{k-1}(v) \leq Ct^{-\sigma_k} \|\nabla u_s\|_{\frac{n}{2+\rho_3}} L_{k-1}(v) & \text{if } k = 2, \\ Ct^{-\frac{1+\rho_3}{2}} \|\nabla u_s\|_{\frac{n}{2+\rho_3}} L_{k-1}(v) \leq Ct^{-\frac{\rho_1}{2}} \|\nabla u_s\|_{\frac{n}{2+\rho_3}} L_{k-1}(v) & \text{if } k \geq 3 \end{cases} \end{aligned}$$

and

$$\int_{\frac{t}{2}}^{t-1} + \int_{t-1}^{t} \le Ct^{-\sigma_{k-1}-\frac{1}{2}}\big(\|\nabla u_s\|_{\frac{n}{2+\rho_3}} + \|\nabla u_s\|_{\frac{n}{2}}\big)L_{k-1}(v),$$

where we have used $\rho_3 > 1$ and $\rho_1 \le 1+\rho_3$. Estimates above imply that

$$\|G_2(t)\|_n \le Ct^{-\sigma_k}\big(\|\nabla u_s\|_{\frac{n}{2+\rho_3}} + \|\nabla u_s\|_{\frac{n}{2}}\big)L_{k-1}(v). \tag{3.55}$$

Similarly, we observe

$$\|G_3(t)\|_n \le Ct^{-\sigma_k}\big(\|u_s\|_{\frac{n}{1+\rho_1}} + \|u_s\|_n\big)L_{k-1}(v). \tag{3.56}$$

Moreover, by the same manner as in the proof of Lemma 3.6, we obtain

$$\|G_4(t)\|_n \le Ct^{-\frac{n}{2q_0}}\sup_{\tau>0}(1+\tau)^\gamma\|v(\tau)\|_{q_0} \le Ct^{-\frac{\rho_1}{2}}\sup_{\tau>0}(1+\tau)^\gamma\|v(\tau)\|_{q_0}, \tag{3.57}$$

$$\|H_1(t)\|_n \le CMt^{-\frac{\rho_1}{2}}\|u_s\|_{\frac{n}{1+\rho_1}}, \tag{3.58}$$

$$\begin{aligned}\|H_2(t)\|_n &\le Ct^{-\frac{2+\kappa}{2}}\|u_s\|_{\frac{n}{1+\kappa}}\|\nabla u_s\|_{\frac{n}{2}} + Ct^{-\frac{1+\rho_3}{2}}\|\nabla u_s\|_{\frac{n}{2+\rho_3}} \\ &\le Ct^{-\frac{\rho_1}{2}}\big(\|u_s\|_{\frac{n}{1+\kappa}}\|\nabla u_s\|_{\frac{n}{2}} + \|\nabla u_s\|_{\frac{n}{2+\rho_3}}\big)\end{aligned} \tag{3.59}$$

for all $t \ge 2$, where κ is chosen such that $\max\{0, \rho_1 - 2\} < \kappa < \min\{n-3, \rho_1\}$. Collecting (3.52)–(3.59), we conclude (3.51) with k. The proof is complete. □

We next consider the uniqueness. We begin with the classical result on the uniqueness of solutions within Y_0 as in Fujita and Kato [15].

Lemma 3.10. *Let ψ be a function on $\mathbb{R}$ satisfying (1.1) and let δ be the constant in Theorem 1.1 with (1.15)–(1.17). Then there exists a constant $\tilde\varepsilon = \tilde\varepsilon(n, D) \in (0, \delta]$ such that if $0 < a^{(n-2)/(n+1)} < \tilde\varepsilon$, (1.21) admits at most one solution within Y_0.*

