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Stochastic Swift-Hohenberg Equation with Degenerate Linear Multiplicative Noise

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Abstract. We study the dynamic transition of the Swift-Hohenberg equation (SHE) when linear multiplicative noise acting on a finite set of modes of the dominant linear flow is introduced. Existence of a stochastic flow and a local stochastic invariant manifold for this stochastic form of SHE are both addressed in this work. We show that the approximate reduced system corresponding to the invariant manifold undergoes a stochastic pitchfork bifurcation, and obtain numerical evidence suggesting that this picture is a good approximation for the full system as well.

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1. Introduction

Randomness and uncertainty are ubiquitous in many branches of natural and engineering sciences. Stochastic modeling taking randomness into account hence plays an important role in obtaining a realistic quantitative and qualitative description of these real-world phenomena, including, for example, climate dynamics [9], pricing and hedging of financial derivatives [22], filtering and control theory in engineering, turbulence theory [1] and population models in biology [19].

The study of the asymptotic behavior and dynamic transitions for the following semi-linear stochastic evolution equation, driven by linear multiplicative white noise in the sense of Stratonovich,

$$du = (L_\lambda u + F(u))dt + \sigma u \circ dW_t, \tag{1.1}$$

has an extensive literature and applications; see e.g. [3,5,7,8,13,24], and the references therein. Here, L_λ is a linear operator parametrized by a scalar control parameter λ, $F(u)$ represents the nonlinear terms, W_t is a two-sided one-dimensional Wiener process, and $\sigma \in \mathbb{R}$ gives a measure of the "amplitude" of the noise. Since σ is a constant, the random noise in (1.1) is also known as scalar multiplicative noise. If we view the noise in Eq. (1.1) as $\sigma \mathrm{Id}(u) \circ dW_t$, where Id is the identity operator, then a natural way to generalize the noise is to consider $Bu \circ dW_t$, where B is a bounded linear operator; see [23]. In this article, we are interested in the case where B acts as a multiplicative operator on the eigenspaces of L_λ, and its dynamic transition is studied using the ideas from the theory of random invariant manifolds and reduction strategies in [6–8].

Before giving a more precise description of the operator B, we begin by describing the general framework. We consider separable Hilbert spaces H and H_1, with the inclusion $H_1 \subset H$ being dense and compact. We are given a family L_λ of linear completely continuous fields from H_1 to H, depending continuously on $\lambda \in \mathbb{R}$, and we assume L_λ is a sectorial operator for each $\lambda \in \mathbb{R}$. Under these assumptions, the spectrum $\sigma(L_\lambda)$ of L_λ consists only of eigenvalues with finite multiplicities, and there are finitely many

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eigenvalues with a given real part. Moreover, from [14], Thm.II.4.18, we know that Re $\sigma(L_\lambda)$ is bounded above. These two properties of Re $\sigma(L_\lambda)$ make lexicographical ordering of $\sigma(L_\lambda)$ possible, and we write $\sigma(L_\lambda) = \{\beta_n(\lambda) : n \in \mathbb{N}\}$, with each eigenvalue $\beta_n(\lambda)$ repeated according to its algebraic multiplicity. We will also assume that for λ in a small neighborhood of some λ_c, denoted by N_{λ_c}, there exists a positive integer m, such that the spectrum $\sigma(L_\lambda)$ splits into the union of $\sigma_c(L_\lambda)$ and $\sigma_s(L_\lambda)$. That is

$$\sigma(L_\lambda) = \sigma_c(L_\lambda) \cup \sigma_s(L_\lambda), \quad \lambda \in N_{\lambda_c}, \text{ with } \eta_c > \eta_s, \tag{1.2}$$

where

$$\begin{aligned} \eta_c &:= \inf_{\lambda \in N_{\lambda_c}} \inf\{\text{Re } \beta(\lambda) : \beta(\lambda) \in \sigma_c(L_\lambda)\}, \\ \eta_s &:= \sup_{\lambda \in N_{\lambda_c}} \sup\{\text{Re } \beta(\lambda) : \beta(\lambda) \in \sigma_s(L_\lambda)\}, \end{aligned} \tag{1.3}$$

and $\sigma_c(L_\lambda)$ consists of the first m eigenvalues (counting multiplicities) in $\sigma(L_\lambda)$. The splitting of the spectrum $\sigma(L_\lambda)$ leads to a decomposition of H_1 as the direct sum of H_1^c and H_1^s, where H_1^c and H_1^s are the spaces spanned by eigenfunctions with eigenvalues in $\sigma_c(L_\lambda)$ and $\sigma_s(L_\lambda)$ respectively. Note that we assume this decomposition to be independent of λ.

Although our theoretical results hold under more general conditions, a prototype of the kind of operators B we can treat is given by

$$Bu = \sum_{k=1}^{\infty} \sigma_k P_k u$$

where $\{\sigma_k\} \subset \mathbb{R}$ is a bounded sequence, and P_k is the orthogonal projection onto the eigenspace corresponding to a given eigenvalue in $\sigma(L_\lambda)$. It might happen that the constants σ_k vanish for infinitely many $k \in \mathbb{N}$, an so it is in this sense that we say $Bu \circ dW_t$ is a degenerate multiplicative noise.

Instead of approaching this problem in full abstraction, we will study it in the context of the Swift-Hohenberg equation (SHE). SHE was first proposed in 1977 by Jack Swift and Pierre Hohenberg as a simple model for the Rayleigh–Benard instability of roll waves [26]. This equation plays an important role in bifurcation analysis and has been widely used as a model for the study of various phenomena in pattern formation; see [2,10,12,16,18] and the references therein. The classical, deterministic SHE, takes the form

$$\frac{\partial u}{\partial t} = -(I + \Delta)^2 u + \lambda u - u^3, \tag{1.4}$$

where λ is a positive real number which serves as a control parameter. We also impose the following odd periodic boundary conditions on the domain $\mathcal{U} = (-l, l) \times (-l, l)$:

$$\begin{aligned} &u(x_1, x_2, t) = u(x_1 + 2l, x_2, t) = u(x_1, x_2 + 2l, t) && \forall (x_1, x_2) \in \mathcal{U}, t \geq 0, \\ &u(-x_1, -x_2, t) = -u(x_1, x_2, t) && \forall (x_1, x_2) \in \mathcal{U}, t \geq 0. \end{aligned} \tag{1.5}$$

The problem analyzed hereafter consists of the following stochastic Swift-Hohenberg equation (SSHE)

$$du = \left(-(I + \Delta)^2 u + \lambda u - u^3\right) dt + Bu \circ dW_t. \tag{1.6}$$

In other words, (1.6) is Eq. (1.4) perturbed by degenerate multiplicative noise in the sense described above.

From the point of view of dynamic transitions [21], it is interesting to see the difference of transitions when degenerate multiplicative noise is used instead of the classical scalar noise. To address the dynamic transition of a given dissipative system, the first step is to study the linear eigenvalue problem, as outlined in Sect. 2. With the linear stability theory established, the detailed information of the transition behavior is then dictated by the nonlinear interactions of the system. In the deterministic case, Han and Hsia [16] have shown that the deterministic SHE undergoes a continuous type transition as λ crosses some critical value. This transition occurs as a result of the stabilizing effect played by the nonlinear cubic term u^3. For the stochastic case, with scalar multiplicative noise, the stabilizing effect of the cubic term also leads to a continuous type transition, see [5] for example. However, by introducing an operator B as described above, or any other bounded linear operator, the nonlinear term might no longer remain a source of stability. This is due in part to the fact that the nonlinear term $e^{-W_tB}F(e^{-W_tB}v)$ loses its stabilizing effect when B is not a multiplication operator, which is the main difference with the deterministic case or the stochastic case with scalar multiplicative noise. Hence, it is not obvious that the dynamic transition of SSHE presented here will again undergo a continuous transition. Nevertheless, as we show at the end of Sect. 5, the reduced equations associated to SSHE with this kind of noise do undergo a continuous transition when λ crosses a critical value.

The pathwise well-possedness of Eq. (1.1) with generalized noise $Bu \circ dW$, which is essential for the applicability of the ideas of [7,8], is also investigated in this paper. Due to the lack of pathwise a priori estimates, which is, in a sense, due to the fact that B does not in general commute with DF, it is not possible to establish global existence of Eq. (3.3) in $\Omega \times H$. Because of this, we present a different approach to address this issue in Sect. 3. More precisely, we prove lower semi-continuity of the blow-up time with respect to (ω, x), and hence establish existence of a local RDS. Moreover, we show that for every $x \in H$ there exists an event of probability one where global solutions of (3.3) exist, which thus contains all the relevant asymptotic dynamics of the problem.

Reducing a nonlinear mathematical model to a set simpler equations which are able to faithfully mimic the main features of the original model is a long established approach in the study of dynamical systems of nonlinear differential equations. In this technique, invariant manifolds, such as the stable, unstable and center manifolds, have been widely used to provide geometric structure for understanding the dynamics of nonlinear systems. Some of the pioneers in this area were Hadamard [15], Liapunov [20] and Perron [25], and, for more recent developments, see [21] and the references therein. There have been fruitful efforts in extending these techniques to stochastic and random dynamical system, see [7,8,13]. However, similar reduction techniques can not be readily applied to the study of stochastic dynamical systems. The main issue here is the incompatibility with large excursions of SPDE solutions caused by white noise, and to the best of our knowledge, this problem has not been fully resolved.

The existence of local invariant manifolds for SPDEs is the first thing one needs to establish in order to apply the reduction techniques mentioned above. The existence of such invariant manifolds for SPDEs with scalar multiplicative noise has been established in [7] Cor 5.1, where the stochastic invariant manifold function is defined on a deterministic ball. Through the relaxation on the assumption of the random noise, the existence of local invariant manifolds for a broader class of SPDEs with multiplicative noise of the form $Bu \circ dW$, where operator $B : H \to H$ is a linear bounded operator that commutes with the semigroup generated by L_λ, is given in Sect. 5. Such type of noise was also considered in [4]. Even though the class of stochastic perturbations included in this case is much wider than what is consider in [7], the result we obtained is only slightly weaker. More precisely, instead of having a stochastic invariant manifold function defined on a deterministic ball, the function we have lives in a random ball, with radius given by a tempered random variable. The results presented in Sects. 5 and 6 provide some justification of the reduction analysis.

