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Exact Solution and Instability for Geophysical Waves with Centripetal Forces and at Arbitrary Latitude

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Abstract. The aim of this paper is to provide, in a β-plane approximation with centripetal forces, an explicit three-dimensional nonlinear solution for geophysical waves propagating at an arbitrary latitude, in the presence of a constant underlying background current. This solution is linearly unstable when the steepness of the wave exceeds a specific threshold.

Keywords. Geophysical waves, Centripetal forces, Exact solution, Short-wavelength method, Localized instability analysis.

1. Introduction

This paper is focused on geophysical ocean waves where both Coriolis and centripetal effects of the Earths rotation play a significant role. In the literature, centripetal forces are typically neglected as they are relatively much smaller than Coriolis forces. The retention of these terms in the appropriate governing equations increases their mathematical complexity but plays a central role in facilitating the admission of a wide-range of depth-invariant underlying currents in their solutions (see [21]). In oceanography, the governing equations are typically simplified by invoking tangent plane-approximations—whereby the Earth's curved surface is locally approximated by a tangent plane—the classical form being the β-plane approximation. The additional centripetal terms will contribute to the standard β-plane approximate equations(in the equatorial region see [21]). Geophysical processes which occur in the equatorial region attract a significant attention and are a fascinating topic in oceanography (see [8,13,18,39]. The β-plane approximation applies in regions within 5° latitude, either side of the Equator. For ease of calculation, within a restricted meridional of approximately 2° latitude, either side of the Equator, it is adequate to use the f-plane approximation in the governing equations. There is a large literature on equatorial wave dynamics. To model various geophysical oceanic waves and wave-current interactions in the equatorial region, nonlinear three-dimensional Gerstner-like solutions have been derived (see [3–6,19–23,26–28,36,37]). The geophysical dynamics of the equatorial region are complicated, the underlying currents being highly depth-dependent; the Gerstner-like solutions do not capture strong depth variations of the flows. The recent papers [9–11,32,33] present some exact nonlinear three-dimensional solutions that capture strong depth variations of the flows.

It is important and very useful, to seek exact and explicit solutions which exist at an arbitrary latitude. An f-plane approximation at an arbitrary latitude was firstly taken into account in [38]. Gerstner-type solution was obtained in the f-plane approximation at an arbitrary latitude in [38] and recently in [12] a depth-invariant mean current was accommodated into this solution. In [15] an extension of the exact solution [26] for equatorial waves in the f-plane approximation was obtained at an arbitrary latitude and in the presence of a constant underlying background current. A β-plane approximation at an arbitrary

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latitude in the presence of an underlying current and a Gerstner-like solution to this problem was very recently provided in [2]. In this paper, we consider the β-plane governing equations at an arbitrary latitude, modified to incorporate centripetal forces. We obtain an exact Gerstner-type solution of this problem, that is, the solution (3.1) with the stated conditions (3.3) and (3.16), which prescribes three-dimensional geophysical wave propagating in a relatively narrow ocean strip at an arbitrary latitude, in the presence of a constant underlying current. The dispersion relation of the waves obtained features contributions from the Coriolis force, the centripetal force and the underlying current. The waves propagate both eastward and westward. We make a detailed discussion of the situations encountered in the Northern Hemisphere and the Southern Hemisphere, for admissible following as well as adverse currents; the solution admits both following and adverse currents of physically plausible magnitude. We mention that we do not deal with the meridional decay of solutions. If we take this into account, then, our study is reduced only to cases N1 and N3 in the Northern Hemisphere and to cases S1 and S3 in the Southern Hemisphere. We remark that in these cases the waves can propagate both eastward and westward too.

The elegant short wavelength instability method (a rigorous mathematical approach to the problem of stability for general three-dimensional inviscid incompressible flows developed independently in [1,16,35]), suitable for Gerstner-like solutions (in the geophysical context was successfully achieved in [2,7,14,17,24, 25,29–31]), is also applied to the exact solution obtained, in order to prove a wave-steepness instability criterion. The critical steepness is very close to $\frac{1}{3}$. The waves which travel from east to west are more prone to instability than those which travel from west to east. An adverse current favours instability in the sense that the threshold on the steepness for the wave to be unstable is decreased compared to the case without current. Conversely, this threshold is increased by a following current.

2. The Governing Equations

We recall the governing equations of geophysical fluid motion in cylindrical coordinates, which was derived by Constantin and Johnson in [10], and from which we derive the appropriate β-plane approximation. The cylindrical coordinates (x, ϕ, z) are chosen such that the origin is located at the centre of the Earth, the generator of the cylinder (which represents the 'straightened-out' equator) is the x-axis with the positive x-direction going from west to east, ϕ is the angle of latitude with $-\frac{\pi}{2} \leqslant \phi \leqslant \frac{\pi}{2}$, and we set $z = r - R$ to be the variation in the locally vertical direction of the radial variable from the Earth's surface. The full governing equations for geophysical fluid dynamics in these cylindrical coordinates are the Euler equations

$$\begin{cases} U_t + UU_x + \frac{VU_\phi}{R+z} + WU_z + 2\Omega(W\cos\phi - V\sin\phi) = -\frac{1}{\rho}P_x, \\ V_t + UV_x + \frac{VV_\phi}{R+z} + \frac{WV}{R+z} + 2\Omega U\sin\phi + (R+z)\Omega^2\sin\phi\cos\phi = -\frac{1}{\rho}\frac{P_\phi}{R+z}, \\ W_t + UW_x + \frac{VW_\phi}{R+z} + WW_z - \frac{V^2}{R+z} - 2\Omega U\cos\phi - (R+z)\Omega^2\cos^2\phi = -\frac{1}{\rho}P_z - g, \end{cases}$$

together with the equation of incompressibility

$$U_x + \frac{1}{R+z}V_\phi + \frac{1}{R+z}\frac{\partial}{\partial z}[(R+z)W] = 0.$$

Here (U, V, W) is the fluid velocity field, P is the pressure, ρ is the water's density, t is the time, $g = 9.8$ m/s^2 is the standard gravitational acceleration at the Earth's surface and the geophysical parameters: $\Omega = 7.29 \times 10^{-5}$ rad/s the rotational speed of the Earth and $R = 6378$ km the radius of the Earth (we assume that the shape of the Earth is a perfect sphere).

