J. Math. Fluid Mech. (2019) 21:22
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1422-6928/19/020001-16
https://doi.org/10.1007/s00021-019-0426-5

Journal of Mathematical Fluid Mechanics

On Wolf's Regularity Criterion of Suitable Weak Solutions to the Navier–Stokes Equations

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Communicated by S. Friedlander

Abstract. In this paper, we consider the local regularity of suitable weak solutions to the 3D incompressible Navier–Stokes equations. By means of the local pressure projection introduced by Wolf (in: Rannacher, Sequeira (eds) Advances in mathematical fluid mechanics, Springer, Berlin, 2010, Ann Univ Ferrara 61:149–171, 2015), we establish a Caccioppoli type inequality just in terms of velocity field for suitable weak solutions to this system

$$\|u\|^2_{L^{\frac{20}{7},\frac{15}{4}}Q(\frac{1}{2})}+\|\nabla u\|^2_{L^2(Q(\frac{1}{2}))}\leq C\|u\|^2_{L^{\frac{20}{7}}(Q(1))}+C\|u\|^4_{L^{\frac{20}{7}}(Q(1))}.$$

This allows us to derive a new ε-regularity criterion: Let u be a suitable weak solution in the Navier–Stokes equations. There exists an absolute positive constant ε such that if u satisfies

$$\iint_{Q(1)}|u|^{20/7}dxdt<\varepsilon,$$

then u is bounded in some neighborhood of point $(0,0)$. This gives an improvement of previous corresponding results obtained in Chae and Wolf (Arch Ration Mech Anal 225:549–572, 2017), in Guevara and Phuc (Calc Var 56:68, 2017) and Wolf (Ann Univ Ferrara 61:149–171, 2015).

Mathematics Subject Classification. 76D03, 76D05, 35B33, 35Q35.

Keywords. Navier–Stokes equations, Suitable weak solutions, Regularity.

1. Introduction

We focus on the following incompressible Navier–Stokes equations in three-dimensional space

$$\begin{cases} u_t-\Delta u+u\cdot\nabla u+\nabla\Pi=0,\ \operatorname{div}u=0,\\ u|_{t=0}=u_0,\end{cases}\tag{1.1}$$

where u stands for the flow velocity field, the scalar function Π represents the pressure. The initial velocity u_0 satisfies $\operatorname{div}u_0=0$.

In this paper, we are concerned with the local regularity of suitable weak solutions to the 3D Navier–Stokes equations (1.1). This kind of weak solutions obeys the local energy inequality below, for a.e. $t\in[-T,0]$,

$$\begin{aligned}&\int_{\mathbb{R}^3}|u(x,t)|^2\phi(x,t)dx+2\int_{-T}^{t}\int_{\mathbb{R}^3}|\nabla u|^2\phi dxds\\ &\quad\leq\int_{-T}^{t}\int_{\mathbb{R}^3}|u|^2(\partial_s\phi+\Delta\phi)dxds+\int_{-T}^{t}\int_{\mathbb{R}^3}u\cdot\nabla\phi(|u|^2+2\Pi)dxds,\end{aligned}\tag{1.2}$$

where non-negative function $\phi(x,s)\in C_0^\infty(\mathbb{R}^3\times(-T,0))$.

Before going further, we shall introduce some notations in what follows. For $p \in [1, \infty]$, the notation $L^p((0, T); X)$ stands for the set of measurable functions on the interval $(0, T)$ with values in X and $\|f(t, \cdot)\|_X$ belongs to $L^p(0, T)$. For simplicity, we write

$$\|f\|_{L^{p,q}(Q(r))} := \|f\|_{L^p(-r^2,0;L^q(B(r)))} \text{ and } \|f\|_{L^p(Q(r))} := \|f\|_{L^{p,p}(Q(r))},$$

where $Q(r) = B(r) \times (t - r^2, t)$ and $B(r)$ denotes the ball of center x and radius r.

Roughly speaking, the local regularity of suitable weak solutions is intimately connected to ε-regularity criteria (see, e.g., [1,3,4,7,8,11–13,15–19]). Particularly, a well-known ε-regularity criterion is the following one with $p = 3$: there is an absolute constant ε such that, if

$$\|u\|^p_{L^p(Q(1))} + \|\Pi\|^{p/2}_{L^{p/2}(Q(1))} < \varepsilon, \tag{1.3}$$

then u is bounded in some neighborhood of point $(0, 0)$. This was proved by Lin in [12] (see also Ladyzenskaja and Seregin [11]). In [10], Kukavica proposed three questions regarding this regularity criteria (1.3)

(1) If this result holds for weak solutions which are not suitable.
(2) It is not known if the regularity criteria holds for $p < 3$ in (1.3).
(3) If the pressure can be removed from the condition (1.3).

Recently, Guevara and Phuc [7] answered Kukavica's issue (2) via establishing following regularity criteria

$$\|u\|_{L^{2p,2q}(Q(1))} + \|\Pi\|_{L^{p,q}(Q(1))} < \varepsilon,\ 3/q + 2/p = 7/2 \text{ with } 1 \le q \le 2. \tag{1.4}$$

Later, He et al. [8] extended Guevara and Phuc's results to

$$\|u\|_{L^{p,q}(Q(1))} + \|\Pi\|_{L^1(Q(1))} < \varepsilon,\ 1 \le 2/p + 3/q < 2, 1 \le p,\ q \le \infty. \tag{1.5}$$

To the question (3), for a given bounded C^2 domain $\Omega \subseteq \mathbb{R}^3$, Wolf introduced the local pressure projection depended on domain (for the detail, see Sect. 2) $\mathcal{W}_{p,\Omega} : W^{-1,p}(\Omega) \to W^{-1,p}(\Omega)$ $(1 < p < \infty)$ in [18,19] and obtained a ε-regularity criterion without pressure below

$$\iint_{Q(1)} |u|^3 dxdt < \varepsilon. \tag{1.6}$$

In addition, very recently, in [3], Wolf and Chae studied Liouville type theorems for self-similar solutions to the Navier–Stokes equations by proving ε-regularity criteria

$$\sup_{-1\le t\le 0} \int_{B(1)} |u|^q dx < \varepsilon,\ \frac{3}{2} < q \le 3. \tag{1.7}$$

Based on Kukavica's questions and recent progresses (1.4)–(1.6), a natural issue is whether the regularity criteria (1.3) holds for $p < 3$ without pressure. The goal of this paper is devoted to this. Before we state our results, we roughly mention the novelty in [3,18,19]. For any ball $B(R) \subseteq \mathbb{R}^3$, by the local pressure projection, Wolf et al. presented the pressure decomposition

$$-\nabla\Pi = -\partial_t \nabla\Pi_h - \nabla\Pi_1 - \nabla\Pi_2,$$

where

$$\nabla\Pi_h = -\mathcal{W}_{p,B(R)}(u),\ \nabla\Pi_1 = \mathcal{W}_{p,B(R)}(\Delta u),\ \nabla\Pi_2 = -\mathcal{W}_{p,B(R)}(u \cdot \nabla u).$$

After denoting $v = u + \nabla\Pi_h$, one gets the local energy inequality, for a.e. $t \in [-T, 0]$ and non-negative function $\phi(x, s) \in C_0^\infty(\mathbb{R}^3 \times (-T, 0))$,

$$\begin{aligned}
&\int_{B(r)} |v|^2\phi(x,t)dx + \int_{-T}^t \int_{B(r)} \left|\nabla v\right|^2 \phi(x,t)dxds \\
&\quad\le \int_{-T}^t \int_{B(r)} |v|^2(\Delta\phi + \partial_t\phi)dxds + \int_{-T}^t \int_{B(r)} |v|^2 u \cdot \nabla\phi dsds \\
&\qquad + \int_{-T}^t \int_{B(r)} \phi(u \otimes v : \nabla^2\Pi_h)dsds + \int_{-T}^t \int_{B(r)} \Pi_1 v \cdot \nabla\phi dxds + \int_{-T}^t \int_{B(r)} \Pi_2 v \cdot \nabla\phi dxds.
\end{aligned} \tag{1.8}$$

It is worth pointing out that any usual suitable weak solutions to the Navier–Stokes system enjoys the local energy inequality (1.8). We refer the reader to [2, Appendix A, p. 1372] for its proof. As stated in [3,18,19], the advantage of local energy inequality (1.8) removed the non-local effect of the pressure term. Based on this, Caccioppoli type inequalities without pressure are derived in [3,18], respectively,

$$\|u\|^2_{L^{3,\frac{18}{5}}Q(\frac12)}+\|\nabla u\|^2_{L^2(Q(\frac12))}\le C\|u\|^2_{L^3(Q(1))}+C\|u\|^3_{L^3(Q(1))}. \tag{1.9}$$

$$\|u\|^2_{L^{3,\frac{18}{5}}Q(\frac12)}+\|\nabla u\|^2_{L^2(Q(\frac12))}\le C\|u\|^2_{L^{\frac{3q}{2q-3},q}(Q(1))}+C\|u\|^{\frac{3q}{2q-3}}_{L^{\frac{3q}{2q-3},q}(Q(1))},\quad \frac32<q\le 3. \tag{1.10}$$

Our first result is to derive a new Caccioppoli type inequality in terms of the velocity field only

Proposition 1.1. *Assume that u is a suitable weak solutions to the Navier–Stokes equations (1.1). There holds*

$$\|u\|^2_{L^{\frac{20}{7},\frac{15}{4}}Q(\frac12)}+\|\nabla u\|^2_{L^2(Q(\frac12))}\le C\|u\|^2_{L^{\frac{20}{7}}(Q(1))}+C\|u\|^4_{L^{\frac{20}{7}}(Q(1))}. \tag{1.11}$$

This Caccioppoli type inequality allows us to obtain our main result

Theorem 1.2. *Let the pair (u,Π) be a suitable weak solution to the 3D Navier–Stokes system (1.1) in $Q(1)$. There exists an absolute positive constant ε such that if u satisfies*

$$\|u\|_{L^{20/7}(Q(1))}<\varepsilon, \tag{1.12}$$

then, $u\in L^\infty(Q(1/16))$.