Proof. The following argument is based on [15]. Suppose that $v, \tilde v \in Y_0$ are solutions. Then we have

$$\|v - \tilde v\|_{Y,t} \le \big\{C_1\big([\nabla v]_{n,t} + [\tilde v]_{n,t}^{\frac{1}{2}}[\tilde v]_{\infty,t}^{\frac{1}{2}}\big) + C_2 a^{\frac{n-2}{n+1}}\big\}\|v - \tilde v\|_{Y,t}, \quad t > 0 \tag{3.60}$$

by applying (1.10) and Lemma 3.3. If we assume

$$a^{\frac{n-2}{n+1}} < \min\Big\{\delta, \frac{1}{2C_2}\Big\} =: \tilde\varepsilon \tag{3.61}$$

and choose $t_0 > 0$ such that

$$C_1\big\{[\nabla v]_{n,t_0} + \big(\sup_{0<\tau<\infty}\|\tilde v(\tau)\|_n\big)^{\frac{1}{2}}[\tilde v]_{\infty,t_0}^{\frac{1}{2}}\big\} < \frac{1}{2},$$

then (3.60) yields $[v - \tilde v]_{Y,t_0} = 0$. Hence, we conclude $v = \tilde v$ on $(0, t_0]$ and obtain

$$\begin{aligned}v(t) - \tilde v(t) = \int_{t_0}^{t} e^{-(t-\tau)A_a}P\Big[&-(v-\tilde v)\cdot\nabla v - \tilde v\cdot\nabla(v-\tilde v) - \psi(\tau)(v-\tilde v)\cdot\nabla u_s \\ &- \psi(\tau)u_s\cdot\nabla(v-\tilde v) + (1-\psi(\tau))a\frac{\partial}{\partial x_1}(v-\tilde v)\Big]\,d\tau.\end{aligned}$$

By the same argument as in the proof of Lemma 3.3 together with (1.10), we see that

$$\|v - \tilde v\|_{Y,t_0,t} \le C_*\|v - \tilde v\|_{Y,t_0,t} \tag{3.62}$$

for all $t > t_0$, where

$$\|v\|_{Y,t_0,t} := \sup_{t_0\le\tau\le t}\|v(\tau)\|_n + \sup_{t_0\le\tau\le t}\|v(\tau)\|_\infty + \sup_{t_0\le\tau\le t}\|\nabla v(\tau)\|_n, \tag{3.63}$$

$$C_* = C\Big[\big(t_0^{-\frac12}\|v\|_Y + t_0^{-\frac14}\|\tilde v\|_Y\big)\big\{(t-t_0)^{\frac34} + (t-t_0)^{\frac14}\big\} + a^{\frac{n-1}{n+1}}\big\{(t-t_0)^{\frac12} + (t-t_0)^{\frac{\rho_2}{2}} + (t-t_0)^{\frac{\rho_4}{2}}\big\} + a\big\{(t-t_0) + (t-t_0)^{\frac12}\big\}\Big] \tag{3.64}$$

and the constant C is independent of v, $\tilde v$, t and t_0. We choose $\eta > 0$ such that

$$\xi := C\Big[\big(t_0^{-\frac12}\|v\|_Y + t_0^{-\frac14}\|\tilde v\|_Y\big)\big(\eta^{\frac34} + \eta^{\frac14}\big) + a^{\frac{n-1}{n+1}}\big(\eta^{\frac12} + \eta^{\frac{\rho_2}{2}} + \eta^{\frac{\rho_4}{2}}\big) + a\big(\eta + \eta^{\frac12}\big)\Big] < 1.$$

On account of (3.62), we have $\|v-\tilde v\|_{Y,t_0,t_0+\eta} \le \xi\|v-\tilde v\|_{Y,t_0,t_0+\eta}$, which leads us to $v = \tilde v$ on $[t_0, t_0+\eta]$. By the same calculation, we can obtain (3.62)–(3.64), in which t_0 should be replaced by $t_0+\eta$ and hence

$$\begin{aligned}\|v-\tilde v\|_{Y,t_0+\eta,t_0+2\eta} &\le C\Big[\big\{(t_0+\eta)^{-\frac12}\|v\|_Y + (t_0+\eta)^{-\frac14}\|\tilde v\|_Y\big\}\big(\eta^{\frac34}+\eta^{\frac14}\big) \\ &\quad + a^{\frac{n-1}{n+1}}\big(\eta^{\frac12} + \eta^{\frac{\rho_2}{2}} + \eta^{\frac{\rho_4}{2}}\big) + a\big(\eta+\eta^{\frac12}\big)\Big]\|v-\tilde v\|_{Y,t_0+\eta,t_0+2\eta} \\ &< \xi\|v-\tilde v\|_{Y,t_0+\eta,t_0+2\eta}\end{aligned}$$

holds. This implies $v = \tilde v$ on $[t_0+\eta, t_0+2\eta]$. Repeating this procedure, we conclude $v = \tilde v$. □

Remark 3.11. It is clear that the Eq. (3.28) admits at most one solution within Y_0 under the same condition as in Lemma 3.10.

