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Global Existence of Solutions to 2-D Navier–Stokes Flow with Non-decaying Initial Data in Exterior Domains

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Abstract. We study the two dimensional Navier–Stokes initial boundary value problem in exterior domains assuming that the initial data u_0 belongs L^∞. The global (in time) unique existence of this problem is furnished. The behavior of $\|u(t)\|_\infty$ for large t is of double exponential kind.

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1. Introduction

We consider the two dimensional Navier–Stokes initial boundary value problem in exterior domains:

$$\begin{cases} u_t - \Delta u + (u\cdot\nabla)u + \nabla\pi_u = 0 & x\in\Omega, \quad t>0,\\ \nabla\cdot u = 0 & x\in\Omega, \quad t>0,\\ u=0 & x\in\partial\Omega, \quad t>0,\\ u=u_0 & x\in\Omega, \quad t=0. \end{cases} \tag{1}$$

The symbol $\Omega\subseteq\mathbb{R}^2$ denotes a smooth exterior domain. The aim is to prove global (in time) existence of smooth solutions with non-decaying initial data. This problem was originally considered by Giga et al. [12] for the Cauchy problem (see Giga et al. [11] local existence). Subsequently, in [23] Sawada and Taniuchi proved an L^∞-norm of the solutions which is an improvement of the result stated in [12]. Based on a result by Zelik [25], a recent contribute given by Gallay [8] establishes that the solutions furnished in [12] enjoy the estimate

$$\|u(t)\|_\infty \le c\|u_0\|_\infty(1+c\|u_0\|_\infty^2 t), \text{ for all } t>0, \tag{2}$$

with c independent of u_0. Estimate (2) represents a remarkable improvement with respect to the one given in [23] where the grow is of exponential type. Instead in the case of the Navier–Stokes initial boundary value problem there is a local existence and uniqueness theorem given by Abe [1], and global existence in [2] for non decaying initial data belonging to the subset $L^\infty\cap D$, where $D:=\{u:\|\nabla u\|_2<\infty\}$. As far as we know, no global existence result is known for data in L^∞. We investigate on this question. The results of the quoted papers are our starting point (see Sect. 2). In order to state our result we introduce the following notations: $C_w([0,T);L^\infty(\Omega))$ denotes the space of functions which are weakly star continuous

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from $[0,T]$ to $L^\infty(\Omega)$; for all $t>0$ the symbol $\big[g(t)\big]^{(\beta)}_\Omega$ is the β-Hölder seminorm in x; and, for all $x\in\Omega$, the symbol $\big[g(x)\big]^{(\gamma)}_{[\delta,T]}$ is the γ-Hölder semi norm in t. For all $R>0$, we set $\Omega_R:=\Omega\cap B(O,R)$ with $B(O,R):=\{|x|<R\}$.

From now on, the symbol c denotes a constant whose numerical value is inessential to our aims. Its value can be changed in the same line and it is considered ≥ 1.

We are able to prove the following result:

Theorem 1. *Let $u_0\in L^\infty(\Omega)$ with divergence free in the weak sense. Then there exists a unique solution (u,π_u) to problem (1) such that*

$$\text{for all } T>0,\ u\in C_w([0,T);L^\infty(\Omega)),\quad u,\ u_t,\ D^2u,\ \nabla\pi_u\in C((0,T)\times\overline{\Omega}),\tag{3}$$

with

$$\|u(t)\|_\infty\leq c\|u_0\|_\infty(1+\|u_0\|^2_\infty t)+A(t,\|u_0\|_\infty),\tag{4}$$

where

$$A(t,\|u_0\|_\infty):=\begin{cases}0 \text{ for } t\in[0,T_0],\\ Q(\|u_0\|_\infty)\exp(Q(\|u_0\|_\infty)\exp(c\|u_0\|^2_\infty(1+\|u_0\|^2_\infty t)^2t) \text{ for } t>T_0,\end{cases}$$

where Q is a function of $\|u_0\|_\infty$, and $T_0\geq c\|u_0\|^{-2}_\infty$ with the constant c independent of u_0. Moreover, on interval $(0,T_0)$ holds

$$\begin{aligned}&\text{for } \beta\in(0,1),\ \sup_{(0,T_0)}\Big[t^{\frac12}\|\nabla u(t)\|_\infty+t^{\frac{1+\beta}{2}}\big[\nabla u(t)\big]^{(\beta)}_\Omega\Big]\leq c\|u_0\|_\infty,\\ &\text{for all } \delta>0,\ \gamma\in(0,1),\quad \sup_\Omega\Big[\big[u(x)\big]^{(\gamma)}_{[\delta,T_0]}+\big[\nabla u(x)\big]^{(\frac{\gamma}{2})}_{[\delta,T_0]}\Big]\leq c\|u_0\|_\infty.\end{aligned}\tag{5}$$

Finally, for the pressure field $\pi_u\in C((0,T)\times\Omega_R)$ we get the pointwise estimate

$$|\pi_u(t,x)|\leq c\Big[|x|^\varepsilon\|u(t)\|_\infty\big[u(t)\big]^{(\varepsilon)}+\|(u\cdot\nabla)u(t)\|_{1,\lambda}+\|u(t)\|^2_\infty+\big[\nabla u(t)\big]^{(\lambda)}\Big],\ \text{for all } t>0.\tag{6}$$

In particular we get

$$|\pi_u(t,x)|\leq c_0|x|^\varepsilon\|u_0\|^2_\infty(1+c\|u_0\|^2_\infty t)t^{-\frac{\varepsilon}{2}}+B(t,\|u_0\|_\infty),\ \text{for all } t\in(0,T_0),\tag{7}$$

where

$$B(t,\|u_0\|_\infty):=c_1\|u_0\|^2_\infty\big[1+t^{-\frac12}+t^{-\frac12-\frac{\lambda}{2}}\big]+c_2\|u_0\|_\infty t^{-\frac12-\frac{\lambda}{2}}.$$

Remark 1. We look for the existence of a global solution in $L^\infty((0,T)\times\Omega)$. Nevertheless, the estimate (4), of the L^∞-norm of the solution by means of the initial data, is important. Actually, we do not have estimate uniform in time. This is also the situation in the case of the Cauchy problem where L^∞-norm of the solution has estimate (2). We justify estimate (4) below.

The Helmholtz decomposition in L^∞ is actually not known. Hence, the pressure field is suitably constructed. However estimates (6)–(7) are of some interest. Via estimate (6) we deduce that for non decaying solutions the pressure field grows at infinity at most as $|x|^\varepsilon$, where $\varepsilon>0$ is arbitrary, for all $t>0$. Instead, for the sake of brevity we omit the behavior for large t, it is easy to image that the pressure field is bounded via a suitable exponential function of t for all $x\in\Omega$. Nevertheless, since the data is only in L^∞, it is of particular interest to understand the behavior of the pressure field in neighborhood of $t=0$. This justify the interest for estimate (7). It is interesting to stress that in (7) in the case of the Cauchy problem function $B(t,\|u_0\|_\infty)=0$ for all $t>0$, as well as in the linear case the constants $c_0=c_1=0$.

The result of existence partially follows the lines of proof given in [10] for the three dimensional case. In [10] the existence is local in time and the initial data belongs to $C^{0,\alpha}(\Omega)$.

