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Global Existence of Martingale Solutions and Large Time Behavior for a 3D Stochastic Nonlocal Cahn–Hilliard–Navier–Stokes Systems with Shear Dependent Viscosity

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Abstract. In this paper, we consider a stochastic version of a nonlinear system which consists of the incompressible Navier–Stokes equations with shear dependent viscosity controlled by a power $p > 2$, coupled with a convective nonlocal Cahn–Hilliard-equations. This is a diffuse interface model which describes the motion of an incompressible isothermal mixture of two (partially) immiscible fluid having the same density. We prove the existence of a weak martingale solutions when $p \in [11/5, 12/5)$, and their exponential decay when the time goes to infinity.

Keywords. Martingale solutions, Nonlocal Cahn–Hilliard equations, Navier–Stokes equations, Exponential decay.

1. Introduction

We consider a mathematical model of two isothermal, incompressible, immiscible fluids evolving in three dimensional bounded domain $\mathcal{M} \subset \mathbb{R}^3$. This system of equations is well-known as a diffuse interface model (see, e.g., [1,25,26]) for the phase separation of an incompressible and isothermal non-Newtonian binary fluid mixture. In a simplified setting where the density of the mixture is supposed to be one as well as the viscosity and the mobility, the model reduces to

$$\begin{cases} \partial_t u + (u.\nabla)u - \operatorname{div} \boldsymbol{T}(\varphi, Du) + \nabla\pi = \mu\nabla\varphi + g_0(t), \\ \operatorname{div} u = 0, \\ \partial_t\varphi + (u.\nabla)\varphi = \Delta\mu, \\ \mu = -\Delta\varphi + F'(\varphi), \end{cases} \tag{1.1}$$

in $(0,T) \times \mathcal{M}$, where $T > 0$ is a given final time, π is the pressure, g_0 is a given volume force applied to the binary mixture fluid, u and φ are unknown variables which represent the (volume) averaged velocity and the (relative) concentration difference of one of the fluids, respectively. The chemical potential μ is the variation of the free energy functional (cf. [9])

$$\mathcal{F}(\varphi) = \int_{\mathcal{M}} \left(\frac{1}{2}|\nabla\varphi|^2 + F(\varphi)\right) dx. \tag{1.2}$$

Here F is a double well potential (e.g., $F(r) = (r^2-1)^2, r \in \mathbb{R}$), which accounts for the presence of two components. The potential can be defined either on the whole real line (smooth potential) or on a bounded interval (singular potential). The latter case (in a logarithmic form) is the most appropriate choice from the modeling viewpoint (see [9]), while the former can be considered as an approximation. In the context of statistical mechanics, the square gradient term in (1.2) arises from attractive long-ranged interactions between the molecules of the fluid (see, e.g., [1] and references therein). The stress tensor $\boldsymbol{T}$, up to the pressure term, dependents on the symmetric gradient $Du := (\nabla u + \nabla^{\mathrm{tr}} u)/2$ of the flow velocity field u and, possibly, on φ, through a suitable constitutive law. In fact, when we are in presence of Newtonian mixture, the stress tensor is defined as

$$\boldsymbol{T}(\varphi, Du) = \nu(\varphi)Du, \tag{1.3}$$

where ν is a given strictly positive function depending only on φ; and in this case, system (1.1) is what we call the Cahn–Hilliard–Navier–Stokes system (CH-NSs) or the H-model (cf. [2,21,35,53]). The CH-NS model describes the chemical interactions between the two phases at the interface, which is achieved using a Cahn–Hilliard approach, and also the hydrodynamic properties of the mixture which is obtained using Navier–Stokes equations with surface tension terms acting at the interface (cf. [21]). Now, when the mixture has non-Newtonian features, then the stress tensor $\boldsymbol{T}$ depends on some power of $|Du|$. For instance, it can be given as follows

$$\boldsymbol{T}(\varphi, Du) = (\nu_1(\varphi) + \nu_2(\varphi)|Du|^{p-2})Du \tag{1.4}$$

where ν_1 and ν_2 are strictly positive functions and $p > 2$. Systems like (1.1)–(1.3), also known as CH-NSs, have been analyzed by many authors and used in several different contexts (see, for instance, [2,21,22,53], cf. also [14,28] for numerical issues).

We note that system (1.1) has been deduced phenomenologically, i.e., as the (conserved) gradient flow associated with the Fréchet derivative of the free energy functional $\mathcal{F}$ defines in (1.2). However in [23,24], starting from a microscopic model, another form of the free energy functional has been proposed and rigorously justified as a macroscopic limit of microscopic phase segregation models with particle conserving dynamics (cf. also [8]). In this case the gradient term is replaced by a nonlocal spatial operator, namely,

$$\mathcal{E}(\varphi) = \frac{1}{4}\int_{\mathcal{M}}\int_{\mathcal{M}} J(x-y)(\varphi(x)-\varphi(y))^2 dxdy + \int_{\mathcal{M}} F(\varphi(x))dx, \tag{1.5}$$

where $J : \mathbb{R}^3 \to \mathbb{R}$ is a sufficiently smooth interaction kernel such that $J(x) = J(-x)$. Taking the first variation of $\mathcal{E}$ the chemical potential becomes

$$\mu = a\varphi - J * \varphi + F'(\varphi),$$

where

$$a(x) = \int_{\mathcal{M}} J(x-y)dy \text{ and } (J*\varphi)(x) = \int_{\mathcal{M}} J(x-y)\varphi(y)dy, \tag{1.6}$$

and consequently, we have the following nonlocal evolution system

$$\begin{cases} \partial_t u + (u.\nabla)u - \operatorname{div}\boldsymbol{T}(\varphi, Du) + \nabla\pi = \mu\nabla\varphi + g_0(t), \\ \operatorname{div} u = 0, \\ \partial_t\varphi + (u.\nabla)\varphi = \Delta\mu, \\ \mu = a\varphi - J*\varphi + F'(\varphi), \end{cases} \tag{1.7}$$

in $(0,T)\times\mathcal{M}$. As mention by Van der Waals in [46], we can observe (formally) that the nonlocal interaction term can be locally approximated by the square gradient, provided that the interaction J is sufficiently concentrated around 0; i.e., the functional $\mathcal{F}$ can be viewed as a local approximation of $\mathcal{E}$. Hence system (1.7) seems well justified and more general than the classical one, though the related literature is far less abundant. In the case (1.3), the solvability of system (1.7) has been analyzed first in [10] and then in [16–19] under various assumptions and generalizations. In [15], assuming that the stress tensor $\boldsymbol{T}$ only depends on Du with a $(p-1)$-power growth, the authors proved the existence of a weak solution when $p \geq 11/5$ and they extend some previous results on time regularity and uniqueness when $p > 11/5$. The aim of this paper is to study a stochastic version of the system (1.7), in the case that $\boldsymbol{T}$ only depends on Du with a $(p-1)$-power growth.

However, in order to consider a more realistic model for our problem, it is sensible to consider some king of noise in the equation (1.7). This may reflect, for instance, some environmental effects on the phenomena, some external random forces, etc. This approach is basically motivated by Reynold's work, which stipulates that the velocity of a fluid particle in turbulent regime is composed of slow (deterministic) and fast (stochastic) components. While this belief was based on empirical and experimental data, Rozovskii and Mikulevicius were able to derive the models rigorously in their recent work [36], thereby confirming the importance of this approach in hydrodynamic turbulence. More precisely, it is mentioned in [29] that some rigorous information on questions in turbulence might be obtained from stochastic version of the equations of fluid dynamics. To the best of our knowledge, the study of the stochastic version of the system (1.7) has not been analyzed

Considering the fact that the majority of work studies in SPDEs assumed that the fluids are Newtonian (since it is well-known that the incompressible Navier–Stokes equation governs the motions of single-phase fluids such as air or water), and that there are some conducting materials appearing in many practical and theoretical situations that cannot be characterized by Newtonian fluids (see for instance the introduction of Biskamp's book [5] for some examples of these non-Newtonian conducting fluids), we will analyzed in this paper a stochastic version of problem (1.7) within a reasonably simple (but meaningful) non-Newtonian setting. Namely, for a final time $T > 0$ and sufficiently smooth bounded domain $\mathcal{M} \subset \mathbb{R}^3$, assuming matched densities equal to unity and that the stress tensor $\boldsymbol{T}$ only depends on Du with a $(p-1)$-power growth, we have to deal with the system of stochastic partial differential equations

$$\begin{cases} \partial_t u + (u.\nabla)u - \operatorname{div} \boldsymbol{T}(Du) + \nabla\pi = \mu\nabla\varphi + g_0(t) + g_1(u,\varphi) + g_2(t,u,\varphi)\dot{W}_t, \\ \operatorname{div} u = 0, \\ \partial_t\varphi + (u.\nabla)\varphi = \Delta\mu, \\ \mu = a\varphi - J * \varphi + F'(\varphi), \\ u = 0,\ \frac{\partial\mu}{\partial\eta} = 0 \text{ on } (0,T)\times\partial\mathcal{M}, \\ u(0) = u_0,\ \varphi(0) = \varphi_0 \text{ in } \mathcal{M}, \end{cases} \tag{1.8}$$

where $\partial\mathcal{M}$ is the boundary of $\mathcal{M}$, η is the outward normal to $\partial\mathcal{M}$, $u = (u_1, u_2, u_3)$, φ and π are unknown random fields defined on $[0,T]\times\mathcal{M}$, representing, respectively, the fluid velocity, the order (phase) parameter and the pressure, at each point of $[0,T]\times\mathcal{M}$. The external volume forces $g_0(t),\ g_1(u,\varphi)$, are given. The term $g_2(t,u,\varphi)\dot{W}_t$ is an external force depending on u and φ, where $\dot{W}_t$ denotes the time derivative of a cylindrical Wiener process. The quantities u_0 and φ_0 are given non-random initial velocity and phase field, respectively. These equations are of the nonlocal type because of the presence of the term J, which is the spatial-dependent internal kernel and $J * \varphi$ denotes the spatial convolution over $\mathcal{M}$. The purpose of the present manuscript is to prove some results related to problem (1.8), which are the stochastic analog of some of those obtained in [15] for the deterministic case. Our main results are the following:

1. We prove the existence of martingale solution for the stochastic system (1.8). We consider a sufficiently general forcing consisting of a regular part and a stochastic part both depending nonlinearly on the velocity of the fluids and the (order) phase parameter φ (i.e. the relative concentration of one fluid or the difference of the two concentration). These forces terms are supposed to be non-Lipschitz. The method for the proof is based in the Galerkin, compactness, and monotonicity methods.
2. Having the existence of a martingale weak solutions in hand, we now move to the study of its asymptotic behavior as the time t is large. Then, we study the decay of the martingale weak solutions as times goes to infinity. More precisely, we prove that under some conditions on the forcing terms g_i, $i = 0,1,2$, the couple (u,φ) converges to zero exponentially in the mean square.

We note that for the proof of item (1) we drew our inspiration from the paper [15,45] and for the proof of item (2) we mainly follow the idea in [6,7].

The layout of the manuscript is as follows. In Section 2, we present the mathematical setting of our model, the stochastic framework and we gather all the necessary tools and the hypotheses. In Section 3 we introduce the notion of weak solutions and we state our first result for the existence of weak probabilistic solution. In Section 4, we derive the proof of our first main result by means of Galerkin methods and probabilistic and analytic compactness results. In Section 5, we prove our second main result concerning the exponential asymptotic behavior of these weak solutions.

2. Functional Setup and Preliminary

Here, we introduce some necessary notations and most of the hypotheses relevant for our analysis.

2.1. The Deterministic Framework

We introduce some notations and background following the mathematical theory of hydrodynamic equations such as Navier–Stokes equations. We denote by $\mathcal{D}(\mathcal{M})$ the space of functions $u \in \mathcal{C}^\infty(\mathcal{M})$ with compact support. Let $p \in (1, \infty)$, we introduce the following spaces

$$\begin{aligned} \mathcal{V} &= \{u \in \mathcal{D}(\mathcal{M})^3 : \operatorname{div} u = 0\}, \\ G_{\mathrm{div}} &= \text{the closure of } \mathcal{V} \text{ in } (L^2(\mathcal{M}))^3, \\ V_{\mathrm{div},p} &= \text{the closure of } \mathcal{V} \text{ in } (W^{1,p}(\mathcal{M}))^3. \end{aligned}$$

We denote by $|\,.\,|$ the $(L^2(\mathcal{M}))^3$-norm, and by $(.\,,\,.)$ the $(L^2(\mathcal{M}))^3$-inner product.

The space G_{div} is equipped with the scalar product and norm induced by $(L^2(\mathcal{M}))^3$ and thanks to Poincaré's inequality we can endow the space $V_{\mathrm{div},p}$ with the norm $\|u\|_{1,p}$ defined by

$$\|u\|_{1,p}^p = \int_{\mathcal{M}} |\nabla u|^p dx.$$

Note that this norm is equivalent to the usual $(W^{1,p}(\mathcal{M}))^3$-norm on $V_{\mathrm{div},p}$.

We equip the space $V_{\mathrm{div}} := V_{\mathrm{div},2}$ with the norm $\|.\|$ generated by the scalar product

$$((u, v)) = \int_{\mathcal{M}} \nabla u \,.\, \nabla v \; dx.$$

Owing to Poincaré's inequality, $\|.\|$ and the usual $(H^1(\mathcal{M}))^3$-norm are equivalent on V_{div}.

For other Hilbert spaces X, the scalar product will be denoted by $(.,.)_X$. The notations $\langle .,. \rangle_Y$ and $\|.\|_Y$ will stand for the duality pairing between a Banach space Y and its duality Y', and for the norm of Y, respectively.

With the view to implement the approximation scheme (see Section 4 below), we introduce the auxiliary Hilbert space $\mathbf{W}_s$ defined by (see [33])

$$\mathbf{W}_s = \text{the closure of } \mathcal{V} \text{ in } (H^s(\mathcal{M}))^3,$$

where $s > \frac{5}{2}$ is fixed and we have the following Gelfand chain

$$\mathbf{W}_s \hookrightarrow V_{\mathrm{div},p} \hookrightarrow G_{\mathrm{div}} \cong G'_{\mathrm{div}} \hookrightarrow V'_{\mathrm{div},p} \hookrightarrow \mathbf{W}'_s, \tag{2.1}$$

where each space is densely and compactly embedded into the next one.

Note that it is enough to take $s \geq \frac{5}{2} - \frac{3}{p}$ so as to (2.1) holds.

We set $H := L^2(\mathcal{M}),\ U := H^1(\mathcal{M})$ and we also introduce the spaces (see [15] for more details)

$$\begin{aligned} H^{s_1}_{(0)}(\mathcal{M}) &:= \{\phi \in H^{s_1}(\mathcal{M}) : \langle \phi, 1\rangle_{H^{s_1}} = 0\}, \\ H^{-1}_{(0)}(\mathcal{M}) &:= H^1_{(0)}(\mathcal{M})' = \{\phi \in U' : \langle \phi, 1\rangle_U = 0\}, \end{aligned}$$

with $s_1 \in \mathbb{R}$.

We set

$$\|\psi\|^2_{H^{s_1}} = \sum_{k\in\mathbb{N}} \iota_k^{s_1} c_k^2,\ c_k = \int_{\mathcal{M}} \psi(x)\psi_k(x)dx,$$

where $\{(\iota_k, \psi_k)\}_{k\in\mathbb{N}}$ are the eigenvalues and the eigenfunctions of the weak Laplace operator A_N with homogeneous Neumann boundary condition, that is, for $f \in U'$ and $\phi \in U$ we have

$$A_N\phi = f \iff \int_{\mathcal{M}} \nabla\phi.\nabla\psi dx = \langle f, \psi\rangle_U\,, \text{ for all } \psi \in U. \tag{2.2}$$

We recall that A_N maps U onto $H^{-1}_{(0)}(\mathcal{M})$ and the restriction of A_N to $H^1_{(0)}(\mathcal{M})$ is an isometry between $H^1_{(0)}(\mathcal{M})$ and the space $H^{-1}_{(0)}(\mathcal{M})$. Further, we denote by $A_N^{-1}: H^{-1}_{(0)}(\mathcal{M}) \to H^1_{(0)}(\mathcal{M})$ the inverse map defined by

$$A_N A_N^{-1} f = f, \ \forall f \in H^{-1}_{(0)}(\mathcal{M}) \text{ and } A_N^{-1} A_N f = f, \ \forall f \in H^1_{(0)}(\mathcal{M}).$$

We know that, for every $f \in H^{-1}_{(0)}(\mathcal{M})$, $A_N^{-1} f$ is the unique solution with zero mean value of the Neumann problem

$$\begin{cases} -\Delta\phi = f, & \text{in } \mathcal{M}, \\ \frac{\partial\phi}{\partial\eta} = 0, & \text{on } \partial\mathcal{M}. \end{cases}$$

In addition, we have

$$\begin{aligned} &\left\langle A_N\phi, A_N^{-1} f\right\rangle_U = \langle \phi, f\rangle_U, \text{ for all } \phi \in U, f \in H^{-1}_{(0)}(\mathcal{M}), \\ &\left\langle f, A_N^{-1} g\right\rangle_U = \left\langle g, A_N^{-1} f\right\rangle_U = \int_{\mathcal{M}} \nabla(A_N^{-1} f).\nabla(A_N^{-1} g)dx, \text{ for all } f, g \in H^{-1}_{(0)}(\mathcal{M}). \end{aligned} \tag{2.3}$$

Note that A_N can be also viewed as an unbounded linear operator on H with domain $D(A_N) = \{\phi \in H^2(\mathcal{M}) : \frac{\partial\phi}{\partial\eta} = 0 \text{ on } \partial\mathcal{M}\}$ and there a positive constant $c > 0$ such that

$$\|A_N^{-1}\phi\|_U \leq c\|\phi\|_{U'}, \ \|A_N^{-1}\phi\|_{H^2} \leq c|\phi|. \tag{2.4}$$

Remark 2.1. The natural no-flux condition $\frac{\partial\mu}{\partial\eta} = 0$ implies the conservation of the following quantity

$$\langle\varphi(t)\rangle = \frac{1}{|\mathcal{M}|}\int_{\mathcal{M}} \varphi(t,x)dx,$$

where $|\mathcal{M}|$ stands for the Lebesgue measure of $\mathcal{M}$. More precisely, we have

$$\langle\varphi(t)\rangle = \langle\varphi(0)\rangle, \quad \forall t \geq 0.$$

Thus, up to a shift of the order parameter field, we can always assume that the mean of φ is zero at the initial time and, therefore it will remain zero for all positive times. Hereafter, we assume that

$$\langle\varphi(t)\rangle = \langle\varphi(0)\rangle = 0 \ \ \forall t \geq 0.$$

Remark 2.2. We have just added a stochastic force in the equation for velocity u, not in the equation for the relative concentration φ since it will involve tedious calculations and will increase significantly the size of the paper. In fact, we need to apply Itô's formula to the functional $\mathcal{E}_{tot}$ defines in (3.3) below, which will require tedious calculations and probably more assumptions.

Let us set

$$\begin{aligned} &\mathbb{H} = G_{\text{div}} \times H, \\ &\mathbb{W}_s = \mathbf{W}_s \times U, \quad \text{for fix} \ \ s > 5/2. \end{aligned}$$

The space $\mathbb{H}$ is a complete metric space with respect to the norm

$$\|(u,\varphi)\|^2_{\mathbb{H}} = |u|^2 + |\varphi|^2. \tag{2.5}$$

The space $\mathbb{W}_s$ will be equipped with the usual scalar product and norm of the cartesian space $H^s(\mathcal{M}) \times H^1(\mathcal{M})$, respectively denoted by $((.,.))_s = ((.,.))_{H^s} + ((.,.))_{H^1}$ and $\|.\|^2_s = \|.\|^2_{H^s} + \|.\|^2_U$.

We define the Banach space $\mathbb{V}$ by

$$\mathbb{V} = V_{\text{div},p} \times U,$$

with norm

$$\|(u,\varphi)\|_{\mathbb{V}} = \|u\|_{1,p} + \|\varphi\|_U. \tag{2.6}$$

Remark 2.3. The norm defines in (2.6) is equivalent to any norm of the form

$$[[(u,\varphi)]] = C_1\|u\|_{1,p} + C_2\|\varphi\|_U, \tag{2.7}$$

where C_1 and C_2 are positive constants depending only on p and $|\mathcal{M}|$.

Hereafter we set

$$\|(u,\varphi)\|_{\mathbb{V}}^{p,2} = \|u\|_{1,p}^p + \|\varphi\|_U^2 \text{ and } L^{p,2}(0,T;\mathbb{V}) = L^p(0,T;V_{\mathrm{div},p}) \times L^2(0,T;U).$$

The space $L^{p,2}(0,T;\mathbb{V})$ is a Banach space with respect to the norm

$$\|(u,\varphi)\|^2_{L^{p,2}(0,T;\mathbb{V})} = \|u\|^p_{L^p(0,T;V_{\mathrm{div},p})} + \|\varphi\|^2_{L^2(0,T;U)} = \int_0^T \|(u(s),\varphi(s))\|_{\mathbb{V}}^{p,2} ds. \tag{2.8}$$

2.2. Nonlinear Operators

For $u,v,w \in V_{\mathrm{div}}$, we define the trilinear operator $b(.,.,.)$ as

$$b(u,v,w) = \int_{\mathcal{M}} (u(x).\ \nabla)v(x).\ w(x)dx = \sum_{i,j=1}^{3} \int_{\mathcal{M}} u_i(x)\partial_{x_i} v_j(x) w_j(x)dx.$$

We recall that

$$\begin{cases} b(u,v,w) = -b(u,w,v),\ \forall\ u,v,w \in V_{\mathrm{div}}, \\ b(u,v,v) = 0,\ \forall\ u,v \in V_{\mathrm{div}}. \end{cases} \tag{2.9}$$

For more properties concerning the nonlinear operator b, we refer the readers to [51].

In order to introduce the weak formulation of problem (1.8), we introduce the following bilinear and trilinear forms as in [15].

$$\begin{aligned} \langle N(u),v\rangle_{V_{\mathrm{div},p}} &= \int_{\mathcal{M}} \boldsymbol{T}(Du).\ Dv\ dx, \\ \langle B_0(u,u),v\rangle_{V_{\mathrm{div},p}} &= \int_{\mathcal{M}} [(u.\nabla)v].\ u\ dx, \\ \langle R_1(\varphi),v\rangle_{V_{\mathrm{div},p}} &= -\frac{1}{2}\int_{\mathcal{M}} \varphi^2\nabla a.\ v\ dx + \int_{\mathcal{M}} (\nabla J * \varphi)\varphi.\ v\ dx, \\ \langle B_1(u,\varphi),\psi\rangle_U &= \int_{\mathcal{M}} \varphi u.\ \nabla\psi\ dx, \end{aligned} \tag{2.10}$$

which are well defined for all $u,v \in V_{\mathrm{div},p}$ and for all $\varphi \in L^{2\kappa+2}(\mathcal{M})$ and $\psi \in U = H^1(\mathcal{M})$, where p and κ are chosen as in Theorem 3.1 below. Here $\boldsymbol{T}$ designates the extra stress tensor of the non-Newtonian fluid, and it only depends on Du with a $(p-1)$-power growth (cf. (1.4) as in the introduction).

We note that $b(u,u,v) := -\langle B_0(u,u),v\rangle$, for all $u,v \in V_{\mathrm{div},p}$. For simplicity we will write $B_0(u) := B_0(u,u)$.

2.3. Stochastic Setting and Assumptions

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete probability space and $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ an increasing and right continuous family of sub σ-algebras of $\mathcal{F}$, such that $\mathcal{F}_0$ contains all the $\mathbb{P}$-null sets of $\mathcal{F}$. Given K_1 and K_2 two separable Banach spaces, we denote by $L(K_1)$ the set of bounded linear map in K_1, by $\mathcal{L}(K_1,K_2)$ the space of continuous linear mapping from K_1 into K_2. By $L_2(K_1,K_2)$, we mean the subspace of $\mathcal{L}(K_1,K_2)$ consisting of Hilbert–Schmidt operators when K_1 and K_2 are separable. It is known that $L_2(K_1,K_2)$ is a Hilbert space, and its norm is denoted by $\|.\|_{L_2(K_1,K_2)}$.

Let $\{\beta_t^j, t \geq 0, j = 1,2,\ldots\}$ be a given sequence of mutually independent standard real $\mathcal{F}_t$-Wiener processes defined on $(\Omega,\mathcal{F},\mathbb{P})$, and suppose given K, a separable Hilbert space, and $\{e_j, j = 1,2,\ldots\}$, an orthonormal basis of K. We denote by $\{W_t, t \geq 0\}$, the cylindrical Wiener process with values in K defined by

$$W_t = \sum_{j=1}^{\infty} \beta_t^j e_j. \tag{2.11}$$

Remark 2.4. Let $\tilde{H}$ be a Hilbert space and $\mathbb{M}^2(\Omega \times [0,T]; L_2(K,\tilde{H}))$ the space of all equivalence classes of $\mathbb{F}$-progressively measurable processes $\psi : \Omega \times [0,T] \to L_2(K,\tilde{H})$ satisfying

$$\mathbb{E} \int_0^T \|\psi(s)\|^2_{L_2(K,\tilde{H})} ds < \infty.$$

(i) For any $s \in [0,T]$ and $\psi \in \mathbb{M}^2(\Omega \times [0,T]; L_2(K,\tilde{H}))$, we have

$$\psi(s) \circ \mathfrak{J}^{-1} \in L_2(Q^{1/2}(K_1), \tilde{H}),$$

where $\mathfrak{J}$ is any one-to-one Hilbert–Schmidt operator from K into another Hilbert space $(K_1, (.,.)_{K_1})$ and $Q = \mathfrak{J}\mathfrak{J}^* \in L(K_1)$, $\mathfrak{J}^*$ the adjoint of $\mathfrak{J}$; since

$$\begin{aligned}\|\psi(s)\|^2_{L_2(K,\tilde{H})} &= \sum_{i\in\mathbb{N}} (\psi(s)e_i, \psi(s)e_i)_{\tilde{H}} \\ &= \sum_{i\in\mathbb{K}} \big(\psi(s)\circ\mathfrak{J}^{-1}(\mathfrak{J}e_i), \psi(s)\circ\mathfrak{J}^{-1}(\mathfrak{J}e_i)\big)_{\tilde{H}} = \|\psi(s)\circ\mathfrak{J}^{-1}\|^2_{L_2(Q^{1/2}(K_1),\tilde{H})}.\end{aligned}$$

Hence

$$\|\psi(s)\circ\mathfrak{J}^{-1}\|_{L_2(Q^{1/2}(K_1),\tilde{H})} = \|\psi(s)\|_{L_2(K,\tilde{H})}.$$

(ii) It follows from the theory of stochastic integration on infinite dimensional Hilbert space, cf. [37, Chapter 5, Section 26] and [11, Chapter 4], that the process χ defined by

$$\chi(t) = \int_0^t \psi(s) dW_s := \int_0^t \psi(s)\circ\mathfrak{J}^{-1} d\bar{W}(s), \ t \in [0,T] \tag{2.12}$$

is a $\tilde{H}$-valued martingale; where

$$\bar{W}(t) := \sum_{j=1}^{\infty} \beta_t^j \mathfrak{J} e_j, \quad t \in [0,T]. \tag{2.13}$$

Moreover, the following Itô isometry holds

$$\mathbb{E}\| \int_0^t \psi(s)\circ\mathfrak{J}^{-1} d\bar{W}(s)\|^2_{\tilde{H}} = \mathbb{E}\int_0^t \|\psi(s)\|^2_{L_2(K,\tilde{H})} ds, \ \forall t \in [0,T], \tag{2.14}$$

and the Burkholder–Davis–Gundy inequality

$$\mathbb{E}\left(\sup_{s\in[0,T]} \left\| \int_0^s \psi(\tau)\circ\mathfrak{J}^{-1} d\bar{W}(\tau) \right\|^q_{\tilde{H}}\right) \leq c_q \mathbb{E}\left(\int_0^t \|\psi(s)\|^2_{L_2(K,\tilde{H})} ds\right)^{q/2}, \quad \forall t \in [0,T], \ \forall q \in (1,\infty). \tag{2.15}$$

(iii) $\bar{W}(t)$, $t \in [0,T]$ is a Q-Wiener process on K_1, see [34, Proposition 2.5.2]; and it is also a K_1-valued continuous, square integrable $\mathcal{F}_t$-martingale, see [34, Proposition 2.2.10].

(iv) The series defines in (2.13) even converges in $L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathcal{C}([0,T]; K_1))$, and thus always has a $\mathbb{P}$-a.s. continuous version.

