J. Math. Fluid Mech. (2020) 22:12

1422-6928/20/010001-13
https://doi.org/10.1007/s00021-019-0468-8

Journal of Mathematical
Fluid Mechanics

Global Smooth Axisymmetic Solutions of the Boussinesq Equations for Magnetohydrodynamics Convection

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Communicated by G. Seregin

Abstract. In this paper, we consider global axisymmetric smooth solutions for the Boussinesq equation for magnetohydrodynamics convection without magnetic diffusion and heat convection. We obtain that for axially symmetric initial data without any smallness restrictions, such a system admits global smooth axially symmetric solutions without swirl.

Mathematics Subject Classification. 35Q35, 76D03.

Keywords. Boussinesq equation for magnetohydrodynamics convection, Axisymmetric solutions, Global regularity.

1. Introduction

In this paper, we consider the following three dimensional incompressible Boussinesq equations for magnetohydrodynamics (MHD) convection

$$u_t + u \cdot \nabla u + \nabla \pi = \mu \Delta u + B \cdot \nabla B + g\theta e_z, \tag{1.1a}$$

$$B_t + u \cdot \nabla B = B \cdot \nabla u, \tag{1.1b}$$

$$\theta_t + u \cdot \nabla \theta = 0, \tag{1.1c}$$

$$\nabla \cdot u = \nabla \cdot B = 0. \tag{1.1d}$$

Here $u = (u_1, u_2, u_3)$ is the velocity, π is the pressure, θ is the temperature fluctuation about a constant, and $B = (B_1, B_2, B_3)$ is the magnetic field, defined on $x \in \mathbb{R}^3$ and $t \in \mathbb{R}^+$. This system can be used to model the large scale cosmic magnetic fields that are maintained by hydromagnetic dynamos. Physically, the first equation describes the conservation law of the momentum with the effect of the buoyancy force, and the constant μ is the viscosity. Here $-ge_z$ denotes the direction of the gravity and the original form of the buoyancy term is $g(\theta - \theta_0)e_z$ with θ_0 denoting the temperature distribution of the reference state which can be absorbed in the pressure term and hence is assumed to be zero in this paper. The second equation shows that the electromagnetic field is governed by the Maxwell equation and the third equation describes the temperature fluctuation about a constant state. Here, we have omitted the magnetic diffusion and heat diffusion. For more physics details and numerical simulations, the interested readers may refer to [5,28,31,32] and the references therein. Hereafter, the system is referred to as the Boussinesq–MHD system or BMHD for short.

Global regularity of such a PDE system for large initial data is widely open even if when $\theta = B \equiv 0$. In this case, the system reduces to the 3D classical incompressible Navier–Stokes equation, whose global well-posedness is widely open for large initial data. But under axially symmetric assumptions, global well-posedness of classical solutions without swirl component of velocity field was solved by Ladyzhenskaya [20] and by Ukhovskii and Yudovich [33] independently. More precisely, the Navier–Stokes system has a unique global axisymmetric solution for initial data $u_0 \in H^1$ and vorticity ω_0 and $r^{-1}\omega_0 \in L^2 \cap L^\infty$, which can be guaranteed when $u_0 \in H^s$ with $s > 7/2$ in 3D. The initial regularity was weakened to

$u_0 \in H^2$ by Leonardi et al. [24] and to $u_0 \in H^{1/2}$ by Abidi [1] later on. The main observation is that under axisymmetric assumptions, the vorticity quantity $r^{-1}\omega$ has maximum principle and hence global regularity can be obtained. See also [17] for the global regularity of the axisymmetric Navier–Stokes equation with swirl for a class of large anisotropic initial data. If furthermore, we let the viscosity μ to be zero, the Navier–Stokes equation reduces to the standard 3D Euler equation describing the motion of an ideal incompressible fluid, whose global in time regularity is a long standing open problem due to possible vortex stretching [4,27]. To gain insight into this challenging problem, many authors turn to study the 2D Boussinesq equation, i.e., the system (1.1) without magnetic field B, which retains some key features of the 3D Euler or Navier–Stokes equations.

When the magnetic field $B \equiv 0$, the BMHD system reduces to the Boussinesq system, whose weak solutions in L^p was studied in $\mathbb{R}^n$ by Cannon and DiBenedetto [10] for general spatial dimensions and the local well-posedness in Sobolev spaces was obtained By Chae and Nam in [13] in $\mathbb{R}^2$. In the 2D case, many global well-posedness results are obtained under various viscous conditions. For example, see [12,18] for global well-posedness in the partial viscosity case and [21] for the initial boundary value problem. See also [2] for global well-posedness in the 2D case with partial vsicosity in Besov spaces. For the 3D case, when the initial data is axisymmetric, global well-posedness was shown by Abidi et al. [3], under the assumption that the initial density/temperature θ_0 does not intersect the Z-axis and the orthogonal projection of the support of θ_0 to the Z-axis is compact.

When the temperature θ vanishes, the BMHD system reduces to the well-known MHD system, for which there are lots of important results up to date. Concerning the local well-posedness, one may refer to the paper of Sermange and Temam [30] in the case of fully viscosity, where the authors also proved the global well-posedness in the 2D case. Global existence of classical solutions is obtained by Lin et al. [26] under smallness conditions in Sobolev spaces of the initial velocity field and the displacement of the magnetic field from a non-zero constant. See also [11,19,29,36] and the references therein for global well-posedness under different conditions. For partial regularity and various blowup conditions, one may refer to [9,14–16,23] and the references therein. For global well-posedness in the 3D case, Lin et al. [25,35] studied global well-posedness of small solutions for MHD-type solutions. For a class of axisymmetric initial data, Lei [22] established the global well-posedness of classical solutions whose the swirl component of the velocity and magnetic vorticity vanish.

For the full BMHD system, there are some theoretical as well as numerical results up to date. Bian et al. [5–7] studied the global existence and uniqueness for the initial boundary value problem to the 2D stratified Boussinesq–MHD system without smallness assumptions on the initial data, with temperature-dependent viscosity, thermal diffusivity and electrical conductivity. But few results are known up to date about global well-posedness in the 3D case. In a recent paper [8], the authors proved a global well-posedness result for large initial data for the BMHD system with a nonlinear damping term, with both fluid viscosity and magnetic diffusion. However, it is not known whether global well-posedness holds without the nonlinear damping term even with full velocity viscosity, magnetic diffusion as well as heat diffusion. Numerically, Schrinner et al. [31,32] studied the global numerical simulations of rotating magnetoconvection and the geodynamo with mean-field description, where mean fields are defined by azimuthal averaging over all values of the zaimuthal coordinate and are axisymmetric about the polar axis. Both the theoretic difficulties and the numerical simulations motivate us to study the radial solutions or the axisymmetric solutions of such a system.