In order to prove the existence of solution (u,π_u) we look for $u:=U+W$, where U solves the Cauchy problem with initial data u_0 and W is the correction of $U\neq 0$ on $(0,T)\times\partial\Omega$. Actually, we construct $W:=V+w$, where V is smooth field with compact support whose L^∞-norm shares the same bound of

U, and w solves a suitable problem. The advantage of the above decomposition is that, for field w, we can employ the L^2-theory that furnishes global existence in 2-D (see e.g [15]). As a consequence of this approach, we "pay" for the L^∞-norm, in the sense that we increase from estimate (2) to estimate (4) for the L^∞-norm of the solution. More precisely, in order to arrive at an estimate of $\|w(t)\|_\infty$, we perform estimates related to $\|w(t)\|_{2,2}$, that are deduced from energy inequalities concerning $\|w(t)\|_2$ and $\|\nabla w(t)\|_2$ pointwise in t (see estimates (64) and (74)). Since we look for $u = U + W$, the nonlinear term produces the term $(w \cdot \nabla)(U + V)$, hence performing the energy relation for $\|w(t)\|_2 + \int_0^t \|\nabla w(\tau)\|_2^2 d\tau$ arises the first exponential function. Analogously when we perform the estimate for $\|\nabla w(t)\|_2$, the nonlinear term $(w \cdot \nabla)w$ produces the second exponential function. The estimate of $\|D^2 w(t)\|_2$ is deduced as consequence of the one related to $\|w_t(t)\|_2$, that, roughly speaking, has a linear character in the derivation. Hence it does not produce a further exponential function.

The authors would like to express special gratitude to the referee who suggested a modification of a previous version. The referee communicated the papers [8] (and [25]) which allow an improvement of the estimate (4). Moreover the referee suggested that we state the result for the initial data $u_0 \in L^\infty$ with divergence free in the weak sense. In the first version the authors limited ourselves to stating Theorem 4 and in a remark simply claim the generalization to data $u_0 \in L^\infty$.

The plan of this paper is as follows. In Sect. 2 we consider Theorems 2–5 as achieved and thus we give the proof of Theorem 1. In Sect. 3, we recall the results obtained in [12,23], and we prove some lemmas that will be subsequently employed. We furnish a suitable extension in Ω of the trace on $\partial\Omega$ of the solution U in Sect. 4. Section 5 is devoted to analyzing the existence of field W. Finally, in Sect. 6 we give the proof of Theorem 4 which is crucial for our goals and in Sect. 7 the proof Theorem 5 crucial to obtain estimate (6).

Notations. We set $\Omega^c := \mathbb{R}^2 - \Omega$. Let $D \subset \mathbb{R}^2$. The symbol $W^{k-\frac{1}{q},q}(\partial D)$ denotes the trace space of elements of $W^{k,q}(D)$. By the symbol h, we indicate a smooth positive cut-off function such that $h(x) = 1$ for $|x| \le \frac{1}{3}$, $h(x) \in [0, 1]$ for $|x| \in [\frac{1}{3}, \frac{2}{3}]$ and $h(x) = 0$ for $|x| \ge \frac{2}{3}$. For $R > 0$, we set $h_R := h(\frac{x}{R})$.

By the symbol $J^q(\Omega)$, $q \in (1, \infty)$, we indicate the completion of $\mathscr{C}_0(\Omega)$ in $L^q(\Omega)$ Lebesgue space. As well we set $J^{1,q}(\Omega) :=$ completion of $\mathscr{C}_0(\Omega)$ in Sobolev space $W^{1,q}(\Omega)$ and $J^{2-\frac{2}{q},q}(\Omega) :=$ completion of $\mathscr{C}_0(\Omega)$ in Besov space $W^{2-\frac{2}{q},q}(\Omega)$.

By the symbol $C^{k,\lambda}(\Omega)$, $k \in \mathbb{N}$ and $\lambda \in (0, 1)$, we denote the Hölder's space of functions continuous differentiable with their derivatives $D^\alpha u$, $|\alpha| \le k$, and with $D^\alpha U$, $|\alpha| = k$ λ-Hölder continuous. The norm in $C^{k,\lambda}$ is indicated by $\|\cdot\|_{k,\lambda}$ and Hölder's seminorm by $[\cdot]_\Omega^{(\lambda)}$. We use the symbol $[\cdot]^{(\lambda)}$ when there is no confusion about the domain.

Let $q \in [1, \infty)$, let X be a Banach space with norm $\|\cdot\|_X$. We denote by $L^q(a, b; X)$ the set of all function $g : (a, b) \to X$ which are measurable and such that the Lebesgue integral $\int_a^b \|g(\tau)\|_X^q d\tau = \|g\|_{L^q(a,b;X)} < \infty$. As well as, if $q = \infty$ we denote by $L^\infty(a, b; X)$ the set of all function $g : (a, b) \to X$ which are measurable and such that $\operatorname{ess\,sup}_{t\in(a,b)} \|g(t)\|_X = \|g\|_{L^\infty(a,b;X)} < \infty$. Finally, we denote by $C(a, b; X)$ the set of functions which are continuous from (a, b) into X and $\sup_{(a,b)} \|g(t)\|_X < \infty$ and by $C([a, b); X) \subset C(a, b; X)$ the set of functions which are continuous up $t = a$.

2. Proof of Theorem 1

Theorem 1 is proved thanks to a local existence theorem of solutions to problem (1), and thanks to a result concerning extension of these solutions. Actually, we employ the local existence theorem proved in [1] by Abe, and Theorem 4 stated below that represents the chief result of this note.

In order to give the statement of the results proved in [1] we introduce the notion of mild solution. In the following the symbol $S(t)$ denotes the Stokes semigroup and $\mathbb{P}$ the Helmholtz projection. In [1] the composition operator $S(t)\mathbb{P}\mathrm{div}$ is studied, and it is proved that $S(t)\mathbb{P}\mathrm{div}$ admits a unique extension

$\overline{S(t)\mathbb{P}\mathrm{div}}$ from non-decaying space $W_0^{1,\infty}$ to L^∞_σ. This extension makes it possible to study the integral equation on L^∞_σ:

$$u(t) = S(t)u_0 - \int_0^t \overline{S(t)\mathbb{P}\mathrm{div}}(u\otimes u)(s)ds. \tag{8}$$

Theorem 2 ([1]). *Let $u_0 \in L^\infty(\Omega)$ with divergence free in the weak sense. Then there exist a T_0 and a unique mild solution to equation (8) such that $u \in C_w([0,T_0); L^\infty(\Omega))$, and u is a weak solution to problem (1), that is such that*

$$\int_0^{T_0}\int_\Omega (u\cdot(\varphi_t+\Delta\varphi) + u\otimes u\cdot\nabla\varphi)dxdt = -\int_\Omega u_0\cdot\varphi(0)dx$$

for all $\varphi \in C_0^\infty([0,T_0)\times\Omega)$ with $\nabla\cdot\varphi = 0$. Moreover, the following estimates hold:

$$\begin{aligned} &\beta\in(0,1),\ \sup_{(0,T_0)}\Big[\|u(t)\|_\infty + t^{\frac12}\|\nabla u(t)\|_\infty + t^{\frac{1+\beta}{2}}\big[\nabla u(t)\big]^{(\beta)}_\Omega\Big] \le c\|u_0\|_\infty \\ &\textit{for all } \delta>0,\ \gamma\in(0,1),\ \sup_\Omega\Big[\big[u(x)\big]^{(\gamma)}_{[\delta,T_0]} + \big[\nabla u(x)\big]^{(\frac{\gamma}{2})}_{[\delta,T_0]}\Big] \le c\|u_0\|_\infty, \end{aligned} \tag{9}$$

with c independent of u_0 and $T_0 \ge \frac{c}{\|u_0\|^2_\infty}$.

Via the integral equation (8) we further find that any solution of Theorem 2 solves the integral equation

$$u(t) = S(t)u(t_0) - \int_{t_0}^t \overline{S(t)\mathbb{P}\mathrm{div}}(u\otimes u)(s)ds \text{ for all } t>t_0\in[0,T_0). \tag{10}$$

Here and in the following the symbol $BUC(\Omega)$, $\Omega\subseteq\mathbb{R}^2$, denotes the set of bounded uniformly continuous functions in Ω. The operator R_i denotes the Riesz transform.

