J. Math. Fluid Mech. 20 (2018), 1123–1135

1422-6928/18/031123-13
https://doi.org/10.1007/s00021-017-0357-y

Journal of Mathematical Fluid Mechanics

Estimates of the Modeling Error of the α-Models of Turbulence in Two and Three Space Dimensions

Argus A. Dunca

Communicated by G.P. Galdi

Abstract. This report investigates the convergence rate of the weak solutions $\mathbf{w}^\alpha$ of the Leray-α, modified Leray-α, Navier–Stokes-α and the zeroth ADM turbulence models to a weak solution $\mathbf{u}$ of the Navier–Stokes equations. It is assumed that this weak solution $\mathbf{u}$ of the NSE belongs to the space $L^4(0,T;H^1)$. It is shown that under this regularity condition the error $\mathbf{u}-\mathbf{w}^\alpha$ is $\mathcal{O}(\alpha)$ in the norms $L^2(0,T;H^1)$ and $L^\infty(0,T;L^2)$, thus improving related known results. It is also shown that the averaged error $\overline{\mathbf{u}}-\overline{\mathbf{w}^\alpha}$ is higher order, $\mathcal{O}(\alpha^{1.5})$, in the same norms, therefore the α-regularizations considered herein approximate better filtered flow structures than the exact (unfiltered) flow velocities.

Mathematics Subject Classification. 35Q30, 76D05.

Keywords. Leray-α, Navier–Stokes equations, Navier–Stokes-α model, Modified Leray-α model, Leray-deconvolution model, Viscous Camassa–Holm, Zeroth ADM model, Simplified Bardina model, Helmholtz filter.

1. Introduction

The scope of this report is to investigate the rate of convergence of several α-models of turbulence to the Navier–Stokes equations (NSE). It is known that in a direct numerical simulation of a turbulent flow at high Reynolds number the NSE although very good as a physical model, is not efficient computationally (as it requires a very fine mesh and very many degrees of freedom for an accurate computation), [23]. For this reason, many regularizations of the NSE have been introduced with the goal of approximating only averages of flow structures and not the true flow. Amongst them, there are the four α-models of turbulence considered herein: the Leray-α, the modified Leray-α model, the Navier–Stokes-α and the zeroth ADM model. These turbulence models have been introduced and investigated mathematically in [4,7,8,12,13,20,26–28,31,32,39] and analyzed numerically and tested in papers such as [9,15,18,21,22,29,30,34,37]. The asymptotic behavior of the above α models of turbulence as the regularization parameter α converges to zero has been investigated in [3,4,6,11,13]. The asymptotic behavior of related models (Voigt-MHD system, inviscid Euler-α system, ADM-MHD system) is studied in papers such as [2,24,33].

In this report we investigate the rate of convergence of the error $\mathbf{u}-\mathbf{w}^\alpha$ as $\alpha\to 0$ in the $L^2(0,T;H^1)$ and $L^\infty(0,T;L^2)$ norms. Here $\mathbf{u}$ is a weak solution of the NSE which is assumed to satisfy $\mathbf{u}\in L^4(0,T;H^1)$ and $\mathbf{w}^\alpha$ is the weak solution of any of the four α-models of turbulence from above. These models take the form

$$\begin{aligned} \mathbf{w}^\alpha_t + N(\mathbf{w}^\alpha) - \nu\Delta\mathbf{w}^\alpha + \nabla p &= \mathbf{f} && \text{in } \Omega\\ \nabla\cdot\mathbf{w}^\alpha &= 0 && \text{in } \Omega\\ \mathbf{w}^\alpha(\mathbf{x},0) &= \mathbf{u}_0(\mathbf{x}) && \text{in } \Omega \end{aligned} \tag{1.1}$$

This work has been funded by University Politehnica of Bucharest, through the Excellence Research Grants Program, UPB GEX. No. 26/2016, Internal No. MA 52.16.01.

where the nonlinear operator N is given by $N(\mathbf{w}^\alpha) = \overline{\mathbf{w}^\alpha} \cdot \nabla \mathbf{w}^\alpha$ in the Leray-α model, $N(\mathbf{w}^\alpha) = \mathbf{w}^\alpha \cdot \nabla \overline{\mathbf{w}^\alpha}$ in the Modified Leray-α model, $N(\mathbf{w}^\alpha) = \overline{\mathbf{w}^\alpha} \cdot \nabla \overline{\mathbf{w}^\alpha}$ in the zeroth ADM model and $N(\mathbf{w}^\alpha) = (\nabla \times \mathbf{w}^\alpha) \times \overline{\mathbf{w}^\alpha}$ in the Navier–Stokes-α model. Here Ω is the 2d or 3d periodic box of size $L > 0$ and periodic boundary conditions are used.

Here we'll adopt the notations from works such as [28,34] and call $\mathbf{w}^\alpha$ the solution of the α-model of turbulence, but it should be pointed out that in several related works such as [4,5,8,20] the function $\overline{\mathbf{w}^\alpha}$ (denoted differently therein) is quoted as the solution of the α-model.

The average $\overline{\mathbf{w}^\alpha}$ of $\mathbf{w}^\alpha$ in the periodic context is the differential filter of $\mathbf{w}^\alpha$ defined as

$$\overline{\mathbf{w}^\alpha} = (I - \alpha^2 \Delta)^{-1} \mathbf{w}^\alpha.$$

The differential filter was introduced by Germano in [16,17] and it is also called the Helmholtz filter, [3], or the exponential filter, [40]. Its connection to the well-known Gaussian filter is explained in [10]. One advantage over the Gaussian filter is a simpler representation of the subgrid scales $\mathbf{w}^\alpha - \overline{\mathbf{w}^\alpha}$, [40]. Its use for LES computations is recommended in [35].

The rates of convergence of the error $\mathbf{u} - \overline{\mathbf{w}^\alpha}$ as $\alpha \to 0$ in the $L^2(H^1)$ and $L^\infty(L^2)$ norms have been proved in the paper [5] in two space dimensions in the general case (i.e. with no extra regularity assumption on the weak solution $\mathbf{u}$ of the NSE). In [5] Cao and Titi have shown that

$$||\mathbf{u} - \overline{\mathbf{w}^\alpha}||_{L^\infty(0,T;L^2)} + ||\mathbf{u} - \overline{\mathbf{w}^\alpha}||_{L^2(0,T;H^1)} \leq C\alpha \log(1/\alpha)^{1/2}. \tag{1.2}$$

In their proof they employ some properties of the Navier–Stokes nonlinear term which are true only in the 2d case and the Brezis-Gallouet inequality thus making their argument valid only in the 2d case.

Another study of the convergence rate has been performed in the work [6] where the authors prove that the convergence rate of the NS-α model in the norm $L^1(L^2)$ is $\mathcal{O}(\alpha)$ (thus removing the logarithmic factor). The result is valid in two and three space dimensions, however, it requires a small data condition to assure the existence and uniqueness of the weak solution of the NSE.

In this report it's shown that the logarithmic factor can be removed in both two and three space dimensions provided that there exists a weak solution $\mathbf{u}$ of the NSE satisfying the extra regularity property $\mathbf{u} \in L^4(0,T;H^1)$. This assumption implies the weaker regularity $\mathbf{u} \in L^4(0,T;L^6)$ which is the well-known Leray–Serrin–Prodi (LSP), 3d uniqueness assumption, [32,36,38], $\mathbf{u} \in L^r(0,T;L^s)$ with $3/s + 2/r = 1, s \geq 3$. In our case we pick $r = 4, s = 6$. The LSP condition is very well documented in Galdi, [14], see also Layton, [25], page 144. Several properties of a solution satisfying the LSP condition presented in Galdi, [14], are listed below.

A weak solution of the NSE satisfying the LSP condition is unique in the set of all Leray-Hopf solutions, see Thm 4.2 in Galdi, [14] and also [32,32,36,38].

The existence of regular solutions for small time intervals is presented in Thm. 6.1 in Galdi, [14], and Heywood, [19], where it's proved that if $\mathbf{f} = 0$ and initial condition $\mathbf{u}_0 \in H^1$, then there exists a weak solution in the space $L^\infty(0,T^*;H^1)$ (thus satisfying the assumption $\mathbf{u} \in L^4(0,T^*;H^1)$), where $T^* = \mathcal{O}(1/||\nabla \mathbf{u}_0||^4)$.

According to Thm 5.2 in Galdi, [14], if $\mathbf{f} = 0$ and initial condition $\mathbf{u}_0 \in H^1$ a weak solution satisfying the LPS is regular, i.e. $\mathbf{u} \in C^\infty((0,T] \times \overline{\Omega})$, see also Leray, [32]. In return, if a weak solution is regular up to a time limit T where it looses regularity, necessarily the norm $||\mathbf{u}||_{L^4(0,T;L^6)}$ (and therefore $||\mathbf{u}||_{L^4(0,T;H^1)}$) has to blow up, (i.e. similar to the $L^1(0,T;L^\infty)$ norm of the vorticity, Beale–Kato–Majda, [1]).

The assumption $\mathbf{u} \in L^4(0,T;H^1)$ alone is discussed in Remark 6.2, page 62, Galdi, [14], (see also Leray, [32]), where it's shown that this regularity condition prevents the formation of epochs of irregularities as a direct consequence of Thm. 6.2, page 59, Galdi, [14].