From the results we have in Sect. 4, and adapting the approximation formulas given in [7], the first order stochastic reduction is performed in Sect. 5, by projecting SSHE onto the subspace H_1^c. For deterministic systems, the justification for the reduction techniques is based on a geometric phase space analysis in which families of nearby trajectories are predicted to collapse onto a low-dimensional invariant manifold. However, due to the large deviations that solutions of SSHE may exhibit, which are unavoidable in stochastic systems, it is not obvious that the stochastic invariant manifold found as above will contain all the relevant dynamics. Hence, the results in Sect. 5 should only be interpreted for those trajectories that manage to lie inside the invariant manifold. Nevertheless, using a priori estimates for the reduced equations, and a result in [11], we show that the reduced equations for SSHE undergo a stochastic pitchfork bifurcation in the pullback sense, which is also known as a type I transition, as introduced for deterministic system in [21] and the references therein.

This article is organized as follows. In Sect. 2, the functional setting of (1.4) is introduced and its linear eigenvalue problem is studied. Well-posedness of the pathwise problem is given in Sect. 3. Existence of local stochastic invariant manifold is addressed in Sect. 4. In Sect. 5, a low order reduced equation is obtained using the reduction strategy in [7]. The study of stochastic transitions associated with (1.6) is also given in Sect. 5. By placing the noise at some relevant fast modes, numerical results of reduced SSHE are given in Sect. 6.

2. Mathematical Setting and Linear Problem

The mathematical setting and the linearized eigenvalue problem associated with (1.4) are introduced in this section. For the mathematical setup, we introduce the following spaces:

$$\begin{aligned} H &= \left\{ u \in L^2(\mathcal{U}, \mathbb{R}) : u \text{ satisfies (1.5) and } \int_{\mathcal{U}} u\, dx = 0 \right\}, \\ H_1 &= H^4(\mathcal{U}; \mathbb{R}) \cap H \end{aligned} \tag{2.1}$$

Note that H_1 and H are Hilbert spaces with compact and dense embedding $H_1 \hookrightarrow H$. We then write (1.4) as

$$\begin{aligned} \frac{du}{dt} &= L_\lambda u + F(u), \\ u(x,0) &= \phi(x), \end{aligned} \tag{2.2}$$

where $L_\lambda u = -(I+\Delta)^2 u + \lambda u$.

Since $(I+\Delta)^2 : H_1 \to H$ is a linear homeomorphism, it is clear that the operators $L_\lambda = -(I+\Delta)^2 + \lambda I$ constitute a family of linear completely continuous fields, depending continuously on $\lambda \in \mathbb{R}$. Note also that $F(u) = -u^3$ is a bounded mapping satisfying $F(u) = o(\|u\|_{H_1})$. Next we consider the eigenvalue problem

$$L_\lambda u = \beta(\lambda) u. \tag{2.3}$$

The eigenvalues and corresponding eigenvectors of L_λ are given by

$$\begin{aligned} \beta_K(\lambda) &= \lambda - \lambda_K, \quad \lambda_K = \left(1 - \frac{|K|^2\pi^2}{l^2}\right)^2, \\ e_K(x) &= \sin\left(\frac{k_1\pi}{l}x_1 + \frac{k_2\pi}{l}x_2\right), \end{aligned} \tag{2.4}$$

where $x = (x_1, x_2)$ and $K = (k_1, k_2) \neq (0,0)$ with $k_i \in \mathbb{Z}$, $i = 1, 2$. Let

$$\mathcal{Z} = \{(k_1, k_2) : k_1 \in \mathbb{N}, k_2 \in \mathbb{Z}\} \cup \{(0, k_2) : k_2 \in \mathbb{N}\}, \tag{2.5}$$

so that $\{\beta_K(\lambda) : K \in \mathcal{Z}\}$ is a complete set of eigenvalues of the operator L_λ while $\{e_K : K \in \mathcal{Z}\}$ forms a basis of H. To simplify the presentation we assume further that $l^2 < \frac{3\pi^2}{2}$, so that with these eigenvalues and with $\lambda_0 = \left(1 - \frac{\pi^2}{l^2}\right)^2$, we have

$$\begin{aligned} \beta_{(1,0)}(\lambda) = \beta_{(0,1)}(\lambda) &= \begin{cases} > 0 & \text{if } \lambda > \lambda_0, \\ = 0 & \text{if } \lambda = \lambda_0, \\ < 0 & \text{if } \lambda < \lambda_0, \end{cases} \\ \beta_K(\lambda_0) &< 0, \quad \forall K \in \mathcal{Z} - \{(1,0),(0,1)\}. \end{aligned} \tag{2.6}$$

We denote the eigenfunctions

$$e_1 = e_{(1,0)} = \sin\left(\frac{\pi}{l}x_1\right) \quad \text{and} \quad e_2 = e_{(0,1)} = \sin\left(\frac{\pi}{l}x_2\right),$$

for the corresponding eigenvalues $\beta_{(1,0)}(\lambda)$ and $\beta_{(0,1)}(\lambda)$. Since $\beta_{(1,0)}(\lambda)$ and $\beta_{(0,1)}(\lambda)$ have the same value, for simplicity of notation, from here onwards, we will denote them by $\beta_1(\lambda)$. We also call the above two eigenfunctions as the resolved modes. Notice that the space H_1 and H can be decomposed as

$$\begin{aligned} H_1 &= H_1^c \oplus H_1^s \\ H &= H_1^c \oplus H^s \end{aligned} \tag{2.7}$$

where H_1^c is the subspace spanned by the first two eigenfunctions (with $\beta_1(\lambda)$ as its corresponding eigenvalues), that is

$$H_1^c = span\{e_{(1,0)}, e_{(0,1)}\} = span\{e_1, e_2\},$$

and

$$H_1^s = span\{e_K : K \in \mathcal{Z} - \{(1,0),(0,1)\}\},$$

whereas H^s is the closure of H_1^s in H, see [8].

3. Well-Posedness for SSHE

In this section, we address the pathwise well-possedness for Stochastic Swift-Hohenberg equation (SSHE). Local existence of SSHE is first established in Theorem 3.1 by using the metric structure of $\Omega = C_0(\mathbb{R};\mathbb{R})$. As a corollary of Lemmas 3.1 and 3.2, we see that for every initial condition, the existence of a solution (3.3) is global for almost every $\omega \in \Omega$. As a consequence of the lower semi-continuity of the blow-up time with respect to $(\omega, x) \subset \Omega \times H$, the subset $D \subset \Omega \times H$ where this system is globally defined thus contains all the relevant dynamics and is a Polish space on its own. The theoretical work done in this section is also more general than (1.6), since the multiplicative noise here is of the form $Bu \circ dW$, where $B : H \to H$ is just a linear bounded operator that commutes with the semigroup generated by L_λ.

When we have such operator B, the solution to the linear equation

$$du = L_\lambda u dt + Bu \circ dW, \quad u(0) = x \tag{3.1}$$

can be given explicitly as $U(t,\omega,x) = e^{W_t(\omega)B} e^{tL_\lambda} x$.

It is then clear that $U : \mathbb{R}_+ \times \Omega \times H \to H$ is $(\mathcal{B}(\mathbb{R}_+) \times \mathcal{F} \times \mathcal{B}(H), \mathcal{B}(H))$ measurable, forms a cocycle over the Metric Dynamical System (MDS) $(\Omega, \theta, \mathbb{P})$, for each ω the map is continuous, and for each x it is an adapted process that solves the SPDE (3.1) in the strong sense. This suggests we formulate the nonlinear equation

$$du = (L_\lambda u + F(u))dt + Bu \circ dW, \quad u(0) = x \tag{3.2}$$

as a pathwise fixed-point problem

$$\phi(t,\omega,x) = U(t,\omega,x) + \int_0^t U(t-s, \theta_s\omega, F(\phi(s,\omega,x)))ds. \tag{3.3}$$

For a similar approach see [23].

Since we intend to use the topological structure of $\Omega = C_0(\mathbb{R};\mathbb{R})$, it is convenient at this stage to recall that Ω is a complete metric space, with the metric

$$d(\omega_1,\omega_2) = \sum_{n=1}^{\infty} 2^{-n} \frac{\|\omega_1 - \omega_2\|_{L^\infty(-n,n)}}{1 + \|\omega_1 - \omega_2\|_{L^\infty(-n,n)}}$$

and then we can reformulate the problem with the corresponding notation,

$$\phi(t,\omega,x) = e^{\omega(t)B} e^{tL_\lambda} x + \int_0^t e^{(\omega(t)-\omega(\tau))B} e^{(t-\tau)L_\lambda} F(\phi(\tau,\omega,x))d\tau. \tag{3.4}$$

Although this abstract approach is possible under more general conditions, we restrict ourselves to the case when F satisfies $F(0) = 0$, $F \in C^1(H_\alpha, H)$ and

$$|DF(u)|_{L(H_\alpha,H)} \le c_1 |u|_\alpha^{p-1}, \quad \forall u \in H_\alpha,$$

for some $p > 1$ and $\alpha \in (0, 1/p)$.

Note that the above condition holds for $F(u) = -u^3$ with $p = 3$ and $\alpha = 1/4$.

In what follows we also fix $\eta_0 > \sup_{\lambda\in\Lambda} \sup_k \Re\beta_k(\lambda)$.

We recall that, given a Banach space X and $\alpha \in \mathbb{R}$,

$$C_\alpha((0,\delta];X) = \left\{ u \in C((0,\delta];X) : \sup_{0<t\leq\delta} t^\alpha |u(t)|_X < \infty \right\}$$

is a Banach space with norm

$$\|u\|_{C_\alpha((0,\delta];X)} = \sup_{0<t\leq\delta} t^\alpha |u(t)|_X.$$

Similarly,

$$E_\alpha = \{u \in C([0,\delta];H) \cap C_\alpha((0,\delta];H_\alpha) : \|u\|_{L^\infty(0,\delta;H)} + \|u\|_{C_\alpha((0,\delta];H_\alpha)} < \infty\}$$

is a Banach space with norm $\|u\|_{E_\alpha} = \|u\|_{L^\infty(0,\delta;H)} + \|u\|_{C_\alpha((0,\delta];H_\alpha)}$.