Theorem 3. *Assume that $U_0\in L^\infty(\mathbb{R}^2)$ with divergence free in the weak sense. Then there exists a unique solution $U\in C_w([0,\infty); L^\infty(\mathbb{R}^2))\cap C^\infty((0,\infty)\times\mathbb{R}^2)$ of (1) enjoying the estimate*

$$\|U(t)\|_\infty \le c\|U_0\|_\infty(1+\|U_0\|^2_\infty t),\ \forall t>0, \tag{11}$$

where c is independent of U_0. Up to a constant the pressure field is given by $\pi_U := \sum_{i,j}^2 R_iR_jU^iU^j$, where $R_i, i=1,2$, is the Riesz operator. Moreover if $U_0\in W^{1,\infty}(\mathbb{R}^2)$, then holds $U\in C([0,\infty); BUC(\mathbb{R}^2))$, and

$$\begin{aligned} &\|\omega(t)\|_\infty \le \|\omega_0\|_\infty\ \forall t>0 \\ &\forall\nu\in(0,1),\ [U(t)]^{(\nu)}_{\mathbb{R}^2} \le ct^{-\frac{\nu}{2}}\|U_0\|_\infty + c\|U_0\|_\infty(1+\|U_0\|^2_\infty t)(1+\|\omega_0\|^\nu_\infty t^{\nu\eta_1}),\ \forall t>0, \end{aligned} \tag{12}$$

where $\eta_1\in(0,1)$ is arbitrary.

In the following theorem we set $G_0(t,u_0) := c\|u_0\|_\infty(1+\|u_0\|^2_\infty t)$ and $Q(u_0,\nabla U_0)$ is a polynomial function of $\|u_0\|_\infty$ and $\|\nabla U_0\|_\infty$.

Theorem 4. *Let $u_0\in BUC(\Omega)$ and $\nabla u_0\in L^\infty(\Omega)$, with divergence free a.e. in Ω and $u_0=0$ on $\partial\Omega$. Then for all $T>0$ there exists a unique solution (u,π_u) to problem (1) such that*

$$\begin{aligned} &u\in C([0,T)\times\overline{\Omega}),\quad u_t, D^2u, \nabla\pi_u\in C((0,T)\times\overline{\Omega}), \\ &\|u(t)\|_\infty \le c\|u_0\|_\infty + c(t,u_0,\nabla u_0)e^{c\left[Q(u_0,\omega_0)(t^{\frac12}+t^{3+\eta})\right]^2 e^{c\int_0^t G_0^2(\tau,u_0)d\tau}}, \end{aligned} \tag{13}$$

with $\lim_{t\to0} c(t,u_0,\nabla u_0)=0$ and $\eta\in(0,1)$. Moreover, we get that, for all $t>0$, the pressure field $\pi_u\in C(\Omega_R)$ with $\pi_u=\pi^1+\pi^2$ where

$$\begin{aligned} &\pi^1 := R_iR_jU^iU^j,\ \textit{and for all } \varepsilon\in(0,1),\ |\pi^1(t,x)|\le c\|U(t)\|_\infty[U(t)]^{(\varepsilon)}|x|^\varepsilon,\ t>0, \\ &t^{\frac12}\nabla\pi^2\in L^\infty(0,T;L^2(\Omega)),\ \textit{and, for all } r\in(1,2), \nabla\pi^2\in L^r((0,T)\times\Omega), \end{aligned} \tag{14}$$

where the field U is the solution to the Navier–Stokes Cauchy problem with initial data U_0, with $U_0=u_0$ on Ω and 0 on $\mathbb{R}^2-\Omega$ furnished by Theorem 3.

Let us consider the Neumann problem

$$\Delta\pi_u = -\nabla\cdot\big[(u\cdot\nabla)u\big] \text{ in } \Omega, \quad \frac{dp_u}{dn} = \operatorname{rot}\operatorname{rot} u\cdot n \text{ on } \partial\Omega. \tag{15}$$

We assume that

$$\text{for all } \lambda\in(0,1), \begin{cases} u\in C^{0,\lambda}(\mathbb{R}^2), & \text{with } u=0 \text{ in } \mathbb{R}^2-\Omega,\\ \nabla u\in C^{0,\lambda}(\overline{\Omega}), & \text{with } \nabla\cdot u=0 \text{ in } \mathbb{R}^2. \end{cases} \tag{16}$$

The following result holds for problem (15):

Theorem 5. *Assume that in problem (15) u satisfies (16). Then, there exist a solution to problem (15) such that $\pi_u\in C^2(\Omega)$, and, for $\lambda>\frac{1}{2}$, we get*

$$|\pi_u(x)| \le c\Big[|x|^{\varepsilon}\|u\|_\infty[u]^{(\varepsilon)} + \|(u\cdot\nabla)u\|_{1,\lambda} + \|u\|_\infty^2 + [\nabla u]^{(\lambda)}\Big]. \tag{17}$$

The solution π_u is unique up to a constant c.

The proof of Theorem 5 is given in Sect. 7.

Assuming that Theorem 2 - Theorem 5 hold, we can prove Theorem 1.

Proof. *of Theorem 1.* The proof of Theorem 1 consists in two different and independent steps. The former concerns the existence and uniqueness of (u,π_u) and its regularity in the sense claimed in (3) for all $t>0$. The latter is in the important pointwise estimates (4)–(7).

Proof of the first step. Let $u_0\in L^\infty(\Omega)$ with divergence free in the weak sense. By virtue of Theorem 2 we establish on some interval $(0,T_0)$ the existence of a field $\overline{u}(t,x)$ enjoying the regularity properties (9). Hence for all $t_0>0$ we get that $\overline{u}(t_0,x)\in BUC(\Omega)$ and $\nabla\overline{u}_0(t_0,x)\in C(\overline{\Omega})\cap L^\infty(\Omega)$. Since Theorem 4 **holds independently of the data size**, assuming as data $\overline{u}(t_0,x)$, we can perform the existence of (u,π_u) enjoying the properties (13)–(14) for all $t>t_0$. On the other hand the field u is a mild solution to the integral equation (10). Since in [1] it is proved that there exists a unique mild solution to (10), we get that the field $u(t,x)$ represents a global in time extension of the mild solution $\overline{u}(t,x)$. Since $t_0>0$ is arbitrary, we have proved global existence of the (u,π_u) for all $t>0$.

Proof of the second step. By virtue of (9) and (13), we realize (4) if we can make (13) independent of $\nabla u(t_0)$. Actually on $(0,T_0)$ we have estimate (9), for $t\ge T_0$ we have to prove that (13) is independent of $\nabla u(T_0)$. This is the case if we take into account that from estimate (9) we get $\|\nabla u(T_0)\|_\infty\le c\|u_0\|_\infty^2$. Hence for $t\ge T_0$ estimate (13) substitutes (4) where we have expressed the dependence on t by means of a suitable exponential function.[1] Now we prove (6)–(7). Since (u,π_u) is a regular solution to problem (1), we deduce that π_u satisfies problem (15) for all $t>0$. Hence by virtue of Theorem 5 we have

$$|\pi_u(t,x)| \le c\Big[|x|^{\varepsilon}\|u(t)\|_\infty[u(t)]^{(\varepsilon)} + \|(u(t)\cdot\nabla)u(t)\|_{1,\lambda} + \|u(t)\|_\infty^2 + [\nabla u(t)]^{(\lambda)}\Big],$$

which proves (6). Since $[u]^{(\varepsilon)}\le c\|u\|_\infty^{1-\varepsilon}\|\nabla u\|_\infty^{\varepsilon}$, then estimate (7) is an immediate consequence of the last one and of estimates (9). The theorem is completely proved. □

Remark 2. Theorem 4 also furnishes a structure theorem for the weak solution to (1) stated in Theorem 2.