Herein, it's shown that if there exists a weak NSE solution $\mathbf{u} \in L^4(0,T;H^1)$ then the weak solution $\mathbf{w}^\alpha$ of any of the four turbulence models in (1.1) will converge to this specific $\mathbf{u}$ as $\alpha \to 0$ and more,

$$||\mathbf{u} - \mathbf{w}^\alpha||_{L^\infty(0,T;L^2)} + ||\nabla \mathbf{u} - \nabla \mathbf{w}^\alpha||_{L^2(0,T;L^2)} \leq C_T \alpha \tag{1.3}$$

i.e. the error estimate is valid without a logarithmic term.

Moreover, it is also shown that the averaged error $\overline{\mathbf{u}} - \overline{\mathbf{w}^\alpha}$ converges with higher rate $\mathcal{O}(\alpha^{1.5})$ in the same norms, i.e.

$$||\overline{\mathbf{u}} - \overline{\mathbf{w}^\alpha}||_{L^\infty(0,T;L^2)} + ||\overline{\mathbf{u}} - \overline{\mathbf{w}^\alpha}||_{L^2(0,T;H^1)} \leq C_T\alpha^{1.5} \tag{1.4}$$

We emphasize that, unlike the Titi and Cao's reference paper [5], where the two authors compare $\mathbf{u}$ to $\overline{\mathbf{w}^\alpha}$, herein we compare $\mathbf{u}$ to $\mathbf{w}^\alpha$ (and $\overline{\mathbf{u}}$ to $\overline{\mathbf{w}^\alpha}$) which seem the natural comparisons for our analysis.

Further on, the result (1.2) of Cao and Titi can then be improved by using the inequalities (1.3, 1.4), the triangle inequality and some more filter estimates proved herein [i.e. inequality (2.12)] to estimate the left term in inequality (1.2) as

$$||\mathbf{u} - \overline{\mathbf{w}^\alpha}||_{L^\infty(0,T;L^2)} + ||\nabla\mathbf{u} - \nabla\overline{\mathbf{w}^\alpha}||_{L^2(0,T;L^2)} \leq C_T\alpha.$$

thus improving the result (1.2) proved in [5].

Estimate (1.4) and the one above show that $\overline{\mathbf{w}^\alpha}$ approximates better the average flow velocity $\overline{\mathbf{u}}$ than the velocity $\mathbf{u}$ itself. This is consistent with other works such as [10] that show that when one seeks to approximate average velocities better accuracy is expected.

2. Estimates of the Weak Solution of the Navier–Stokes Equations

For simplicity, we consider only the 3d case and we emphasize that all inequalities presented herein are also valid in the 2d case. $\Omega = (0,L)^3$ denotes the periodic box of size $L > 0$ and $\mathcal{T}_3$ is given by

$$\mathcal{T}_3 := 2\pi\mathbb{Z}^3/L.$$

For a real $s > 0$, $\mathbf{H}^{\mathrm{s}}$ will denote the Hilbert space

$$\mathbf{H}^{\mathrm{s}} = \{\mathbf{v}(\mathbf{x}) = \sum_{\mathbf{k}\in\mathcal{T}_3^*} \hat{\mathrm{v}}_{\mathbf{k}} \mathrm{e}^{\mathbf{i}\mathbf{k}\cdot\mathbf{x}} | \nabla\cdot\mathbf{v} = 0, \hat{\mathbf{v}}_{-\mathbf{k}} = \overline{\hat{\mathbf{v}}}_{\mathbf{k}}, \sum_{\mathbf{k}\in\mathcal{T}_3^*} |\mathbf{k}|^{2\mathrm{s}} |\hat{\mathbf{v}}_{\mathbf{k}}|^2 < \infty\} \tag{2.1}$$

with corresponding norm

$$||\mathbf{v}||_s = \left(\sum_{\mathbf{k}\in\mathcal{T}_3^\star} |\mathbf{k}|^{2s} |\mathbf{v}_{\mathbf{k}}|^2 \right)^{\frac{1}{2}}. \tag{2.2}$$

We let $||\cdot|| := ||\cdot||_0$ denote the $L^2(\Omega)$ norm. For reals $T > 0, p \geq 1$ and Banach space X, the space of vector valued functions $L^p(0,T;X)$ is given by

$$L^p(0,T;X) = \left\{ f : [0,T] \to X | \int_0^T ||f(t)||_X^p dt < \infty \right\}.$$

Our analysis uses standard inequalities such as the Sobolev inequality

$$||\mathbf{u}||_{L^6} \leq C||\nabla\mathbf{u}|| \tag{2.3}$$

and the inequality

$$||\mathbf{u}||_{L^3} \leq ||\mathbf{u}||_{L^2}^{1/2} ||\mathbf{u}||_{L^6}^{1/2} \leq C||\mathbf{u}||_{L^2}^{1/2} ||\nabla\mathbf{u}||_{L^2}^{1/2} \tag{2.4}$$

for $\mathbf{u} \in \mathbf{H}^1$. In the above two inequalities C is a general constant not depending on $\mathbf{u}$.

Definition 2.1. We let $A : \mathbf{H}^{\mathrm{s}+2} \subset \mathbf{H}^{\mathrm{s}} \to \mathbf{H}^{\mathrm{s}}$ and $G : \mathbf{H}^{\mathrm{s}} \to \mathbf{H}^{\mathrm{s}+2} \subset \mathbf{H}^{\mathrm{s}}$

$$A\mathbf{u} = (I - \alpha^2\Delta)\mathbf{u}, \quad G\mathbf{u} = (I - \alpha^2\Delta)^{-1}\mathbf{u} = \overline{\mathbf{u}}.$$

G is the differential filter, see [3,17] and has the properties

$$||\overline{\mathbf{u}}||_s \leq ||\mathbf{u}||_s, \;\; \alpha||\overline{\mathbf{u}}||_{s+1} \leq C||\mathbf{u}||_s, \;\; \alpha^2||\overline{\mathbf{u}}||_{s+2} \leq C||\mathbf{u}||_s \tag{2.5}$$

for a general constant C which does not depend on $\alpha, \mathbf{u}$.

From the above inequalities it follows that

$$||\mathbf{u} - \overline{\mathbf{u}}|| = \alpha^2||\Delta\overline{\mathbf{u}}|| \leq C\alpha||\nabla\mathbf{u}||.$$

Furthermore, one can show using this inequality, the inequality (2.4) and the first inequality in (2.5) that

$$||\mathbf{u}-\overline{\mathbf{u}}||_{L^3} \leq C||\mathbf{u}-\overline{\mathbf{u}}||^{1/2}||\nabla\mathbf{u}-\nabla\overline{\mathbf{u}}||^{1/2} \leq C\alpha^{1/2}||\nabla\mathbf{u}|| \tag{2.6}$$

We next consider the Navier–Stokes equations

$$\begin{aligned} \mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p &= \mathbf{f} && \text{in } \Omega \\ \nabla\cdot\mathbf{u} &= 0 && \text{in } \Omega \\ \mathbf{u}(\mathbf{x},0) &= \mathbf{u}_0(\mathbf{x}) && \text{in } \Omega \end{aligned} \tag{2.7}$$

with data $\mathbf{f} \in L^2(0,T;\mathbf{H}^0)$ and $\mathbf{u}_0 \in \mathbf{H}^1$ and periodic boundary conditions. We assume that the NSE posses a weak solution $\mathbf{u}$ such that

$$\nabla\mathbf{u} \in L^4([0,T],L^2). \tag{2.8}$$

In the sequel this weak solution $\mathbf{u}$ will be called the regular weak solution.

It follows from Galdi, [14], Theorem 4.2, that the regular weak solution $\mathbf{u}$ is unique in the set of all weak solutions satisfying the energy inequality.

In the next theorem we provide some estimates of the average $\overline{\mathbf{u}}$ of the regular weak solution $\mathbf{u}$.

Theorem 2.1. *The regular weak solution $\mathbf{u}$ of the NSE satisfies the following inequality*

$$||\nabla\overline{\mathbf{u}}||_{L^\infty([0,T],L^2)} + \alpha||\Delta\overline{\mathbf{u}}||_{L^\infty([0,T],L^2)} + ||\Delta\overline{\mathbf{u}}||_{L^2([0,T],L^2)} + \alpha||\nabla\Delta\overline{\mathbf{u}}||_{L^2([0,T],L^2)} \leq C_T \tag{2.9}$$

where C_T does not depend on α. The dependence of C in $||\nabla\mathbf{u}||^4_{L^4(L^2)}$ is

$$C_T = \mathcal{O}(e^{\frac{C}{\nu^3}||\nabla\mathbf{u}||^4_{L^4(0,T;L^2)}}) \tag{2.10}$$

where C is a Sobolev constant.