In this paper, we will show that the BMHD system (1.1) in $\mathbb{R}^3$ is globally well-posed for a class of large axially symmetric initial data without swirl, even if there is no magnetic diffusion and heat convection. The case when swirl is present will be pursued in short future. Before we state the main result, we introduce the axisymmetric solutions for the BMHD system (1.1).

Let $x = (x_1, x_2, z) \in \mathbb{R}^3$ and $r = \sqrt{x_1^2 + x_2^2}$. We define the axially symmetric coordinate system (e_r, e_ϕ, e_z) by

$$e_r = (x_1/r, x_2/r, 0)^\top, \quad e_\phi = (-x_2/r, x_1/r, 0)^\top, \quad e_z = (0,0,1)^\top,$$

where ϕ denotes the angle variable. Considering the BMHD system (1.1) in the axially symmetric coordinate (e_r, e_ϕ, e_z), but letting the unknowns depend only on the variables (t, r, z) and be independent of the angular variable ϕ, we can write

$$\begin{cases} u(t,x) = u^r(t,r,z)e_r + u^\phi(t,r,z)e_\phi + u^z(t,r,z)e_z, \\ B(t,x) = B^r(t,r,z)e_r + B^\phi(t,r,z)e_\phi + B^z(t,r,z)e_z, \\ \theta(t,x) = \theta(t,r,z), \quad \pi(t,x) = \pi(t,r,z). \end{cases} \tag{1.2}$$

Then the BMHD system (1.1) can be equivalently written in the axially symmetric coordinate (e_r, e_ϕ, e_z),

$$\begin{cases} \partial_t u^r + u^r\partial_r u^r + u^z\partial_z u^r - \frac{(u^\phi)^2}{r} + \partial_r \pi = \left(\Delta - \frac{1}{r^2}\right) u^r + B^r\partial_r B^r + B^z\partial_z B^r - \frac{(B^\phi)^2}{r}, \\ \partial_t u^\phi + u^r\partial_r u^\phi + u^z\partial_z u^\phi + \frac{u^r u^\phi}{r} = \left(\Delta - \frac{1}{r^2}\right) u^\phi + B^r\partial_r B^\phi + B^z\partial_z B^\phi + \frac{B^r B^\phi}{r}, \\ \partial_t u^z + u^r\partial_r u^z + u^z\partial_z u^z + \partial_z \pi = \Delta u^z + B^r\partial_r B^z + B^z\partial_z B^z + \theta, \\ \partial_t B^r + u^r\partial_r B^r + u^z\partial_z B^r = B^r\partial_r u^r + B^z\partial_z u^r, \\ \partial_t B^\phi + u^r\partial_r B^\phi + u^z\partial_z B^\phi + \frac{B^r u^\phi}{r} = B^r\partial_r u^\phi + B^z\partial_z u^\phi + \frac{u^r B^\phi}{r}, \\ \partial_t B^z + u^r\partial_r B^z + u^z\partial_z B^z = B^r\partial_r u^z + B^z\partial_z u^z, \\ \partial_t \theta + u^r\partial_r\theta + u^z\partial_z\theta = 0. \end{cases} \tag{1.3}$$

For such a system, it is not difficult to have the following local existence and uniqueness result.

Lemma 1.1. *Let $(u_0, B_0, \theta_0) \in H^2(\mathbb{R}^3)$ be axially symmetric and u_0 and B_0 are divergence free. Then there exists exactly one solution (u, B, θ, π) such that*

$$(u, B, \theta) \in L^\infty(0,T; H^2(\mathbb{R}^3)), \quad u \in L^2(0,T; H^3(\mathbb{R}^3)),$$
$$\left(\frac{\partial u}{\partial t}, \frac{\partial B}{\partial t}, \frac{\partial \theta}{\partial t}\right) \in L^2(0,t; H^1(\mathbb{R}^3)), \quad \nabla\pi \in L^\infty(0,T; L^2(\mathbb{R}^3)),$$

for some $T > 0$. Moreover, (u, B, θ, π) is axially symmetric.

The proof can be adapted from a similar local existence and uniqueness result for the incompressible Navier–Stokes equations in $\mathbb{R}^3$ in [24]. By uniqueness of local classical solutions, it is clear that if $u^\phi = B^r = B^z = 0$ for all later times if they vanish initially. In this case, we have the following simplified system for $(u^r, u^z, B^\phi, \theta)$:

$$\begin{cases} \partial_t u^r + u^r\partial_r u^r + u^z\partial_z u^r + \partial_r\pi = (\Delta - \frac{1}{r^2})u^r - \frac{(B^\phi)^2}{r}, \\ \partial_t u^z + u^r\partial_r u^z + u^z\partial_z u^z + \partial_z\pi = \Delta u^z + \theta, \\ \partial_t B^\phi + u^r\partial_r B^\phi + u^z\partial_z B^\phi = \frac{u^r B^\phi}{r}, \\ \partial_t\theta + u^r\partial_r\theta + u^z\partial_z\theta = 0. \end{cases} \tag{1.4}$$

In this case, the incompressible condition is equivalent to

$$\partial_r u^r + \frac{u^r}{r} + \partial_z u^z = 0,$$

and the divergence free condition is automatically satisfied since $B^r = B^z = 0$ for all times $t \geq 0$.