Let us close the paper with completion of the proof of Theorem 1.2.

Proof of Theorem 1.2. Since we know $\varepsilon \le \tilde\varepsilon$ from (3.26) and (3.61), Proposition 3.4 and Lemma 3.10 yield the unique existence of solutions in Y_0 when $(M+1)a^{(n-2)/(n+1)} < \varepsilon$. Moreover, Propositions 3.7 and 3.9 give us sharp decay properties of the solution provided a is still smaller. We finally show the uniqueness of the solution constructed above within Y by following the argument due to Brezis [5]. It suffices to show that if $v \in Y$ is a solution, it necessarily satisfies

$$\lim_{t\to 0}[v]_t = 0, \tag{3.65}$$

where

$$[v]_t := \sup_{0<\tau<t}\tau^{\frac12}(\|v(\tau)\|_\infty + \|\nabla v(\tau)\|_n).$$

We assume

$$(M+1)a^{\frac{n-2}{n+1}} < \min\left\{\delta, \frac{1}{2C_2}, \frac{1}{16C_1C_0}, \frac{1}{16C_1C_3}\right\} =: \hat\varepsilon(n,D)\ (\le \varepsilon) \tag{3.66}$$

and let $v \in Y$ be a solution. Here, the constants C_i are as in Remark 3.5 as well as in the proof of Proposition 3.4. Since $v \in BC([0,\infty); L^n_\sigma(D))$ with $v(0) = 0$, there exists $s_0 > 0$ such that

$$\|v(s)\|_n + (M+1)a^{\frac{n-2}{n+1}} < \hat\varepsilon$$

for all $0 < s \le s_0$. Hence by Remark 3.5, the integral equation (3.28) with $b = v(s)$ admits a solution $T(t)v(s) \in Y_0$ along with

$$\|T(\cdot)v(s)\|_Y \le 4\big(C_0\|v(s)\|_n + C_3(M+1)a^{\frac{n-2}{n+1}}\big) < 4(C_0+C_3)\hat\varepsilon \le \frac{1}{2C_1}. \tag{3.67}$$

On the other hand, given $s \in (0, s_0]$, we can see that $z_s(t) := v(t+s)$ for $t \ge 0$ also satisfies (3.28) with $b = v(s)$ and $z_s \in Y_0$. In view of Remark 3.11, we have $z_s(t) = T(t)v(s)$ for $s \in (0, s_0]$, which implies

$$t^{\frac12}\big(\|v(t+s)\|_\infty + \|\nabla v(t+s)\|_n\big) \le \sup_{f\in K}[T(\cdot)f]_t, \quad K = v((0,s_0]) := \{v(t) \in L^n_\sigma(D) \mid t \in (0,s_0]\}$$

for all $s \in (0, s_0]$ and $t > 0$. Passing to the limit $s \to 0$, we find

$$[v(\cdot)]_t \le \sup_{f \in K} [T(\cdot)f]_t. \tag{3.68}$$

Furthermore, applying Lemma 3.3 to (3.28) with $b = f \in v((0, s_0])$ as well as Proposition 3.1 and (1.10), we have

$$[T(\cdot)f]_t \le C_0[S(\cdot)f]_t + \Big(C_1 \sup_{f\in K} \|T(\cdot)f\|_Y + C_2 a^{\frac{n-2}{n+1}}\Big)[T(\cdot)f]_t + \|H_1\|_{Y,t} + \|H_2\|_{Y,t},$$

where $S(t)f := e^{-tA_a} f$, and deduce from (3.66) and (3.67) that

$$[T(\cdot)f]_t \le \frac{C_0[S(\cdot)f]_t + \|H_1\|_{Y,t} + \|H_2\|_{Y,t}}{1 - \Big(C_1 \sup_{f\in K} \|T(\cdot)f\|_Y + C_2 a^{\frac{n-2}{n+1}}\Big)} \tag{3.69}$$

for all $f \in K$ and $t > 0$. Collecting (3.69), (3.4), (3.68) and $H_1, H_2 \in Y_0$ leads to (3.65). The proof is complete. □

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Compliance with Ethical Standards

Conflict of interest The author declares that there is no conflict of interest.

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