Proof. Set $-\Delta\overline{\mathbf{u}}$ (which is a valid multiplier) as a test function in (2.7) and integrate. It follows that

$$\frac{1}{2}\frac{d}{dt}(||\nabla\overline{\mathbf{u}}||^2 + \alpha^2||\Delta\overline{\mathbf{u}}||^2) + \nu||\Delta\overline{\mathbf{u}}||^2 + \nu\alpha^2||\nabla\Delta\overline{\mathbf{u}}||^2 = (\mathbf{u}\cdot\nabla\mathbf{u},\Delta\overline{\mathbf{u}}) - (f,\Delta\overline{\mathbf{u}}). \tag{2.11}$$

The first term on the right side above is written as:

$$(\mathbf{u}\cdot\nabla\mathbf{u},\Delta\overline{\mathbf{u}}) = (\mathbf{u}\cdot\nabla(\overline{\mathbf{u}}-\alpha^2\Delta\overline{\mathbf{u}}),\Delta\overline{\mathbf{u}}) = (\mathbf{u}\cdot\nabla\overline{\mathbf{u}},\Delta\overline{\mathbf{u}})$$

and then estimated as

$$\begin{aligned} (\mathbf{u}\cdot\nabla\mathbf{u},\Delta\overline{\mathbf{u}}) = (\mathbf{u}\cdot\nabla\overline{\mathbf{u}},\Delta\overline{\mathbf{u}}) &\leq ||\mathbf{u}||_{L^6}||\nabla\overline{\mathbf{u}}||_{L^3}|\Delta\overline{\mathbf{u}}||_{L^2} \\ &\leq C||\nabla\mathbf{u}||||\nabla\overline{\mathbf{u}}||^{1/2}||\Delta\overline{\mathbf{u}}||^{1/2}||\Delta\overline{\mathbf{u}}|| \\ &\leq \tfrac{C}{\nu^3}||\nabla\mathbf{u}||^4||\nabla\overline{\mathbf{u}}||^2 + \frac{\nu}{4}||\Delta\overline{\mathbf{u}}||^2. \end{aligned}$$

The first inequality above uses the Hölder's inequality with the pairing $1/6+1/3+1/2=1$. In the second inequality, the L^6 norm is handled using inequality (2.3), whereas the L^3 norm is handled using inequality (2.4). In the third inequality we use the Young's inequality with the pairing $1/4+3/4=1$.

The second term on the right hand side of (2.11) is estimated as

$$(\mathbf{f},\Delta\overline{\mathbf{u}}) \leq ||\mathbf{f}||||\Delta\overline{\mathbf{u}}|| \leq \frac{1}{\nu}||\mathbf{f}||^2 + \frac{\nu}{4}||\Delta\overline{\mathbf{u}}||^2.$$

In the first inequality from above we use the Cauchy's inequality and in the second we use Young's inequality with the pairing $1/2+1/2=1$.

Replacing in equality (2.11) and applying Gronwall's inequality finishes the proof. Since the integrating factor in Gronwall's inequality is $a(t) = \frac{C}{\nu^3}||\nabla\mathbf{u}(t)||^4$, there follows (2.10). □

Remark 2.2. For the rest of the paper C will always denote a general Sobolev constant, possibly changing from one calculation to another, whereas C_T will denote a constant which is not dependent on α and whose dependence on $||\nabla\mathbf{u}||^4_{L^4(L^2)}$ is given by relation (2.10) above.

Theorem 2.3. *For the regular weak solution* $\mathbf{u}$ *of the NSE the following estimates hold*

$$||\mathbf{u}-\overline{\mathbf{u}}||_{L^\infty(0,T;L^2)}+||\nabla\mathbf{u}-\nabla\overline{\mathbf{u}}||_{L^2(0,T;L^2)}\le C_T\alpha \tag{2.12}$$

$$||\mathbf{u}-\overline{\mathbf{u}}||_{L^2(0,T;L^2)}\le C_T\alpha^2 \tag{2.13}$$

$$||\mathbf{u}-\overline{\mathbf{u}}||_{L^2(0,T;L^3)}\le C_T\alpha^{1.5}. \tag{2.14}$$

Proof. Using the definition of $\overline{\mathbf{u}}$ we can write

$$||\mathbf{u}-\overline{\mathbf{u}}||=\alpha^2||\Delta\overline{\mathbf{u}}||=\alpha(\alpha||\Delta\overline{\mathbf{u}}||)\le C_T\alpha$$

since $\alpha||\Delta\overline{\mathbf{u}}||$ is bounded uniformly in α, see the previous theorem.

Similarly,

$$||\nabla\mathbf{u}-\nabla\overline{\mathbf{u}}||_{L^2(0,T;L^2)}=\alpha^2||\nabla\Delta\overline{\mathbf{u}}||_{L^2(0,T;L^2)}=\alpha(\alpha||\nabla\Delta\overline{\mathbf{u}}||_{L^2(0,T;L^2)})\le C_T\alpha$$

because $\alpha||\nabla\Delta\overline{\mathbf{u}}||_{L^2(L^2)}$ is bounded uniformly in α, see the previous theorem.

To prove (2.13) we write

$$||\mathbf{u}-\overline{\mathbf{u}}||_{L^2(0,T;L^2)}=\alpha^2||\Delta\overline{\mathbf{u}}||_{L^2(0,T;L^2)}\le C_T\alpha^2$$

since $||\Delta\overline{\mathbf{u}}||_{L^2(L^2)}$ is uniformly bounded in α, see the previous theorem.

For the last inequality (2.14) we use inequality (2.4) to write

$$\begin{aligned}||\mathbf{u}-\overline{\mathbf{u}}||^2_{L^2(L^3)}&=\int_0^T||\mathbf{u}-\overline{\mathbf{u}}||^2_{L^3}dt\\&\le C\int_0^T||\mathbf{u}-\overline{\mathbf{u}}||||\nabla\mathbf{u}-\nabla\overline{\mathbf{u}}||dt\\&\le C\left(\int_0^T||\mathbf{u}-\overline{\mathbf{u}}||^2dt\right)^{1/2}\left(\int_0^T||\nabla\mathbf{u}-\nabla\overline{\mathbf{u}}||^2dt\right)^{1/2}\\&=C||\mathbf{u}-\overline{\mathbf{u}}||_{L^2(L^2)}||\nabla\mathbf{u}-\nabla\overline{\mathbf{u}}||_{L^2(L^2)}\\&\le C_T\alpha^3\end{aligned}$$

□

3. The Error Analysis of the α-Models of Turbulence

We consider the general form of the α-models of turbulence

$$\begin{aligned}\mathbf{w}^\alpha_t+N(\mathbf{w}^\alpha)-\nu\Delta\mathbf{w}^\alpha+\nabla p=\mathbf{f}&\quad\text{in }\Omega\\\nabla\cdot\mathbf{w}^\alpha=0&\quad\text{in }\Omega\\\mathbf{w}^\alpha(\mathbf{x},0)=\mathbf{u}_0(\mathbf{x})&\quad\text{in }\Omega\end{aligned} \tag{3.1}$$

where the nonlinear operator N is given by

$$N(\mathbf{w}^\alpha)=\overline{\mathbf{w}^\alpha}\cdot\nabla\mathbf{w}^\alpha\qquad\text{in the Leray-}\alpha\text{ model} \tag{3.2}$$

$$N(\mathbf{w}^\alpha)=\mathbf{w}^\alpha\cdot\nabla\overline{\mathbf{w}^\alpha}\qquad\text{in the Modified Leray-}\alpha\text{ model} \tag{3.3}$$

$$N(\mathbf{w}^\alpha)=\overline{\mathbf{w}^\alpha}\cdot\nabla\overline{\mathbf{w}^\alpha}\qquad\text{in the zeroth ADM model} \tag{3.4}$$

$$N(\mathbf{w}^\alpha)=(\nabla\times\mathbf{w}^\alpha)\times\overline{\mathbf{w}^\alpha}\qquad\text{in the Navier–Stokes-}\alpha\text{ model.} \tag{3.5}$$

Any of the four α-models from above is known to have a unique weak solution $\mathbf{w}^\alpha\in L^\infty(\mathbf{H}^0)\cap \mathrm{L}^2(\mathbf{H}^1)$.

The main result of the paper is stated as:

A weak solution $\mathbf{w}^\alpha$ *of any of the four* α*-models from above satisfies*

$$||\mathbf{u}-\mathbf{w}^\alpha||^2_{L^\infty(0,T;L^2)}+||\nabla\mathbf{u}-\nabla\mathbf{w}^\alpha)||_{L^2(0,T;L^2)}\le C_T\alpha, \tag{3.6}$$

and

$$\begin{aligned}&||\overline{\mathbf{u}} - \overline{\mathbf{w}^\alpha}||_{L^\infty(0,T;L^2)} + \alpha||\nabla\overline{\mathbf{u}} - \nabla\overline{\mathbf{w}^\alpha}||_{L^\infty(0,T;L^2)} \\ &\quad + ||\nabla\overline{\mathbf{u}} - \nabla\overline{\mathbf{w}^\alpha}||_{L^2(0,T;L^2)} + \alpha||\Delta\overline{\mathbf{u}} - \Delta\overline{\mathbf{w}^\alpha}||_{L^2(0,T;L^2)} \le C_T\alpha^{1.5},\end{aligned} \tag{3.7}$$

where C_T is as in Remark 2.2.

