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On Weighted Estimates for the Stokes Flows, with Application to the Navier–Stokes Equations

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Abstract. Weighted estimates on the Stokes flows are given by means of the Stokes solution formula in the half space, which can be regarded as a complement and improvement on the previous known results. There are two main difficulties: in weighted cases, usual $L^q - L^r$ estimates for the Stokes flows do not work any more, and the projection operator becomes unbounded possibly. Finally, as an application, employing these weighted estimates on the Stokes solution, we establish some weighted decay results for the Navier–Stokes flows.

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1. Introduction and Main results

We are concerned with the weighted decay properties of solutions to the Stokes initial-value problem in the half space

$$\begin{cases} \partial_t u - \Delta u + \nabla p = 0 & \text{in } \mathbb{R}^n_+ \times (0,\infty), \\ \nabla \cdot u = 0 & \text{in } \mathbb{R}^n_+ \times (0,\infty), \\ u(x,t) = 0 & \text{on } \partial\mathbb{R}^n_+ \times (0,\infty), \\ u(x,t) \longrightarrow 0 & \text{as } |x| \longrightarrow \infty, \ \ t>0, \\ u(x,0) = a(x) & \text{in } \mathbb{R}^n_+, \end{cases} \tag{1.1}$$

where $n \geq 2$, and $\mathbb{R}^n_+ = \{x = (x', x_n) \in \mathbb{R}^n \mid x_n > 0\}$ is the upper-half space of $\mathbb{R}^n$; $u = (u_1(x,t), u_2(x,t), \ldots, u_n(x,t))$ and $p = p(x,t)$ denote unknown velocity vector and the pressure respectively, while initial data $a(x)$ are assumed to satisfy a *compatibility condition*: $\nabla \cdot a = 0$ in $\mathbb{R}^n_+$ and the normal component of a equals to zero on $\partial\mathbb{R}^n_+$; and

$$\partial_t = \frac{\partial}{\partial t}, \quad \partial_j = \frac{\partial}{\partial x_j} \ (j = 1, 2, \ldots, n), \quad \nabla \cdot u = \sum_{j=1}^n \partial_j u_j, \quad \Delta u = \sum_{j=1}^n \partial_j \partial_j u.$$

Throughout this article, $C_0^\infty(\mathbb{R}^n_+)$ denotes the set of all C^∞ real functions with compact support in $\mathbb{R}^n_+$, and

$$C^\infty_{0,\sigma}(\mathbb{R}^n_+) = \{\phi = (\phi_1, \ldots, \phi_n) \in C_0^\infty(\mathbb{R}^n_+);\ \nabla \cdot \phi = 0 \ \text{ in } \ \mathbb{R}^n_+\};$$

$L^q_\sigma(\mathbb{R}^n_+)$ $(1 < q < \infty)$ is the closure of $C^\infty_{0,\sigma}(\mathbb{R}^n_+)$ with respect to $\|\cdot\|_{L^q(\mathbb{R}^n_+)}$, where $L^q(\mathbb{R}^n_+)$ represents the usual Lebesgue space of vector-valued functions. The norm of $L^q(\mathbb{R}^n_+)$ $(1 \leq q < \infty)$ is denoted by $\|u\|_{L^q(\mathbb{R}^n_+)} = (\int_{\mathbb{R}^n_+} |u(x)|^q dx)^{\frac{1}{q}}$; $\||\omega|^\alpha u(t)\|_{L^q(\mathbb{R}^n_+)} = (\int_{\mathbb{R}^n_+} (|\omega(x)|^\alpha |u(x,t)|)^q dx)^{\frac{1}{q}}$. $\overrightarrow{m} = (m_1, m_2, \ldots, m_n)$, m_i $(1 \leq i \leq n)$ are nonnegative integers, $|\overrightarrow{m}| = \sum_{i=1}^n m_i$. By symbol C, it means a generic positive constant which may vary from line to line.

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Recall the Helmholtz decomposition [13]:

$$L^r(\mathbb{R}^n_+) = L^r_\sigma(\mathbb{R}^n_+) \oplus L^r_\pi(\mathbb{R}^n_+), \quad 1 < r < \infty,$$

where

$$L^r_\sigma(\mathbb{R}^n_+) = \{u = (u_1, u_2, \ldots, u_n) \in L^r(\mathbb{R}^n_+);\ \ \nabla \cdot u = 0,\ u_n|_{\partial \mathbb{R}^n_+} = 0\},$$
$$L^r_\pi(\mathbb{R}^n_+) = \{\nabla p \in L^r(\mathbb{R}^n_+);\ \ p \in L^r_{loc}(\overline{\mathbb{R}^n_+})\}.$$

Let $A = -P\Delta$ denote the Stokes operator in $\mathbb{R}^n_+$, where $P = P_r$ is the associated bounded projection: $L^r(\mathbb{R}^n_+) \longrightarrow L^r_\sigma(\mathbb{R}^n_+), 1 < r < \infty$. Then (see [13]) the operator $-A$ generates a bounded analytic semigroup $\{e^{-tA}\}_{t\geq 0}$ in $L^r_\sigma(\mathbb{R}^n_+)$. So the function $e^{-tA}a$ gives a unique solution of the Stokes problem (1.1) in $L^r_\sigma(\mathbb{R}^n_+)$.

There is a great literature on the decay properties for the Stokes flow $e^{-tA}a$. Here we collect some important known results. The following classical $L^q - L^r$ estimates are from [22]:

$$\|\nabla^k e^{-tA}a\|_{L^q(\mathbb{R}^n_+)} \leq Ct^{-\frac{k}{2}-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\|a\|_{L^r(\mathbb{R}^n_+)}, \quad \forall\, a \in L^r_\sigma(\mathbb{R}^n_+)$$

with $k = 0, 1, \ldots$, provided that either $1 \leq r < q \leq \infty$ or $1 < r \leq q < \infty$.

Fujigaki and Miyakawa [22] derived more rapid L^r-estimate of the Stokes flow $e^{-tA}a$ with initial data a in the weighted L^1 space:

$$\|e^{-tA}a\|_{L^r(\mathbb{R}^n_+)} \leq Ct^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{r})}\|x_n a\|_{L^1(\mathbb{R}^n_+)},$$

whenever $1 < r \leq \infty$.

Bae [1] considered more rapid L^1- and L^∞-estimates with an initial data under the special assumption: $\int_{-\infty}^{\infty} u_0(y)dy_i = 0$ for some $i = 1, 2, \ldots, n-1$, and in this case

$$\|e^{-tA}u_0\|_{L^r(\mathbb{R}^n_+)} \leq Ct^{-\frac{1}{2}}\|x_n u_0\|_{L^r(\mathbb{R}^n_+)} \quad for \quad r = 1, \infty.$$

Under the symmetric assumption on the initial data a, Han [26] proved that for each $0 < \alpha < 1$ and $t > 0$

$$\|e^{-tA}a\|_{L^1(\mathbb{R}^n_+)} \leq Ct^{-\frac{\alpha}{2}}\int_{\mathbb{R}^n_+}|y|^\alpha|a(y)|dy.$$

Han [23] established the weighted $L^q - L^r$ estimates of the Stokes flows in the half space, that is

$$\|\nabla^k e^{-tA}a\|_{L^q(\mathbb{R}^n_+)} \leq Ct^{-\frac{\alpha}{2}-\frac{k}{2}-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\|x_n^\alpha a\|_{L^r(\mathbb{R}^n_+)},\ \ 0 \leq \alpha \leq 1,\ \ k = 0, 1, 2, \ldots,$$

provided that $1 \leq r < q \leq \infty$. For the cases of $q = r = 1, +\infty$, Han [25] obtained the following weighted estimates: Let $k = 1, 2, \ldots$, $0 \leq \alpha \leq 1$. Then

$$\|\nabla^k e^{-tA}a\|_{L^1(\mathbb{R}^n_+)} \leq Ct^{-\frac{\alpha}{2}-\frac{k}{2}}\||x|^\alpha a\|_{L^1(\mathbb{R}^n_+)},$$

and

$$\|\nabla^k e^{-tA}a\|_{L^\infty(\mathbb{R}^n_+)} \leq Ct^{-\frac{k+1}{2}}\||x|a\|_{L^\infty(\mathbb{R}^n_+)}.$$

Jin [33,34] obtained the weighted $L^q - L^1$ estimates of the Stokes flows in the half space: Let $1 < q \leq \infty$. Then

$$\||x'|^r e^{-tA}a\|_{L^q(\mathbb{R}^n_+)} \leq Ct^{-\frac{n}{2}(1-\frac{1}{q})-\frac{1}{2}}\||x'|^r x_n a\|_{L^1(\mathbb{R}^n_+)} + Ct^{-\frac{n}{2}(1-\frac{1}{q})+\frac{r-1}{2}}\|x_n a\|_{L^1(\mathbb{R}^n_+)}$$

for $0 \leq r < (n-1)(1-\frac{1}{q})$; and

$$\|x_n^r e^{-tA}a\|_{L^q(\mathbb{R}^n_+)} \leq Ct^{-\frac{n}{2}(1-\frac{1}{q})-\frac{1}{2}}\|x_n^{r+1}a\|_{L^1(\mathbb{R}^n_+)} + Ct^{-\frac{n}{2}(1-\frac{1}{q})+\frac{r-1}{2}}\|x_n a\|_{L^1(\mathbb{R}^n_+)}$$

for $0 \leq r < n(1-\frac{1}{q})$. Using embedding results for $L^q(\mathbb{R}^n_+)-$space without weight in the weighted $L^q_\omega(\mathbb{R}^n_+)$ context for the general weight function ω from the Muckenhoupt class, Fröhlich [21] established a kind of estimates on $e^{-tA_{p,\omega}}$ in $D^{\alpha,p}_\omega(\mathbb{R}^n_+) := D((I + A_{p,\omega})^\alpha)$. Recently Chang and Jin [20] derived a rapid decay rate of the Stokes flows in space and time when the initial data decreases fast enough and satisfies some additional condition. Kobayashi and Kubo [30] considered the Navier–Stokes equations in half-space and in L^p space with Muckenhoupt weight and showed the $L^p - L^q$ estimates of Stokes semigroup with a type of weight. The range of p, q in Theorem 2.1 in [30] is $1 < p \leq q < \infty$, and the weighted estimates of only zero order and first spatial derivatives of the Stokes flows are treated. While in the following Theorem 1.1, p, q can be relaxed to be $1 < p \leq q < \infty$, $1 \leq p < q < \infty$, and the weighted estimates of arbitrary spatial derivatives of the Stokes flows are given. In addition, The method in [30] is different from ours. Presicely, Ukai's solution formula on the Stokes problem in the half space is used for proving Theorem 2.1 in [30], however Solonnikov's solution representation on the Stokes flows in the half space is employed to establish Theorem 1.1 in our paper. The three kinds of classical weight functions defined in (1.2) clearly show how the decay rates of the Stokes flow are affected, which is needed in the study of viscous fluids. For further results on the Stokes flows, please refer to relevant literature [2–4,27–29,31,45] and the references therein. The main result (i.e. Theorem 1.1) in this article can be regarded as a complement, extension and improvement of some mentioned known conclusions on the Stokes flows.