The Coriolis parameters, defined by:

$$f = 2\Omega\sin\phi, \quad \hat{f} = 2\Omega\cos\phi$$

depend on the variable latitude ϕ. At the Equator $f = 0$, $\hat{f} = 2\Omega$, at the North Pole we have $f = 2\Omega$, $\hat{f} = 0$, close to 45° latitude in the Northern Hemisphere the values are $f = \hat{f} = 10^{-4}$ s^{-1} and close to 45° latitude in the Southern Hemisphere $f = -\hat{f} = -10^{-4}$ s^{-1} (see [18]). For surface water waves propagating zonally in a relatively narrow ocean strip less than a few degrees of latitude wide, that is, $\phi+\alpha$, with α very small, it is adequate to use the f- or β-plane approximations. Within the f-plane approximation the Coriolis parameters are treated as constants. Within the traditional β-plane approximation, we consider that, at the fixed latitude ϕ, $\hat{f}$ is constant and f has a linear variation with the latitude. Defining $y=R\alpha$ and retaining only terms of linear order in the expansion of $\sin(\phi+\alpha)$, this linear variation has the form $f+\beta y$, with

$$\beta=\frac{\hat{f}}{R}=\frac{2\Omega\cos\phi}{R}.$$

Furthermore the radius R is very large relative to the vertical variations z, hence $\frac{z}{R}\to 0$. In terms of the Cartesian coordinate system (x,y,z), we get the following β-plane approximation equations for geophysical fluid dynamics with centripetal terms:

$$\begin{cases} U_t+UU_x+VU_y+WU_z+\hat{f}W-(f+\beta y)V=-\frac{1}{\rho}P_x, \\ V_t+UV_x+VV_y+WV_z+(f+\beta y)U+\frac{\hat{f}^2}{4}y+\frac{\hat{f}f}{4}R=-\frac{1}{\rho}P_y, \\ W_t+UW_x+VW_y+WW_z-\hat{f}U-\frac{\hat{f}^2}{4}R=-\frac{1}{\rho}P_z-g, \end{cases} \tag{2.1}$$

together with the equation of mass conservation

$$\rho_t+U\rho_x+V\rho_y+W\rho_z=0, \tag{2.2}$$

and with the condition of incompressibility

$$U_x+V_y+W_z=0. \tag{2.3}$$

Denoting the free surface by $\eta(x,y,t)$ and letting P_{atm} be the constant atmospheric pressure, the relevant boundary conditions at the free surface are the kinematic boundary condition

$$W=\eta_t+U\eta_x+V\eta_y \quad \text{on} \quad z=\eta(x,y,t), \tag{2.4}$$

which implies that fluid particles on the free surface remain on the surface for all time, and the dynamic boundary condition

$$P=P_{\text{atm}} \quad \text{on} \quad z=\eta(x,y,t), \tag{2.5}$$

which decouples the water flow from the motion of the air above. Finally, we assume that the water is infinitely deep, with the flow converging rapidly with depth to a uniform zonal current, that is,

$$(U,V,W)\to(-c_0,0,0) \quad \text{as} \quad z\to-\infty. \tag{2.6}$$

3. Exact Solution

We will use the Lagrangian framework for the exact solution. In the Lagrangian framework, the Eulerian coordinates of fluid particles $\mathbf{x}=(x,y,z)$ at the time t are expressed as functions of Lagrangian labelling variables (q,s,r) which specify the fluid particle. We suppose that the position of a particle at time t is given as

$$\begin{cases} x=q-c_0t-\frac{1}{k}e^{k[r-h(s)]}\sin[k(q-ct)], \\ y=s, \\ z=r+\frac{1}{k}e^{k[r-h(s)]}\cos[k(q-ct)], \end{cases} \tag{3.1}$$

in which k is the wavenumber and c_0 is a constant underlying current such that for $cc_0>0$ the current is adverse, while for $cc_0<0$ the current is following. For later considerations, we take, on physical grounds,

$$|c_0|<\frac{g}{\hat{f}}-\frac{\hat{f}R}{4}, \quad \hat{f}\neq 0. \tag{3.2}$$

The right hand side in (3.2) is always positive. Indeed, the parameter $\hat{f}$ is positive and since $\Omega^2R\approx 3\times 10^{-2}$ m/s^2, we have $g\gg\Omega^2R>\Omega^2\cos\phi^2R$.

We will prove that the system (3.1) defines an exact solution of the β-plane governing equations (2.1)–(2.6), where the travelling speed c and the function h depending on s are determined below. The Lagrangian labelling variables are given by real values of $q \in \mathbb{R}, s \in [-s_0, s_0]$ and $r \in (-\infty, r_0]$ such that

$$r - h(s) \leq r_0 < 0 \tag{3.3}$$

to ensure that the flow has the appropriate decay properties.
For notational convenience, we set

$$\xi = k[r - h(s)], \quad \theta = k(q - ct).$$

The Jacobian matrix of the transformation (3.1) is given by

$$J = \begin{pmatrix} \frac{\partial x}{\partial q} & \frac{\partial y}{\partial q} & \frac{\partial z}{\partial q} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \end{pmatrix} = \begin{pmatrix} 1 - e^{\xi}\cos\theta & 0 & -e^{\xi}\sin\theta \\ h_s e^{\xi}\sin\theta & 1 & -h_s e^{\xi}\cos\theta \\ -e^{\xi}\sin\theta & 0 & 1 + e^{\xi}\cos\theta \end{pmatrix},$$

from which we know that the determinant of the Jacobian is $1 - e^{2\xi}$, which is non-zero under the condition (3.3), and hence the transformation (3.1) is well defined.

3.1. The Pressure Function

Let us now write the Euler equation (2.1) in the form

$$\begin{cases} \frac{\mathrm{D}U}{\mathrm{D}t} + \hat{f}W - (f + \beta y)V = -\frac{1}{\rho}P_x, \\ \frac{\mathrm{D}V}{\mathrm{D}t} + (f + \beta y)U + \frac{\hat{f}^2}{4}y + \frac{\hat{f}f}{4}R = -\frac{1}{\rho}P_y, \\ \frac{\mathrm{D}W}{\mathrm{D}t} - \hat{f}U - \frac{\hat{f}^2}{4}R = -\frac{1}{\rho}P_z - g, \end{cases} \tag{3.4}$$

where $\frac{\mathrm{D}}{\mathrm{D}t}$ stands for the material derivative. From (3.1) we can compute the velocity and acceleration of a particle as

$$\begin{cases} U = \frac{\mathrm{D}x}{\mathrm{D}t} = -c_0 + ce^{\xi}\cos\theta, \\ V = \frac{\mathrm{D}y}{\mathrm{D}t} = 0, \\ W = \frac{\mathrm{D}z}{\mathrm{D}t} = ce^{\xi}\sin\theta, \end{cases} \tag{3.5}$$

and

$$\begin{cases} \frac{\mathrm{D}U}{\mathrm{D}t} = kc^2 e^{\xi}\sin\theta, \\ \frac{\mathrm{D}V}{\mathrm{D}t} = 0, \\ \frac{\mathrm{D}W}{\mathrm{D}t} = -kc^2 e^{\xi}\cos\theta, \end{cases}$$

respectively. We can therefore write (3.4) as

$$\begin{cases} P_x = -\rho(kc^2 e^{\xi}\sin\theta + \hat{f}ce^{\xi}\sin\theta), \\ P_y = -\rho\left[(f + \beta s)(-c_0 + ce^{\xi}\cos\theta) + \frac{\hat{f}^2}{4}s + \frac{\hat{f}f}{4}R\right], \\ P_z = -\rho\left(-kc^2 e^{\xi}\cos\theta + \hat{f}c_0 - \hat{f}ce^{\xi}\cos\theta - \frac{\hat{f}^2}{4}R + g\right). \end{cases} \tag{3.6}$$