Remark 1.1. This theorem is an improvement of corresponding results in (1.4)–(1.7).

We give some comments on the proof of Proposition 1.1 and Theorem 1.2. Though the non-local pressure disappears in the local energy inequality in (1.8), the velocity field u losses the kinetic energy $\|u\|_{L^{\infty,2}}$. In contrast with works [3,18], owing to $\|u\|^2_{L^{3,\frac{18}{5}}(Q(\frac12))}$ appearing in Caccioppoli type inequalities in (1.9)–(1.10) and without the kinetic energy of u, it seems to be difficult to apply the argument used in [3,7,8,18] directly to obtain (1.11). To circumvent these difficulties, first, we observe that every nonlinear term contains at least v in the local energy inequality (1.8). Meanwhile, $v=u+\nabla\Pi_h$ enjoys all the energy, namely, $\|v\|_{L^\infty L^2}$ and $\|\nabla v\|_{L^2L^2}$, where Π_h is a harmonic function. Hence, it would be natural to absorb v by the left hand of local energy inequality (1.8). Second, this together with the iteration Lemma 2.2 allows us to establish the Caccioppoli type inequality for $\|u\|^2_{L^{\frac{20}{7},\frac{15}{4}}Q(\frac12)}+\|\nabla u\|^2_{L^2(Q(\frac12))}$ instead of $\|u\|^2_{L^{3,\frac{18}{5}}Q(\frac12)}+\|\nabla u\|^2_{L^2(Q(\frac12))}$. However, this is not enough to yield the desired result, which is completely different from that in [7,8]. To this end, in the spirit of [3], we adopt Caccioppoli type inequality (1.8) and induction arguments developed in [1,3,13,16] to complete the proof of Theorem 1.2. Third, to the knowledge of authors, all previous authors in [1,3,13,16] invoked induction arguments for $⨏⨏_{\tilde{Q}_k}|v|^3\le\varepsilon_1^{2/3}$. To bound the term $\iint|v|^2\nabla\Pi_h\cdot\nabla\phi d\tau$ in local energy inequality (1.8) by $⨏⨏_{\tilde{Q}_k}|v|^3\le\varepsilon_1^{2/3}$, one needs $\nabla\Pi_h\in L^p(I;\|\cdot\|)$ with $p\ge 3$, where I is an time interval. Since $\nabla\Pi_h$ is controlled by u, we have to get $u\in L^p(I;\|\cdot\|)$ with $p\ge 3$. However, from (1.11), we have $u\in L^p(I;\|\cdot\|)$ with $p<3$, therefore, induction arguments with $⨏⨏_{\tilde{Q}_k}|v|^3\le\varepsilon_1^{2/3}$, seems to break down in our case. As said above, since we have all the energy of v, we work with

$$⨏⨏_{\tilde{Q}_k}|v|^{\frac{10}{3}}\le\varepsilon_1^{2/3},$$

instead of $⨏⨏_{\tilde{Q}_k}|v|^3\le\varepsilon_1^{2/3}$ in induction arguments. Finally, this enables us to achieve the proof of Theorem 1.2.

The remainder of this paper is structured as follows. In Sect. 2, we explain the detail of Wolf's the local pressure projection $\mathcal{W}_{p,\Omega}$ and present the definition of local suitable weak solutions. Then, we recall some interior estimates of harmonic functions, an interpolation inequality, two classical iteration lemmas and establish an auxiliary lemma utilized in induction arguments. The Caccioppoli type inequality (1.11) is derived in Sect. 3. Section 4 is devoted to the proof of Theorem 1.2.

Notations Throughout this paper, we denote

$$B(x,\mu) := \{y \in \mathbb{R}^n | |x-y| \leq \mu\}, \qquad B(\mu) := B(0,\mu), \qquad \tilde{B}(\mu) := B(x_0,\,\mu),$$
$$Q(x,t,\mu) := B(x,\,\mu) \times (t-\mu^{2\alpha},t), \qquad Q(\mu) := Q(0,0,\mu), \qquad \tilde{Q}(\mu) := Q(x_0,t_0,\mu),$$
$$r_k = 2^{-k}, \qquad \tilde{B}_k := \tilde{B}(r_k), \qquad \tilde{Q}_k := \tilde{Q}(r_k).$$

Denote the average of f on the set Ω by $\overline{f}_\Omega$. For convenience, $\overline{f}_r$ represents $\overline{f}_{B(r)}$ and $\overline{\Pi}_{\tilde{B}_k}$ is denoted by $\tilde{\Pi}_k$. $|\Omega|$ represents the Lebesgue measure of the set Ω. We will use the summation convention on repeated indices. C is an absolute constant which may be different from line to line unless otherwise stated in this paper.

2. Preliminaries

We begin with Wolf's the local pressure projection $\mathcal{W}_{p,\Omega} : W^{-1,p}(\Omega) \to W^{-1,p}(\Omega)$ $(1 < p < \infty)$. More precisely, for any $f \in W^{-1,p}(\Omega)$, we define $\mathcal{W}^{-1,p}(f) = \nabla\Pi$, where Π satisfies (2.1). Let Ω be a bounded domain with $\partial\Omega \in C^1$. According to the L^p theorem of Stokes system in [5, Theorem 2.1, p. 149], there exists a unique pair $(b,\Pi) \in W^{1,p}(\Omega) \times L^p(\Omega)$ such that

$$-\Delta b + \nabla\Pi = f,\ \operatorname{div} b = 0,\ b|_{\partial\Omega} = 0,\ \int_\Omega \Pi dx = 0. \tag{2.1}$$

Moreover, this pair is subject to the inequality

$$\|b\|_{W^{1,q}(\Omega)} + \|\Pi\|_{L^q(\Omega)} \leq C\|f\|_{W^{-1,q}(\Omega)}.$$

Let $\nabla\Pi = \mathcal{W}_{p,\Omega}(f)$ $(f \in L^p(\Omega))$, then $\|\Pi\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)}$, where we used the fact that $L^p(\Omega) \hookrightarrow W^{-1,p}(\Omega)$. Moreover, from $\Delta\Pi = \operatorname{div} f$, we see that $\|\nabla\Pi\|_{L^p(\Omega)} \leq C(\|f\|_{L^p(\Omega)} + \|\Pi\|_{L^p(\Omega)}) \leq C\|f\|_{L^p(\Omega)}$. Now, we present the definition of suitable weak solutions of Navier–Stokes equations (1.1).

Definition 2.1. A pair (u, Π) is called a suitable weak solution to the Navier–Stokes equations (1.1) provided the following conditions are satisfied,

(1) $u \in L^\infty(-T,\,0;\,L^2(\mathbb{R}^3)) \cap L^2(-T,\,0;\,\dot{H}^1(\mathbb{R}^3))$, $\Pi \in L^{3/2}(-T,\,0;L^{3/2}(\mathbb{R}^3))$;
(2) $(u,\ \Pi)$ solves (1.1) in $\mathbb{R}^3 \times (-T,0)$ in the sense of distributions;
(3) The local energy inequality (1.8) is valid and $\nabla\Pi_h$ is a harmonic function. In addition, $\nabla\Pi_h, \nabla\Pi_1$ and $\nabla\Pi_2$ meet the following fact

$$\|\nabla\Pi_h\|_{L^p(B(R))} \leq \|u\|_{L^p(B(R))}, \tag{2.2}$$
$$\|\nabla\Pi_1\|_{L^2(B(R))} \leq \|\nabla u\|_{L^2(B(R))}, \tag{2.3}$$
$$\|\nabla\Pi_2\|_{L^{p/2}(B(R))} \leq \||u|^2\|_{L^{p/2}(B(R))}. \tag{2.4}$$