The two main estimates from above are proved in several steps. We first prove in Theorem 3.1 that the Leray-α model satisfies the first main estimate (3.6). Then we prove in Theorem 3.3 that the Leray-α model satisfies the second main estimate (3.7). Next we prove in Theorem 3.4 that the Modified Leray-α, zeroth ADM, Navier–Stokes-α models satisfy the second main estimate (3.7) and lastly we prove in Theorem 3.4 that the Modified Leray-α, zeroth ADM, Navier–Stokes-α models satisfy the first main estimate (3.6).

Theorem 3.1. *The Leray-α model (3.1,3.2) satisfies the first main estimate (3.6).*

Proof. We subtract (3.1) from (2.7) and set $\mathbf{e} = \mathbf{u} - \mathbf{w}^\alpha$ as a test function to get

$$\frac{1}{2}\frac{d}{dt}||\mathbf{e}||^2 + \nu||\nabla\mathbf{e}||^2 = -(\mathbf{u}\cdot\nabla\mathbf{u},\mathbf{e}) + (\overline{\mathbf{w}^\alpha}\cdot\nabla\mathbf{w}^\alpha,\mathbf{e}). \tag{3.8}$$

The nonlinear terms on the right side are written in the following form

$$-(\mathbf{u}\cdot\nabla\mathbf{u},\mathbf{e}) + (\overline{\mathbf{w}^\alpha}\cdot\nabla\mathbf{w}^\alpha,\mathbf{e}) = -((\mathbf{u}-\overline{\mathbf{u}})\cdot\nabla\mathbf{u},\mathbf{e}) - (\overline{\mathbf{e}}\cdot\nabla\mathbf{u},\mathbf{e}). \tag{3.9}$$

The first term on the right side above is written as

$$((\mathbf{u}-\overline{\mathbf{u}})\cdot\nabla\mathbf{u},\mathbf{e}) = ((\mathbf{u}-\overline{\mathbf{u}})\cdot\nabla(\mathbf{u}-\overline{\mathbf{u}}),\mathbf{e}) + ((\mathbf{u}-\overline{\mathbf{u}})\cdot\nabla\overline{\mathbf{u}},\mathbf{e}) \tag{3.10}$$

The first term on the right is bounded as

$$\begin{aligned}|((\mathbf{u}-\overline{\mathbf{u}})\cdot\nabla(\mathbf{u}-\overline{\mathbf{u}}),\mathbf{e})| &\le ||\mathbf{u}-\overline{\mathbf{u}}||_{L^3}||\nabla(\mathbf{u}-\overline{\mathbf{u}})||_{L^2}||\mathbf{e}||_{L^6} \\ &\le C||\mathbf{u}-\overline{\mathbf{u}}||_{L^2}^{1/2}||\nabla(\mathbf{u}-\overline{\mathbf{u}})||_{L^2}^{3/2}||\nabla\mathbf{e}|| \\ &\le C_T\alpha^{1/2}||\nabla(\mathbf{u}-\overline{\mathbf{u}})||_{L^2}^{3/2}||\nabla\mathbf{e}|| \\ &\le C_T\alpha||\nabla(\mathbf{u}-\overline{\mathbf{u}})||^3 + \tfrac{\nu}{10}||\nabla\mathbf{e}||^2\end{aligned}$$

The first inequality from above is due to the Hölder's inequality with the pairing $1/3+1/2+1/6=1$. In the second inequality the L^3 norm is handled using inequality (2.6), whereas the L^6 norm is handled using inequality (2.3). In the third inequality we use (2.12). The last inequality is due to the Young's inequality with the pairing $1/2+1/2=1$. In the last inequality the second term on the right will be absorbed in the left side of (3.8) whereas the time integral of the first term on the right is bounded using Cauchy inequality and (2.12).

$$\begin{aligned}\alpha\int_0^T ||\nabla(\mathbf{u}-\overline{\mathbf{u}})||^3 dt &\le \alpha||\nabla(\mathbf{u}-\overline{\mathbf{u}})||_{L^2(0,T;L^2)}||\nabla(\mathbf{u}-\overline{\mathbf{u}})||_{L^4(0,T;L^2)}^2 \\ &\le C_T\alpha^2.\end{aligned} \tag{3.11}$$

The second term on the right of (3.10) is bounded as follows.

$$\begin{aligned}|((\mathbf{u}-\overline{\mathbf{u}})\cdot\nabla\overline{\mathbf{u}},\mathbf{e})| &\le ||\mathbf{u}-\overline{\mathbf{u}}||_{L^3}||\nabla\overline{\mathbf{u}}||_{L^2}||\mathbf{e}||_{L^6} \\ &\le C_T||\mathbf{u}-\overline{\mathbf{u}}||_{L^3}^2 + \tfrac{\nu}{10}||\nabla\mathbf{e}||^2\end{aligned}$$

In the first inequality we use the Hölder's inequality with the pairing $1/3+1/2+1/6=1$. In the second inequality the L^2 norm is handled using inequality (2.9), the L^6 norm is handled using inequality (2.3) and the Young's inequality is applied with the pairing $1/2+1/2=1$. In the last inequality second term on the right will be absorbed in the left side of (3.8) whereas the time integral of the first term on the right is bounded using (2.14).

$$\int_0^T ||\mathbf{u}-\overline{\mathbf{u}}||_{L^3}^2 dt \le C_T\alpha^3 \tag{3.12}$$

We now estimate the second term on the right of formula (3.9) as

$$\begin{aligned}|(\overline{\mathbf{e}}\cdot\nabla\mathbf{u},\mathbf{e})| \le ||\overline{\mathbf{e}}||_{L^6}||\nabla\mathbf{u}||_{L^2}||\mathbf{e}||_{L^3} &\le ||\nabla\mathbf{u}||\,||\mathbf{e}||^{1/2}||\nabla\mathbf{e}||^{3/2} \\ &\le \tfrac{C}{\nu^3}||\nabla\mathbf{u}||^4\,||\mathbf{e}||^2 + \frac{\nu}{4}||\nabla\mathbf{e}||^2.\end{aligned} \tag{3.13}$$

In the first inequality from above we use Hölder's inequality with the pairing $1/6+1/2+1/3=1$ and in the second inequality the L^6 norm is bounded using inequality (2.3) and the L^3 norm using inequality (2.4). In the third inequality from above we apply Young's inequality with the pairing $1/4+3/4=1$.

Finally, applying Gronwall's lemma together with (3.11), (3.12) in (3.8) gives

$$||\mathbf{u}-\mathbf{w}^\alpha||^2_{L^\infty(L^2)}+\nu||\nabla\mathbf{u}-\nabla\mathbf{w}^\alpha||^2_{L^2(L^2)}\le C_T\alpha^2.$$

□

The next theorem provides a stability estimate of $\mathbf{w}^\alpha$ that will be later used to show that the weak solution $\mathbf{w}^\alpha$ of the Leray-α model satisfies the second main estimate (3.7).

Theorem 3.2. *The weak solution $\mathbf{w}^\alpha$ of the Leray-α model satisfies the inequality*

$$||\nabla\overline{\mathbf{w}^\alpha}||_{L^\infty(L^2)}+\alpha||\Delta\overline{\mathbf{w}^\alpha}||_{L^\infty(L^2)}+||\nabla\overline{\mathbf{w}^\alpha}||_{L^2(L^2)}+\alpha||\nabla\Delta\overline{\mathbf{w}^\alpha}||_{L^2(L^2)}\le C_T \tag{3.14}$$

Proof. Using formulas $\mathbf{u}=\overline{\mathbf{u}}-\alpha^2\Delta\overline{\mathbf{u}}$, $\mathbf{w}^\alpha=\overline{\mathbf{w}^\alpha}-\alpha^2\Delta\overline{\mathbf{w}^\alpha}$ in the first main estimate (3.6) gives

$$\begin{aligned}&||\nabla\overline{\mathbf{u}}-\nabla\overline{\mathbf{w}^\alpha}||_{L^\infty(L^2)}+\alpha||\Delta\overline{\mathbf{u}}-\Delta\overline{\mathbf{w}^\alpha}||_{L^\infty(L^2)}\le C_T\\&||\Delta\overline{\mathbf{u}}-\nabla\overline{\mathbf{w}^\alpha}||_{L^2(L^2)}+\alpha||\nabla\Delta\overline{\mathbf{u}}-\nabla\Delta\overline{\mathbf{w}^\alpha}||_{L^2(L^2)}\le C_T\end{aligned}$$

and therefore using inequality (2.9) gives (3.14). □

Theorem 3.3. *The weak solution $\mathbf{w}^\alpha$ of the Leray-α model* (3.1, 3.2) *satisfies the second main estimate (3.7).*

Proof. We subtract (3.1) from (2.7) and set $\overline{\mathbf{e}}=\overline{\mathbf{u}}-\overline{\mathbf{w}^\alpha}$ as a test function to get

$$\frac{1}{2}\frac{d}{dt}(||\overline{\mathbf{e}}||^2+\alpha^2||\nabla\overline{\mathbf{e}}||^2)+\nu||\nabla\overline{\mathbf{e}}||^2+\nu\alpha^2||\Delta\overline{\mathbf{e}}||^2=-(\mathbf{u}\cdot\nabla\mathbf{u},\overline{\mathbf{e}})+(\overline{\mathbf{w}^\alpha}\cdot\nabla\mathbf{w}^\alpha,\overline{\mathbf{e}}). \tag{3.15}$$