Theorem 1.1. *Let $a = (a_1, a_2, \dots, a_n)$ satisfy $\nabla \cdot a = 0$ in $\mathbb{R}^n_+$ $(n \geq 2)$, $a_n |_{\partial \mathbb{R}^n_+} = 0$. Then for $i = 1, 2, 3$, and $t > 0$*

$$\|\omega_{\beta i} \nabla^{\vec{m}} e^{-tA} a\|_{L^q(\mathbb{R}^n_+)} \leq C t^{-\frac{|\vec{m}|}{2} - \frac{\alpha-\beta}{2} - \frac{n}{2}(\frac{1}{r} - \frac{1}{q})} \|\omega_{\alpha i} a\|_{L^r(\mathbb{R}^n_+)}, \quad |\vec{m}| = 0, 1, 2, \dots,$$

provided that either $1 < r \leq q < \infty$ or $1 \leq r < q < \infty$. Here $C = C(n, q, r, \vec{m}, \alpha, \beta)$, and the weighted functions $\omega_{\gamma i}$ $(\gamma = \alpha, \beta,\ i = 1, 2, 3)$ are defined as follows for $x = (x', x_n) \in \mathbb{R}^n_+$

$$\begin{cases} \omega_{\gamma 1}(x) = |x|^\gamma, & \gamma = \alpha, \beta, \quad \text{for } -\frac{n}{q} < \beta \leq \alpha < n(1 - \frac{1}{r}); \\ \omega_{\gamma 2}(x) = |x'|^\gamma, & \gamma = \alpha, \beta, \quad \text{for } -\frac{n-1}{q} < \beta \leq \alpha < (n-1)(1 - \frac{1}{r}); \\ \omega_{\gamma 3}(x) = x_n^\gamma, & \gamma = \alpha, \beta, \quad \text{for } -\frac{1}{q} < \beta \leq \alpha < 1 - \frac{1}{r}. \end{cases} \tag{1.2}$$

The second aim is to apply the results of Theorem 1.1 to establish the weighted decay properties on solutions to the Navier–Stokes problem:

$$\begin{cases} \partial_t u - \Delta u + (u \cdot \nabla) u + \nabla p = 0 & \text{in } \mathbb{R}^n_+ \times (0, \infty), \\ \nabla \cdot u = 0 & \text{in } \mathbb{R}^n_+ \times (0, \infty), \\ u(x, t) = 0 & \text{on } \partial \mathbb{R}^n_+ \times (0, \infty), \\ u(x, t) \longrightarrow 0 & \text{as } |x| \longrightarrow \infty,\ t > 0, \\ u(x, 0) = u_0 & \text{in } \mathbb{R}^n_+, \end{cases} \tag{1.3}$$

where the convection term $(u \cdot \nabla) u = \sum_{j=1}^n u_j \partial_j u$.

A vector function u is called a weak solution of (1.3) if $u \in L^\infty(0, \infty; L^2_\sigma(\mathbb{R}^n_+)) \bigcap L^2_{loc}([0, \infty); H^1_0(\mathbb{R}^n_+))$ satisfies problem (1.3) in the sense of distributions. Moreover, the energy inequality holds for almost all $t \in [0, \infty)$ including $t = 0$:

$$\|u(t)\|^2_{L^2(\mathbb{R}^n_+)} + 2 \int_0^t \|\nabla u(s)\|^2_{L^2(\mathbb{R}^n_+)} ds \leq \|u_0\|^2_{L^2(\mathbb{R}^n_+)}.$$

Furthermore we call u is a strong solution of (1.3) if the Serrin's condition holds: $u \in L^q(0, \infty; L^r(\mathbb{R}^n_+))$ with $\frac{2}{q} + \frac{n}{r} \leq 1$, $2 \leq q < \infty$, $n < r \leq \infty$.

The (weak or strong) solution u of problem (1.3) can be written as follows:

$$u(t) = e^{-tA} u_0 - \int_0^t e^{-(t-s)A} P(u(s) \cdot \nabla) u(s) ds.$$

Note that for $t \geq s > 0$, $e^{-(t-s)A} P(u(s) \cdot \nabla) u(s)$ is also a Stokes flow with the initial data $P(u(s) \cdot \nabla) u(s)$. That is, set $w(t) = e^{-(t-s)A} P(u(s) \cdot \nabla) u(s)$, $t \geq s > 0$, then

$$\begin{cases} \partial_t w - \Delta w + \nabla \pi = 0 & \text{in } \mathbb{R}^n_+ \times (s, \infty), \\ \nabla \cdot w = 0 & \text{in } \mathbb{R}^n_+ \times (s, \infty), \\ w(x, t) = 0 & \text{on } \partial \mathbb{R}^n_+ \times (s, \infty), \\ w(x, t)|_{t=s} = P(u(s) \cdot \nabla) u(s) & \text{in } \mathbb{R}^n_+. \end{cases}$$

The above arguments reveal that the conclusions on the Stokes flows are very necessary in studying problem (1.3). In recent years, much attention has been paid to the Navier–Stokes equations. Caffarelli, Kohn and Nirenberg [14], Lin [32], Chae [15–19], Bae and Choe [4], Bae and Jin [5–9], Brandolese [10,11], Brandolese and Vigneron[12], Fujigaki and Miyakawa [22], Schonbek [35–40] studied problem (1.3), and many important and interesting results on regularity and decay properties are obtained, which brings about a deeper understanding on the internal structure and interaction mechanism to this kind of classical system.

Theorem 1.2. *Assume $u_0 \in L^2_\sigma(\mathbb{R}^n_+)$ $(n \geq 2)$. There exists a number $\eta_0 > 0$ such that if $\|u_0\|_{L^n(\mathbb{R}^n_+)} \leq \eta_0$ (smallness condition is unnecessary if $n = 2$), problem (1.3) possesses a unique strong solution u. Suppose u_0 satisfies*

$$\|(1+|x|)(|u_0| + |\nabla u_0|)\|_{L^1(\mathbb{R}^n_+)} + \|(1+|x|)(|u_0| + |\nabla u_0|)\|_{L^\infty(\mathbb{R}^n_+)} < \infty.$$

Let $1 < r \leq q < \infty$, $0 \leq \alpha - \beta < 1 - n(\frac{1}{r} - \frac{1}{q})$, $\alpha < \min\{1, n(1-\frac{1}{r})\}$. Additionally assume $\|\omega_{\alpha i} u_0\|_{L^r(\mathbb{R}^n_+)} + \|\omega_{\alpha i}\nabla u_0\|_{L^r(\mathbb{R}^n_+)} < \infty,\ \ i = 1,2,3$. Then for $t > 0$

$$\|\omega_{\beta i}\nabla^{\overrightarrow{m}} u(t)\|_{L^q(\mathbb{R}^n_+)} \leq Ct^{-\frac{|\overrightarrow{m}|}{2} - \frac{\alpha-\beta}{2} - \frac{n}{2}(\frac{1}{r} - \frac{1}{q})},$$

where $|\overrightarrow{m}| = 0, 1,$ and $\omega_{\gamma i}$ $(\gamma = \alpha, \beta,\ i = 1, 2, 3)$ are defined in (1.2).

Remark. The assumption: $0 \leq \alpha - \beta < 1 - n(\frac{1}{r} - \frac{1}{q})$ implies that $\frac{1}{2} - \frac{\alpha-\beta}{2} - \frac{n}{2}(\frac{1}{r} - \frac{1}{q}) > 0$, which guarantees that the strong singularity does not appear in calculating the term in (3.5) for $|\overrightarrow{m}| = 1$. That is

$$\int_{\frac{t}{2}}^{t} (t-s)^{-\frac{|\overrightarrow{m}|}{2} - \frac{\alpha-\beta}{2} - \frac{n}{2}(\frac{1}{r} - \frac{1}{q})} ds < +\infty \quad \text{with} \quad |\overrightarrow{m}| = 1.$$

2. Weighted Estimates on the Stokes flows

The Stokes flow $e^{-tA}a$ of problem (1.1) can be represented as follows for $x \in \mathbb{R}^n_+$ and $t > 0$ (see [41,42])

$$[e^{-tA}a](x) = \int_{\mathbb{R}^n_+} \mathcal{M}(x, y, t) a(y) dy, \tag{2.1}$$

where $\mathcal{M} = (M_{ij})_{i,j=1,2,\ldots,n}$ is defined as follows for $1 \leq i, j \leq n$

$$M_{ij}(x, y, t) = \delta_{ij}(G_t(x-y) - G_t(x-y^*)) + M^*_{ij}(x, y, t).$$

Here

$$M^*_{ij}(x, y, t) = 4(1-\delta_{jn}) \frac{\partial}{\partial x_j} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \frac{\partial E(x-z)}{\partial x_i} G_t(z - y^*) dz,$$

$y^* = (y_1, y_2, \ldots, -y_n)$, $G_t(x) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$ is the Gaussian kernel, and

$$E(z) = \begin{cases} -\dfrac{\Gamma(\frac{n}{2})}{2(n-2)\pi^{\frac{n}{2}}} \dfrac{1}{|z|^{n-2}} & \text{if } n \geq 3, \\ \dfrac{1}{2\pi} \log|z| & \text{if } n = 2, \end{cases}$$

is the fundamental solution of the Laplace equation. In addition, the following estimate holds for $\mathcal{M}^*$ $(x, y, t) = (M^*_{ij}(x, y, t))_{n\times n},\ \ x = (x', x_n),\ y = (y', y_n) \in \mathbb{R}^n_+$:

$$|\partial_t^s \partial_x^{\overrightarrow{\ell}} \partial_y^{\overrightarrow{m}} \mathcal{M}^*(x, y, t)| \leq Ct^{-s-\frac{m_n}{2}} (x_n + \sqrt{t})^{-\ell_n} (|x'-y'| + x_n + y_n + \sqrt{t})^{-(n+|\ell'|+|m'|)}, \tag{2.2}$$

where $\overrightarrow{m} = (m_1, m_2, \ldots, m_{n-1}, m_n) = (m', m_n),\ \ \ell = (\ell_1, \ell_2, \ldots, \ell_{n-1}, \ell_n) = (\ell', \ell_n)$.

Define the linear operator T_λ in the whole space $\mathbb{R}^m$ $(m \geq 1)$ as follows:

$$(T_\lambda f)(x) = \int_{\mathbb{R}^m} \frac{f(y)}{|x-y|^\lambda} dy, \quad 0 < \lambda \leq m. \tag{2.3}$$

Let $\lambda = m$, there holds for $1 < p < \infty$ and $-\frac{m}{p} < \beta < m(1-\frac{1}{p})$ (see [43])

$$\left(\int_{\mathbb{R}^m} \left(|(T_m f)(x)||x|^\beta\right)^p dx\right)^{\frac{1}{p}} \le A_{p,\beta}\left(\int_{\mathbb{R}^m} \left(|f(x)||x|^\beta\right)^p dx\right)^{\frac{1}{p}}. \tag{2.4}$$

On the operator T_λ with $0 < \lambda < m$, we have the following weighted inequality, which can be found in [44].