Let $\omega = \nabla \times u$ be the vorticity. Then it is computed that $\omega = \omega^\phi e^\phi$, where $\omega^\phi = \partial_z u^r - \partial_r u^z$. From the first two equations of (1.4), we have the following equation for ω^ϕ,

$$\partial_t\omega^\phi + u\cdot\nabla\omega^\phi - \frac{u^r}{r}\omega^\phi = \left(\partial_{rr} + \frac{1}{r}\partial_r + \partial_{zz} - \frac{1}{r^2}\right)\omega^\phi - \frac{2}{r}B^\phi\partial_z B^\phi - \partial_r\theta. \tag{1.5}$$

Further, let $\Pi = B^\phi/r$ and $\Omega = \omega^\phi/r$, the system (1.4) gives the following system

$$\begin{cases} \partial_t\Omega + u\cdot\nabla\Omega = (\Delta + \frac{2}{r}\partial_r)\Omega - \partial_z\Pi^2 - \frac{\partial_r\theta}{r}, \\ \partial_t\Pi + u\cdot\nabla\Pi = 0, \\ \partial_t\theta + u\cdot\nabla\theta = 0. \end{cases} \tag{1.6}$$

We also note that by definition of ω and Ω, there exists some ψ^ϕ such that

$$-\left(\Delta+\frac{2}{r}\partial_r\right)(r^{-1}\psi^\phi)=\Omega,\quad u=\nabla\times(\psi^\phi e_\phi).$$

The main result is stated in the following.

Theorem 1.1. *Suppose that u_0, B_0 and θ_0 are all axially symmetric and u_0, B_0 are divergence free vectors with $u_0^\phi = B_0^r = B_0^z = 0$. Moreover, we assume that $u_0, B_0 \in H^s(\mathbb{R}^3)$ with $s \geq 2$ and $r^{-1}B_0^\phi \in L^\infty(\mathbb{R}^3)$. Suppose also that $\theta_0 \in H^s(\mathbb{R}^3)$ with $s \geq 2$ such that $\operatorname{spt}\theta_0$, the support of θ_0, does not intersect the Z-axis and the projection of $\operatorname{spt}\theta_0$ to the Z-axis is compact. Then there exists a unique global solution to the system (1.1) with initial data (u_0, B_0, θ_0) that satisfies*

$$\|\nabla^s u(t)\|_{L^2}^2+\|\nabla^s B(t)\|_{L^2}^2+\|\nabla^s\theta(t)\|_{L^2}^2)+\mu\int_0^t\|\nabla^{s+1}u(\tau)\|_{L^2}^2d\tau\lesssim C(t),$$

for some $C(t) > 0$.

Remark 1.1. Here, we indeed assumed that $\theta_0 \in L^\infty(\mathbb{R}^3)$ thanks to Sobolev embedding. We also remark that as pointed out in [3], the assumption that $\operatorname{spt}\theta_0$ is away from the Z-axis can be relaxed to by assuming that θ is a constant c_0 near the Z-axis, by taking a change of variable $\bar\theta = \theta - c_0$ and $\bar\pi = \pi - c_0 z$. We will not go into the details of this point.

Compared to the MHD system considered in [22], we have an extra transport equation of the temperature and an extra singular term $r^{-1}\partial_r\theta$ in the momentum equation in (1.6). This singular term causes difficulties in estimating the $\|\Omega(t)\|_{L^2}$ in Lemma 2.3, due to the exponential growth of the quantity $\|r^{-1}\theta(t)\|_{L^2}$ in terms of $\int_0^t \|r^{-1}u^r\|_{L^\infty}d\tau$. More precisely, from (1.6), one has

$$\partial_t(r^{-1}\theta)+u\cdot\nabla(r^{-1}\theta)+(r^{-1}\theta)(r^{-1}u^r)=0,$$

which gives the estimate

$$\|r^{-1}\theta(t)\|_{L^p}\lesssim\|r^{-1}\theta_0\|_{L^p}e^{\int_0^t\|r^{-1}u^r\|_{L^\infty}d\tau}.$$

To avoid this difficulty, we assume as in [3] that $\operatorname{spt}\theta_0$ is away from Z-axis and its projection to Z-axis is compact, and this property is maintained due to the transport equation satisfied by θ in (1.6). Therefore, not involved in much technicalities, we assume that $\operatorname{spt}\theta_0$ is away from the Z-axis to avoid the singularities of last term $r^{-1}\partial_r\theta$ in (1.6) near $r = 0$.

In the next section, we will prove theorem 1.1. Throughout this paper, $A \lesssim B$ means there exists some constant $C > 0$ such that $A \leq CB$.

2. Proof of Theorem 1.1

2.1. Basic Estimates

From the Biot–Savart law, we have the following Lemma.

Lemma 2.1. *Let u be a smooth axisymmetric divergence free vector field and $\omega = \omega^\phi e_\phi$ be its curl, then*

$$\|u\|_{L^\infty}\lesssim\|\omega^\phi\|_{L^2}^{1/2}\|\nabla\omega^\phi\|_{L^2}^{1/2},$$
$$\|r^{-1}u^r\|_{L^\infty}\lesssim\|\Omega\|_{L^2}^{1/2}\|\nabla\Omega\|_{L^2}^{1/2}.$$

This lemma was proved in [3]. See also similar estimates in [22] in integral form.

2.2. The Flow Map

First, we cite the following proposition concerning the transport equation satisfied by the temperature θ, which was proved in [3].

Proposition 2.1. *Let u be a smooth axisymmetric vector field and θ be a solution of the transport equation*

$$\partial_t \theta + u \cdot \nabla \theta = 0, \tag{2.1}$$

with initial data $\theta(t=0) = \theta_0$.

(a) Assume that $d(\operatorname{spt}\theta_0, \{OZ\}) = r_0 > 0$. Then one has for every $t \geq 0$ that

$$d(\operatorname{spt}\theta(t), \{OZ\}) \geq r_0 e^{-\int_0^t \|r^{-1}u^r\|_{L^\infty} d\tau}.$$

(b) Denote by Π_z the orthogonal projector over the z-axis $\{OZ\}$, and assume that $\Pi_z(\operatorname{spt}\theta_0)$ is a compact set with diameter d_0. Then for every $t \geq 0$, one has $\Pi_z(\operatorname{spt}\theta(t))$ is a compact set with diameter $d(t)$ such that

$$d(t) \leq d_0 + 2\int_0^t \|u(\tau)\|_{L^\infty} d\tau.$$

With this proposition, one has the following Corollary, which was also proved in [3].

Corollary 2.1. *Let u be a smooth axisymmetric divergence free vector field, and θ be a solution of the transport equation (2.1) with initial data $\theta_0 \in L^2 \cap L^\infty$. Assume further that*

$$r_0 := d(\operatorname{spt}\theta_0, \{OZ\}) > 0, \quad d_0 := diam(\Pi_z(\operatorname{spt}\theta_0)) < \infty,$$

then we have

$$\begin{aligned}\int_{\mathbb{R}^3} r^{-2}\theta^2(t,x)dx \leq r_0^{-2}\|\theta_0\|_{L^2}^2 \\ +2\pi\|\theta_0\|_{L^\infty}^2 \int_0^t \|r^{-1}u^r(\tau)\|_{L^\infty} d\tau \left(d_0 + 2\int_0^t \|u(\tau)\|_{L^\infty} d\tau\right).\end{aligned} \tag{2.2}$$

2.3. Energy Estimates

Here, we first give some L^2-estimates for the solutions of the Boussinesq system (1.1).