The change of variables

$$\begin{pmatrix} P_q \\ P_s \\ P_r \end{pmatrix} = J \begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix}$$

transforms (3.6) into

$$\begin{cases} P_q = -\rho\left(kc^2 + \hat{f}c - \hat{f}c_0 + \frac{\hat{f}^2}{4}R - g\right)e^{\xi}\sin\theta, \\ P_s = -\rho\left[h_s(kc^2 + \hat{f}c)e^{2\xi} + H(s)e^{\xi}\cos\theta - \left(fc_0 + \beta c_0 s - \frac{\hat{f}^2}{4}s - \frac{\hat{f}f}{4}R\right)\right], \\ P_r = -\rho\left[-(kc^2 + \hat{f}c)e^{2\xi} - \left(kc^2 + \hat{f}c - \hat{f}c_0 + \frac{\hat{f}^2}{4}R - g\right)e^{\xi}\cos\theta \right. \\ \qquad \left. + \left(\hat{f}c_0 + g - \frac{\hat{f}^2}{4}R\right)\right], \end{cases} \tag{3.7}$$

where

$$H(s) = fc + c\beta s - \hat{f}c_0 h_s + \frac{\hat{f}^2}{4}Rh_s - gh_s.$$

Now we give some suitable conditions on the pressure function P such that (3.7) holds and (3.1) is indeed an exact solution of the governing equations (2.1)–(2.6). Since the condition (2.5) enforces a time independence in the pressure function at the surface, it is necessary to eliminate terms containing ϕ in (3.7) by setting

$$kc^2 + \hat{f}c - \hat{f}c_0 + \frac{\hat{f}^2}{4}R - g = 0, \tag{3.8}$$

and

$$H(s) = fc + c\beta s - \hat{f}c_0 h_s + \frac{\hat{f}^2}{4}Rh_s - gh_s = 0. \tag{3.9}$$

It follows from (3.9) that we can choose

$$h(s) = \frac{c\beta}{2\left(\hat{f}c_0 + g - \frac{\hat{f}^2}{4}R\right)}s^2 + \frac{fc}{\hat{f}c_0 + g - \frac{\hat{f}^2}{4}R}s. \tag{3.10}$$

From (3.2), we get that

$$\hat{f}c_0 + g - \frac{\hat{f}^2}{4}R > 0 \tag{3.11}$$

With the constraints (3.8) and (3.9), we can solve the pressure function as

$$\begin{aligned} P(r,s) = {} & \rho\left(\hat{f}c_0 + g - \frac{\hat{f}^2}{4}R\right)\left[\frac{e^{2\xi}}{2k} + \frac{c_0}{c}h(s) - \frac{\hat{f}^2 s^2 + 2\hat{f}fRs}{8\left(\hat{f}c_0 + g - \frac{\hat{f}^2}{4}R\right)} - r\right] \\ & - \rho\left(\hat{f}c_0 + g - \frac{\hat{f}^2}{4}R\right)\left(\frac{e^{2kr_0}}{2k} - r_0\right) + P_{\text{atm}}. \end{aligned} \tag{3.12}$$

The constant terms in (3.12) have been chosen to ensure the conditions (2.4) and (2.5) hold on the free surface.

From the dispersion relation (3.8), for $c_0 \neq c$, we get

$$c_{\pm} = \frac{-\hat{f} \pm \sqrt{\hat{f}^2 + 4k\left(\hat{f}c_0 + g - \frac{\hat{f}^2}{4}R\right)}}{2k}, \tag{3.13}$$

in which $\Delta = \hat{f}^2 + 4k\left(\hat{f}c_0 + g - \frac{\hat{f}^2}{4}R\right) > 0$ following (3.11). If $c = c_+ > 0$, the wave travels from west to east and if $c = c_- < 0$, the wave travels from east to west.

3.2. The Free-Surface Interface

We now focus on the expression (3.12) of the pressure function and the fulfilment of the boundary conditions (2.4) and (2.5). In Lagrangian variables, the kinematic boundary condition (2.4) holds if at each fixed latitude s the free surface is given by specifying a value of r, the label q being the free parameter of the curve that represents the wave profile at this latitude. With (3.12) in view, this is achieved if we show that, at each fixed s, there exists a unique solution $r(s) \leq r_0 < 0$ such that $P(r(s), s) = P_{\text{atm}}$, which is equivalent to

$$\eta(r(s), s) = \frac{e^{2kr_0}}{2k} - r_0, \tag{3.14}$$

in which

$$\eta(r, s) = \frac{e^{2\xi}}{2k} + \frac{c_0}{c}h(s) - \frac{\hat{f}^2 s^2 + 2\hat{f}fRs}{8\left(\hat{f}c_0 + g - \frac{\hat{f}^2}{4}R\right)} - r.$$

For $s = 0$, the choice $r(0) = r_0$ works in (3.14). For the case $s \neq 0$, by (3.3), we have

$$\eta_r = e^{2k[r-h(s)]} - 1 < 0,$$

which means that η is decreasing in r. Moreover, it is obvious that

$$\lim_{r \to -\infty} \eta(r, s) = +\infty.$$

Thus equation (3.14) has a unique solution if the following inequality holds

$$\begin{aligned} \lim_{r \to r_0} \eta(r, s) = \frac{e^{2k[r_0 - h(s)]}}{2k} + \frac{c_0}{c}h(s) - \frac{\hat{f}^2 s^2 + 2\hat{f}fRs}{8\left(\hat{f}c_0 + g - \frac{\hat{f}^2}{4}R\right)} - r_0 \\ < \frac{e^{2kr_0}}{2k} - r_0, \end{aligned}$$

which is equivalent to the inequality

$$A(s) := \frac{e^{2kr_0}}{2k}\left[e^{-2kh(s)} - 1\right] + \frac{c_0}{c}h(s) - \frac{\hat{f}^2 s^2 + 2\hat{f}fRs}{8\left(\hat{f}c_0 + g - \frac{\hat{f}^2}{4}R\right)} < 0. \tag{3.15}$$

We will look for the geophysical waves satisfying

$$\begin{cases} r'(s) < 0, & s > 0, \\ r'(s) > 0, & s < 0. \end{cases} \tag{3.16}$$