Remark 2.5. for any $g \in \tilde{H}$, one has

$$\left(\int_0^t \psi(s) dW(s), g\right)_{\tilde{H}} = \left(\int_0^t \psi(s)\circ\mathfrak{J}^{-1} d\bar{W}(s), g\right)_{\tilde{H}} = \sum_{j=1}^{\infty} \int_0^t (\psi(s)e_j, g)_{\tilde{H}}\, d\beta_s^j, \quad t \in [0,T],$$

where each stochastic integral in the series is understood as an Itô's stochastic integral with respect to the corresponding real valued Wiener process β_s^j. The above series converges in $L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathcal{C}([0,t]; \tilde{H}))$, for each $0 < t \leq T$; see [11] for details. In particular, we note that if $\psi \in \mathbb{M}^2(\Omega \times [0,T]; L_2(K, \tilde{H}))$ and $g \in L^2(\Omega, \mathcal{F}, \mathbb{P}; L^\infty(0,T;\tilde{H}))$ is $\mathcal{F}_t$-progressively measurable, then the series

$$\sum_{j=1}^{\infty} \int_0^t (\psi(s)e_j, g(s)) d\beta_s^j,\ t \in [0,T],$$

converges in $L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathcal{C}([0,t];\mathbb{R}))$, and defined a real-valued continuous $\mathcal{F}_t$-martingale.

We will use the notation

$$\int_0^t (\psi(s)dW(s), g) := \int_0^t \left(\psi(s) \circ \mathfrak{J}^{-1} d\bar{W}(s), g\right) := \sum_{j=1}^{\infty} \int_0^t (\psi(s)e_j, g)_{\tilde{H}} d\beta_s^j,\ t \in [0,T].$$

Hereafter, we shall fix one such $\mathfrak{J}$ and $(K_1, (.,.)_{K_1})$ as in Remark 2.4 and for the process W_t, $t \in [0,T]$, given by (2.11) we define $\bar{W}(t)$, $t \in [0,T]$ as in (2.13) for the fixed $\mathfrak{J}$.

The stochastic integral of $g_2(s, u(s), \varphi(s))$ (which is the unique G_{div}-valued $\mathcal{F}_t$-martingale) with respect to the K-cylindrical Wiener process W_t, $t \in [0,T]$ is given by

$$\begin{aligned}\int_0^t g_2(s,u(s),\varphi(s))dW_s &= \int_0^t g_2(s,u(s),\varphi(s)) \circ \mathfrak{J}^{-1} d\bar{W}(s)\\ &:= \int_0^t \bar{g}_2(s,u(s),\varphi(s))d\bar{W}(s) \quad t \in [0,T].\end{aligned} \tag{2.16}$$

Using the notations above, we rewrite problem (1.8) as follows

$$\begin{cases} \dfrac{du}{dt} + N(u) - B_0(u) - R_1(\varphi) = g_0(t) + g_1(u,\varphi) + g_2(t,u,\varphi)\dot{W}_t \text{ in } V'_{\text{div},p},\\ \dfrac{d\varphi}{dt} - B_1(u,\varphi) = \Delta\mu \text{ in } U' = (H^1(\mathcal{M}))',\\ \mu = a\varphi - J * \varphi + F'(\varphi),\\ (u,\varphi)(0) = (u_0,\varphi_0), \end{cases} \tag{2.17}$$

or equivalently

$$\begin{cases} u(t) + \displaystyle\int_0^t (N(u(s)) - B_0(u(s)) - R_1(\varphi(s)))ds = u_0 + \int_0^t g_0(s)ds\\ \quad + \displaystyle\int_0^t g_1(u(s),\varphi(s))ds + \int_0^t \bar{g}_2(s,u(s),\varphi(s))d\bar{W}(s),\\ \varphi(t) - \displaystyle\int_0^t B_1(u(s),\varphi(s))ds = \varphi_0 + \int_0^t \Delta\mu(s)ds,\\ \mu = a\varphi - J * \varphi + F'(\varphi), \end{cases} \tag{2.18}$$

$\mathbb{P}$-a.s, and for all $t \in [0,T]$, where

$$g_0 \in L^2(0,\infty; G_{\text{div}}),\ g_1 : G_{\text{div}} \times H \to V'_{\text{div},p},\ g_2 : [0,\infty) \times G_{\text{div}} \times H \to L_2(K, G_{\text{div}}). \tag{2.19}$$

Remark 2.6. The pressure is excluded from (2.17) as usual; in fact, one (formally) has

$$\mu\nabla\varphi = \nabla\left(F(\varphi) + \tfrac{a}{2}\varphi^2 - (J * \varphi)\varphi\right) - \frac{\nabla a}{2}\varphi^2 + (\nabla J * \varphi)\varphi.$$

This explains that $\mu\nabla\varphi = R_1(\varphi)$ in $V'_{\text{div},p}$. Here $J \in W^{1,1}_{\text{loc}}(\mathbb{R}^3)$. More assumptions on the kernel J will be given below (see $(H5)$).

We also introduce additional notations frequently used throughout the work. The mathematical expectation with respect to the probability measure $\mathbb{P}$ is denoted by $\mathbb{E}$. For a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a Banach space X, we denote by $L^\gamma(\Omega, \mathcal{F}, \mathbb{P}; L^q(0,T;X))$ $(1 \leq \gamma, q < \infty)$ the space of random functions $u : [0,T] \times \mathcal{M} \times \Omega \to L^q(0,T;X)$ such that u is measurable w.r.t. (t,ω) and for all t, u is measurable w.r.t. $\mathcal{F}_t$, with $(\mathcal{F}_t)_{0\leq t\leq T}$ be a filtration of nondecreasing and right continuous family of sub σ-algebra of $\mathcal{F}$ with $\mathcal{F}_0$ containing all the $\mathbb{P}$-null sets. We furthermore endow this space with the norm

$$\|u\|_{L^\gamma(\Omega,\mathcal{F},\mathbb{P};L^q(0,T;X))} = \left[\mathbb{E}\|u\|^\gamma_{L^q(0,T;X)}\right]^{1/\gamma}.$$

If $q = \infty$, we write

$$\|u\|_{L^\gamma(\Omega,\mathcal{F},\mathbb{P};L^\infty(0,T;X))} = [\mathbb{E}\text{ess}\sup_{t\in[0,T]} \|u(t)\|_X^\gamma]^{1/\gamma}.$$

Let us introduce the hypotheses on $g_1(u,\varphi), \boldsymbol{T}, J$ and F that are relevant for the major part of the paper.

(H_1) We assume that $g_1 : G_{\text{div}} \times H \to V'_{\text{div},p}$ is nonlinear mapping such that

(a) $(u,\varphi) \mapsto g_1(u,\varphi)$ is continuous; there exists a positive constant C such that

$$\|g_1(u,\varphi)\|_{V'_{\text{div},p}} \le C(1+|u|+|\varphi|), \quad \forall(u,\varphi) \in G_{\text{div}} \times H.$$

(H_2) We suppose that $g_2 : [0,T]\times G_{\text{div}}\times H \to L_2(K,G_{\text{div}})$ is nonlinear mapping such that it is continuous in both variables. We require that, for any $t \in [0,T]$ and $(u,\varphi) \in G_{\text{div}} \times H$, $g_2(t,u,\varphi)$ satisfy

$$\|g_2(t,u,\varphi)\|_{L_2(K,G_{\text{div}})} = \|\bar{g}_2(t,u,\varphi)\|_{L_2(Q^{1/2}(K),G_{\text{div}})} \le C(1+|u|+|\varphi|),$$

with $\bar{g}_2$ defined as in (2.16).

As in ([15]), our assumption on the stress tensor $\boldsymbol{T}$, the potential F and the kernel J are the following:

(H_3) $\boldsymbol{T}(.)$ continuously depends on a symmetric tensor $e \in \mathbb{R}^{3\times3}$ and satisfies the following conditions

$$(\boldsymbol{T}(E) - \boldsymbol{T}(S)) .(E-S) \ge \begin{cases} c_1(1+|E|+|S|)^{p-2}|E-S|^2 \\ c_2|E-S|^2 + c_2|E-S|^p \end{cases} \tag{2.20}$$

$$|\boldsymbol{T}(E) - \boldsymbol{T}(S)| \le c_3(1+|E|+|S|)^{p-2}|E-S|, \ \boldsymbol{T}(0) = 0,$$

for all $E, S \in \mathbb{R}^{3\times3}$, for some $c_i > 0$, $i = 1,2,3$ and some $p > 2$. Here $|\ .\ |$ stands for a Euclidean norm of a tensor and "." at the left hand side of (2.20) denotes the scalar product of two tensors.

(H_4) $F \in \mathcal{C}^2(\mathbb{R})$ has a polynomially controlled growth

$$|F'(s)|^r \le c_4(1+|F(s)|), \ r \in (1,2], \tag{2.21}$$

for some $c_4 > 0$ and satisfies the coercivity condition:

$$F''(s) + a(x) \ge c_5 \max\{1, |s|^{2\kappa}\}, \tag{2.22}$$

for all $s \in \mathbb{R}$, almost any $x \in \mathcal{M}$, some $c_5 > 0$ and some $\kappa \ge 0$.

(H_5) $J \in W^{1,1}_{\text{loc}}(\mathbb{R}^3)$, $J(x) = J(-x)$ and $a(x) = \int_{\mathcal{M}} J(x-y)dy \ge 0$ a.e., in $\mathcal{M}$. Moreover, we set

$$a^* := \sup_{x\in\mathcal{M}} \int_{\mathcal{M}} |J(x-y)|dy < \infty, \ b^* := \sup_{x\in\mathcal{M}} \int_{\mathcal{M}} |\nabla J(x-y)|dy < \infty.$$

Remark 2.7. Assumption $J \in W^{1,1}_{\text{loc}}(\mathbb{R}^3)$ can be weakened. Indeed, it can be replaced by $J \in W^{1,1}(B_\delta)$, where $B_\delta := \{z \in \mathbb{R}^3 : |z| < \delta\}$ with $\delta := \text{diam}(\mathcal{M}) = \sup_{x,y\in\mathcal{M}} d_*(x,y)$, where $d_*(.,.)$ is the Euclidean metric on $\mathbb{R}^3$.

Remark 2.8. The hypothesis (2.22) is physically justified and relevant (see [20] for more details). From the mathematical viewpoint assumption (2.22) is satisfied, in particular, if $a \equiv 0$ and F is strictly convex.

Remark 2.9. (2.22) implies the existence of positive constants $c_6 > 0$ and $c_7 > 0$ such that

$$F(s) \ge c_6|s|^{2\kappa+2} - c_7, \ \text{for all } s \in \mathbb{R}.$$

In order to formulate problem (1.8) in the framework of the proof of existence theorem in [27, Chapter IV, Section 2, pp. 167–177], which do not require the Lipschitz condition on the coefficients, we need some preliminaries which we state below.

Lemma 2.1. (Korn's inequalities) *Let $1 < m < \infty$ and let $\mathcal{M} \subset \mathbb{R}^3$ be of class $\mathcal{C}^1$. Then, there exist two positive constants $\kappa^i_m = \kappa^i_m(\mathcal{M})$, $i = 1,2$ such that*

$$\kappa^1_m\|v\|_{1,m} \le \left(\int_{\mathcal{M}} |Dv|^m dx\right)^{1/m} \le \kappa^2_m\|v\|_{1,m}, \ \forall v \in V_{\text{div},m}.$$

Proof. The proof of this lemma can be found in [39, Chapter 5, Theorem 1.10]. □

Let X be a Banach space and X' be its topological dual. Let $\mathbb{T}$ be a function from X to X' with domain $\mathbb{D} = \mathbb{D}(\mathbb{T}) \subseteq X$.

Definition 2.1. [44, Definition 2.3] The function $\mathbb{T}$ is said to be

(a) demicontinuous if for a sequence $v_n \in \mathbb{D}$, $v \in \mathbb{D}$ and $v_n \to v$ in X implies that $\mathbb{T}(v_n) \overset{\text{weakly}}{\rightharpoonup} \mathbb{T}(v)$ in X',
(b) hemicontinuous if $v \in \mathbb{D}$, $u \in X$ and $v + t_n u \in \mathbb{D}$ for a sequence of positive real numbers t_n such that $t_n \to 0$ implies $\mathbb{T}(v + t_n u) \overset{\text{weakly}}{\rightharpoonup} \mathbb{T}(v)$ in X',
(c) locally bounded if for a sequence $v_n \in \mathbb{D}$, $v \in \mathbb{D}$ and $v_n \to v$ in X imply that $\mathbb{T}(v_n)$ is bounded in X'.

From the above definition, it is clear that a demicontinuous function is hemicontinuous and locally bounded.

We now introduce the following result concerning the operator $N : V_{\text{div},p} \to V'_{\text{div},p}$.

Proposition 2.1. *We assume that $\boldsymbol{T}$ satisfies (H_3) with $p \geq 2$. Then,*

(a) *the operator N is demicontinuous,*
(b) *the operator N is monotone; that is, $\langle N(u) - N(v), u - v\rangle_{V_{\text{div},p}} \geq 0, \quad \forall u, v \in V_{\text{div},p}$.*
(c) *There exists a positive constant C such that*

$$\|N(u)\|^{p'}_{V'_{\text{div},p}} \leq C(1 + \|u\|^p_{1,p}), \quad \forall u \in V_{\text{div},p}, \text{ with } p' \text{ the conjugate index to } p. \tag{2.23}$$

Proof. Let $p > 2$. The item (b) of proposition 2.1 follows easily from $(2.10)_1$ and $(2.20)_1$.

Proof of item (a). Let $(v_n)_{n\geq 1}$ be a sequence of points of $V_{\text{div},p}$ and $v \in V_{\text{div},p}$ be such that $v_n \to v$ in $V_{\text{div},p}$. Let $u \in V_{\text{div},p}$. We have

$$\begin{aligned}\langle N(v_n) - N(v), u\rangle_{V_{\text{div},p}} &(\leq) \int_{\mathcal{M}} |\boldsymbol{T}(Dv_n) - \boldsymbol{T}(Dv)||Du|dx \\ &(\leq)\ c_3 \int_{\mathcal{M}} (1 + |Dv_n| + |Dv|)^{p-2} |D(v_n - v)||Du|dx \\ &\leq c_{c_3,p} \left(\int_{\mathcal{M}} |Du|^p dx\right)^{\frac{1}{p}} \left(\int_{\mathcal{M}} (1 + |Dv_n|^p + |Dv|^p)dx\right)^{\frac{p-2}{p}} \left(\int_{\mathcal{M}} |D(v_n - v)|^p dx\right)^{\frac{1}{p}},\end{aligned} \tag{2.24}$$

where we have used Hölder's inequality and $(2.20)_1$. Here $c_{c_3,p}$ is a positive constant depending on p and c_3. It then follows from (2.24) and Lemma (2.1) that

$$\langle N(v_n) - N(v), u\rangle_{V_{\text{div},p}} \leq c_{c_3,p}(\kappa_p^2)^2 \|u\|_{1,p}[1 + (\kappa_p^2)^p(\|v_n\|^p_{1,p} + \|v\|^p_{1,p})]\|v_n - v\|_{1,p} \tag{2.25}$$

for all $u, v \in V_{\text{div},p}$. Therefore from (2.25), the fact that $v_n \to v$ in $V_{\text{div},p}$; i.e., $\|v_n - v\|_{1,p} \to 0$ as $n \to \infty$, $v \in V_{\text{div},p}$ and every convergent sequence is bounded, we deduce that

$$\langle N(v_n) - N(v), u\rangle_{V_{\text{div},p}} \to 0,$$

as $n \to \infty$ for every $u \in V_{\text{div},p}$. This proves that the operator N is demicontinuous; and hence hemicontinuous and locally bounded.

By definition we have

$$\|N(u)\|_{V'_{\text{div},p}} = \sup_{\|v\|_{1,p}=1} |\langle N(u), v\rangle_{V_{\text{div},p}}|.$$

Hence, thanks to $(2.10)_1$, using Hölder's and Korn's inequalities we have

$$\|N(u)\|_{V'_{\text{div},p}} \leq C\left[\int_{\mathcal{M}} |\boldsymbol{T}(Du)|^{p'} ds\right]^{\frac{1}{p'}} \leq C\left[\int_{\mathcal{M}} (1 + |Du|^p)ds\right]^{1/p'}, \tag{2.26}$$

where we have also used $(2.20)_2$.

Finally, thanks to (2.26) in conjunction with Korn's inequality, we obtain (2.23). □

3. Statement of the Main Result

We introduce the concept of solution of the problem (2.17) or (2.18) that is interest to us.

Definition 3.1. By a solution of the problem (2.17) or (2.18), we mean a system $((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}}), \tilde{W}_t, \tilde{u}, \tilde{\varphi})$, where

(1) $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ is a complete probability space; $\tilde{\mathbb{F}} := \{\tilde{\mathcal{F}}_t : t \geq 0\}$ is a filtration on the probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, i.e., a nondecreasing family $\{\tilde{\mathcal{F}}_t : t \geq 0\}$ of sub σ-fields of $\tilde{\mathcal{F}}$: $\tilde{\mathcal{F}}_s \subset \tilde{\mathcal{F}}_t \subset \tilde{\mathcal{F}}$ for $0 \leq s < t < \infty$;
(2) $\tilde{W}_t$ is a $\tilde{\mathcal{F}}_t$-cylindrical Wiener process on G_{div};
(3) for almost every $t \in [0, T]$, $\tilde{u}(t)$ and $\tilde{\varphi}(t)$ are $\tilde{\mathcal{F}}_t$ measurable;
(4) for almost every t, $\tilde{u}(t) \in L^q(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; L^p(0, T; V_{\text{div},p})) \cap L^q(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; L^\infty(0, T; G_{\text{div}}))$, $\tilde{\varphi}(t) \in L^q(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; L^2(0, T; U)) \cap L^q(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; L^\infty(0, T; L^{2\kappa+2}(\mathcal{M})))$, $2 \leq q < \infty$;
(5) $\tilde{\mathbb{P}}$-a.s the following integral equations of Itô type hold:

$$\begin{aligned}
&(\tilde{u}(t) - u_0, v) + \int_0^t \langle (N(\tilde{u}(s)) - B_0(\tilde{u}(s)) - R_1(\tilde{\varphi}(s))), v \rangle_{V_{\text{div},p}} ds \\
&\quad = \int_0^t \langle g_0(s), v \rangle_{V_{\text{div},p}} + \int_0^t \langle g_1(\tilde{u}(s), \tilde{\varphi}(s)), v \rangle_{V_{\text{div},p}} ds + \int_0^t (g_2(s, \tilde{u}(s), \tilde{\varphi}(s)), v) d\tilde{W}_s, \\
&(\tilde{\varphi}(t) - \varphi_0, \psi) - \int_0^t \langle B_1(\tilde{u}(s), \tilde{\varphi}(s)), \psi \rangle_U ds = \int_0^t \langle \Delta \tilde{\mu}(s), \psi \rangle_U ds, \\
&\tilde{\mu} = a\tilde{\varphi} - J * \tilde{\varphi} + F'(\tilde{\varphi}),
\end{aligned} \tag{3.1}$$

for any $t \in [0, T]$ and $(v, \psi) \in \mathbf{W}_s \times U := \mathbf{W}_s \times H^1(\mathcal{M})$, $s \geq 5/2 > 5/2 - 3/p$.

Now we can state our first result in the following theorem.

Theorem 3.1. *Let $p \geq 11/5$ and*

$$\begin{cases} \kappa \geq \dfrac{2(3-p)}{5p-6}, & \text{if } p < 3, \\ 0 < \kappa \leq 2, & \text{if } p = 3, \\ \kappa > 0 & \text{if } p > 3. \end{cases} \tag{3.2}$$

Assume that $u_0 \in G_{\text{div}}$, $\varphi_0 \in H$ with $F(\varphi_0) \in L^1(\mathcal{M})$ and $g_0 \in L^{p'}(0, T; V'_{\text{div},p})$. Suppose also that all the assumptions, namely, (H_1) and (H_5) are satisfied. Then problem (2.17) or (2.18) has a solution in the sense of the above definition. Moreover, we note that a solution $(\tilde{u}, \tilde{\varphi})$ in the sense of definition 3.1 belongs to $L^q(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; \mathcal{C}(0, T; \mathbb{H}))$, $q \in [2, \infty)$.

Proof. The proof will be carried out in Sects. 4.1–4.4. □

In the rest of the paper, we set

$$\mathcal{E}_{tot}(u, \varphi) := |u|^2 + 2\mathcal{E}(\varphi), \text{ where } \mathcal{E} \text{ is given by (1.5)}. \tag{3.3}$$

4. Auxiliary Results

In this section, we introduce the Galerkin approximation scheme to reduce the original system to a system of finite-dimensional ordinary stochastic differential equations (SDEs). We derive crucial a priori estimates from the Galerkin approximation which will serve as a toolkit for the proof of Theorem 3.1. More precisely, the priori estimates will be used to prove the tightness of the family of laws of the sequence of solutions of the system of SDEs on appropriate topological spaces.

4.1. The Approximate Solution

We first assume that $\varphi_0 \in D(\mathcal{B}) = \{\varphi \in H^2(\mathcal{M}) : \partial_\eta \varphi = 0 \text{ on } \partial\mathcal{M}\} \subset H$ instead of $\varphi_0 \in H$, where $\mathcal{B} = -\Delta + I$. The general case $\varphi_0 \in H$ with $F(\varphi_0) \in L^1(\mathcal{M})$ can be dealt in the same fashion as in [10], by means of a density argument and by relying on the form of the potential F as a quadratic perturbation of a convex function.

As a Galerkin base in $V_{\text{div},p}$ we employ the family $\{w_j, j = 1, 2, \dots\}$, where each w_j solves

$$(w_j, v)_{W_s} = \lambda_j (w_j, v), \quad \forall v \in W_s.$$

In U we choose as Galerkin base the family $\{\psi_j, j = 1, 2, \dots\}$, where ψ_j are the eigenfunctions of the operator $\mathcal{B}$. We set $\mathrm{W}_s^n = \operatorname{span}\{w_1, w_2, \dots, w_n\}$ and $H_n = \operatorname{span}\{\psi_1, \psi_2, \dots, \psi_n\}$. Let $\mathcal{P}_n^1$ be the operator from $\mathbf{W}_s'$ to $\mathbf{W}_s^n$ defined by $\mathcal{P}_n^1 u^* := \sum_{j=1}^n \langle u^*, w_j \rangle_{W_s} w_j,\ u^* \in \mathbf{W}_s'$. We will consider the restriction of the operator $\mathcal{P}_n^1$ to the space G_{div} (still) denoted by $\mathcal{P}_n^1$. More precisely, we have $G_{\text{div}} \cong G'_{\text{div}} \hookrightarrow V'_{\text{div},p} \hookrightarrow \mathbf{W}_s'$ $(s > 5/2)$, i.e. every element $u \in G_{\text{div}}$ induces a functional $u^* \in W_s'$ by the formula

$$\langle u^*, v \rangle_{\mathbf{W}_s} = (u, v),\ v \in \mathbf{W}_s.$$

Thus the restriction of $\mathcal{P}_n^1$ to G_{div} is given by $\mathcal{P}_n^1 u = \sum_{j=1}^n (u, w_j) w_j,\ u \in G_{\text{div}}$. Hence in particular, $\mathcal{P}_n^1$ is the $(.,.)$-orthogonal projection from G_{div} onto $\mathbf{W}_s^n$. Similarly, we define by $\mathcal{P}_n^2$ the orthogonal projection from H onto H_n. Let $(\Omega, \mathcal{F}, \mathbb{P}, W_t)$ (W_t is a cylindrical Wiener processes evolving on K). We equip the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the natural filtration of W_t which is denoted by $\mathcal{F}_t$. We then look for the three functions of the form

$$u_n(t) = \sum_{k=1}^n a_k^{(n)}(t) w_k,\ \varphi_n(t) = \sum_{k=1}^n b_k^{(n)}(t) \psi_k,\ \mu_n(t) = \sum_{k=1}^n c_k^{(n)}(t) \psi_k,$$

which solves the following approximating problem

$$\begin{cases} d(u_n, w_j) + [(\boldsymbol{T}(Du_n), Dw_j) + b(u_n, u_n, w_j)]dt = -(\varphi_n \nabla \mu_n, w_j)dt + (\mathcal{P}_n^1 g_0, w_j)dt \\ \quad +(g_1(u_n, \varphi_n), w_j)dt + \sum_{i=1}^n (g_2(t, u_n, \varphi_n)e_i, w_j) d\beta_t^i, \\ d(\varphi_n, \psi_j) + (\nabla \mu_n, \nabla \psi_j)dt = (u_n \varphi_n, \nabla \psi_j)dt, \\ \mu_n = \mathcal{P}_n^2 (a\varphi_n - J * \varphi_n + F'(\varphi_n)), \\ u_n(0) = \mathcal{P}_n^1 u_0 := u_{0n},\ \varphi_n(0) = \mathcal{P}_n^2 \varphi_0 := \varphi_{0n}, \end{cases} \tag{4.1}$$

where u_{0n}, φ_{0n} and $\mathcal{P}_n^1 g_0$ are such that $u_{0n} \to u_0$ in G_{div}, $\varphi_{0n} \to \varphi_0$ in $H^2(\mathcal{M})$ and $\mathcal{P}_n^1 g_0 \to g_0$ in $L^{p'}(0, T; V'_{\text{div},p})$ as $n \to \infty$ respectively.

We first note that since the operator N is hemicontinuous and monotone (see Proposition 2.1), we infer from [33, Chapitre II, page 171] that N is continuous from $V_{\text{div},p}$ into $V'_{\text{div},p}$ and locally bounded. System (4.1) is then a system of stochastic differential equations in a finite dimensional Banach spaces with continuous and locally bounded coefficients. From the existence theorem in [49, Chapter 3, Section 3, p. 59] (see also [27, Chapter 4, Section 2, pp. 167-177]), which do not require the Lipschitz condition on the coefficients, there exists on a short interval $[0, T_n)$, $T_n \le T$ a sequence of continuous functions (u_n, φ_n) solving (4.1). It will follows from a priori estimates that (u_n, φ_n) exists on $[0, T]$.

4.2. A Priori Estimates

In this subsection, we derive some basic energy estimates for the sequence of the approximate solutions u_n, φ_n and for the sequence μ_n, $F(\varphi_n)$ and $F'(\varphi_n)$.

First, we prove the following lemma.

Lemma 4.1. *the sequence $(u_n, \varphi_n, \mu_n, F(\varphi_n), F'(\varphi_n), n = 1, 2, \ldots)$ satisfies*

$$\begin{aligned} &\mathbb{E} \sup_{s\in[0,T]} |u_n(s)|^q < C, \quad \mathbb{E} \sup_{s\in[0,T]} \|\varphi_n(s)\|^q_{L^{2\kappa+2}(\mathcal{M})} < C, \\ &\mathbb{E} \sup_{s\in[0,T]} \|F(\varphi_n(s))\|^{q/2}_{L^1(\mathcal{M})} < C, \quad \mathbb{E} \sup_{s\in[0,T]} \|F'(\varphi_n(s))\|^q_{L^r(\mathcal{M})} < C, \end{aligned} \tag{4.2}$$

and

$$\mathbb{E}\left(\int_0^T \|u_n(s)\|^p_{1,p} ds\right)^{\frac{q}{p}} < C, \mathbb{E}\left(\int_0^T \|\varphi_n(s)\|^2_U ds\right)^{\frac{q}{2}} < C, \mathbb{E}\left(\int_0^T \|\mu_n(s)\|^2_U ds\right)^{\frac{q}{2}} < C, \tag{4.3}$$

for any $q \in [2, \infty)$, $r \in (1, 2]$ and $p \geq 11/5$, with $U := H^1(\mathcal{M})$. Here C is a positive constant depending on the parameters $T, q, p, \mathcal{M}, \kappa_2^1, \kappa_p^1, |J|_{L^1(\mathbb{R}^3)}, \kappa, r, c_2, c_4, c_5, c_6$ and the initial data u_0, φ_0 and $|F(\varphi_0)|_{L^1(\mathcal{M})}$. We recall that the constant c_3 is given by (H_3), the constants c_4 and c_5 are given by (H_4), the constant c_6 is defined as in Remark 2.9, and the constants κ_2^1 and κ_p^1 are given by Korn's inequality (see Lemma 2.1).