Lemma 2.1. *Let $0 < \lambda < m$ $(m \ge 1)$, $1 < r < \infty$, $-\frac{m}{q} < \beta \le \alpha < m(1-\frac{1}{r})$, and $\frac{1}{q} = \frac{1}{r} + \frac{\lambda+\alpha-\beta}{m} - 1$. If $r \le q < \infty$, then there exists $C = C(m, q, r, \lambda, \alpha, \beta)$ such that the operator T_λ defined in (2.3) satisfies*

$$\left(\int_{\mathbb{R}^m} \left(|(T_\lambda f)(x)||x|^\beta\right)^q dx\right)^{\frac{1}{q}} \le C\left(\int_{\mathbb{R}^m} \left(|f(x)||x|^\alpha\right)^r dx\right)^{\frac{1}{r}}.$$

Remark. In this article, the integer m in (2.3), (2.4), (2.21), (2.22) and in Lemma 2.1 will be sometimes valued at n, sometimes $n-1$ and sometimes 1 as required.

Set

$$(Sf)(x) = \int_{\mathbb{R}^n_+} \frac{f(y)dy}{(|x'-y'| + x_n + y_n + 1)^n}, \quad x = (x', x_n) \in \mathbb{R}^n_+.$$

Lemma 2.2. *Let $n \ge 2$. Then the operator S satisfies for $i = 1, 2, 3$*

$$\|\omega_{\beta i} S f\|_{L^q(\mathbb{R}^n_+)} \le C\|\omega_{\alpha i} f\|_{L^r(\mathbb{R}^n_+)} \tag{2.5}$$

provided that either $1 < r \le q < \infty$ or $1 \le r < q < \infty$. Here $C = C(n, q, r, \alpha, \beta)$, and the weighted functions $\omega_{\gamma i}$ ($\gamma = \alpha, \beta, i = 1, 2, 3$) are defined in (1.2).

Proof. **Case 1.** $r = q \in (1, \infty)$. Note that there holds for $x \in \mathbb{R}^n_+$

$$\begin{aligned} |(Sf)(x)| &\le \int_{\mathbb{R}^n_+} \frac{|f(y)|dy}{(|x'-y'| + x_n + y_n + 1)^n} \\ &\le \int_{\mathbb{R}^n_+} \frac{|f(y)|dy}{(|x-y| + 1)^n} \\ &\le \int_{\mathbb{R}^n} \frac{|f_*(y)|dy}{(|x-y| + 1)^n} \\ &\le \int_{\mathbb{R}^n} \frac{|f_*(y)|dy}{|x-y|^{n-(\alpha-\beta)}}, \end{aligned} \tag{2.6}$$

where f_* denotes the even extension of f from $\mathbb{R}^n_+$ to $\mathbb{R}^n$:

$$f_*(y', y_n) = \begin{cases} f(y', y_n) & \text{if } y_n \ge 0, \\ f(y', -y_n) & \text{if } y_n < 0. \end{cases}$$

□

If $\alpha > \beta$.

Set $\lambda_1 = n - (\alpha - \beta)$, then $\lambda_1 \in (0, n)$ for $-\frac{n}{q} < \beta < \alpha < n(1-\frac{1}{q})$. Using (2.6) and Lemma 2.1 with $m = n$ yields

$$\begin{aligned} \|\omega_{\beta 1} S f\|_{L^q(\mathbb{R}^n_+)} &= \||x|^\beta S f\|_{L^q(\mathbb{R}^n_+)} \le \||x|^\beta T_{\lambda_1} f_*\|_{L^q(\mathbb{R}^n)} \\ &\le C\||x|^\alpha f_*\|_{L^q(\mathbb{R}^n)} \le C\||x|^\alpha f\|_{L^q(\mathbb{R}^n_+)} = C\|\omega_{\alpha 1} f\|_{L^q(\mathbb{R}^n_+)}. \end{aligned} \tag{2.7}$$

Using the Young inequality yields that

$$\begin{aligned}\|\omega_{\beta 2}Sf\|_{L^q(0,+\infty)} &= |x'|^\beta\left(\int_0^\infty |Sf(x)|^q dx_n\right)^{\frac{1}{q}}\\ &\le C|x'|^\beta \int_{\mathbb{R}^{n-1}} \frac{\|f(y',\cdot)\|_{L^q(0,+\infty)}}{(|x'-y'|+1)^{n-1}}dy'\\ &\le C|x'|^\beta \int_{\mathbb{R}^{n-1}} \frac{\|f(y',\cdot)\|_{L^q(0,+\infty)}}{|x'-y'|^{n-1-(\alpha-\beta)}}dy' \qquad (2.8)\end{aligned}$$

for $-\frac{n-1}{q} < \beta < \alpha < (n-1)(1-\frac{1}{q})$; and

$$\begin{aligned}\|\omega_{\beta 3}Sf\|_{L^q(\mathbb{R}^{n-1})} &= x_n^\beta\left(\int_{\mathbb{R}^{n-1}} |Sf(x)|^q dx'\right)^{\frac{1}{q}}\\ &\le Cx_n^\beta \int_0^\infty \frac{\|f(\cdot,y_n)\|_{L^q(\mathbb{R}^{n-1})}}{x_n+y_n+1}dy_n\\ &\le Cx_n^\beta \int_0^\infty \frac{\|f(\cdot,y_n)\|_{L^q(\mathbb{R}^{n-1})}}{|x_n-y_n|+1}dy_n\\ &\le Cx_n^\beta \int_{\mathbb{R}^1} \frac{[\|f(\cdot,y_n)\|_{L^q(\mathbb{R}^{n-1})}]_*}{|x_n-y_n|^{1-(\alpha-\beta)}}dy_n \qquad (2.9)\end{aligned}$$

for $-\frac{1}{q} < \beta < \alpha < 1-\frac{1}{q}$.

It follows from (2.8), (2.9) and Lemma 2.1 that

$$\begin{aligned}\|\omega_{\beta 2}Sf\|_{L^q(\mathbb{R}^n_+)} &\le C\||x'|^\beta T_{\lambda_2}\|f(x',\cdot)\|_{L^q(0,+\infty)}\|_{L^q(\mathbb{R}^{n-1})}\\ &\le C\||x'|^\alpha\|f(x',\cdot)\|_{L^q(0,+\infty)}\|_{L^q(\mathbb{R}^{n-1})}\\ &\le C\||x'|^\alpha f\|_{L^q(\mathbb{R}^n_+)} = C\|\omega_{\alpha 2}f\|_{L^q(\mathbb{R}^n_+)}, \qquad (2.10)\end{aligned}$$

where $\lambda_2 := n-1-(\alpha-\beta) \in (0,n-1)$ with $-\frac{n-1}{q} < \beta < \alpha < (n-1)(1-\frac{1}{q})$; and

$$\begin{aligned}\|\omega_{\beta 3}Sf\|_{L^q(\mathbb{R}^n_+)} &\le C\|x_n^\beta T_{\lambda_3}[\|f(\cdot,y_n)\|_{L^q(\mathbb{R}^{n-1})}]_*\|_{L^q(\mathbb{R}^1)}\\ &\le C\||x_n|^\alpha[\|f(\cdot,y_n)\|_{L^q(\mathbb{R}^{n-1})}]_*\|_{L^q(\mathbb{R}^1)}\\ &\le C\|x_n^\alpha\|f(\cdot,y_n)\|_{L^q(\mathbb{R}^{n-1})}\|_{L^q(\mathbb{R}^n_+)} = C\|\omega_{\alpha 3}f\|_{L^q(\mathbb{R}^n_+)}, \qquad (2.11)\end{aligned}$$

where $\lambda_3 := 1-(\alpha-\beta) \in (0,1)$ with $-\frac{1}{q} < \beta < \alpha < (1-\frac{1}{q})$.

If $\alpha = \beta$.

Using (2.4), (2.6), (2.8) and (2.9), we find for $i = 1,2,3$

$$\|\omega_{\beta i}Sf\|_{L^q(\mathbb{R}^n_+)} \le \|\omega_{\beta i}T_n f_*\|_{L^q(\mathbb{R}^n)} \le C\|\omega_{\alpha i}f_*\|_{L^q(\mathbb{R}^n)} \le C\|\omega_{\alpha i}f\|_{L^q(\mathbb{R}^n_+)}. \qquad (2.12)$$

Therefore from (2.7), (2.10)–(2.12), we conclude that for $1 < r = q < \infty$, and $i = 1,2,3$

$$\|\omega_{\beta i}Sf\|_{L^q(\mathbb{R}^n_+)} \le C\|\omega_{\alpha i}f\|_{L^q(\mathbb{R}^n_+)}.$$

Case 2. $1 < r < q < \infty$.

If $-\frac{n}{q} < \beta \le \alpha < n(1-\frac{1}{r})$.

Then $\alpha-\beta < n(1-\frac{1}{r}) + \frac{n}{q} = n(1+\frac{1}{q}-\frac{1}{r})$, and $\mu_1 := n(1+\frac{1}{q}-\frac{1}{r}) - (\alpha-\beta) > 0$. On the other hand, $\frac{1}{q} = \frac{1}{r} + \frac{\mu_1+\alpha-\beta}{n} - 1$ yields $\mu_1 < n$ due to $r < q$. Whence $\mu_1 \in (0,n)$. Note that it holds for $x = (x',x_n) \in \mathbb{R}^n_+$

$$\begin{aligned}|(Sf)(x)| &\le \int_{\mathbb{R}^n_+} \frac{|f(y)|dy}{(|x'-y'|+x_n+y_n+1)^n}\\ &\le \int_{\mathbb{R}^n_+} \frac{|f(y)|dy}{(|x-y|+1)^n}\\ &\le \int_{\mathbb{R}^n} \frac{|f_*(y)|dy}{(|x-y|+1)^n}\\ &\le \int_{\mathbb{R}^n} \frac{|f_*(y)|dy}{|x-y|^{\mu_1}}, \end{aligned} \tag{2.13}$$

Combining (2.13) and Lemma 2.1 with $m=n$, $\lambda=\mu_1$, we have

$$\begin{aligned}\|\omega_{\beta 1}Sf\|_{L^q(\mathbb{R}^n_+)} &= \||x|^\beta Sf\|_{L^q(\mathbb{R}^n_+)} \le \||x|^\beta T_{\mu_1}f_*\|_{L^q(\mathbb{R}^n)}\\ &\le C\||x|^\alpha f_*\|_{L^r(\mathbb{R}^n)} \le C\||x|^\alpha f\|_{L^r(\mathbb{R}^n_+)} = C\|\omega_{\alpha 1}f\|_{L^r(\mathbb{R}^n_+)}. \end{aligned} \tag{2.14}$$

Direct computations show that for any $\ell>1$

$$\begin{aligned}&\sup_{y_n>0}\left(\int_0^\infty \frac{dx_n}{(|x'-y'|+x_n+y_n+1)^{n\ell}}\right)^{\frac{1}{\ell}}\\ &\quad+\sup_{x_n>0}\left(\int_0^\infty \frac{dy_n}{(|x'-y'|+x_n+y_n+1)^{n\ell}}\right)^{\frac{1}{\ell}}\\ &\le C(|x'-y'|+1)^{-n+\frac{1}{\ell}}, \quad \forall x',y'\in\mathbb{R}^{n-1}; \end{aligned} \tag{2.15}$$

and

$$\begin{aligned}&\sup_{x'\in\mathbb{R}^{n-1}}\left(\int_{\mathbb{R}^{n-1}} \frac{dy'}{(|x'-y'|+x_n+y_n+1)^{n\ell}}\right)^{\frac{1}{\ell}}\\ &\quad+\sup_{y'\in\mathbb{R}^{n-1}}\left(\int_{\mathbb{R}^{n-1}} \frac{dx'}{(|x'-y'|+x_n+y_n+1)^{n\ell}}\right)^{\frac{1}{\ell}}\\ &\le C(x_n+y_n+1)^{-n+\frac{n-1}{\ell}}, \quad \forall x_n,y_n>0; \end{aligned} \tag{2.16}$$

If $-\frac{n-1}{q}<\beta\le\alpha<(n-1)(1-\frac{1}{r})$.