3. The Cauchy Problem

The first step to prove Theorem 4 is a result of global existence of solutions to the Navier–Stokes Cauchy problem, that is Theorem 3. For the convenience of the reader, below we state again Theorem 6. Hence we claim that for the 2-D Navier–Stokes Cauchy problem the following result holds:

[1] Actually, in (4) we look for a qualitative estimate and not quantitative being in any case the grow of exponential kind.

Theorem 6. ([6,25]). *Assume that $U_0 \in L^\infty(\mathbb{R}^2)$ with divergence free in the weak sense. Then there exists unique solution $U \in C_w([0,\infty); L^\infty(\mathbb{R}^2)) \cap C^\infty((0,\infty)\times\mathbb{R}^2)$ of (1) enjoying the estimates*

$$\|U(t)\|_\infty \le c\|U_0\|_\infty(1+\|U_0\|_\infty^2 t),\ \forall t>0, \tag{18}$$

where c is independent of U_0. Up to a constant the pressure field is given by $\pi_U := \sum_{i,j}^2 R_iR_jU^iU^j$, where $R_i, i=1,2,$ is the Riesz operator. Moreover if $U_0 \in W^{1,\infty}(\mathbb{R}^2)$, then holds $U \in C([0,\infty); BUC(\mathbb{R}^2))$, and, for all $\nu \in (0,1)$ and $\eta_1 \in (0,1)$, we get

$$\begin{aligned}\|\omega(t)\|_\infty &\le \|\omega_0\|_\infty,\\ \big[U(t)\big]^{(\nu)}_{\mathbb{R}^2} &\le ct^{-\frac{\nu}{2}}\|U_0\|_\infty + G_0(t,U_0)(1+\|\omega_0\|_\infty)^\nu t^{\nu\eta_1},\end{aligned} \tag{19}$$

for all $t>0$.

Proof. The existence result is proved in [12]. Then, employing a local estimate of the L^2-norm of the solutions, furnished in [25] by Zelik, Gallay proves estimate (18) in [8] (p. 5 estimate (1.13)). Estimate $(19)_1$ is proved in [12]. Finally, estimate $(19)_2$ is a classical one provided that (18) and $(19)_1$ hold. □

We need some special estimates of the solutions in Theorem 6. These estimates concern the behavior in a neighborhood of $t=0$ and of infinity. To this end, we recall that the kinetic field U is the solution to mild equation

$$U(t) = e^{t\Delta}U_0 - \int_0^t e^{(t-\tau)\Delta}P\nabla\cdot(U\otimes U)(\tau)d\tau. \tag{20}$$

In (20) $e^{t\Delta}(\cdot)$ denotes the convolution between the heat kernel $H(x,s) = (4\pi t)^{-1}exp(-\frac{|x|^2}{4t})$ and the data, and P denotes the projector between $L^\infty(\mathbb{R}^2)$ onto the subspace of divergence free functions contained in $BMO(\mathbb{R}^2)$ whose i,j component is defined by $\delta_{ij}+R_iR_j$, $i,j=1,2$, where δ_{ij} is the Kronecker delta and R_i is the Riesz operator. As stated in [11] the operators $P, \nabla\cdot$ and $e^{t\Delta}$ commute, so that the integral equation (20) becomes

$$U(t) = e^{t\Delta}U_0 - \int_0^t \nabla\cdot e^{(t-\tau)\Delta}P(U\otimes U)(\tau)d\tau. \tag{21}$$

Moreover, the equation of the vorticity $\omega := \operatorname{rot} u$:

$$\omega_t - \Delta\omega = -U\cdot\nabla\omega,\ \text{in } (0,T)\times\mathbb{R}^2,\quad \omega=\omega_0 \text{ on } \{0\}\times\mathbb{R}^2, \tag{22}$$

is crucial for the estimate and the integral equation deduced from (21)

$$\nabla U = \nabla e^{t\Delta}U_0 + \int_0^t (\nabla E(t-\tau),(U\times\omega)(\tau)d\tau, \tag{23}$$

where, taking U and ω as three dimensional vectors ($U\equiv(U_1,U_2,0)$ and $\omega\equiv(0,0,\omega)$), we employed the formula $(U\cdot\nabla)U = \frac{1}{2}\nabla|U|^2 + U\times\omega$, and function E is the Oseen tensor with $(\nabla E(t-\tau),b) = \int\limits_{\mathbb{R}^2}\nabla_x E_{ij}(t-\tau,x-y)b_j(\tau,y)dy$.

For the sake of simplicity and of brevity we set

$$\begin{aligned}&Q(a,b) \textit{ is a function of } a \textit{ and } b \textit{ whose}\\ &\textit{expression is inessential to our goals.}\end{aligned} \tag{24}$$

The following result holds:[2]

[2] We set

$$\nabla^k := \overbrace{\nabla\cdots\nabla}^{k-times}.$$

Lemma 1. *Let (U,π_U) the solution given in Theorem 6 with $U_0\in W^{1,\infty}(\mathbb{R}^2)$. Then we have*

$$\begin{aligned}&\|U(t)\|_\infty\le c\|U_0\|_\infty+Q(U_0)t,\ t>0\\ \eta_0\in(0,1),\ &\|\nabla U(t)\|_\infty\le c\|\nabla U_0\|_\infty+Q(U_0,\omega_0)(1+t^{1+\eta_0}),\ t>0\\ &\|\nabla^2U(t)\|_\infty\le Q(U_0,\nabla U_0)(t^{-\frac12}+1+t^{\frac43+\eta_0}),\ t>0\\ &\|\nabla^4U(t)\|_\infty\le c\|\nabla U_0\|_\infty t^{-\frac32}+Q(U_0,\omega_0)(1+t^{2+\eta_0}),\ t>0.\end{aligned}\tag{25}$$

Assume that $\|\nabla U_0\|_\infty\le c\|U_0\|_\infty^2$, then the estimates (25) hold with a function $Q(U_0)$.

Proof. Estimates (25) are a suitable modification of the classical ones given with an initial $u_0\in L^\infty$. The modification is due to the fact that we employ estimates (18) and (19), as well as we combine the equation of the vorticity (22) and integral equation (23). Actually in order to give estimates ∇^iU, $i=2,3,4$, we use (23) whose singularity in E is controlled by Hölder's seminorm of $\nabla^k(U\times\omega)$ that gets involved the ones of $\nabla\omega$. The latter are deduced employing the equation of vorticity. We omit the technical details and in Appendix we give an idea of the lines of argument. □

For the sake of simplicity and brevity we denote by

$$\begin{aligned}G_0(t,U_0):&=\text{the right hand side of }(25)_1,\\ G_1(t,U_0,\omega_0):&=\text{the right hand side of }(25)_2,\\ G_2(t,U_0,\nabla U_0):&=\text{the right hand side of }(25)_3,\\ G_4(t,U_0,\omega_0):&=\text{the right hand side of }(25)_4.\end{aligned}\tag{26}$$

Moreover, in order to simplify the notations, if there is no confusion, in some estimates G_i, $i=0,1,4$, is meant up to multiplicatives constants c which are independent of U_0 and t, that is in some computations we set $cG_i\equiv G_i$.