We list some interior estimates of harmonic functions $\Delta h = 0$, which will be frequently utilized later. Let $1 \leq p,q \leq \infty$ and $0 < r < \rho$, then, it holds

$$\|\nabla^k h\|_{L^q(B(r))} \leq \frac{Cr^{\frac{n}{q}}}{(\rho-r)^{\frac{n}{p}+k}}\|h\|_{L^p(B(\rho))}. \tag{2.5}$$

$$\|h-\overline{h}_r\|_{L^q(B(r))} \leq \frac{Cr^{\frac{n}{q}+1}}{(\rho-r)^{\frac{n}{q}+1}}\|h-\overline{h}_\rho\|_{L^q(B(\rho))}. \tag{2.6}$$

The proof of estimate (2.5) rests on the mean value property of harmonic functions. This together with mean value theorem leads to inequality (2.6). We leave the detail to the readers. In addition, for readers' convenience, we recall an interpolation inequality. For each $2 \le l \le \infty$ and $2 \le k \le 6$ satisfying $\frac{2}{l}+\frac{3}{k}=\frac{3}{2}$, according to the Hölder inequality and the Young inequality, we know that

$$\begin{aligned}\|u\|_{L^{k,l}(Q(\mu))} &\le C\|u\|_{L^{2,\infty}(Q(\mu))}^{1-\frac{2}{l}}\|u\|_{L^{6,2}(Q(\mu))}^{\frac{2}{l}}\\ &\le C\|u\|_{L^{2,\infty}(Q(\mu))}^{1-\frac{2}{l}}(\|u\|_{L^{2,\infty}(Q(\mu))}+\|\nabla u\|_{L^2(Q(\mu))})^{\frac{2}{l}}\\ &\le C(\|u\|_{L^{2,\infty}(Q(\mu))}+\|\nabla u\|_{L^2(Q(\mu))}). \end{aligned} \tag{2.7}$$

Next, we present two well-known iteration lemmas.

Lemma 2.1 [6, Lemma 2.1, p.86]. *Let $\phi(t)$ be a nonegative and nondecreasing function on $[0,R]$. Suppose that*

$$\phi(r)\le A\Big[\Big(\frac{r}{\rho}\Big)^{\alpha}+\kappa\Big]\phi(\rho)+B\rho^{\beta}$$

for any $0<r\le\rho\le R$, with A,B,α,β nonnegative constants and $\beta<\alpha$. Then for any $\gamma\in(\beta,\alpha)$, there exists a constant κ_0 such that if $\kappa<\kappa_0$ we have for all $0<r\le\rho\le R$

$$\phi(r)\le C\Big\{\Big(\frac{r}{\rho}\Big)^{\beta}\phi(\rho)+Br^{\beta}\Big\}.$$

where C is a positive constant depending on A,α,β,γ.

Lemma 2.2 [6, Lemma V.3.1, p.161]. *Let $I(s)$ be a bounded nonnegative function in the interval $[r,R]$. Assume that for every $\sigma,\rho\in[r,R]$ and $\sigma<\rho$ we have*

$$I(\sigma)\le A_1(\rho-\sigma)^{-\alpha_1}+A_2(\rho-\sigma)^{-\alpha_2}+A_3+\ell I(\rho)$$

for some non-negative constants A_1,A_2,A_3, non-negative exponents $\alpha_1\ge\alpha_2$ and a parameter $\ell\in[0,1)$. Then there holds

$$I(r)\le c(\alpha_1,\ell)[A_1(R-r)^{-\alpha_1}+A_2(R-r)^{-\alpha_2}+A_3].$$

The following lemma is motivated by [3, Lemma 2.9, p. 558].

Lemma 2.3. *Let $f\in L^q(Q(1))$ with $q>1$ and $0<r_0<1$. Suppose that for all $(x_0,t_0)\in Q(1/2)$ and $r_0\le r\le\frac{1}{2}$*

$$\iint_{\tilde{Q}(r)}|f-\overline{f}_{\tilde{B}(r)}|^q\le Cr^4. \tag{2.8}$$

Let $\nabla\Pi=\mathcal{W}_{q,B(1)}(\nabla\cdot f)$. Then for all $(x_0,t_0)\in Q(1/2)$ and $r_0\le r\le\frac{1}{4}$, it holds

$$\iint_{\tilde{Q}(r)}|\Pi-\overline{\Pi}_{\tilde{B}(r)}|^q\le Cr^4$$

Proof. From the definition of pressure projection $\mathcal{W}_{q,B(1)}$, we know that

$$\|\Pi\|_{L^q(B(1))}\le C\|f-\overline{f}_{B(1)}\|_{L^q(B(1))}. \tag{2.9}$$

Let $\phi(x)=1, x\in\tilde{B}(\frac{3r}{4}), \phi(x)=0, x\in\tilde{B}^c(r)$.
Note that

$$\Delta\Pi=\mathrm{div}\mathcal{W}_{q,B(1)}(\nabla\cdot f).$$

We set $\Pi=\Pi_{(1)}+\Pi_{(2)}$, where

$$\Delta\Pi_{(1)}=\mathrm{div}\mathcal{W}_{q,B(1)}(\nabla\cdot[\phi(f-\overline{f}_{\tilde{B}(r)})]),$$

therefore, as a consequence, it holds

$$\Delta\Pi_{(2)}=0, x\in\tilde{B}(3r/4).$$

In view of classical Calderón–Zygmund theorem, we have

$$\|\Pi_{(1)}-\overline{\Pi_{(1)}}_{\tilde{B}(r)}\|_{L^q(\tilde{B}(r))}\le C\|f-\overline{f}_{\tilde{B}(r)}\|_{L^q(\tilde{B}(r))}. \tag{2.10}$$

Combining this and hypothesis (2.8), we get

$$\|\Pi_{(1)} - \overline{\Pi_{(1)}}_{\tilde{B}(r)}\|_{L^q(\tilde{Q}(r))} \le Cr^{\frac{4}{q}}.$$

The interior estimates of harmonic functions (2.6) and the triangle inequality guarantee that, for $\theta < 1/2$,

$$\begin{aligned}
&\int_{\tilde{B}(\theta r)} |\Pi_{(2)} - \overline{\Pi_{(2)}}_{\tilde{B}(\theta r)}|^q dx \\
&\quad \le \frac{C(r\theta)^{3+q}}{(\frac{r}{2})^{3+q}} \int_{\tilde{B}(r/2)} |\Pi_{(2)} - \overline{\Pi_{(2)}}_{\tilde{B}(r/2)}|^q dx \\
&\quad \le C\theta^{3+q} \int_{\tilde{B}(r/2)} |\Pi - \overline{\Pi}_{\tilde{B}(r/2)}|^q dx + C \int_{\tilde{B}(r/2)} |\Pi_{(1)} - \overline{\Pi_{(1)}}_{\tilde{B}(r/2)}|^q dx.
\end{aligned}$$

This and inequality (2.10) imply

$$\iint_{\tilde{Q}(\theta r)} |\Pi_{(2)} - \overline{\Pi_{(2)}}_{\tilde{B}(\theta r)}|^q dxds \le C\theta^{3+q} \iint_{\tilde{Q}(r/2)} |\Pi - \overline{\Pi}_{\tilde{B}(r/2)}|^q dxds + Cr^4.$$

Utilizing the triangle inequality again, (2.10) and the last inequality, we have

$$\begin{aligned}
&\iint_{\tilde{Q}(\theta r)} |\Pi - \overline{\Pi}_{\tilde{B}(\theta r)}|^q dxds \\
&\quad \le \iint_{\tilde{Q}(\theta r)} |\Pi_{(1)} - \overline{\Pi_{(1)}}_{\tilde{B}(\theta r)}|^q dxds + \iint_{\tilde{Q}(\theta r)} |\Pi_{(2)} - \overline{\Pi_{(2)}}_{\tilde{B}(\theta r)}|^q dxds \\
&\quad \le \iint_{\tilde{Q}(r)} |f - \overline{f}_{\tilde{B}(r)}|^q dxds + C\theta^{3+q} \iint_{\tilde{Q}(r/2)} |\Pi - \overline{\Pi}_{\tilde{B}(r/2)}|^q dxds + Cr^4 \\
&\quad \le C\theta^{3+q} \iint_{\tilde{Q}(r)} |\Pi - \overline{\Pi}_{\tilde{B}(r)}|^q dxds + Cr^4,
\end{aligned} \tag{2.11}$$

where we used the fact that $\|g - \overline{g}_{B(r)}\|_{L^p(B(r))} \le C\|g - c\|_{L^p(B(r))}$ with $p \ge 1$.

Now, applying Lemma 2.1 to (2.11) and (2.9), we see that

$$\begin{aligned}
\iint_{\tilde{Q}(r)} |\Pi - \overline{\Pi}_{\tilde{B}(r)}|^q dx &\le Cr^4 \iint_{\tilde{Q}(1/4)} |\Pi - \overline{\Pi}_{\tilde{B}(1/4)}|^q dx + Cr^4 \\
&\le Cr^4 \iint_{Q(1)} |f - \overline{f}_{B(1)}|^q dx + Cr^4 \\
&\le Cr^4.
\end{aligned}$$

This completes the proof of this lemma. □

3. Proof of Proposition 1.1

This section contains the proof of Proposition 1.1.