Then $\alpha-\beta<(n-1)(1-\frac{1}{r})+\frac{n-1}{q}$, and $\mu_2:=(n-1)(1+\frac{1}{q}-\frac{1}{r})-(\alpha-\beta)>0$. On the other hand, $\frac{1}{q}=\frac{1}{r}+\frac{\mu_2+\alpha-\beta}{n-1}-1$ yields $\mu_2<n-1$ due to $r<q$. Whence $\mu_2\in(0,n-1)$.

If $-\frac{1}{q}<\beta\le\alpha<1-\frac{1}{r}$.

Then $\alpha-\beta<1-\frac{1}{r}+\frac{1}{q}$, and $\mu_3:=1+\frac{1}{q}-\frac{1}{r}-(\alpha-\beta)>0$. In addition, $\frac{1}{q}=\frac{1}{r}+\mu_3+\alpha-\beta-1$ yields $\mu_3<1$ due to $r<q$. Whence $\mu_3\in(0,1)$.

Choose $\ell>1$ such that $1+\frac{1}{q}=\frac{1}{\ell}+\frac{1}{r}$. Note that $n-\frac{1}{\ell}>n-1>\mu_2>0$, $n-\frac{n-1}{\ell}>1>\mu_3>0$. Using (2.15), (2.16) and the Young inequality, we conclude that

$$\begin{aligned}\|\omega_{\beta 2}Sf\|_{L^q(0,+\infty)} &= |x'|^\beta\left(\int_0^\infty |Sf(x)|^q dx_n\right)^{\frac{1}{q}}\\ &\le C|x'|^\beta\int_{\mathbb{R}^{n-1}} \frac{\|f(y',\cdot)\|_{L^r(0,+\infty)}}{(|x'-y'|+1)^{n-\frac{1}{\ell}}}dy'\\ &\le C|x'|^\beta\int_{\mathbb{R}^{n-1}} \frac{\|f(y',\cdot)\|_{L^r(0,+\infty)}}{|x'-y'|^{\mu_2}}dy'; \end{aligned} \tag{2.17}$$

and

$$\begin{aligned}\|\omega_{\beta 3}Sf\|_{L^q(\mathbb{R}^{n-1})} &= x_n^\beta\biggl(\int_{\mathbb{R}^{n-1}}|Sf(x)|^q dx'\biggr)^{\frac{1}{q}}\\ &\le Cx_n^\beta\int_0^\infty\frac{\|f(\cdot,y_n)\|_{L^r(\mathbb{R}^{n-1})}}{(x_n+y_n+1)^{n-\frac{n-1}{\ell}}}dy_n\\ &\le Cx_n^\beta\int_0^\infty\frac{\|f(\cdot,y_n)\|_{L^r(\mathbb{R}^{n-1})}}{(x_n+y_n+1)^{n-\frac{n-1}{\ell}}}dy_n\\ &\le Cx_n^\beta\int_{\mathbb{R}^1}\frac{[\|f(\cdot,y_n)\|_{L^r(\mathbb{R}^{n-1})}]_*}{|x_n-y_n|^{\mu_3}}dy_n, \qquad (2.18)\end{aligned}$$

where $[\cdot]_*$ denotes the even extension of $[\cdot]$ from $\mathbb{R}^n_+$ to $\mathbb{R}^n$.

Applying Lemma 2.1 to (2.17), (2.18) respectively, we find that

$$\begin{aligned}\|\omega_{\beta 2}Sf\|_{L^q(\mathbb{R}^n_+)} &\le C\||x'|^\beta T_{\mu_2}\|f(x',\cdot)\|_{L^r(0,+\infty)}\|_{L^q(\mathbb{R}^{n-1})}\\ &\le C\||x'|^\alpha\|f(x',\cdot)\|_{L^r(0,+\infty)}\|_{L^r(\mathbb{R}^{n-1})}\\ &\le C\||x'|^\alpha f\|_{L^r(\mathbb{R}^n_+)} = C\|\omega_{\alpha 2}f\|_{L^r(\mathbb{R}^n_+)} \qquad (2.19)\end{aligned}$$

for $-\frac{n-1}{q}<\beta\le\alpha<(n-1)(1-\frac{1}{r})$; and

$$\begin{aligned}\|\omega_{\beta 3}Sf\|_{L^q(\mathbb{R}^n_+)} &\le C\|x_n^\beta T_{\mu_3}[\|f(\cdot,y_n)\|_{L^r(\mathbb{R}^{n-1})}]_*\|_{L^r(\mathbb{R}^1)}\\ &\le C\||x_n|^\alpha[\|f(\cdot,y_n)\|_{L^r(\mathbb{R}^{n-1})}]_*\|_{L^r(\mathbb{R}^1)}\\ &\le C\|x_n^\alpha\|f(\cdot,y_n)\|_{L^r(\mathbb{R}^{n-1})}\|_{L^r(\mathbb{R}^n_+)} = C\|\omega_{\alpha 3}f\|_{L^r(\mathbb{R}^n_+)} \qquad (2.20)\end{aligned}$$

for $-\frac{1}{q}<\beta\le\alpha<1-\frac{1}{r}$.

Whence from (2.14), (2.19) and (2.20), we conclude that for $1<r<q<\infty$, and $i=1,2,3$

$$\|\omega_{\beta i}Sf\|_{L^q(\mathbb{R}^n_+)}\le C\|\omega_{\alpha i}f\|_{L^q(\mathbb{R}^n_+)}.$$

Case 3. $1=r<q<\infty$.

We first establish a useful estimate. Let $m\ge 1$, $1<q<\infty$, $-\frac{m}{q}<\beta\le\alpha<0$, $\lambda>\frac{m}{q}-(\alpha-\beta)$. We claim that there exists a constant $C=C(m,\lambda,\alpha,\beta,q)$ such that for $f\in L^1(\mathbb{R}^m)$, $g\in L^{\frac{q}{q-1}}(\mathbb{R}^m)$

$$\left|\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\frac{f(y)g(x)}{|x|^{-\beta}(|x-y|+1)^\lambda|y|^\alpha}dydx\right|\le C\|f\|_{L^1(\mathbb{R}^m)}\|g\|_{L^{\frac{q}{q-1}}(\mathbb{R}^m)}. \qquad (2.21)$$

The inequality (2.21) can be written equivalently as follows

$$\left\|\int_{\mathbb{R}^m}\frac{f(y)dy}{|x|^{-\beta}(|x-y|+1)^\lambda|y|^\alpha}\right\|_{L^q(\mathbb{R}^m)}\le C\|f\|_{L^1(\mathbb{R}^m)}. \qquad (2.22)$$

We limit ourselves to non-negative f and g, without loss of generality. Write the produce space $\mathbb{R}^m\times\mathbb{R}^m$ into three disjoint regions E_1,E_2,E_3, where

$$\begin{cases}E_1=\{(x,y)\in\mathbb{R}^m\times\mathbb{R}^m;\ \frac{1}{2}|y|<|x|<2|y|\},\\ E_2=\{(x,y)\in\mathbb{R}^m\times\mathbb{R}^m;\ |x|\le\frac{1}{2}|y|\},\\ E_3=\{(x,y)\in\mathbb{R}^m\times\mathbb{R}^m;\ |y|\le\frac{1}{2}|x|\}.\end{cases}$$

Set

$$L=\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\frac{f(y)g(x)}{|x|^{-\beta}(|x-y|+1)^\lambda|y|^\alpha}dydx,$$

and

$$L_k = \int\int_{E_k} \frac{f(y)g(x)}{|x|^{-\beta}(|x-y|+1)^{\lambda}|y|^{\alpha}} dydx, \quad k=1,2,3.$$

Then $L = L_1 + L_2 + L_3$. Whence it is sufficient to prove that

$$L_k \le C\|f\|_{L^1(\mathbb{R}^m)}\|g\|_{L^{\frac{q}{q-1}}(\mathbb{R}^m)}, \quad k=1,2,3.$$

Since L_1 is taken over E_1, $\frac{1}{2}|y| < |x| < 2|y|$ in this case. Therefore $|x-y| \le |x|+|y| < 3|x|$. By the hypothesis of $-\frac{m}{q} < \beta \le \alpha < 0$, we have $0 \le \alpha - \beta < \frac{m}{q}$, and

$$|x-y|^{\alpha-\beta} \le 3^{\alpha-\beta}|x|^{\alpha-\beta} \le 3^{\alpha-\beta}2^{-\alpha}|x|^{-\beta}|y|^{\alpha}.$$