Let us consider the Poisson equation

$$\Delta\pi=-\nabla\cdot((u\cdot\nabla)u)\quad\text{ in }\mathbb{R}^2.\tag{27}$$

In order to establish some properties of the pressure field of the solution (u,π_u) given in Theorem 6, we have to study Eq. (27). This is made by means of the following Lemmas 3 and 5. The former lemma is proved in [16]. The results of the former lemma are close to the results contained in [18] for the Cauchy problem in the case of $n\ge3$. Since we need more precise estimates related to the pressure field, we reproduce the proof of the Lemma 3 for the sake of completeness. We premise a classical result of potential theory:

Lemma 2. *Let $K(z)\in C^1(\mathbb{R}^n-\{0\})$, homogeneous function of $(1-n)$-order, and $g(x)\in C^{0,\mu}(\mathbb{R}^n)$, $\mu\in(0,1)$, with compact support. Then, the transformation $\frac{\partial}{\partial x_h}T(g)\in C^{0,\mu}(\mathbb{R}^n)$ with*

$$\begin{aligned}\frac{\partial}{\partial x_h}T(g)=k_hg(x)-\int^*_{\mathbb{R}^n}\frac{\partial}{\partial y_h}K(x-y)g(y)dy,\\ [\nabla T(g)]^{(\mu)}\le c[g]^{(\mu)},\end{aligned}\tag{28}$$

where the constant c depends only on the Euclidean dimension n. Here $k_h:=\int\limits_{|\xi|=1}K(\xi)\xi_hd\sigma$, and the symbol $\int^*$ denotes p.v. $\int$.

Proof. See in [16] Ch. 3 Lemma 2.2. □

Lemma 3. *Let $u\in C^{3,\lambda}(\mathbb{R}^2)$. Then there exists a unique solution to problem (27) such that*

$$\begin{aligned}&\textit{for all }\varepsilon\in(0,\lambda],\ |\pi(x)|\le c|x|^\varepsilon\|u\|_\infty[u]^{(\varepsilon)},\\ &\textit{for all }\lambda'\in(0,\lambda),\ [\nabla^i\pi]^{(\lambda')}\le c[\nabla^{i-1}(u\cdot\nabla)u]^{(\lambda)},\ \textit{for }i=1,2,3,\end{aligned}\tag{29}$$

and

$$\begin{aligned}&\|\nabla^i\pi\|_\infty \le c\Big[\|\nabla^{i-\imath(i)}u\|_\infty[\nabla^{\imath(i)}u]^{(\lambda)}+[\nabla^{i-\imath(i)}u]^{(\lambda)}\|\nabla^{\imath(i)}u\|_\infty+\|u\|^2_\infty\Big],\\ &\qquad\qquad\qquad\qquad \textit{for } i=1,3 \textit{ and } \imath(1)=1,\imath(3)=2,\\ &\|\nabla^2\pi\|_\infty\le c\|\nabla u\|_\infty[\nabla u]^{(\lambda)},\end{aligned}\tag{30}$$

where c is independent of u.

Proof. The uniqueness of π up to a constant in the class of existence is a classical result. Since $(29)_1$ implies that $\pi(0)=0$, then the solution π is fixed. We define the sequence of problems

$$\Delta\pi_k=-\nabla\cdot((u\cdot\nabla)h_ku))\quad \text{in } \mathbb{R}^2,\tag{31}$$

where $\{h_k\}$ is a sequence of smooth cut-off functions. It is well known that

$$\widetilde{\pi}_k(x):=\nabla\int\limits_{\mathbb{R}^2}\nabla_y\mathscr{E}(x-y)\cdot(u\otimes h_ku)(y)dy,\tag{32}$$

where $\mathscr{E}(z)$ is the fundamental solution of 2-D Laplacian operator. By virtue of Lemma 2 we obtain for $i=1,2,3$,

$$\begin{aligned}&\text{for all } \varepsilon\in(0,\lambda],\ \ [\widetilde{\pi}_k]^{(\varepsilon)}\le c[u\otimes h_ku]^{(\varepsilon)}\le c\|u\|_\infty[u]^{(\varepsilon)}+o(k),\\ &[\nabla^i\widetilde{\pi}_k]^{(\lambda)}\le c[\nabla^{i-1}(u\cdot\nabla)(h_ku)]^{(\lambda)}\le c[\nabla^{i-1}(u\cdot\nabla)u]^{(\lambda)}+o(k)\quad \text{for } i=1,2,3,\end{aligned}\tag{33}$$

where c is independent of $k\in\mathbb{N}$ and u. From the representation formula (32) we also obtain

$$\nabla\widetilde{\pi}_k(x)=\nabla_x\int\limits_{\mathbb{R}^2}\nabla_z\mathscr{E}(z)\cdot(u\cdot\nabla)(h_ku)(x-z)dz.$$

By virtue of the properties of the kernel $\nabla\nabla E(z)$ and via an integration by parts, we easily deduce

$$\begin{aligned}|\nabla\widetilde{\pi}_k(x)|&=|\nabla\int\limits_{B(0,1)}\nabla_z\mathscr{E}(z)(u\cdot\nabla)(h_ku)(x-z)dz+\nabla\int\limits_{\mathbb{R}^2-B(0,1)}\nabla_z\mathscr{E}(z)(u\cdot\nabla)(h_ku)(x-z)dz|\\ &=:|I_1(x)+I_2(x)|.\end{aligned}$$

Hence we get

$$|I_1(x)|\le c\|(u\cdot\nabla)(h_ku)\|_\infty+[(u\cdot\nabla)(h_ku)]^{(\lambda)}+o(k),$$

and

$$\begin{aligned}|I_2(x)|&\le|\int\limits_{\mathbb{R}^2-B(0,1)}\nabla_z\nabla_z\nabla_z\mathscr{E}(z)\cdot u\otimes h_ku(x-z)dz|+|\int\limits_{|z|=1}\nabla_z\nabla_z\mathscr{E}(z)\cdot h_ku(x-z)u\cdot n d\sigma_z|\\ &\le c\|u\|^2_\infty.\end{aligned}$$

We have proved that

$$\|\nabla\widetilde{\pi}_k\|_\infty\le c\Big[\|u\|_\infty[\nabla u]^{(\lambda)}+[u]^{(\lambda)}\|\nabla u\|_\infty+\|u\|^2_\infty\Big]+o(k)\ ,\quad k\in\mathbb{N},\tag{34}$$

where c is independent of k and u. Analogously for the second and third derivatives one proves[3]

$$\begin{aligned}&\|\nabla^2\widetilde{\pi}_k\|_\infty\le c\|\nabla u\|_\infty[\nabla u]^{(\lambda)}+o(k),\\ &\|\nabla^3\widetilde{\pi}_k\|_\infty\le c\Big[\|\nabla u\|_\infty[D^2u]^{(\lambda)}+[\nabla u]^{(\lambda)}\|D^2u\|_\infty\Big]+o(k),\end{aligned}\tag{35}$$

[3] Here we can consider

$$\widetilde{\pi}_k=\int\limits_{\mathbb{R}^2}\mathscr{E}(x-y)\nabla u(y)\cdot(\nabla(h_ku(y))^T)dy.$$

This is consequence of divergence free of u. Hence, for all terms that does not contain derivatives of h_k, we have a simplified expression for ∇^2 and for ∇^3 of $\widetilde{\pi}$.

where c is independent of k and u. We define the function $\pi_k(x)$ by means of the line integral of $\nabla\widetilde{\pi}_k$, with end points x and O. It is well known that

$$\nabla\widetilde{\pi}_k = \nabla\pi_k, \quad [\widetilde{\pi}_k]^{(\lambda)} = [\pi_k]^{(\lambda)}.$$