Proof. Let $\tau_n^R, R, n \in \mathbb{N}$, be stopping time defined by

$$\tau_n^R = \inf\{t \in [0, T]; |u_n(t)|^2 + \|\varphi_n(t)\|^2_{L^{2\kappa+2}} + \textstyle\int_0^t (\|u_n(s)\|^p_{1,p} + \|\varphi_n(s)\|^2_U) ds \geq R^2\} \wedge T. \tag{4.4}$$

Let $t \in [0, \tau_n^R \wedge T]$. By applying Itô's formula to the process $|u_n(t)|^2$, taking μ_n as test function in $(4.1)_2$, recalling that $b(u_n, u_n, u_n) = 0$ (see (2.9)) and by summing the ensuing identities, we obtain

$$\begin{aligned} &[\mathcal{E}_{tot}(u_n(t), \varphi_n(t)) + \bar{\kappa}_1] + 2\int_0^t (\boldsymbol{T}(Du_n), Du_n) ds + 2\int_0^t |\nabla \mu_n|^2 ds \\ &= \bar{\kappa}_1 + \mathcal{E}_{tot}(u_{0n}, \varphi_{0n}) + 2\int_0^t \langle \mathcal{P}_n^1 g_0, u_n \rangle_{V_{\mathrm{div},p}} ds + 2\int_0^t \langle g_1(u_n, \varphi_n), u_n \rangle_{V_{\mathrm{div},p}} ds \\ &\quad + \sum_{j,k=1}^n \int_0^t [(g_2(s, u_n, \varphi_n)e_j, w_k)]^2 ds + 2\sum_{j=1}^n \int_0^t (g_2(s, u_n, \varphi_n)e_j, u_n) d\beta_s^j, \end{aligned} \tag{4.5}$$

with

$$\bar{\kappa}_1 = \frac{\kappa|\mathcal{M}|}{c_6^{1/\kappa}} \left(\frac{|J|_{L^1(\mathbb{R}^3)}}{\kappa + 1}\right)^{(\kappa+1)/\kappa} + 2c_7|\mathcal{M}|. \tag{4.6}$$

Note that the constant $\bar{\kappa}_1$ is such that $[\mathcal{E}_{tot}(u_n(t), \varphi_n(t)) + \bar{\kappa}_1] \geq 0$. In fact, we have

$$\begin{aligned} 2\mathcal{E}(\varphi_n(t)) &= 2|\sqrt{a}\varphi_n(t)|^2 + 2\textstyle\int_{\mathcal{M}} F(\varphi_n(t, x)) dx - (\varphi_n(t), J * \varphi_n(t)) \\ &\geq \textstyle\int_{\mathcal{M}} (a(x) - |J|_{L^1(\mathbb{R}^3)})(\varphi_n(t, x))^2 dx + 2c_6\|\varphi_n(t)\|^{2\kappa+2}_{L^{2\kappa+2}(\mathcal{M})} - 2c_7|\mathcal{M}| \\ &\geq c_6\|\varphi_n(t)\|^{2\kappa+2}_{L^{2\kappa+2}(\mathcal{M})} - \bar{\kappa}_1, \end{aligned} \tag{4.7}$$

where we have used Hölder's inequality, Young's inequality for convolutions, Young's inequality and Remark 2.9. It then follows from (4.7) that

$$\begin{aligned} \mathcal{E}_{tot}(u_n(t), \varphi_n(t)) + \bar{\kappa}_1 &= |u_n(t)|^2 + 2\mathcal{E}(\varphi_n(t)) + \bar{\kappa}_1 \\ &\geq |u_n(t)|^2 + c_6\|\varphi_n(t)\|^{2\kappa+2}_{L^{2\kappa+2}(\mathcal{M})} \geq 0. \end{aligned} \tag{4.8}$$

Now since

$$\sum_{j,k=1}^n [(g_2(s, u_n(s), \varphi_n(s))e_j, w_k)]^2 \leq \|g_2(s, u_n(s), \varphi_n(s))\|^2_{L_2(K, G_{\mathrm{div}})},$$

and

$$
\begin{aligned}
\mathcal{E}_{tot}(u_{0n},\varphi_{0n}) &\leq |u_{0n}|^2 + 2|J|_{L^1(\mathbb{R}^3)}|\varphi_{0n}|^2 + 2\int_{\mathcal{M}} F(\varphi_{0n}(x))dx \\
&\leq |u_0|^2 + 2|J|_{L^1(\mathbb{R}^3)}|\varphi_0|^2 + 2\int_{\mathcal{M}} F(\varphi_0(x))dx \\
&\equiv \mathcal{K}(u_0,\varphi_0),
\end{aligned} \tag{4.9}
$$

(where in (4.9), we have used the fact that, since $\varphi_0 \in D(\mathcal{B})$ a.s., then we have $\varphi_{0n} \to \varphi_0$ in $H^2(\mathcal{M})$ a.s. and hence also in $L^\infty(\mathcal{M})$ a.s.), it follows from (4.5) that

$$
\begin{aligned}
&[\mathcal{E}_{tot}(u_n(t),\varphi_n(t)) + \bar{\kappa}_1] + 2\mathcal{Z}_p\int_0^t [\|u_n(s)\|^2 + \|u_n(s)\|_{1,p}^p]ds + 2\int_0^t |\nabla\mu_n|^2 ds \\
&\leq \bar{\kappa}_1 + \mathcal{K}(u_0,\varphi_0) + 2\int_0^t [\|g_0(s)\|_{V'_{\mathrm{div},p}} + \|g_1(u_n(s),\varphi_n(s))\|_{V'_{\mathrm{div},p}}]\|u_n(s)\|_{1,p}ds \\
&\quad + \int_0^t \|g_2(s,u_n(s),\varphi_n(s))\|^2_{L_2(K,G_{\mathrm{div}})}ds + 2\int_0^t \sum_{j=1}^n (g_2(s,u_n(s),\varphi_n(s))e_j,u_n)d\beta_s^j,
\end{aligned} \tag{4.10}
$$

with $\mathcal{Z}_p = c_2 \min\left((\kappa_2^1)^2,(\kappa_p^1)^p\right)$. Note that in (4.10) we have also used the assumption (H_3) (see (2.20)) in conjunction with Lemma 2.1.

Hereafter, we set (for the sake of simplicity):

$$
\begin{aligned}
&\chi_n(t) := \mathcal{E}_{tot}(u_n(t),\varphi_n(t)) + \bar{\kappa}_1, \quad \vartheta_{p,n} := [\|u_n(t)\|^2 + \|u_n(t)\|_{1,p}^p], \\
&\varpi_n(t) := \int_0^t \sum_{j=1}^n (g_2(s,u_n(s),\varphi_n(s))e_j,u_n(s))d\beta_s^j, \quad t\in[0,T].
\end{aligned} \tag{4.11}
$$

Setting $a_1 = (\frac{p\mathcal{Z}_p}{2})^{\frac{1}{p}}\|u_n(s)\|_{1,p}, b = 2(\frac{1}{p})^{\frac{1}{p}}(\frac{1}{\mathcal{Z}_p})^{\frac{1}{p}}\|g_0(s)\|_{V'_{\mathrm{div},p}} \times 2^{\frac{1}{p}}$, using Young's inequality, we see that

$$
2\|g_0(s)\|_{V'_{\mathrm{div},p}}\|u_n(s)\|_{1,p} = a_1 b \leq \tfrac{1}{p}a_1^p + \tfrac{1}{p'}b^{p'} = \tfrac{\mathcal{Z}_p}{2}\|u_n(s)\|_{1,p}^p + \frac{(2)^{\frac{p'(p+1)}{p}}}{p'\mathcal{Z}_p^{\frac{p'}{p}}p^{\frac{p'}{p}}}\|g_0(s)\|^{p'}_{V'_{\mathrm{div},p}}. \tag{4.12}
$$

Here $p' = \frac{p}{p-1}$. One can easily see that

$$
2\|g_1(u_n(s),\varphi_n(s))\|_{V'_{\mathrm{div},p}}\|u_n(s)\|_{1,p} \leq \tfrac{\mathcal{Z}_p}{2}\|u_n(s)\|_{1,p}^2 + \tfrac{2}{\mathcal{Z}_p}\|g_1(u_n(s),\varphi_n(s))\|^2_{V'_{\mathrm{div},p}}. \tag{4.13}
$$

Now setting $r_1 = \frac{p}{2}, r_2 = \frac{p}{p-2}, a_1 = (\frac{p}{2})^{\frac{2}{p}}\|u_n(s)\|_{1,p}^2, b = (\frac{2}{p})^{\frac{2}{p}}$, using Young's inequality, we have

$$
\|u_n(s)\|_{1,p}^2 = a_1 b \leq \tfrac{1}{r_1}a_1^{r_1} + \tfrac{1}{r_2}b^{r_2} = \|u_n(s)\|_{1,p}^p + \tfrac{(p-2)\times 2^{\frac{2}{p-2}}}{p[p]^{\frac{2}{p-2}}}. \tag{4.14}
$$

It then follows from estimates (4.12)–(4.14) that

$$
\begin{aligned}
&2[\|g_0(s)\|_{V'_{\mathrm{div},p}} + \|g_1(u_n(s),\varphi_n(s))\|_{V'_{\mathrm{div},p}}]\|u_n(s)\|_{1,p} \\
&\leq \mathcal{Z}_p\|u_n\|_{1,p}^p + \tfrac{\mathcal{Z}_p(p-2)}{2p}\left(\tfrac{2}{p}\right)^{\frac{2}{p-2}} + \tfrac{2}{\mathcal{Z}_p}\|g_1(u_n,\varphi_n)\|^2_{V'_{\mathrm{div},p}} + \frac{(2)^{\frac{p'(p+1)}{p}}}{p'\mathcal{Z}_p^{\frac{p'}{p}}p^{\frac{p'}{p}}}\|g_0\|^{p'}_{V'_{\mathrm{div},p}}.
\end{aligned} \tag{4.15}
$$

By Young inequality's, we infer that

$$
|\varphi_n|^2 \leq \frac{c_6}{\kappa+1}\|\varphi_n\|^{2\kappa+2}_{L^{2\kappa+2}(\mathcal{M})} + \frac{\kappa|\mathcal{M}|}{(\kappa+1)c_6^{1/\kappa}}. \tag{4.16}
$$

Now, owing to the assumption on g_1 and g_2, we can derive from the estimates (4.8), (4.15)–(4.16) and (4.10) that

$$\begin{aligned}&\chi_n(t)+2\mathcal{Z}_p\int_0^t\vartheta_{p,n}(s)ds+2\int_0^t|\nabla\mu_n(s)|^2ds\\&\leq\bar{\kappa}_1+\mathcal{K}(u_0,\varphi_0)+C_1\int_0^t\|g_0(s)\|_{V'_{\mathrm{div},p}}^{p'}ds+C_2\int_0^t(1+\chi_n(s))ds+2\varpi_n(t),\end{aligned}\tag{4.17}$$

with $C_1:=C_1(\kappa_2^1,\kappa_p^1,p)$ and $C_2:=C_2(\kappa_2^1,\kappa_p^1,c_2,c_6,\kappa,p,|\mathcal{M}|)$. $\chi_n(t),\vartheta_{p,n}(t)$ and $\varpi_n(t)$ are defined as in (4.11).

From (4.17), we infer that

$$\begin{aligned}&\mathbb{E}\sup_{s\in[0,t\wedge\tau_n^R]}[\chi_n(s)]+2\mathbb{E}\int_0^{t\wedge\tau_n^R}[\mathcal{Z}_p\vartheta_{p,n}(s)+|\nabla\mu_n(s)|^2]ds\\&\leq\bar{\kappa}_1+\mathcal{K}(u_0,\varphi_0)+C_1\int_0^{t\wedge\tau_n^R}\|g_0(s)\|_{V'_{\mathrm{div},p}}^{p'}ds+C_2\mathbb{E}\int_0^{t\wedge\tau_n^R}(1+\chi_n(s))ds+2\mathbb{E}\sup_{s\in[0,t\wedge\tau_n^R]}|\varpi_n(s)|.\end{aligned}\tag{4.18}$$

By Burkholder–Davis–Gundy's inequality and Hölder's inequality, we have

$$\begin{aligned}2\mathbb{E}\sup_{s\in[0,t\wedge\tau_n^R]}|\varpi_n(s)|&=2\mathbb{E}\sup_{s\in[0,t\wedge\tau_n^R]}\left|\int_0^s\sum_{j=1}^n(g_2(\tau,u_n(\tau),\varphi_n(\tau))e_j,u_n(\tau))d\beta_\tau^j\right|\\&\leq C\mathbb{E}\left(\int_0^{t\wedge\tau_n^R}\sum_{j=1}^n(g_2(s,u_n(s),\varphi_n(s))e_j,u_n)^2ds\right)^{1/2}\\&\leq C\mathbb{E}\left(\int_0^{t\wedge\tau_n^R}\|g_2(s,u_n(s),\varphi_n(s))\|_{L_2(K,G_{\mathrm{div}})}^2|u_n(s)|^2ds\right)^{1/2}\\&\leq C\left[\mathbb{E}\sup_{s\in[0,t\wedge\tau_n^R]}|u_n(s)|^2\right]^{1/2}\left[\mathbb{E}\int_0^{t\wedge\tau_n^R}\|g_2(s,u_n,\varphi_n)\|_{L_2(K,G_{\mathrm{div}})}^2ds\right]^{1/2}.\end{aligned}\tag{4.19}$$

Thanks to (4.19), using Young's inequality and the assumption on g_2 (see (H_2)), we obtain

$$\begin{aligned}2\mathbb{E}\sup_{s\in[0,t\wedge\tau_n^R]}|\varpi_n(s)|&\leq\frac{1}{2}\mathbb{E}\sup_{s\in[0,t\wedge\tau_n^R]}|u_n(s)|^2+C\mathbb{E}\int_0^{t\wedge\tau_n^R}\|g_2(s,u_n,\varphi_n)\|_{L_2(K,G_{\mathrm{div}})}^2ds\\&\leq\frac{1}{2}\mathbb{E}\sup_{s\in[0,t\wedge\tau_n^R]}|u_n(s)|^2+C\mathbb{E}\int_0^{t\wedge\tau_n^R}(1+|u_n(s)|^2+|\varphi_n(s)|^2)ds.\end{aligned}$$

Hence, from this previous inequality and (4.16), there exists a positive constant C_3 depending on c_6,κ and $\mathcal{M}$ such that

$$\begin{aligned}2\mathbb{E}\sup_{s\in[0,t\wedge\tau_n^R]}|\varpi_n(s)|&\leq\frac{1}{2}\mathbb{E}\sup_{s\in[0,t\wedge\tau_n^R]}|u_n(s)|^2+C_3\mathbb{E}\int_0^{t\wedge\tau_n^R}(1+[|u_n|^2+c_6\|\varphi_n\|_{L^{2\kappa+2}}^{2\kappa+2}])ds\\&\leq\frac{1}{2}\mathbb{E}\sup_{s\in[0,t\wedge\tau_n^R]}[\chi_n(s)]+C_3\mathbb{E}\int_0^{t\wedge\tau_n^R}(1+[\chi_n(s)])ds,\end{aligned}\tag{4.20}$$

where we have also used (4.8).

Thanks to (4.18) and (4.20), we obtain

$$\mathbb{E}\sup_{s\in[0,t\wedge\tau_n^R]}[\chi_n(s)]+2\mathbb{E}\int_0^{t\wedge\tau_n^R}[2\mathcal{Z}_p\vartheta_{p,n}(s)+2|\nabla\mu_n|^2]ds\leq\mathcal{K}_1+\tilde{C}_2\mathbb{E}\int_0^{t\wedge\tau_n^R}\sup_{0\leq s\leq\tau}[\chi_n(s)]d\tau,\tag{4.21}$$

with

$$\begin{aligned} &\tilde{C}_2 := \tilde{C}_2(\kappa_2^1, \kappa_p^1, c_2, \kappa, c_6, p, |\mathcal{M}|), \\ &\mathcal{K}_1 := 2\bar{\kappa}_1 + 2\mathcal{K}(u_0, \varphi_0) + C_1(\kappa_2^1, \kappa_p^1, p) \int_0^T \|g_0(s)\|_{V'_{\mathrm{div},p}}^{p'} ds + \tilde{C}_2 T, \end{aligned} \tag{4.22}$$

χ_n and $\vartheta_{p,n}$ are defined in (4.11). Now, we define

$$\mathcal{Z}(t) := \mathbb{E} \int_0^{t\wedge\tau_n^R} \sup_{0\le s\le\tau} [\chi_n(s)] d\tau \tag{4.23}$$

and then (4.21), implies

$$\mathcal{Z}'(t) \le \mathcal{K}_1 + \tilde{C}_2 \mathcal{Z}(t). \tag{4.24}$$

This gives

$$\mathcal{Z}(t) \le \frac{\mathcal{K}_1}{\tilde{C}_2}(e^{\tilde{C}_2 t} - 1). \tag{4.25}$$

Owing to (4.21), (4.23) and (4.25), we derive that

$$\mathbb{E} \sup_{s\in[0,t\wedge\tau_n^R]} [\chi_n(s)] + 2\mathbb{E} \int_0^{t\wedge\tau_n^R} [2\mathcal{Z}_p \vartheta_{p,n}(s) + 2|\nabla\mu_n(s)|^2] ds \le \mathcal{K}_1 + \mathcal{K}_1 e^{\tilde{C}_2 t}. \tag{4.26}$$

Arguing similarly as in [13, Inequality (4.57)], using (4.16) we get

$$|\nabla\varphi_n|^2 \le \tfrac{8}{c_5^2}|\varphi_n|^2 + \tfrac{4}{c_5^2}|\nabla\mu_n|^2 \le \tfrac{8c_6}{c_5^2(\kappa+1)}\|\varphi_n\|_{L^{2\kappa+2}(\mathcal{M})}^{2\kappa+2} + \tfrac{8\kappa|\mathcal{M}|}{c_5^2(\kappa+1)c_6^{1/\kappa}} + \tfrac{4}{c_5^2}|\nabla\mu_n|^2. \tag{4.27}$$

Thanks to (4.8), (4.16), (4.27) and (4.26), we infer that

$$\begin{aligned} \mathbb{E} \int_0^{t\wedge\tau_n^R} \|\varphi_n\|_U^2 ds &= \mathbb{E} \int_0^{t\wedge\tau_n^R} (|\varphi_n|^2 ds + |\nabla\varphi_n|^2) ds \\ &\le [\tilde{\kappa}_1 + \tilde{\kappa}_3]t + [\tilde{\kappa}_2 + \tilde{\kappa}_4] t \mathbb{E} \sup_{s\in[0,t\wedge\tau_n^R]} \|\varphi_n(s)\|_{L^{2\kappa+2}}^{2\kappa+2} + \tilde{\kappa}_5 \mathbb{E} \int_0^{t\wedge\tau_n^R} |\nabla\mu_n|^2 ds \\ &\le [\tilde{\kappa}_1 + \tilde{\kappa}_3]t + [\tilde{\kappa}_2 + \tilde{\kappa}_4]\tilde{\kappa}_6 t(1 + e^{\tilde{C}_2 t}) + \tilde{\kappa}_7(1 + e^{\tilde{C}_2 t}) < C, \end{aligned} \tag{4.28}$$

where $\tilde{\kappa}_1 = \frac{\kappa|\mathcal{M}|}{(\kappa+1)c_6^{1/\kappa}}, \tilde{\kappa}_2 = \frac{c_6}{\kappa+1}, \tilde{\kappa}_3 = \frac{8\kappa|\mathcal{M}|}{c_5^2(\kappa+1)c_6^{1/\kappa}}, \tilde{\kappa}_4 = \frac{8c_6}{c_5^2(\kappa+1)}, \tilde{\kappa}_5 = \frac{4}{c_5^2}, \tilde{\kappa}_6 = \frac{\mathcal{K}_1}{c_6}$ and $\tilde{\kappa}_7 = \frac{\mathcal{K}_1}{c_5^2}$. Now, we will prove that

$$\tau_n^R \nearrow T \ \mathbb{P} - \text{almost surely as } R \to \infty.$$

Indeed, since $(u_n, \varphi_n)(. \wedge \tau_n^R) : [0,T] \to \mathbf{W}_s^n \times H_n$ is continuous, we have

$$\begin{aligned} R^2\mathbb{P}(\tau_n^R < t) &\le \mathbb{E}[1_{\tau_n^R<t}(\rho_n(\tau_n^R) + \textstyle\int_0^{\tau_n^R} \varrho_n(s)ds)] \\ &\le \mathbb{E}[1_{\tau_n^R<t}(\rho_n(\tau_n^R) + \textstyle\int_0^{\tau_n^R} \varrho_n(s)ds)] + \mathbb{E}[1_{\tau_n^R\ge t}(\rho_n(\tau_n^R) + \textstyle\int_0^{\tau_n^R} \varrho_n(s)ds)] \\ &= \mathbb{E}[\rho_n(\tau_n^R \wedge t) + \textstyle\int_0^{\tau_n^R\wedge t} \varrho_n(s)ds], \end{aligned} \tag{4.29}$$

with $t \wedge \tau_n^R = \tau_n^R$, since $\tau_n^R < t$, and for any $n \in \mathbb{N}$ and $t \in [0,T]$. Here $\rho_n(.) = |u_n(.)|^2 + \|\varphi_n(.)\|_{L^{2\kappa+2}(\mathcal{M})}^2$ and $\varrho_n(.) = \|u_n\|_{1,p}^p(.) + \|\varphi_n\|_U^2(.)$.

From (4.29) and the inequalities (4.8), (4.26) and (4.28), we infer that

$$\mathbb{P}(\tau_n^R < t) \le \frac{C}{R^2}. \tag{4.30}$$

Since the constant C in (4.30) does not depend on n and R, it then follows that

$$\lim_{R\to\infty} \mathbb{P}(\tau_n^R < t) = 0 \text{ for all } t \in [0,T] \text{ and } n \in \mathbb{N},$$

which implies that there exists a subsequence $\tau_n^{R_k}$, such that $\tau_n^{R_k} \to T$ a.s., which along with the fact that $(\tau_n^R, n)_{R\in\mathbb{N}}$ is increasing, yields that $\tau_n^R \nearrow T$ a.s. for any $n \in \mathbb{N}$. Therefore $T_n = T$.

Now, since the constant $\mathcal{K}_1+\mathcal{K}_1 e^{\tilde{C}_2 t}$ in (4.26) does not depend on n and R, and since $\tau_n^R \nearrow T$ $\mathbb{P}$-almost surely as $R\to\infty$, we can conclude by passing to the limit in (4.26) that

$$\mathbb{E}\sup_{s\in[0,T]}[\chi_n(s)] + 2\mathbb{E}\int_0^T [2\mathcal{Z}_p\vartheta_{p,n}(s)+2|\nabla\mu_n(s)|^2]ds \le \mathcal{K}_1+\mathcal{K}_1 e^{\tilde{C}_2 T} \equiv C. \tag{4.31}$$

We infer from (4.17) that

$$\begin{aligned}\sup_{s\in[0,T]}[\chi_n(s)] + \int_0^T [2\mathcal{Z}_p\vartheta_{p,n}(s)+2|\nabla\mu_n|^2]ds \le \bar{\kappa}_1 + \mathcal{K}(u_0,\varphi_0) + C_1\int_0^T \|g_0(s)\|^{p'}_{V'_{\mathrm{div},p}}ds \\ + C_2T + C_2\int_0^T [\chi_n(s)]ds + 2\sup_{s\in[0,T]}|\varpi_n(s)|.\end{aligned} \tag{4.32}$$

Now squaring both sides of the above inequality to the power $q/2$, $q>2$, we obtain thanks to the Minkowski inequality and after taking the expected values

$$\begin{aligned}&\mathbb{E}\sup_{s\in[0,T]}[\chi_n(s)]^{\frac{q}{2}} + \left(\int_0^T [2\mathcal{Z}_p\vartheta_{p,n}(s)+2|\nabla\mu_n(s)|^2]ds\right)^{\frac{q}{2}} \\ &\le C_5(q)\left[\bar{\kappa}_1^{\frac{q}{2}} + \tilde{\mathcal{K}}_q(u_0,\varphi_0) + C_1^{\frac{q}{2}}\left(\int_0^T \|g_0(s)\|^{p'}_{V'_{\mathrm{div},p}}ds\right)^{\frac{q}{2}} + (C_2T)^{\frac{q}{2}}\right] \\ &\quad + C_5(q)C_2^{\frac{q}{2}}T^{\frac{q-2}{2}}\mathbb{E}\int_0^T [\chi_n(s)]^{q/2}ds + C_5(q)\mathbb{E}\sup_{s\in[0,T]}|\varpi_n(s)|^{q/2},\end{aligned} \tag{4.33}$$

with

$$\tilde{\mathcal{K}}_q(u_0,\varphi_0) = |u_0|^q + 2^{\frac{q}{2}}|J|^{\frac{q}{2}}_{L^1(\mathbb{R}^3)}|\varphi_0|^q + 2^{\frac{q}{2}}|F(\varphi_0)|^{\frac{q}{2}}_{L^1(\mathcal{M})}. \tag{4.34}$$

Using again the Burkholder–Davis–Gundy inequality as we did in the proof of Eq. (4.20) and the assumption on g_2, we can check that

$$\begin{aligned}C_5(q)\mathbb{E}\sup_{s\in[0,T]}|\varpi_n(s)|^{q/2} &\le \tilde{C}_5(q)\mathbb{E}\left(\int_0^T \sum_{j=1}^n (g_2(s,u_n,\varphi_n)e_j,u_n)^2 ds\right)^{q/4} \\ &\le \tfrac{1}{2}\mathbb{E}\sup_{s\in[0,T]}|u_n(s)|^q + \bar{C}_5T + \bar{C}_5\mathbb{E}\int_0^T \left(|u_n|^2 + c_6|\varphi_n|^{2\kappa+2}_{L^{2\kappa+2}}\right)^{q/2}ds,\end{aligned} \tag{4.35}$$

with $\bar{C}_5 := \bar{C}_5(q,\kappa,c_6,|\mathcal{M}|)$ and where we have also used (4.16).

From (4.35) and (4.8), we infer that

$$C_5(q)\mathbb{E}\sup_{s\in[0,T]}|\varpi_n(s)|^{q/2} \le \frac{1}{2}\mathbb{E}\sup_{s\in[0,T]}[\chi_n(s)]^{q/2} + \bar{C}_5T + \bar{C}_5\mathbb{E}\int_0^T [\chi_n(s)]^{q/2}ds. \tag{4.36}$$

Inserting (4.36) in (4.33) and multiplying the resulting inequality by 2, we obtain

$$\begin{aligned}&\mathbb{E}\sup_{s\in[0,T]}[\chi_n(s)]^{\frac{q}{2}} + 2\mathbb{E}\left(\int_0^T [2\mathcal{Z}_p\vartheta_{p,n}(s)+2|\nabla\mu_n(s)|^2]ds\right)^{\frac{q}{2}} \\ &\le 2C_5(q)\left[\bar{\kappa}_1^{\frac{q}{2}} + \tilde{\mathcal{K}}_q(u_0,\varphi_0) + C_1^{\frac{q}{2}}\left(\int_0^T \|g_0(s)\|^{p'}_{V'_{\mathrm{div},p}}ds\right)^{\frac{q}{2}} + (C_2T)^{\frac{q}{2}}\right] \\ &\quad + 2\bar{C}_5T + C_6\mathbb{E}\int_0^T \sup_{\tau\in[0,s]}[\chi_n(\tau)]^{\frac{q}{2}}ds,\end{aligned} \tag{4.37}$$

with $C_6 = 2C_5(q)C_2^{\frac{q}{2}}T^{\frac{q-2}{2}} + 2\bar{C}_5(q,\kappa,c_6,|\mathcal{M}|)$. Now dropping the integral term in the left-hand side of (4.37) and applying the deterministic Gronwall lemma, we arrive at

$$\mathbb{E}\sup_{s\in[0,T]}[\chi_n(s)]^{q/2} \le C_7\left[1+\tilde{\mathcal{K}}_q(u_0,\varphi_0)+\left(\int_0^T \|g_0(s)\|_{V'_{\mathrm{div},p}}^{p'}ds\right)^{\frac{q}{2}}\right], \tag{4.38}$$

with $C_7 := C_7(q,T,\kappa,c_6,|\mathcal{M}|,\kappa_2^1,\kappa_p^1,|J|_{L^1(\mathbb{R}^3)})$ and $\tilde{\mathcal{K}}_q(u_0,\varphi_0)$ is given by (4.34).

Therefore the first two estimates in (4.2) follow from (4.38), (4.31), (4.8) and the fact that the domain $\mathcal{M}$ is bounded.