Differentiating (3.14) with respect to s, we obtain that

$$r'(s)\left[e^{2k[r(s)-h(s)]} - 1\right] - h'(s)\left[e^{2k[r(s)-h(s)]} - \frac{c_0}{c}\right] - \frac{\hat{f}^2 s + \hat{f}fR}{4\left(\hat{f}c_0 + g - \frac{\hat{f}^2}{4}R\right)} = 0,$$

and therefore we get

$$\begin{aligned} r'(s) &= \frac{\left[c_0 - ce^{2k[r(s)-h(s)]}\right]\frac{h'(s)}{c} - \frac{\hat{f}^2 s + \hat{f}fR}{4\left(\hat{f}c_0 + g - \frac{\hat{f}^2}{4}R\right)}}{1 - e^{2k[r(s)-h(s)]}} \\ &= \frac{\left[c_0 - ce^{2k[r(s)-h(s)]}\right](\beta s + f) - \frac{\hat{f}^2}{4}s - \frac{\hat{f}fR}{4}}{\left[1 - e^{2k[r(s)-h(s)]}\right]\left(\hat{f}c_0 + g - \frac{\hat{f}^2}{4}R\right)}. \end{aligned}$$

Then, the inequalities (3.16) become

$$\begin{cases} \left[c_0 - ce^{2k[r(s)-h(s)]}\right](\beta s + f) - \frac{\hat{f}^2}{4}s - \frac{\hat{f}fR}{4} < 0, & s > 0, \\ \left[c_0 - ce^{2k[r(s)-h(s)]}\right](\beta s + f) - \frac{\hat{f}^2}{4}s - \frac{\hat{f}fR}{4} > 0, & s < 0, \end{cases} \tag{3.17}$$

since (3.3) and (3.11) hold.

It follows from (3.16) and the fact $r(0) = r_0$ that $r(s) < r_0$ for $s \neq 0$. Since (3.3) holds and $h(0) = 0$, we obtain that for $s \neq 0$,

$$h(s) > 0, \text{or } h(s) < 0 \text{ close enough to zero.} \tag{3.18}$$

Note that $h(s) > 0$ is equivalent to

$$cs(\beta s + 2f) > 0, \tag{3.19}$$

and $h(s) < 0$ is equivalent to

$$cs(\beta s + 2f) < 0. \tag{3.20}$$

Up to now, we have obtained the restrictions (3.15)–(3.17)–(3.18) required for the hydrodynamical possibility of the flow.

Let us see for which values of the uniform current c_0 the required restrictions (3.15) and (3.17) are satisfied. We will make a separate case discussion in the Northern Hemisphere and in the Southern Hemisphere.

I. Northern Hemisphere

For the Northern Hemisphere, we have that $\hat{f} > 0$, $f > 0$ and $\beta > 0$. We have four different cases.

Case N1. $cc_0 > 0$, that is, the current is adverse, and $h(s) > 0$.

The condition $cc_0 > 0$ is obtained if:

(a) the solution $c = c_+ > 0$ and the uniform current $0 < c_0 < \frac{g}{\hat{f}} - \frac{\hat{f}R}{4}$.

It follows from (3.19) that $h(s) > 0$ if and only if $s > 0$ or $s < -\frac{2f}{\beta}$. From the physical viewpoint, the possibility $s < -\frac{2f}{\beta}$ can be excluded, therefore $s > 0$. The inequalities (3.15) and (3.17) become

$$A(s) < 0, \quad s > 0 \tag{3.21}$$

$$\left[c_0 - c_+ e^{2k[r(s)-h(s)]}\right](\beta s + f) - \frac{\hat{f}^2}{4}s - \frac{\hat{f}fR}{4} < 0, \quad s > 0, \tag{3.22}$$

respectively. We note that $A(0) = 0$. Thus, to ensure the inequality (3.21) holds, the necessary condition is

$$A'(s) = \frac{\beta s + f}{\hat{f}c_0 + g - \frac{\hat{f}^2}{4}R}\left[c_0 - c_+ e^{2k[r_0 - h(s)]}\right] - \frac{\hat{f}^2 s + \hat{f}fR}{4\left(\hat{f}c_0 + g - \frac{\hat{f}^2}{4}R\right)} < 0, \tag{3.23}$$

for $s > 0$ small enough. The inequality (3.23) is equivalent to

$$c_0 - c_+ e^{2k[r_0 - h(s)]} < \frac{\hat{f}^2 s + \hat{f}fR}{4(\beta s + f)}. \tag{3.24}$$

For s small enough, the right hand side of (3.24) becomes $\frac{\hat{f}R}{4}$. Therefore, for a given uniform current c_0 with

$$0 < c_0 < c_+ e^{2kr_0} + \frac{\hat{f}R}{4},$$

(3.24) holds for some $s \in (0, s_1]$ and accordingly (3.22) holds for $s \in (0, s_0]$ with $s_0 < s_1$. We observe that, at the Equator, the right hand side of the inequality (3.24) becomes $\frac{\Omega R}{2} \approx 2.33 \times 10^{-2}$ m/s.

(b) the solution $c = c_- < 0$ and the uniform current $-\frac{g}{\hat{f}} + \frac{\hat{f}R}{4} < c_0 < 0$.

In this case, $-\frac{2f}{\beta} < s < 0$. Since we consider a relatively narrow ocean strip, we can restrict to $-\frac{f}{\beta} < s < 0$, and therefore $\beta s + f > 0$. The inequalities (3.15) and (3.17) become

$$A(s) < 0, \quad -\frac{f}{\beta} < s < 0 \tag{3.25}$$

$$\left[c_0 - c_- e^{2k[r(s)-h(s)]}\right](\beta s + f) - \frac{\hat{f}^2}{4}s - \frac{\hat{f} f R}{4} > 0, \quad -\frac{f}{\beta} < s < 0, \tag{3.26}$$

respectively. To ensure the validity of (3.25), the necessary condition is

$$A'(s) = \frac{\beta s + f}{\hat{f}c_0 + g - \frac{\hat{f}^2}{4}R}\left[c_0 - c_- e^{2k[r_0 - h(s)]}\right] - \frac{\hat{f}^2 s + \hat{f} f R}{4\left(\hat{f}c_0 + g - \frac{\hat{f}^2}{4}R\right)} > 0, \tag{3.27}$$

for $-\frac{f}{\beta} < s < 0$ with $|s|$ small enough. The inequality (3.27) is equivalent to

$$c_0 - c_- e^{2k[r_0 - h(s)]} > \frac{\hat{f}^2 s + \hat{f} f R}{4(\beta s + f)}. \tag{3.28}$$

For $|s|$ small enough, the right hand side of (3.28) becomes $\frac{\hat{f}R}{4}$. Therefore, if

$$c_- e^{2kr_0} + \frac{\hat{f}R}{4} < 0, \tag{3.29}$$

then, for a given uniform current c_0 with

$$c_- e^{2kr_0} + \frac{\hat{f}R}{4} < c_0 < 0,$$

(3.28) holds for some $s \in [-s_1, 0)$ and accordingly (3.26) holds for $s \in [-s_0, 0)$ with $s_0 < s_1$.

Case N2. $cc_0 > 0$, that is, the current is adverse, and $h(s) < 0$.
The condition $cc_0 > 0$ is obtained if:

(a) the solution $c = c_+ > 0$ and the uniform current $0 < c_0 < \frac{g}{\hat{f}} - \frac{\hat{f}R}{4}$.