Hence, for $i = 1, 2, 3$, and for all $\varepsilon > 0$ we get for $\{\pi^k\}$ estimates (33)–(35) uniformly in $k \in \mathbb{N}$. Since $\pi_k(O) = 0$ in particular it follows that

$$|\pi_k(x)| \le [\pi_k]^{(\lambda)}|x|^\lambda, \text{ for all } x \in \mathbb{R}^2 \text{ and } k \in \mathbb{N}.$$

Considering $\mathbb{R}^2 = \cup_{j\in\mathbb{N}} B_j$, with $\{B_j\}$ sequence of balls of radius j such that $\overline{B}_j \subset B_{j+1}$, we have that, for all $j \in \mathbb{N}$, $\{\pi_k\} \subset C^{i,\lambda}(B_j)$ is an equi-continuous and equi-bounded sequence of functions, so that it is relatively compact in $C^{i,\lambda'}(B_j)$, where $\lambda' \in (0, \lambda)$ is independent of B_j. Then, by means of a classical diagonal procedure, we deduce the convergence of a subsequence on the whole space $\mathbb{R}^2$ to a limit π and the limit satisfies estimates (29)–(30). □

Lemma 4. *Assume that the hypotheses of Lemma 3. Let $u(t, x)$ be a one parameter family such that $u_t \in C^{1,\lambda}(\delta, T; C^1(\Omega))$. Then there exists π_t with the estimates*

$$\begin{aligned} &\text{for all } \varepsilon \in (0,1) \;\; |\pi_t(t,x)| \le c|x|^\varepsilon \Big[[u]^{(\varepsilon)}\|u_t\|_\infty + \|u\|_\infty[u_t]^{(\varepsilon)}\Big] \;\; \text{for all } t \in (\delta, T),\\ &\|\nabla\pi_t\|_\infty \le c\sum_{i=0}^{1}\Big[\|D_t^i u\|_\infty[\nabla D_t^{1-i}u]^{(\lambda)} + [D_t^i u]^{(\lambda)}\|\nabla D_t^{1-i}u\|_\infty\Big],\end{aligned} \tag{36}$$

where c is independent of t and u.

Proof. The proof follows the line arguments of the above lemma, hence it is omitted. □

Lemma 5. *Let (U, π_U) be the solution furnished by Theorem 6. Then, the pressure field $\pi_U = R_iR_jU_iU_j$ coincides up to a constant with the solution π of Lemma 3 provided that the data of Poisson equation (27) are the kinetic field u.*

Proof. See Lemma 3.1 of [19]. □

Corollary 1. *Let (U, π_U) be the solution furnished by Theorem 6. Then, for the pressure field we have the estimate, for all $\varepsilon \in (0, 1)$,*

$$|\pi_U(x)| \le ct^{-\frac{\varepsilon}{2}}|x|^\varepsilon\big[1 + ct^{\frac12}(1 + \|U_0\|^2_\infty t)\|\omega_0\|_\infty\big]\|U_0\|^2_\infty(1 + \|U_0\|_\infty t). \tag{37}$$

If we assume $U_0 \in W^{1,\infty}(\mathbb{R}^2)$, via Lemma 1, we get

$$\begin{aligned}\|\nabla^2\pi\|_\infty &\le c\big(G_1(t, U_0, \omega_0)\big)^{2-\frac{\lambda}{3}}\big(G_4(t, U_0, \omega_0)\big)^{\frac{\lambda}{3}},\\ \|\nabla^3\pi\|_\infty &\le c\big(G_1(t, U_0, \omega_0)\big)^{\frac53-\frac{\lambda}{3}}\big(G_4(t, U_0, \omega_0)\big)^{\frac13+\frac{\lambda}{3}}.\end{aligned} \tag{38}$$

In estimates (37)–(38) the constant c is independent of u and t.

Proof. From (29) of Lemma 3 via estimates (18) and $(19)_2$ we get (37). Instead $(38)_{1,2}$ are a consequence of (30) and estimates $(25)_{2,4}$. Of course, estimating the above quantities is tacitly assumed to be the interpolation of all the terms between the first derivative and the fourth derivatives of u. □

Lemma 6. *Let (U, π_U) be the solution of Theorem 6. Assume that $U_0 \in W^{1,\infty}(\mathbb{R}^2)$. Then, for all $\lambda \in (0, 1)$ the following estimates hold:*

$$\begin{aligned}\|U_t(t)\|_\infty &\le G_2 + G_0^{1-\lambda}G_1^{1-\frac{\lambda}{3}}\big[G_0^\lambda(G_1^{\frac{\lambda}{3}} + G_4^{\frac{\lambda}{3}}) + G_1^{\frac43\lambda}\big],\\ \|\nabla U_t(t)\|_\infty &\le G_1^{\frac13}\big[G_4^{\frac23} + G_1^{\frac53} + G_0G_1^{\frac13}G_4^{\frac13} + G_1^{\frac53-\frac{\lambda}{3}}G_4^{\frac{\lambda}{3}}\big],\\ \|\nabla^2U_t(t)\|_\infty &\le G_4 + G_1^{\frac13}\big[G_0G_4^{\frac23} + G_1^{\frac43}G_4^{\frac13} + G_1^{\frac43-\frac{\lambda}{3}}G_4^{\frac13+\frac{\lambda}{3}}\big].\end{aligned} \tag{39}$$

We recall that $G_0 = G_0(t, U_0)$, $G_1 = G_1(t, U_0, \omega_0)$, $G_4 = G_4(t, U_0, \omega_0)$.

Proof. We note that from the Eq. $(1)_1$ proceeds the following identity:

$$\begin{aligned}\|U_t(t)\|_\infty &\le \|\Delta U(t)\|_\infty + \|U(t)\|_\infty \|\nabla U(t)\|_\infty + \|\nabla \pi_U(t)\|_\infty \ , t>0,\\ \|\nabla U_t\|_\infty &\le \|\nabla^3 U(t)\|_\infty + \|\nabla U\|^2_\infty + \|U\|_\infty \|\nabla^2 U\|_\infty + \|\nabla^2 \pi_U\|_\infty,\ t>0,\end{aligned} \tag{40}$$

and

$$\|\nabla^2 U_t\|_\infty \le \|\nabla^4 U(t)\|_\infty + \|U\|_\infty \|\nabla^3 U\|_\infty + 3\|\nabla U\|_\infty \|\nabla^2 U\|_\infty + \|\nabla^3 \pi_U\|_\infty,\ t>0.$$

Hence via estimates $(25)_{1,2,3}$ we can estimate the terms in U on the right hand side of (40). Then, employing again $(25)_{1,2,3}$ we estimate Hölder's seminorm of $(30)_1$, evaluated for $i=1$. Therefore one arrives at $(39)_1$. Estimating $(40)_2$ we employ the same arguments. Hence for the terms in U we employ $(25)_{1,2,3,4}$, for the term $\nabla \pi_U$ via estimate $(38)_1$ one arrives at $(39)_2$ for ∇U_t. Finally, via estimates $(25)_{1,2,3}$ and $(38)_2$, one arrives at $(39)_3$ for $\nabla^2 U_t$. The lemma is completely proved. □

For the sake of simplicity and brevity we denote by

- $D_1(t, U_0, \omega_0)$ the right hand side of $(39)_1$,
- $D_2(t, U_0, \omega_0)$ the right hand side of $(39)_2$,
- $D_3(t, U_0, \omega_0)$ the right hand side of $(39)_3$.