By using (4.37) and (4.38), it is straightforward to check that

$$\begin{aligned}&\mathbb{E}\left(\int_0^T \|u_n(s)\|_{1,p}^p ds\right)^{\frac{q}{2}} + \mathbb{E}\left(\int_0^T \|u_n(s)\|^2 ds\right)^{\frac{q}{2}} + \mathbb{E}\left(\int_0^T |\nabla\mu_n(s)|^2 ds\right)^{\frac{q}{2}}\\ &\quad\le \tilde{C}_7\left[1+\tilde{\mathcal{K}}_q(u_0,\varphi_0)+\left(\int_0^T \|g_0(s)\|_{V'_{\mathrm{div},p}}^{p'}ds\right)^{\frac{q}{2}}\right],\end{aligned} \tag{4.39}$$

with $\tilde{C}_7 := \tilde{C}_7(q,T,\kappa,c_6,|\mathcal{M}|,\kappa_2^1,\kappa_p^1,|J|_{L^1(\mathbb{R}^3)})$.

So as to proved $(4.3)_1$, we make the following observation: For any $a_1>0$, $q>2$ and $p\ge 11/5$, we have using the Young inequality

$$a_1^{q/p} \le \frac{2}{p}a_1^{q/2} + \frac{p-2}{2}.$$

Now, applying this previous inequality with $a_1 = \left(\int_0^T \|u_n(s)\|_{1,p}^p ds\right)^{q/p}$ in conjunction with (4.39), we infer that

$$\begin{aligned}\mathbb{E}\left(\int_0^T \|u_n(s)\|_{1,p}^p ds\right)^{q/p} &\le \tfrac{2}{p}\mathbb{E}\left(\int_0^T \|u_n(s)\|_{1,p}^p ds\right)^{q/2} + \tfrac{p-2}{2}\\ &\le \tilde{C}_7\left[1+\tilde{\mathcal{K}}_q(u_0,\varphi_0)+\left(\int_0^T \|g_0(s)\|_{V'_{\mathrm{div},p}}^{p'}ds\right)^{\frac{q}{2}}\right] + \tfrac{p-2}{2}.\end{aligned}$$

This proves $(4.3)_1$.

We will now prove that $\mathbb{E}\sup_{s\in[0,T]}\|F(\varphi_n(s))\|_{L^1(\mathcal{M})}^{q/2} < C$. The proof will be done in two cases:

First case: One can have $F(\varphi_n)>0$.

From the first line of (4.7) in conjunction with (4.32), we obtain

$$\begin{aligned}2\sup_{s\in[0,T]}\|F(\varphi_n(s))\|_{L^1(\mathcal{M})} &\le \sup_{s\in[0,T]}|(\varphi_n(s),J*\varphi_n(s))| + \bar{\kappa}_1 + \mathcal{K}(u_0,\varphi_0)\\ &\quad + C_1\int_0^T \|g_0(s)\|_{V'_{\mathrm{div},p}}^{p'}ds + C_2T + C_2\int_0^T[\chi_n(s)]ds + 2\sup_{s\in[0,T]}|\varpi_n(s)|.\end{aligned} \tag{4.40}$$

Using young's inequality for convolutions, we obtain

$$\sup_{s\in[0,T]}|(\varphi_n(s),J*\varphi_n(s))| \le |J|_{L^1(\mathbb{R}^3)}\sup_{s\in[0,T]}|\varphi_n(s)|^2. \tag{4.41}$$

Making similar reasoning as in (4.20), we obtain

$$2\mathbb{E}\sup_{s\in[0,T]}|\varpi_n(s)| \le \frac{1}{2}\mathbb{E}\sup_{s\in[0,T]}[\chi_n(s)] + C_3(c_6,\kappa,|\mathcal{M}|)\int_0^T(1+[\chi_n(s)])ds. \tag{4.42}$$

Taking the expected values in (4.40), using (4.41)–(4.42) and dividing the resulting inequality by 2, we obtain

$$\begin{aligned}\mathbb{E}\sup_{s\in[0,T]}\|F(\varphi_n(s))\|_{L^1(\mathcal{M})} &\le 2|J|_{L^1(\mathbb{R}^3)}\mathbb{E}\sup_{s\in[0,T]}|\varphi_n(s)|^2 + 2\bar{\kappa}_1 + 2\mathcal{K}(u_0,\varphi_0)\\ &\quad + 2C_1\int_0^T\|g_0\|_{V'_{\mathrm{div},p}}^{p'}ds + C_{2,3}T + (1+C_{2,3}T)\mathbb{E}\sup_{s\in[0,T]}[\chi_n(s)],\end{aligned} \tag{4.43}$$

where $C_1 := C_1(\kappa_2^1,\kappa_p^1,p)$, $C_2 := C_2(\kappa_2^1,\kappa_p^1,c_2,c_6,\kappa,p,|\mathcal{M}|)$, $C_3 := C_3(c_6,\kappa,|\mathcal{M}|)$ and $C_{2,3} = 2C_2 + 2C_3$.

Now, from (4.43), (4.16), (4.8) and (4.31), we obtain

$$\mathbb{E}\sup_{s\in[0,T]}\|F(\varphi_n(s))\|_{L^1(\mathcal{M})}\leq \bar{C}_7\left[1+\mathcal{K}(u_0,\varphi_0)+\int_0^T\|g_0(s)\|^{p'}_{V'_{\mathrm{div},p}}ds\right], \tag{4.44}$$

with $\bar{C}_7:=\bar{C}_7(T,\kappa,c_6,|\mathcal{M}|,\kappa_2^1,\kappa_p^1,|J|_{L^1(\mathbb{R}^3)})$ and $\mathcal{K}(u_0,\varphi_0)$ is given by the last inequality of (4.9). Inserting (4.41) in (4.40), raising both sides to the power $q/2>1$, and taking the expectation in the resulting inequality, we obtain

$$\begin{aligned}\sup_{s\in[0,T]}\|F(\varphi_n(s))\|^{\frac{q}{2}}_{L^1(\mathcal{M})}\leq{}& C(q)[|J|^{\frac{q}{2}}_{L^1}\sup_{s\in[0,T]}|\varphi_n(s)|^q+\bar{\kappa}_1^{\frac{q}{2}}+\tilde{\mathcal{K}}_q(u_0,\varphi_0)]\\&+C(q)C_1^{\frac{q}{2}}\left(\int_0^T\|g_0(s)\|^{p'}_{V'_{\mathrm{div},p}}ds\right)^{\frac{q}{2}}+C(q)(C_2T)^{\frac{q}{2}}\\&+C(q)C_2^{\frac{q}{2}}\left(\int_0^T[\chi_n(s)]ds\right)^{\frac{q}{2}}+C(q)\sup_{s\in[0,T]}|\varpi_n(s)|^{\frac{q}{2}}.\end{aligned} \tag{4.45}$$

Arguing similarly as in (4.36), we can check that

$$C(q)\mathbb{E}\sup_{s\in[0,T]}|\varpi_n(s)|^{q/2}\leq\frac{1}{2}\mathbb{E}\sup_{s\in[0,T]}[\chi_n(s)]^{q/2}+\bar{C}_qT+\bar{C}_q\mathbb{E}\int_0^T[\chi_n(s)]^{q/2}ds,$$

with $\bar{C}_q:=\bar{C}_q(q,\kappa,c_6,|\mathcal{M}|)$. Inserting now this previous inequality to (4.45) (after taking the expectation), using the inequality (4.38) and the fact that $\mathcal{M}$ is a bounded domain, we arrive at

$$\mathbb{E}\sup_{s\in[0,T]}\|F(\varphi_n(s))\|^{q/2}_{L^1(\mathcal{M})}\leq\iota_q\left[1+\tilde{\mathcal{K}}_q(u_0,\varphi_0)+\left(\int_0^T\|g_0(s)\|^{p'}_{V'_{\mathrm{div},p}}ds\right)^{q/2}\right], \tag{4.46}$$

where $\iota_q:=\tilde{C}_q(q,T,\kappa,c_6,|\mathcal{M}|,\kappa_2^1,\kappa_p^1,|J|_{L^1(\mathbb{R}^3)})$ and $\tilde{\mathcal{K}}_q(u_0,\varphi_0)$ is given by (4.34).

Second case: or $F(\varphi_n)<0$.

From (4.7), we infer that

$$\begin{aligned}2\|F(\varphi_n(t))\|_{L^1(\mathcal{M})}&\leq 2|\sqrt{a}\varphi_n(t)|^2+|J|_{L^1(\mathbb{R}^3)}|\varphi_n(t)|^2+\bar{\kappa}_1\\&\leq 3|J|_{L^1(\mathbb{R}^3)}|\varphi_n(t)|^2+\bar{\kappa}_1\leq\tilde{\kappa}_8\|\varphi_n(t)\|^{2\kappa+2}_{L^{2\kappa+2}(\mathcal{M})}+\tilde{\kappa}_9,\end{aligned} \tag{4.47}$$

where we have also used Young's inequality for convolutions, the inequality (4.16), and the fact that $|a|_{L^\infty(\mathcal{M})}\leq|J|_{L^1(\mathbb{R}^3)}$. Here $\tilde{\kappa}_8=\frac{3c_6|J|_{L^1(\mathbb{R}^3)}}{\kappa+1}$ and $\tilde{\kappa}_9=\frac{3\kappa|\mathcal{M}||J|_{L^1(\mathbb{R}^3)}}{(\kappa+1)c_6^{1/\kappa}}+\bar{\kappa}_1$.

Since (4.47) holds for every $t\in[0,T]$, we also infer that

$$2\sup_{t\in[0,T]}\|F(\varphi_n(t))\|_{L^1(\mathcal{M})}\leq\tilde{\kappa}_8\sup_{t\in[0,T]}\|\varphi_n(t)\|^{2\kappa+2}_{L^{2\kappa+2}}+\tilde{\kappa}_9\leq\tilde{\kappa}_8\sup_{t\in[0,T]}[\chi_n(s)]+\tilde{\kappa}_9. \tag{4.48}$$

We note that in (4.48), we have also used (4.8).

Taking now the mathematical expectation in (4.48), making used of (4.31), we infer that

$$2\mathbb{E}\sup_{t\in[0,T]}\|F(\varphi_n(t))\|_{L^1(\mathcal{M})}\leq\tilde{\kappa}_8[\mathcal{K}_1+\mathcal{K}_1e^{\tilde{C}_2T}]+\tilde{\kappa}_9, \tag{4.49}$$

with $\mathcal{K}_1$, $\tilde{C}_2$ given by (4.22); and $\tilde{\kappa}_8,\tilde{\kappa}_9$ given by (4.47).

Also from (4.48), it is straightforward to check that

$$\mathbb{E}\sup_{t\in[0,T]}\|F(\varphi_n(t))\|^{q/2}_{L^1(\mathcal{M})}\leq C(q)(\tilde{\kappa}_9)^{q/2}+C(q)(\tilde{\kappa}_8)^{q/2}\mathbb{E}\sup_{t\in[0,T]}[\chi_n(t)]^{q/2}.$$

From this previous inequality and (4.38), we get

$$\mathbb{E}\sup_{t\in[0,T]}\|F(\varphi_n(t))\|^{q/2}_{L^1(\mathcal{M})}\leq\iota_7\left[1+\tilde{\mathcal{K}}_q(u_0,\varphi_0)+\left(\int_0^T\|g_0(s)\|^{p'}_{V'_{\mathrm{div},p}}ds\right)^{\frac{q}{2}}\right], \tag{4.50}$$

with $\iota_7 := \iota_7(q, T, \kappa, c_6, |\mathcal{M}|, \kappa_2^1, \kappa_p^1, |J|_{L^1(\mathbb{R}^3)})$ and $\tilde{\mathcal{K}}_q(u_0, \varphi_0)$ is defined as in (4.34). This completes the proof of estimate $(4.2)_3$, i.e.,

$$\mathbb{E} \sup_{t\in[0,T]} \|F(\varphi_n(t))\|_{L^1(\mathcal{M})}^{q/2} < C, \text{ for both cases.} \tag{4.51}$$

In view to prove estimate $(4.2)_4$, we begin by making the following observation: from assumption (H_4) and the Young inequality, it is straightforward to check that

$$|F'(\varphi_n(s))| \le \frac{1}{r}|F'(\varphi_n(s))|^r + \frac{r-1}{r} \le \left(\frac{c_4}{r} + 1\right)|F(\varphi_n(s))| + \frac{r-1}{r}, \ \forall s \in [0,T]. \tag{4.52}$$

Hence, $(4.2)_4$ follows from (4.52), (4.51) and (4.49).

We now give the proof of estimates $(4.3)_2$ and $(4.3)_3$.

From (4.28) and the fact that $\tau_n^R \nearrow T$ $\mathbb{P}$-almost surely as $R \to \infty$, we obtain

$$\mathbb{E}\int_0^T \|\varphi_n(s)\|_U^2 ds \le C, \tag{4.53}$$

Thanks to (4.16), (4.27), (4.38) and (4.39), we infer that

$$\begin{aligned} \mathbb{E}\left(\int_0^T \|\varphi_n(s)\|_U^2 ds\right)^{\frac{q}{2}} &\le C(q)\left[\mathbb{E}\left(\int_0^T |\varphi_n(s)|^2 ds\right)^{q/2} + \mathbb{E}\left(\int_0^T |\nabla\varphi_n(s)|^2 ds\right)^{\frac{q}{2}}\right] \\ &\le \bar{\iota}_7\left[1 + \tilde{\mathcal{K}}_q(u_0,\varphi_0) + \left(\int_0^T \|g_0(s)\|_{V'_{\mathrm{div},p}}^{p'} ds\right)^{\frac{q}{2}}\right], \end{aligned} \tag{4.54}$$

with $\bar{\iota}_7 := \bar{\iota}_7(q, T, \kappa, c_6, |\mathcal{M}|, \kappa_2^1, \kappa_p^1, |J|_{L^1(\mathbb{R}^3)})$. So, by (4.53) and (4.54), we obtain $(4.3)_2$.

Arguing similarly as in [12, Inequality (3.65)], we check that

$$\begin{aligned} |\mu_n(s)|^2 &\le C\left[|\nabla\mu_n(s)|^2 + 4|\mathcal{M}|^{-1}|J|_{L^1(\mathbb{R}^3)}^2|\varphi_n(s)|^2\right] \\ &\le C\left[c_4^2 r^{-2}|\mathcal{M}|^{-2}\|F(\varphi_n(s))\|_{L^1(\mathcal{M})}^2 + \left(\frac{c_4 + r - 1}{r}\right)^2\right]. \end{aligned} \tag{4.55}$$

Now, owing to (4.55), (4.49), (4.16) and (4.31), it follows that

$$\mathbb{E}\int_0^T \|\mu_n(s)\|_U^2 ds = \mathbb{E}\int_0^T |\mu_n(s)|^2 ds + \mathbb{E}\int_0^T |\nabla\mu_n(s)|^2 ds < C. \tag{4.56}$$

Also, from (4.55), (4.51), (4.39) and (4.38), we get

$$\mathbb{E}\left(\int_0^T \|\mu_n(s)\|_U^2 ds\right)^{q/2} \le \mathbb{E}\left(\int_0^T |\mu_n(s)|^2 ds\right)^{q/2} + \mathbb{E}\left(\int_0^T |\nabla\mu_n(s)|^2 ds\right)^{q/2} < C, \tag{4.57}$$

where $U = H^1(\mathcal{M})$.

The estimate $(4.3)_3$ follows from (4.56) and (4.57). This completes the proof of Lemma 4.1. □

In the next propositions, we prove two uniform estimates for u_n and φ_n which are very crucial for our purpose.

Proposition 4.1. *In addition to assumptions of Theorem 3.1, Let $s \in \mathbb{R}$ such that $s > 5/2$. We assume that $t \mapsto u_n(t)$ is extended to zero outside the interval $[0,T]$. Then, there exists a positive constant C such that*

$$\mathbb{E} \sup_{0<|\theta|\le\delta<1} \|u_n(t+\theta) - u_n(t)\|_{\boldsymbol{W}'_s}^{p'} \le C\delta^{\frac{p'}{p}}, \ \forall t \in [0,T], \ n \in \mathbb{N}^* \text{ and } p' = p/(p-1).$$

Proposition 4.2. *Let the assumptions of Theorem 3.1 be satisfied. We assume that $t \mapsto \varphi_n(t)$ is extended to zero outside the interval $[0,T]$. Then, there exists a positive constant C such that*

$$\mathbb{E} \sup_{0<|\theta|\le\delta<1} \|\varphi_n(t+\theta) - \varphi_n(t)\|_{V'_s}^{2} \le C\delta^{\frac{p'}{p}}, \ \forall t \in [0,T], \ n \in \mathbb{N}^* \text{ and } p' = p/(p-1).$$

Proof of Proposition 4.1. We rewrite the equation for u_n as

$$\begin{aligned} d(u_n, v) + [(\boldsymbol{T}(Du_n), Dv) + b(u_n, u_n, v)]dt = -(\varphi_n \nabla \mu_n, v)dt + (\mathcal{P}_n^1 g_0, v)dt \\ + (g_1(u_n, \varphi_n), v)dt + \textstyle\sum_{i=1}^n (g_2(t, u_n, \varphi_n)e_i, v)d\beta_t^i, \quad \forall v \in \mathbf{W}_s^n. \end{aligned} \tag{4.58}$$

Let us take $v \in \mathbf{W}_s$, and decompose it as $v = v_I + v_{II}$, where $v_I \in \mathbf{W}_s^n$ and $v_{II} \in (\mathbf{W}_s^n)^\perp$, and notice that v_I and v_{II} are orthogonal also in $\mathbf{W}_s$. Then, from (4.58) we can write

$$\begin{aligned} d(u_n, v) = d\langle u_n, v\rangle_{\mathbf{W}_s} &= d\langle u_n, v_I\rangle_{\mathbf{W}_s} \\ &= -[(\boldsymbol{T}(Du_n), Dv_I) + b(u_n, u_n, v_I)]dt \\ &\quad - (\varphi_n \nabla \mu_n, v_I)dt + (\mathcal{P}_n^1 g_0, v_I)dt + (g_1(u_n, \varphi_n), v_I)dt \\ &\quad + \sum_{i=1}^n (g_2(t, u_n, \varphi_n)e_i, v_I)d\beta_t^i. \end{aligned} \tag{4.59}$$

Let us set $\tilde{\tilde{u}}_n(t) = u_n(t+\theta) - u_n(t)$, for any $\theta \in (0, \delta)$ and $\delta \in (0, 1)$. Hence from (4.59), we have

$$\begin{aligned} \langle \tilde{\tilde{u}}_n(t), v\rangle_{\mathbf{W}_s} = &-\int_t^{t+\theta} [(\boldsymbol{T}(Du_n), Dv_I) + b(u_n, u_n, v_I)]d\tau - \int_t^{t+\theta} (\varphi_n \nabla \mu_n, v_I)d\tau \\ &+ \int_t^{t+\theta} (\mathcal{P}_n^1 g_0, v_I)d\tau + \int_t^{t+\theta} (g_1(u_n, \varphi_n), v_I)d\tau + \int_t^{t+\theta} (\mathcal{P}_n^1 g_2(\tau, u_n, \varphi_n)e_i, v_I)d\beta_\tau^i, \end{aligned} \tag{4.60}$$

where, for the sake of simplicity, we have set

$$\int_t^{t+\theta} (\mathcal{P}_n^1 g_2(\tau, u_n, \varphi_n)e_i, v_I)d\beta_\tau^i := \sum_{j=1}^n \int_t^{t+\theta} (\mathcal{P}_n^1 g_2(\tau, u_n, \varphi_n)e_i, v_I)d\beta_\tau^i.$$

We set

$$\begin{aligned} y_t(\theta) = |\textstyle\int_t^{t+\theta} &[(\boldsymbol{T}(Du_n), Dv_I) + b(u_n, u_n, v_I)]d\tau \\ &- \textstyle\int_t^{t+\theta} (\varphi_n \nabla \mu_n, v_I)d\tau + \int_t^{t+\theta} (\mathcal{P}_n^1 g_0, v_I)d\tau \\ &+ \textstyle\int_t^{t+\theta} (g_1(u_n, \varphi_n), v_I)d\tau + \int_t^{t+\theta} (\mathcal{P}_n^1 g_2(\tau, u_n, \varphi_n)e_i, v_I)d\beta_\tau^i|. \end{aligned}$$

It follows from this that

$$\begin{aligned} y_t(\theta) \le\ & |\int_t^{t+\theta} (\boldsymbol{T}(Du_n), Dv_I)d\tau| + |\int_t^{t+\theta} b(u_n, u_n, v_I)d\tau| \\ &+ |\int_t^{t+\theta} (\varphi_n \nabla \mu_n, v_I)d\tau| + |\int_t^{t+\theta} (\mathcal{P}_n^1 g_0, v_I)d\tau| \\ &+ |\int_t^{t+\theta} (g_1(u_n, \varphi_n), v_I)d\tau| + |\int_t^{t+\theta} (\mathcal{P}_n^1 g_2(\tau, u_n, \varphi_n)e_i, v_I)d\beta_\tau^i|. \end{aligned} \tag{4.61}$$

We have

$$\begin{aligned} |\textstyle\int_t^{t+\theta} (\boldsymbol{T}(Du_n), Dv_I)d\tau| &= |\textstyle\int_t^{t+\theta} \langle N(u_n(\tau)), v_I\rangle_{V_{\mathrm{div},p}}\, d\tau| \\ &\le \textstyle\int_t^{t+\theta} |\langle N(u_n(\tau)), v_I\rangle_{V_{\mathrm{div},p}}|d\tau \\ &\le \|v_I\|_{V_{\mathrm{div},p}} \textstyle\int_t^{t+\theta} \|N(u_n(\tau))\|_{V'_{\mathrm{div},p}} d\tau \\ &\le C\|v_I\|_{\mathbf{W}_s} \textstyle\int_t^{t+\theta} \|N(u_n(\tau))\|_{V'_{\mathrm{div},p}} d\tau, \end{aligned}$$

where we have used the Cauchy–Schwarz inequality and the fact $\mathbf{W}_s \hookrightarrow V_{\mathrm{div},p}$ continuously.

Now, from this last inequality and by Hölder's inequality, we infer that

$$|\textstyle\int_t^{t+\theta} (\boldsymbol{T}(Du_n), Dv_I)d\tau| \le C\|v_I\|_{\mathbf{W}_s} \theta^{\frac{1}{p}} \left(\int_t^{t+\theta} \|N(u_n(\tau))\|_{V'_{\mathrm{div},p}}^{p'} d\tau\right)^{\frac{1}{p'}}.$$

Hence,

$$\begin{aligned}\left|\int_t^{t+\theta}(\boldsymbol{T}(Du_n),Dv_I)d\tau\right|^{p'} &\leq C\|v_I\|_{\mathbf{W}_s}^{p'}\theta^{\frac{p'}{p}}\int_t^{t+\theta}\|N(u_n(s))\|_{V'_{\mathrm{div},p}}^{p'}ds\\ &\leq C\|v\|_{\mathbf{W}_s}^{p'}\theta^{\frac{p'}{p}}\int_t^{t+\theta}\|N(u_n(s))\|_{V'_{\mathrm{div},p}}^{p'}ds := C\|v\|_{\mathbf{W}_s}^{p'}y_t^1(\theta,p).\end{aligned} \tag{4.62}$$

Note that (see [15, Inequalities (4.14)])

$$|b(u_n,u_n,v_I)| \leq C|u_n|^2\|v_I\|_{\mathbf{W}_s} \leq C|u_n|^2\|v\|_{\mathbf{W}_s}, \tag{4.63}$$

and the following holds

$$\begin{aligned}|\int_t^{t+\theta}b(u_n,u_n,v_I)d\tau| \leq \int_t^{t+\theta}|b(u_n,u_n,v_I)|d\tau &\leq C\|v\|_{\mathbf{W}_s}\int_t^{t+\theta}|u_n(\tau)|^2d\tau\\ &\leq C\|v\|_{\mathbf{W}_s}\theta^{\frac{1}{p}}\left(\int_t^{t+\theta}|u_n(\tau)|^{2p'}d\tau\right)^{\frac{1}{p'}}.\end{aligned}$$

Therefore, we have

$$\left|\int_t^{t+\theta}b(u_n,u_n,v_I)d\tau\right|^{p'} \leq C\|v\|_{\mathbf{W}_s}^{p'}\theta^{\frac{p'}{p}}\int_t^{t+\theta}|u_n(\tau)|^{2p'}d\tau := C\|v\|_{\mathbf{W}_s}^{p'}y_t^2(\theta,p). \tag{4.64}$$

As in [15, Inequalities (4.15)], we also have

$$|(\varphi_n\nabla\mu_n,v_I)| \leq C|\varphi_n||\nabla\mu_n|\|v_I\|_{\mathbf{W}_s} \leq C|\varphi_n||\nabla\mu_n|\|v\|_{\mathbf{W}_s}. \tag{4.65}$$

By (4.65) and the Hölder inequality, we have

$$\begin{aligned}|\int_t^{t+\theta}(\varphi_n\nabla\mu_n,v_I)d\tau| \leq \int_t^{t+\theta}|(\varphi_n\nabla\mu_n,v_I)|d\tau &\leq C\|v\|_{\mathbf{W}_s}\int_t^{t+\theta}|\varphi_n||\nabla\mu_n|d\tau\\ &\leq C\|v\|_{\mathbf{W}_s}\theta^{\frac{1}{p}}\left(\int_t^{t+\theta}|\varphi_n|^{p'}|\nabla\mu_n|^{p'}d\tau\right)^{\frac{1}{p'}}.\end{aligned}$$

Thus,

$$\begin{aligned}\left|\int_t^{t+\theta}(\varphi_n\nabla\mu_n,v_I)d\tau\right|^{p'} &\leq C\|v\|_{\mathbf{W}_s}^{p'}\theta^{\frac{p'}{p}}\int_t^{t+\theta}|\varphi_n|^{p'}|\nabla\mu_n|^{p'}d\tau\\ &\leq C\|v\|_{\mathbf{W}_s}^{p'}\theta^{\frac{p'}{p}}\left(\int_t^{t+\theta}|\varphi_n|^{\frac{2p'}{2-p'}}d\tau\right)^{\frac{2-p'}{2}}\left(\int_t^{t+\theta}|\nabla\mu_n|^2d\tau\right)^{\frac{p'}{2}}\\ &:= C\|v\|_{\mathbf{W}_s}^{p'}y_t^3(\theta,p).\end{aligned} \tag{4.66}$$

We have by Cauchy–Schwarz's inequality

$$\begin{aligned}|\int_t^{t+\theta}(\mathcal{P}_n^1g_0,v_I)d\tau| &\leq \int_t^{t+\theta}|(\mathcal{P}_n^1g_0,v_I)|d\tau\\ &\leq \|v_I\|_{\mathbf{W}_s}\int_t^{t+\theta}\|\mathcal{P}_n^1g_0(\tau)\|_{\mathbf{W}'_s}d\tau\\ &\leq \|v_I\|_{\mathbf{W}_s}\int_t^{t+\theta}\|g_0(\tau)\|_{\mathbf{W}'_s}d\tau\\ &\leq C\|v\|_{\mathbf{W}_s}\int_t^{t+\theta}\|g_0(\tau)\|_{V'_{\mathrm{div},p}}d\tau,\end{aligned}$$

where we have also used the fact that $\|\mathcal{P}_n^1\|_{\mathcal{L}(\mathbf{W}'_s,\mathbf{W}_{(s)})} \leq 1$ and $V'_{\mathrm{div},p} \hookrightarrow \mathbf{W}'_s$ continuously. Hence, by Hölder's inequality, we obtain

$$\left|\int_t^{t+\theta}(\mathcal{P}_n^1g_0,v_I)d\tau\right| \leq C\|v\|_{\mathbf{W}_s}\theta^{\frac{1}{p}}\left(\int_t^{t+\theta}\|g_0(\tau)\|_{V'_{\mathrm{div},p}}^{p'}d\tau\right)^{\frac{1}{p'}}$$

From this previous inequality, we have

$$\left|\int_t^{t+\theta}(\mathcal{P}_n^1g_0,v_I)d\tau\right|^{p'} \leq C\|v\|_{\mathbf{W}_s}^{p'}\theta^{\frac{p'}{p}}\int_t^{t+\theta}\|g_0(\tau)\|_{V'_{\mathrm{div},p}}^{p'}d\tau := C\|v\|_{\mathbf{W}_s}^{p'}y_t^4(\theta,p). \tag{4.67}$$

Making similar reasoning as in (4.67), we obtain

$$\left|\int_t^{t+\theta}(g_1(u_n(\tau),\varphi_n(\tau)),v_I)d\tau\right|^{p'} \leq C\|v\|_{\mathbf{W}_s}^{p'}\theta^{\frac{p'}{p}}\int_t^{t+\theta}\|g_1(u_n,\varphi_n)\|_{V'_{\mathrm{div},p}}^{p'}d\tau := C\|v\|_{\mathbf{W}_s}^{p'}y_t^5(\theta,p). \tag{4.68}$$