Lemma 2.2. *Let $u_0, B_0 \in L^2$ be divergence free, $\theta_0 \in L^2 \cap L^\infty$. Then for every smooth solution (u, B, θ), it holds that,*

$$\begin{aligned}&\|\theta(t)\|_{L^p} \leq \|\theta_0\|_{L^p}, \quad \forall p \in [1,\infty], \\ &\|\Pi(t)\|_{L^p} \leq \|\Pi_0\|_{L^p}, \quad \forall p \in [1,\infty], \\ &\|u(t)\|_{L^2}^2 + \|B(t)\|_{L^2}^2 + \int_0^t \|\nabla u(\tau)\|_{L^2}^2 d\tau \lesssim (1+t)e^t.\end{aligned} \tag{2.3}$$

Proof. The first two inequalities are standard. Since u is divergence free, by taking L^2-estimates for the first two equations, one has

$$\frac{1}{2}\frac{d}{dt}\|u(t), B(t)\|_{L^2}^2 + \mu\|\nabla u(\tau)\|_{L^2}^2 \leq \|u(t)\|_{L^2}\|\theta(t)\|_{L^2} \lesssim 1 + \|u(t)\|_{L^2}^2, \tag{2.4}$$

which implies immediately that

$$\|u(t), B(t)\|_{L^2}^2 + \mu\int_0^t \|\nabla u(\tau)\|_{L^2}^2 d\tau \leq e^t(\|u_0, B_0\|_{L^2}^2 + t) \lesssim (1+t)e^t, \tag{2.5}$$

thanks to the Gronwall inequality. □

Next, we give some estimates for ω^ϕ and $\Omega = \omega^\phi/r$.

Lemma 2.3. *Suppose that (u, B, θ) is a smooth solution of the Boussinesq–MHD system (1.6) with initial data $(u_0, B_0, \theta_0) \in H^2$, which satisfies the conditions of Theorem 1.1. Then there holds,*

$$\|\omega^\phi(t)\|_{L^2}^2 + \int_0^t \|\nabla\omega^\phi(\tau)\|_{L^2}^2 + \|r^{-1}\omega^\phi\|_{L^2}^2 d\tau \lesssim C(t),$$

and

$$\|\Omega(t)\|_{L^2}^2 + \int_0^t \|\nabla\Omega(\tau)\|_{L^2}^2 d\tau + 4\pi\int_0^t\int_{\mathbb{R}} |\Omega(\tau,0,z)|^2 dz d\tau \lesssim C(t),$$

for some constant $C(t)$.

Proof. (i). Recall the equation (1.5) for ω^ϕ. Take the L^2-inner product with ω^ϕ to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\omega^\phi(t)\|_{L^2}^2 = &\int \omega^\phi\left(\partial_{rr} + \frac{1}{r}\partial_r + \partial_{zz} - \frac{1}{r^2}\right)\omega^\phi dx - \int \omega^\phi (u\cdot\nabla\omega^\phi)dx \\ &+ \int \frac{u^r}{r}|\omega^\phi|^2 dx - \int 2r\Pi\partial_z\Pi\omega^\phi dx - \int \omega^\phi\partial_r\theta dx.\end{aligned}$$

The first integral on the RHS equals to

$$\begin{aligned}-\int \omega^\phi\left(\partial_{rr} + \frac{1}{r}\partial_r + \partial_{zz} - \frac{1}{r^2}\right)\omega^\phi dx &= \|\partial_r\omega^\phi\|_{L^2}^2 + \|\partial_z\omega^\phi\|_{L^2}^2 + \|r^{-1}\omega^\phi\|_{L^2}^2 \\ &= \|\nabla\omega^\phi\|_{L^2}^2 + \|r^{-1}\omega^\phi\|_{L^2}^2.\end{aligned}$$

The second integral vanishes, and the third and fourth terms can be estimated as

$$\begin{aligned}\left|\int \frac{u^r}{r}|\omega^\phi|^2 dx\right| \le &\|u^r\|_{L^6}\|\omega^\phi\|_{L^3}\|\Omega\|_{L^2} \le C\|u^r\|_{L^6}\|\omega^\phi\|_{L^2}^{1/2}\|\nabla\omega^\phi\|_{L^2}^{1/2}\|\Omega\|_{L^2} \\ \le &\frac{1}{4}\|\nabla\omega^\phi\|_{L^2}^2 + C\|\omega^\phi\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2\|\Omega\|_{L^2}^2,\end{aligned}$$

and

$$\begin{aligned}\left|\int 2r\Pi\partial_z\Pi\omega^\phi dx\right| = \left|\int B^\phi\Pi\partial_z\omega^\phi dx\right| &\le \|\Pi\|_{L^\infty}\|B^\phi\|_{L^2}\|\partial_z\omega^\phi\|_{L^2} \\ &\le \|\Pi_0\|_{L^\infty}\|B^\phi\|_{L^2}\|\partial_z\omega^\phi\|_{L^2} \le C(1+t)e^t + \frac{1}{4}\|\partial_z\omega^\phi\|_{L^2}^2,\end{aligned}$$

where we have used Lemma 2.2. For the last integral, it follows from integration by parts that

$$\begin{aligned}\left|\int \omega^\phi\partial_r\theta dx\right| = \left|2\pi\int \omega^\phi\partial_r\theta r dr dz\right| &\le \left|2\pi\int \theta\Omega r dr dz\right| + \left|2\pi\int \theta\partial_r\omega^\phi r dr dz\right| \\ &\le \|\theta\|_{L^2}\|\Omega\|_{L^2} + \|\theta\|_{L^2}^2 + \frac{1}{4}\|\partial_r\omega^\phi\|_{L^2}^2 \\ &\le C\|\Omega\|_{L^2}^2 + C\|\theta_0\|_{L^2}^2 + \frac{1}{4}\|\partial_r\omega^\phi\|_{L^2}^2.\end{aligned}$$