It follows from (3.20) that $h(s) < 0$ if and only if $-\frac{2f}{\beta} < s < 0$. We can restrict $-\frac{f}{\beta} < s < 0$, and therefore $\beta s + f > 0$. The inequalities (3.15) and (3.17) become (3.25) and (3.26), with c_+ instead of c_-. The same reasoning as in the **N1(b)** applies to this case, hence, we get that, for a given uniform current c_0 with

$$c_+ e^{2kr_0} + \frac{\hat{f}R}{4} < c_0 < \frac{g}{\hat{f}} - \frac{\hat{f}R}{4},$$

the corresponding inequalities hold for $s \in [-s_0, 0)$ with s_0 small enough.

(b) the solution $c = c_- < 0$ and the uniform current $-\frac{g}{\hat{f}} + \frac{\hat{f}R}{4} < c_0 < 0$.

In the same manner as in **N1(a)**, we get that, for a given uniform current c_0 with

$$-\frac{g}{\hat{f}} + \frac{\hat{f}R}{4} < c_0 < c_- e^{2kr_0} + \frac{\hat{f}R}{4},$$

(3.15) and (3.17) hold for $s \in (0, s_0]$ with s_0 small enough.

Case N3. $cc_0 < 0$, that is, the current is following, and $h(s) > 0$.
In this case, (3.15) is obviously satisfied. Now we see what happens with the inequality (3.17). The condition $cc_0 < 0$ is obtained if:

(a) the solution $c = c_+ > 0$ and the uniform current $-\frac{g}{\hat{f}} + \frac{\hat{f}R}{4} < c_0 < 0$.

In this situation, the term $c_0 - c_+ e^{2k[r(s)-h(s)]} < 0$. Just as in the case **N1(a)**, we can exclude $s < -\frac{2f}{\beta}$, and we get $s > 0$. We thus get that the inequality (3.17) is satisfied for all $s > 0$.

(b) the solution $c = c_- < 0$ and the uniform current $0 < c_0 < \frac{g}{\hat{f}} - \frac{\hat{f}R}{4}$.

Then, the term $c_0 - c_- e^{2k[r(s)-h(s)]} > 0$ and from (3.10) and (3.11) it follows that $h(s) > 0$ if and only if $-\frac{2f}{\beta} < s < 0$. Since we consider a relatively narrow ocean strip, we can restrict to $-\frac{f}{\beta} < s < 0$, where the inequality (3.17) holds.

Case N4. $cc_0 < 0$, that is, the current is following, and $h(s) < 0$.
The condition $cc_0 < 0$ is obtained if:

(a) the solution $c = c_+ > 0$ and the uniform current $-\frac{g}{\hat{f}} + \frac{\hat{f}R}{4} < c_0 < 0$.

In this case, $-\frac{2f}{\beta} < s < 0$, we can restrict to $-\frac{f}{\beta} < s < 0$, and therefore $\beta s + f > 0$. The inequalities (3.15) and (3.17) become (3.25) and (3.26), with c_+ instead of c_-. In order to satisfy these inequalities, we get that, for s small enough, the uniform current c_0 has to satisfy

$$0 < c_+ e^{2kr_0} + \frac{\hat{f}R}{4} < c_0,$$

which is in contradiction with $c_0 < 0$. Thus, this is a nonvalid case.

(b) the solution $c = c_- < 0$ and the uniform current $0 < c_0 < \frac{g}{\hat{f}} - \frac{\hat{f}R}{4}$.

In the same manner as in **N1(a)**, $s > 0$ and the inequalities (3.15) and (3.17) become (3.21) and (3.22), with c_- instead of c_+. For s small enough, these inequalities hold for uniform currents c_0 with

$$c_0 < c_- e^{2kr_0} + \frac{\hat{f}R}{4}.$$

If (3.29) is satisfied, then, the above inequality is in contradiction with $c_0 > 0$, and the case is invalid.

II. Southern Hemisphere

In this case, we have that $\hat{f} > 0$, $f < 0$ and $\beta > 0$. Similar arguments to that in the Northern Hemisphere apply in the four cases considered.

Case S1. $cc_0 > 0$, that is, the current is adverse, and $h(s) > 0$.
The condition $cc_0 > 0$ is obtained if:

(a) the solution $c = c_+ > 0$ and the uniform current $0 < c_0 < \frac{g}{\hat{f}} - \frac{\hat{f}R}{4}$. By (3.19), $h(s) > 0$ if and only if $s < 0$ or $s > -\frac{2f}{\beta}$. From the physical viewpoint, the possibility $s > -\frac{2f}{\beta}$ can be excluded, therefore $s < 0$. The inequalities (3.15) and (3.17) become

$$A(s) < 0, \quad s < 0 \tag{3.30}$$

$$\left[c_0 - c_+ e^{2k[r(s)-h(s)]}\right](\beta s + f) - \frac{\hat{f}^2}{4}s - \frac{\hat{f}fR}{4} > 0, \quad s < 0, \tag{3.31}$$

respectively. To ensure the validity of (3.30), the necessary condition is

$$A'(s) = \frac{\beta s + f}{\hat{f}c_0 + g - \frac{\hat{f}^2}{4}R}\left[c_0 - c_+ e^{2k[r_0-h(s)]}\right] - \frac{\hat{f}^2 s + \hat{f}fR}{4\left(\hat{f}c_0 + g - \frac{\hat{f}^2}{4}R\right)} > 0, \tag{3.32}$$

for $s < 0$ with $|s|$ small enough. The inequality (3.32) is equivalent to

$$(\beta s + f)\left[c_0 - c_+ e^{2k[r_0-h(s)]}\right] > \frac{\hat{f}^2 s + \hat{f}fR}{4}. \tag{3.33}$$

For $|s|$ small enough, (3.33) tends to $f\left(c_0 - c_+ e^{2kr_0}\right) > \frac{\hat{f}fR}{4}$, and taking into account that now $f < 0$, we get that for a given uniform current c_0 with

$$0 < c_0 < c_+ e^{2kr_0} + \frac{\hat{f}R}{4},$$

(3.33) and (3.31) hold for some $s \in [-s_0, 0)$.

(b) the solution $c = c_- < 0$ and the uniform current $-\frac{g}{\hat{f}} + \frac{\hat{f}R}{4} < c_0 < 0$. The inequalities (3.15) and (3.17) become

$$A(s) < 0, \quad 0 < s < -\frac{f}{\beta} \tag{3.34}$$

$$\left[c_0 - c_+ e^{2k[r(s)-h(s)]}\right](\beta s + f) - \frac{\hat{f}^2}{4}s - \frac{\hat{f}fR}{4} < 0, \quad 0 < s < -\frac{f}{\beta}, \tag{3.35}$$

respectively. Then, if (3.29) is satisfied, for a given uniform current c_0 with

$$c_- e^{2kr_0} + \frac{\hat{f}R}{4} < c_0 < 0,$$

(3.15) and (3.17) hold for $s \in (0, s_0]$ with s_0 small enough.