We now prove that (3.4) has a local solution.

Theorem 3.1. *For every $r_1 > 0, r_2 \in (0,1)$ there exist $\delta(r_1,r_2), K(r_1,r_2) > 0$ such that*

(1) *For every $(x,\omega) \in B_H(0,r_1) \times B_\Omega(0,r_2)$, (3.3) has a unique solution*

$$\phi(\cdot,\omega,x) \in C([0,\delta];H) \cap C_\alpha((0,\delta];H_\alpha).$$

(2) *For $x_1, x_2 \in B_H(0,r_1)$ and $\omega_1,\omega_2 \in B_\Omega(0,r_2)$ it holds*

$$\|\phi(\cdot,\omega_1,x_1) - \phi(\cdot,\omega_2,x_2)\|_{E_\alpha} \leq K(|x_1-x_2| + d(\omega_1,\omega_2)).$$

(3) *If $\omega \in C^\gamma$ for some $\gamma \leq (p-1)\alpha$, then*

$$\phi(\cdot,\omega,x) - e^{\cdot L + \omega(\cdot)B}x \in C_{p(p\alpha-\gamma)}((0,\delta];H_1).$$

In particular, $\phi(\cdot,\omega,x) \in L^2(0,\delta;H_{1/2})$ provided $p(p\alpha-\gamma) < 1$, which holds when $p=3$, $\alpha = 1/4$ and $\gamma \in (5/12, 1/2)$.

The techniques used in the proof are rather standard in the theory of parabolic equations. We present here a proof in order to show the role the random parameter ω plays in the resolution.

Proof. Existence

Fix $T > 0$ so that $r_2 < 2^{-T}$, and let $R_2 > 0$ be large enough so that $r_2 \leq 2^{-T}\frac{R_2}{1+R_2}$. This guarantees that $\sup_{t\in[0,T]} |\omega(t)| \leq R_2$ for all $\omega \in \Omega$ with $d(\omega,0) \leq r_2$.

For every $\delta \in (0,T]$ and $R_1 > 0$, to be chosen, we consider the complete metric spaces $X \subset E_\alpha$ and $Y \subset \Omega$ given by

$$X = \{v \in E_\alpha \ \|v\|_{E_\alpha} \leq R_1\}.$$

$$Y = \left\{ \omega \in \Omega : \sup_{t\in[0,T]} |\omega(t)| \leq R_2 \right\}.$$

Let the operator $\Gamma = \Gamma_x : X \times Y \to E_\alpha$ be given by

$$\Gamma(u,\omega)(t) = e^{tL_\lambda} e^{\omega(t)B}x + \int_0^t e^{(t-s)L_\lambda} e^{(\omega(t)-\omega(s))B} F(u(s))ds.$$

where $x \in B_H(0,r_1)$ is arbitrary but fixed.

Note that, for $(u_i,\omega) \in X \times Y$, $i = 1,2$, and $t \in (0,\delta]$, we have

$$t^{p\alpha}|e^{-\omega(t)B}(F(u_1(t)) - F(u_2(t)))|_0$$
$$\leq e^{|\omega(t)||B|} t^{p\alpha} \int_0^1 |DF(ru_1(t) + (1-r)u_2(t))|_{L(H_\alpha,H)} dr |u_1(t) - u_2(t)|_\alpha,$$

where $|B| = \max\{|B|_{L(H)}, |B|_{L(H_\alpha)}\}$, and consequently

$$\|e^{-\omega(\cdot)B}(F(u_1(\cdot)) - F(u_2(\cdot))\|_{C_{p\alpha}((0,\delta];H)} \leq c_1 e^{R_2|B|} R_1^{p-1} \|u_1 - u_2\|_{C_\alpha((0,\delta];H_\alpha)},$$

for all $u_1, u_2 \in X$, $\omega \in Y$.

Next we make use of the fact that, for every $\alpha \in (0, 1/p)$, there exists $c_2 > 0$ such that the function $\psi(t) = \int_0^t e^{(t-s)L_\lambda}\varphi(s)ds$ satisfies

$$\delta^{1-p\alpha}\|\psi\|_{C^{1-p\alpha}([0,\delta];H)} + \|\psi\|_{C_\alpha((0,\delta];H_\alpha)} \leq c_2\delta^{1-p\alpha}e^{\eta_0 T}\|\varphi\|_{C_{p\alpha}((0,\delta];H)}.$$

Thus we get, from the previous estimate,

$$\|\Gamma(u_1,\omega) - \Gamma(u_2,\omega)\|_{E_\alpha} \leq c_1c_2\delta^{1-p\alpha}R_1^{p-1}e^{2R_2|B|+\eta_0 T}\|u_1 - u_2\|_{E_\alpha}.$$

Similarly, making use of the fact that $|e^{tL_\lambda}x|_\beta \leq K_\beta e^{\eta_0 T}t^{-\beta}|x|_0$, for $\beta = 0, \alpha$, we see that

$$\|\Gamma(u,\omega)\|_{E_\alpha} \leq (K_0 + K_\alpha)e^{R_2|B|+\eta_0 T}r_1 + c_1c_2\delta^{1-p\alpha}R_1^p e^{2R_2|B|+\eta_0 T}.$$

So it suffices to choose R_1 so that

$$(K_0 + K_\alpha)e^{R_2|B|+\eta_0 T}r_1 \leq \frac{R_1}{2}$$

and $\delta \in (0, T]$ so that

$$c_1c_2\delta^{1-p\alpha}R_1^{p-1}e^{2R_2|B|+\eta_0 T} \leq \frac{1}{2}.$$

With this choice of R_1 and δ the operator $\Gamma(\cdot,\omega)$ becomes a 1/2-contraction that maps X into itself, and thus it has a unique fixed point in X, for all $\omega \in Y$.

Continuity

Let $\phi(\cdot,\omega,x) \in X$ denote the fixed point of $\Gamma_x(\cdot,\omega)$. For $x_1, x_2 \in B_H(0,r_1)$ we have

$$\phi(t,\omega,x_1) - \phi(t,\omega,x_2) = e^{tL_\lambda+\omega(t)B}(x_1 - x_2) + \Gamma_{x_2}(\phi(\cdot,\omega,x_1))(t) - \Gamma_{x_2}(\phi(\cdot,\omega,x_2))(t)$$

and thus

$$\|\phi(\cdot,\omega,x_1) - \phi(\cdot,\omega,x_2)\|_{E_\alpha} \leq 2(K_0 + K_\alpha)e^{R_2|B|+\eta_0\delta}|x_1 - x_2|_0.$$

On the other hand, for $x \in B_H(0,r_1)$ fixed and $\omega_1, \omega_2 \in Y$, we have

$$\begin{aligned}\Gamma(u,\omega_1)(t) - \Gamma(u,\omega_2)(t) &= (e^{(\omega_1(t)-\omega_2(t))B} - I)\Gamma(u,\omega_2)\\ &\quad + e^{\omega_1(t)B}\int_0^t e^{(t-s)L_\lambda}(e^{-\omega_1(s)B} - e^{-\omega_2(s)B})F(u(s))ds\end{aligned}$$

This can be estimated using the same arguments as above, whence

$$\begin{aligned}\|\phi(\cdot,\omega,x_1) - \phi(\cdot,\omega,x_2)\|_{E_\alpha} &\leq e^{2R_2|B|}|B|\|\omega_1 - \omega_2|_{L^\infty(0,\delta)}(R_1 + c_1c_2\delta^{1-p\alpha}R_1^p e^{\eta_0 T})\\ &\leq (e^{2R_2|B|} + 1)|B|R_1\|\omega_1 - \omega_2\|_{L^\infty(0,\delta)}\end{aligned}$$

and, for $\omega_1, \omega_2 \in Y$, it holds that $|\omega_1 - \omega_2|_{L^\infty(0,\delta)} \leq (2R_2 + 1)2^\delta d(\omega_1,\omega_2)$, from where the result follows.

Uniqueness

This is a standard argument so we only give a sketch. Let v_1 and v_2 be two solutions defined on $[0,\delta]$. Let

$$t_0 = \sup\{t \in [0,\delta]\ v_1(t,\omega,x) = v_2(t,\omega,x)\}.$$

Suppose $t_0 < \delta$, and let $y = v_1(t_0,\omega,x) = v_2(t_0,\omega,x)$.

By leaving R_2 unchanged, and choosing $R_1' > \sup_{[t_0,t_0+\epsilon]}|v_i(t)|$, for $0 < \epsilon < \delta - t_0$ fixed, we can repeat the argument above with R_1 replaced by R_1' and r_1 replaced by $r_1' = |y|$, so that the problem has a unique solution among all functions bounded in $[t_0, t_0 + \delta']$ by R_1', at least for some short time $\delta' > 0$. This implies that $v_1(t) = v_2(t)$ in $[t_0, t_0 + \delta']$, which contradicts the definition of t_0.

Higher regularity

We make use of the following weighted Hölder spaces, defined for $\gamma \in (0,1)$ and $\mu \in \mathbb{R}$, by

$$C^\gamma_{\gamma+\mu}((0,\delta];X) = \{u \in C_\mu((0,\delta];X) :\ \|u\|_{C^\gamma_{\gamma+\mu}((0,\delta];X)} < \infty\}$$

where X is a Banach space, and

$$\|u\|_{C^{\gamma}_{\gamma+\mu}((0,\delta];X)} = \|u\|_{C_\mu((0,\delta];X)} + \sup_{0<\epsilon<\delta} \epsilon^{\mu+\gamma}[u]_{C^\gamma([\epsilon,\delta];X)}.$$

These spaces have the property that, for $\varphi \in C^{\gamma}_{\gamma+\mu}((0,\delta];H) \cap C_{p\alpha}((0,\delta];H)$, the function $\psi(t) = \int_0^t e^{(t-s)L_\lambda}\varphi(s)ds$ satisfies

$$\|\psi\|_{C^{1-\alpha}_{p\alpha}((0,\delta];H_\alpha)} \le c_3 e^{\eta_0 T}\|\varphi\|_{C_{p\alpha}((0,\delta];H)}$$

and

$$\|\psi\|_{C^{\gamma}_{\gamma+\mu}((0,\delta];H_1)} \le c_3 e^{\eta_0 T}\|\varphi\|_{C^{\gamma}_{\gamma+\mu}((0,\delta];H)},$$

where c_3 depends only on $\alpha \in (0,1/p)$, $\gamma,\mu \in (0,1)$.