Raising both sides of the inequality (4.61) to the power p' (p' is the conjugate index to p), and thanks to previous inequalities, we obtain

$$y_t(\theta)^{p'} \leq C\|v\|_{\mathbf{W}_s}^{p'} \sum_{i=1}^{5} y_t^i(\theta,p) + \left|\int_t^{t+\theta} (\mathcal{P}_n^1 g_2(\tau,u_n,\varphi_n)e_i,v_I)d\beta_\tau^i\right|^{p'}. \tag{4.69}$$

Thanks to (2.23), we have

$$\mathbb{E}\sup_{\theta\in(0,\delta)} y_t^1(\theta,p) \leq C\delta^{\frac{p'}{p}}\left(\delta + \mathbb{E}\int_0^T \|u_n(\tau)\|_{1,p}^p d\tau\right).$$

Owing to Lemma 4.1 (see inequality $(4.3)_1$), we derive from this last inequality that

$$\mathbb{E}\sup_{\theta\in(0,\delta)} y_t^1(\theta,p) \leq C\delta^{\frac{p'}{p}}. \tag{4.70}$$

Also, thanks to Lemma 4.1, we obtain

$$\mathbb{E}\sup_{\theta\in(0,\delta)} y_t^2(\theta,p) \leq C\delta^{\frac{p'}{p}}\mathbb{E}\sup_{\tau\in[0,T]} |u_n(\tau)|^{2p'} \leq C\delta^{\frac{p'}{p}}. \tag{4.71}$$

By Hölder's inequality, we have

$$\begin{aligned}
\mathbb{E}\sup_{\theta\in(0,\delta)} y_t^3(\theta,p) &\leq \delta^{\frac{p'}{p}}\mathbb{E}\left(\int_t^{t+\delta} |\varphi_n|^{\frac{2p'}{2-p'}}d\tau\right)^{\frac{2-p'}{2}}\left(\int_t^{t+\delta} |\nabla\mu_n|^2 d\tau\right)^{\frac{p'}{2}} \\
&\leq \delta^{\frac{p'}{p}}\left[\mathbb{E}\int_t^{t+\delta} |\varphi_n|^{\frac{2p'}{2-p'}}d\tau\right]^{\frac{2-p'}{2}}\left[\mathbb{E}\int_t^{t+\delta} |\nabla\mu_n|^2 d\tau\right]^{\frac{p'}{2}} \\
&\leq C\delta^{\frac{p'}{p}}\left[\mathbb{E}\sup_{\tau\in[0,T]} |\varphi_n(\tau)|^{\frac{2p'}{2-p'}}\right]^{\frac{2-p'}{2}}\left[\mathbb{E}\int_0^T |\nabla\mu_n|^2 d\tau\right]^{\frac{p'}{2}}.
\end{aligned}$$

Therefore, thanks to (4.16) in conjunction with Lemma 4.1, we have

$$\mathbb{E}\sup_{\theta\in(0,\delta)} y_t^3(\theta,p) \leq C\delta^{\frac{p'}{p}}. \tag{4.72}$$

By the assumption on g_0, it follows that

$$\mathbb{E}\sup_{\theta\in(0,\delta)} y_t^4(\theta,p) \leq C\delta^{\frac{p'}{p}}\mathbb{E}\int_0^T \|g_0(\tau)\|_{V'_{\mathrm{div},p}}^{p'} d\tau \leq C\delta^{\frac{p'}{p}}. \tag{4.73}$$

Using the assumptions on g_1 in conjunction with Lemma 4.1, we obtain

$$\mathbb{E}\sup_{\theta\in(0,\delta)} y_t^5(\theta,p) \leq C\delta^{\frac{p'}{p}}. \tag{4.74}$$

By the Burkholder–Davis–Gundy lemma, we have

$$\begin{aligned}
&\mathbb{E}\sup_{\theta\in(0,\delta)}\left|\int_t^{t+\theta}\sum_{i=1}^n (\mathcal{P}_n^1 g_2(\tau,u_n,\varphi_n)e_i,v_I)d\beta_\tau^i\right|^{p'} \\
&\leq C\mathbb{E}\left(\int_t^{t+\delta}\sum_{i=1}^n (\mathcal{P}_n^1 g_2(\tau,u_n(\tau),\varphi_n(\tau))e_i,v_I)^2 d\tau\right)^{\frac{p'}{2}} \\
&\leq C|v_I|^{p'}\mathbb{E}\left(\int_t^{t+\delta}\sum_{i=1}^n |\mathcal{P}_n^1 g_2(\tau,u_\tau,\varphi_n(\tau))|^2 d\tau\right)^{\frac{p'}{2}} \\
&\leq C\|v\|_{\mathbf{W}_s}^{p'}\mathbb{E}\left(\int_t^{t+\delta} \|g_2(\tau,u_n(\tau),\varphi_n(\tau))\|_{L_2(K,G_{\mathrm{div}})}^2 d\tau\right)^{\frac{p'}{2}}.
\end{aligned}$$

From this last estimate and the assumption on g_2, we derive that

$$\begin{aligned}
&\mathbb{E}\sup_{\theta\in(0,\delta)}\left|\int_t^{t+\theta}\sum_{i=1}^n(\mathcal{P}_n^1 g_2(\tau,u_n,\varphi_n)e_i,v_I)d\beta_\tau^i\right|^{p'}\\
&\quad\le C\|v\|_{\mathbf{W}_s}^{p'}\mathbb{E}\left(\delta+\int_t^{t+\delta}(|u_n(\tau)|^2+|\varphi_n(\tau)|^2)d\tau\right)^{\frac{p'}{2}}\\
&\quad\le C\|v\|_{\mathbf{W}_s}^{p'}\left(\delta+\delta\mathbb{E}\sup_{\tau\in[t,t+\delta]}|u_n(\tau)|^2+\delta\mathbb{E}\sup_{\tau\in[t,t+\delta]}|\varphi_n(\tau)|^2\right)^{\frac{p'}{2}}.
\end{aligned} \tag{4.75}$$

Thanks to (4.16), (4.75), and Lemma 4.1, we see that

$$\mathbb{E}\sup_{\theta\in(0,\delta)}\left|\int_t^{t+\theta}\sum_{i=1}^n(\mathcal{P}_n^1 g_2(\tau,u_n,\varphi_n)e_i,v_I)d\beta_\tau^i\right|^{p'}\le C\|v\|_{\mathbf{W}_s}^{p'}\delta^{\frac{p'}{2}}\le C\|v\|_{\mathbf{W}_s}^{p'}\delta^{\frac{p'}{p}}. \tag{4.76}$$

Now, from (4.69) and the estimates (4.70)–(4.76), we have

$$\mathbb{E}\sup_{\theta\in(0,\delta)}\sup_{v\in\mathbf{W}_s,\|v\|_{\mathbf{W}_s}=1}y_t(\theta)^{p'}\le C\delta^{\frac{p'}{p}}. \tag{4.77}$$

Hence

$$\mathbb{E}\sup_{\theta\in(0,\delta)}\|\tilde{\tilde{u}}_n(t)\|_{\mathbf{W}_s'}^{p'}=\mathbb{E}\sup_{\theta\in(0,\delta)}\sup_{v\in\mathbf{W}_s,\|v\|_{\mathbf{W}_s}=1}|\left\langle\tilde{\tilde{u}}_n(t),v\right\rangle_{\mathbf{W}_s}|^{p'}\le C\delta^{\frac{p'}{p}}, \tag{4.78}$$

with $\tilde{\tilde{u}}_n(t)=u_n(t+\theta)-u_n(t)$, and for any positive integer n, $t\in[0,T]$ and $\delta\in(0,1)$. The Proposition 4.1 follows readily from this last inequality and noting that a similar argument can be carried out to find a similar estimate for negative values of θ.

Proof of Proposition 4.2. The second equations for the Galerkin approximation is written as

$$d(\varphi_n,\psi)+(\nabla\mu_n,\nabla\psi)dt=(u_n\varphi_n,\nabla\psi)dt,\quad\forall\psi\in H_n. \tag{4.79}$$

Let $\psi\in U:=H^1(\mathcal{M})$ and decompose it as $\psi=\psi_I+\psi_{II}$, where $\psi_I\in H_n$ and $\psi_{II}\in H_n^\perp$. Recall that ψ_I and ψ_{II} are orthogonal also in U. Then, from (4.79), we deduce

$$d(\varphi_n,\psi)=d\left\langle\varphi_n,\psi\right\rangle_U=d\left\langle\varphi_n,\psi_I\right\rangle_U=-(\nabla\mu_n,\nabla\psi_I)dt+(u_n\varphi_n,\nabla\psi_I)dt. \tag{4.80}$$

It follows from (4.80) that

$$\begin{aligned}
\left\langle\varphi_n(t+\theta)-\varphi_n(t),\psi\right\rangle_U&=-\int_t^{t+\theta}(\nabla\mu_n(\tau),\nabla\psi_I)d\tau+\int_t^{t+\theta}(u_n(\tau)\varphi_n(\tau),\nabla\psi_I)d\tau\\
&:=X_t(\theta),
\end{aligned} \tag{4.81}$$

for any $0<\theta<\delta<1$.

We have

$$(X_t(\theta))^2\le C\left|\int_t^{t+\theta}(\nabla\mu_n(\tau),\nabla\psi_I)d\tau\right|^2+C\left|\int_t^{t+\theta}(u_n(\tau)\varphi_n(\tau),\nabla\psi_I)d\tau\right|^2. \tag{4.82}$$

Note that

$$\begin{aligned}
C\left|\int_t^{t+\theta}(\nabla\mu_n(\tau),\nabla\psi_I)d\tau\right|^2&\le C\left(\int_t^{t+\theta}|(\nabla\mu_n(\tau),\nabla\psi_I)|d\tau\right)^2\\
&\le C|\nabla\psi_I|^2\theta\int_t^{t+\theta}|\nabla\mu_n(\tau)|^2d\tau\\
&\le C\|\psi\|_U^2\theta\int_t^{t+\theta}|\nabla\mu_n(\tau)|^2d\tau:=X_t^1(\theta).
\end{aligned} \tag{4.83}$$

Owing to Lemma 4.1, we derive that

$$\mathbb{E}\sup_{\theta\in(0,\delta)}X_t^1(\theta)\le C\|\psi\|_U^2\delta\le C\|\psi\|_U^2\delta^{\frac{p'}{p}}. \tag{4.84}$$

As far as the second term in the right hand side of (4.82) is concerned we notice that when $p < 3$ and $\kappa \geq \frac{2(3-p)}{5p-6}$, due to the embedding $W^{1,p}(\mathcal{M}) \hookrightarrow L^{\frac{3p}{3-p}}(\mathcal{M}) \hookrightarrow L^{\frac{2\kappa+2}{\kappa}}(\mathcal{M})$, we can check that (see [15, Inequality (4.18)])

$$|(u_n(\tau)\varphi_n(\tau), \nabla\psi_I)| \leq C\|u_n(\tau)\|_{V_{\mathrm{div},p}}\|\varphi_n(\tau)\|_{L^{2\kappa+2}}\|\psi\|_U. \tag{4.85}$$

When $p = 3$ and $\kappa \in (0, 2]$ or $p > 3$ and $\kappa > 0$, due to the embedding $W^{1,p}(\mathcal{M}) \hookrightarrow L^{\frac{2\kappa+2}{\kappa}}(\mathcal{M})$, we have also

$$|(u_n(\tau)\varphi_n(\tau), \nabla\psi_I)| \leq C\|u_n(\tau)\|_{1,p}\|\varphi_n(\tau)\|_{L^{2\kappa+2}}\|\psi\|_U. \tag{4.86}$$

Now from (4.85), we estimate the second term in the right hand side of (4.82) as follows

$$\begin{aligned} C\left|\int_t^{t+\theta}(u_n(\tau)\varphi_n(\tau), \nabla\psi_I)d\tau\right|^2 &\leq C\theta\int_t^{t+\theta}|(u_n(\tau)\varphi_n(\tau), \nabla\psi_I)|^2 d\tau \\ &\leq C\theta\|\psi\|_U^2\int_t^{t+\theta}\|u_n(\tau)\|_{1,p}^2\|\varphi_n(\tau)\|_{L^{2\kappa+2}}^2 d\tau \\ &\leq C\theta\|\psi\|_U^2\left(\int_t^{t+\theta}\|u_n\|_{1,p}^p d\tau\right)^{\frac{2}{p}}\left(\int_t^{t+\theta}\|\varphi_n\|_{L^{2\kappa+2}}^{\frac{2p}{p-2}} d\tau\right)^{\frac{p-2}{p}} = X_t^2(\theta), \end{aligned}$$

where we have also used the Hölder inequality.

Hence, from this previous inequality, we infer that

$$\begin{aligned} \mathbb{E}\sup_{\theta\in(0,\delta)} X_t^2(\theta) &\leq C\delta\|\psi\|_U^2\left[\mathbb{E}\int_t^{t+\delta}\|u_n\|_{1,p}^p d\tau\right]^{\frac{2}{p}}\left[\mathbb{E}\int_t^{t+\delta}\|\varphi_n\|_{L^{2\kappa+2}}^{\frac{2p}{p-2}} d\tau\right]^{\frac{p-2}{p}} \\ &\leq C\delta^{\frac{2(p-1)}{p}}\|\psi\|_U^2\mathbb{E}\sup_{\tau\in[t,t+\delta]}\|\varphi_n(\tau)\|_{L^{2\kappa+2}}^2\left[\mathbb{E}\int_t^{t+\delta}\|u_n\|_{1,p}^p d\tau\right]^{\frac{2}{p}}. \end{aligned} \tag{4.87}$$

Owing to Lemma 4.1, we derive from (4.87) that

$$\mathbb{E}\sup_{\theta\in(0,\delta)} X_t^2(\theta) \leq C\delta^{\frac{2(p-1)}{p}}\|\psi\|_U^2 \leq C\delta^{\frac{p'}{p}}\|\psi\|_U^2, \quad \text{with} \quad p' = \frac{p}{p-1}. \tag{4.88}$$

Collecting (4.84), (4.88), from (4.82) we then get

$$\mathbb{E}\sup_{\theta\in(0,\delta)}\sup_{\psi\in\mathbf{W}_s,\|\psi\|_V=1}(X_t(\theta))^2 \leq C\delta^{\frac{p'}{p}}. \tag{4.89}$$

Therefore

$$\mathbb{E}\sup_{\theta\in(0,\delta)}\|\varphi_n(t+\theta)-\varphi_n(t)\|_{U'}^2 = \mathbb{E}\sup_{\theta\in(0,\delta)}\sup_{\psi\in V,\|\psi\|_U=1}|\langle\varphi_n(t+\theta)-\varphi_n(t),\psi\rangle_U|^2 \leq C\delta^{\frac{p'}{p}}, \tag{4.90}$$

for any positive integer n, $t \in [0, T]$ and $\delta \in (0, 1)$.

Finally, collecting all the estimates and making a similar reasoning with $\theta < 0$, we thus deduce

$$\mathbb{E}\sup_{0<|\theta|\leq\delta<1}\|\varphi_n(t+\theta)-\varphi_n(t)\|_{U'}^2 \leq C\delta^{\frac{p'}{p}},$$

for any positive integer n, $t \in [0, T]$. This completes the proof of Proposition 4.2. □

4.3. Tightness and Compactness Results

In this subsection, we study the tightness property of the Galerkin solutions and derive several weak convergence results. The estimates from the previous Propositions (Propositions 4.1–4.2) play an important role in this part of the paper.

Throughout this subsection, we fix $s \in \mathbb{R}$ such that $s > 5/2$. Let us consider the spaces

$$\begin{aligned} \mathfrak{X}_1 &= L^2(0, T; \mathbb{H}) \cap \mathcal{C}(0, T; \mathbb{W}_s'), \\ \mathfrak{X}_2 &= \mathcal{C}(0, T; K) \end{aligned}$$

and we denote by $\mathfrak{B}(\mathfrak{X}_1)$ and $\mathfrak{B}(\mathfrak{X}_2)$ their borel σ-algebras, respectively.

Now, before proving that the family of laws $\{\mathcal{L}(u_n,\varphi_n) : n \in \mathbb{N}\}$ is tight on the Polish space $\mathfrak{X}_1$, we recall the following result which will be needed in the sequel. The proof of this result can be found in the book of Métivier [38, Chapter VI, Lemma 2 and Lemma 3].

Lemma 4.2. *Let $\mathbb{B}$, $\mathbb{B}_0$ and $\mathbb{B}_1$ be three reflexive Banach spaces satisfying the compact embedding $\mathbb{B}_0 \hookrightarrow \mathbb{B} \hookrightarrow \mathbb{B}_1$. Let $q \in (1,\infty)$ and $\mathfrak{Q}$ be a subset of $L^q(0,T;\mathbb{B})$, which is included in a compact set of $L^q(0,T;\mathbb{B}_1)$ and*

$$\sup_{v\in\mathfrak{Q}} \int_0^T \|u(s)\|_{\mathbb{B}_0}^q ds < \infty.$$

Then, $\mathfrak{Q}$ is relatively compact in $L^q(0,T;\mathbb{B})$.

We shall prove the following important result.

Lemma 4.3. *The family of laws $\{\mathcal{L}(u_n,\varphi_n) : n \in \mathbb{N}\}$ is tight in $\mathfrak{X}_1$.*

Proof. We firstly prove that $\{\mathcal{L}(u_n,\varphi_n) : n \in \mathbb{N}\}$ is tight in $\mathcal{C}(0,T;\mathbb{W}_s')$. For this aim, we first observe that for a fixed number $R > 0$ we have

$$\mathbb{P}(\|(u_n(t),\varphi_n(t))\|_{\mathbb{H}} > R) \leq \frac{1}{R^2}\mathbb{E}\sup_{t\in[0,T]} \|(u_n(t),\varphi_n(t))\|_{\mathbb{H}}^2,$$

from which along with $(4.2)_1$, $(4.2)_2$ and the fact that the domain $\mathcal{M}$ is bounded we infer that

$$\sup_{n\in\mathbb{N}} \mathbb{P}(\|(u_n(t),\varphi_n(t))\|_{\mathbb{H}} > R) \leq \frac{C}{R^2}, \tag{4.91}$$

for any $t \in [0,T]$.

Since, by the compact embedding $\mathbb{H} \subset \mathbb{W}_s'$, balls in $\mathbb{H}$ are compact for the strong topology in $\mathbb{W}_s'$, then this implies that the family $\{(u_n(t),\varphi_n(t)) : n \in \mathbb{N}\}$ is relatively compact in $\mathbb{W}_s'$ for any $t \in [0,T]$. Therefore, by Propositions 4.1–4.2 and [47, Lemma 1, page 71] we derive that the laws of the family $\{(u_n,\varphi_n) : n \in \mathbb{N}\}$ are tight in $\mathcal{C}(0,T;\mathbb{W}_s')$. This means that for any $\epsilon > 0$ there exists a compact subset K_ϵ of $\mathcal{C}(0,T;\mathbb{W}_s')$ such that

$$\mathbb{P}((u_n,\varphi_n) \in K_\epsilon) \geq 1 - \frac{\epsilon}{2}, \quad n \in \mathbb{N}. \tag{4.92}$$

We also observe that for a fixed number $R > 0$, we have

$$\mathbb{P}\left(\|(u_n,\varphi_n)\|_{L^{p,2}(0,T;\mathbb{V})} > R\right) \leq \tfrac{1}{R^2}\mathbb{E}\|(u_n,\varphi_n)\|_{L^{p,2}(0,T;\mathbb{V})}^2 = \tfrac{1}{R^2}\left(\mathbb{E}\int_0^T [\|u_n\|_{1,p}^p + \|\varphi_n\|_U^2] ds\right),$$

from which along with $(4.3)_1$, $(4.3)_2$ we derive that

$$\mathbb{P}\left(\|(u_n,\varphi_n)\|_{L^{p,2}(0,T;\mathbb{V})} > R\right) \leq \frac{C}{R^2}. \tag{4.93}$$

Now, taking $R = \sqrt{\frac{2C}{\epsilon}} := \varepsilon_1$, where C is the constant appearing in (4.93), we infer that

$$\begin{aligned}\mathbb{P}(\|(u_n,\varphi_n)\|_{L^{p,2}(0,T;\mathbb{V})} \leq \varepsilon_1) &= 1 - \mathbb{P}(\|(u_n,\varphi_n)\|_{L^{p,2}(0,T;\mathbb{V})} > \varepsilon_1)\\ &\geq 1 - \tfrac{C}{\varepsilon_1^2} = 1 - \tfrac{\epsilon}{2}, \quad n \in \mathbb{N}.\end{aligned} \tag{4.94}$$

Now, let

$$\mathfrak{Q}_\epsilon = \{(u,\varphi) \in L^{p,2}(0,T;\mathbb{V}) : \|(u,\varphi)\|_{L^{p,2}(0,T;\mathbb{V})} \leq \varepsilon_1\} \cap K_\epsilon.$$

Since $L^{p,2}(0,T;\mathbb{V}) \cap \mathcal{C}(0,T;\mathbb{W}_s')$ is compactly embedded in $L^2(0,T;\mathbb{H}) \cap L^2(0,T;\mathbb{W}_s')$, then $\mathfrak{Q}_\epsilon$ satisfies the conditions of Lemma 4.2. Hence $\mathfrak{Q}_\epsilon$ is relatively compact in $L^2(0,T;\mathbb{H})$. Moreover, $\mathbb{P}((u_n,\varphi_n) \in \mathfrak{Q}_\epsilon) \geq 1 - \epsilon$, $n \in \mathbb{N}$. This proves that the family of laws $\{\mathcal{L}(u_n,\varphi_n) : n \in \mathbb{N}\}$ is tight in $L^2(0,T;\mathbb{H})$ and we can easily conclude the proof of the lemma. □

Hereafter, the law of the cylindrical Brownian motion $W = (W_t)_{t\in[0,T]}$ is denoted by Π and we mention that it is possible to find a set $\bar{\Omega} \in \mathcal{F}$ of measure zero such that $W(\omega) \in \mathcal{C}(0,T;K)$ for any $\omega \in \Omega/\bar{\Omega}$. For any $n \in \mathbb{N}$, we construct a family of probability laws on $\mathfrak{X}_2 = \mathcal{C}(0,T;K)$ by setting

$$\Pi_n(.) = \mathbb{P}(W \in .) \in P_r(\mathfrak{X}_2) = \Pi, \quad \forall n \geq 1 \tag{4.95}$$

and where $P_r(\mathfrak{X}_2)$ denotes the set of all probability measures on $(\mathfrak{X}_2, \mathfrak{B}(\mathfrak{X}_2))$.

We now prove the following important result.

Theorem 4.1. *The family of laws of $((u_n,\varphi_n);W)$ is tight on the Polish space $\mathfrak{X}_1 \times \mathfrak{X}_2$.*

Proof. We have already proved that the family of laws $\{\mathcal{L}(u_n,\varphi_n) : n \in \mathbb{N}\}$ is tight in $\mathfrak{X}_1$ (cf. Lemma 4.3). Now we shall proved that the family $\{\Pi_n : n = 1,2,\ldots\}$ is tight in $P_r(\mathfrak{X}_2)$. For this, we endow the space $\mathfrak{X}_2$ with the uniform convergence, and then $\mathfrak{X}_2$ is now a Polish space. Hence, it follows from [3, Theorem 6.8] that $P_r(\mathfrak{X}_2)$ endowed with the Prohorov's metric is a separable and complete metric space. By construction, the family of probability laws $\{\Pi_n : n = 1,2,\ldots\}$ is reduced to one element which is the law of W and belongs to $P_r(\mathfrak{X}_2)$. Thus, by [43, Chapter II, Theorem 3.2] we infer that the family $\{\Pi_n : n = 1,2,\ldots\}$ is tight on $P_r(\mathfrak{X}_2)$. Finally from the fact that $\{\mathcal{L}(u_n,\varphi_n) : n \in \mathbb{N}\}$ is tight in $\mathfrak{X}_1$, the family $\{\Pi_n : n = 1,2,\ldots\}$ is tight in $P_r(\mathfrak{X}_2)$ in conjunction with [31, Corollary 1.3], we infer that the family of laws of the joint processes $((u_n,\varphi_n),W)$ is tight in $\mathfrak{X}_1 \times \mathfrak{X}_2$. □

Proposition 4.3. *Let $\mathfrak{X} = \mathfrak{X}_1 \times \mathcal{C}(0,T;K)$. There exist a Borel probability measure μ_1 on $\mathfrak{X}$ and a subsequence of $((u_n,\varphi_n),W)$ such that their laws weakly converge to μ_1.*

Proof. Thanks to the theorem 4.1, the laws of $((u_n,\varphi_n),W)$ form a tight family on $\mathfrak{X}$. Since $\mathfrak{X}$ is a Polish space, we get the result from the application of Prohorov's theorem (cf. [3, Theorem I. 5. 1, page 59]). □

The following result relates the above convergence in law to almost sure convergence.

Proposition 4.4. *There exist a complete probability space $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}})$ and a sequence of $\mathfrak{X}$-valued random variables, denoted by $\{(\tilde{u}_n,\tilde{\varphi}_n,\tilde{W}_n) : n \in \mathbb{N}\}$, defined on $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}})$ such that their laws are equal to the laws of $\{(u_n,\varphi_n,W) : n \in \mathbb{N}\}$ on $\mathfrak{X}$. Also, there exists an $\mathfrak{X}$-random variable $\{(\tilde{u},\tilde{\varphi}),\tilde{W}\}$ defined on $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}})$ such that*

$$\begin{aligned}
&\mathcal{L}(\tilde{u},\tilde{\varphi},\tilde{W}) = \mu_1,\\
&\tilde{W}_n \to \tilde{W} \text{ in } \mathcal{C}(0,T;K)\ \tilde{\mathbb{P}}\text{-a.s.},\\
&(\tilde{u}_n,\tilde{\varphi}_n) \to (\tilde{u},\tilde{\varphi}) \text{ in } L^2(0,T;\mathbb{H})\ \tilde{\mathbb{P}}\text{-a.s.},\\
&(\tilde{u}_n,\tilde{\varphi}_n) \to (\tilde{u},\tilde{\varphi}) \text{ in } \mathcal{C}(0,T;\mathbb{W}_s').
\end{aligned} \tag{4.96}$$

Proof. The proof of Proposition 4.4 is a consequence of Proposition 4.3 and Skorokhod's Theorem [50]. □

Proposition 4.5. *Let $Q = \mathfrak{I}\mathfrak{I}^*$ where $\mathfrak{I}$ is the canonical injection, which is Hilbert–Schmidt, from K into K_1. Then, the stochastic process $\tilde{W} = (\tilde{W}_t)_{t\in[0,T]}$ is a K_1-valued Q-Wiener process on $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}})$. Furthermore, if $0 \leq s < t \leq T$, then the increments $\tilde{W}_t - \tilde{W}_s$ are independent of the σ-algebra $\tilde{\mathcal{F}}_s$ generated by $\tilde{u}(\tau)$, $\tilde{\varphi}(\tau)$, $\tilde{W}_\tau$ for $\tau \in [0,s]$.*

Proof. The proof is a verbatim reproduction of similar result in [45, Proposition 3.11]. □

Now, since $(\tilde{u}_n, \tilde{\varphi}_n, \tilde{W}_n)$ and (u_n, φ_n, W) have the same law (cf. Proposition 4.4), it follows from Lemma 4.1 that $(\tilde{u}_n, \tilde{\varphi}_n)$ satisfies the estimates

$$\begin{aligned} &\tilde{\mathbb{E}} \sup_{s\in[0,T]} |\tilde{u}_n(s)|^q < C,\\ &\tilde{\mathbb{E}} \sup_{s\in[0,T]} \|\tilde{\varphi}_n(s)\|^q_{L^{2\kappa+2}(\mathcal{M})} < C,\\ &\tilde{\mathbb{E}} \sup_{s\in[0,T]} \|F(\tilde{\varphi}_n(s))\|^{q/2}_{L^1(\mathcal{M})} < C,\\ &\tilde{\mathbb{E}} \sup_{s\in[0,T]} \|F'(\tilde{\varphi}_n(s))\|^q_{L^r(\mathcal{M})} < C,\\ &\tilde{\mathbb{E}} \left(\int_0^T \|\tilde{u}_n(s)\|^p_{1,p} ds\right)^{q/p} < C,\\ &\tilde{\mathbb{E}} \left(\int_0^T \|\tilde{\varphi}_n(s)\|^2_U ds\right)^{q/2} < C,\\ &\tilde{\mathbb{E}} \left(\int_0^T \|\tilde{\mu}_n(s)\|^2_U ds\right)^{q/2} < C, \end{aligned} \tag{4.97}$$

for any $n \in \mathbb{N}$, $q \in [2, \infty)$ and $p \geq 11/5$. Here $\tilde{\mu}_n = a\tilde{\varphi}_n - J * \tilde{\varphi}_n + F'(\tilde{\varphi}_n)$ and $U = H^1(\mathcal{M})$.