Proof of Proposition 1.1. It suffices to show, for any $R > 0$,

$$\|u\|^2_{L^{\frac{20}{7},\frac{15}{4}}Q(\frac{R}{2})} + \|\nabla u\|^2_{L^2(Q(\frac{R}{2}))} \le CR^{-\frac{1}{2}}\|u\|^2_{L^{\frac{20}{7}}(Q(R))} + CR^{-2}\|u\|^4_{L^{\frac{20}{7}}(Q(R))}. \tag{3.1}$$

Consider $0 < R/2 \le r < \frac{3r+\rho}{4} < \frac{r+\rho}{2} < \rho \le R$. Let $\phi(x,t)$ be non-negative smooth function supported in $Q(\frac{r+\rho}{2})$ such that $\phi(x,t) \equiv 1$ on $Q(\frac{3r+\rho}{4})$, $|\nabla\phi| \le C/(\rho - r)$ and $|\nabla^2\phi| + |\partial_t\phi| \le C/(\rho - r)^2$.

Let $\nabla\Pi_h = \mathcal{W}_{20/7,B(\rho)}(u)$, then, there holds

$$\|\nabla\Pi_h\|_{L^{20/7}(Q(\rho))} \le C\|u\|_{L^{20/7}(Q(\rho))}, \tag{3.2}$$

$$\|\Pi_1\|_{L^2(Q(\rho))} \le C\|\nabla u\|_{L^2(Q(\rho))}, \tag{3.3}$$

$$\|\Pi_2\|_{L^{\frac{10}{7}}(Q(\rho))} \leq C\||u|^2\|_{L^{\frac{10}{7}}(Q(\rho))}. \tag{3.4}$$

Thanks to $v = u + \nabla\Pi_h$, the Hölder inequality and (3.2), we arrive at

$$\begin{aligned}\iint_{Q(\rho)} |v_{_B}|^2\Big|\Delta\phi^4 + \partial_t\phi^4\Big| \leq& \frac{C}{(\rho-r)^2}\iint_{Q(\frac{r+\rho}{2})} |u|^2 + |\nabla\Pi_h|^2\\ \leq& \frac{C\rho^{3/2}}{(\rho-r)^2}\Big(\iint_{Q(\frac{r+\rho}{2})} |u|^{\frac{20}{7}} + |\nabla\Pi_h|^{\frac{20}{7}}\Big)^{\frac{7}{10}}\\ \leq& \frac{C\rho^{3/2}}{(\rho-r)^2}\Big(\iint_{Q(\rho)} |u|^{\frac{20}{7}}\Big)^{\frac{7}{10}}.\end{aligned} \tag{3.5}$$

From now on, we drop the symbols ds and $dxds$ for simple presentation. Combining Hölder's inequality with interpolation inequality (2.7) and Young's inequality yields

$$\begin{aligned}&\iint_{Q(\rho)} |v|^2\phi^3 u\cdot\nabla\phi\\ &\leq \frac{C}{(\rho-r)}\|v\phi^2\|_{L^{10/3}(Q(\frac{r+\rho}{2}))}\|v\|_{L^{20/7}(Q(\frac{r+\rho}{2}))}\|u\|_{L^{20/7}(Q(\frac{r+\rho}{2}))}\\ &\leq \frac{1}{16}\|v\phi^2\|^2_{L^{10/3}(Q(\frac{r+\rho}{2}))} + \frac{C}{(\rho-r)^2}\|v\|^2_{L^{20/7}(Q(\frac{r+\rho}{2}))}\|u\|^2_{L^{20/7}(Q(\frac{r+\rho}{2}))}\\ &\leq \frac{1}{16}\Big(\|v\phi^2\|^2_{L^{2,\infty}(Q(\rho))} + \|\nabla(v\phi^2)\|^2_{L^2(Q(\rho))}\Big) + \frac{C}{(\rho-r)^2}\|u\|^4_{L^{20/7}(Q(\rho))}.\end{aligned} \tag{3.6}$$

By virtue of interior estimate of harmonic function (2.5) and (3.2), we conclude that

$$\begin{aligned}\|\nabla^2\Pi_h\|_{L^{20/7}(Q(\frac{r+\rho}{2}))} \leq& \frac{(r+\rho)^{\frac{21}{20}}}{(\rho-r)^{\frac{41}{20}}}\|\nabla\Pi_h\|_{L^{20/7}(Q(\rho))}\\ \leq& \frac{C\rho^{\frac{21}{20}}}{(\rho-r)^{\frac{41}{20}}}\|u\|_{L^{20/7}(Q(\rho))},\end{aligned}$$

which leads to

$$\begin{aligned}&\iint_{Q(\rho)} \phi^4(u\otimes v:\nabla^2\Pi_h)\\ &\leq \|v\phi^2\|_{L^{10/3}(Q(\frac{r+\rho}{2}))}\|u\|_{L^{20/7}(Q(\frac{r+\rho}{2}))}\|\nabla^2\Pi_h\|_{L^{20/7}(Q(\frac{r+\rho}{2}))}\\ &\leq \frac{1}{16}\Big(\|v\phi^2\|^2_{L^{2,\infty}(Q(\rho))} + \|\nabla(\phi^2 v)\|^2_{L^2(Q(\rho))}\Big) + \frac{C\rho^{\frac{21}{10}}}{(\rho-r)^{\frac{41}{10}}}\|u\|^4_{L^{20/7}(Q(\rho))}.\end{aligned} \tag{3.7}$$

In light of Hölder inequality, (3.3) and Young's inequality, we deduce that

$$\begin{aligned}\iint_{Q(\rho)} \phi^3\Pi_1 v\cdot\nabla\phi \leq& \frac{C}{(\rho-r)}\|v\|_{L^2(Q(\frac{r+\rho}{2}))}\|\Pi_1\|_{L^2(Q(\frac{r+\rho}{2}))}\\ \leq& \frac{C}{(\rho-r)^2}\|v\|^2_{L^2(Q(\frac{r+\rho}{2}))} + \frac{1}{16}\|\Pi_1\|^2_{L^2(Q(\rho))}\\ \leq& \frac{C\rho^{3/2}}{(\rho-r)^2}\Big(\iint_{Q(\rho)} |u|^{\frac{20}{7}}\Big)^{\frac{7}{10}} + \frac{1}{16}\|\nabla u\|^2_{L^2(Q(\rho))}.\end{aligned} \tag{3.8}$$

We derive from the Hölder inequality, (2.4) and Young's inequality that

$$\begin{aligned}\iint_{Q(\rho)} \phi^3\Pi_2 v\cdot\nabla\phi \leq& \frac{C}{(\rho-r)}\|v\phi^2\|_{L^{\frac{10}{3}}(Q(\frac{r+\rho}{2}))}\|\Pi_2\|_{L^{\frac{10}{7}}(Q(\frac{r+\rho}{2}))}\\ \leq& \frac{1}{16}\|v\|^2_{L^{\frac{10}{3}}(Q(\frac{r+\rho}{2}))} + \frac{C}{(\rho-r)^2}\|\Pi_2\|^2_{L^{\frac{10}{7}}(Q(\rho))}\end{aligned}$$

$$\leq \frac{1}{16}\Big(\|v\phi^2\|^2_{L^{2,\infty}(Q(\rho))}+\|\nabla(\phi^2 v)\|^2_{L^2(Q(\rho))}\Big)+\frac{C}{(\rho-r)^2}\|u\|^4_{L^{20/7}(Q(\rho))}. \tag{3.9}$$

The Cauchy–Schwarz inequality and (3.5) allows us to obtain that

$$\begin{aligned}\iint_{Q(\rho)}|\nabla(v\phi^2)|^2 &\leq 2\Big(\iint_{Q(\rho)}|\nabla v|^2\phi^4+4\iint_{Q(\rho)}|\nabla\phi|^2|v|^2\phi^2\Big)\\ &\leq 2\iint_{Q(\rho)}|\nabla v|^2\phi^4+\frac{C\rho^{3/2}}{(\rho-r)^2}\Big(\iint_{Q(\rho)}|u|^{\frac{20}{7}}\Big)^{\frac{7}{10}}.\end{aligned} \tag{3.10}$$

Substituting (3.5)–(3.9) into (1.8) and using (3.10), we infer that

$$\begin{aligned}&\sup_{-\rho^2\leq t\leq 0}\int_{B(\rho)}|v\phi^2|^2+\iint_{Q(\rho)}\big|\nabla(v\phi^2)\big|^2\\ &\leq \frac{1}{4}\Big(\|v\phi^2\|^2_{L^{2,\infty}(Q(\rho))}+\|\nabla(v\phi^2)\|^2_{L^2(Q(\rho))}\Big)+\left\{\frac{C}{(\rho-r)^2}+\frac{C\rho^{\frac{21}{10}}}{(\rho-r)^{\frac{41}{10}}}\right\}\|u\|^4_{L^{20/7}(Q(\rho))}\\ &\quad+\frac{C\rho^{3/2}}{(\rho-r)^2}\|u\|^2_{L^{20/7}(Q(\rho))}+\frac{1}{16}\|\nabla u\|^2_{L^2(Q(\rho))},\end{aligned}$$

that is,

$$\begin{aligned}\sup_{-\rho^2\leq t\leq 0}\int_{B(\rho)}|v\phi^2|^2+\iint_{Q(\rho)}\big|\nabla(v\phi^2)\big|^2 &\leq \left\{\frac{C}{(\rho-r)^2}+\frac{C\rho^{\frac{21}{10}}}{(\rho-r)^{\frac{41}{10}}}\right\}\|u\|^4_{L^{20/7}(Q(\rho))}\\ &\quad+\frac{C\rho^{3/2}}{(\rho-r)^2}\|u\|^2_{L^{20/7}(Q(\rho))}+\frac{1}{16}\|\nabla u\|^2_{L^2(Q(\rho))},\end{aligned} \tag{3.11}$$