Therefore, we have

$$\frac{d}{dt}\|\omega^\phi(t)\|_{L^2}^2 + \|\nabla\omega^\phi\|_{L^2}^2 + \|r^{-1}\omega^\phi\|_{L^2}^2 \lesssim (1+t)e^t + \|\omega^\phi\|_{L^2}^2 + (1+\|\nabla u\|_{L^2}^2)\|\Omega\|_{L^2}^2.$$

Integrating over $[0, t]$, one has

$$\begin{aligned}&\|\omega^\phi(t)\|_{L^2}^2 + \int_0^t \|\nabla\omega^\phi\|_{L^2}^2 + \|r^{-1}\omega^\phi\|_{L^2}^2 d\tau \\ &\lesssim te^t + \int_0^t \|\omega^\phi(\tau)\|_{L^2}^2 d\tau + \int_0^t (1+\|\nabla u(\tau)\|_{L^2}^2)\|\Omega(\tau)\|_{L^2}^2 d\tau.\end{aligned} \tag{2.6}$$

(ii). On the other hand, from the transport equation for Π in (1.6), we have for any $\Pi_0 \in L^2 \cap L^\infty$

$$\|\Pi(t)\|_{L^p} \le \|\Pi_0\|_{L^p}, \quad \forall\, p \in [2, \infty]. \tag{2.7}$$

Note also that

$$|\nabla B|^2 = |(e_r\partial_r + \frac{1}{r}e_\phi\partial_\phi + e_z\partial_z)(B^\phi e_\phi)|^2 = |\nabla B^\phi|^2 + |\Pi|^2.$$

In particular, one has

$$\|\Pi(t)\|_{L^2} \le \|\Pi_0\|_{L^2} \lesssim \|B_0\|_{H^1}, \qquad \|\Pi(t)\|_{L^4} \le \|\Pi_0\|_{L^4} \le \|\nabla B_0\|_{L^4} \lesssim \|B_0\|_{H^2}.$$

(iii). Taking L^2-inner product of the equation for Ω in (1.6), we obtain

$$\frac{1}{2}\frac{d}{dt}\|\Omega(t)\|_{L^2}^2 = \int \Omega(\Delta + \frac{2}{r}\partial_r)\Omega dx - \int \Omega(u\cdot\nabla\Omega)dx - \int \Omega\partial_z\Pi^2 dx - \int \Omega\frac{\partial_r\theta}{r}dx.$$

For the terms on the RHS, it follows from integration by parts that

$$\begin{aligned} -\int \Omega(\Delta + \frac{2}{r}\partial_r)\Omega dx &= \|\nabla\Omega\|_{L^2}^2 + 2\pi\int_{\mathbb{R}}|\Omega(t,0,z)|^2 dz \\ &= \|\partial_r\Omega\|_{L^2}^2 + \|\partial_z\Omega\|_{L^2}^2 + 2\pi\int_{\mathbb{R}}|\Omega(t,0,z)|^2 dz, \\ \int \Omega(u\cdot\nabla\Omega)dx &= 0, \\ \left|\int \Omega\partial_z\Pi^2 dx\right| &\le \|\Pi\|_{L^4}^2\|\partial_z\Omega\|_{L^2} \le \frac{1}{2}\|\Pi\|_{L^4}^4 + \frac{1}{2}\|\partial_z\Omega\|_{L^2}^2, \\ \left|\int \Omega\frac{\partial_r\theta}{r}dx\right| &= \left|2\pi\int \Omega\partial_r\theta drdz\right| = \left|2\pi\int \partial_r\Omega\frac{\theta}{r}rdrdz\right| \le \frac{1}{2}\|\theta/r\|_{L^2}^2 + \frac{1}{2}\|\partial_r\Omega\|_{L^2}^2. \end{aligned}$$

Therefore, we have

$$\frac{d}{dt}\|\Omega(t)\|_{L^2}^2 + \|\nabla\Omega\|_{L^2}^2 + 4\pi\int_{\mathbb{R}}|\Omega(t,0,z)|^2 dz \le \|\theta/r\|_{L^2}^2 + \|B_0\|_{H^2}^4.$$

Integrating over $[0,t]$, one then has

$$\begin{aligned} &\|\Omega(t)\|_{L^2}^2 + \int_0^t \|\nabla\Omega(\tau)\|_{L^2}^2 d\tau + 4\pi\int_0^t\int_{\mathbb{R}}|\Omega(\tau,0,z)|^2 dz d\tau \\ &\le \|\Omega(0)\|^2 + \int_0^t \|\theta/r(\tau)\|_{L^2}^2 d\tau + t\|B_0\|_{H^2}^4 \\ &\lesssim 1 + t + t\int_0^t \|r^{-1}u^r(\tau)\|_{L^\infty}d\tau + t\int_0^t \|r^{-1}u^r(\tau)\|_{L^\infty}d\tau\int_0^t \|u(\tau)\|_{L^\infty}d\tau \\ &\lesssim 1 + t + t\int_0^t \|r^{-1}u^r(\tau)\|_{L^\infty}d\tau + t^{3/2}\int_0^t \|r^{-1}u^r(\tau)\|_{L^\infty}d\tau\left(\int_0^t \|u(\tau)\|_{L^\infty}^2 d\tau\right)^{1/2} \\ &\lesssim 1 + t + t\int_0^t \|r^{-1}u^r(\tau)\|_{L^\infty}d\tau + t^3\left(\int_0^t \|r^{-1}u^r(\tau)\|_{L^\infty}d\tau\right)^2 + \int_0^t \|u(\tau)\|_{L^\infty}^2 d\tau, \end{aligned} \tag{2.8}$$

where in the above inequality, we have used Corollary 2.1. For the first two integrals, we use Hölder and Young's inequalities and Lemma 2.1 to obtain

$$\begin{aligned} t^\alpha\int_0^t \|r^{-1}u^r(\tau)\|_{L^\infty}d\tau &\le t^\alpha\int_0^t \|\Omega(\tau)\|_{L^2}^{1/2}\|\nabla\Omega(\tau)\|_{L^2}^{1/2}d\tau, \\ &\le t^{\alpha+\frac{1}{2}}\left(\int_0^t \|\Omega(\tau)\|_{L^2}^2 d\tau\right)^{1/4}\left(\int_0^t \|\nabla\Omega(\tau)\|_{L^2}^2 d\tau\right)^{1/4}. \end{aligned}$$