Moreover, the map $t \mapsto e^{tA}x$ belongs to $C^{\gamma}_{\gamma+\alpha}((0,\delta];H_\alpha)$.

The above properties imply that, when $\omega \in C^\gamma$, $u = \phi(\cdot,\omega,x)$ can be written as $u = e^{\omega(\cdot)B}(u_1+u_2)$, where $u_1 \in C^{\gamma}_{\gamma+\alpha}((0,\delta];H_\alpha)$ and $u_2 \in C^{1-\alpha}_{p\alpha}((0,\delta];H_\alpha)$ are given by

$$u_1(t) = e^{tL}x, \quad u_2(t) = \int_0^t e^{(t-s)L}e^{-\omega(s)B}F(u(s))ds.$$

Since $\gamma \le (p-1)\alpha = (p\alpha-1)+(1-\alpha) < 1-\alpha$, we have the inclusion $C^{1-\alpha}_{p\alpha} \subset C^{\gamma}_{p\alpha}$, and since $p\alpha-\gamma \ge \alpha$, we also have $C^{\gamma}_{\gamma+\alpha} \subset C^{\gamma}_{p\alpha}$. Thus $u_1, u_2 \in C^{\gamma}_{p\alpha}((0,\delta];H_\alpha)$, and $e^{\omega(\cdot)B} \in C^\gamma([0,\delta];L(H_\alpha))$, so we get $u \in C^{\gamma}_{p\alpha}$.

It is easily verified that $\|F(u(\cdot))\|_{C^{\gamma}_{\gamma+p\sigma}((0,\delta];H)} \le c_1\|u\|^p_{C^{\gamma}_{\gamma+\sigma}((0,\delta];H_\alpha)}$, and $e^{-\omega(\cdot)B} \in C^\gamma([0,\delta];L(H_1))$, so, with $\sigma = p\alpha-\gamma$, we obtain $u_2 \in C^{\gamma}_{\gamma+p(p\alpha-\gamma)}((0,\delta];H_1)$. □

The existence of a maximally defined solution is now standard: for a fixed $x \in H$ and $\omega \in \Omega$, let $I(\omega,x)$ be the union of all the intervals $[0,a]$ for which there exists a solution ϕ_a of (3.3) in $C([0,a];H)$. Let $\delta(\omega,x) = \sup I(\omega,x)$, and define $\phi(t,\omega,x) = \phi_a(t,\omega,x)$ for $t \in [0,a]$ if $[0,a] \subset I(\omega,x)$. By uniqueness $\phi(\cdot,\omega,x) : [0,\delta(\omega,x)) \to H$ is well defined.

A close examination of the proof above shows that if a solution is defined in some interval $[0,a)$, and $\lim_{t\uparrow a} u(t)$ exists in H, then it can be extended to an interval $[0,a+\delta)$. In fact, a standard argument shows that whenever $\delta(\omega,x) < \infty$ it must be the case that $\lim_{t\uparrow\delta(\omega,x)} |\phi(t,\omega,x)|_0 = \infty$. That is to say, if a solution is strictly local then it must blow up in H norm.

The following result is concerned with the continuity of the maximally defined solution with respect to the data, and the semi-continuity of the blow-up time.

Lemma 3.1. *For every $(\omega,x) \in \Omega \times H$ and $a < \delta(\omega,x)$ there exist $r_1, r_2 > 0$ such that $a < \delta(\omega',y)$ for all $y \in B_{H_\alpha}(x,r_1)$ and $\omega' \in B_\Omega(\omega,r_2)$. Furthermore, there exists $L > 0$ such that the maximally defined solution satisfies*

$$|\phi(t,\omega,x)-\phi(t,\omega',y)| + t^\alpha|\phi(t,\omega,x)-\phi(t,\omega',y)|_\alpha \le L(|x-y|_0 + d(\omega,\omega'))$$
$$\forall t \in [0,a],\ y \in B_H(x,r_1),\ \omega' \in B_\Omega(\omega,r_2).$$

In particular, $\delta : \Omega \times H \to (0,\infty]$ is lower semi-continuous.

Proof. Since the set $\{\phi(t,\omega,x) :\ t \in [0,a]\}$ is compact, we can choose $R_1 > 0$ large enough so that it is contained in $B_H(0,R_1)$. We then fix any $R_2 \in (0,1)$ such that that $\omega \in B_\Omega(0,R_2)$. By choosing R_1 larger if necessary, we can assume that $\mathrm{dist}(\{\phi(t,\omega,x) :\ t \in [0,a]\}, \partial B_H(0,R_1)) = 2d > 0$.

In this setting we can apply (3.1), thus obtaining, for every $y \in B_H(0,R_1)$ and $\omega' \in B_\Omega(0,R_2)$, the existence of a solution $\phi(t,\omega',y)$ defined for $t \in [0,\delta(R_1,R_2)] \subset [0,a]$.

Let $t_i = i\delta$, for $i = 0,\dots,n$, with $(n-1)\delta \le a < n\delta$, and assume $n > 2$ (otherwise there is nothing to prove). For $x_i = \phi(t_{i+1},\omega,x)$, $i = 0,\dots,n-1$, consider the balls $B_i = B_H(x_i,d)$. By choice of d, all these balls are inside $B_H(0,R_1)$. Let $r_0 = \max\{K,1\}^{-n}d/2$, where K is the constant given by (3.1).

If $|x-y| + d(\omega,\omega') \le r_0$, then by (3.1), part(2):

$$|\phi(t,\omega,x)-\phi(t,\omega',y)| \le K(|x-y| + d(\omega,\omega')) \le \max\{K,1\}^{-n+1}d/2 \le d/2,$$

for all $t \in [0, \delta]$. So $\phi(\delta, \omega', y) \in B_i$ for some i, and then $\phi(\delta, \omega', y) \in B_H(0, R_1)$ as well. By (3.1), $\phi(\cdot, \omega', y)$ can be extended uniquely to $[0, 2\delta)$.

Because of the choice of r_0, this procedure can be repeated until $\phi(\cdot, \omega', y)$ has been extended to $[0, (n-1)\delta)$, at which point we get that $\phi((n-1)\delta, \omega', y)$ still lies $B_H(0, R_1)$, so it can be extended by δ once more, which gives that $\delta(\omega', y) > a$.

Choosing, say, $r_1 = r_2 = \frac{r_0}{2}$, the estimate holds with $L = K(R_1, R_2)$. □

For fixed $(\omega, x) \in \Omega \times H$, it is readily verified that

$$\phi(t+s, \omega, x) = e^{tL_\lambda} e^{W_t(\theta_s \omega)} \phi(s, \omega, x) + \int_0^t e^{(t-\tau)L_\lambda} e^{W_{t-\tau}(\theta_s \omega)B} F(\phi(\tau + s, \omega, x)) d\tau,$$

for $t + s < \delta(\omega, x)$. By uniqueness, this implies that $\phi(t+s, \omega, x) = \phi(t, \theta_s \omega, \phi(s, \omega, x))$ whenever $0 \le t + s < \delta(\omega, x)$, and moreover

$$\delta(\omega, x) = t + \delta(\theta_t \omega, \phi(t, \omega, x)), \quad \forall t \in [0, \delta(\omega, x))$$

so the cocycle property holds under the natural restrictions.

It follows immediately from the previous lemma that, for each $x \in H$, the set $\Omega(x) = \{\omega \in \Omega : \delta(\omega, x) = \infty\}$ is a G_δ (hence, in particular, measurable). Furthermore, we have the next lemma.

Lemma 3.2. *Assume $F(u) = -u^3$. Then $\mathbb{P}(\Omega(x)) = 1$ for every $x \in H$.*

Proof. By part (3) of Theorem 1, and the almost sure Hölder continuity of paths the Wiener process, we can assume without loss of generality that $\omega \in C^\gamma$, where $\gamma \in (5/12, 1/2)$.

By replacing the nonlinearity F by $F_n(u) = \zeta\left(\frac{|u|}{n}\right) F(u)$, where $\zeta : \mathbb{R} \to [0, 1]$ is smooth and satisfies $\zeta(s) = 1$ for $|s| \le 1$ and $\zeta(s) = 0$ for $|s| \ge 2$, we obtain a globally defined family of solutions to the fixed point problems

$$\phi_n(t, \omega, x) = e^{tL_\lambda} e^{W_t(\omega)B} x + \int_0^t e^{(t-s)L_\lambda} e^{W_{t-s}(\theta_s \omega)B} F_n(\phi_n(s, \omega, x)) ds.$$

Indeed, F_n satisfies

$$|F_n(u)| \le c_1 \zeta\left(\frac{|u|}{n}\right) (|u|_{1/2}|u| + |u|^4 |u|_{1/2}) \le c_1(n^4 + n)|u|_{1/2}, \quad \forall u \in H_{1/2}$$

and such a linear growth condition can be used in a standard fashion to derive exponential estimates in time from Gronwall's inequality.

Now, for fixed $x \in H$ and $n > |x|$, it is clear that $\phi(t, \omega, x) = \phi_n(t, \omega, x)$ for all $t \in [0, \tau_n(\omega, x))$, where

$$\tau_n(\omega, x) = \inf\{t > 0 : \ |\phi_n(t, \omega, x)| \ge n\}.$$

Note that this implies that $\delta(\omega, x) \ge \sup_n \tau_n(\omega, x)$.

The function $(t, \omega) \to |\phi_n(t, \omega, x)|$ is continuous, hence measurable. Moreover, since ϕ_n is obtained using a contraction mapping argument, it can also be given by an iteration procedure, which implies that it is adapted to the natural filtration of the Wiener process. In particular, τ_n is a stopping time.