Case S2. $cc_0 > 0$, that is, the current is adverse, and $h(s) < 0$.
The condition $cc_0 > 0$ is obtained if:

(a) the solution $c = c_+ > 0$ and the uniform current $0 < c_0 < \frac{g}{\hat{f}} - \frac{\hat{f}R}{4}$. It follows from (3.20) that $h(s) < 0$ if and only if $0 < s < -\frac{2f}{\beta}$. We can restrict $0 < s < -\frac{f}{\beta}$, and therefore $\beta s + f < 0$. The inequalities (3.15) and (3.17) become

$$A(s) < 0, \quad 0 < s < -\frac{f}{\beta} \tag{3.36}$$

$$\left[c_0 - c_+ e^{2k[r(s)-h(s)]}\right](\beta s + f) - \frac{\hat{f}^2}{4}s - \frac{\hat{f}fR}{4} < 0, \quad 0 < s < -\frac{f}{\beta}, \tag{3.37}$$

respectively. In the same manner, (3.36) holds if

$$A'(s) = \frac{\beta s + f}{\hat{f}c_0 + g - \frac{\hat{f}^2}{4}R}\left[c_0 - c_+ e^{2k[r_0-h(s)]}\right] - \frac{\hat{f}^2 s + \hat{f}fR}{4\left(\hat{f}c_0 + g - \frac{\hat{f}^2}{4}R\right)} < 0, \tag{3.38}$$

for $s > 0$ small enough. The inequality (3.38) is equivalent to

$$(\beta s + f)\left[c_0 - c_+ e^{2k[r_0-h(s)]}\right] < \frac{\hat{f}^2 s + \hat{f}fR}{4}. \tag{3.39}$$

For s small enough, (3.39) tends to $f\left(c_0 - c_+ e^{2kr_0}\right) < \frac{\hat{f}fR}{4}$, and we get that for a given uniform current c_0 with

$$c_+ e^{2kr_0} + \frac{\hat{f}R}{4} < c_0 < \frac{g}{\hat{f}} - \frac{\hat{f}R}{4},$$

(3.39) and (3.37) hold for some $s \in (0, s_0]$, with s_0 small enough.

(b) the solution $c = c_- < 0$ and the uniform current $-\frac{g}{\hat{f}} + \frac{\hat{f}R}{4} < c_0 < 0$.
We get that, for a given uniform current c_0 with

$$-\frac{g}{\hat{f}} + \frac{\hat{f}R}{4} < c_0 < c_- e^{2kr_0} + \frac{\hat{f}R}{4},$$

(3.15) and (3.17) hold for $s \in [-s_0, 0)$ with s_0 small enough.

Case S3. $cc_0 < 0$, that is, the current is following, and $h(s) > 0$.
The condition $cc_0 < 0$ is obtained if:

(a) the solution $c = c_+ > 0$ and the uniform current $-\frac{g}{\hat{f}} + \frac{\hat{f}R}{4} < c_0 < 0$.
In this case, just as in the case **S1(a)**, we can exclude $s > -\frac{2f}{\beta}$, and we get $s < 0$. Then the inequalities (3.15) and (3.17) are obviously satisfied for all $s < 0$.

(b) the solution $c = c_- < 0$ and the uniform current $0 < c_0 < \frac{g}{\hat{f}} - \frac{\hat{f}R}{4}$.

In this case, $0 < s < -\frac{2f}{\beta}$, and we can restrict $0 < s < -\frac{f}{\beta}$, therefore $\beta s + f < 0$. Then the inequalities (3.15) and (3.17) are obviously satisfied for all $0 < s < -\frac{f}{\beta}$.

Case S4. $cc_0 < 0$, that is, the current is following, and $h(s) < 0$.
This case can be handled in the same way as the case **N4**. It follows that this is a non valid case.
Summing up, we have got the following theorem.

Theorem 3.1. *For a narrow strip $[-s_0, s_0]$, with $s_0 > 0$ small enough, at an arbitrary latitude s in the Northern Hemisphere or in the Southern Hemisphere, the solution (3.1) with the stated conditions (3.3) and (3.16), defines an exact solution to the governing equations (2.1)–(2.6) if:*

- *the current c_0 is adverse, it satisfies the inequality $|c_0| < |c|e^{2kr_0} + \frac{\hat{f}R}{4}$ and the function $h(s)$ (which enforces a strong exponential decay of particle oscillation in the meridional direction) is positive;*
- *the current c_0 is adverse, it satisfies the inequality $|c|e^{2kr_0} + \frac{\hat{f}R}{4} < |c_0| < \frac{g}{\hat{f}} - \frac{\hat{f}R}{4}$ and the function $h(s)$ is negative;*
- *the current c_0 is following, it satisfies $|c_0| < \frac{g}{\hat{f}} - \frac{\hat{f}R}{4}$ and the function $h(s)$ is positive; in the Northern Hemisphere, for flows with positive wave speed $c = c_+ > 0$, the range for the fixed latitude is $s > 0$, and for flows with negative wave speed $c = c_- < 0$, we are restricted to latitudes in the region $-\frac{f}{\beta} < s < 0$; in the Southern Hemisphere for flows with positive wave speed $c = c_+ > 0$, the range for the fixed latitude is $s < 0$, and for flows with negative wave speed $c = c_- < 0$, we are restricted to latitudes in the region $0 < s < -\frac{f}{\beta}$.*

This solution represents a wave-current interaction propagating eastward or westward above a flow which accommodates a constant underlying background current of magnitude $|c_0|$. The free surface $z = \eta(x, y, t)$ is implicitly prescribed at $s = 0$ by setting $r = r_0$ in (3.1), and for any other fixed latitude $s \in [-s_0, s_0]$ there exists a unique value $r(s) < r_0$ which implicitly prescribes the free surface $z = \eta(x, s, t)$ by setting $r = r(s)$ in (3.1).

3.3. The Vorticity

The velocity gradient tension is given as

$$\nabla \mathbf{U} = \frac{cke^{\xi}}{1 - e^{2\xi}} \begin{pmatrix} -\sin\theta & h_s(e^{\xi} - \cos\theta) & -e^{\xi} + \cos\theta \\ 0 & 0 & 0 \\ e^{\xi} + \cos\theta & -h_s \sin\theta & \sin\theta \end{pmatrix}. \tag{3.40}$$

Thus the vorticity $\omega = (w_y - v_z, u_z - w_x, v_x - u_y)$ is given as

$$\omega = \Big(-\frac{c^2k(\beta s + f)}{\hat{f}c_0 + g - \frac{\hat{f}^2}{4}R} \frac{e^{\xi}\sin\theta}{1 - e^{2\xi}}, -\frac{2cke^{2\xi}}{1 - e^{2\xi}}, -\frac{c^2k(\beta s + f)}{\hat{f}c_0 + g - \frac{\hat{f}^2}{4}R} \frac{e^{2\xi} - e^{\xi}\cos\theta}{1 - e^{2\xi}} \Big). \tag{3.41}$$

We can see that the vorticity (3.41) is three-dimensional away from the equator, although the velocity field (3.5) is two-dimensional. Moreover, the first and third components in (3.41) depends on the latitude and the underlying current c_0.