We will now prove the following important lemma:

Lemma 4.4. *We can extract a subsequence $\{(\tilde{u}_{n_k}, \tilde{\varphi}_{n_k}) : k \in \mathbb{N}\}$ from $\{(\tilde{u}_n, \tilde{\varphi}_n) : n \in \mathbb{N}\}$ such that*

$$\begin{aligned} &(\tilde{u}_{n_k}, \tilde{\varphi}_{n_k}) \to (\tilde{u}, \tilde{\varphi}) \quad in \quad L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; L^2(0,T;\mathbb{H}),\\ &J * \tilde{\varphi}_{n_k} \to J * \tilde{\varphi} \quad in \quad L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; L^2(0,T;U)),\\ &\tilde{\mu}_{n_k} \rightharpoonup \tilde{\mu} = a\tilde{\varphi} - J * \tilde{\varphi} + F'(\tilde{\varphi}) \quad in \quad L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; L^2(0,T;U)). \end{aligned} \tag{4.98}$$

Also the processes $\tilde{u}$, $\tilde{\varphi}$ and $\tilde{\mu}$ satisfy the estimates (4.97).

Proof. Thanks to the estimate $(4.97)_5$ and the Eberlein–Smulian theorem (see [52, Chapter 21, Proposition 21.23-(h)]), we infer that there exists a subsequence $\tilde{u}_{n_k}$ of $\tilde{u}_n$ satisfying

$$\begin{aligned} &\tilde{u}_{n_k} \rightharpoonup \tilde{u} \quad \text{in} \quad L^q(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; L^p(0,T;V_{\text{div},p})),\\ &\tilde{u}_{n_k}(T) \rightharpoonup \eta_1 \quad \text{in} \quad L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; G_{\text{div}}), \quad \text{for any} \quad q \in [2,\infty). \end{aligned} \tag{4.99}$$

We claim that $\eta_1 = \tilde{u}(T)$ in $L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; G_{\text{div}})$. The prove of this will be given later.

Thanks to $(4.99)_1$, reasoning similarly as in [13, Equation (4.88)], we check that

$$\tilde{u}_{n_k} \to \tilde{u} \quad \text{in} \quad L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; L^2(0,T;G_{\text{div}})).$$

Thus, modulo the extraction of a subsequence (still) denoted $(\tilde{u}_{n_k})_{k\geq 1}$ we have

$$\tilde{u}_{n_k} \to \tilde{u} \quad d\tilde{\mathbb{P}} \otimes dt\text{-a.e. in} \quad G_{\text{div}}. \tag{4.100}$$

Thanks to $(4.96)_2$ and $(4.97)_2$, [13, Equation (4.89)] we obtain

$$\tilde{\varphi}_{n_k} \to \tilde{\varphi} \quad \text{in} \quad L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; L^2(0,T;H)),$$

and thus, modulo the extraction of a subsequence (still) denoted $(\tilde{\varphi}_{n_k})_{k\geq 1}$ one has

$$\tilde{\varphi}_{n_k} \to \tilde{\varphi} \quad d\tilde{\mathbb{P}} \otimes dt\text{-a.e. in} \quad H. \tag{4.101}$$

Naturally the convergent $(4.98)_1$ follows from the previous convergence.

Later, we will show that

$$\tilde{\varphi}_{n_k} \rightharpoonup \eta_2 = \tilde{\varphi}(T) \quad \text{in} \quad L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; H). \tag{4.102}$$

Due to $(4.98)_1$ and the fact that the map $J* : H \to U$ is linear and bounded, we easily derive the convergence $(4.98)_2$.

Now, we will show that $\tilde{\mu}_{n_k} \rightharpoonup \tilde{\mu} = a\tilde{\varphi} - J * \tilde{\varphi} + F'(\tilde{\varphi})$ in $L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; L^2(0,T;U))$.

First of all, from the estimate $(4.97)_7$ and the Eberlein–Smulian theorem (see [52, Chapter 21, Proposition 21.23-(h)]), we can extract a subsequence of $\tilde{\mu}_n$ denoted by $\tilde{\mu}_{n_k}$ such that for any $q \in [2,\infty)$

$$\tilde{\mu}_{n_k} \rightharpoonup \tilde{\mu} \ \text{ in } \ L^q(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; L^2(0,T;U)). \tag{4.103}$$

From the estimates $(4.97)_{2,4}$ and the fact that the domain $\mathcal{M}$ is bounded, we infer that $\tilde{\rho}(.,\tilde{\varphi}_n) = a(.)\tilde{\varphi}_n + F'(\tilde{\varphi}_n)$ is bounded in $L^q(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; L^\infty(0,T;L^r(\mathcal{M})))$, for any $q \in [2,\infty)$. Therefore from the Banach–Alaoglu theorem, we conclude there exits a subsequence of $\rho(.,\tilde{\varphi}_n)$ denoted by $\rho(.,\tilde{\varphi}_{n_k})$ such that for any $q \in [2,\infty)$

$$\rho(.,\tilde{\varphi}_{n_k}) \overset{*}{\rightharpoonup} \rho \ \text{ in } \ L^q(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; L^\infty(0,T;L^r(\mathcal{M}))). \tag{4.104}$$

From the pointwise convergence (4.101) we have $\rho(.,\tilde{\varphi}_{n_k}) \to a(.)\tilde{\varphi} + F'(\tilde{\varphi})$ $d\tilde{\mathbb{P}}$-almost everywhere in $(0,T) \times \mathcal{M}$ and therefore from (4.104) we have $\rho = a\tilde{\varphi} + F'(\tilde{\varphi})$, i.e., $\rho(x,\varphi) = a(x)\tilde{\varphi} + F'(\tilde{\varphi})$; $x \in \mathcal{M}$.

Now, we introduce the following set

$$\boldsymbol{D} = \{\boldsymbol{\Phi} = \phi(\omega)\chi(t)v : \phi \in L^\infty(\tilde{\Omega}, \tilde{\mathbb{P}}), \chi \in \mathcal{D}(0,T) \text{ and } v \in H_n\},$$

where H_n is defined in the Sect. 4.1. This set is dense in $L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; L^2(0,T;H))$.

For any $\boldsymbol{\Phi} = \phi(\omega)\chi(t)v \in \boldsymbol{D}$ and every $n_k \geq n$ (n is fixed), we have

$$\tilde{\mathbb{E}}\left(\phi(\omega)\int_0^T (\tilde{\mu}_{n_k}(t), v)\,\chi(t)dt\right) = \tilde{\mathbb{E}}\left(\phi(\omega)\int_0^T (\rho(.,\tilde{\varphi}_{n_k}(t)) - J * \tilde{\varphi}_{n_k}(t), v)\chi(t)dt\right).$$

By passing to the limit as $n_k \to \infty$ in this identity and using the convergence (4.103), (4.104) and $(4.98)_2$, on account of the density of $\boldsymbol{D}$ in $L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; L^2(0,T;H))$ we get $\tilde{\mu} = a\tilde{\varphi} + F'(\tilde{\varphi}) - J * \tilde{\varphi}$ i.e. $(4.98)_3$. In particular we obtain $\rho(.,\tilde{\varphi}) = a(.)\tilde{\varphi} + F'(\tilde{\varphi}) \in L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; L^2(0,T;U))$.

Finally, making similar reasoning as in [13, Proposition 4.2, Inequality (4.86)] we can check that the stochastic processes $\tilde{u}, \tilde{\varphi}$ and $\tilde{\mu}$ satisfies the same estimates as in (4.97). This ends the proof of Lemma 4.4. □

Proposition 4.6. *Let $p \in [11/5, 12/5)$ and $T > 0$. There exits five processes $\boldsymbol{N}, \boldsymbol{B}_0, \boldsymbol{R}_1 \in L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; L^{p'}(0,T;V'_{\mathrm{div},p})$, $\boldsymbol{B}_1 \in L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; L^2(0,T;U')$ and $\boldsymbol{g}_1 \in L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; L^2(0,T;V'_{\mathrm{div},p})$ such that*

$$\begin{aligned}
&\mathcal{P}^1_{n_k} N(\tilde{u}_{n_k}) \rightharpoonup \boldsymbol{N} \quad in \ \ L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; L^{p'}(0,T;V'_{\mathrm{div},p}),\\
&\mathcal{P}^1_{n_k} B_0(\tilde{u}_{n_k}) \rightharpoonup \boldsymbol{B}_0 \quad in \ \ L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; L^{p'}(0,T;V'_{\mathrm{div},p}),\\
&\mathcal{P}^1_{n_k} \tilde{\varphi}_{n_k} \nabla\tilde{\mu}_{n_k} \rightharpoonup \boldsymbol{R}_1 \quad in \ \ L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; L^{p'}(0,T;V'_{\mathrm{div},p}),\\
&\mathcal{P}^2_{n_k} \tilde{u}_{n_k}.\nabla\tilde{\varphi}_{n_k} \rightharpoonup \boldsymbol{B}_1 \quad in \ \ L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; L^2(0,T;U'),\\
&\mathcal{P}^1_{n_k} g_1(\tilde{u}_{n_k}, \tilde{\varphi}_{n_k}) \rightharpoonup \boldsymbol{g}_1 \quad in \ \ L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; L^2(0,T;V'_{\mathrm{div},p})).
\end{aligned} \tag{4.105}$$

Proof. Using the assumption (H_1) for g_1, the estimates $(4.97)_{1,2}$, and applying also the Banach–Alaoglu theorem we easily derive $(4.105)_5$.

Owing to (2.23), $(4.97)_5$ and application of Banach–Alaoglu's theorem, we get $(4.105)_1$.

By mean of Hölder's and Gagliardo–Nirenberg's inequalities, we have for all $v \in V_{\mathrm{div},p}$

$$\begin{aligned}
|b(\tilde{u}_{n_k}, \tilde{u}_{n_k}, v)| &\leq c\|\tilde{u}_{n_k}\|_{L^{3p/(4p-6)}} \|\tilde{u}_{n_k}\|_{1,p} \|v\|_{L^{3p/(3-p)}}\\
&\leq c|\tilde{u}_{n_k}|^{(10p-18)/(5p-6)} \|\tilde{u}_{n_k}\|^{(12-5p)/(5p-6)}_{W^{1,p}} \|\tilde{u}_{n_k}\|_{1,p} \|v\|_{L^{3p/(3-p)}}\\
&\leq c|\tilde{u}_{n_k}|^{(10p-18)/(5p-6)} \|\tilde{u}_{n_k}\|^{(12-5p)/(5p-6)}_{W^{1,p}} \|\tilde{u}_{n_k}\|_{1,p} \|v\|_{W^{1,p}}\\
&\leq c|\tilde{u}_{n_k}|^{(10p-18)/(5p-6)} \|\tilde{u}_{n_k}\|^{6/(5p-6)}_{1,p} \|v\|_{1,p},
\end{aligned}$$

where we have also used the fact that $W^{1,p} \hookrightarrow L^{\frac{3p}{3-p}}$ and that $V_{\mathrm{div},p}$-norm is equivalent to the $W^{1,p}$-norm. Hence,

$$\|B_0(\tilde{u}_{n_k})\|_{V'_{\mathrm{div},p}} = \sup_{v \in V_{\mathrm{div},p}, \|v\|_{1,p} \leq 1} |b(\tilde{u}_{n_k}, \tilde{u}_{n_k}, v)| \leq c|\tilde{u}_{n_k}|^{\frac{10p-18}{5p-6}} \|\tilde{u}_{n_k}\|^{\frac{6}{5p-6}}_{1,p}. \tag{4.106}$$

Thanks to (4.106), using Hölder's inequality, we see that

$$\begin{aligned}\tilde{\mathbb{E}}\left[\int_0^T \|\mathcal{P}^1_{n_k} B_0(\tilde{u}_{n_k})\|^{p'}_{V'_{\mathrm{div},p}} dt\right]^{2/p'} &\leq c\tilde{\mathbb{E}}\left[\left(\int_0^T |\tilde{u}_{n_k}|^{\frac{2(5p-9)}{5p-11}} dt\right)^{\frac{2(5p-11)}{5p-6}} \left(\int_0^T \|\tilde{u}_{n_k}\|^p_{1,p} dt\right)^{\frac{12}{p(5p-6)}}\right]\\ &\leq c\left[\tilde{\mathbb{E}} \sup_{t\in[0,T]} |\tilde{u}_{n_k}(t)|^{\frac{4(5p-9)}{5p-8}}\right]^{\frac{5p-8}{5p-6}} \left[\tilde{\mathbb{E}}\left(\int_0^T \|\tilde{u}_{n_k}\|^p_{1,p} dt\right)^{\frac{6}{p}}\right]^{\frac{2}{5p-6}}.\end{aligned}$$

Therefore, from this previous inequality and the estimates $(4.97)_1$–$(4.97)_5$, we infer that

$$\tilde{\mathbb{E}}\left[\int_0^T \|\mathcal{P}^1_{n_k} B_0(\tilde{u}_{n_k}(t))\|^{p'}_{V'_{\mathrm{div},p}} dt\right]^{2/p'} < C. \tag{4.107}$$

Thanks to (4.107) and the Banach–Alaoglu theorem, we derive that $(4.105)_2$ holds.

By using the Sobolev embedding $W^{1,p} \hookrightarrow L^{\frac{3p}{3-p}}$, the Hölder inequality and the fact that $\kappa \geq \frac{2(3-p)}{5p-6}$, we have

$$|(\tilde{\varphi}_{n_k}\nabla\tilde{\mu}_{n_k}, v)| \leq c\|\tilde{\varphi}_{n_k}\|_{L^{6p/(5p-6)}}|\nabla\tilde{\mu}_{n_k}|\|v\|_{L^{3p/(3-p)}} \leq c\|\tilde{\varphi}_{n_k}\|_{L^{2\kappa+2}}|\nabla\tilde{\mu}_{n_k}|\|v\|_{1,p},$$

for all $v \in V_{\mathrm{div},p}$. Hence

$$\|\tilde{\varphi}_{n_k}\nabla\tilde{\mu}_{n_k}\|_{V'_{\mathrm{div},p}} \leq c\|\tilde{\varphi}_{n_k}\|_{L^{2\kappa+2}}|\nabla\tilde{\mu}_{n_k}|. \tag{4.108}$$

From (4.108), $(4.97)_2$–$(4.97)_7$ and the Hölder inequality, we obtain

$$\begin{aligned}\tilde{\mathbb{E}}[\int_0^T \|\tilde{\varphi}_{n_k}(t)\nabla\tilde{\mu}_{n_k}(t)\|^{p'}_{V'_{\mathrm{div},p}} dt]^{\frac{2}{p'}} &\leq c\tilde{\mathbb{E}}\left[\left(\int_0^T \|\tilde{\varphi}_{n_k}(t)\|^{\frac{2p}{p-2}}_{L^{2\kappa+2}} dt\right)^{\frac{p-2}{p}} \int_0^T |\nabla\tilde{\mu}_{n_k}(t)|^2 dt\right]\\ &\leq c[\tilde{\mathbb{E}} \sup_{t\in[0,T]} \|\tilde{\varphi}_{n_k}(t)\|^4_{L^{2\kappa+2}}]^{\frac{1}{2}} \left[\tilde{\mathbb{E}}\left(\int_0^T |\nabla\tilde{\mu}_{n_k}|^2 dt\right)^2\right]^{\frac{1}{2}} < C.\end{aligned} \tag{4.109}$$

Thus from (4.109) and the Banach–Alaoglu theorem, we derive $(4.105)_3$.

Now, we complete our proof by proving $(4.105)_4$. For this we first give the following estimate:

$$\begin{aligned}|(\tilde{u}_{n_k}.\nabla\tilde{\varphi}_{n_k}, \psi)| = |(\tilde{u}_{n_k}, \tilde{\varphi}_{n_k}\nabla\psi)| &\leq c\|\tilde{\varphi}_{n_k}\|_{L^{2\kappa+2}}\|\tilde{u}_{n_k}\|_{L^{2(\kappa+1)/\kappa}}\|\psi\|_U\\ &\leq c\|\tilde{\varphi}_{n_k}\|_{L^{2\kappa+2}}\|\tilde{u}_{n_k}\|_{L^{3p/(3-p)}}\|\psi\|_U\\ &\leq c\|\tilde{\varphi}_{n_k}\|_{L^{2\kappa+2}}\|\tilde{u}_{n_k}\|_{1,p}\|\psi\|_U,\end{aligned} \tag{4.110}$$

where we have used the fact that, since $\kappa \geq \frac{2(3-p)}{5p-6}$, then the following Sobolev embedding hold: $W^{1,p} \hookrightarrow L^{3p/(3-p)} \hookrightarrow L^{(2\kappa+2)/\kappa}$.

Using now (4.110), $(4.97)_{2,5}$ and the Hölder inequality, we get

$$\tilde{\mathbb{E}}\int_0^T \|\mathcal{P}^2_{n_k}\tilde{u}_{n_k}(t).\nabla\tilde{\varphi}_{n_k}(t)\|^2_{U'} dt \leq c[\tilde{\mathbb{E}} \sup_{t\in[0,T]} \|\tilde{\varphi}_{n_k}(t)\|^4_{L^{2\kappa+2}}]^{\frac{1}{2}} \left[\tilde{\mathbb{E}}\left(\int_0^T \|\tilde{u}_{n_k}\|^p_{1,p} dt\right)^{\frac{4}{p}}\right]^{\frac{1}{2}} < C. \tag{4.111}$$

Finally $(4.105)_4$ follows from (4.111) and an application of Banach–Alaoglu's theorem. The proof of Proposition 4.6 is now complete. □

Hereafter, $\langle .,. \rangle$ denotes the dual pairing between $V_{\mathrm{div},p}$ and $V'_{\mathrm{div},p}$ relative to G_{div}, and that between $U = H^1(\mathcal{M})$ and U' relative to H.

4.4. Passage to the Limit and the End of Proof of Theorem 3.1

Here we prove several convergence which will enable us to conclude that the limiting objects that we found in Proposition 4.4 are in fact a weak martingale solution to our problem.

Lemma 4.4 will be used to prove the following convergence results.

Proposition 4.7.

$$\begin{aligned}
&\int_0^{\cdot} \mathcal{P}^1_{n_k} g_1(\tilde{u}_{n_k}(s), \tilde{\varphi}_{n_k}(s))ds \to \int_0^{\cdot} \boldsymbol{g}_1(s)ds \\
&\quad = \int_0^{\cdot} g_1(\tilde{u}(s), \tilde{\varphi}(s))ds \quad in \quad L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; L^2(0,T; V'_{\mathrm{div}})); \\
&\mathcal{P}^1_{n_k} B_0(\tilde{u}_{n_k}) \rightharpoonup \boldsymbol{B}_0 = B_0(\tilde{u}) \quad in \quad L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; L^{p'}(0,T; V'_{\mathrm{div},p})); \\
&\mathcal{P}^1_{n_k} \tilde{\varphi}_{n_k} \nabla \tilde{\mu}_{n_k} \rightharpoonup \boldsymbol{R}_1 = \tilde{\varphi} \nabla \tilde{\mu} \quad in \quad L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; L^{p'}(0,T; V'_{\mathrm{div},p})); \\
&\mathcal{P}^2_{n_k} \tilde{u}_{n_k}.\nabla \tilde{\varphi}_{n_k} \rightharpoonup \boldsymbol{B}_1 = \tilde{u}.\nabla \tilde{\varphi} \quad in \quad L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; L^2(0,T; U')).
\end{aligned} \tag{4.112}$$

Proof. We have already proved in Proposition 4.6 that $\mathcal{P}^1_{n_k} g_1(\tilde{u}_{n_k}, \tilde{\varphi}_{n_k})$ belongs to a bounded set of $L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; L^{p'}(0,T; V'_{\mathrm{div},p}))$; $\mathcal{P}^1_{n_k} K_0(\tilde{u}_{n_k})$; $\mathcal{P}^1_{n_k} \tilde{\varphi}_{n_k} \nabla \tilde{\mu}_{n_k}$ belong to a bounded set of $L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; L^{p'}(0,T; V'_{\mathrm{div},p}))$ and $\mathcal{P}^2_{n_k} \tilde{u}_{n_k}.\nabla \tilde{\varphi}_{n_k}$ belongs to a bounded set of $L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; L^2(0,T; U'))$.

Hereafter we denote by (for the sake of simplicity) $\tilde{\tilde{u}}_{n_k}(.) := \tilde{u}_{n_k}(.) - \tilde{u}(.)$ and $\tilde{\tilde{\varphi}}_{n_k}(.) := \tilde{\varphi}_{n_k}(.) - \tilde{\varphi}(.)$. In order to prove $(4.112)_{2,3}$, we introduce the following set

$$\mathbb{D} = \{\boldsymbol{\Phi} = \phi(\omega)\chi(t)w_j : \phi \in L^\infty(\tilde{\Omega}, \tilde{\mathbb{P}}), \chi \in \mathcal{D}(0,T) \text{ and } j = 1, 2, \dots\},$$

where $\{w_j : j = 1, 2, \dots\}$ is defined in Sect. 4.1. Since this set is dense in $L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; L^p(0,T; V_{\mathrm{div},p}))$, it then follows from [52, Proposition 21.23] that the claims $(4.112)_{2,3}$ are achieved if we prove that

$$\begin{aligned}
&\tilde{\mathbb{E}}(\phi(\omega) \textstyle\int_0^T \langle B_0(\tilde{u}_{n_k}(s)) - B_0(\tilde{u}(s)), w_j \rangle \chi(s)ds) \to 0; \\
&\tilde{\mathbb{E}}(\phi(\omega) \textstyle\int_0^T \langle \tilde{\varphi}_{n_k}(s) \nabla \tilde{\mu}_{n_k}(s) - \tilde{\varphi}(s) \nabla \tilde{\mu}(s), w_j \rangle \chi(s)ds) \to 0
\end{aligned}$$

for any $\boldsymbol{\Phi} = \phi(\omega)\chi(t)w_j \in \mathbb{D}$. For this purpose we first note that

$$\begin{aligned}
\tilde{\mathbb{E}}(\phi(\omega) \textstyle\int_0^T \langle B_0(\tilde{u}_{n_k}) - B_0(\tilde{u}), w_j \rangle \chi(s)ds) &= -\tilde{\mathbb{E}}(\phi(\omega) \textstyle\int_0^T b(\tilde{\tilde{u}}_{n_k}, \tilde{u}_{n_k}, w_j)\chi(s)ds) \\
&\quad + \tilde{\mathbb{E}}(\phi(\omega) \textstyle\int_0^T b(\tilde{u}, \tilde{u} - \tilde{u}_{n_k}, w_j)\chi(s)ds) \\
&= I_1 + I_2
\end{aligned} \tag{4.113}$$

and

$$\begin{aligned}
\tilde{\mathbb{E}}(\phi(\omega) \textstyle\int_0^T \langle \tilde{\varphi}_{n_k} \nabla \tilde{\mu}_{n_k} - \tilde{\varphi} \nabla \tilde{\mu}, w_j \rangle \chi(s)ds) &= \tilde{\mathbb{E}}(\phi(\omega) \textstyle\int_0^T ([\tilde{\varphi}_{n_k} - \tilde{\varphi}] \nabla(\tilde{\mu}_{n_k} - \tilde{\mu}), w_j) \chi(s)ds) \\
&\quad - \tilde{\mathbb{E}}(\phi(\omega) \textstyle\int_0^T ([\tilde{\mu}_{n_k} - \tilde{\mu}] \nabla \tilde{\varphi}, w_j) \chi(s)ds) \\
&\quad + \tilde{\mathbb{E}}(\phi(\omega) \textstyle\int_0^T ([\tilde{\varphi}_{n_k} - \tilde{\varphi}] \nabla \tilde{\mu}, w_j) \chi(s)ds) \\
&= I_3 + I_4 + I_5.
\end{aligned} \tag{4.114}$$

The mapping $b(\tilde{u}, ., w_j)$ from $L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; L^2(0,T; V_{\mathrm{div},p}))$ into $L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; L^2(0,T; \mathbb{R}))$ is linear and continuous. Therefore, by invoking (4.99), $b(\tilde{u}, \tilde{u} - \tilde{u}_{n_k}, w_j) \to 0$ weakly in $L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; L^2(0,T; \mathbb{R}))$; and then $I_2 \to 0$ as $n_k \to \infty$. Next, from the properties of the operator b and Hölder's inequality, we see that

$$|\tilde{\mathbb{E}}(\phi(\omega) \textstyle\int_0^T b(\tilde{u}_{n_k} - \tilde{u}, \tilde{u}_{n_k}, w_j)\chi(s)ds)| \le c\|\boldsymbol{\Phi}\|_{L^\infty} \left[\tilde{\mathbb{E}} \left(\int_0^T \|\tilde{u}_{n_k}\|^p_{1,p} ds\right)^{\frac{2}{p}}\right]^{\frac{1}{2}} \left[\int_0^T |\tilde{u}_{n_k} - \tilde{u}|^2 ds\right]^{\frac{1}{2}},$$

with $\|\boldsymbol{\Phi}\|_{L^\infty} = \|\boldsymbol{\Phi}\|_{L^\infty(\tilde{\Omega} \times [0,T] \times \mathcal{M})}$.

Thanks to $(4.97)_5$ in conjunction with $(4.98)_1$ we see that the right-hand side of the above inequality converges to 0 as $n_k \to \infty$. Hence, I_1 converges to 0 and then $(4.112)_2$ holds.

By Lemma 4.4 and Hölder's inequality, we have

$$|\tilde{\mathbb{E}}(\phi(\omega)\int_0^T([\tilde{\tilde{\varphi}}_{n_k}(s)]\nabla\tilde{\mu},w_j)\chi(s)ds)| \le c\|\mathbf{\Phi}\|_{L^\infty}[\tilde{\mathbb{E}}\int_0^T|\nabla\tilde{\mu}|^2ds]^{\frac{1}{2}}[\tilde{\mathbb{E}}\int_0^T|\tilde{\tilde{\varphi}}_{n_k}|^2ds]^{\frac{1}{2}} \to 0, \tag{4.115}$$

and therefore $I_5 \to 0$ as $n_k \to \infty$. We recall that $\tilde{\tilde{\varphi}}_{n_k} = \tilde{\varphi}_{n_k} - \tilde{\varphi}$.

For fixed $\mathbf{\Phi} \in \mathbb{D}$ and $\tilde{\varphi} \in L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; L^2(0,T;U))$, the mapping

$$\Gamma \mapsto \tilde{\mathbb{E}}\left(\phi(\omega)\int_0^T(\Gamma(s)\nabla\tilde{\varphi}(s),w_j)\,\chi(s)ds\right)$$

is a continuous linear functional on $L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; L^2(0,T;U))$. Then, by invoking (4.103), we infer that $I_4 \to 0$ as n_k goes to infinity.

Arguing similarly as in (4.115), we have

$$\begin{aligned}&|\tilde{\mathbb{E}}(\phi(\omega)\int_0^T\left([\tilde{\tilde{\varphi}}_{n_k}(s)]\nabla(\tilde{\mu}_{n_k}(s)-\tilde{\mu}(s)),w_j\right)\chi(s)ds)|\\ &\quad\le c\|\mathbf{\Phi}\|_{L^\infty}\left[\tilde{\mathbb{E}}\int_0^T\left[|\nabla\tilde{\mu}(s)|^2+|\nabla\tilde{\mu}_{n_k}(s)|^2\right]ds\right]^{\frac{1}{2}}\left[\tilde{\mathbb{E}}\int_0^T|\tilde{\tilde{\varphi}}_{n_k}(s)|^2ds\right]^{\frac{1}{2}}.\end{aligned} \tag{4.116}$$

Using now (4.116), $(4.97)_7$ in conjunction with Lemma 4.4, it follows that I_3 converges to 0 as n_k goes to infinity; and then we get the convergence $(4.112)_3$.