The nonlinear terms on the right side are written in the following form

$$-(\mathbf{u}\cdot\nabla\mathbf{u},\overline{\mathbf{e}})+(\overline{\mathbf{w}^\alpha}\cdot\nabla\mathbf{w}^\alpha,\overline{\mathbf{e}})=-((\mathbf{u}-\overline{\mathbf{u}})\cdot\nabla\mathbf{u},\overline{\mathbf{e}})-(\overline{\mathbf{e}}\cdot\nabla\mathbf{u},\overline{\mathbf{e}})-(\overline{\mathbf{w}^\alpha}\cdot\nabla\mathbf{e},\overline{\mathbf{e}}). \tag{3.16}$$

The estimation of the second term on the right side is done as in inequality (3.13) to obtain

$$|(\overline{\mathbf{e}}\cdot\nabla\mathbf{u},\overline{\mathbf{e}})|\le\frac{C}{\nu^3}||\nabla\mathbf{u}||^4\,||\overline{\mathbf{e}}||^2+\frac{\nu}{4}||\nabla\overline{\mathbf{e}}||^2. \tag{3.17}$$

The third is written as

$$|(\overline{\mathbf{w}^\alpha}\cdot\nabla\mathbf{e},\overline{\mathbf{e}})|=\alpha^2|(\overline{\mathbf{w}^\alpha}\cdot\nabla\Delta\overline{\mathbf{e}},\overline{\mathbf{e}})|=\alpha^2|(\overline{\mathbf{w}^\alpha}\cdot\nabla\overline{\mathbf{e}},\Delta\overline{\mathbf{e}})|$$

and then bounded as

$$\begin{aligned}|(\overline{\mathbf{w}^\alpha}\cdot\nabla\mathbf{e},\overline{\mathbf{e}})|&\le\alpha^2||\overline{\mathbf{w}^\alpha}||_{L^6}||\nabla\overline{\mathbf{e}}||_{L^3}||\Delta\overline{\mathbf{e}}||_{L^2}\\&\le C_T\alpha^2||\nabla\overline{\mathbf{e}}||^{1/2}||\Delta\overline{\mathbf{e}}||^{3/2}\\&\le C_T\alpha^2||\nabla\overline{\mathbf{e}}||^2+\frac{\nu}{4}\alpha^2||\Delta\overline{\mathbf{e}}||^2.\end{aligned} \tag{3.18}$$

In the second inequality the L^6 norm is handled using the inequality (2.3) and the resulting term $||\nabla\overline{\mathbf{w}^\alpha}||$ is bounded uniformly in α, see inequality (3.14). The L^3 norm is handled using the inequality (2.4). Lastly, in the third inequality we use Young's inequality with the pairing $1/4+3/4=1$.

In the last inequality, the second term on the right will be absorbed in the left side of (3.15) and the time integral of the first term is bounded as

$$\alpha^2\int_0^T||\nabla\overline{\mathbf{e}}||^2=\alpha^2\int_0^T||\nabla\overline{\mathbf{u}}-\nabla\overline{\mathbf{w}^\alpha}||^2\le C_T\alpha^4$$

due to the first main estimate (3.6).

The first term on the right side in formula (3.16) will be estimated as

$$|((\mathbf{u}-\overline{\mathbf{u}})\cdot\nabla\mathbf{u},\overline{\mathbf{e}})|\le|((\mathbf{u}-\overline{\mathbf{u}})\cdot\nabla(\mathbf{u}-\overline{\mathbf{u}}),\overline{\mathbf{e}})|+|((\mathbf{u}-\overline{\mathbf{u}})\cdot\nabla\overline{\mathbf{u}},\overline{\mathbf{e}})|. \tag{3.19}$$

Then the first term on the right side is estimated as

$$\begin{aligned}|((\mathbf{u}-\overline{\mathbf{u}})\cdot\nabla(\mathbf{u}-\overline{\mathbf{u}}),\overline{\mathbf{e}})| &= |((\mathbf{u}-\overline{\mathbf{u}})\cdot\nabla\overline{\mathbf{e}},\mathbf{u}-\overline{\mathbf{u}})| \\ &\leq ||\mathbf{u}-\overline{\mathbf{u}}||_{L^2}||\mathbf{u}-\overline{\mathbf{u}}||_{L^6}||\nabla\overline{\mathbf{e}}||_{L^3} \\ &\leq C_T\alpha||\nabla\mathbf{u}-\nabla\overline{\mathbf{u}}||||\nabla\overline{\mathbf{e}}||^{1/2}||\Delta\overline{\mathbf{e}}||^{1/2} \\ &\leq C_T\alpha||\nabla\mathbf{u}-\nabla\overline{\mathbf{u}}||^2+\frac{\nu}{4}||\nabla\overline{\mathbf{e}}||^2+\alpha^2\frac{\nu}{4}||\Delta\overline{\mathbf{e}}||^2.\end{aligned} \tag{3.20}$$

In the second inequality from above the L^6 norm is handled using the Sobolev inequality (2.3), the L^3 norm is handled using the inequality (2.4) and the first term $||\mathbf{u}-\overline{\mathbf{u}}||_{L^2}$ is bounded by $C_T\alpha$ due to inequality (2.12). In the third inequality we use the Young's inequality with the pairing $1/2+1/4+1/4=1$.

Upon applying Gronwall's lemma the last two terms on the right will be absorbed in the left side of (3.15) whereas the first term is estimated as

$$\alpha\int_0^T||\nabla\mathbf{u}-\nabla\overline{\mathbf{u}}||^2dt\leq C_T\alpha^3$$

using inequality (2.12) in Theorem 2.3.

Going back to inequality (3.19), the second term on the right is bounded as

$$\begin{aligned}|((\mathbf{u}-\overline{\mathbf{u}})\cdot\nabla\overline{\mathbf{u}},\overline{\mathbf{e}})| &\leq ||\overline{\mathbf{u}}-\mathbf{u}||_{L^3}||\nabla\overline{\mathbf{u}}||_{L^2}||\overline{\mathbf{e}}||_{L^6} \\ &\leq C_T||\overline{\mathbf{u}}-\mathbf{u}||_{L^3}||\nabla\overline{\mathbf{e}}|| \\ &\leq \frac{\nu}{4}||\nabla\overline{\mathbf{e}}||^2+C_T||\nabla\overline{\mathbf{u}}-\nabla\mathbf{u}||^2_{L^3}.\end{aligned} \tag{3.21}$$

In the second inequality above the term $||\nabla\overline{\mathbf{u}}||$ is bounded by C_T using inequality (2.9). The L^6 norm is bounded using the Sobolev inequality (2.3). In the third inequality we use Young's inequality with the pairing $1/2+1/2=1$.

The term containing $\overline{\mathbf{e}}$ will be absorbed in the left side and upon time integration, the second term is estimated as

$$\int_0^T||\mathbf{u}-\overline{\mathbf{u}}||^2_{L^3}\mathrm{d}t\leq C_T\alpha^3$$

using inequality (2.14) in Theorem 2.3.

Collecting the terms from the inequalities (3.17, 3.18, 3.20, 3.21), replacing in (3.15) and applying Gronwall's inequality proves that the weak solution $\mathbf{w}^\alpha$ of the Leray-α model satisfies the second main estimate (3.7). □

Theorem 3.4. *The weak solution $\mathbf{w}^\alpha$ of any of the three models the Modified Leray-α model* (3.1, 3.3)*, the zeroth ADM model* (3.1, 3.4) *and the Navier–Stokes-α model* (3.1, 3.5) *satisfies the second main estimate* (3.7).

Proof. Let's assume that $\mathbf{w}^\alpha$ is the weak solution of the modified Leray-α turbulence model (3.1, 3.3).

We subtract (3.1) from (2.7) and set $\overline{\mathbf{e}}=\overline{\mathbf{u}}-\overline{\mathbf{w}^\alpha}$ as a test function to get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(||\overline{\mathbf{e}}||^2+\alpha^2||\nabla\overline{\mathbf{e}}||^2)+\nu||\nabla\overline{\mathbf{e}}||^2+\nu\alpha^2||\Delta\overline{\mathbf{e}}||^2 \\ =-(\mathbf{u}\cdot\nabla\mathbf{u},\overline{\mathbf{e}})+(\mathbf{w}^\alpha\cdot\nabla\overline{\mathbf{w}^\alpha},\overline{\mathbf{e}}).\end{aligned} \tag{3.22}$$

The terms on the right side are written in the following form

$$-(\mathbf{u}\cdot\nabla\mathbf{u},\overline{\mathbf{e}})+(\mathbf{w}^\alpha\cdot\nabla\overline{\mathbf{w}^\alpha},\overline{\mathbf{e}})=-(\mathbf{u}\cdot\nabla\overline{\mathbf{e}},\overline{\mathbf{u}}-\mathbf{u})-(\mathbf{e}\cdot\nabla\overline{\mathbf{u}},\overline{\mathbf{e}}). \tag{3.23}$$

The first term is written as

$$(\mathbf{u}\cdot\nabla\overline{\mathbf{e}},\overline{\mathbf{u}}-\mathbf{u})=((\mathbf{u}-\overline{\mathbf{u}})\cdot\nabla\overline{\mathbf{e}},\overline{\mathbf{u}}-\mathbf{u})+(\overline{\mathbf{u}}\cdot\nabla\overline{\mathbf{e}},\overline{\mathbf{u}}-\mathbf{u}).$$

The first term on the right side above is estimated as in formula (3.20) whereas the second is estimated as

$$|(\overline{\mathbf{u}} \cdot \nabla \overline{\mathbf{e}}, \overline{\mathbf{u}} - \mathbf{u})| \leq ||\overline{\mathbf{u}} - \mathbf{u}||_{L^3} ||\nabla \overline{\mathbf{e}}||_{L^2} ||\overline{\mathbf{u}}||_{L^6}$$

using Hölder's inequality with the pairing $1/3 + 1/6 + 1/2 = 1$ and from here on the inequality proceeds as in inequality (3.21).