Part (3) of Theorem 1 implies that $\phi_n(\cdot, \omega, x) \in L^2(0, T; H_{1/2}) \cap C([0, T]; H)$ for all $\omega \in C^\gamma$, so the process $u_n(t) = \phi_n(t, \cdot, x)$ is a weak solution of the SPDE

$$du_n = (L_\lambda u_n + F(u_n))dt + Bu_n \circ dW, \quad u_n(0) = x.$$

Thus we can apply Ito formula to $|u_n|^2$ to obtain

$$\mathbb{E}|u_n(t \wedge \tau_n)|^2 = |x|^2 + 2\mathbb{E} \int_0^{t \wedge \tau_n} [(L_\lambda u_n, u_n) + \sigma^2 |Pu_n|^2 + (F_n(u_n), u_n)] ds$$

Using that $(L_\lambda u, u) \le \beta_1(\lambda)|u|^2 \le \eta_0 |u|^2$, and $(F_n(u), u) \le -|D|^{-1}|u|^4$ for all u with $|u| \le n$, we see that

$$\mathbb{E}|u_n(t \wedge \tau_n)|^2 \le |x|^2 + 2\mathbb{E} \int_0^{t \wedge \tau_n} [(\eta_0 + \sigma^2)|u_n|^2 - |D|^{-1}|u_n|^4] ds$$

and then

$$n^2\mathbb{P}(\tau_n \le t) \le |x|^2 + (\eta_0 + \sigma^2)^2|D|t.$$

Since τ_n is non-decreasing in n, an application of Borel Cantelli shows that

$$\mathbb{P}\left(\sup_n \tau_n \le t\right) = 0,$$

and since $t > 0$ was arbitrary, we get that $\delta(\cdot, x) = \infty$ almost surely. □

This result ensures that the set $D = \{(\omega, x) \in \Omega \times H : \delta(\omega, x) = \infty\}$ contains all the relevant dynamics of the system. Note that D is a Polish space, and is invariant under the action of the skew-flow $\Theta_t(\omega, x) = (\theta_t\omega, \phi(t, \omega, x))$, $t \ge 0$. It would be desirable to show that D actually contains a set of the form $\Omega_0 \times H$, where Ω_0 is θ-invariant and has full measure. However, the lack of pathwise a priori estimates precludes us from obtaining such result.

4. Existence of a Local Stochastic Invariant Manifold

In this section, we will study the existence of local invariant manifolds for a broad class of SPDEs with multiplicative noise of the form $Bu \circ dW$, where $B : H \to H$ is a linear bounded operator that commutes with the semigroup generated by L_λ. Hence, the noise we consider here is more general than what is studied in [7] and [8]. Because of this, one can expect the result presented here to be weaker than what is in the references mentioned above. However, we will show that the result obtained for this boarder class of random noise is only slightly weaker than those with scalar multiplicative noise. The existence of such local invariant manifold leads us to an explicit reduction procedure for Eq. (1.6) to the corresponding local invariant manifold. This lower-dimensional reduced system describes the long time dynamics of the original system in a lower dimensional space, which will be the subject of discussion in the next section.

The notion of a local random invariant manifold given here is a natural generalization of the classical local invariant manifold for deterministic dynamical systems, see [17], Def 6.1.1 for the latter concept. For the random case, "local" has to be understood in terms of both the time variable t, as well as the phase-space variable x. To be precise, a RDS $\phi(t, \omega)$ on a Hilbert space H, associated to a random differential equation is locally invariant in the sense that for almost all ω and each $u_0 \in \mathring{\mathcal{M}}^{loc}(\omega)$, with $\mathring{\mathcal{M}}^{loc}(\omega)$ being the interior of $\mathcal{M}^{loc}(\omega)$, there exists $t_{u_0,\omega} > 0$, such that $\phi(t, \omega)u_0 \in \mathring{\mathcal{M}}^{loc}(\theta_t\omega)$ for all $t \in [0, t_{u_0,\omega})$.

The result on the existence of local invariant manifolds is stated as follows.

Theorem 4.1. *Assume that L_λ is the linear operator specified in (2.2), which satisfies (1.2)–(1.3). Suppose that the nonlinearity $F \in C^1(H_\alpha; H)$ is such that $F(u) = O(|u|^p)$ and $DF(u) = O(|u|^{p-1})$ with $p > 1$. Then there exists a positive random variable $\rho : \Omega \to (0, \infty)$, tempered below, and a family of Lipschitz functions $h(\omega, \cdot) : B(0, \rho(\omega)) \cap H_c \to H_s$, such that*

$$du = (L_\lambda u + F(u))dt + Bu \circ dW, \quad u(0) = x,$$

admits a local random invariant Lipschitz manifold $\mathcal{M}^{loc}_\lambda$ of dimension m defined as the graph of a local random invariant manifold function $h_\lambda(\omega, x)$ that is

$$\mathcal{M}^{loc}_\lambda(\omega) = \{x + h_\lambda(\omega, x) : x \in H^c_1, \|x\|_\alpha \le \rho(\omega)\} \tag{4.1}$$

for all $\lambda \in N_{\lambda_c}$.

Proof. We study the fixed point problem directly in the original system. Namely, we work with the operator

$$\Gamma_\xi(u)(t) = e^{tL_c + W_tB}P_c\xi - \int_t^0 e^{(t-\tau)L_c + (W_t - W_\tau)B}P_cF(u(\tau))d\tau + \int_{-\infty}^t e^{(t-\tau)L_s + (W_t - W_\tau)B}P_sF(u(\tau))d\tau, \quad t \in (-\infty, 0].$$

In order to obtain existence of a local invariant manifold, we replace the non-linear term by one with a random cut-off, namely, we consider

$$F_\rho(\omega,u)=\zeta\left(\frac{|u|_\alpha}{\rho(\omega)}\right)F(u),$$

where $\rho(\cdot)>0$ is to be chosen, and $\zeta\in C_c^\infty(\mathbb{R})$, with $0\le\zeta\le 1$, $|\zeta'|\le 2$, $\zeta(s)=1$ for $|s|\le 1$ and $\zeta(s)=0$ for $|s|>2$.

Then, for $F\in C^1(H_\alpha;H)$, with $F(0)=0$, $DF(0)=0$, we see that $F_\rho\in C^1(H_\alpha;H)$, with

$$DF_\rho(\omega,u)v=\frac{1}{\rho(\omega)}\zeta'\left(\frac{|u|_\alpha}{\rho(\omega)}\right)\frac{(u,v)_\alpha}{|u|_\alpha}F(u)+\zeta\left(\frac{|u|_\alpha}{\rho(\omega)}\right)DF(u)v.$$

From the estimates $|F(u)|_0\le c_1|u|_\alpha^p$, and $|DF(u)|_{\alpha,0}\le c_2|u|_\alpha^{p-1}$, with $p>1$, we deduce that

$$\begin{aligned}&|F_\rho(\omega,u)|_0\le c_1\rho(\omega)^{p-1}|u|_\alpha,\\&|F_\rho(\omega,u_1)-F_\rho(\omega,u_2)|_0\le(2c_1+c_2)\rho(\omega)^{p-1}|u_1-u_2|_\alpha.\end{aligned}$$

We introduce the Banach space

$$C_\eta=\left\{u\in C((-\infty,0];H_\alpha):\ \|u\|_\eta=\sup_{t\le 0}e^{-\eta t}|u(t)|_\alpha\right\}.$$

Then we have

$$\begin{aligned}e^{-\eta t}|\Gamma_\xi(u)(t)|_\alpha\le{}&Ke^{(\eta_c-\eta)t+|W_t(\omega)|B|}|P_c\xi|_\alpha\\&+K\int_t^0e^{(\eta_c-\eta)(t-\tau)}e^{|W_{t-\tau}(\theta_\tau\omega)||B|}e^{-\eta\tau}|P_cF_\rho(\theta_\tau\omega,u(\tau))|_0d\tau\\&+K\int_{-\infty}^te^{(\eta_s-\eta)(t-\tau)}(t-\tau)^{-\alpha}e^{|W_{t-\tau}(\theta_\tau\omega)||B|}e^{-\eta\tau}|P_sF_\rho(\theta_\tau\omega,u(\tau))|_0d\tau\end{aligned}$$

Consider the tempered random variables $r_\epsilon^\pm:\Omega\to[1,\infty)$ given by

$$r_\epsilon^+(\omega)=\sup_{t\ge 0}e^{-\epsilon t+|W_t(\omega)||B|},\quad r_\epsilon^-(\omega)=\sup_{t\le 0}e^{\epsilon t+|W_t(\omega)||B|}.$$

Using the previous estimates for F_ρ and the above definitions, with $0<\epsilon<\min\{\eta_c-\eta,\eta-\eta_s\}$, we find that

$$\begin{aligned}e^{-\eta t}|\Gamma_\xi(u)(t)|_\alpha\le{}&Ke^{(\eta_c-\eta-\epsilon)t}r_\epsilon^-(\omega)|P_c\xi|\\&+c_1K\|u\|_\eta\int_t^0e^{(\eta_c-\eta-\epsilon)(t-\tau)}r_\epsilon^-(\theta_\tau\omega)\rho(\theta_\tau\omega)^{p-1}d\tau\\&c_1K\|u\|_\eta\int_{-\infty}^te^{(\eta_s-\eta+\epsilon)(t-\tau)}(t-\tau)^{-\alpha}r_\epsilon^+(\theta_\tau\omega)\rho(\theta_\tau\omega)^{p-1}d\tau.\end{aligned}$$

A similar computation gives

$$\begin{aligned}&e^{-\eta t}|(\Gamma_\xi(u_1)(t)-\Gamma_\xi(u_2)(t))|_\alpha\\&\le(2c_1+c_2)K\|u_1-u_2\|_\eta\int_t^0e^{(\eta_c-\eta-\epsilon)(t-\tau)}r_\epsilon^-(\theta_\tau\omega)\rho(\theta_\tau\omega)^{p-1}d\tau\\&\quad+(2c_1+c_2)K\|u_1-u_2\|_\eta\int_{-\infty}^te^{(\eta_s-\eta+\epsilon)(t-\tau)}(t-\tau)^{-\alpha}r_\epsilon^+(\theta_\tau\omega)\rho(\theta_\tau\omega)^{p-1}d\tau.\end{aligned}$$

This suggests that we choose a ρ of the form

$$\rho(\omega)=\delta^{1/(p-1)}\min\{r_\epsilon^+(\omega)^{-1/(p-1)},r_\epsilon^-(\omega)^{-1/(p-1)}\}$$

for some $\delta>0$.