Let us moves to the proof of $(4.112)_4$. For this let us introduce the following set

$$\tilde{\mathbb{D}} = \{\tilde{\Phi} = \phi(\omega)\chi(t)\psi_j : \phi \in L^\infty(\tilde{\Omega}, \tilde{\mathbb{P}}), \chi \in \mathcal{D}(0,T) \text{ and } j = 1,2,\ldots\},$$

where $\{\psi_j : j = 1,2,\ldots\}$ is defined in Sect. 4.1. Since this set is dense in $L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; L^2(0,T;U))$, it then follows from [52, Proposition 21.23] that the claims $(4.112)_4$ are achieved if we prove that

$$\tilde{\mathbb{E}}(\phi(\omega)\int_0^T\langle\tilde{u}_{n_k}(s).\nabla\tilde{\varphi}_{n_k}(s)-\tilde{u}(s).\nabla\tilde{\varphi}(s),\psi_j\rangle\,\chi(s)ds) \to 0,$$

for any $\tilde{\Phi} = \phi(\omega)\chi(t)\psi_j \in \tilde{\mathbb{D}}$. For this purpose, we begin by rewriting the last identity as follows

$$\begin{aligned}&\tilde{\mathbb{E}}(\phi(\omega)\int_0^T\langle\tilde{u}_{n_k}(s).\nabla\tilde{\varphi}_{n_k}(s)-\tilde{u}(s).\nabla\tilde{\varphi}(s),\psi_j\rangle\,\chi(s)ds)\\ &\quad= -\tilde{\mathbb{E}}(\phi(\omega)\int_0^T([\tilde{u}_{n_k}-\tilde{u}]\tilde{\varphi}_{n_k},\nabla\psi_j)\,\chi ds) - \tilde{\mathbb{E}}(\phi(\omega)\int_0^T(\tilde{u}[\tilde{\varphi}_{n_k}-\tilde{\varphi}],\nabla\psi_j)\,\chi ds)\\ &\quad= I_6 + I_7.\end{aligned}$$

By the Hölder inequality and Lemma 4.4, we have

$$|\tilde{\mathbb{E}}(\phi(\omega)\int_0^T\left(\tilde{u}[\tilde{\tilde{\varphi}}_{n_k}],\nabla\psi_j\right)\chi ds)| \le c\|\tilde{\Phi}\|_{L^\infty}[\tilde{\mathbb{E}}\int_0^T|\tilde{u}(s)|^2ds]^{\frac{1}{2}}[\tilde{\mathbb{E}}\int_0^T|\tilde{\tilde{\varphi}}_{n_k}(s)|^2ds]^{1/2} \to 0,$$

as $n_k \to \infty$. Hence, $I_7 \to 0$ as $n_k \to \infty$.

By Hölder's inequality, we obtain

$$|\tilde{\mathbb{E}}(\phi(\omega)\int_0^T\left([\tilde{\tilde{u}}_{n_k}(s)]\tilde{\varphi}_{n_k}(s),\nabla\psi_j\right)\chi(s)ds)| \le c\|\tilde{\Phi}\|_{L^\infty}[\tilde{\mathbb{E}}\int_0^T|\tilde{\varphi}_{n_k}|^2ds]^{\frac{1}{2}}[\tilde{\mathbb{E}}\int_0^T|\tilde{\tilde{u}}_{n_k}(s)|^2ds]^{\frac{1}{2}},$$

from which in conjunction with $(4.97)_6$ and Lemma 4.4, we infer that $I_6 \to 0$ as n_k goes to infinity. Hence $(4.112)_4$ holds.

Thanks to (4.100)–(4.101), the continuity of $\mathcal{P}^1_{n_k}g_1(u_{n_k},\varphi_{n_k})$ and the applicability of dominated convergence theorem, we infer that $(4.112)_1$ holds. The proof of Proposition 4.7 is now complete. □

Remark 4.1. Almost surely the paths of the process $(\tilde{u},\tilde{\varphi})$ are $\mathbb{H} = G_{\text{div}} \times H$-valued weakly continuous. Indeed, we note that from $(4.96)_4$ and the fact that the processes $\tilde{u}$ and $\tilde{\varphi}$ satisfy the estimates $(4.97)_1$ and $(4.97)_2$, respectively, see Lemma 4.4; it follows that almost surely $(\tilde{u},\tilde{\varphi}) \in \mathcal{C}(0,T;\mathbb{W}'_s) \cap L^2(0,T;\mathbb{H})$. Hence, we infer from [48, Theorem 2.1] that $\tilde{\mathbb{P}}$-a.s. $(\tilde{u},\tilde{\varphi}) \in \mathcal{C}(0,T;\mathbb{H}_w)$, where $\mathcal{C}(0,T;\mathbb{H}_w)$ denotes the space of weakly continuous functions $\mathbf{u} : [0,T] \to \mathbb{H}$. By closely follows the proof of [48, Theorem 2.1], we derive from [48, Eq. (2.1), p. 544] that $(\tilde{u}(t),\tilde{\varphi}(t)) \in \mathbb{H}$ for all $t \in [0,T]$. We can used the same argument to prove that $\tilde{\mathbb{P}}$-a.s. $(\tilde{u}_{n_k}(t),\tilde{\varphi}_{n_k}(t)) \in \mathbb{H}$ for all $t \in [0,T]$.

Now, to simplify notation let us define the processes $\mathfrak{M}_{n_k}(t)$ and $\mathfrak{R}_{n_k}(t)$, $t \in (0,T]$ by

$$\mathfrak{M}_{n_k}(t) := \tilde{u}_{n_k}(t) - \tilde{u}_{0n_k} + \int_0^t \mathcal{P}^1_{n_k} N(u_{n_k}) ds - \int_0^t \mathcal{P}^1_{n_k} B_0(\tilde{u}_{n_k}) ds + \int_0^t \mathcal{P}^1_{n_k} \tilde{\varphi}_{n_k} \nabla \tilde{\mu}_{n_k} ds - \int_0^t \mathcal{P}^1_{n_k} g_0(s) ds - \int_0^t \mathcal{P}^1_{n_k} g_1(\tilde{u}_{n_k}, \tilde{\varphi}_{n_k}) ds$$

and

$$\mathfrak{R}_{n_k}(t) := \tilde{\varphi}_{n_k}(t) - \tilde{\varphi}_{0n_k} - \int_0^t \mathcal{P}^2_{n_k} \Delta \tilde{\mu}_{n_k}(s) ds + \int_0^t \mathcal{P}^2_{n_k} \tilde{u}_{n_k}(s).\nabla \tilde{\varphi}_{n_k}(s) ds = 0.$$

Proposition 4.8. *Let $11/5 \le p < 12/5$, $\mathfrak{M}(t)$ and $\mathfrak{R}(t)$, $t \in (0,T]$ be two processes define by*

$$\mathfrak{M}(t) = \tilde{u}(t) - u_0 + \int_0^t (\boldsymbol{N}(s) - B_0(\tilde{u}(s)) + \tilde{\varphi}(s)\nabla\tilde{\mu}(s) - g_0(s) - g_1(\tilde{u}(s), \tilde{\varphi}(s)))ds,$$
$$\mathfrak{R}(t) = \tilde{\varphi}(t) - \varphi_0 - \int_0^t \Delta\tilde{\mu}(s) ds + \int_0^t \tilde{u}(s).\nabla\tilde{\varphi}(s) ds.$$

Then, for any $t \in (0,T]$, we have

$$\mathfrak{M}_{n_k}(t) \rightharpoonup \mathfrak{M}(t) \quad in \quad L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; L^{p'}(0,T; V'_{\mathrm{div},p})),$$
$$\mathfrak{R}_{n_k}(t) \rightharpoonup \mathfrak{R}(t) \quad in \quad L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; L^2(0,T; U')),$$

as $n_k \to \infty$.

Proof. Thanks to Remark 4.1, the convergence $(4.105)_1$ and Proposition 4.7, we see that

$$\mathfrak{M}_{n_k}(t) \rightharpoonup \mathfrak{M}(t) \ \text{ in } \ L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; L^{p'}(0,T; V'_{\mathrm{div},p}))$$

as $n_k \to \infty$.

Also, thanks to Remark 4.1, $(4.98)_3$ in Lemma 4.4 and $(4.112)_4$ in Proposition 4.7, we see that

$$\mathfrak{R}_{n_k}(t) \rightharpoonup \mathfrak{R}(t) \ \text{ in } \ L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; L^2(0,T; U'))$$

as $n_k \to \infty$. □

Let $\mathcal{N}$ be the set of null sets of $\tilde{\mathcal{F}}$ and for any $t \ge 0$ and $k \in \mathbb{N}$, let

$$\tilde{\mathcal{F}}^{n_k}_t := \sigma(\sigma((\tilde{u}_{n_k}(s), \tilde{\varphi}_{n_k}(s), \tilde{W}_{n_k}(s)); s \le t) \cup \mathcal{N}),$$
$$\tilde{\mathcal{F}}_t := \sigma(\sigma((\tilde{u}(s), \tilde{\varphi}(s), \tilde{W}(s)); s \le t) \cup \mathcal{N}),$$

be the completion of the natural filtration generated by $(\tilde{u}_{n_k}, \tilde{\varphi}_{n_k}, \tilde{W}_{n_k})$ and $(\tilde{u}, \tilde{\varphi}, \tilde{W})$, respectively.

We infer from Proposition 4.4 that the law of (u_n, φ_n, W) are equal to those of $(\tilde{u}_n, \tilde{\varphi}_n, \tilde{W}_n)$ on $\mathfrak{X} = \mathfrak{X}_1 \times \mathcal{C}(0,T;K)$, with $\mathfrak{X}_1 = L^2(0,T;\mathbb{H}) \cap \mathcal{C}(0,T;\mathbb{W}'_s)$. Hence, it is easy to check that $\tilde{W}_n$ is a sequence of K_1-valued Wiener process adapted to the filtration $\tilde{\mathbb{F}}^{n_k} := \{\tilde{\mathcal{F}}^{n_k}_t : t \in [0,T]\}$. Also from Proposition 4.5, we see that $\tilde{W}$ is a K_1-valued Wiener process adapted to the filtration $\tilde{\mathbb{F}} := \{\tilde{\mathcal{F}}_t : t \in [0,T]\}$. The $\mathbb{W}'_s$-valued stochastic processes $(\tilde{u}_{n_k}, \tilde{\varphi}_{n_k})$ and $(\tilde{u}, \tilde{\varphi})$ are adapted with respect to $\tilde{\mathbb{F}}^{n_k}$ and $\tilde{\mathbb{F}}$ as well. Hence, since their sample paths are continuous in $\mathbb{W}'_s$, we infer that there are also predictable in $\mathbb{W}'_s$.

We now give the following important result.

Proposition 4.9. *For each $t \in (0,T]$ we have*

$$\mathcal{M}_{n_k}(t) := \int_0^t \mathcal{P}^1_{n_k} g_2(s, \tilde{u}_{n_k}(s), \tilde{\varphi}_{n_k}(s)) d\tilde{W}_{n_k} \to \int_0^t g_2(s, \tilde{u}(s), \tilde{\varphi}(s)) d\tilde{W} \tag{4.117}$$

in $L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; L^2(0,T; G_{\mathrm{div}}))$ and the following identity holds $\tilde{\mathbb{P}}$-a.s

$$\mathfrak{M}(t) = \int_0^t g_2(s, \tilde{u}(s), \tilde{\varphi}(s)) d\tilde{W}(s). \tag{4.118}$$

Proof. The proof of (4.117) and (4.118) is similar to that of Lemma 3.13 and Proposition 3.16 of [45]. □

The process $(\tilde{u}, \tilde{\varphi})$ satisfies the following property

Proposition 4.10. *For any $q \in [2,\infty)$, we have $(\tilde{u},\tilde{\varphi}) \in L^q(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}};\mathcal{C}(0,T;\mathbb{H}))$.*

Proof. In fact, from (4.118) in Proposition 4.9 and the last convergence in Proposition 4.8, we have for each $t \in (0,T]$

$$\begin{aligned}\tilde{u}(t) &= u_0 + \textstyle\int_0^t G(s)ds + \int_0^t S(s)d\tilde{W}(s),\\ \tilde{\varphi}(t) &= \varphi_0 + \textstyle\int_0^t \Delta\tilde{\mu}(s)ds - \int_0^t \tilde{u}(s).\nabla\tilde{\varphi}(s)ds,\end{aligned} \tag{4.119}$$

where

$$G(.) := -\boldsymbol{N}(.) + B_0(\tilde{u}(.)) - \tilde{\varphi}(.)\nabla\tilde{\mu}(.) + g_0(.) + g_1(\tilde{u}(.),\tilde{\varphi}(.)),\ \ S(.) := g_2(.,\tilde{u}(.),\tilde{\varphi}(.)).$$

From the properties of g_0 and thanks to $(4.105)_1$ and $(4.112)_{1,2,3}$, we have $G(.) \in L^2(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}};L^{p'}(0,T;V'_{\mathrm{div},p}))$. By $\tilde{\mathbb{E}}\sup_{s\in[0,T]}|\tilde{u}(s)|^q < C, \tilde{\mathbb{E}}\sup_{s\in[0,T]}\|\tilde{\varphi}(s)\|^q_{L^{2\kappa+2}} < C$ (see Lemma 4.4) and assumption (H_2), we obtain $S(.) \in L^2(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}};L^\infty(0,T;L_2(K,G_{\mathrm{div}})))$. Thus, we infer from [30, Chapter I, Theorem 3.2] that there exists $\tilde{\Omega}_1 \in \tilde{\mathbb{F}}$ such that $\tilde{\mathbb{P}}(\tilde{\Omega}_1) = 1$ and for each $\omega \in \tilde{\Omega}_1$ the function $\tilde{u}(\omega,.)$ takes values in G_{div}, and it is continuous in G_{div} with respect to t.

Since $\tilde{\mu}$ also satisfies the estimate $(4.97)_7$ (cf. Lemma 4.4), we can easily show that $\Delta\tilde{\mu} \in L^q(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}};L^2(0,T;U'))$ and thanks to $(4.112)_4$ in Proposition 4.7 and $(4.119)_2$, we derive that $\tilde{\varphi}_t \in L^q(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}};L^2(0,T;U'))$. Also since $\tilde{\varphi}$ satisfies the estimate $(4.97)_6$ (cf. Lemma 4.4), we infer from [51, Lemma 1.2, page 261] that for $\tilde{\mathbb{P}}$-almost all $\omega \in \tilde{\Omega}$ the trajectory $\tilde{\varphi}(\omega,.)$ is equal almost everywhere to a continuous H-valued functions defined in $[0,T]$. Now, since $\tilde{\mathbb{E}}\sup_{s\in[0,T]}|\tilde{u}(s)|^q < C$, $\tilde{\mathbb{E}}\sup_{s\in[0,T]}\|\tilde{\varphi}(s)\|^q_{L^{2\kappa+2}} < C$ we infer that $(\tilde{u},\tilde{\varphi}) \in L^q(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}};\mathcal{C}(0,T;\mathbb{H}))$. □

To complete the proof of Theorem 3.1, we need to prove some additionally results.

Proposition 4.11. *We have the following identity*

$$\boldsymbol{N}(.) = N(\tilde{u}) \quad in \quad L^2(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}};L^{p'}(0,T;V'_{\mathrm{div},p})).$$

Before proving this proposition, we first state and prove the following important result.

Proposition 4.12. *The following energy identity holds*

$$\begin{aligned}&\tilde{\mathbb{E}}\mathcal{E}_{tot}(\tilde{u}(T),\tilde{\varphi}(T)) + 2\tilde{\mathbb{E}}\int_0^T [\langle\boldsymbol{N}(s),\tilde{u}(s)\rangle + |\nabla\tilde{\mu}(s)|^2]ds\\ &\quad= \mathcal{E}_{tot}(u_0,\varphi_0) + \tilde{\mathbb{E}}\int_0^T \|g_2(s,\tilde{u},\tilde{\varphi})\|^2_{L_2(K,G_{\mathrm{div}})} + 2[\langle g_0(s) + g_1(\tilde{u},\tilde{\varphi}),\tilde{u}\rangle]ds.\end{aligned} \tag{4.120}$$

Proof. Since the processes $G(.)$ and $S(.)$ define in (4.119) belong to $L^2(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}};L^{p'}(0,T;V'_{\mathrm{div},p}))$ and $L^2(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}};L^\infty(0,T;L_2(K,G_{\mathrm{div}})))$, respectively, we can apply the Itô formula to the process $|\tilde{u}|^2$ (see, for instance, [42, Theorem 4.2.5]) and derive that

$$\begin{aligned}2\tilde{\mathbb{E}}\int_0^T \langle\boldsymbol{N}(s),\tilde{u}(s)\rangle\,ds &= |\tilde{u}_0|^2 - \tilde{\mathbb{E}}|\tilde{u}(T)|^2 - 2\tilde{\mathbb{E}}\int_0^T \langle\tilde{\varphi}(s)\nabla\tilde{\mu}(s),\tilde{u}(s)\rangle\,ds\\ &\quad + \tilde{\mathbb{E}}\int_0^T 2[\langle g_0(s) + g_1(\tilde{u}(s),\tilde{\varphi}(s)),\tilde{u}(s)\rangle] + \|g_2(s,\tilde{u}(s),\tilde{\varphi}(s))\|^2_{L_2(K,G_{\mathrm{div}})}ds,\end{aligned} \tag{4.121}$$

where we have also used $(2.9)_1$.

We note that $(4.119)_2$ can be rewritten in the following form

$$\langle\partial_t\tilde{\varphi},\psi_j\rangle + (\nabla\tilde{\mu},\nabla\psi_j) = (\tilde{u}\tilde{\varphi},\nabla\psi_j). \tag{4.122}$$

Now taking $\tilde{\mu}$ as a test function in (4.122) and multiplying the resulting equality by 2, we get

$$2\frac{d}{dt}\mathcal{E}(\tilde{\varphi}(.)) + 2|\nabla\tilde{\mu}|^2 = 2(\tilde{u}\tilde{\varphi},\nabla\tilde{\mu}). \tag{4.123}$$

Integrating now (4.123) between 0 and T and adding the resulting equality to (4.121), we obtain (4.120). $\square$

We now give the proof of Proposition 4.11.

Proof. For the proof, we will use the method of monotonicity (see, for instance, [40, Chapitre 3, Section 3, p. 103]).

From Propositions 4.1–4.2, we have for any $j = 1, 2, \dots, n_k$

$$\begin{aligned}(\tilde{u}_{n_k}(T), w_j) = (\tilde{u}_{0n_k}, w_j) &- \int_0^T \left\langle \tilde{\mathcal{P}}^1_{n_k}[N(\tilde{u}_{n_k}) - B_0(\tilde{u}_{n_k})], w_j \right\rangle ds \\ &- \int_0^T \left\langle \tilde{\mathcal{P}}^1_{n_k} \tilde{\varphi}_{n_k} \nabla \tilde{\mu}_{n_k}, w_j \right\rangle ds + \int_0^T (\tilde{\mathcal{P}}^1_{n_k} g_0, w_j) ds \\ &+ \int_0^T \left\langle \tilde{\mathcal{P}}^1_{n_k} g_1(\tilde{u}_{n_k}, \tilde{\varphi}_{n_k}), w_j \right\rangle ds + \int_0^T (\tilde{\mathcal{P}}^1_{n_k} g_2(s, \tilde{u}_{n_k}, \tilde{\varphi}_{n_k}), w_j) d\tilde{W}_{n_k}\end{aligned} \tag{4.124}$$

and

$$(\tilde{\varphi}_{n_k}(T), \psi_j) = (\tilde{\varphi}_{0n_k}, \psi_j) + \int_0^T (\mathcal{P}^2_{n_k} \Delta \tilde{\mu}_{n_k}, \psi_j) ds + \int_0^T \left\langle \mathcal{P}^2_{n_k} \tilde{u}_{n_k}.\nabla \tilde{\varphi}_{n_k}, \psi_j \right\rangle ds \tag{4.125}$$

where (w_j, ψ_j) are introduce in Sect. 4.1.

Now from (4.124), $(4.112)_{1,2,3}$ (see Proposition 4.7), using (4.117), Remark 4.1 and (4.100), we derive that

$$(\tilde{u}(T), w_j) = (u_0, w_j) + \int_0^T \langle G(s), w_j \rangle ds + \int_0^T (S(s), w_j) d\tilde{W}(s),$$

where $G(.)$ and $S(.)$ are defined as in the proof of Proposition 4.10. Also from $(4.99)_2$ and (4.124), we infer that

$$(\eta_1, w_j) = (\tilde{u}_0, w_j) + \int_0^T \langle G(s), w_j \rangle ds + \int_0^T (S(s), w_j) d\tilde{W}(s),$$

for all $j \geq 1$. Hence, from this two previous equalities, we infer that $\eta_1 = \tilde{u}(T)$ in $L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; G_{\text{div}})$.

Thanks to (4.125), using Remark 4.1, $(4.112)_4$ in Proposition 4.7 and (4.103), we get

$$(\tilde{\varphi}(T), \psi_j) = (\varphi_0, \psi_j) + \int_0^T (\Delta \tilde{\mu}, \psi_j) ds + \int_0^T \langle \tilde{u}.\nabla \tilde{\varphi}, \psi_j \rangle ds. \tag{4.126}$$

Owing to (4.125) and (4.102), we get

$$(\eta_2, \psi_j) = (\tilde{\varphi}_0, \psi_j) + \int_0^T (\Delta \tilde{\mu}, \psi_j) ds + \int_0^T \langle \tilde{u}.\nabla \tilde{\varphi}, \psi_j \rangle ds.$$

Thus, from this two previous equalities, we derive that $\eta_2 = \tilde{\varphi}(T)$ in $L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; H)$.

Applying also the Itô formula to the process $|\tilde{u}_{n_k}|^2$, we derive that

$$\begin{aligned}2\tilde{\mathbb{E}} \int_0^T \langle N(\tilde{u}_{n_k}), \tilde{u}_{n_k} \rangle ds &= |\tilde{u}_{0n_k}|^2 - \tilde{\mathbb{E}}|\tilde{u}_{n_k}(T)|^2 - 2\tilde{\mathbb{E}} \int_0^T \left\langle \mathcal{P}^1_{n_k} \tilde{\varphi}_{n_k} \nabla \tilde{\mu}_{n_k}, \tilde{u}_{n_k} \right\rangle ds \\ &+ \tilde{\mathbb{E}} \int_0^T 2[\langle \mathcal{P}^1_{n_k}[g_0(s) + g_1(\tilde{u}_{n_k}, \tilde{\varphi}_{n_k})], \tilde{u}_{n_k} \rangle] + \|\mathcal{P}^1_{n_k} g_2(s, \tilde{u}_{n_k}, \tilde{\varphi}_{n_k})\|^2_{L_2(K, G_{\text{div}})} ds.\end{aligned} \tag{4.127}$$

Making similar reasoning as in the proof of (4.120), we can easily check that the processes $\tilde{u}_{n_k}$ and $\tilde{\varphi}_{n_k}$ satisfy

$$\begin{aligned}&\tilde{\mathbb{E}}\mathcal{E}_{tot}(\tilde{u}_{n_k}(T), \tilde{\varphi}_{n_k}(T)) + 2\tilde{\mathbb{E}} \int_0^T [\langle N(\tilde{u}_{n_k}(s)), \tilde{u}_{n_k}(s) \rangle + |\nabla \tilde{\mu}_{n_k}(s)|^2] ds \\ &= \mathcal{E}_{tot}(\tilde{u}_{0n_k}, \tilde{\varphi}_{0n_k}) + \tilde{\mathbb{E}} \int_0^T \|\mathcal{P}^1_{n_k} g_2(s, \tilde{u}_{n_k}(s), \tilde{\varphi}_{n_k}(s))\|^2_{L_2(K, G_{\text{div}})} \\ &\quad + 2\tilde{\mathbb{E}} \int_0^T [\langle \mathcal{P}^1_{n_k}[g_0(s) + g_1(\tilde{u}_{n_k}(s), \tilde{\varphi}_{n_k}(s))], \tilde{u}_{n_k}(s) \rangle] ds.\end{aligned} \tag{4.128}$$

Take now an arbitrary $v \in L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; L^p(0, T; V_{\text{div},p}))$ and set

$$\begin{aligned}\mathcal{Z}_{n_k}(T) &:= \tilde{\mathbb{E}}\mathcal{E}_{tot}(\tilde{u}_{n_k}(T), \tilde{\varphi}_{n_k}(T)) + 2\tilde{\mathbb{E}} \int_0^T \langle N(\tilde{u}_{n_k}) - N(v), \tilde{u}_{n_k} - v \rangle ds \\ &+ \tilde{\mathbb{E}} \int_0^T 2|\nabla(\tilde{\mu}_{n_k} - \tilde{\mu})|^2 + \|\mathcal{P}^1_{n_k} g_2(s, \tilde{u}_{n_k}, \tilde{\varphi}_{n_k}) - g_2(s, \tilde{u}, \tilde{\varphi})\|^2_{L_2(K, G_{\text{div}})} ds.\end{aligned} \tag{4.129}$$

Using (4.128), we see that $\mathcal{Z}_{n_k}(T)$ can be rewritten in the form

$$\begin{aligned}\mathcal{Z}_{n_k}(T) = \mathcal{E}_{tot}(\tilde{u}_{0n_k}, \tilde{\varphi}_{0n_k}) - 2\tilde{\mathbb{E}}\int_0^T \langle N(\tilde{u}_{n_k}), v\rangle\, ds - 2\tilde{\mathbb{E}}\int_0^T \langle N(v), \tilde{u}_{n_k} - v\rangle\, ds \\ - 4\tilde{\mathbb{E}}\int_0^T (\nabla\tilde{\mu}_{n_k}, \nabla\tilde{\mu}) ds + 2\tilde{\mathbb{E}}\int_0^T |\nabla\tilde{\mu}|^2 ds - \tilde{\mathbb{E}}\int_0^T \|g_2(s,\tilde{u},\tilde{\varphi})\|^2_{L_2(K,G_{\mathrm{div}})} ds \\ + 2\tilde{\mathbb{E}}\int_0^T \|\mathcal{P}^1_{n_k} g_2(s,\tilde{u}_{n_k},\tilde{\varphi}_{n_k})\|^2_{L_2(K,G_{\mathrm{div},p})} ds + 2\tilde{\mathbb{E}}\int_0^T \left\langle \mathcal{P}^1_{n_k} g_0(s), \tilde{u}_{n_k}\right\rangle ds \\ + 2\int_0^T \left(g_2(s,\tilde{u},\tilde{\varphi}) - \mathcal{P}^1_{n_k} g_2(s,\tilde{u}_{n_k},\tilde{\varphi}_{n_k}), g_2(s,\tilde{u},\tilde{\varphi})\right)_{L_2(K,G_{\mathrm{div}})} ds \\ + 2\tilde{\mathbb{E}}\int_0^T \left\langle \mathcal{P}^1_{n_k} g_1(\tilde{u}_{n_k},\tilde{\varphi}_{n_k}) - g_1(\tilde{u},\tilde{\varphi}), \tilde{u}_{n_k}\right\rangle ds + 2\tilde{\mathbb{E}}\int_0^T \langle g_1(\tilde{u},\tilde{\varphi}), \tilde{u}_{n_k}\rangle\, ds.\end{aligned} \tag{4.130}$$

Now, from the fact that $\tilde{u}_{0n_k} \to u_0$ in G_{div} and $\tilde{\varphi}_{0n_k} \to \varphi_0$ in $H^2(\mathcal{M})$ and hence also in $L^\infty(\mathcal{M})$, we derive that

$$\mathcal{E}_{tot}(\tilde{u}_{0n_k}, \tilde{\varphi}_{0n_k}) \to \mathcal{E}_{tot}(u_0, \varphi_0) \quad \text{as} \quad n_k \to \infty.$$

From $(4.105)_1$ and $(4.99)_1$ we obtain

$$\begin{aligned}&\tilde{\mathbb{E}}\int_0^T \langle N(\tilde{u}_{n_k}(s)), v\rangle\, ds \to \tilde{\mathbb{E}}\int_0^T \langle \boldsymbol{N}(s), v\rangle\, ds,\\ &\tilde{\mathbb{E}}\int_0^T \langle N(v), \tilde{u}_{n_k}(s) - v\rangle\, ds \to \tilde{\mathbb{E}}\int_0^T \langle N(v), \tilde{u}(s) - v\rangle\, ds,\\ &\tilde{\mathbb{E}}\int_0^T \langle g_1(\tilde{u}(s),\tilde{\varphi}(s)), \tilde{u}_{n_k}(s)\rangle\, ds \to \tilde{\mathbb{E}}\int_0^T \langle g_1(\tilde{u}(s),\tilde{\varphi}(s)), \tilde{u}(s)\rangle\, ds\end{aligned}$$

as $n_k \to \infty$.