Combining with the assumption: $\lambda > \frac{m}{q} - (\alpha-\beta)$, we find that

$$\begin{aligned} L_1 &\le C\int\int_{E_1} \frac{f(y)g(x)}{|x-y|^{\alpha-\beta}(|x-y|+1)^{\lambda}} dydx \\ &\le C\int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \frac{f(y)g(x)}{|x-y|^{\alpha-\beta}(|x-y|+1)^{\lambda}} dydx \\ &\le C\|f\|_{L^1(\mathbb{R}^m)} \sup_{y\in\mathbb{R}^m}\int_{\mathbb{R}^m} \frac{g(x)}{|x-y|^{\alpha-\beta}(|x-y|+1)^{\lambda}} dx \\ &\le C\|f\|_{L^1(\mathbb{R}^m)}\|g\|_{L^{\frac{q}{q-1}}(\mathbb{R}^m)} \sup_{y\in\mathbb{R}^m}\left(\int_{\mathbb{R}^m} |x-y|^{-q(\alpha-\beta)}(|x-y|+1)^{-q\lambda}dx\right)^{\frac{1}{q}} \\ &\le C\|f\|_{L^1(\mathbb{R}^m)}\|g\|_{L^{\frac{q}{q-1}}(\mathbb{R}^m)} \left(\int_{|z|\le 1} |z|^{-q(\alpha-\beta)}dz + \int_{|z|>1}|z|^{-q(\alpha-\beta)-q\lambda}dz\right)^{\frac{1}{q}} \\ &\le C\|f\|_{L^1(\mathbb{R}^m)}\|g\|_{L^{\frac{q}{q-1}}(\mathbb{R}^m)} \left(\int_0^1 s^{m-1-q(\alpha-\beta)}ds + \int_1^{\infty} s^{m-1-q(\alpha-\beta)-q\lambda}ds\right)^{\frac{1}{q}} \\ &\le C\|f\|_{L^1(\mathbb{R}^m)}\|g\|_{L^{\frac{q}{q-1}}(\mathbb{R}^m)}. \end{aligned} \tag{2.23}$$

In E_2, $|x| \le \frac{1}{2}|y|$; thus $|x-y| \ge |y| - |x| \ge \frac{1}{2}|y|$, and $(|x-y|+1)^{-\lambda'} \le 2^{\lambda'}(|y|+1)^{-\lambda'}$, where $\frac{m}{q} - (\alpha-\beta) = \lambda' \in (0,\lambda)$. Therefore

$$\begin{aligned} L_2 &\le C\int\int_{E_2} \frac{f(y)g(x)}{|x|^{-\beta}(|y|+1)^{\lambda'}|y|^{\alpha}} dydx \\ &\le C\int_{\mathbb{R}^m} f(y)(|y|+1)^{-\lambda'}|y|^{-\alpha}\left(\int_{|x|\le\frac{1}{2}|y|} g(x)|x|^{\beta}dx\right)dy \\ &\le C\|f\|_{L^1(\mathbb{R}^m)} \sup_{y\in\mathbb{R}^m}\left((|y|+1)^{-\lambda'}|y|^{-\alpha}\int_{|x|\le|y|} g(x)|x|^{\beta}dx\right) \\ &\le C\|f\|_{L^1(\mathbb{R}^m)}\|g\|_{L^{\frac{q}{q-1}}(\mathbb{R}^m)} \sup_{y\in\mathbb{R}^m}\left((|y|+1)^{-\lambda'}|y|^{-\alpha}\left(\int_{|x|\le|y|}|x|^{\beta q}dx\right)^{\frac{1}{q}}\right) \\ &\le C\|f\|_{L^1(\mathbb{R}^m)}\|g\|_{L^{\frac{q}{q-1}}(\mathbb{R}^m)} \sup_{y\in\mathbb{R}^m}\left((|y|+1)^{-\lambda'}|y|^{-\alpha}\left(\int_0^{|y|} s^{m-1+\beta q}ds\right)^{\frac{1}{q}}\right) \end{aligned}$$

$$\leq C\|f\|_{L^1(\mathbb{R}^m)}\|g\|_{L^{\frac{q}{q-1}}(\mathbb{R}^m)} \sup_{y\in\mathbb{R}^m}\left((|y|+1)^{-\lambda'}|y|^{\frac{m}{q}-(\alpha-\beta)}\right)$$

$$\leq C\|f\|_{L^1(\mathbb{R}^m)}\|g\|_{L^{\frac{q}{q-1}}(\mathbb{R}^m)} \quad \text{due to} \quad \frac{m}{q}-(\lambda'+\alpha-\beta)=0. \tag{2.24}$$

In E_3, $|y| \leq \frac{1}{2}|x|$; so $|x-y| \geq |x|-|y| \geq \frac{1}{2}|x|$, and $(|x-y|+1)^{-\lambda'} \leq 2^{\lambda'}(|x|+1)^{-\lambda'}$, where $\lambda > \lambda' = \frac{m}{q}-(\alpha-\beta) > \frac{m}{q}+\beta > 0$. Whence

$$\begin{aligned}
L_3 &\leq C\int\int_{E_2}\frac{f(y)g(x)}{|x|^{-\beta}(|x|+1)^{\lambda'}|y|^{\alpha}}dydx\\
&\leq C\int_{\mathbb{R}^m} f(y)|y|^{-\alpha}\left(\int_{|x|\geq 2|y|} g(x)|x|^{\beta}(|x|+1)^{-\lambda'}dx\right)dy\\
&\leq C\|f\|_{L^1(\mathbb{R}^m)}\sup_{y\in\mathbb{R}^m}\left(|y|^{-\alpha}\int_{|x|\geq|y|} g(x)|x|^{\beta-\lambda'}dx\right)\\
&\leq C\|f\|_{L^1(\mathbb{R}^m)}\|g\|_{L^{\frac{q}{q-1}}(\mathbb{R}^m)}\sup_{y\in\mathbb{R}^m}\left(|y|^{-\alpha}\left(\int_{|y|}^{\infty} s^{m-1-q(\lambda'-\beta)}ds\right)^{\frac{1}{q}}\right)\\
&\leq C\|f\|_{L^1(\mathbb{R}^m)}\|g\|_{L^{\frac{q}{q-1}}(\mathbb{R}^m)}\sup_{y\in\mathbb{R}^m}\left(|y|^{-\alpha}|y|^{\frac{m}{q}-\lambda'+\beta}\right)\\
&\leq C\|f\|_{L^1(\mathbb{R}^m)}\|g\|_{L^{\frac{q}{q-1}}(\mathbb{R}^m)} \quad \text{due to} \quad \frac{m}{q}-(\lambda'+\alpha-\beta)=0.
\end{aligned} \tag{2.25}$$

From (2.23)–(2.25), we conclude that the inequality (2.21) is valid.

Now we proceed to consider **Case 3**. $1 = r < q < \infty$.

Suppose $-\frac{n}{q} < \beta \leq \alpha < 0$. Note that for $x = (x', x_n) \in \mathbb{R}^n_+$

$$\begin{aligned}
\omega_{\beta 1}|(Sf)(x)| &\leq \int_{\mathbb{R}^n_+}\frac{|y|^{\alpha}|f(y)|dy}{|x|^{-\beta}(|x'-y'|+x_n+y_n+1)^n|y|^{\alpha}}\\
&\leq \int_{\mathbb{R}^n_+}\frac{|y|^{\alpha}|f(y)|dy}{|x|^{-\beta}(|x-y|+1)^n|y|^{\alpha}}\\
&\leq \int_{\mathbb{R}^n}\frac{|y|^{\alpha}|f_*(y)|dy}{|x|^{-\beta}(|x-y|+1)^n|y|^{\alpha}}.
\end{aligned} \tag{2.26}$$

From (2.26), (2.22) with $m = n$, $\lambda = n$, we derive that

$$\begin{aligned}
\|\omega_{\beta 1}Sf\|_{L^q(\mathbb{R}^n_+)} &\leq \left\|\int_{\mathbb{R}^n}\frac{|y|^{\alpha}|f_*(y)|dy}{|x|^{-\beta}(|x-y|+1)^n|y|^{\alpha}}\right\|_{L^q(\mathbb{R}^n)}\\
&\leq C\||y|^{\alpha}f_*\|_{L^1(\mathbb{R}^n)} \leq C\||y|^{\alpha}f\|_{L^1(\mathbb{R}^n_+)} = C\|\omega_{\alpha 1}f\|_{L^1(\mathbb{R}^n_+)}.
\end{aligned} \tag{2.27}$$

Now suppose $-\frac{n-1}{q} < \beta \leq \alpha < 0$. Observe that for $x = (x', x_n) \in \mathbb{R}^n_+$

$$\omega_{\beta 2}|(Sf)(x)| \leq \int_{\mathbb{R}^n_+}\frac{|y'|^{\alpha}|f(y',y_n)|dy'dy_n}{|x'|^{-\beta}(|x'-y'|+x_n+y_n+1)^n|y'|^{\alpha}}.$$

Using the Young inequality yields for any $x' \in \mathbb{R}^{n-1}$

$$\begin{aligned}
\omega_{\beta 2}\|(Sf)(x',\cdot)\|_{L^q(0,\infty)} &\leq \int_{\mathbb{R}^{n-1}}\frac{|y'|^{\alpha}\|f(y',\cdot)\|_{L^1(0,\infty)}dy'}{|x'|^{-\beta}(|x'-y'|+1)^{n-\frac{1}{q}}|y'|^{\alpha}}\\
&\leq C\int_{\mathbb{R}^{n-1}}\frac{|y'|^{\alpha}\|f(y',\cdot)\|_{L^1(0,\infty)}dy'}{|x'|^{-\beta}(|x'-y'|+1)^{n-1}|y'|^{\alpha}}.
\end{aligned} \tag{2.28}$$

Combining (2.28), (2.22) with $m=n-1$, $\lambda=n-1$, we derive

$$\begin{aligned}\|\omega_{\beta 2}Sf\|_{L^q(\mathbb{R}^n_+)} &\le \left\| \int_{\mathbb{R}^{n-1}} \frac{|y'|^\alpha \|f(y',\cdot)\|_{L^1(0,\infty)}dy'}{|x'|^{-\beta}(|x'-y'|+1)^{n-1}|y'|^\alpha}\right\|_{L^q(\mathbb{R}^{n-1})} \\ &\le C\||y'|^\alpha f\|_{L^1(\mathbb{R}^n_+)} = C\|\omega_{\alpha 2} f\|_{L^1(\mathbb{R}^n_+)}. \end{aligned} \tag{2.29}$$

Finally if $-\frac{1}{q}<\beta\le\alpha<0$. Then

$$\omega_{\beta 3}|(Sf)(x)| \le \int_{\mathbb{R}^n_+} \frac{y_n^\alpha |f(y',y_n)|dy}{x_n^{-\beta}(|x'-y'|+x_n+y_n+1)^n y_n^\alpha}, \quad \forall\, x=(x',x_n)\in\mathbb{R}^n_+.$$

Applying the Young inequality to the above inequality yields for any $x_n>0$

$$\begin{aligned}\omega_{\beta 3}\|(Sf)(\cdot,x_n)\|_{L^q(\mathbb{R}^{n-1})} &\le \int_{\mathbb{R}^1_+} \frac{y_n^\alpha\|f(\cdot,y_n)\|_{L^1(\mathbb{R}^{n-1})}dy_n}{x_n^{-\beta}(x_n+y_n+1)^{n-\frac{n-1}{q}}y_n^\alpha} \\ &\le C\int_{\mathbb{R}^1_+} \frac{y_n^\alpha\|f(\cdot,y_n)\|_{L^1(\mathbb{R}^{n-1})}dy_n}{x_n^{-\beta}(|x_n-y_n|+1)y_n^\alpha} \\ &\le C\int_{\mathbb{R}^1} \frac{[y_n^\alpha\|f(\cdot,y_n)\|_{L^1(\mathbb{R}^{n-1})}]_* dy_n}{|x_n|^{-\beta}(|x_n-y_n|+1)|y_n|^\alpha}. \end{aligned} \tag{2.30}$$