From the definition of G_i, $i = 0, 1, 2, 4$, by simple computation we determine that for $\eta \in (0, 1)$

$$\begin{aligned} D_1 &\le Q(U_0, \nabla U_0)(t^{-\frac12} + 1 + t^{2+\eta}),\\ D_2 &\le Q(U_0, \nabla U_0)(t^{-1} + 1 + t^{\frac73+\eta}),\\ D_3 &\le Q(U_0, \nabla U_0)(t^{-\frac32} + 1 + t^{\frac83+\eta}).\end{aligned} \tag{41}$$

Lemma 7. *Let (u, π_u) be the solution furnished by Theorem 6. Then for $\eta \in (0, \frac12)$ we get*

$$\begin{aligned} \|\nabla \pi_{U\,t}\|_\infty &\le D_4(t, U_0, \omega_0),\\ \|u_{tt}\|_\infty &\le D_5(t, U_0, \omega_0),\end{aligned} \tag{42}$$

where $D_4 \le Q(U_0, \omega_0)(t^{-1-\eta} + 1 + t^{3+\eta})$ and $D_5 \le Q(U_0, \omega_0)(t^{-\frac32} + 1 + t^{3+\eta})$.

Proof. From Lemma 4 in particular we obtain

$$\|\nabla \pi_{U\,t}\|_\infty \le c \sum_{i=0}^{1} \Big[\|D_t^i U\|_\infty [\nabla D_t^{1-i} U]^{(\lambda)} + [D_t^i U]^{(\lambda)} \|\nabla D_t^{1-i} U\|_\infty \Big].$$

By virtue of estimate (41), interpolating the seminorm $[\nabla D_t u]^{(\lambda)}$ between $\nabla^2 u_t$ and ∇u_t, we arrive at $(42)_1$. Estimate $(42)_2$ is deduced by derivating $(1)_1$ and then applying the above estimate related to $\nabla \pi_{U\,t}$ and estimate (41) for U_t. □

4. Auxiliary Results

4.1. A Suitable Extension of a Field Given on $\partial\Omega$

Let us consider the Stokes problem:

$$\begin{cases} \Delta \widetilde V - \nabla \pi_{\widetilde V} = 0 & x \in B_R \cap \Omega,\\ \nabla \cdot \widetilde V = 0 & x \in B_R \cap \Omega,\\ \widetilde V|_{\partial\Omega} = a|_{\partial\Omega} & \widetilde V|_{|x|=R} = 0. \end{cases} \tag{43}$$

We have

Lemma 8. *Let $q \in (1, \infty)$ and assume $a \in W^{k-\frac{1}{q},q}(\partial\Omega)$, $k = 1, 2$. Then problem (43) admits a unique solution $\widetilde{V} \in W^{k,q}(\Omega)$ with the estimate*

$$\|\widetilde{V}\|_{k,q} \le c\|a\|_{k-\frac{1}{q},q}, \tag{44}$$

where c is independent of a. Moreover, if $a(t, x)$ is a one parameter family such that, for all $\delta > 0$, $a_t(t, x) \in C(\delta, T; W^{k-\frac{1}{q},q}(\partial\Omega))$, then there exists V_t with the estimate

$$\|\widetilde{V}_t(t)\|_{k,q} \le c\|a_t(t)\|_{k-\frac{1}{q},q},\ t \in (\delta, T), \tag{45}$$

where c is independent of t. Finally, if $a \in C(\partial\Omega)$ (resp. $a_t \in C(\partial\Omega)$ $t \in (\delta, T)$), then the maximum modulus estimate holds:

$$\|\widetilde{V}\|_\infty \le c\|a\|_{C(\partial\Omega)}, \quad (\textit{resp. } \|\widetilde{V}_t(t)\|_\infty \le c\|a_t(t)\|_{C(\partial\Omega)}, t \in (\delta, T)). \tag{46}$$

Proof. The results are classical. For estimates (44) see e.g [5] or [9]. For estimate (46) see [17,20]. □

Let h_R be a smooth cut-off function in $B_R \cap \Omega$. The function $\widetilde{V} h_R$ has a compact support in $\overline{\Omega}$ and

$$\gamma(\widetilde{V} h_R) = a \quad \text{on } \partial\Omega,$$
$$\nabla \cdot (\widetilde{V} h_R) = \widetilde{V} \cdot \nabla h_R \quad \text{in } D_R = \{x \in \mathbb{R}^2; \frac{R}{3} < |x| < \frac{2}{3}R\},$$

where γ is the trace operator onto $\partial\Omega$. Let

$$\widetilde{W}^{m,r}(D_R) = \begin{cases} L^r(D_R), & m = 0,\\ \{u \in W^{m,r}(D_R);\ \partial_x^\alpha u|_{\partial D_R} = 0,\ |\alpha| \le m-1\}, & m \ge 1, \end{cases}$$
$$\widetilde{W}^{m,r}_M(D_R) = \{u \in W^{m,r}(D_R); \int_{D_R} u\, dx = 0\}.$$

There exists the Bogosvkiĭ operator

$$\mathbb{B} : \widetilde{W}^{m,r}_M(D_R) \to \widetilde{W}^{m+1,r}(D_R)$$

such that the following properties hold:

$$\begin{aligned} &\nabla \cdot \mathbb{B}[u] = u \quad \text{for } u \in \widetilde{W}^{m,r}_M(D_R),\\ &\|\mathbb{B}[u]\|_{W^{k+1,r}(D_R)} \le C\|u\|_{W^{k,r}(D_R)}, \quad k = 0, 1, \ldots, m. \end{aligned} \tag{47}$$

(cf. [3,4], and also [9]).

Lemma 9. *Let $q \in (1, \infty)$ and assume $a \in W^{k-\frac{1}{q},q}(\partial\Omega) \cap C(\partial\Omega)$, $k = 1, 2$. Then there exists a field $V \in W^{k,q}(\Omega)$ with compact support in $\overline{\Omega}$ such that $\gamma(V) = a$ and*

$$\begin{aligned} &\|V\|_\infty && \le c\|a\|_{C(\partial\Omega)},\\ &\|V\|_{1,q} && \le c\|a\|_{1-\frac{1}{q},q},\\ &\|V\|_{2,q} && \le c\|a\|_{2-\frac{1}{q},q},\\ &\|\nabla V\|_\infty && \le c\big[\|a\|^\alpha_{2-\frac{1}{q},q}\|a\|^{1-\alpha}_\infty + \|a\|_\infty\big],\ \textit{with } q > 2 \textit{ and } \alpha = \tfrac{2+q}{2q}, \end{aligned} \tag{48}$$

where c is independent of a. Moreover, if $a(t, x)$ is a one parameter family such that $a_t(t, x) \in C(\delta, T; W^{k-\frac{1}{q},q}(\partial\Omega) \cap C(\partial\Omega))$, then there exists V_t and it enjoys estimates (48) with a_t in place of a, and c is independent of t.

Proof. Let h_R be a smooth cut-off function in $B_R \cap \Omega$. Setting $V := \widetilde{V} h_R - \mathbb{B}[\widetilde{V} \cdot \nabla h_R]$, the existence is an easy consequence of Lemma 8 and properties of Bogosvkiĭ operator. Estimate $(48)_1$ is obtained by employing estimate $(46)_1$ (resp. $(46)_2$) for the term $\widetilde{V} h_R$ (resp. for the term $\widetilde{V}_t h_R$) and, via Sobolev embedding, estimate $(47)_2$, for $k = 0$ and $r > 2$, for the L^∞-norm of the term $\mathbb{B}[\widetilde{V} \cdot \nabla h_R]$. □

4.2. Stokes Problem

Let us consider the Stokes initial boundary value problem in exterior domains:

$$\begin{aligned} &v_t - \Delta v = -\nabla \pi_v + f,\ \nabla \cdot v = 0 \text{ in } (0,T) \times \Omega,\\ &v = 0 \text{ on } (0,T) \times \partial\Omega \text{ and } v = v_0 \text{ on } \{0\} \times \partial\Omega. \end{aligned} \tag{49}$$

We recall some well known results.