Then the above estimates show that $\Gamma_\xi : C_\eta \to C_\eta$ is well defined, and moreover

$$\|\Gamma_\xi(u)\|_\eta \le K r_\epsilon^-(\omega)|P_c\xi| + c_1 K\delta\|u\|_\eta \left(\frac{1}{(\eta_c-\eta-\epsilon)} + \frac{\Gamma(1-\alpha)}{(\eta-\eta_s-\epsilon)^{1-\alpha}}\right),$$

$$\|\Gamma_\xi(u_1)-\Gamma_\xi(u_2)\|_\eta \le (2c_1+c_2)K\delta\|u_1-u_2\|_\eta \left(\frac{1}{(\eta_c-\eta-\epsilon)} + \frac{\Gamma(1-\alpha)}{(\eta-\eta_s-\epsilon)^{1-\alpha}}\right).$$

Then Γ_ξ is a 1/2- contraction provided

$$\delta \le \frac{1}{2(2c_1+c_2)K}\left(\frac{1}{(\eta_c-\eta-\epsilon)} + \frac{\Gamma(1-\alpha)}{(\eta-\eta_s-\epsilon)^{1-\alpha}}\right)^{-1}.$$

For this choice of δ and hence ρ, it follows that Γ_ξ has a unique fixed point in C_η, say $u_\xi(\cdot,\omega)$, and then the local invariant manifold is given by the graph of $h(\omega,\cdot) : B(0,\rho(\omega))\cap H_c \to H_s$, where $h(\omega,\xi) = P_s u_\xi(\omega,0)$.

Note that the fixed point of the operator found above satisfies

$$|u_{\xi_1}(t,\omega) - u_{\xi_2}(t,\omega)|_\alpha \le 2Ke^{\eta_c t}|e^{W_t(\omega)B}P_c(\xi_1-\xi_2)|_\alpha,$$

which shows that $h(\omega,\cdot)$ is $2K$-Lipschitz.

Measurability of $h(\cdot,\xi)$ is immediate from the fact that the fixed point of a contracting map can be realized through any sequence of iterates, which allows to express $h(\cdot,\xi)$ as a pointwise limit of measurable functions.

In fact, it can be shown, using the same tools as in the local existence theorem, that h depends continuously in ω in the appropriate topology, and restricted to the subspace of Ω where $r_\epsilon^\pm$ are finite. □

With the same ideas it can also be shown that the stochastic invariant manifold $\mathcal{M}_\lambda(\omega)$ has the exponentially forward and pullback attractiveness properties, see [7], Sect. 4.2.

5. Stochastic Reduced Equations

In [7], the authors have derived theorems on the existence of local random invariant manifolds for Eq. (1.1). Approximation formulas for the local random invariant manifold functions have also been given, see Theorem 6.1 in [7]. We adapted the approximation formulas and performed a low order stochastic reduction based on the strategy proposed in the above reference. The low order stochastic reduction equations obtained in this section will shed light on the stochastic bifurcations associated with Eq. (1.6). Note that, from Sect. 2,the critical space H_1^c is two dimensional, hence the reduced system is also of dimension 2.

We write the projection operator $B : H_1 \to H_1^s$ as

$$Bu = \sum_{K\in\mathcal{Z}} \sigma_K \frac{(u,e_K)}{|e_K|^2} e_K, \tag{5.1}$$

Writing $u = u_1e_1 + u_2e_2 + u_s$, where $u_s \in H_1^s$, we obtain

$$\begin{aligned} du_1 &= \beta_1(\lambda)u_1 + \frac{1}{2l^2}\int_{\mathcal{U}} F(u_1e_1+u_2e_2+u_s)e_1dx + \sigma_{(1,0)}u_1 \circ dW \\ du_2 &= \beta_1(\lambda)u_2 + \frac{1}{2l^2}\int_{\mathcal{U}} F(u_1e_1+u_2e_2+u_s)e_2dx + \sigma_{(0,1)}u_2 \circ dW. \end{aligned} \tag{5.2}$$

The next step in the reduction procedure is to approximate u_s. Adapting Theorem 6.1 in [7] to our setting, the approximation formula for u_s is given by

$$h_\lambda^{app}(\xi,\omega) = \int_{-\infty}^0 e^{-\tau L_s - W_\tau(\omega)B} P_s F(e^{\tau L_c + W_\tau P_c(B)}\xi)d\tau, \tag{5.3}$$

where L_s and L_c are the projections of the linear operator L_λ to H_1^s and H_1^c, respectively. Using the properties of the exponential and projection operators, we can write (5.3) as

$$h_\lambda^{app}(\xi,\omega) = \sum_{K\in G}\left[\int_{-\infty}^{0} e^{-\sigma_K W_\tau(\omega)-\tau P_K L} P_K F(e^{\tau L_c}\xi)d\tau\right] e_K, \tag{5.4}$$

where $G = \mathcal{Z} - \{(1,0),(0,1)\}$.

Hence the approximate stochastic reduced Eqs. (5.2) have the form

$$\begin{aligned} du_1 &= \beta_1(\lambda)u_1 + \frac{1}{2l^2}\int_{\mathcal{U}} F(u_1e_1+u_2e_2+h_\lambda^{app}(u_1e_1+u_2e_2,\theta_t\omega))e_1dx + \sigma_{(1,0)}u_1\circ dW \\ du_2 &= \beta_1(\lambda)u_2 + \frac{1}{2l^2}\int_{\mathcal{U}} F(u_1e_1+u_2e_2+h_\lambda^{app}(u_1e_1+u_2e_2,\theta_t\omega))e_2dx + \sigma_{(0,1)}u_2\circ dW. \end{aligned} \tag{5.5}$$

In order to study the dynamics capture by (5.5), we need to compute the approximation formula given by

$$h_\lambda^{app}(u_1e_1+u_2e_2,\theta_t\omega). \tag{5.6}$$

Notice that the orthogonal projection of F onto the stable space is given by

$$P_sF\left(e^{\tau L_c}(u_1e_1+u_2e_2)\right) = \sum_{K\in G}\frac{\left(F\left(e^{\tau L_c}(u_1e_1+u_2e_2)\right),e_K\right)}{|e_K|^2}e_K. \tag{5.7}$$

Since, $F(u) = -u^3$, we obtain

$$\begin{aligned} \left(F\left(e^{\tau L_c}(u_1e_1+u_2e_2)\right),e_K\right) = &- e^{3\tau\beta_1(\lambda)}[u_1^3\left(e_1^3,e_K\right) \\ &+ 3u_1^2u_2\left(e_1^2e_2,e_K\right) + 3u_1u_2^2\left(e_1e_2^2,e_K\right) + u_2^3\left(e_2^3,e_K\right)]. \end{aligned} \tag{5.8}$$

Together with these L^2-inner product for $K\in G$

$$\left(e_1^3,e_K\right) = \begin{cases} -\frac{1}{2}l^2 & \text{if } K=(3,0) \\ 0 & \text{otherwise,} \end{cases} \tag{5.9}$$

$$\left(e_2^3,e_K\right) = \begin{cases} -\frac{1}{2}l^2 & \text{if } K=(0,3) \\ 0 & \text{otherwise,} \end{cases} \tag{5.10}$$

$$\left(e_1^2e_2,e_K\right) = \begin{cases} -\frac{1}{2}l^2 & \text{if } K=(2,1) \\ \frac{1}{2}l^2 & \text{if } K=(2,-1) \\ 0 & \text{otherwise,} \end{cases} \tag{5.11}$$

$$\left(e_1e_2^2,e_K\right) = \begin{cases} -\frac{1}{2}l^2 & \text{if } K=(1,2) \\ -\frac{1}{2}l^2 & \text{if } K=(1,-2) \\ 0 & \text{otherwise,} \end{cases} \tag{5.12}$$

we deduce that the orthogonal projection of F onto the stable space (5.7) has only 6 nonzero terms, namely when e_K are

$$e_{(3,0)},\quad e_{(2,1)},\quad e_{(2,-1)},\quad e_{(1,2)},\quad e_{(1,-2)}\quad \text{and}\quad e_{(0,3)}.$$

We then obtain the approximation formula as

$$\begin{aligned} h_\lambda^{app}(u_1\frac{e_1}{|e_1|}+u_2\frac{e_2}{|e_2|},\omega) = \frac{1}{8\sqrt{2}\,l^3}&\left[I_{(3,0)}u_1^3e_{(3,0)} + 3I_{(2,1)}u_1^2u_2e_{(2,1)}\right. \\ &- 3I_{(2,-1)}u_1^2u_2e_{(2,-1)} + 3I_{(1,2)}u_1u_2^2e_{(1,2)} \\ &\left.+3I_{(1,-2)}u_1u_2^2e_{(1,-2)} + I_{(0,3)}u_2^3e_{(0,3)}\right], \end{aligned} \tag{5.13}$$

where

$$I_{(m,n)} = \int_{-\infty}^{0} e^{-\sigma_{(m,n)} W_\tau(\omega) + 3\tau\beta_1(\lambda) - \tau\beta_{(m,n)}(\lambda)} d\tau. \tag{5.14}$$

For simplicity of notation, we are going to let

$$M_6 = I_{(3,0)}, \quad M_5 = I_{(2,1)}, \quad M_4 = I_{(2,-1)}, \quad M_3 = I_{(1,2)}, \quad M_2 = I_{(1,-2)}, \quad \text{and} \quad M_1 = I_{(0,3)}.$$