The last term on the right side of 3.23 is written as

$$-(\mathbf{e} \cdot \nabla \overline{\mathbf{u}}, \overline{\mathbf{e}}) = -(\overline{\mathbf{e}} \cdot \nabla \overline{\mathbf{u}}, \overline{\mathbf{e}}) + \alpha^2 (\Delta \overline{\mathbf{e}} \cdot \nabla \overline{\mathbf{u}}, \overline{\mathbf{e}}). \tag{3.24}$$

The first term on the right side above is handled similar to the term in inequality (3.13)

$$(\overline{\mathbf{e}} \cdot \nabla \overline{\mathbf{u}}, \overline{\mathbf{e}}) \leq \frac{C}{\nu^3} ||\nabla \overline{\mathbf{u}}||^4 ||\overline{\mathbf{e}}||^2 + \frac{\nu}{8} ||\nabla \overline{\mathbf{e}}||^2.$$

The second term in (3.24) is estimated as

$$\alpha^2 (\Delta \overline{\mathbf{e}} \cdot \nabla \overline{\mathbf{u}}, \overline{\mathbf{e}}) \leq \alpha^2 ||\Delta \overline{\mathbf{e}}|| ||\nabla \overline{\mathbf{u}}||_{L^6} ||\overline{\mathbf{e}}||_{L^3} \leq C\alpha ||\Delta \overline{\mathbf{e}}|| ||\nabla \mathbf{u}|| ||\overline{\mathbf{e}}||^{1/2} ||\nabla \overline{\mathbf{e}}||^{1/2}. \tag{3.25}$$

The first inequality from above uses Hölder's inequality with the duality pairing $1/2 + 1/6 + 1/3 = 1$. In the second inequality we handle the L^3 norm using inequality (2.4) and the L^6 norm using the Sobolev inequality (2.3) and Theorem 2.1 to write

$$\alpha ||\nabla \overline{\mathbf{u}}||_{L^6} \leq C ||\nabla \mathbf{u}||.$$

Going back to inequality (3.25) we may apply Young's inequality with the pairing $1/2 + 1/4 + 1/4 = 1$ to write

$$\alpha^2 (\Delta \overline{\mathbf{e}} \cdot \nabla \overline{\mathbf{u}}, \overline{\mathbf{e}}) \leq \frac{\nu}{2} \alpha^2 ||\Delta \overline{\mathbf{e}}||^2 + \frac{C}{\nu^3} ||\nabla \mathbf{u}||^4 ||\overline{\mathbf{e}}||^2 + \frac{\nu}{8} ||\nabla \overline{\mathbf{e}}||^2.$$

Collecting terms in (3.22) and applying Gronwall's inequality will finish the proof.

If $\mathbf{w}^\alpha$ is the weak solution of the zeroth ADM turbulence model (3.1, 3.4) then subtracting (3.1) from (2.7) and setting $\overline{\mathbf{e}} = \overline{\mathbf{u}} - \overline{\mathbf{w}^\alpha}$ as a test function gives

$$\frac{1}{2} \frac{d}{dt} (||\overline{\mathbf{e}}||^2 + \alpha^2 ||\nabla \overline{\mathbf{e}}||^2) + \nu ||\nabla \overline{\mathbf{e}}||^2 + \nu \alpha^2 ||\Delta \overline{\mathbf{e}}||^2 = -(\mathbf{u} \cdot \nabla \mathbf{u}, \overline{\mathbf{e}}) + (\overline{\mathbf{w}^\alpha} \cdot \nabla \overline{\mathbf{w}^\alpha}, \overline{\mathbf{e}}).$$

The nonlinear terms on the right side are written in the following form

$$-(\mathbf{u} \cdot \nabla \mathbf{u}, \overline{\mathbf{e}}) + (\overline{\mathbf{w}^\alpha} \cdot \nabla \overline{\mathbf{w}^\alpha}, \overline{\mathbf{e}}) = -((\mathbf{u} - \overline{\mathbf{u}}) \cdot \nabla \mathbf{u}, \overline{\mathbf{e}}) - (\overline{\mathbf{u}} \cdot \nabla (\mathbf{u} - \overline{\mathbf{u}}), \overline{\mathbf{e}}) - (\overline{\mathbf{e}} \cdot \nabla \overline{\mathbf{u}}, \overline{\mathbf{e}}).$$

All terms will be bounded as in the modified Leray-α case and then Gronwall's inequality will finish the proof in the case of the second main estimate (3.7) for the zeroth ADM model.

The proof in the NS-α case follows similar lines. (3.1) is subtracted from the NSE (2.7) (which is written in the rotational form) and $\overline{\mathbf{e}} = \overline{\mathbf{u}} - \overline{\mathbf{w}^\alpha}$ is set as a test function in the resulting equation to get

$$\frac{1}{2} \frac{d}{dt} (||\overline{\mathbf{e}}||^2 + \alpha^2 ||\nabla \overline{\mathbf{e}}||^2) + \nu ||\nabla \overline{\mathbf{e}}||^2 + \nu \alpha^2 ||\Delta \overline{\mathbf{e}}||^2 = -((\nabla \times \mathbf{u}) \times \mathbf{u}, \overline{\mathbf{e}}) + ((\nabla \times \mathbf{w}^\alpha) \times \overline{\mathbf{w}^\alpha}, \overline{\mathbf{e}}).$$

The term on the right is written in the following form

$$-((\nabla \times \mathbf{u}) \times \mathbf{u}, \overline{\mathbf{e}}) + ((\nabla \times \mathbf{w}^\alpha) \times \overline{\mathbf{w}^\alpha}, \overline{\mathbf{e}}) = -((\mathbf{u} - \overline{\mathbf{u}}) \times (\nabla \times \mathbf{u}), \overline{\mathbf{e}}) - ((\nabla \times \mathbf{e}) \times \overline{\mathbf{u}}, \overline{\mathbf{e}}).$$

In the above equality the first term on the right side is written as

$$-((\nabla \times \mathbf{u}) \times (\mathbf{u} - \overline{\mathbf{u}}), \overline{\mathbf{e}}) = -((\mathbf{u} - \overline{\mathbf{u}}) \cdot \nabla \mathbf{u}, \overline{\mathbf{e}}) + (\overline{\mathbf{e}} \cdot \nabla \mathbf{u}, \mathbf{u} - \overline{\mathbf{u}}).$$

and both terms are estimated as in inequality (3.20, 3.21).

Gronwall's inequality is then applied to obtain the second main estimate (3.7). □

Theorem 3.5. *A weak solution $\mathbf{w}^\alpha$ of any of the three models Modified Leray-α model* (3.3)*, zeroth ADM model* (3.4)*, Navier–Stokes -α model* (3.5) *satisfies*

$$||\nabla \overline{\mathbf{w}^\alpha}||_{L^\infty([0,T],L^2)} + \alpha ||\Delta \overline{\mathbf{w}^\alpha}||_{L^\infty([0,T],L^2)} + ||\Delta \overline{\mathbf{w}^\alpha}||_{L^2([0,T],L^2)} + \alpha ||\nabla \Delta \overline{\mathbf{w}^\alpha}||_{L^2([0,T],L^2)} \leq C_T \tag{3.26}$$

Proof. As a direct consequence of the fact that $\mathbf{w}^\alpha$ satisfies (3.7) and $\mathbf{u}$ satisfies (2.9) we obtain that

$$||\nabla\overline{\mathbf{w}^\alpha}||_{L^\infty([0,T],L^2)} + ||\Delta\overline{\mathbf{w}^\alpha}||_{L^2([0,T],L^2)} \leq C_T. \tag{3.27}$$

In case $\mathbf{w}^\alpha$ is the solution of the Zeroth ADM model (3.1, 3.4) we multiply (3.1) by $-\Delta\overline{\mathbf{w}^\alpha}$ and integrate on Ω to get

$$\frac{1}{2}\frac{d}{dt}(||\nabla\overline{\mathbf{w}^\alpha}||^2 + \alpha^2||\Delta\overline{\mathbf{w}^\alpha}||^2) + \nu||\Delta\overline{\mathbf{w}^\alpha}||^2 + \nu\alpha^2||\nabla\Delta\overline{\mathbf{w}^\alpha}||^2 = (N(\mathbf{w}^\alpha), \Delta\overline{\mathbf{w}^\alpha}) - (\mathbf{f}, \Delta\overline{\mathbf{w}^\alpha}), \tag{3.28}$$

where $N(\mathbf{w}^\alpha)$ is given by formula (3.4).