Together with interior estimate of harmonic function (2.5) and (3.2) implies that

$$\begin{aligned}\|\nabla\Pi_h\|^2_{L^{\frac{20}{7},\frac{15}{4}}(Q(r))} &\leq \frac{Cr^{\frac{8}{5}}}{(\rho-r)^{\frac{21}{10}}}\|\nabla\Pi_h\|^2_{L^{\frac{20}{7}}Q(\rho)}\\ &\leq \frac{Cr^{\frac{8}{5}}}{(\rho-r)^{\frac{21}{10}}}\|u\|^2_{L^{\frac{20}{7}}(Q(\rho))}.\end{aligned}$$

With the help of the triangle inequality, interpolation inequality (2.7) and the last inequality, we get

$$\begin{aligned}\|u\|^2_{L^{\frac{20}{7},\frac{15}{4}}(Q(r))} &\leq \|v\|^2_{L^{\frac{20}{7},\frac{15}{4}}(Q(r))}+\|\nabla\Pi_h\|^2_{L^{\frac{20}{7},\frac{15}{4}}(Q(r))}\\ &\leq C\Big\{\|v\|^2_{L^{2,\infty}(Q(r))}+\|\nabla v\|^2_{L^2(Q(r))}\Big\}+\frac{r^{\frac{8}{5}}}{(\rho-r)^{\frac{21}{10}}}\|u\|^2_{L^{\frac{20}{7}}(Q(\rho))}\\ &\leq \Big\{\frac{C}{(\rho-r)^2}+\frac{C\rho^{\frac{21}{10}}}{(\rho-r)^{\frac{41}{10}}}\Big\}\|u\|^4_{L^{20/7}(Q(\frac{r+\rho}{2}))}\\ &\quad+\Big\{\frac{C\rho^{3/2}}{(\rho-r)^2}+\frac{Cr^{\frac{8}{5}}}{(\rho-r)^{\frac{21}{10}}}\Big\}\|u\|^2_{L^{20/7}(Q(\rho))}+\frac{1}{16}\|\nabla u\|^2_{L^2(Q(\rho))}.\end{aligned}$$

Employing (2.5) and (2.3) once again, we have the estimate

$$\|\nabla^2\Pi_h\|^2_{L^2(Q(r))}\leq \frac{Cr^3}{(\rho-r)^{3+2\cdot 1}}\|\nabla\Pi_h\|^2_{L^2(Q(\frac{r+\rho}{2}))}\leq \frac{Cr^3\rho^{3/2}}{(\rho-r)^5}\|u\|^2_{L^{20/7}(Q(\rho))}.$$

This together with the triangle inequality and (3.11) leads to

$$\|\nabla u\|^2_{L^2(Q(r))}\leq \|\nabla v\|^2_{L^2(Q(r))}+\|\nabla^2\Pi_h\|^2_{L^2(Q(r))}$$

$$\leq \Big\{\frac{C}{(\rho-r)^2}+\frac{C\rho^{\frac{21}{10}}}{(\rho-r)^{\frac{41}{10}}}\Big\}\|u\|^4_{L^{20/7}(Q(\rho))} + \Big\{\frac{C\rho^{3/2}}{(\rho-r)^2}+\frac{Cr^3\rho^{3/2}}{(\rho-r)^5}\Big\}\|u\|^2_{L^{20/7}(Q(\rho))}+\frac{1}{16}\|\nabla u\|^2_{L^2(Q(\rho))}. \tag{3.12}$$

Eventually, we infer that

$$\begin{aligned}&\|u\|^2_{L^{\frac{20}{7},\frac{15}{4}}(Q(r))}+\|\nabla u\|^2_{L^2(Q(r))}\\ &\leq \Big\{\frac{C}{(\rho-r)^2}+\frac{C\rho^{\frac{21}{10}}}{(\rho-r)^{\frac{41}{10}}}\Big\}\|u\|^4_{L^{20/7}(Q(\rho))}\\ &\quad+\Big\{\frac{C\rho^{3/2}}{(\rho-r)^2}+\frac{Cr^3\rho^{3/2}}{(\rho-r)^{3+2\cdot 1}}+\frac{Cr^{\frac{8}{5}}}{(\rho-r)^{\frac{21}{10}}}\Big\}\|u\|^2_{L^{20/7}(Q(\rho))}+\frac{3}{16}\|\nabla u\|^2_{L^2(Q(\rho))}.\end{aligned}$$

Now, we are in a position to apply lemma 2.2 to the latter estimate to find that

$$\|u\|^2_{L^{\frac{20}{7},\frac{15}{4}}(Q(\frac{R}{2}))}+\|\nabla u\|^2_{L^2(Q(\frac{R}{2}))}\leq CR^{-\frac{1}{2}}\|u\|^2_{L^{\frac{20}{7}}(Q(R))}+CR^{-2}\|u\|^4_{L^{\frac{20}{7}}(Q(R))}.$$

This achieves the proof of this proposition. □

4. Induction Arguments and Proof of Theorem 1.2

In this section, we begin with a critical proposition, which can be seen as the bridge between the previous step and the next step for the given statement in the induction arguments. Next, we finish the proof of Theorem 1.2.

Proposition 4.1. *Assume that $\iint_{\tilde{Q}(r)}|v|^{\frac{10}{3}}\leq r^5N$ with $r_k\leq r\leq r_{k_0}$. There is a constant C such that the following result holds. For any given $(x_0,\,t_0)\in\mathbb{R}^n\times\mathbb{R}^-$ and $k_0\in\mathbb{N}$, we have for any $k>k_0$,*

$$\begin{aligned}&\sup_{-r_k^2\leq t-t_0\leq 0}\fint_{\tilde{B}_k}|v|^2+r_k^{-3}\iint_{\tilde{Q}_k}|\nabla v|^2\\ &\leq C\sup_{-r_{k_0}^2\leq t-t_0\leq 0}\fint_{\tilde{B}_{k_0}}|v|^2+C\sum_{l=k_0}^{k}r_l\Big(\fint\!\!\fint_{\tilde{Q}_l}|v|^{\frac{10}{3}}\Big)^{\frac{9}{10}}+C\sum_{l=k_0}^{k}r_l^{\frac{3}{10}}\Big(\fint\!\!\fint_{\tilde{Q}_l}|v|^{\frac{10}{3}}\Big)^{\frac{3}{5}}\Big(\iint_{\tilde{Q}_3}|u|^{\frac{20}{7}}\Big)^{\frac{7}{20}}\\ &\quad+C\sum_{l=k_0}^{k}r_l^{\frac{13}{10}}\Big(\fint\!\!\fint_{\tilde{Q}_l}|v|^{\frac{10}{3}}\Big)^{\frac{3}{5}}\Big(\iint_{\tilde{Q}_3}|u|^{\frac{20}{7}}\Big)^{\frac{7}{20}}+C\sum_{l=k_0}^{k}r_l^{\frac{3}{5}}\Big(\fint\!\!\fint_{\tilde{Q}_l}|v|^{\frac{10}{3}}\Big)^{\frac{3}{10}}\Big(\iint_{\tilde{Q}_3}|u|^{\frac{20}{7}}\Big)^{\frac{7}{10}}\\ &\quad+C\sum_{l=k_0}^{k}r_l\Big(\fint\!\!\fint_{\tilde{Q}_l}|v|^{\frac{10}{3}}\Big)^{\frac{3}{10}}\Big(\iint_{\tilde{Q}_3}|\nabla u|^2\Big)^{\frac{1}{2}}+C\sum_{l=k_0}^{k}r_l^{\frac{3}{2}}\Big(\fint\!\!\fint_{\tilde{Q}_l}|v|^{\frac{10}{3}}\Big)^{\frac{3}{10}}\Big\{N^{3/5}\\ &\quad+N^{\frac{3}{10}}\Big(\iint_{\tilde{Q}_3}|u|^{\frac{20}{7}}\Big)^{\frac{7}{20}}+\Big(\iint_{\tilde{Q}_3}|u|^{20/7}\Big)^{\frac{7}{10}}\Big\}.\end{aligned}$$

Proof. Without loss of generality, we suppose $(x_0,t_0)=(0,0)$. We denote the backward heat kernel

$$\Gamma(x,t)=\frac{1}{4\pi(r_k^2-t)^{3/2}}e^{-\frac{|x|^2}{4(r_k^2-t)}}.$$

In addition, consider the smooth cut-off functions below

$$\phi(x,t)=\begin{cases}1,\ (x,t)\in Q(r_{k_0+1}),\\ 0,\ (x,t)\in Q^c\left(\frac{3}{2}r_{k_0+1}\right);\end{cases}$$

satisfying

$$0 \leq \phi,\ \phi_2 \leq 1 \text{ and } r_{k_0}^2|\partial_t\phi(x,t)| + r_{k_0}^l|\partial_x^l\phi(x,t)| \leq C.$$

To proceed further, we list some properties of the test function $\phi(x,t)\Gamma(x,t)$. A detailed proof can be found in Kukavica [9, Proof of Lemma 1, p.720].