Therefore, when $\alpha = 1$, we obtain

$$
\begin{aligned}
t\int_0^t \|r^{-1}u^r(\tau)\|_{L^\infty} d\tau &\le t^{3/2}\left(\int_0^t \|\Omega(\tau)\|_{L^2}^2 d\tau\right)^{1/4}\left(\int_0^t \|\nabla\Omega(\tau)\|_{L^2}^2 d\tau\right)^{1/4} \\
&\lesssim t^2\left(\int_0^t \|\Omega(\tau)\|_{L^2}^2 d\tau\right)^{1/3} + \frac{1}{4}\int_0^t \|\nabla\Omega(\tau)\|_{L^2}^2 d\tau \\
&\lesssim 1 + t^6\int_0^t \|\Omega(\tau)\|_{L^2}^2 d\tau + \frac{1}{4}\int_0^t \|\nabla\Omega(\tau)\|_{L^2}^2 d\tau,
\end{aligned}
$$

and when $\alpha = 3/2$, we obtain

$$
\begin{aligned}
t^3\left(\int_0^t \|r^{-1}u^r(\tau)\|_{L^\infty} d\tau\right)^2 &\le t^4\left(\int_0^t \|\Omega(\tau)\|_{L^2}^2 d\tau\right)^{1/2}\left(\int_0^t \|\nabla\Omega(\tau)\|_{L^2}^2 d\tau\right)^{1/2} \\
&\lesssim t^8\int_0^t \|\Omega(\tau)\|_{L^2}^2 d\tau + \frac{1}{4}\int_0^t \|\nabla\Omega(\tau)\|_{L^2}^2 d\tau.
\end{aligned}
$$

Thanks again to Lemma 2.1, the last integral in (2.8) can be estimated as

$$
\begin{aligned}
\int_0^t \|u(\tau)\|_{L^\infty}^2 d\tau &\le \int_0^t \|\omega^\phi(\tau)\|_{L^2}\|\nabla\omega^\phi(\tau)\|_{L^2} d\tau \\
&\le \frac{1}{2}\int_0^t \|\omega^\phi(\tau)\|_{L^2}^2 d\tau + \frac{1}{2}\int_0^t \|\nabla\omega^\phi(\tau)\|_{L^2}^2 d\tau.
\end{aligned}
$$

Therefore, we arrive at the following inequality

$$
\begin{aligned}
&\|\Omega(t)\|_{L^2}^2 + \int_0^t \|\nabla\Omega(\tau)\|_{L^2}^2 d\tau + 4\pi\int_0^t\int_{\mathbb{R}} |\Omega(\tau,0,z)|^2 dz d\tau \\
&\quad\le C(1+t^8)\left(1+\int_0^t \|\Omega(\tau)\|_{L^2}^2 d\tau\right) + C\int_0^t \|\omega^\phi(\tau)\|_{L^2}^2 d\tau + \frac{1}{2}\int_0^t \|\nabla\omega^\phi(\tau)\|_{L^2}^2 d\tau.
\end{aligned}
\tag{2.9}
$$

(iv). Combining the inequalities (2.6) and (2.9), one has

$$
\begin{aligned}
&\|\omega^\phi(t)\|_{L^2}^2 + \|\Omega(t)\|_{L^2}^2 + \int_0^t \|\nabla\omega^\phi\|_{L^2}^2 + \|r^{-1}\omega^\phi\|_{L^2}^2 d\tau \\
&\quad + \int_0^t \|\nabla\Omega(\tau)\|_{L^2}^2 d\tau + 4\pi\int_0^t\int_{\mathbb{R}} |\Omega(\tau,0,z)|^2 dz d\tau \\
&\quad\lesssim (1+t^8)\left(1+\int_0^t (1+\|\nabla u(\tau)\|_{L^2}^2)(\|\omega^\phi(\tau)\|_{L^2}^2 + \|\Omega(\tau)\|_{L^2}^2) d\tau\right).
\end{aligned}
\tag{2.10}
$$

Recalling the third inequality in Lemma 2.2, we have by integral Gronwall inequality that

$$
\|\omega^\phi(t)\|_{L^2}^2 + \|\Omega(t)\|_{L^2}^2 \lesssim C(t).
$$

It follows from (2.10) that

$$
\begin{aligned}
&\int_0^t \|\partial_r\omega^\phi\|_{L^2}^2 + \|\partial_z\omega^\phi\|_{L^2}^2 + \|r^{-1}\omega^\phi\|_{L^2}^2 d\tau \\
&\quad + \int_0^t \|\nabla\Omega(\tau)\|_{L^2}^2 d\tau + 4\pi\int_0^t\int_{\mathbb{R}} |\Omega(\tau,0,z)|^2 dz d\tau \lesssim C(t).
\end{aligned}
$$

The proof is complete. □

By using the Biot–Savart law [24], we have the following two corollaries.

Corollary 2.2. *Under the assumption of Lemma 2.3, there exists some constant $C(t)$ such that*

$$\|u(t)\|_{H^1}^2 + \int_0^t \|u(\tau)\|_{H^2}^2 d\tau \lesssim C(t).$$

Corollary 2.3. *Under the assumption of Lemma 2.3, there exists some constant $C(t)$ such that*

$$\int_0^t \|r^{-1}u^r(\tau)\|_{L^\infty} d\tau \lesssim C(t).$$

Proof. By combing the estimates in Lemma 2.3 and Lemma 2.1,

$$\int_0^t \|r^{-1}u^r(\tau)\|_{L^\infty} d\tau \le C \sup_{0\le\tau\le t} \|\Omega(\tau,\cdot)\|_{L^2}^{1/2} \int_0^t \|\nabla\Omega(\tau)\|_{L^2}^{1/2} d\tau \lesssim C(t).$$

□

Lemma 2.4. *Suppose that (u, B, θ) is a smooth solution of the Boussinesq–MHD system (1.6) with initial data $(u_0, B_0, \theta_0) \in H^2$, which satisfies the conditions of Theorem 1.1. Then there holds*

$$\|B^\phi(t)\|_{L^p} \lesssim C(t), \qquad \forall\, p \in [1, +\infty].$$

Proof. By multiplying the third equation in (1.4) with $p|B^\phi|^{p-2}B^\phi$ and integrating over $\mathbb{R}^3$, one has

$$\frac{d}{dt}\|B^\phi(t)\|_{L^p} \le \|r^{-1}u^r\|_{L^\infty}\|B^\phi(t)\|_{L^p}, \quad \forall\, p \in [1,\infty].$$