It follows from (2.30), (2.22) with $m=1$, $\lambda=1$ that

$$\begin{aligned}\|\omega_{\beta 3}Sf\|_{L^q(\mathbb{R}^n_+)} &\le \left\| \int_{\mathbb{R}^1} \frac{[y_n^\alpha\|f(\cdot,y_n)\|_{L^1(\mathbb{R}^{n-1})}]_* dy_n}{|x_n|^{-\beta}(|x_n-y_n|+1)|y_n|^\alpha}\right\|_{L^q(\mathbb{R}^1)} \\ &\le C\|[y_n^\alpha\|f(\cdot,y_n)\|_{L^1(\mathbb{R}^{n-1})}]_*\|_{L^1(\mathbb{R}^1)} \\ &\le C\|y_n^\alpha\|f(\cdot,y_n)\|_{L^1(\mathbb{R}^{n-1})}\|_{L^1(\mathbb{R}^1_+)} = C\|\omega_{\alpha 3}f\|_{L^1(\mathbb{R}^n_+)}. \end{aligned} \tag{2.31}$$

Therefore from (2.27), (2.29) and (2.31), we conclude that for $1=r<q<\infty$, and $i=1,2,3$

$$\|\omega_{\beta i}Sf\|_{L^q(\mathbb{R}^n_+)} \le C\|\omega_{\alpha i}f\|_{L^q(\mathbb{R}^n_+)}.$$

From the above arguments on the three cases, we complete the proof of Lemma 2.2. □

Set

$$v(x,t)=\int_{\mathbb{R}^n_+}\mathcal{M}^*(x,y,t)a(y)dy, \quad x=(x',x_n)\in\mathbb{R}^n_+, \quad t>0. \tag{2.32}$$

Lemma 2.3. *Let $n\ge 2$, then there exists $C=C(n,q,r,\alpha,\beta)$ such that the function v defined in (2.32) satisfies for $t>0$, $i=1,2,3$*

$$\|\omega_{\beta i}\nabla^{\vec{m}}v(\cdot,t)\|_{L^q(\mathbb{R}^n_+)} \le Ct^{-\frac{\alpha-\beta}{2}-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})-\frac{|\vec{m}|}{2}}\|\omega_{\alpha i}a\|_{L^r(\mathbb{R}^n_+)}, \quad |\vec{m}|=0,1,2,\ldots,$$

provided that either $1<r\le q<\infty$ or $1\le r<q<\infty$. Here the weighted functions $\omega_{\beta i}$ ($i=1,2,3$) are given in (1.2).

Proof. Let $\vec{m}=(m_1,m_2,\ldots,m_n)=(m',m_n)$, $\widetilde{x}=(\widetilde{x'},\widetilde{x_n})=(\frac{x'}{\sqrt{t}},\frac{x_n}{\sqrt{t}})$. Using the estimate (2.2), we have for $x=(x',x_n)\in\mathbb{R}^n_+$, $t>0$

$$\begin{aligned}|\nabla_x^{\vec{m}}v(x,t)| &\le \int_{\mathbb{R}^n_+}|\nabla_x^{\vec{m}}\mathcal{M}^*(x,y,t)||a(y)|dy \\ &\le C\int_{\mathbb{R}^n_+}(x_n+\sqrt{t})^{-m_n}(|x'-y'|+x_n+y_n+\sqrt{t})^{-n-|m'|}|a(y)|dy \\ &= Ct^{-\frac{n+|\vec{m}|}{2}}\int_{\mathbb{R}^n_+}\left(\frac{x_n}{\sqrt{t}}+1\right)^{-m_n}\left(\left|\frac{x'}{\sqrt{t}}-\frac{y'}{\sqrt{t}}\right|+\frac{x_n}{\sqrt{t}}+\frac{y_n}{\sqrt{t}}+1\right)^{-n-|m'|}|a(y)|dy\end{aligned}$$

$$
\begin{aligned}
&= Ct^{-\frac{|\vec{m}|}{2}} \int_{\mathbb{R}^n_+} (\widetilde{x_n}+1)^{-m_n} \left(|\widetilde{x}' - \widetilde{y}'| + \widetilde{x_n} + \widetilde{y_n} + 1\right)^{-n-|m'|} |a(\sqrt{t}\widetilde{y})| d\widetilde{y} \\
&\le Ct^{-\frac{|\vec{m}|}{2}} \int_{\mathbb{R}^n_+} \frac{|a(\sqrt{t}\widetilde{y})| d\widetilde{y}}{(|\widetilde{x}' - \widetilde{y}'| + \widetilde{x_n} + \widetilde{y_n} + 1)^n} \\
&\le Ct^{-\frac{|\vec{m}|}{2}} Sa(\sqrt{t}\widetilde{x}),
\end{aligned} \tag{2.33}
$$

where the operator S is defined in Lemma 2.2.

Combining (2.33) and Lemma 2.2, we conclude that for $i = 1, 2, 3$, and $t > 0$

$$
\begin{aligned}
\|\omega_{\beta i} \nabla^{\vec{m}} v(\cdot, t)\|_{L^q(\mathbb{R}^n_+)} &= \left(\int_{\mathbb{R}^n_+} |\omega_{\beta i}(x) \nabla^{\vec{m}} v(x,t)|^q dx \right)^{\frac{1}{q}} \\
&\le Ct^{-\frac{|\vec{m}|}{2}} \left(\int_{\mathbb{R}^n_+} |\omega_{\beta i}(x) Sa(\sqrt{t}\widetilde{x})|^q dx \right)^{\frac{1}{q}} \\
&= Ct^{-\frac{|\vec{m}|}{2} + \frac{\beta}{2} + \frac{n}{2q}} \left(\int_{\mathbb{R}^n_+} |\omega_{\beta i}(\widetilde{x}) Sa(\sqrt{t}\widetilde{x})|^q d\widetilde{x} \right)^{\frac{1}{q}} \\
&\le Ct^{-\frac{|\vec{m}|}{2} + \frac{\beta}{2} + \frac{n}{2q}} \left(\int_{\mathbb{R}^n_+} |\omega_{\alpha i}(\widetilde{y}) a(\sqrt{t}\widetilde{y})|^r d\widetilde{y} \right)^{\frac{1}{r}} \\
&= Ct^{-\frac{|\vec{m}|}{2} + \frac{\beta-\alpha}{2} + \frac{n}{2q} - \frac{n}{2r}} \left(\int_{\mathbb{R}^n_+} |\omega_{\alpha i}(y) a(y)|^r dy \right)^{\frac{1}{r}} \\
&= Ct^{-\frac{|\vec{m}|}{2} - \frac{\alpha-\beta}{2} - \frac{n}{2}(\frac{1}{r} - \frac{1}{q})} \|\omega_{\alpha i} a\|_{L^r(\mathbb{R}^n_+)}.
\end{aligned}
$$

□

Now we consider the following standard linear parabolic problem in the half space:

$$
\begin{cases}
\partial_t w - \Delta w = 0 & \text{in } \mathbb{R}^n_+ \times (0,\infty), \\
w(x,t) = 0 & \text{on } \partial\mathbb{R}^n_+ \times (0,\infty), \\
w(x,t) \longrightarrow 0 & \text{as } |x| \longrightarrow \infty, \quad t > 0, \\
w(x,0) = b(x) & \text{in } \mathbb{R}^n_+.
\end{cases} \tag{2.34}
$$

A direct calculation shows that the solution of problem (2.34) can be represented as $w(x,t) = [e^{t\Delta} b](x)$, and for $x = (x', x_n) \in \mathbb{R}^n_+$, $t > 0$

$$
\begin{aligned}
[e^{t\Delta} b](x) &= \int_{\mathbb{R}^n_+} (G_t(x-y) - G_t(x'-y', x_n + y_n)) b(y) dy \\
&= \int_{\mathbb{R}^n} G_t(x-y) b_*(y) dy \\
&= \int_{\mathbb{R}^n} G_1(\widetilde{x} - \widetilde{y}) b_*(\sqrt{t}\widetilde{y}) d\widetilde{y}, \qquad \widetilde{x} = \frac{x}{\sqrt{t}}, \ \ \widetilde{y} = \frac{y}{\sqrt{t}}.
\end{aligned} \tag{2.35}
$$

To proceed, we need a variant inequality of (2.4). Set

$$
(\widetilde{S} f)(x) = \int_{\mathbb{R}^n} \frac{f(y) dy}{(|x-y|+1)^n}, \quad x \in \mathbb{R}^n.
$$

Checking the proof process of Lemma 2.2, we find that the similar result holds. That is,
Let $n \ge 2$, then there exists $C = C(n, q, r, \alpha, \beta)$ such that the operator $\widetilde{S}$ satisfies for $i = 1, 2, 3$

$$
\|\omega_{\beta i} \widetilde{S} f\|_{L^q(\mathbb{R}^n)} \le C \|\omega_{\alpha i} f\|_{L^r(\mathbb{R}^n)} \tag{2.36}
$$

provided that either $1 < r \le q < \infty$ or $1 \le r < q < \infty$. Here $\omega_{\gamma i}$ $(\gamma = \alpha, \beta, i = 1, 2, 3)$ are given in (1.2).

Lemma 2.4. *Let $n \ge 2$, then the solution $[e^{t\Delta} b](x)$ of problem (2.34) satisfies for $t > 0$, $i = 1, 2, 3$*

$$\|\omega_{\beta i} \nabla^{\vec{m}} [e^{t\Delta} b]\|_{L^q(\mathbb{R}^n_+)} \le C t^{-\frac{\alpha-\beta}{2} - \frac{n}{2}(\frac{1}{r} - \frac{1}{q}) - \frac{|\vec{m}|}{2}} \|\omega_{\alpha i} b\|_{L^r(\mathbb{R}^n_+)}, \quad |\vec{m}| = 0, 1, 2, \ldots,$$

provided that either $1 < r \le q < \infty$ or $1 \le r < q < \infty$. Here the weighted functions $\omega_{\gamma i}$ $(\gamma = \alpha, \beta, i = 1, 2, 3)$ are given in (1.2).