4. Instability

Small perturbations $(\mathbf{u}(t,\mathbf{x}),\ p(t,\mathbf{x}))$ of the geophysical flow $(\mathbf{U}(t,\mathbf{x}),\ P(t,\mathbf{x}))$ which solve the problem (2.1), (2.3) (we set the water density $\rho=1$), are governed by the following linearized equations:

$$\mathbf{u}_t+(\mathbf{U}\cdot\nabla)\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{U}+\mathcal{L}_{f,\hat{f},\beta}\mathbf{u}=-\nabla p \tag{4.1}$$

$$\nabla\cdot\mathbf{u}=0 \tag{4.2}$$

with $\mathcal{L}_{f,\hat{f},\beta}$ given by

$$\mathcal{L}_{f,\hat{f},\beta}:=\begin{pmatrix}0 & -(f+\beta y) & \hat{f}\\ (f+\beta y) & 0 & 0\\ -\hat{f} & 0 & 0\end{pmatrix}.$$

No centripetal terms appear in the linearized equations (4.1)–(4.2), they have the same form as the linearized equations governing the dynamic of the small perturbations of the geophysical flow at an arbitrary latitude on a zonal current obtained in [2] .

The theory of short-wave instabilities consists of considering the evolution of a rapidly varying WKB wave packet:

$$\mathbf{u}(t,\mathbf{x})=[\mathbf{A}(t,\mathbf{x})+\epsilon\boldsymbol{\mathcal{A}}(t,\mathbf{x})]\,e^{\frac{i}{\epsilon}\Phi(t,\mathbf{x})}+\epsilon\mathbf{u}_{rem}(t,\mathbf{x},\epsilon) \tag{4.3}$$

$$p(t,\mathbf{x})=[B(t,\mathbf{x})+\epsilon\mathcal{B}(t,\mathbf{x})]\,e^{\frac{i}{\epsilon}\Phi(t,\mathbf{x})}+\epsilon p_{rem}(t,\mathbf{x},\epsilon), \tag{4.4}$$

with a sharply-peaked initial disturbance $\mathbf{u}_0$ of the form:

$$\mathbf{u}_0:=\mathbf{u}(0,\mathbf{x})=\mathbf{A}(0,\mathbf{x})e^{\frac{i}{\epsilon}\Phi(0,\mathbf{x})}=:\mathbf{A}_0(\mathbf{x})e^{\frac{i}{\epsilon}\Phi_0(\mathbf{x})},$$

ϵ being a small parameter, $\mathbf{A}$, $\boldsymbol{\mathcal{A}}$ vector functions and Φ, B, $\mathcal{B}$ scalar functions, The evolution in time of the solutions of the linearized system (4.1)–(4.2) in the form (4.3)–(4.4), is governed, at leading order in powers of ϵ, by a system of partial differential equations (the eikonal equation for the wave phase Φ and the so-called transport equation for the amplitude vector $\mathbf{A}$) (see [2] and the references therein). The next-order terms $\boldsymbol{\mathcal{A}}$ and $\mathcal{B}$ depend only on $\mathbf{A}$ and $\nabla\Phi$, and the remainder terms $\mathbf{u}_{rem}$, p_{rem} are bounded, in an appropriate norm, at any time t by functions that can depend on t but are independent of ϵ, they being dominated by the growth of the leading order terms. See, for example, [29] for details. By defining the local wave vector $\boldsymbol{\xi}:=\nabla\Phi$, it may be shown that the stability problem is reduced to a system of ordinary differential equations that evolves along the trajectories of the basic flow $\mathbf{U}(t,\mathbf{x})$ (3.5):

$$\frac{d\mathbf{x}}{dt}=\mathbf{U}(t,\mathbf{x}), \tag{4.5}$$

$$\frac{d\boldsymbol{\xi}}{dt}=-(\nabla\mathbf{U})^T\boldsymbol{\xi}, \tag{4.6}$$

$$\frac{d\mathbf{A}}{dt}=-(\mathbf{A}\cdot\nabla)\mathbf{U}-\mathcal{L}_{f,\hat{f},\beta}\mathbf{A}+\frac{\boldsymbol{\xi}\cdot[2(\mathbf{A}\cdot\nabla)\mathbf{U}+\mathcal{L}_{f,\hat{f},\beta}\mathbf{A}]}{\|\boldsymbol{\xi}\|^2}\boldsymbol{\xi}, \tag{4.7}$$

where $\nabla\mathbf{U}$ is the velocity gradient matrix (3.40). The initial conditions for the ODE system (4.5)–(4.7) are

$$\mathbf{x}(0)=\mathbf{x}_0,\quad \boldsymbol{\xi}(0)=\boldsymbol{\xi}_0,\quad \mathbf{A}(0)=\mathbf{A}_0,\quad \text{with}\quad \mathbf{A}_0\cdot\boldsymbol{\xi}_0=0. \tag{4.8}$$

To prove the instability of the geophysical fluid flow (3.1), it is not necessary to investigate the system above for all initial data. We only need to chose an initial disturbance which can lead to an exponentially growing amplitude $\mathbf{A}$. Let us choose the latitudinal wave vector $\xi_0=(0\ 1\ 0)^T$. For this initial condition, taking into account the expression (3.40) of the velocity gradient matrix, (4.6) yields

$$\xi(t)=(0\ 1\ 0)^T\quad\text{for all time}\quad t\geq 0. \tag{4.9}$$

Hence, from (4.7), it follows that $\mathbf{A} = (A_1, A_2, A_3)$ satisfies

$$\begin{cases} \dot{A}_1 = -\hat{f}A_3 + (f+\beta s)A_2 - \frac{cke^{\xi}}{1-e^{2\xi}}[-A_1\sin\theta + A_2 h_s(e^{\xi}-\cos\theta) + A_3(\cos\theta - e^{\xi})], \\ \dot{A}_2 = 0, \\ \dot{A}_3 = \hat{f}A_1 - \frac{cke^{\xi}}{1-e^{2\xi}}[A_1(e^{\xi}+\cos\theta) - A_2 h_s\sin\theta + A_3\sin\theta)]. \end{cases}$$

For the chosen initial vector $\xi_0 = (0\ 1\ 0)^T$ and the condition of orthogonality in (4.8), we must have $A_2(0) = 0$. Thus, the second equation in the above system yields

$$A_2(t) = 0 \quad \text{for all} \quad t \geq 0.$$

Thus, the system reduces to the following two-dimensional system

$$\begin{pmatrix} \dot{A}_1 \\ \dot{A}_3 \end{pmatrix} = M(t)\begin{pmatrix} A_1 \\ A_3 \end{pmatrix}, \tag{4.10}$$

with

$$M(t) = \begin{pmatrix} \frac{cke^{\xi}}{1-e^{2\xi}}\sin\theta & -\hat{f} + \frac{cke^{\xi}}{1-e^{2\xi}}(e^{\xi}-\cos\theta) \\ \hat{f} - \frac{cke^{\xi}}{1-e^{2\xi}}(e^{\xi}+\cos\theta) & -\frac{cke^{\xi}}{1-e^{2\xi}}\sin\theta \end{pmatrix}.$$