By Gronwall inequality, one has

$$\|B^\phi(t)\|_{L^p} \le \|B_0^\phi\|_{L^p} e^{\int_0^t \|r^{-1}u^r\|_{L^\infty} d\tau} \lesssim C(t),$$

independent of $p > 0$. Letting $p \to \infty$, then we finishe the proof. □

Lemma 2.5. *Suppose that (u, B, θ) is a smooth solution of the Boussinesq–MHD system (1.6) with initial data $(u_0, B_0, \theta_0) \in H^2$, which satisfies the conditions of Theorem 1.1. Then there exists some constant $C(t)$ such that*

$$\|\omega^\phi(t)\|_{L^4}^4 + \int_0^t \|\omega^\phi(\tau)\|_{L^{12}}^4 d\tau \lesssim C(t).$$

Proof. Now, we consider the L^4-estimate of the vorticity ω^ϕ. For this, we multiply the equation (1.5) with $|\omega^\phi|^2\omega^\phi$ and then integrating over $\mathbb{R}^3$ to obtain

$$\begin{aligned} \frac{1}{4}\frac{d}{dt}\|\omega^\phi\|_{L^4}^4 = &\int |\omega^\phi|^2\omega^\phi \left(\partial_{rr} + \frac{1}{r}\partial_r + \partial_{zz} - \frac{1}{r^2}\right)\omega^\phi - \int (u\cdot\nabla\omega^\phi)|\omega^\phi|^2\omega^\phi \\ &+ \int \frac{u^r}{r}\omega^\phi|\omega^\phi|^2\omega^\phi - 2\int |\omega^\phi|^2\omega^\phi r\Pi\partial_z\Pi - \int |\omega^\phi|^2\omega^\phi\partial_r\theta. \end{aligned} \tag{2.11}$$

For the first integral, we can show by integration by parts that

$$\int |\omega^\phi|^2\omega^\phi \left(\partial_{rr} + \frac{1}{r}\partial_r + \partial_{zz} - \frac{1}{r^2}\right)\omega^\phi dx = -\int \left(\frac{3}{4}|\nabla|\omega^\phi|^2|^2 + \frac{|\omega^\phi|^4}{r^2}\right) dx.$$

By Hölder and Young's inequality, (2.7), Lemma 2.3 and 2.4

$$\begin{aligned} \left|2\int |\omega^\phi|^2\omega^\phi r\Pi\partial_z\Pi dx\right| \le& C\|\Pi\|_{L^\infty}\|B^\phi\|_{L^\infty}\|\partial_z|\omega^\phi|^2\|_{L^2}\|\omega^\phi\|_{L^2} \\ \le& \frac{1}{4}\|\partial_z|\omega^\phi|^2\|_{L^2}^2 + C(t). \end{aligned}$$

From integration by parts, it holds that

$$-\int |\omega^\phi|^2\omega^\phi\partial_r\theta dx = 6\pi\int \theta|\omega^\phi|^2\partial_r\omega^\phi rdrdz + 2\pi\int\theta|\omega^\phi|^2\frac{\omega^\phi}{r}rdrdz$$
$$= \frac{3}{2}\int\theta\omega^\phi\partial_r(|\omega^\phi|^2)2\pi rdrdz + \int\theta|\omega^\phi|^2\frac{\omega^\phi}{r}2\pi rdrdz,$$

and hence

$$\left|\int |\omega^\phi|^2\omega^\phi\partial_r\theta dx\right| \le \frac{1}{4}\|\partial_r|\omega^\phi|^2\|_{L^2}^2 + 2\|\theta\|_{L^\infty}^2\|\omega^\phi\|_{L^2}^2 + \|\omega^\phi\|_{L^4}^4 + \|\theta\|_{L^\infty}^2\|r^{-1}\omega^\phi\|_{L^2}^2$$
$$\le \frac{1}{4}\|\partial_r|\omega^\phi|^2\|_{L^2}^2 + \|\omega^\phi\|_{L^4}^4 + C(t).$$

Noting that

$$|\partial_r(|\omega^\phi|^2)|^2 + |\partial_z(|\omega^\phi|^2)|^2 \le |\nabla(|\omega^\phi|^2)|^2$$

by direct computation and that the second integral on the right side of (2.11) vanishes by integration by parts, we obtain

$$\frac{d}{dt}\|\omega^\phi\|_{L^4}^4 + \int\left(|\nabla|\omega^\phi|^2|^2 + \frac{|\omega^\phi|^4}{r^2}\right)dx \lesssim (1+\|r^{-1}u^r\|_{L^\infty})\|\omega^\phi\|_{L^4}^4 + C(t).$$

Using Gronwall inequality, one has

$$\|\omega^\phi(t)\|_{L^4}^4 + \int_0^t\int\left(|\nabla|\omega^\phi(\tau)|^2|^2 + \frac{|\omega^\phi(\tau)|^4}{r^2}\right)dxd\tau$$
$$\lesssim e^{\int_0^t(1+\|r^{-1}u^r\|_{L^\infty})d\tau}\left(\|\omega_0^\phi\|_{L^4}^4 + \int_0^t C(\tau)d\tau\right) \lesssim C(t),$$

thanks to Corollary 2.3. □

Lemma 2.6. *Under the same conditions of Lemma 2.3, there exists some constant $C(t)$*

$$\int_0^t\|\nabla u(\tau)\|_{L^\infty}d\tau \lesssim C(t), \quad \|\nabla B(t)\|_{L^\infty} \lesssim C(t).$$

Proof. Recalling Lemma 2.5, we have by interpolation that

$$\|u\|_{L^\infty([0,t];L^\infty(\mathbb{R}^3))} \lesssim \|u\|_{L^\infty([0,t];L^2(\mathbb{R}^3))} + \|\omega^\phi\|_{L^\infty([0,t];L^4(\mathbb{R}^3))} \lesssim C(t),$$

and hence

$$\|\omega\times u\|_{L^4([0,t];L^{12}(\mathbb{R}^3))} \lesssim C(t).$$

Rewriting the equation for $\omega = \nabla\times u$, we have

$$\partial_t\omega + \nabla\times(\omega\times u) = \mu\Delta\omega - \partial_z(\Pi B^\phi e_\phi) + \nabla\times(\theta e_z).$$