(i) There is a constant $c > 0$ independent of r_k such that, for any $(x,t) \in Q(r_k)$,

$$\Gamma(x,t) \geq c r_k^{-3}.$$

(ii) For any $(x,t) \in Q(r_{k_0})$, we have

$$|\Gamma(x,t)\phi(x,t)| \leq C r_k^{-3},\ |\nabla\phi(x,t)\Gamma(x,t)| \leq C r_k^{-4},\ |\phi(x,t)\nabla\Gamma(x,t)| \leq C r_k^{-4}.$$

(iii) For any $(x,t) \in Q(3r_{k_0}/4)\backslash Q(r_{k_0}/2)$, one can deduce that

$$\Gamma(x,t) \leq C r_{k_0}^{-3},\ \partial_i\Gamma(x,t) \leq C r_{k_0}^{-4},$$

which yields that

$$|\Gamma(x,t)\partial_t\psi(x,t)| + |\Gamma(x,t)\Delta\psi(x,t)| + |\nabla\psi(x,t)\nabla\Gamma(x,t)| \leq C r_{k_0}^{-5}.$$

(iv) For any $(x,t) \in Q_l\backslash Q_{l+1}$,

$$\Gamma \leq C r_{l+1}^{-3},\ \nabla\Gamma \leq C r_{l+1}^{-4}.$$

Now, setting $\varphi_1 = \phi\Gamma$ in the local energy inequality (1.8) and utilizing the fact that $\Gamma_t + \Delta\Gamma = 0$, we see that

$$\begin{aligned}
&\int_{B_1} |v|^2\phi(x,t)\Gamma + \int_{-r_{k_0}^2}^t \int_{B_1} \big|\nabla v\big|^2\phi(x,s)\Gamma \\
&\quad \leq \int_{-r_{k_0}^2}^t \int_{B_1} |v|^2(\Gamma\Delta\phi + \Gamma\partial_t\phi + 2\nabla\Gamma\nabla\phi) \\
&\qquad + \int_{-r_{k_0}^2}^t \int_{B_1} |v|^2 v\cdot\nabla(\phi\Gamma) - |v|^2\nabla\Pi_h\cdot\nabla\phi \\
&\qquad + \int_{-r_{k_0}^2}^t \int_{B_1} \Gamma\phi(v\otimes v - v\otimes\nabla\Pi_h : \nabla^2\Pi_h) + \int_{-r_{k_0}^2}^t \int_{B_1} \Pi_1 v\cdot\nabla(\Gamma\phi) + \int_{-r_{k_0}^2}^t \int_{B_1} \Pi_2 v\cdot\nabla(\Gamma\phi)
\end{aligned}$$

where

$$\nabla\Pi_1 = \mathcal{W}_{2,B_1}(\Delta u),\ \nabla\Pi_2 = -\mathcal{W}_{\frac{20}{7},B_1}(\nabla\cdot(u\otimes u)).$$

First, we present the low bound estimates of the terms on the left hand side of this inequality. Indeed, with the help of (iv), we find

$$\int_{B_k} |v|^2\phi\Gamma \geq C \fint_{B_k} |v|^2,$$

and

$$\int_{-r_{k_0}^2}^t \int_{B_1} \phi\Gamma|\nabla v|^2 \geq r_k^{-3} \iint_{Q_k} |\nabla v|^2.$$

Having observed that the support of $\partial_t\phi$ is included in $Q(\frac{3r_{k_0}}{4})/Q(\frac{r_{k_0}}{2})$, we get

$$\int_{-r_{k_0}^2}^t \int_{B_1} |v|^2\Big|\Gamma\Delta\phi + \Gamma\partial_t\phi + 2\nabla\Gamma\nabla\phi\Big| \leq C \sup_{-r_{k_0}^2\leq t\leq 0} \fint_{B_{k_0}} |v|^2.$$

Hölder's inequality and (iv) enable us to write that

$$\iint_{Q_{k_0}} |v|^2 v\cdot\nabla(\phi\Gamma)d\tau$$

$$\leq \sum_{l=k_0}^{k-1} \iint_{Q_l/Q_{l+1}} |v|^3 |\nabla(\phi\Gamma)| + \iint_{Q_k} |v|^3 |\nabla(\phi\Gamma)|$$

$$\leq \sum_{l=k_0}^{k} r_l^{-4} \iint_{Q_l} |v|^3$$

$$\leq C \sum_{l=k_0}^{k} r_l \Big(-\!\!\!\!\!\!\iint_{Q_l} |v|^{\frac{10}{3}} \Big)^{\frac{9}{10}}.$$

Following the lines of reasoning which led to the last inequality, we have

$$\iint_{Q_{k_0}} |v|^2 \nabla\Pi_h \cdot \nabla(\phi\Gamma)$$

$$\leq \sum_{l=k_0}^{k} r_l^{-4} \iint_{Q_l} |v|^2 |\nabla\Pi_h|$$

$$\leq C \sum_{l=k_0}^{k} r_l^{\frac{3}{10}} \Big(-\!\!\!\!\!\!\iint_{Q_l} |v|^{\frac{10}{3}} \Big)^{\frac{3}{5}} \Big(\iint_{Q_3} |u|^{\frac{20}{7}} \Big)^{\frac{7}{20}}. \tag{4.1}$$

Likewise, we have

$$\iint_{Q_{k_0}} |v|^2 |\nabla^2\Pi_h| (\phi\Gamma)$$

$$\leq \sum_{l=k_0}^{k} r_l^{-3} \iint_{Q_l} |v|^2 |\nabla^2\Pi_h|$$

$$\leq C \sum_{l=k_0}^{k} r_l^{\frac{13}{10}} \Big(-\!\!\!\!\!\!\iint_{Q_l} |v|^{\frac{10}{3}} \Big)^{\frac{3}{5}} \Big(\iint_{Q_3} |u|^{\frac{20}{7}} \Big)^{\frac{7}{20}}.$$

Using Hölder's inequality again, (iv), (2.5) and (2.3), we infer that

$$\iint_{Q_{k_0}} \phi\Gamma |v| |\nabla\Pi_h| |\nabla^2\Pi_h|$$

$$\leq C \sum_{l=k_0}^{k} r_l^{-3} \Big(\iint_{Q_l} |v|^{\frac{10}{3}} \Big)^{\frac{3}{10}} \Big(\iint_{Q_l} |\nabla\Pi_h|^{\frac{20}{7}} \Big)^{\frac{7}{20}} \Big(\iint_{Q_l} |\nabla^2\Pi_h|^{\frac{20}{7}} \Big)^{\frac{7}{20}}$$

$$\leq C \sum_{l=k_0}^{k} r_l^{\frac{3}{5}} \Big(-\!\!\!\!\!\!\iint_{Q_l} |v|^{\frac{10}{3}} \Big)^{\frac{3}{10}} \Big(\iint_{Q_3} |u|^{\frac{20}{7}} \Big)^{\frac{7}{10}}.$$

Set $\chi_l = 1$ on $|x| \leq 7/8 r_l$ and $\chi_l = 0$ if $|x| \geq r_l$. $\chi_{k_0}\Gamma = \Gamma$ on Q_{k_0} By the support of $(\chi_l - \chi_{l+1})$, we derive from (iv) that $|\nabla((\chi_l - \chi_{l+1})\phi\Gamma)| \leq C r_{l+1}^{-4}$. With the help of (iv) again, we see that $|\nabla(\chi_k \phi\Gamma)| \leq C r_k^{-4}$. Therefore, it holds

$$\iint_{Q_{k_0}} v \cdot \nabla(\phi\Gamma)\Pi_1 = \sum_{l=k_0}^{k-1} \iint_{Q_l} v \cdot \nabla((\chi_l - \chi_{l+1})\phi\Gamma)\Pi_1 + \iint_{Q_k} v \cdot \nabla(\chi_k \phi\Gamma)\Pi_1$$

$$= \sum_{l=k_0}^{k-1} \iint_{Q_l} v \cdot \nabla((\chi_l - \chi_{l+1})\phi\Gamma)(\Pi_1 - \overline{\Pi_1}_l) + \iint_{Q_k} u \cdot \nabla(\chi_k \phi\Gamma)(\Pi_1 - \overline{\Pi_1}_k)$$

$$
\begin{aligned}
&\le C\sum_{l=k_0}^{k-1} r_{l+1}^{-4}\iint_{Q_l}|v||\Pi_1-\overline{\Pi_1}_l| + r_k^{-4}\iint_{Q_k}|v||\Pi_1-\overline{\Pi_1}_k|\\
&=: I+II. \qquad (4.2)
\end{aligned}
$$