By rotating the canonical Cartesian basis with the angle $\alpha = \frac{kct}{2}$ about the vector $\boldsymbol{\xi}(t)$ from (4.9), the system (4.10) can be transformed to an autonomous linear system:

$$\begin{pmatrix} \dot{\tilde{A}}_1 \\ \dot{\tilde{A}}_3 \end{pmatrix} = D\begin{pmatrix} \tilde{A}_1 \\ \tilde{A}_3 \end{pmatrix}, \tag{4.11}$$

where D the time-independent matrix

$$D = \begin{pmatrix} \frac{cke^{\xi}}{1-e^{2\xi}}\sin(kq) & -\hat{f} - \frac{cke^{\xi}}{1-e^{2\xi}}\cos(kq) + \frac{cke^{2\xi}}{1-e^{2\xi}} - \frac{kc}{2} \\ \hat{f} - \frac{cke^{\xi}}{1-e^{2\xi}}\cos(kq) - \frac{cke^{2\xi}}{1-e^{2\xi}} + \frac{kc}{2} & -\frac{cke^{\xi}}{1-e^{2\xi}}\sin(kq) \end{pmatrix},$$

and $\tilde{A}_1$, $\tilde{A}_3$, the components of the vector $\mathbf{A}(t)$ in the new basis. The solution to the non-autonomous system (4.10) is obtained by multiplying the rotation matrix with the solution to the autonomous system (4.11). The rotation matrix being time periodic, the behaviour in time of the amplitude vector $\mathbf{A}$ is determined by the eigenvalues of the matrix D, which satisfy the following equation

$$\lambda^2 = \frac{(2\hat{f}+3kc)^2 e^{2\xi} - (2\hat{f}+kc)^2}{4(1-e^{2\xi})}.$$

Therefore, taking into account (3.3), if

$$e^{\xi} > \frac{2\hat{f}+kc}{2\hat{f}+3kc} \overset{(3.13)}{=} \frac{3\hat{f} \pm \sqrt{\hat{f}^2 + 4k\left(\hat{f}c_0 + g - \frac{\hat{f}^2}{4}R\right)}}{\hat{f} \pm 3\sqrt{\hat{f}^2 + 4k\left(\hat{f}c_0 + g - \frac{\hat{f}^2}{4}R\right)}}, \tag{4.12}$$

then, the amplitude $\mathbf{A}$ increases unboundedly in time, the exponential growth rate being

$$\lambda = \frac{1}{2}\sqrt{\frac{(2\hat{f}+3kc)^2 e^{2\xi} - (2\hat{f}+kc)^2}{1-e^{2\xi}}}.$$

By (3.1), the steepness of the longitudinal wave profile, defined as the amplitude multiplied by the wave number, is e^{ξ}. We have proved the following wave-steepness instability criterion:

Theorem 4.1. *At arbitrary latitude, the geophysical waves (3.1) are linearly unstable if their steepness exceeds the value* $\frac{3\hat{f} \pm \sqrt{\hat{f}^2 + 4k\left(\hat{f}c_0 + g - \frac{\hat{f}^2}{4}R\right)}}{\hat{f} \pm 3\sqrt{\hat{f}^2 + 4k\left(\hat{f}c_0 + g - \frac{\hat{f}^2}{4}R\right)}}$.

In the absence of the underlying current, for $c_0 = 0$, this value becomes

$$\frac{3\hat{f} \pm \sqrt{\hat{f}^2 + 4k\left(g - \frac{\hat{f}^2}{4}R\right)}}{\hat{f} \pm 3\sqrt{\hat{f}^2 + 4k\left(g - \frac{\hat{f}^2}{4}R\right)}}. \tag{4.13}$$

Since $\hat{f} \ll \sqrt{\hat{f}^2 + 4k\left(g - \frac{\hat{f}^2}{4}R\right)}$, the threshold (4.13) is

$$\frac{3\frac{\hat{f}}{\sqrt{\hat{f}^2+4k\left(g-\frac{\hat{f}^2}{4}R\right)}} + 1}{\frac{\hat{f}}{\sqrt{\hat{f}^2+4k\left(g-\frac{\hat{f}^2}{4}R\right)}} + 3} \gtrapprox \frac{1}{3} \quad \text{for} \quad c = c_+ > 0,$$

$$\frac{3\frac{\hat{f}}{\sqrt{\hat{f}^2+4k\left(g-\frac{\hat{f}^2}{4}R\right)}} - 1}{\frac{\hat{f}}{\sqrt{\hat{f}^2+4k\left(g-\frac{\hat{f}^2}{4}R\right)}} - 3} \lessapprox \frac{1}{3} \quad \text{for} \quad c = c_- < 0.$$

These considerations suggest that waves which travel from east to west are more prone to instability than those which travel from west to east.
In the presence of the current $c_0 \neq 0$, we get

$$\frac{3\frac{\hat{f}}{\sqrt{\hat{f}^2+4k\left(\hat{f}c_0+g-\frac{\hat{f}^2}{4}R\right)}} + 1}{\frac{\hat{f}}{\sqrt{\hat{f}^2+4k\left(\hat{f}c_0+g-\frac{\hat{f}^2}{4}R\right)}} + 3} \gtrapprox \frac{1}{3} \quad \text{for} \quad c = c_+ > 0,$$

$$\frac{3\frac{\hat{f}}{\sqrt{\hat{f}^2+4k\left(\hat{f}c_0+g-\frac{\hat{f}^2}{4}R\right)}} - 1}{\frac{\hat{f}}{\sqrt{\hat{f}^2+4k\left(\hat{f}c_0+g-\frac{\hat{f}^2}{4}R\right)}} - 3} \lessapprox \frac{1}{3} \quad \text{for} \quad c = c_- < 0.$$

In particular, we deduce that an adverse current with $cc_0 > 0$ favours instability in the sense that the threshold on the steepness for the wave to be unstable is decreased compared to the case without current. Conversely, this threshold is increased by a following current with $cc_0 < 0$.

Let us also note that, for equatorial waves, $\hat{f} = 2\Omega$, we recover the result obtained in [17], that is, the right-hand side of (4.12) has the expression

$$\frac{3\Omega \pm \sqrt{\Omega^2 + k(2\Omega c_0 + g - \Omega^2 R)}}{\Omega \pm 3\sqrt{\Omega^2 + k(\Omega c_0 + g - \Omega^2 R)}}.$$

For waves near the North Pole, $\hat{f} = 0$, the right-hand side of (4.12) becomes $\frac{1}{3}$ and we recover the result [34] for Gerstner's wave.

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Compliance with ethical standards
Conflict of interest The authors declare that they have no conflict of interest.

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