Standard estimates show that [34]

$$\|\nabla\omega\|_{L^4([0,t];L^{12}(\mathbb{R}^3))} \lesssim C(t).$$

Sobolev embedding then implies that

$$\|\nabla u\|_{L^4([0,t];L^\infty(\mathbb{R}^3))} \lesssim C(t). \tag{2.12}$$

Applying ∇ to the third equation in (1.4), we have

$$\partial_t\nabla B^\phi + u\cdot\nabla\nabla B^\phi = -\nabla u\cdot\nabla B^\phi + \frac{u^r}{r}\nabla B^\phi + \nabla u^r\Pi - \frac{u^r}{r}\Pi e_r.$$

Multiplying the equation with $|\nabla B^\phi|^{p-2}\nabla B^\phi$, and then integrating over $\mathbb{R}^3$, one has

$$\frac{1}{p}\frac{d}{dt}\|\nabla B^\phi\|_{L^p}^p + \int u\cdot\nabla\nabla B^\phi|\nabla B^\phi|^{p-2}\nabla B^\phi dx \le \|\nabla u\|_{L^\infty}\|\nabla B^\phi\|_{L^p}^p$$
$$+ \|r^{-1}u^r\|_{L^\infty}\|\nabla B^\phi\|_{L^p}^p + \left(\|\nabla u^r\|_{L^\infty} + \|r^{-1}u^r\|_{L^\infty}\right)\|\Pi\|_{L^p}\|\nabla B^\phi\|_{L^p}^{p-1}.$$

Since u is divergence free, from integration by parts, one has

$$\frac{d}{dt}\|\nabla B^\phi\|_{L^p} \le \left(\|\nabla u\|_{L^\infty} + \|r^{-1}u^r\|_{L^\infty}\right)\|\nabla B^\phi\|_{L^p} + \left(\|\nabla u^r\|_{L^\infty} + \|r^{-1}u^r\|_{L^\infty}\right)\|\Pi\|_{L^p}.$$

Using Gronwall inequality then gives that

$$\|\nabla B^\phi(t)\|_{L^p} \le e^{\int_0^t \left(\|\nabla u\|_{L^\infty}+\|r^{-1}u^r\|_{L^\infty}\right)d\tau}$$
$$\times\left(\|\nabla B_0^\phi\|_{L^p} + \int_0^t \left(\|\nabla u^r\|_{L^\infty} + \|r^{-1}u^r\|_{L^\infty}\right)d\tau\right) \lesssim C(t).$$

where we have used (2.7), (2.12) and Corollary 2.3. Letting $p\to\infty$ then implies the result. □

2.4. Proof of Theorem 1.1

Applying the H^2 estimate for the system (1.1), we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\nabla^2 u(t)\|_{L^2}^2 + \|\nabla^2 B(t)\|_{L^2}^2 + \|\nabla^2\theta(t)\|_{L^2}^2) + \mu\|\nabla^3 u(t)\|_{L^2}^2\\ &\quad= \int \nabla^2 u\nabla^2(B\cdot\nabla B - u\cdot\nabla u)dx + \int \nabla^2\theta\nabla^2 u^z dx\\ &\qquad+ \int \nabla^2 B\nabla^2(B\cdot\nabla u - u\cdot\nabla B)dx + \int\nabla^2\theta\nabla^2(u\cdot\nabla\theta)dx.\end{aligned} \tag{2.13}$$

Note that

$$\begin{aligned}&\int \nabla^2 u\nabla^2(B\cdot\nabla B)dx + \int\nabla^2 B\nabla^2(B\cdot\nabla u)dx\\ &= \int \nabla^2 u\cdot[\nabla^2, B\cdot]\nabla B + \nabla^2 B\cdot[\nabla^2, B\cdot]\nabla u dx + \int \nabla^2 u B\cdot\nabla\nabla^2 B + \nabla^2 B B\cdot\nabla\nabla^2 u dx\\ &= \int \nabla^2 u\cdot[\nabla^2, B\cdot]\nabla B + \nabla^2 B\cdot[\nabla^2, B\cdot]\nabla u dx\\ &\lesssim (\|\nabla u\|_{L^\infty} + \|\nabla B\|_{L^\infty})\|\nabla^2 u\|_{L^2}\|\nabla^2 B\|_{L^2},\end{aligned}$$

where $[\cdot,\cdot]$ denotes the commutation and the last integral in the second line cancels thanks to integration by parts and divergence free condition of B. Other terms in (2.13) can be treated similarly, thanks to integration by parts and the divergence free condition of u and B, leading to the estimates

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\nabla^2 u(t)\|_{L^2}^2 + \|\nabla^2 B(t)\|_{L^2}^2 + \|\nabla^2\theta(t)\|_{L^2}^2) + \mu\|\nabla^3 u(t)\|_{L^2}^2\\ &\lesssim (\|\nabla u(t)\|_{L^\infty} + \|\nabla B(t)\|_{L^\infty})(\|\nabla^2 u(t)\|_{L^2}^2 + \|\nabla^2 B(t)\|_{L^2}^2)\\ &\quad+ (1 + \|\nabla u(t)\|_{L^\infty} + \|\nabla\theta(t)\|_{L^\infty})(\|\nabla^2 u(t)\|_{L^2}^2 + \|\nabla^2\theta(t)\|_{L^2}^2).\end{aligned}$$

Gronwall inequality then implies that

$$\|\nabla^2 u(t)\|_{L^2}^2 + \|\nabla^2 B(t)\|_{L^2}^2 + \|\nabla^2\theta(t)\|_{L^2}^2 + \mu\int_0^t \|\nabla^3 u(\tau)\|_{L^2}^2 d\tau \lesssim C(t).$$

Similarly, one can get H^s estimates as follows

$$\|\nabla^s u(t)\|_{L^2}^2 + \|\nabla^s B(t)\|_{L^2}^2 + \|\nabla^s\theta(t)\|_{L^2}^2) + \mu\int_0^t \|\nabla^{s+1} u(\tau)\|_{L^2}^2 d\tau \lesssim C(t).$$

This completes the proof of Theorem 1.1.

Acknowledgements. The first author D. Bian was partially supported by NSFC under the Contracts 11871005 and 11771041. The second author X. Pu was partially supported by NSFC under the contract 11871172 and the Natural Science Foundation of Guangdong Province of China under 2019A1515012000.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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(accepted: October 25, 2019; published online: December 26, 2019)