Putting all these together into Eq. (5.5), we obtain the stochastic reduced equations as

$$\begin{aligned} du_1 = \Bigg[&\beta_1(\lambda)u_1 - \left(\frac{3}{4l^2}u_1u_2^2 + \frac{3}{8l^2}u_1^3\right) + \left(\frac{3}{64l^4}M_6u_1^5 + \frac{9}{64l^4}M_2u_1u_2^4 + \frac{9}{64l^4}M_3u_1u_2^4\right. \\ &\left. + \frac{9}{32l^4}M_4u_1^3u_2^2 + \frac{9}{32l^4}M_5u_1^3u_2^2\right) - \left(\frac{27}{256l^6}M_2M_3u_1^3u_2^4 + \frac{27}{256l^6}M_2M_4u_1^3u_2^4\right. \\ &+ \frac{9}{256l^6}M_1M_3u_1u_2^6 + \frac{9}{256l^6}M_1M_2u_1u_2^6 + \frac{9}{256l^6}M_4M_6u_1^5u_2^2 + \frac{27}{256l^6}M_3M_5u_1^3u_2^4 \\ &+ \frac{9}{256l^6}M_5M_6u_1^5u_2^2 + \frac{3}{256l^6}M_6^2u_1^7 + \frac{27}{256l^6}M_2^2u_1^3u_2^4 + \frac{3}{256l^6}M_1^2u_1u_2^6 \\ &\left. + \frac{27}{256l^6}M_4^2u_1^5u_2^2 + \frac{27}{256l^6}M_3^2u_1^3u_2^4 + \frac{27}{256l^6}M_5^2u_1^5u_2^2\right) + o(\|u\|^8)\Bigg] dt + \sigma_{(1,0)}u_1 \circ dW, \end{aligned} \tag{5.15}$$

and

$$\begin{aligned} du_2 = \Bigg[&\beta_1(\lambda)u_2 - \left(\frac{3}{8l^2}u_2^3 + \frac{3}{4l^2}u_1^2u_2\right) + \left(\frac{3}{64l^4}M_1u_2^5 + \frac{9}{64l^4}M_5u_1^4u_2 + \frac{9}{64l^4}M_4u_1^4u_2\right. \\ &\left. + \frac{9}{32l^4}M_3u_1^2u_2^3 + \frac{9}{32l^4}M_2u_1^2u_2^3\right) - \left(\frac{3}{256l^6}M_1^2u_2^7 + \frac{27}{256l^6}M_3M_5u_1^4u_2^3\right. \\ &+ \frac{27}{256l^6}M_2M_4u_1^4u_2^3 + \frac{9}{256l^6}M_1M_3u_1^2u_2^5 + \frac{9}{256l^6}M_1M_2u_1^2u_2^5 + \frac{27}{256l^6}M_4M_5u_1^4u_2^3 \\ &+ \frac{9}{256l^6}M_5M_6u_1^6u_2 + \frac{9}{256l^6}M_4M_6u_1^6u_2 + \frac{3}{256l^6}M_6^2u_1^6u_2 + \frac{27}{256l^6}M_5^2u_1^4u_2^3 \\ &\left. + \frac{27}{256l^6}M_4^2u_1^4u_2^3 + \frac{27}{256l^6}M_3^2u_1^2u_2^5 + \frac{27}{256l^6}M_2^2u_1^2u_2^5\right) + o(\|u\|^8)\Bigg] dt + \sigma_{(0,1)}u_2 \circ dW. \end{aligned} \tag{5.16}$$

We conclude this section by analyzing, from a geometric point of view, the stochastic bifurcation associated with the explicit reduced Eqs. (5.15), (5.16). The result obtained is stated as the theorem below.

Theorem 5.1. *Suppose $\sigma_{(1,0)} = \sigma_{(0,1)} = \sigma_0$. Then, the reduced system (5.15), (5.16) undergoes a local stochastic pitchfork bifurcation in the pullback sense. More precisely, for $\lambda < \lambda_c$ the origin is a global attractor, whereas, for $\lambda > \lambda_c$, there exists a a random pullback attractor whose size is proportional to*

$$\left(\int_{-\infty}^{0} e^{2\beta_1(\lambda)s + 2\sigma_0 W_s(\omega)} ds\right)^{-\frac{1}{2}}.$$

Proof. Applying Ito formula we get that

$$\begin{aligned} &\frac{1}{2}dt(u_1^2 + u_2^2) \\ &= \Bigg[\beta_1(\lambda)(u_1^2 + u_2^2) - 3\left(\frac{1}{4}u_1^4 + u_1^2u_2^2 + \frac{1}{4}u_2^4\right) + \frac{3}{16}\left(M_6u_1^6 + M_1u_2^6\right) \\ &\quad - \frac{3}{32}\left(M_6^2u_1^8 + M_1^2u_2^8\right) + \frac{27}{16}u_1^2u_2^2\left((M_5 + M_4)u_1^2 + (M_3 + M_2)u_2^2\right) \\ &\quad - \frac{27}{32}u_1^2u_2^2\left((M_5u_1^2 + M_3u_2^2)^2 + (M_4u_1^2 + M_2u_2^2)^2\right) + R\Bigg] dt + \sigma_0(u_1^2 + u_2^2) \circ dW \end{aligned}$$

where $R \le 0$.

The terms with a factor of $3u_1^2u_2^2$ add up to

$$-1+\frac{9}{16}\left((M_5u_1^2+M_3u_2^2)+(M_4u_1^2+M_2u_2^2)\right)$$
$$-\frac{9}{32}\left((M_5u_1^2+M_3u_2^2)^2+(M_4u_1^2+M_2u_2^2)^2\right)$$

and the function

$$(x,y)\mapsto -1+\frac{9}{16}\left(x+y\right)-\frac{9}{32}\left(x^2+y^2\right)$$

is bounded above by $-\frac{7}{16}$.

The terms with a factor of $\frac{3}{4}u_1^4$ add up to

$$-1+\frac{1}{4}M_6u_1^2-\frac{1}{8}u_1^4$$

which is bounded above by $-\frac{7}{8}$.

Arguing similarly with u_2^4, we get in the end

$$\frac{1}{2}d(u_1^2+u_2^2)\le \beta_1(\lambda)(u_1^2+u_2^2)-\frac{21}{32}(u_1^2+u_2^2)^2+\sigma_0(u_1^2+u_2^2)\circ dW.$$

The above can be integrated explicitly to give

$$|u(t,\theta_{-t}\omega)|^2\le\left(e^{-2\beta_1(\lambda)t-2\sigma_0W_t(\omega)}|u_0(\omega)|^{-2}+\frac{21}{16}\int_{-t}^0 e^{2\beta_1(\lambda)s+2\sigma_0W_s(\omega)}\right)^{-1}$$

and this leads to the desired conclusion. More precisely, for $\beta_1(\lambda)<0$ we see that

$$\lim_{t\to\infty}|u(t,\theta_{-t}\omega)|=0 \quad \text{a.s.},$$

and for $\beta_1(\lambda)>0$ we get

$$\limsup_{t\to\infty}|u(t,\theta_{-t}\omega)|\le\left(\frac{16}{21}\right)^{1/2}\left(\int_{-\infty}^0 e^{2\beta_1(\lambda)s+2\sigma_0W_s(\omega)}ds\right)^{-\frac{1}{2}}.$$

□

It is noteworthy to mention that for (1.6) with scalar multiplicative noise, the attractor obtained has a random size, however the center manifold is defined on a deterministic ball, see [7]. But as shown in the above theorem, when the noise is concentrated on the fast modes, the opposite holds, that is, the size of the attractor is not random (Theorem 5.1 with $\sigma_0=0$), but the center manifold is defined on a random ball (Theorem 4.1).

6. Numerical Results and Discussion

In this section, numerical simulations of the reduced Stochastic Swift-Hohenberg Eqs. (5.15), (5.16) are performed by placing the noise at the most relevant fast modes and one of the slow modes. Using the notation in Sect. 5, multiplicative noise is placed on the following modes with the following intensities.

Mode	Noise intensity
$e_{(1,0)}$	$\sigma_{(1,0)}=0.01$
$e_{(1,1)}$	$\sigma_{(1,1)}=-0.5$
$e_{(0,2)}$	$\sigma_{(0,2)}=0.8$
$e_{(2,0)}$	$\sigma_{(2,0)}=-0.9$
$e_{(1,-1)}$	$\sigma_{(1,-1)}=0.5$

Recall that, in Sect. 2, the length l of the interval is taken to be between zero and $\sqrt{3/2}\,\pi$, so that we have the required PES as indicated in (2.6). For the numerical simulations l is taken to be 3.835, namely, l is below but close to its maximum value $\sqrt{3/2}\,\pi$, which makes the spectral gap small and, hence, we expect then that the interaction between high and slow modes will be more significant.

Both the full system and the reduced equations are solved using a semi-implicit Milstein method. The full SPDE is approximated by projecting the equation onto the eigenvectors of $(I+\Delta)^2$, and truncating the resulting system so that we keep only the first 40 modes. Note, however, that because of the polynomial

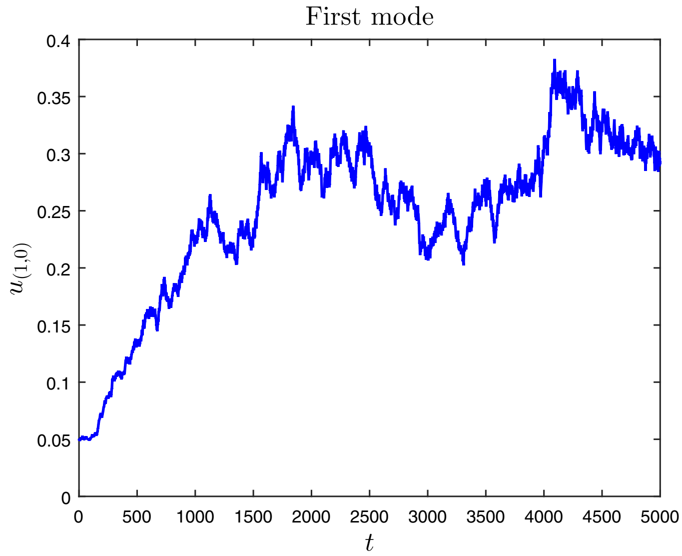

Fig. 1. Time evolution of the projection of the solution on $e_{(1,0)}$

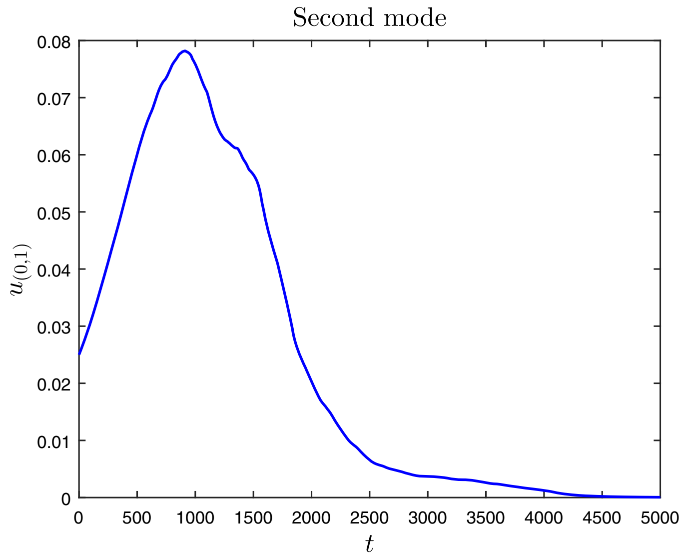

Fig. 2. Time evolution of the projection of the solution on $e_{(0,1)}$. The relative smoothness of this curve reflects the fact that $\sigma_{(0,1)} = 0$