The Hölder inequality, (2.4) and (2.6) give

$$
\begin{aligned}
I &\le C\sum_{l=k_0}^{k-1} r_{l+1}^{-4}\Big(\iint_{Q_l}|v|^{\frac{10}{3}}\Big)^{\frac{3}{10}}\Big(\iint_{Q_l}|\Pi_1-\overline{\Pi_1}_l|^2\Big)^{\frac{1}{2}} r_l\\
&\le C\sum_{l=k_0}^{k-1} r_l\Big(⨌_{Q_l}|v|^{\frac{10}{3}}\Big)^{\frac{3}{10}}\Big(\iint_{Q_3}|\Pi_1-\overline{\Pi_1}_3|^2\Big)^{\frac{1}{2}}\\
&\le C\sum_{l=k_0}^{k-1} r_l\Big(⨌_{Q_l}|v|^{\frac{10}{3}}\Big)^{\frac{3}{10}}\Big(\iint_{Q_3}|\Pi_1|^2\Big)^{\frac{1}{2}}\\
&\le C\sum_{l=k_0}^{k-1} r_l\Big(⨌_{Q_l}|v|^{\frac{10}{3}}\Big)^{\frac{3}{10}}\Big(\iint_{Q_3}|\nabla u|^2\Big)^{\frac{1}{2}}, \qquad (4.3)
\end{aligned}
$$

and

$$
II \le C r_k\Big(⨌_{Q_l}|v|^{\frac{10}{3}}\Big)^{\frac{3}{10}}\Big(\iint_{Q_3}|\nabla u|^2\Big)^{\frac{1}{2}},
$$

which turns out that

$$
\iint_{\tilde{Q}_{k_0}} v\cdot\nabla(\phi\Gamma)\Pi_1 \le C\sum_{l=k_0}^{k} r_l\Big(⨌_{Q_l}|v|^{\frac{10}{3}}\Big)^{\frac{3}{10}}\Big(\iint_{Q_3}|\nabla u|^2\Big)^{\frac{1}{2}}.
$$

Note that

$$
u\otimes u = v\otimes v - v\otimes\nabla\Pi_h - \nabla\Pi_h\otimes v + \nabla\Pi_h\otimes\nabla\Pi_h. \qquad (4.4)
$$

For $r_k\le r\le r_{k_0}$, we compute directly that

$$
\iint_{Q(r)}|v\otimes v-\overline{v\otimes v}_l|^{10/7} \le \iint_{Q(r)}|v|^{20/7} \le Cr^5\Big(⨌_{Q(r)}|v|^{\frac{10}{3}}\Big)^{6/7} \le Cr^5N^{6/7}. \qquad (4.5)
$$

The Hölder inequality and (2.5) ensure that

$$
\begin{aligned}
\iint_{Q(r)}|v\otimes\nabla\Pi_h-\overline{v\otimes\nabla\Pi_h}_l|^{10/7} &\le C\iint_{Q(r)}|v\otimes\nabla\Pi_h|^{10/7}\\
&\le C\Big(\iint_{Q(r)}|v|^{\frac{10}{3}}\Big)^{\frac{3}{7}}\Big(\iint_{Q(r)}|\nabla\Pi_h|^{\frac{20}{7}}\Big)^{\frac{1}{2}} r^{\frac{5}{14}}\\
&\le Cr^4\Big(⨌_{Q(r)}|v|^{\frac{10}{3}}\Big)^{\frac{3}{7}}\Big(\iint_{Q_3}|u|^{\frac{20}{7}}\Big)^{\frac{1}{2}}\\
&\le Cr^4N^{\frac{3}{7}}\Big(\iint_{Q_3}|u|^{\frac{20}{7}}\Big)^{\frac{1}{2}}. \qquad (4.6)
\end{aligned}
$$

In view of Poincaré inequality for a ball, Hölder's inequality, (2.5) and (2.3), we arrive at

$$
\begin{aligned}
&\iint_{Q(r)}|\nabla\Pi_h\otimes\nabla\Pi_h-\overline{\nabla\Pi_h\otimes\nabla\Pi_h}_l|^{10/7}\\
&\quad\le Cr^{10/7}\Big(\iint_{Q(r)}|\nabla\Pi_h|^{20/7}\Big)^{\frac{1}{2}}\Big(\iint_{Q(r)}|\nabla^2\Pi_h|^{20/7}\Big)^{\frac{1}{2}}
\end{aligned}
$$

$$\le Cr^{\frac{31}{7}}\Big(\iint_{Q_1}|\nabla\Pi_h|^{20/7}\Big)^{\frac12}\Big(\iint_{Q_3}|\nabla\Pi_h|^{20/7}\Big)^{\frac12}$$
$$\le Cr^{\frac{31}{7}}\Big(\iint_{Q_3}|u|^{20/7}\Big). \tag{4.7}$$

We deduce from (4.4)–(4.7) that

$$\begin{aligned}\iint_{Q(r)}|u\otimes u-\overline{(u\otimes u)_r}|^{10/7}&\le Cr^5N^{6/7}+r^4N^{\frac37}\Big(\iint_{Q_3}|u|^{\frac{20}{7}}\Big)^{\frac12}+Cr^{\frac{31}{7}}\Big(\iint_{Q_3}|u|^{20/7}\Big)\\&\le Cr^4\Big\{N^{6/7}+N^{\frac37}\Big(\iint_{Q_3}|u|^{\frac{20}{7}}\Big)^{\frac12}+C\Big(\iint_{Q_3}|u|^{20/7}\Big)\Big\}.\end{aligned} \tag{4.8}$$

With (4.8) in hand, we can apply Lemma 2.3 to obtain that

$$\iint_{Q(r)}|\Pi_2-\overline{\Pi_2}_{B(r)}|^{10/7}\le Cr^4\Big\{N^{6/7}+N^{\frac37}\Big(\iint_{Q_3}|u|^{\frac{20}{7}}\Big)^{\frac12}+C\Big(\iint_{Q_3}|u|^{20/7}\Big)\Big\}. \tag{4.9}$$

Particulary, for any $k\le l\le k_0$, it holds

$$\iint_{Q_l}|\Pi_2-\overline{\Pi_2}_{B_l}|^{10/7}\le Cr_l^4\Big\{N^{6/7}+N^{\frac37}\Big(\iint_{Q_3}|u|^{\frac{20}{7}}\Big)^{\frac12}+C\Big(\iint_{Q_3}|u|^{20/7}\Big)\Big\}. \tag{4.10}$$

By the Hölder inequality, we see that

$$r_l^{-4}\iint_{Q_l}|v||\Pi_2-\overline{\Pi_2}_{B(r)}|\le Cr_l^{-4}\Big(\iint_{Q_l}|v|^{\frac{10}{3}}\Big)^{\frac{3}{10}}\Big(\iint_{Q_l}|\Pi_2-\overline{(\Pi_2)}_{B_l}|^{\frac{10}{7}}\Big)^{\frac{7}{10}}. \tag{4.11}$$

Plugging (4.9) into (4.11), we have

$$\begin{aligned}&\iint_{Q_{k_0}}v\cdot\nabla(\phi\Gamma)\Pi_2\\&\le C\sum_{l=k_0}^{k-1}r_{l+1}^{-4}\iint_{Q_l}|v||\Pi_2-\overline{\Pi_2}_l|+r_k^{-4}\iint_{Q_k}|v||\Pi_2-\overline{\Pi_2}_k|\\&\le C\sum_{l=k_0}^{k}r_l^{\frac{3}{10}}\Big(⨌_{Q_l}|v|^{\frac{10}{3}}\Big)^{\frac{3}{10}}\Big\{N^{6/7}+N^{\frac37}\Big(\iint_{Q_3}|u|^{\frac{20}{7}}\Big)^{\frac12}+C\Big(\iint_{Q_3}|u|^{20/7}\Big)\Big\}^{\frac{7}{10}}.\end{aligned}$$

Finally, collected these estimates leads to (4.17). □

With Proposition 4.1 at our disposal, we will now present the proof of Theorem 1.2.