Proof. Note that for each $\lambda > 0$, there exists $C = C(n, \vec{m}, \lambda)$ such that

$$|\nabla_z^{\vec{m}} G_1(z)| = (4\pi)^{-\frac{n}{2}} |\nabla_z^{\vec{m}} e^{-\frac{|z|^2}{4}}| \le C e^{-\frac{|z|^2}{8}} \sum_{\gamma \le |\vec{m}|} |z|^\gamma \le C(1 + |z|)^{-\lambda}, \quad \forall\, z \in \mathbb{R}^n.$$

Whence there holds for any $x = (x', x_n) \in \mathbb{R}^n_+$ and $t > 0$

$$\begin{aligned} |\nabla_x^{\vec{m}} [e^{t\Delta} b](x)| &\le \int_{\mathbb{R}^n} |\nabla_x^{\vec{m}} G_1(\widetilde{x} - \widetilde{y})| |b_*(\sqrt{t}\widetilde{y})| d\widetilde{y} \\ &= t^{-\frac{|\vec{m}|}{2}} \int_{\mathbb{R}^n} |\nabla_{\widetilde{x}}^{\vec{m}} G_1(\widetilde{x} - \widetilde{y})| |b_*(\sqrt{t}\widetilde{y})| d\widetilde{y} \\ &\le C t^{-\frac{|\vec{m}|}{2}} \int_{\mathbb{R}^n} \frac{|b_*(\sqrt{t}\widetilde{y})| d\widetilde{y}}{(1 + |\widetilde{x} - \widetilde{y}|)^n} \\ &= C t^{-\frac{|\vec{m}|}{2}} (\widetilde{S} |b_*(\sqrt{t}\cdot)|)(\widetilde{x}). \end{aligned} \tag{2.37}$$

Combining (2.36) and (2.37), we conclude that for $i = 1, 2, 3$, and $t > 0$

$$\begin{aligned} \|\omega_{\beta i} \nabla^{\vec{m}} [e^{t\Delta} b]\|_{L^q(\mathbb{R}^n_+)} &\le C t^{-\frac{|\vec{m}|}{2}} \left(\int_{\mathbb{R}^n} |\omega_{\beta i}(x) (\widetilde{S} |b_*(\sqrt{t}\cdot)|)(\widetilde{x})|^q dx \right)^{\frac{1}{q}} \\ &\le C t^{-\frac{|\vec{m}|}{2} + \frac{\beta}{2} + \frac{n}{2q}} \left(\int_{\mathbb{R}^n} |\omega_{\beta i}(\widetilde{x}) (\widetilde{S} |b_*(\sqrt{t}\cdot)|)(\widetilde{x})|^q d\widetilde{x} \right)^{\frac{1}{q}} \\ &\le C t^{-\frac{|\vec{m}|}{2} + \frac{\beta}{2} + \frac{n}{2q}} \left(\int_{\mathbb{R}^n} |\omega_{\alpha i}(\widetilde{y})| |b_*(\sqrt{t}\widetilde{y})|^r d\widetilde{y} \right)^{\frac{1}{r}} \\ &= C t^{-\frac{|\vec{m}|}{2} + \frac{\beta - \alpha}{2} + \frac{n}{2q} - \frac{n}{2r}} \left(\int_{\mathbb{R}^n} |\omega_{\alpha i}(y)| |b_*(y)|^r dy \right)^{\frac{1}{r}} \\ &\le C t^{-\frac{|\vec{m}|}{2} - \frac{\alpha - \beta}{2} - \frac{n}{2}(\frac{1}{r} - \frac{1}{q})} \|\omega_{\alpha i} b\|_{L^r(\mathbb{R}^n_+)}. \end{aligned}$$

□

Proof of Theorem 1.1. Note that the solution of the Stokes problem (1.1) can be written as follows (see [41,42])

$$e^{-tA} a = e^{t\Delta} a - v(x, t),$$

where $e^{t\Delta} a$ is the solution of the parabolic problem (2.34) with $b = a$, and v is defined in (2.32) with the initial data a.

By means of Lemmas 2.3, 2.4, we conclude that for $i = 1, 2, 3$, and $t > 0$

$$\begin{aligned} \|\omega_{\beta i} \nabla^{\vec{m}} e^{-tA} a\|_{L^q(\mathbb{R}^n_+)} &\le \|\omega_{\beta i} \nabla^{\vec{m}} e^{t\Delta} a\|_{L^q(\mathbb{R}^n_+)} + \|\omega_{\beta i} \nabla^{\vec{m}} v(\cdot, t)\|_{L^q(\mathbb{R}^n_+)} \\ &\le C t^{-\frac{|\vec{m}|}{2} - \frac{\alpha - \beta}{2} - \frac{n}{2}(\frac{1}{r} - \frac{1}{q})} \|\omega_{\alpha i} a\|_{L^r(\mathbb{R}^n_+)}. \end{aligned}$$

□

3. Application to the Navier–Stokes Flows

In this section, we apply the estimates on the Stokes flows to solutions of problem (1.3), and establish the weighted decay properties of the nonstationary Navier–Stokes flows.

Let $g = \mathcal{N} f$ denote the solution of the elliptic problem with the homogeneous Neumann boundary condition:

$$-\Delta g = f \quad \text{in} \quad \mathbb{R}^n_+, \quad \partial_\nu g\,|_{\partial\mathbb{R}^n_+} = 0.$$

Then (see [24])

$$\mathcal{N} = \int_0^\infty F(\tau) d\tau, \tag{3.1}$$

where

$$F(t)f(x) = \int_{\mathbb{R}^n_+} [G_t(x' - y', x_n - y_n) + G_t(x' - y', x_n + y_n)] f(y) dy.$$

Moreover for any $u \in L^2_\sigma(\mathbb{R}^n_+) \bigcap H^1_0(\mathbb{R}^n_+)$,

$$P(u \cdot \nabla)u = (u \cdot \nabla)u + \sum_{i,j=1}^n \nabla \mathcal{N} \partial_i \partial_j (u_i u_j). \tag{3.2}$$

Lemma 3.1. *Let $n \geq 2$. There holds for any $u \in C^\infty_{0,\sigma}(\mathbb{R}^n_+)$*

$$\| \sum_{i,j=1}^n \omega_{\beta k} \nabla \mathcal{N} \partial_i \partial_j (u_i u_j) \|_{L^q(\mathbb{R}^n_+)} \leq C(\| \sqrt{\omega_{\alpha k}} \nabla u \|^2_{L^{2r}(\mathbb{R}^n_+)} + \| \sqrt{\omega_{\alpha k}} u \|^2_{L^{2r}(\mathbb{R}^n_+)}), \quad k = 1, 2, 3,$$

provided that either $1 < r \leq q < \infty$ or $1 \leq r < q < \infty$. Moreover $0 \leq \alpha - \beta < 1 - n(\frac{1}{r} - \frac{1}{q})$. And the weighted functions $\omega_{\gamma k}$ $(\gamma = \alpha, \beta,\ k = 1, 2, 3)$ are given in (1.2).

Proof. In the proof of Lemma 2.4, we have proved that for $n \geq 2$, and $i = 1, 2, 3$

$$\| \omega_{\beta i} \nabla^{\overrightarrow{m}} [G_t * f] \|_{L^q(\mathbb{R}^n)} \leq C t^{-\frac{\alpha-\beta}{2} - \frac{n}{2}(\frac{1}{r} - \frac{1}{q}) - \frac{|\overrightarrow{m}|}{2}} \| \omega_{\alpha i} f \|_{L^r(\mathbb{R}^n)}, \quad |\overrightarrow{m}| = 0, 1, 2, \ldots, \tag{3.3}$$

provided that either $1 < r \leq q < \infty$ or $1 \leq r < q < \infty$. And the weighted functions $\omega_{\gamma i}$ $(\gamma = \alpha, \beta,\ i = 1, 2, 3)$ are given in (1.2).

Note that $\frac{1}{2} - \frac{\alpha-\beta}{2} - \frac{n}{2}(\frac{1}{r} - \frac{1}{q}) > 0$ by the assumption: $0 \leq \alpha - \beta < 1 - n(\frac{1}{r} - \frac{1}{q})$. Combining (3.1) and (3.3), we get for $k = 1, 2, 3$

$$\begin{aligned} \left\| \sum_{i,j=1}^n \omega_{\beta k} \nabla \mathcal{N} \partial_i \partial_j (u_i u_j) \right\|_{L^q(\mathbb{R}^n_+)} &= \left\| \sum_{i,j=1}^n \omega_{\beta k} \nabla \int_0^\infty F(\tau) \partial_i \partial_j (u_i u_j) d\tau \right\|_{L^q(\mathbb{R}^n_+)} \\ &\leq \int_0^1 \left\| \sum_{i,j=1}^n \omega_{\beta k} \nabla G_\tau * [\partial_i \partial_j (u_i u_j)]_* \right\|_{L^q(\mathbb{R}^n_+)} d\tau \\ &\quad + \int_1^\infty \left\| \sum_{i,j=1}^n \omega_{\beta k} \partial_i \partial_j \nabla G_\tau * w_{ij} \right\|_{L^q(\mathbb{R}^n_+)} d\tau \end{aligned}$$

$$\le C\int_0^1 \tau^{-\frac{\alpha-\beta}{2}-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})-\frac{1}{2}}d\tau \left\| \omega_{\alpha k}\sum_{i,j=1}^n [\partial_i\partial_j(u_iu_j)]_* \right\|_{L^r(\mathbb{R}^n)}$$

$$+C\sum_{i,j=1}^n \int_1^\infty \tau^{-\frac{\alpha-\beta}{2}-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})-\frac{3}{2}}d\tau \|\omega_{\alpha k}w_{ij}\|_{L^r(\mathbb{R}^n)}$$

$$= C(\|\sqrt{\omega_{\alpha k}}\nabla u\|^2_{L^{2r}(\mathbb{R}^n_+)} + \|\sqrt{\omega_{\alpha k}}u\|^2_{L^{2r}(\mathbb{R}^n_+)}),$$

where $w_{ij}=(u_iu_j)_*$ if $1\le i,j\le n-1$ or $i=j=n$; $w_{in}=(u_iu_n)^*$ if $1\le i\le n-1$; $w_{nj}=(u_nu_j)^*$ if $1\le j\le n-1$. $(\cdot)^*$, $(\cdot)_*$ denote the odd and even extensions of $(\cdot)$ from $\mathbb{R}^n_+$ to $\mathbb{R}^n$. □

Remark. Taking $\alpha=\beta$, $q=r\in(1,\infty)$ in Lemma 3.1, we get for $n\ge 2$, $k=1,2,3$

$$\left\|\sum_{i,j=1}^n \omega_{\alpha k}\nabla\mathcal{N}\partial_i\partial_j(u_iu_j)\right\|_{L^r(\mathbb{R}^n_+)} \le C(\|\sqrt{\omega_{\alpha k}}\nabla u\|^2_{L^{2r}(\mathbb{R}^n_+)} + \|\sqrt{\omega_{\alpha k}}u\|^2_{L^{2r}(\mathbb{R}^n_+)}), \quad \forall\, u\in C^\infty_{0,\sigma}(\mathbb{R}^n_+),$$

where the weighted functions $\omega_{\alpha k}(x)$ $(k=1,2,3)$ are given as follows for $x=(x',x_n)\in\mathbb{R}^n_+$:

$$\begin{cases} \omega_{\alpha 1}(x)=|x|^\alpha \ \text{ for } \ -\frac{n}{r}<\alpha<n(1-\frac{1}{r}); \\ \omega_{\alpha 2}(x)=|x'|^\alpha \ \text{ for } \ -\frac{n-1}{r}<\alpha<(n-1)(1-\frac{1}{r}); \\ \omega_{\alpha 3}(x)=x_n^\alpha \quad \text{ for } \ -\frac{1}{r}<\alpha<1-\frac{1}{r}. \end{cases}$$