The first term on the right side of (3.28) is bounded as

$$\begin{aligned}(\overline{\mathbf{w}^\alpha}\cdot\nabla\overline{\mathbf{w}^\alpha}, \Delta\overline{\mathbf{w}^\alpha}) &\leq ||\overline{\mathbf{w}^\alpha}||_{L^6}||\nabla\overline{\mathbf{w}^\alpha}||_{L^3}||\Delta\overline{\mathbf{w}^\alpha}||_{L^2}\\ &\leq ||\nabla\overline{\mathbf{w}^\alpha}||||\nabla\overline{\mathbf{w}^\alpha}||^{1/2}||\Delta\overline{\mathbf{w}^\alpha}||^{1/2}||\Delta\overline{\mathbf{w}^\alpha}||\\ &\leq \tfrac{C}{\nu^3}||\nabla\overline{\mathbf{w}^\alpha}||^6 + \frac{\nu}{4}||\Delta\overline{\mathbf{w}^\alpha}||^2\end{aligned}$$

The first inequality from above follows from Hölder's inequality with the pairing $1/6+1/3+1/2=1$. In the second inequality the L^6 norm is handled using inequality (2.3) and the L^3 norm is treated using inequality (2.4). In the third inequality we use Young's inequality with the pairing $1/4+3/4=1$.

The time integral of the first term on the right side of the last inequality above is bounded as

$$\int_0^T ||\nabla\overline{\mathbf{w}^\alpha}||^6 dt \leq C_T$$

due to inequality (3.27).

The second term on the right side of (3.28) is bounded using Cauchy and Young's inequalities.

$$|(\mathbf{f}, \Delta\overline{\mathbf{w}^\alpha})| \leq ||\mathbf{f}||||\Delta\overline{\mathbf{w}^\alpha}|| \leq C||\mathbf{f}||^2 + \frac{\nu}{4}||\Delta\overline{\mathbf{w}^\alpha}||^2$$

Collecting terms in (3.28) and integrating in time finishes the proof.

In the case of the Modified Leray-α model (3.1, 3.3), upon setting $-\Delta\overline{\mathbf{w}^\alpha}$ as test function in (3.3) the weak form of the nonlinear term will be

$$(\mathbf{w}^\alpha\cdot\nabla\overline{\mathbf{w}^\alpha}, \Delta\overline{\mathbf{w}^\alpha}) = (\overline{\mathbf{w}^\alpha}\cdot\nabla\overline{\mathbf{w}^\alpha}, \Delta\overline{\mathbf{w}^\alpha}) - \alpha^2(\Delta\overline{\mathbf{w}^\alpha}\cdot\nabla\overline{\mathbf{w}^\alpha}, \Delta\overline{\mathbf{w}^\alpha}). \tag{3.29}$$

The first term on the right side above is treated exactly as in the zeroth ADM case. The second term in the right side above is estimated as

$$\begin{aligned}\alpha^2|(\Delta\overline{\mathbf{w}^\alpha}\cdot\nabla\overline{\mathbf{w}^\alpha}, \Delta\overline{\mathbf{w}^\alpha})| &\leq \alpha^2||\Delta\overline{\mathbf{w}^\alpha}||_{L^3}||\nabla\overline{\mathbf{w}^\alpha}||_{L^2}||\Delta\overline{\mathbf{w}^\alpha}||_{L^6}\\ &\leq ||\nabla\overline{\mathbf{w}^\alpha}||(\alpha||\Delta\overline{\mathbf{w}^\alpha}||)^{1/2}(\alpha||\nabla\Delta\overline{\mathbf{w}^\alpha}||)^{3/2}\\ &\leq \tfrac{C}{\nu^3}||\nabla\overline{\mathbf{w}^\alpha}||^4(\alpha||\Delta\overline{\mathbf{w}^\alpha}||)^2 + \frac{\nu}{4}(\alpha||\nabla\Delta\overline{\mathbf{w}^\alpha}||)^2.\end{aligned}$$

The first inequality from above is due to Hölder's inequality with the pairing $1/3+1/2+1/6=1$. In the second inequality the L^3 norm is handled using inequality (2.4), the L^6 norm is handled using inequality (2.3). In the second inequality we use Young's inequality with the pairing $1/4+3/4=1$.

We have that the time integral of the first term on the right side of the last inequality above is bounded as

$$\int_0^T ||\nabla\overline{\mathbf{w}^\alpha}||^4(\alpha||\Delta\overline{\mathbf{w}^\alpha}||)^2 \leq C_T$$

due to inequality (3.27).

Collecting terms in (3.28) and integrating in time finishes the proof.

In the case of the NS-α model (3.1, 3.5), upon setting $-\Delta\overline{\mathbf{w}^\alpha}$ as test function in (3.1) the weak form of the nonlinear term will be

$$((\nabla\times\mathbf{w}^\alpha)\times\overline{\mathbf{w}^\alpha}, \Delta\overline{\mathbf{w}^\alpha}) = (\overline{\mathbf{w}^\alpha}\cdot\nabla\mathbf{w}^\alpha, \Delta\overline{\mathbf{w}^\alpha}) - (\Delta\overline{\mathbf{w}^\alpha}\cdot\nabla\mathbf{w}^\alpha, \overline{\mathbf{w}^\alpha}),$$

both terms being further handled similarly to the term (3.29) in the Modified Leray-α case. □

Theorem 3.6. *A weak solution of any of the three models, Modified Leray-α model (3.1, 3.3), the zeroth ADM model (3.1, 3.4) and the Navier–Stokes-α model (3.1, 3.5), satisfies the first main estimate (3.6).*

Proof. Due to Theorem 3.5, the arguments in Theorem 2.3 apply also to the weak solution $\mathbf{w}^\alpha$ of any of the three α-models (3.3, 3.4, 3.5), therefore it satisfies the following inequality

$$||\mathbf{w}^\alpha - \overline{\mathbf{w}^\alpha}||_{L^\infty(L^2)} + ||\nabla \mathbf{w}^\alpha - \nabla \overline{\mathbf{w}^\alpha}||_{L^2(L^2)} \leq C_T \alpha.$$

Since $\mathbf{w}^\alpha$ satisfies the second main estimate (3.7) it follows that

$$||\overline{\mathbf{u}} - \overline{\mathbf{w}^\alpha}||_{L^\infty(L^2)} + ||\nabla \overline{\mathbf{u}} - \nabla \overline{\mathbf{w}^\alpha}||_{L^2(L^2)} \leq C_T \alpha^{1.5}.$$

Upon applying the triangle inequality we obtain that the weak solution of any of the three turbulence models (3.3, 3.4, 3.5) satisfies the first main estimate 3.6,

$$||\mathbf{u} - \mathbf{w}^\alpha||_{L^\infty(L^2)} + ||\nabla \mathbf{u} - \nabla \mathbf{w}^\alpha||^2_{L^2(L^2)} \leq C_T \alpha^2.$$

□

Remark 3.7. Although the above rates have been proved for the 3d case, all the results are also true in the 2d case as all inequalities used herein are valid also in two space dimensions.

Remark 3.8. The constants C_T obtained in the estimates (3.6, 3.7) depend continuously on $||\mathbf{u}||_{L^4(0,T;H^1)}$ and grow exponentially fast in the parameter $||\mathbf{u}||^4_{L^4(0,T;H^1)} \to \infty$, see Remark 2.2.

If we assume that T is an epoch of irregularity of $\mathbf{u}$, see Definition 6.1 on page 58, Galdi, [14], necesarily the norm $||\mathbf{u}||_{L^4(0,T;H^1)}$ has to blow-up, see Remark 6.2, page 62, [14], and therefore in this case the main estimates (3.6, 3.7) will not necesarily hold. The application of the theory on intervals $[0, T']$ with $T' \to T, T' < T$, will lead to $||\mathbf{u}||_{L^4(0,T';H^1)} \to \infty$ and as such the constants $C_{T'}$ (corresponding to T') in (3.6, 3.7) will grow exponentially fast as $T' \to T$. We emphasize that this is due to the use of Gronwall's inequality in the underlying analysis. Alternative ways to avoid its use will be considered in future investigations.

4. Conclusions

In this report it is shown that under the regularity assumption $\mathbf{u} \in L^4(H^1)$ on a weak solution $\mathbf{u}$ of the NSE the weak solution $\mathbf{w}^\alpha$ of any of the Leray-α, modified Leray-α, Navier–Stokes-α and the zeroth ADM turbulence models will converge to $\mathbf{u}$ with the convergence rate $\mathcal{O}(\alpha)$ in the norms $L^2(H^1)$ and $L^\infty(L^2)$ thus improving the convergence rate $\mathcal{O}(\alpha \log(1/\alpha)^{1/2})$ proved in two space dimensions in [5].

Another important result is that the averaged error $\overline{\mathbf{u}} - \overline{\mathbf{w}^\alpha}$ converges with higher rate $\mathcal{O}(\alpha^{1.5})$ than the error $\mathbf{u} - \mathbf{w}^\alpha$ itself, i.e. the filtered flow structures are approximated better than the exact (unfiltered) flow velocities.