Proof of Theorem 1.2. By the interior estimate (2.5) of harmonic function and (3.2), we have

$$\|\nabla\Pi_h\|_{L^\infty(\tilde B(1/8))}\le C\|\nabla\Pi_h\|_{L^{20/7}(B(1))}\le C\|u\|_{L^{20/7}(B(1))}. \tag{4.12}$$

Assume for a while we have proved that, for any Lebesgue point $(x_0,t_0)\in Q(1/8)$,

$$|v(x_0,t_0)|\le C. \tag{4.13}$$

We derive from (4.12) and (4.13) that

$$\|u\|_{L^{20/7,\infty}(\tilde Q(1/8))}\le\|\nabla\Pi_h\|_{L^{20/7,\infty}(\tilde Q(1/8))}+\|v\|_{L^{20/7,\infty}(\tilde Q(1/8))}\le C\|u\|_{L^{20/7}(Q(1))}.$$

By the well-known Serrin regularity criteria in [14], we know that $(0,0)$ is a regular point. Therefore, it remains to prove (4.13). In what follows, let $(x_0,t_0)\in Q(1/8)$ and $r_k=2^{-k}$. According to the Lebesgue differentiation theorem, it suffices to show

$$⨌_{\tilde Q_k}|v|^{\frac{10}{3}}\le\varepsilon_1^{2/3},\ k\ge3. \tag{4.14}$$

First, we show that (4.14) is valid for $k = 3$. Indeed, from (3.11) in Sect. 3, (3.1) and hypothesis 1.12, we know that

$$\begin{aligned} &\sup_{-(\frac{3}{8})^2\le t\le 0}\int_{B(\frac{3}{8})}|v|^2+\iint_{Q(\frac{3}{8})}|\nabla v|^2\\ &\le C\|u\|^4_{L^{20/7}(Q(\frac{1}{2}))}+C\|u\|^2_{L^{20/7}(Q(\frac{1}{2}))}+\frac{1}{16}\|\nabla u\|^2_{L^2(Q(\frac{1}{2}))}\\ &\le C\varepsilon^{7/10}. \end{aligned} \tag{4.15}$$

In light of Sobolev embeddings and the Young inequality, we see that

$$\Big(\iint_{Q(\frac{3}{8})}|v|^{\frac{10}{3}}\Big)^{\frac{3}{10}}\le C\Big(\sup_{-(\frac{3}{8})^2\le t<0}\int_{B(\frac{3}{8})}|v|^2\Big)^{1/2}+C\Big(\iint_{Q(\frac{3}{8})}|\nabla v|^2\Big)^{1/2}. \tag{4.16}$$

It turns out that

$$⨌_{\tilde{Q}_3}|v|^{\frac{10}{3}}\le C\varepsilon^{\frac{7}{6}}.$$

This proves (4.14) in the case $k = 3$. Now, we assume that, for any $3 \le l \le k$,

$$⨌_{\tilde{Q}_l}|v|^{\frac{10}{3}}\le \varepsilon^{2/3}.$$

Furthermore, there holds, for any $r_k \le r \le r_3$

$$⨌_{\tilde{Q}(r)}|v|^{\frac{10}{3}}\le C\varepsilon^{2/3}.$$

For any $3 \le i \le k$, by Proposition 4.1 with $N = C\varepsilon^{2/3}$, (1.12) and the above induction hypothesis, we find that

$$\begin{aligned} &\sup_{-r_i^2\le t-t_0\le 0}⨍_{\tilde{B}_i}|v|^2+r_i^{-3}\iint_{\tilde{Q}_i}|\nabla v|^2\\ &\le C\sup_{-r_3^2\le t-t_0\le 0}⨍_{\tilde{B}_3}|v|^2+C\sum_{l=3}^{k}r_l\Big(⨌_{\tilde{Q}_l}|v|^{\frac{10}{3}}\Big)^{\frac{9}{10}}\\ &\quad+C\sum_{l=3}^{i}r_l^{\frac{3}{10}}\Big(⨌_{\tilde{Q}_l}|v|^{\frac{10}{3}}\Big)^{\frac{3}{5}}\Big(\iint_{Q_1}|u|^{\frac{20}{7}}\Big)^{\frac{7}{20}}\\ &\quad+C\sum_{l=3}^{i}r_l^{\frac{13}{10}}\Big(⨌_{\tilde{Q}_l}|v|^{\frac{10}{3}}\Big)^{\frac{3}{5}}\Big(\iint_{Q_1}|u|^{\frac{20}{7}}\Big)^{\frac{7}{20}}\\ &\quad+C\sum_{l=3}^{i}r_l^{\frac{3}{5}}\Big(⨌_{\tilde{Q}_l}|v|^{\frac{10}{3}}\Big)^{\frac{3}{10}}\Big(\iint_{Q_1}|u|^{\frac{20}{7}}\Big)^{\frac{7}{10}}\\ &\quad+C\sum_{l=3}^{i}r_l\Big(⨌_{\tilde{Q}_l}|v|^{\frac{10}{3}}\Big)^{\frac{3}{10}}\Big(\iint_{Q_1}|\nabla u|^2\Big)^{\frac{1}{2}}\\ &\quad+C\sum_{l=3}^{i}r_l^{\frac{3}{10}}\Big(⨌_{\tilde{Q}_l}|v|^{\frac{10}{3}}\Big)^{\frac{3}{10}}\Big\{N^{3/5}+N^{\frac{3}{10}}\Big(\iint_{Q_1}|u|^{\frac{20}{7}}\Big)^{\frac{7}{20}}+\Big(\iint_{Q_1}|u|^{\frac{20}{7}}\Big)^{\frac{7}{10}}\Big\}\\ &\le C\varepsilon^{\frac{7}{10}}+C\sum_{l=3}^{i}r_l\varepsilon^{\frac{3}{5}}+C\sum_{l=3}^{i}r_l^{\frac{3}{10}}\varepsilon^{\frac{2}{5}}\varepsilon^{\frac{7}{20}}+C\sum_{l=3}^{i}r_l^{\frac{13}{10}}\varepsilon^{\frac{2}{5}}\varepsilon^{\frac{7}{20}} \end{aligned}$$

$$
+ C\sum_{l=3}^{i} r_l^{\frac{3}{5}}\varepsilon^{\frac{1}{5}}\varepsilon^{\frac{7}{10}} + C\sum_{l=3}^{i} r_l\varepsilon^{\frac{1}{5}}\varepsilon^{\frac{7}{20}} + C\sum_{l=3}^{i} r_l^{\frac{3}{10}}\ \varepsilon^{\frac{1}{5}}\Big\{\varepsilon^{\frac{2}{5}} + \varepsilon^{\frac{11}{20}} + \varepsilon^{\frac{7}{10}}\Big\}
$$
$$
\leq C\varepsilon^{\frac{11}{20}}. \tag{4.17}
$$

Invoking the Gagliardo–Nirenberg inequality, we deduce that

$$
\int_{\tilde{B}_{k+1}} |v|^{\frac{10}{3}}dx \leq C\Big(\int_{\tilde{B}_k}|v|^2\Big)^{\frac{2}{3}}\Big[\Big(\int_{\tilde{B}_k}|\nabla v|^2\Big)^{1/2} + r_k^{-1}\Big(\int_{\tilde{B}_k}|v|^2\Big)^{1/2}\Big]^2,
$$

which means

$$
\iint_{\tilde{Q}_{k+1}} |v|^{\frac{10}{3}} \leq C\Big(\sup_{-r_k^2\leq t-t_0<0}\int_{\tilde{B}_k}|v|^2\Big)^{\frac{2}{3}}\Big(\iint_{\tilde{Q}_k}|\nabla v|^2\Big) + \Big(\sup_{-r_k^2\leq t-t_0<0}\int_{\tilde{B}_k}|v|^2\Big)^{5/3}.
$$

This inequality, combined with (4.17), implies that

$$
\begin{aligned}
\frac{1}{r_{k+1}^5}\iint_{\tilde{Q}_{k+1}} |v|^{\frac{10}{3}} \leq& C\Big(\frac{1}{r_k^3}\sup_{-r_k^2\leq t-t_0<0}\int_{\tilde{B}_k}|v|^2\Big)^{\frac{5}{3}}\\
&+ C\Big(\frac{1}{r_k^3}\sup_{-r_k^2\leq t-t_0<0}\int_{\tilde{B}_k}|v|^2\Big)^{\frac{3}{2}}\Big(r_k^{-3}\iint_{\tilde{Q}_k}|\nabla v|^2\Big)\\
\leq& C\varepsilon^{\frac{11}{12}}.
\end{aligned} \tag{4.18}
$$

Collecting the above bounds, we eventually conclude that

$$
⨌_{\tilde{Q}_{k+1}} |v|^{\frac{10}{3}} \leq \varepsilon^{2/3}.
$$

This completes the proof of this theorem. □

Acknowledgements. We are deeply grateful to the anonymous referee and the associated editor for the invaluable comments and suggestions which helped to improve the paper significantly. Jiu was partially supported by the National Natural Science Foundation of China (No. 11671273) and by Beijing Natural Science Foundation (No. 1192001). The research of Wang was partially supported by the National Natural Science Foundation of China under Grant No. 11601492 and the Youth Core Teachers Foundation of Zhengzhou University of Light Industry. The research of Zhou is supported in part by the China Scholarship Council for one year study at Mathematical Institute of University of Oxford and Doctor Fund of Henan Polytechnic University (No. B2012-110). Part of this work was done when D. Zhou is visiting Mathematical Institute of University of Oxford. Zhou appreciates Prof. G. Seregin's warm hospitality and support.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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(accepted: March 13, 2019; published online: March 22, 2019)