Thanks to $(4.98)_3$, we have

$$\tilde{\mathbb{E}}\int_0^T (\nabla\tilde{\mu}_{n_k}(s), \nabla\tilde{\mu}(s) ds \to \tilde{\mathbb{E}}\int_0^T |\nabla\tilde{\mu}(s)|^2 ds \quad \text{as} \quad n_k \to \infty.$$

It follows from $(4.99)_1$ and $(4.112)_1$ that

$$\tilde{\mathbb{E}}\int_0^T \left\langle \mathcal{P}^1_{n_k} g_1(\tilde{u}_{n_k}(s),\tilde{\varphi}_{n_k}(s)) - g_1(\tilde{u}(s),\tilde{\varphi}(s)), \tilde{u}_{n_k}(s)\right\rangle ds \to 0 \quad \text{as} \quad n_k \to \infty.$$

Note that $\left\langle \mathcal{P}^1_{n_k} g_0(s), \tilde{u}_{n_k}\right\rangle = \left\langle \mathcal{P}^1_{n_k} g_0(s), \tilde{u}_{n_k} - \tilde{u}\right\rangle + \left\langle \mathcal{P}^1_{n_k} g_0(s), \tilde{u}\right\rangle$. From this observation and making use of the convergence $(4.98)_1$ and $\mathcal{P}^1_{n_k} g_0 \to g_0$ in $L^{p'}(0,T;V'_{\mathrm{div},p})$, we infer that

$$\tilde{\mathbb{E}}\int_0^T \left\langle \mathcal{P}^1_{n_k} g_0(s), \tilde{u}_{n_k}(s)\right\rangle ds \to \tilde{\mathbb{E}}\int_0^T \langle g_0(s), \tilde{u}(s)\rangle\, ds \quad \text{as} \quad n_k \to \infty.$$

Now, thanks to the continuity of $\mathcal{P}^1_{n_k} g_2$ and $(4.96)_4$, we can arguing as in the proof of (4.100) to derive that

$$\mathcal{P}^1_{n_k} g_2(s,\tilde{u}_{n_k},\tilde{\varphi}_{n_k}) \to g_2(s,\tilde{u}(s),\tilde{\varphi}(s)) \quad \text{in} \quad L_2(K,G_{\mathrm{div}}) \quad d\tilde{\mathbb{P}}\otimes dt\text{-a.e..} \tag{4.131}$$

From (4.131), $(4.97)_{1,2}$ and (H_2), we can apply the Vitali convergence theorem to derive that

$$\mathcal{P}^1_{n_k} g_2(s,\tilde{u}_{n_k},\tilde{\varphi}_{n_k}) \to g_2(s,\tilde{u}(s),\tilde{\varphi}(s)) \quad \text{in} \quad L^4(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}}; L^4(0,T;L_2(K,G_{\mathrm{div}}))). \tag{4.132}$$

It follows from (4.132) that

$$\begin{aligned}&\tilde{\mathbb{E}}\int_0^T [2\|\mathcal{P}^1_{n_k} g_2(s,\tilde{u}_{n_k},\tilde{\varphi}_{n_k})\|^2_{L_2(K,G_{\mathrm{div},p})} - \|g_2(s,\tilde{u},\tilde{\varphi})\|^2_{L_2(K,G_{\mathrm{div}})}] ds\\ &\to \tilde{\mathbb{E}}\int_0^T \|g_2(s,\tilde{u},\tilde{\varphi})\|^2_{L_2(K,G_{\mathrm{div}})} ds,\\ &\int_0^T \left(g_2(s,\tilde{u},\tilde{\varphi}) - \mathcal{P}^1_{n_k} g_2(s,\tilde{u}_{n_k},\tilde{\varphi}_{n_k}), g_2(s,\tilde{u},\tilde{\varphi})\right)_{L_2(K,G_{\mathrm{div}})} ds \to 0,\end{aligned}$$

as $n_k \to \infty$.

Letting $n_k \to \infty$ in (4.130) and using all the previous convergences, we see that

$$\begin{aligned}\mathcal{Z}_{n_k}(T) \to \mathcal{E}_{tot}(u_0, \varphi_0) - 2\tilde{\mathbb{E}} \int_0^T \langle \boldsymbol{N}(s), v \rangle \, ds - 2\tilde{\mathbb{E}} \int_0^T \langle N(v), \tilde{u}(s) - v \rangle \, ds \\ -2\tilde{\mathbb{E}} \int_0^T |\nabla \tilde{\mu}(s)|^2 ds + \tilde{\mathbb{E}} \int_0^T \| g_2(s, \tilde{u}(s), \tilde{\varphi}(s)) \|^2_{L_2(K, G_{\mathrm{div}})} ds \\ +2\tilde{\mathbb{E}} \int_0^T \langle g_0(s), \tilde{u}(s) \rangle \, ds + 2\tilde{\mathbb{E}} \int_0^T \langle g_1(\tilde{u}(s), \tilde{\varphi}(s)), \tilde{u}(s) \rangle \, ds.\end{aligned} \tag{4.133}$$

On the other hand, thanks to item (b) in Proposition 2.1, to the lower semicontinuity of the norms, using Fatou's Lemma we have

$$\lim_{n_k \to \infty} \inf \mathcal{Z}_{n_k}(T) \geq \tilde{\mathbb{E}} \mathcal{E}_{tot}(\tilde{u}(T), \tilde{\varphi}(T)). \tag{4.134}$$

Hence, we obtain

$$\begin{aligned}\tilde{\mathbb{E}} \mathcal{E}_{tot}(\tilde{u}(T), \tilde{\varphi}(T)) \leq \mathcal{E}_{tot}(u_0, \varphi_0) - 2\tilde{\mathbb{E}} \int_0^T \langle \boldsymbol{N}(s), v \rangle \, ds - 2\tilde{\mathbb{E}} \int_0^T \langle N(v), \tilde{u}(s) - v \rangle \, ds \\ - 2\tilde{\mathbb{E}} \int_0^T |\nabla \tilde{\mu}(s)|^2 ds + \tilde{\mathbb{E}} \int_0^T \| g_2(s, \tilde{u}(s), \tilde{\varphi}(s)) \|^2_{L_2(K, G_{\mathrm{div}})} ds \\ +2\tilde{\mathbb{E}} \int_0^T \langle g_0(s), \tilde{u}(s) \rangle \, ds + 2\tilde{\mathbb{E}} \int_0^T \langle g_1(\tilde{u}(s), \tilde{\varphi}(s)), \tilde{u}(s) \rangle \, ds,\end{aligned}$$

which, combined with (4.120) in Proposition 4.12, yields the variational inequality

$$2\tilde{\mathbb{E}} \int_0^T \langle \boldsymbol{N}(s) - N(v(s)), v(s) - \tilde{u}(s) \rangle \, ds \leq 0 \tag{4.135}$$

for any $v \in L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; L^p(0, T; V_{\mathrm{div},p}))$. Let $\zeta \in L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; L^p(0, T; V_{\mathrm{div},p}))$ and $\epsilon > 0$. By taking $v = \tilde{u} \pm \epsilon\zeta$, we derive from (4.135) that

$$2\tilde{\mathbb{E}} \int_0^T \langle \boldsymbol{N}(s) - N(\tilde{u}(s) \pm \epsilon\zeta(s)), \pm\epsilon\zeta(s) \rangle \, ds \leq 0, \tag{4.136}$$

from which along the hemicontinuity of N we conclude the proof of Proposition 4.11. □

We can now give the proof of Theorem 3.1, which concerns the existence of a weak martingale solution.

Proof. Endowing the complete probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ with the filtration $\tilde{\mathbb{F}} = \{\tilde{\mathcal{F}}_t : t \in [0, T]\}$, where the σ-algebra $\tilde{\mathcal{F}}_t$ is defined by

$$\tilde{\mathcal{F}}_t := \sigma(\sigma((\tilde{u}(s), \tilde{\varphi}(s), \tilde{W}(s)); s \leq t) \cup \mathcal{N}),$$

and combining Propositions 4.5, 4.9, 4.10 and 4.11 , we derive that the system $\{(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}}), (\tilde{u}, \tilde{\varphi}, \tilde{W})\}$ is a martingale solution to (2.17) or (2.18) which satisfy all the items of Definition 3.1. This ends the proof of the existence theorem. □

5. Exponential Decay of the Weak Solution

In this section, we will prove that any weak solution $(\tilde{u}, \tilde{\varphi})$ to (2.17) or (2.18) converges to zero exponentially in the mean square. So in the rest of this section, we will assume the existence of such solution.

We first note from (2.20) that

$$\langle N(\tilde{u}), \tilde{u} \rangle_{V_{\mathrm{div},p}} = \int_{\mathcal{M}} \boldsymbol{T}(D\tilde{u}).\, D\tilde{u} \, dx \geq c_1 \int_{\mathcal{M}} \left(|D\tilde{u}|^2 + |D\tilde{u}|^p \right) dx.$$

Owing to Korn's inequalities, we infer from this previous inequality that

$$\langle N(\tilde{u}), \tilde{u} \rangle_{V_{\mathrm{div},p}} \geq \tilde{c}_1 (\|\tilde{u}\|^2_{1,2} + \|\tilde{u}\|^p_{1,p}) \geq \tilde{c}_1 \tilde{c}_2 |\tilde{u}|^2,$$

where $\tilde{c}_2$ is the constant in Poincaré's inequality. Setting $\tilde{c}_3 = \tilde{c}_1 \tilde{c}_2$, then

$$\langle N(\tilde{u}), \tilde{u} \rangle_{V_{\mathrm{div},p}} \geq \tilde{c}_3 |\tilde{u}|^2. \tag{5.1}$$

We also remark that by setting $\langle \tilde{\mu} \rangle = \frac{1}{|\mathcal{M}|} \int_{\mathcal{M}} \tilde{\mu} dx$ and since the mean of $\tilde{\varphi}$ is zero (cf. Remark 2.1), we have

$$(\tilde{\mu}, \tilde{\varphi}) = (\tilde{\mu} - \langle \tilde{\mu} \rangle, \tilde{\varphi}) \leq C_p |\nabla \tilde{\mu}| |\tilde{\varphi}|, \tag{5.2}$$

where C_p is the Poincaré–Wirtinger constant.

To prove exponential stability, furthermore we assume that the constant c_5 is such that

$$\frac{c_5}{2} \geq |J|_{L^1(\mathbb{R}^3)}. \tag{5.3}$$

Theorem 5.1. *We assume that $g_1 = 0$, $F'(0) = 0$ and there exists a constant $\zeta > 0$ such that*

$$\|g_2(t, \tilde{u}(t), \tilde{\varphi}(t))\|^2_{L_2(K,G_{\mathrm{div}})} \leq \gamma(t) + (\zeta + \delta(t)) \left(|\tilde{u}(t)|^2 + |\tilde{\varphi}(t)|^2\right), \tag{5.4}$$

where $\gamma(t)$ and $\delta(t)$ are nonnegative integrable functions such that there exist real numbers $\theta > 0$, $M_\gamma \geq 1$, $M_\delta \geq 1$ with

$$\gamma(t) \leq M_\gamma e^{-\theta t}, \quad \delta(t) \leq M_\delta e^{-\theta t}, \quad t \geq 0. \tag{5.5}$$

We also suppose that there exist positive constants c_{g_1}, M_α, M_β and two integrable functions $\alpha(.)$ and $\beta(.)$ satisfying

$$0 < \alpha(t) \leq M_\alpha e^{-\theta t}, \quad 0 < \beta(t) \leq M_\beta e^{-\theta t}, \tag{5.6}$$

and

$$\langle g_1(\tilde{u}(t), \tilde{\varphi}(t)), \tilde{u}(t)\rangle_{V_{\mathrm{div},p}} \leq \alpha(t) + (c_{g_1} + \beta(t)) \left(|\tilde{u}(t)|^2 + |\tilde{\varphi}(t)|^2\right), \tag{5.7}$$

for any $t \geq 0$ and $(\tilde{u}, \tilde{\varphi}) \in \mathbb{H}$.

Furthermore, we assume that $F''(\varphi_0) \in L^2(\mathcal{M})$ and we suppose that

$$2\tilde{c}_3 > 2c_{g_1} + \zeta \quad \text{and} \quad \frac{2(c_5 - |J|_{L^1(\mathbb{R}^3)})}{C_p^2} > \frac{2c_{g_1} + \zeta}{(c_5 - |J|_{L^1(\mathbb{R}^3)})}. \tag{5.8}$$

Then any weak solution $(\tilde{u}(t), \tilde{\varphi}(t))$ to (2.17) or (2.18) converges to zero exponentially in the mean square. That is, there exist real numbers $b \in (0, \theta)$, $M_0 = M_0(u_0, \varphi_0) > 0$ such that

$$\tilde{\mathbb{E}}\|(\tilde{u}(t), \tilde{\varphi}(t))\|^2_{\mathbb{H}} \leq M_0 e^{-bt}, \quad t \geq 0.$$

Proof. We recall that in Theorem 3.1, we have proved that the process $(\tilde{u}, \tilde{\varphi}, \tilde{W})$ is a weak martingale solution of problem (2.17) in the sense of Definition 3.1. Now from $(3.1)_1$ and Itô's formula (see, for instance, [41, Theorem I. 3. 3. 2, page 147]) we obtain

$$\begin{aligned} |\tilde{u}(t)|^2 &= |u_0|^2 - 2\int_0^t \langle N(\tilde{u}(s)) - R_1(\tilde{\varphi}(s)) - g_1(\tilde{u}(s), \tilde{\varphi}(s)), \tilde{u}(s)\rangle \, ds \\ &\quad + \int_0^t \|g_2(s, \tilde{u}(s), \tilde{\varphi}(s))\|^2_{L_2(K,G_{\mathrm{div}})} ds + 2\int_0^t (g_2(s, \tilde{u}(s), \tilde{\varphi}(s)), \tilde{u}(s)) \, d\tilde{W}(s), \end{aligned} \tag{5.9}$$

where we have also used the fact that $\langle B_0(\tilde{u}(s)), \tilde{u}(s)\rangle = -b_0(\tilde{u}(s), \tilde{u}(s), \tilde{u}(s)) = 0$. Here $\langle ., . \rangle$ denotes the dual pairing between $V_{\mathrm{div},p}$ and $V'_{\mathrm{div},p}$ relative to G_{div} .

Thanks to $(3.1)_2$, we have

$$(\tilde{\varphi}_t(t), \tilde{\mu}(t)) = -|\nabla \tilde{\mu}(t)|^2 + (B_1(\tilde{u}(t), \tilde{\varphi}(t)), \tilde{\mu}(t)) \quad t \in [0, T].$$

Note that since $F'(0) = 0$ and $\frac{d}{dt}(\int_{\mathcal{M}} F(0, x)dx) = 0$, we have

$$\begin{aligned} (\tilde{\varphi}_t(t), \tilde{\mu}(t)) &= (\tilde{\varphi}_t(t), a\tilde{\varphi}(t) - J * \tilde{\varphi}(t) + F'(\tilde{\varphi}(t))) \\ &= \frac{d}{dt}\left\{\frac{1}{2}|\sqrt{a}\tilde{\varphi}(t)|^2 - \frac{1}{2}(J * \tilde{\varphi}(t), \tilde{\varphi}(t)) + \int_{\mathcal{M}} [F(\tilde{\varphi}(t, x)) - F(0) - F'(0)\tilde{\varphi}(t, x)]dx\right\}. \end{aligned} \tag{5.10}$$

Using Taylor's formula, we have

$$\int_{\mathcal{M}} [F(\tilde{\varphi}(t, x)) - F(0) - F'(0)\tilde{\varphi}(t, x)]dx = \frac{1}{2}\int_{\mathcal{M}} F''(\xi\tilde{\varphi}(t, x))(\tilde{\varphi}(t, x))^2 dx$$

for some $0 < \xi < 1$. Thus, from (5.10), we get

$$\begin{aligned} (\tilde{\varphi}_t(t), \tilde{\mu}(t)) &= \frac{1}{2}\frac{d}{dt}\left\{|\sqrt{a}\tilde{\varphi}(t)|^2 - (J * \tilde{\varphi}(t), \tilde{\varphi}(t)) + \int_{\mathcal{M}} F''(\xi\tilde{\varphi}(t, x))(\tilde{\varphi}(t, x))^2 dx\right\} \\ &= \frac{1}{2}\frac{d}{dt}\left\{\int_{\mathcal{M}} (a(x) + F''(\xi\tilde{\varphi}(t, x)))\, (\tilde{\varphi}(t, x))^2 dx - (J * \tilde{\varphi}(t), \tilde{\varphi}(t))\right\}. \end{aligned}$$

Therefore, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\{\int_{\mathcal{M}}\left(a(x)+F''(\xi\tilde{\varphi}(t,x))\right)(\tilde{\varphi}(t,x))^2dx-(J*\tilde{\varphi}(t),\tilde{\varphi}(t))\right\}\\ =-|\nabla\tilde{\mu}(t)|^2+(B_1(\tilde{u}(t),\tilde{\varphi}(t)),\tilde{\mu}(t)).\end{aligned}\tag{5.11}$$

Now integrating (5.11) between 0 and t, multiplying the resulting equality by 2 and adding it to (5.9), we obtain

$$\begin{aligned}&|\tilde{u}(t)|^2+\int_{\mathcal{M}}(a(x)+F''(\xi\tilde{\varphi}(t,x)))(\tilde{\varphi}(t,x))^2dx-(J*\tilde{\varphi}(t),\tilde{\varphi}(t))+2\int_0^t[\langle N(\tilde{u}),\tilde{u}\rangle+|\nabla\tilde{\mu}|^2]ds\\ &=|u_0|^2+\int_{\mathcal{M}}\left(a(x)+F''(\xi\varphi_0(x))(\varphi_0(x))^2dx-(J*\varphi_0,\varphi_0)\right)+2\int_0^t\langle g_1(\tilde{u},\tilde{\varphi}),\tilde{u}\rangle\,ds\\ &\quad+\int_0^t\|g_2(s,\tilde{u},\tilde{\varphi})\|^2_{L_2(K,G_{\text{div}})}ds+2\int_0^t(g_2(s,\tilde{u},\tilde{\varphi}),\tilde{u})\,d\tilde{W}(s).\end{aligned}\tag{5.12}$$

Since (5.8) is satisfied, we can choose a constant $b\in(0,\theta)$ such that

$$2\tilde{c}_3>2c_{g_1}+\zeta+b\ \text{ and }\ \frac{2(c_5-|J|_{L^1(\mathbb{R}^3)})}{C_p^2}\geq\frac{\zeta+2c_{g_1}}{c_5-|J|_{L^1(\mathbb{R}^3)}}+b.\tag{5.13}$$

Hence, applying again the Itô formula to the real process $e^{bt}\left[|\tilde{u}(t)|^2+\left\{\int_{\mathcal{M}}\left(a(x)+F''(\xi\tilde{\varphi}(t,x))\right)(\tilde{\varphi}(t,x))^2dx-(J*\tilde{\varphi}(t),\tilde{\varphi}(t))\right\}\right]$, using (5.12) and since the mathematical expectation of the stochastic integral vanishes, we obtain

$$\begin{aligned}&e^{bt}\tilde{\mathbb{E}}\left[|\tilde{u}(t)|^2+\left\{\int_{\mathcal{M}}\left(a(x)+F''(\xi\tilde{\varphi}(t,x))\right)(\tilde{\varphi}(t,x))^2dx-(J*\tilde{\varphi}(t),\tilde{\varphi}(t))\right\}\right]\\ &\quad+2\tilde{\mathbb{E}}\int_0^te^{bs}\langle N(\tilde{u}(s)),\tilde{u}(s)\rangle\,ds+2\tilde{\mathbb{E}}\int_0^te^{bs}|\nabla\tilde{\mu}(s)|^2ds\\ &=|u_0|^2+\left\{\int_{\mathcal{M}}\left(a(x)+F''(\xi\varphi_0(x))(\varphi_0(x))^2dx-(J*\varphi_0,\varphi_0)\right)\right\}\\ &\quad+2\tilde{\mathbb{E}}\int_0^te^{bs}\langle g_1(\tilde{u}(s),\tilde{\varphi}(s)),\tilde{u}(s)\rangle\,ds+\tilde{\mathbb{E}}\int_0^te^{bs}\|g_2(s,\tilde{u}(s),\tilde{\varphi}(s))\|^2_{L_2(K,G_{\text{div}})}ds\\ &\quad+b\tilde{\mathbb{E}}\int_0^te^{bs}\left[|\tilde{u}(s)|^2+\left\{\int_{\mathcal{M}}\left(a+F''(\xi\tilde{\varphi}(s,x))\right)(\tilde{\varphi}(s,x))^2dx-(J*\tilde{\varphi}(s),\tilde{\varphi}(s))\right\}\right]ds.\end{aligned}\tag{5.14}$$

Using Assumption (H_4), (5.3) and Young's inequality for convolutions, we obtain

$$\textstyle\int_{\mathcal{M}}\left(a+F''(\xi\tilde{\varphi})\right)\tilde{\varphi}^2dx\geq c_5|\tilde{\varphi}|^2\geq|J|_{L^1(\mathbb{R}^3)}|\tilde{\varphi}|^2\geq(J*\tilde{\varphi},\tilde{\varphi}).$$

Therefore, we have

$$\textstyle\int_{\mathcal{M}}\left(a+F''(\xi\tilde{\varphi})\right)\tilde{\varphi}^2dx-(J*\tilde{\varphi},\tilde{\varphi})\geq0.\tag{5.15}$$

Using the Taylor series expansion, Assumption (H_4) and the fact that $F'(0)=0$, we see that

$$\begin{aligned}(\tilde{\mu},\tilde{\varphi})=(a\tilde{\varphi}-J*\tilde{\varphi}+F'(\tilde{\varphi}),\tilde{\varphi})&=(a\tilde{\varphi}-J*\tilde{\varphi}+F'(\tilde{\varphi})-F'(0),\tilde{\varphi})\\ &=(a\tilde{\varphi}+F''(\xi\tilde{\varphi})\tilde{\varphi},\tilde{\varphi})-(J*\tilde{\varphi},\tilde{\varphi})\\ &\geq(c_5-|J|_{L^1(\mathbb{R}^3)})|\tilde{\varphi}|^2,\end{aligned}\tag{5.16}$$

for some $0<\xi<1$ and where we have also used the Young inequality for convolutions. From the above relation in conjunction with (5.2), it follows that

$$|\tilde{\varphi}|\leq\delta_1|\nabla\tilde{\mu}|,\tag{5.17}$$

where $\delta_1=\frac{C_p}{[c_5-|J|_{L^1(\mathbb{R}^3)}]}$. Using Cauchy–Schwarz's inequality, (5.2) and (5.17), one has

$$(a\tilde{\varphi}+F''(\xi\tilde{\varphi})\tilde{\varphi},\tilde{\varphi})-(J*\tilde{\varphi},\tilde{\varphi})=(\tilde{\mu},\tilde{\varphi})\leq\delta_1C_p|\nabla\tilde{\mu}|^2.\tag{5.18}$$

Thanks to Hölder's inequality, Young's inequality for convolutions and (2.22), we obtain

$$\begin{aligned} &\int_{\mathcal{M}} \left(a(x) + F''(\xi\varphi_0(x))(\varphi_0(x))^2 dx - (J * \varphi_0, \varphi_0)\right) \\ &\quad \le \left(|a|_{L^\infty(\mathcal{M})} + |J|_{L^1(\mathbb{R}^3)} +\right) |\varphi_0|^2 + |F''(\xi\varphi_0)|_{L^2(\mathcal{M})} |\varphi_0|_{L^\infty(\mathcal{M})} |\varphi_0| \\ &\quad \le 2|J|_{L^1(\mathbb{R}^3)} |\varphi_0|^2 + |F''(\xi\varphi_0)|_{L^2(\mathcal{M})} |\varphi_0|_{L^\infty(\mathcal{M})} |\varphi_0|, \end{aligned} \tag{5.19}$$

where we used the fact that $\varphi_0 \in H^2(\mathcal{M}) \hookrightarrow L^\infty(\mathcal{M})$ and that $|a|_{L^\infty(\mathcal{M})} \le |J|_{L^1(\mathbb{R}^3)}$.

Inserting now these estimates (5.18)–(5.19) in (5.14), using also (5.17), (5.7), (5.4) and (5.1), we get

$$\begin{aligned} &e^{bt}\tilde{\mathbb{E}}\left[|\tilde{u}(t)|^2 + \left\{\int_{\mathcal{M}} (a(x) + F''(\xi\tilde{\varphi}(t,x)))\,(\tilde{\varphi}(t,x))^2 dx - (J * \tilde{\varphi}(t), \tilde{\varphi}(t))\right\}\right] \\ &\quad + 2\tilde{c}_3\tilde{\mathbb{E}}\int_0^t e^{bs}|\tilde{u}(s)|^2 ds + 2\tilde{\mathbb{E}}\int_0^t e^{bs}|\nabla\tilde{\mu}(s)|^2 ds \\ &\quad \le |u_0|^2 + 2|J|_{L^1(\mathbb{R}^3)}|\varphi_0|^2 + |F''(\xi\varphi_0)|_{L^2(\mathcal{M})}|\varphi_0|_{L^\infty(\mathcal{M})}|\varphi_0| \\ &\quad + \int_0^t (2M_\alpha + M_\gamma)\, e^{(b-\theta)s} ds + (2c_{g_1} + \zeta + b)\tilde{\mathbb{E}}\int_0^t e^{bs}|\tilde{u}(s)|^2 ds \\ &\quad + [\tilde{c}_4 + \tilde{c}_5 b]\tilde{\mathbb{E}}\int_0^t e^{bs}|\nabla\tilde{\mu}(s)|^2 ds + \tilde{\mathbb{E}}\int_0^t (2M_\beta + M_\delta)\, e^{(b-\theta)s}\|(\tilde{u}(s), \tilde{\varphi}(s))\|_{\mathbb{H}}^2 ds, \end{aligned} \tag{5.20}$$

where $\tilde{c}_4 = (2c_{g_1} + \zeta)\delta_1^2$, $\tilde{c}_5 = \delta_1 C_p$ and δ_1 is given by (5.17). Hence, from (5.13) and (5.20) we have

$$\begin{aligned} &e^{bt}\tilde{\mathbb{E}}\left[|\tilde{u}(t)|^2 + \left\{\int_{\mathcal{M}} (a(x) + F''(\xi\tilde{\varphi}(t,x)))\,(\tilde{\varphi}(t,x))^2 dx - (J * \tilde{\varphi}(t), \tilde{\varphi}(t))\right\}\right] \\ &\quad \le |u_0|^2 + 2|J|_{L^1(\mathbb{R}^3)}|\varphi_0|^2 + |F''(\xi\varphi_0)|_{L^2(\mathcal{M})}|\varphi_0|_{L^\infty(\mathcal{M})}|\varphi_0| \\ &\quad + \int_0^t (2M_\alpha + M_\gamma)\, e^{(b-\theta)s} ds + \tilde{\mathbb{E}}\int_0^t (2M_\beta + M_\delta)\, e^{(b-\theta)s}\|(\tilde{u}(s), \tilde{\varphi}(s))\|_{\mathbb{H}}^2 ds. \end{aligned} \tag{5.21}$$

By using (5.16) and (5.21), we can easily see that

$$\begin{aligned} e^{bt}\tilde{\mathbb{E}}\|(\tilde{u}(t), \tilde{\varphi}(t))\|_{\mathbb{H}}^2 &\le \frac{|u_0|^2}{\delta_2} + 2|J|_{L^1(\mathbb{R}^3)}\frac{|\varphi_0|^2}{\delta_2} + \frac{|\varphi_0|}{\delta_2}|\varphi_0|_{L^\infty(\mathcal{M})}|F''(\xi\varphi_0)|_{L^2(\mathcal{M})} \\ &\quad + \frac{1}{\delta_2}\int_0^t (2M_\alpha + M_\gamma)\, e^{(b-\theta)s} ds + \frac{1}{\delta_2}\tilde{\mathbb{E}}\int_0^t (2M_\beta + M_\delta)\, e^{(b-\theta)s}\|(\tilde{u}, \tilde{\varphi})\|_{\mathbb{H}}^2 ds, \end{aligned} \tag{5.22}$$

where $\delta_2 = \min(1, (c_5 - |J|_{L^1(\mathbb{R}^3)}))$. Now, by applying the deterministic Gronwall's lemma, we can infer the existence of $M_0 \equiv M_0(|u_0|^2 + 2|J|_{L^1(\mathbb{R}^3)}|\varphi_0|^2 + |F''(\xi\varphi_0)|_{L^2(\mathcal{M})}|\varphi_0|_{L^\infty(\mathcal{M})}|\varphi_0|)$ such that

$$\tilde{\mathbb{E}}\|(\tilde{u}(t), \tilde{\varphi}(t))\|_{\mathbb{H}}^2 \le M_0 e^{-bt} \quad \text{for all} \quad t > 0.$$

This completes the proof of Theorem 5.1. □

Compliance with Ethical Standards

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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