It is also proved herein that the weak solution $\mathbf{w}^\alpha$ of the α-models of turbulence satisfies the uniform (in α) stability estimate

$$\underset{t\in[0,T]}{sup} ||\nabla \overline{\mathbf{w}^\alpha}|| \leq C_T.$$

It would be interesting to see how this stability property transfers to discrete finite element solutions of the α-models of turbulence in case the filter radius α is chosen equal to the mesh width h. This problem will be investigated in an upcoming paper.

Acknowledgements. The author thanks Edriss Titi for a helpful communication on the convergence of α-models of turbulence.

Compliance with ethical standards
Conflict of interest The author declares that he has no conflict of interests.

References

[1] Beale, J.T., Kato, T., Majda, A.: Remarks on the breakdown of smooth solutions for the 3-D Euler solutions. Commun. Math. Phys. **94**, 61–66 (1984)
[2] Berselli, L., Catania, D., Lewandowski, R.: Convergence of approximate deconvolution models to the mean magnetohydrodynamics equations: analysis of two models. J. Math. Anal. Appl. **401**, 864–880 (2013)
[3] Berselli, L., Lewandowski, R.: Convergence of approximate deconvolution models to the filtered Navier–Stokes equations. Annales de l'Institut Henri Poincare (C), Non Linear Anal. **29**(2), 171–198 (2012)
[4] Cao, Y., Lunasin, E.M., Titi, E.S.: Global well-posedness of the three-dimensional viscous and inviscid simplified Bardina turbulence models. Commun. Math. Sci. **4**(4), 823–848 (2006)
[5] Cao, Y., Titi, E.S.: On the rate of convergence of the two-dimensional α-models of turbulence to the Navier–Stokes equations. Numer. Funct. Anal. Optim. **30**, 1231–1271 (2009)
[6] Chen, L., Guenther, R.B., Kim, S., Thomann, E.A., Waymire, E.C.: A rate of convergence for the LANS-α regularization of Navier–Stokes equations. J. Math. Anal. Appl. **348**, 637–649 (2008)
[7] Chepyzhov, V.V., Titi, E.S., Vishik, M.I.: On the convergence of solutions of the Leray-α model to the trajectory attractor of the 3D Navier–Stokes system. J. Discrete Contin. Dyn. Syst. Ser. A **17**, 481–500 (2007)
[8] Cheskidov, A., Holm, D., Olson, E., Titi, E.: On a Leray-alpha model of turbulence. In Proceedings, Series A, Mathematical, Physical and Engineering Sciences, vol. 461. Royal Society, London, pp. 629–649 (2005)
[9] Cuff, V., Dunca, A., Manica, C., Rebholz, L.: The reduced order NS-α model for incompressible flow: theory, numerical analysis and benchmark testing. M2AN **49**(3), 641–662 (2015)
[10] Dunca, A., John, V.: Finite element error analysis of space averaged flow fields defined by a differential filter. Math. Models Methods Appl. Sci. **14**(4), 603–618 (2004)
[11] Dunca, A., Lewandowski, R.: Modeling error in approximate deconvolution models. Commun. Math. Sci. **12**(4), 757–778 (2014)
[12] Foias, C., Holm, D., Titi, E.: The Navier–Stokes-alpha model of fluid turbulence. Phys. D **152–153**, 505–519 (2001)
[13] Foias, C., Holm, D., Titi, E.: The three dimensional viscous Camassa–Holm equations, and their relation to the Navier–Stokes equations and turbulence theory. J. Dyn. Differ. Equ. **14**, 1–35 (2002)
[14] Galdi, G.P.: An introduction to the Navier–Stokes initial boundary value problem. In: Galdi, G.P., Heywood, J., Rannacher, R. (eds.) Fundamental Directions in Mathematical Fluid Mechanics. Advances in Mathematical Fluid Mechanics, vol. 1, pp. 1–98. Birkhauser, Basel (2000)
[15] Galvin, K., Rebholz, L., Trenchea, C.: Efficient, unconditionally stable, and optimally accurate FE algorithms for approximate deconvolution models. SIAM J. Numer. Anal. **52**(2), 678–707 (2014)
[16] Germano, M.: Differential filters for the large eddy numerical simulation of turbulent flows. Phys. Fluids **29**, 1755–1757 (1986)
[17] Germano, M.: Differential filters of elliptic type. Phys. Fluids **29**, 1757–1758 (1986)
[18] Morales Hernandez, M., Rebholz, L., Tone, C., Tone, F., Tone, F.: On the stability of the crank–Nicolson–Adams–Bashforth scheme for the 2D Leray-alpha model. Numer. Methods Partial Differ. Equ. **32**(4), 1155–1183 (2016)
[19] Heywood, J.G.: The Navier–Stokes equations: on the existence, regularity and decay of solutions. Indiana Univ. Math. J. **29**, 639–681 (1980)
[20] Ilyin, A., Lunasin, E., Titi, E.: A modified-Leray-α subgrid scale model of turbulence. Nonlinearity **19**, 879–897 (2006)
[21] Kaya, S., Manica, C., Rebholz, L.: On Crank–Nicolson Adams–Bashforth timestepping for approximate deconvolution models in two dimensions. Appl. Math. Comput. **246**, 23–38 (2014)
[22] Kaya, S., Manica, C.C.: Convergence analysis of the finite element method for a fundamental model in turbulence. Math. Models Methods Appl. Sci. **22**, 1250033 (2012)
[23] Kolmogorov, A.V.: The local structure of turbulence in incompressible viscous fluids for very large Reynolds number. Dokl. Akad. Nauk. SSR **30**, 9–13 (1941)
[24] Larios, A., Titi, E.: On the higher order global regularity of the inviscid Voigt-regularization three-dimensional hidrodynamic models. Discrete Contin. Dyn. Syst. Ser. B **14**(2), 603–627 (2010)
[25] Layton, W.: Introduction to the Numerical Analysis of Incompressible Viscous Flows. SIAM Publications, Philadelphia. ISBN:978-0-898716-57-3 (2008)
[26] Layton, W., Lewandowski, R.: A simple and stable scale similarity model for large scale eddy simulation: energy balance and existence of weak solutions. Appl. Math. Lett. **16**, 1205–1209 (2003)
[27] Layton, W., Lewandowski, R.: On a well-posed turbulence model. Discrete Contin. Dyn. Syst. Seri. B **6**(1), 111–128 (2006)
[28] Layton, W., Lewandowski, R.: A high accuracy Leray-deconvolution model of turbulence and its limiting behavior. Anal. Appl. (Singap.) **6**(1), 23–49 (2008)
[29] Layton, W., Manica, C., Neda, M., Rebholz, L.: Numerical analysis and computational testing of a high-order Leray-deconvolution turbulence model. Numer. Methods Partial Differ. Equ. **24**(2), 555–582 (2008)
[30] Layton, W., Manica, C., Neda, M., Rebholz, L.: Numerical analysis and computational comparisons of the NS-alpha and NS-omega regularizations. Comput. Methods Appl. Mech. Eng. **199**, 916–931 (2010)
[31] Leray, J.: Essay sur les mouvements plans d'une liquide visqueux que limitent des parois. J. Math. Pure Appl. Paris Ser. IX **13**, 331–418 (1934)
[32] Leray, J.: Sur les mouvements d'une liquide visqueux emplissant l'espace. Acta Math. **63**, 193–248 (1934)

[33] Linshiz, J., Titi, E.S.: On the convergence rate of the Euler-α, inviscid second-grade fluid, model to the Euler equations. J. Stat. Phys. **138**(1), 305–332 (2010)
[34] Miles, W., Rebholz, L.: An enhanced physics based scheme for the NS-alpha turbulence model. Numer. Methods Partial Differ. Equ. **26**, 1530–1555 (2010)
[35] Mullen, J.S., Fischer, P.F.: Filtering techniques for complex geometry fluid flows. Commun. Numer. Methods Eng. **15**, 9–18 (1999)
[36] Prodi, G.: Un teorema di unicit'a per le equazioni di Navier–Stokes. Ann. Mat. Pura Appl. **48**, 173–182 (1959)
[37] Rebholz, L., Kim, T.-Y., Byon, Young-Li: On an accurate α model for coarse mesh turbulent channel flow simulation. Appl. Math. Model. **43**, 139–154 (2017)
[38] Serrin, J.: The initial value problem for the Navier–Stokes equations. In: Langer, R.E. (ed.), Nonlinear Problems, vol. 9. University of Wisconsin Press, Madison (1963)
[39] Vishik, M.I., Titi, E.S., Chepyzhov, V.V.: Trajectory attractor approximations of the 3D Navier–Stokes system by the Leray-α model. Russ. Math. Dokladi **71**, 91–95 (2005)
[40] Zhou, Y., Hossain, M., Vahala, G.: A critical look at the use of filters in large eddy simulation. Phys. Lett. A **139**, 330–332 (1989)

Argus A. Dunca
Department of Mathematics and Informatics
University Politehnica of Bucharest
Bucharest
Romania
e-mail: argus_dunca@yahoo.com

(accepted: December 8, 2017; published online: December 29, 2017)