Lemma 3.2 [22,28]. *Assume $u_0\in L^2_\sigma(\mathbb{R}^n_+)$ $(n\ge 2)$. There exists a number $\eta_0>0$ such that if $\|u_0\|_{L^n(\mathbb{R}^n_+)}\le \eta_0$ (smallness condition is unnecessary if $n=2$), problem (1.1) admits a unique strong solution u. Additionally if u_0 satisfies*

$$\|(1+|x|)(|u_0|+|\nabla u_0|)\|_{L^1(\mathbb{R}^n_+)} + \|(1+|x|)(|u_0|+|\nabla u_0|)\|_{L^\infty(\mathbb{R}^n_+)}<\infty.$$

Then there holds for $1<r<\infty$, $k=0,1$, and $t>0$

$$\|\omega_{\gamma i}\nabla^k u(t)\|_{L^r(\mathbb{R}^n_+)}\le C(1+t)^{\frac{\gamma}{2}-\frac{k}{2}-\frac{n}{2}(1-\frac{1}{r})}, \quad i=1,2,3,$$

here $\omega_{\gamma i}$ $(i=1,2,3)$ are defined as follows for $x=(x',x_n)\in\mathbb{R}^n_+$:

$$\omega_{\gamma 1}(x)=|x|^\gamma, \quad \omega_{\gamma 2}(x)=|x'|^\gamma, \quad \omega_{\gamma 3}(x)=x_n^\gamma,$$

where

$$0\le\gamma<\min\left\{1, n\left(1-\frac{1}{r}\right)\right\} \quad \text{if} \quad k=0; \quad \text{and} \quad 0\le\gamma<1 \quad \text{if} \quad k=1.$$

Proof of Theorem 1.2. The strong solution u of problem (1.3), which is given in Lemma 3.2, can be represented as follows

$$u(t)=e^{-tA}u_0-\int_0^t e^{-(t-s)A}P(u(s)\cdot\nabla)u(s)ds.$$

Let $0\le\alpha-\beta<1-n(\frac{1}{r}-\frac{1}{q})$, where $1<r\le q<\infty$.

From Theorem 1.1, and the Remark in Lemma 3.1, we have for $k=1,2,3$, and $t>0$

$$\|\omega_{\beta k}\nabla^{\overrightarrow{m}}u(t)\|_{L^q(\mathbb{R}^n_+)}\le\|\omega_{\beta k}\nabla^{\overrightarrow{m}}e^{-tA}u_0\|_{L^q(\mathbb{R}^n_+)}$$

$$+\int_0^t\|\omega_{\beta k}\nabla^{\overrightarrow{m}}e^{-(t-s)A}P(u(s)\cdot\nabla)u(s)\|_{L^q(\mathbb{R}^n_+)}ds$$

$$\le Ct^{-\frac{|\overrightarrow{m}|}{2}-\frac{\alpha-\beta}{2}-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\|\omega_{\alpha k}u_0\|_{L^r(\mathbb{R}^n_+)}$$

$$+C\int_0^t(t-s)^{-\frac{|\overrightarrow{m}|}{2}-\frac{\alpha-\beta}{2}-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\|\omega_{\alpha k}P(u(s)\cdot\nabla)u(s)\|_{L^r(\mathbb{R}^n_+)}ds$$

$$\le Ct^{-\frac{|\overrightarrow{m}|}{2}-\frac{\alpha-\beta}{2}-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\|\omega_{\alpha k}u_0\|_{L^r(\mathbb{R}^n_+)}$$

$$
\begin{aligned}
&+C\int_0^t (t-s)^{-\frac{|\vec{m}|}{2}-\frac{\alpha-\beta}{2}-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\big(\|\omega_{\alpha k}(u(s)\cdot\nabla)u(s)\|_{L^r(\mathbb{R}^n_+)}\\
&+\Big\|\omega_{\alpha k}\sum_{i,j=1}^n \nabla\mathcal{N}\partial_i\partial_j(u_iu_j)(s)\Big\|_{L^r(\mathbb{R}^n_+)}\big)ds\\
&\le Ct^{-\frac{|\vec{m}|}{2}-\frac{\alpha-\beta}{2}-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}+\left(\int_0^{\frac{t}{2}}+\int_{\frac{t}{2}}^t\right)(t-s)^{-\frac{|\vec{m}|}{2}-\frac{\alpha-\beta}{2}-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\\
&\times\big(\|\sqrt{\omega_{\alpha k}}u(s)\|^2_{L^{2r}(\mathbb{R}^n_+)}+\|\sqrt{\omega_{\alpha k}}\nabla u(s)\|^2_{L^{2r}(\mathbb{R}^n_+)}\big)ds\\
&\le Ct^{-\frac{|\vec{m}|}{2}-\frac{\alpha-\beta}{2}-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}+\left(\int_0^{\frac{t}{2}}+\int_{\frac{t}{2}}^t\right)(t-s)^{-\frac{|\vec{m}|}{2}-\frac{\alpha-\beta}{2}-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\\
&\times\big(\|\omega_{\alpha k}u(s)\|_{L^r(\mathbb{R}^n_+)}\|u(s)\|_{L^\infty(\mathbb{R}^n_+)}+\|\omega_{\alpha k}\nabla u(s)\|_{L^r(\mathbb{R}^n_+)}\|\nabla u(s)\|_{L^\infty(\mathbb{R}^n_+)}\big)ds. \qquad (3.4)
\end{aligned}
$$

Case 1. $0\le\alpha<\min\{1,n(1-\frac{1}{r})\}$. Applying Lemma 3.2 to estimate the two terms: $\|\omega_{\alpha k}u(s)\|_{L^r(\mathbb{R}^n_+)}$, $\|\omega_{\alpha k}\nabla u(s)\|_{L^r(\mathbb{R}^n_+)}$ in (3.4). It follows from Theorem 1.1, and Lemma 3.2 that for $i=1,2,3$, and $t>0$

$$
\begin{aligned}
&\|\omega_{\beta i}\nabla^{\vec{m}}u(t)\|_{L^q(\mathbb{R}^n_+)}\\
&\le Ct^{-\frac{|\vec{m}|}{2}-\frac{\alpha-\beta}{2}-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}+Ct^{-\frac{|\vec{m}|}{2}-\frac{\alpha-\beta}{2}-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\int_0^{\frac{t}{2}}(1+s)^{\frac{\alpha}{2}-\frac{n}{2}-\frac{n}{2}(1-\frac{1}{r})}ds\\
&+C(1+t)^{\frac{\alpha}{2}-\frac{n}{2}-\frac{n}{2}(1-\frac{1}{r})}\int_{\frac{t}{2}}^t(t-s)^{-\frac{|\vec{m}|}{2}-\frac{\alpha-\beta}{2}-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}ds\\
&\le Ct^{-\frac{|\vec{m}|}{2}-\frac{\alpha-\beta}{2}-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}. \qquad (3.5)
\end{aligned}
$$

Case 2. $\alpha<0$. Taking $\alpha=0$ in (3.5), we find for $\beta\le0$, $i=1,2,3$, and $t>0$

$$
\|\omega_{\beta i}\nabla^{\vec{m}}u(t)\|_{L^q(\mathbb{R}^n_+)}\le Ct^{-\frac{|\vec{m}|}{2}-\frac{|\beta|}{2}-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})},\quad |\vec{m}|=0,1. \qquad (3.6)
$$

Especially taking $\frac{1}{r}=1-\epsilon$ $(0<\epsilon<1)$ in (3.6) yields $\beta\le0$, $i=1,2,3$, and $t>0$

$$
\|\omega_{\beta i}\nabla^{\vec{m}}u(t)\|_{L^q(\mathbb{R}^n_+)}\le Ct^{-\frac{|\vec{m}|}{2}-\frac{|\beta|}{2}-\frac{n}{2}(1-\frac{1}{q})+\frac{n\epsilon}{2}},\quad |\vec{m}|=0,1.
$$

Recall the assumption on the initial data u_0, we have for $i=1,2,3$, and $s>0$

$$
\begin{aligned}
&\|\omega_{\alpha k}u(s)\|_{L^r(\mathbb{R}^n_+)}\|u(s)\|_{L^\infty(\mathbb{R}^n_+)}+\|\omega_{\alpha k}\nabla u(s)\|_{L^r(\mathbb{R}^n_+)}\|\nabla u(s)\|_{L^\infty(\mathbb{R}^n_+)}\\
&\le C(1+s)^{-\frac{|\alpha|}{2}-\frac{n}{2}-\frac{n}{2}(1-\frac{1}{r})+\frac{n\epsilon}{2}}. \qquad (3.7)
\end{aligned}
$$

Combining (3.4) and (3.7) yields for $\beta\le\alpha<0$, $i=1,2,3$, and $t>0$

$$
\begin{aligned}
&\|\omega_{\beta i}\nabla^{\vec{m}}u(t)\|_{L^q(\mathbb{R}^n_+)}\\
&\le Ct^{-\frac{|\vec{m}|}{2}-\frac{\alpha-\beta}{2}-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\\
&+\left(\int_0^{\frac{t}{2}}+\int_{\frac{t}{2}}^t\right)(t-s)^{-\frac{|\vec{m}|}{2}-\frac{\alpha-\beta}{2}-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}(1+s)^{-\frac{|\alpha|}{2}-\frac{n}{2}-\frac{n}{2}(1-\frac{1}{r})+\frac{n\epsilon}{2}}ds\\
&\le Ct^{-\frac{|\vec{m}|}{2}-\frac{\alpha-\beta}{2}-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}+Ct^{-\frac{|\vec{m}|}{2}-\frac{\alpha-\beta}{2}-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\int_0^{\frac{t}{2}}(1+s)^{-\frac{|\alpha|}{2}-\frac{n}{2}-\frac{n}{2}(1-\frac{1}{r})+\frac{n\epsilon}{2}}ds\\
&+C(1+t)^{-\frac{|\alpha|}{2}-\frac{n}{2}-\frac{n}{2}(1-\frac{1}{r})+\frac{n\epsilon}{2}}\int_{\frac{t}{2}}^t(t-s)^{-\frac{|\vec{m}|}{2}-\frac{\alpha-\beta}{2}-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}ds\\
&\le Ct^{-\frac{|\vec{m}|}{2}-\frac{\alpha-\beta}{2}-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\quad\text{by taking}\quad \epsilon>0\quad\text{sufficiently small.} \qquad (3.8)
\end{aligned}
$$

From (3.5) and (3.8), we complete the proof of Theorem 1.2. □

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Compliance with ethical standards

Conflict of interest The author declares that the work described in the above article complied with the ethical standards, has also not been submitted elsewhere for publication, in whole or in part. Moreover, there is no financial or competing interests to disclose in relation to this work.

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