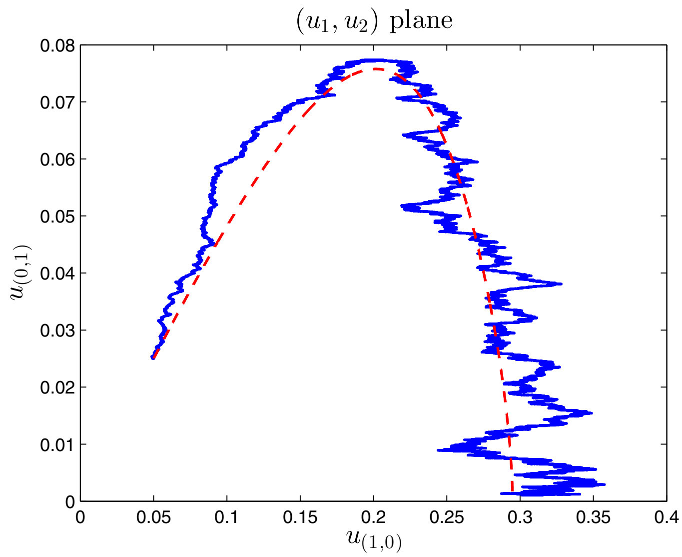

FIG. 3. Trajectories of the stochastic and deterministic systems projected onto the H_c plane

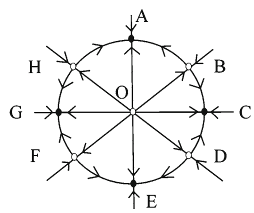

FIG. 4. Schematic representation of the deterministic attractor on the $e_{(1,0)} - e_{(0,1)}$ plane

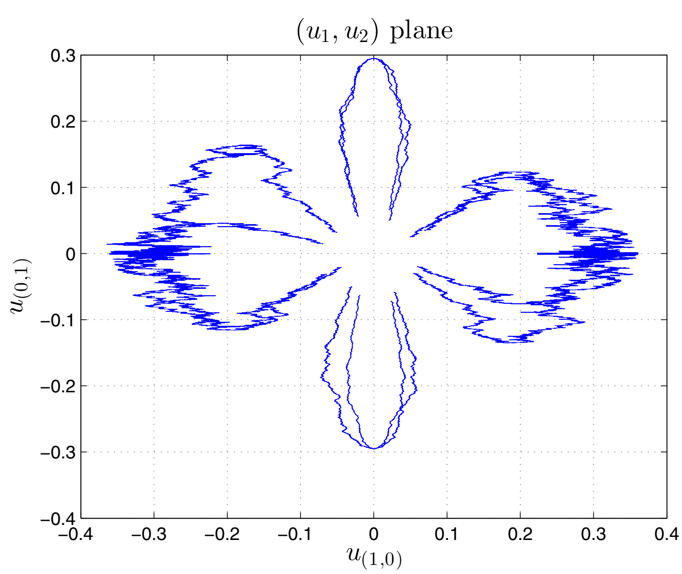

FIG. 5. Time evolution of the SPDE from an ensemble of different initial conditions projected on the $e_{(1,0)} - e_{(0,1)}$ plane. The long time behavior of the solutions resembles the structure of the deterministic system shown in Fig. 4

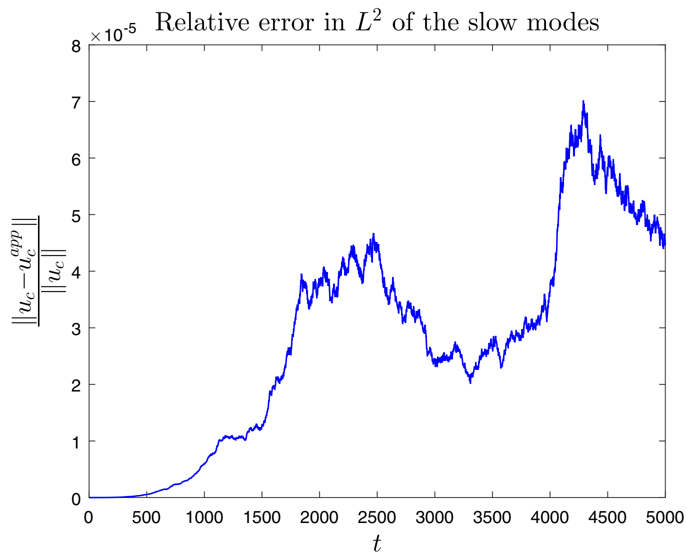

Fig. 6. Time evolution of the difference between solutions of the full SPDE and the reduced SDE using the same initial conditions

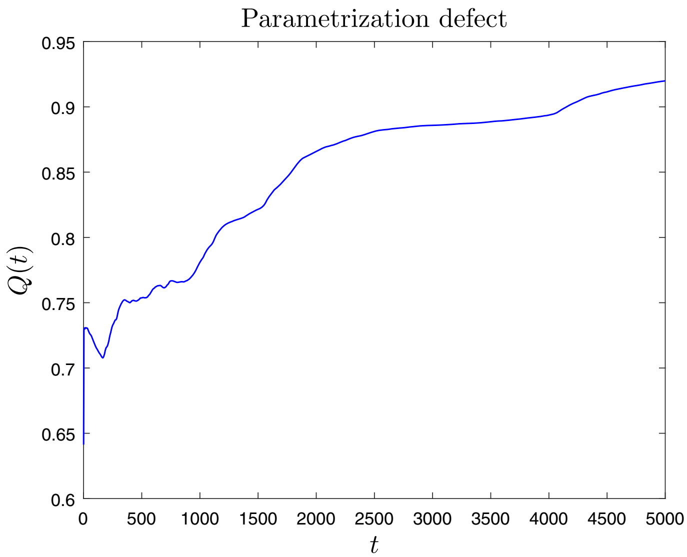

Fig. 7. Time evolution of the parametrization defect corresponding to solutions of the full SPDE and the reduced SDE using the same initial conditions

growth of the eigenvalues the degree of accuracy of this strategy can be achieved with even less modes; in fact, all the results obtained below remain essentially unchanged if we keep only as few as 20 modes.

Figures 1 and 2 show the amplitudes of the first and second modes of the reduced SSHE. Recall that the first mode has multiplicative noise with intensity 0.01, while second mode is free of noise. Notice that even though the second mode is free of noise, Fig. 1 shows that it still manages to get some feedback from the high modes, on which the noise is intentionally chosen with a higher intensity.

Figure 3 shows a typical trajectory of the system. We set the initial data to be a point on the approximated invariant manifold, with coordinates on the H_c plane given by $u_c(0) = 0.05e_{(1,0)} + 0.025e_{(0,1)}$. For comparison, the dashed line shows the trajectory of the deterministic system corresponding to the same initial data.

Since the reduced equations cannot be solved in closed form, we are not able to give a detailed description of the geometric structure of the bifurcated random attractor. Nevertheless, the numerical simulations show that the structure of the deterministic attractor, shown in Fig. 4, is not entirely lost. For instance, by taking an initial condition at a point close to the origin O in the sector OCB, Fig. 3 shows that the corresponding trajectory has the same qualitative behavior of its deterministic counterpart: at first it is influenced by the repulsive nature of the origin, and travels in the direction of OB before the effect of the attractive point C changes the trajectory, after which the path moves closer to the x-axis. This suggests that the structure of the random attractor will resemble that of the deterministic case. Figure 5 shows that this is indeed the case, in the sense that, by fixing a realization of the noise and choosing a set initial data on different points of the H_c plane, the corresponding trajectories approach the regions where the deterministic attractive points would have been.

Next we give two types of error analysis: first, the relative error incurred by approximating the dynamics of the slow modes by the reduced system, and, second, the error incurred by expressing the high modes of the full SPDE in terms of the slow modes via the approximate invariant manifold function. In the first case, we consider $u_c(t) = P_c u(t)$, where $u(t)$ is the solution of (1.6), with initial datum $u(0)$ taken as a point on the approximate invariant manifold $\mathcal{M}_\lambda^{loc}$. Namely, $u_c(t)$ consists of the first two modes of the full solution (the slow modes). We compare this to $u_c^{\mathrm{app}}(t)$, the solution of (5.15), (5.16), that is, the reduced SSHE equations. Figure 6 shows that the relative L^2-error is very small. This shows that the reduced system captures the behavior of the full SPDE with good accuracy.

Next, we investigate the effectiveness of h^{app} in approximating the true invariant manifold function h whose existence is given in Sect. 4. This is measured by the so called *parametrization defect*, $Q(T, \omega; u_0)$, given by

$$Q(T, \omega; u_0) = \frac{\int_0^T \|u_s(t, \omega; u_0) - h^{\mathrm{app}}(u_c(t, \omega; u_0), \theta_t \omega)\|^2 dt}{\int_0^T \|u_s(t, \omega; u_0)\|^2 dt}, \tag{6.1}$$

see (4.29) [8]. Notice that $Q = 1$ when $h^{\mathrm{app}} \equiv 0$, so that $Q < 1$ implies that h^{app} is a better approximation compared to the Galerkin method, which amounts to neglect all interactions between the fast and slow modes (Fig. 7).

Compliance with ethical standards

Conflict of interest All authors declare that they have no conflict of interest.

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