Theorem 7. *Assume that $f \in L^q(0,T;L^q(\Omega))$ and $v_0 \in J^{2-\frac{2}{q},q}(\Omega) \cap J^{1,2}(\Omega)$, $q \in (1,\infty)$. Then there exists a unique solution (v,π_v) to problem (49) such that*

$$\begin{aligned} &v \in C([0,T);J^q(\Omega)) \cap L^q(0,T;J^{1,q}(\Omega)),\\ &\int_0^T \big[\|v_t\|_q^q + \|D^2 v\|_q^q + \|\nabla \pi_v\|_q^q\big] d\tau \le c(T)\left[\int_0^T \|f\|_q^q d\tau + \|v_0\|^q_{2-\frac{2}{q},q}\right], \end{aligned} \tag{50}$$

where, for all $\varepsilon > 0$, $c(T) := C(1+T^{1+\varepsilon q - \frac{n}{2q}})$, and C is independent of f and v_0.

Proof. The assumption $v_0 \in J^{2-\frac{2}{q},q}(\Omega) \cap J^{1,2}(\Omega)$ makes the theorem a special case of the ones proved (e.g.) in [13,22]. □

Theorem 8. *Let $v_0 \in J^p(\Omega)$, for some $p \in (1,\infty)$, and $f = 0$ in (49). Then, there exists a unique solution (v,π_v) to problem (49) such that*

$$\begin{aligned} &\eta \in (0,T),\ v \in C([0,T);J^p(\Omega)) \cap L^p(\eta,T;J^{1,p}(\Omega) \cap W^{2,p}(\Omega)),\\ &\nabla \pi_v, v_t \in L^p(\eta,T;L^p(\Omega)). \end{aligned} \tag{51}$$

Moreover, for $q \in [p,\infty]$, it holds that

$$\begin{aligned} &\|v(t)\|_q \le c\|v_0\|_p t^{-\mu}, \quad \mu = \tfrac{n}{2}\big(\tfrac{1}{p} - \tfrac{1}{q}\big),\ t > 0;\\ &\|\nabla v(t)\|_q \le c\|v_0\|_p t^{-\mu_1},\ \mu_1 = \begin{cases} \frac{1}{2} + \mu \ \textit{if}\ t \in (0,1],\\ \frac{1}{2} + \mu \ \textit{if}\ t > 0 \textit{ and } q \in [p,n],\\ \frac{n}{2p} \quad\ \textit{if}\ t > 1 \textit{ and } q \ge n; \end{cases}\\ &\|v_t(t)\|_q \le c\|v_0\|_p t^{-\mu_2},\ \ \mu_2 = 1 + \mu,\ t > 0; \end{aligned} \tag{52}$$

where the constant c is independent of v_0 and the exponent μ_1 is sharp.

Proof. See [6,13,22]. □

4.3. Some Inequalities

We recall some inequalities:

Lemma 10. *Let Ω be an exterior domain with the cone property. Let $m \in \mathbb{N}$ and $q, r \in [1,\infty)$. Let $u \in L^q(\Omega)$ and, for $|\alpha| = m$, $D^\alpha u \in L^r(\Omega)$. Then there exists a constant c independent of u such that*

$$\|D^\beta u\|_p \le c\|D^\alpha u\|_r^a \|u\|_q^{1-a}, \tag{53}$$

provided that for $|\beta| = j$ the following relation holds:

$$\frac{1}{p} = \frac{j}{n} + a\left(\frac{1}{r} - \frac{m}{n}\right) + (1-a)\frac{1}{q},$$

with $a \in [\frac{j}{m},1]$ either if $p = 1$ or if $p > 1$ and $m - j - \frac{n}{p} \notin \mathbb{N} \cup \{0\}$, while $a \in [\frac{j}{m},1)$ if $p > 1$ and $m - j - \frac{n}{p} \in \mathbb{N} \cup \{0\}$. If Ω is bounded then (53) holds in the following form

$$\|D^\beta u\|_p \le c\big[\|D^\alpha u\|_r^a \|u\|_q^{1-a} + \|u\|_q\big]. \tag{54}$$

In the case of the exterior domain the above lemma, proved in [7], gives an interpolation inequality of Gagliardo–Nirenberg's type. The difference with respect to the usual result is the fact that the function u does not belong to a completion space of $C_0^\infty(\Omega)$. This is interesting for our tasks.

Lemma 11. *Let $q \in (1,\infty)$. Assume that $D^2u \in L^q(\Omega)$ and $u \in J^{1,q}(\Omega)$. Then there exists a pressure field π_u such that*

$$\|D^2u\|_q + \|\nabla\pi_u\|_q + \|u\|_{W^{1,q}(\Omega')} \le c(\|P_q\Delta u\|_q + \|u\|_{L^q(\Omega')}), \tag{55}$$

where Ω' is a bounded domain such that $\Omega' \subset \Omega$ with $\partial\Omega \cap \partial(\Omega - \Omega') = \emptyset$ and c is independent of u.

Proof. For the proof see for example [9] or [14,21]. □

5. The Solution for a Special Initial Boundary Value Problem

5.1. A Special Initial Boundary Value Problem

We consider the following initial boundary value problem:

$$\begin{aligned}
&w_t - \Delta w + w\cdot\nabla w + (U+V)\cdot\nabla w + w\cdot\nabla(U+V) + V\cdot\nabla(U+V) + U\cdot\nabla V\\
&\quad +\nabla\pi_w = -V_t + \Delta V \text{ in } (0,T)\times\Omega,\\
&\nabla\cdot w = 0 \text{ in } (0,T)\times\Omega,\ w = 0 \text{ on } (0,T)\times\partial\Omega,\ w = 0 \text{ on } \{0\}\times\Omega.
\end{aligned}\tag{56}$$

Assumption 1. *i. Here U denotes the kinetic field of the pair (U,π_U) solution to the Cauchy problem (1) with an initial data $U_0 := \begin{cases} u_0 \text{ if } x \in \overline{\Omega} \\ 0 \text{ if } x \in \mathbb{R}^2 - \overline{\Omega}\end{cases}$, with $u_0 \in W_0^{1,\infty}(\Omega)$. We have $U_0 \in BUC(\mathbb{R}^2)$ and $U_0 \in W^{1,\infty}(\mathbb{R}^2)$.*

ii. The term V is the extension given in Sect. 3 with the data $a \equiv -U_{|\partial\Omega}$.

iii. In (56) we set

$$F = -V_t + \Delta V - (V\cdot\nabla)(U+V) - (U\cdot\nabla)V. \tag{57}$$

Concerning Assumption 1 we recall:

1. The existence and related estimates for the pair (U,π_U) were discussed in Sect. 2.
2. Setting $a = -U$ in Lemma 9 we increase the trace norm as follows

$$\|D_t^k a\|_{2-\frac{1}{q},q} \le c\|D_t^k U\|_{W^{2,q}(\Omega^c)}, \text{ for all } t > 0.$$

Hence from (48), for $k = 0,1,2$, and $q \in (1,\infty)$ we deduce

$$\begin{aligned}
&\|D_t^k V(t)\|_\infty \le c\|D_t^k U(t)\|_{L^\infty(\partial\Omega)}, \text{ for all } t > 0,\\
&\|D_t^k V(t)\|_{1,q} \le c\|D_t^k U(t)\|_{W^{1,q}(\Omega^c)}, \text{ for all } t > 0,\\
&\|D_t^k V(t)\|_{2,q} \le c\|D_t^k U(t)\|_{W^{2,q}(\Omega^c)}, \text{ for all } t > 0.
\end{aligned}\tag{58}$$

Finally, employing the Gagliardo–Nirenberg inequality and (58), for $q > 2$, we get

$$\begin{aligned}
\|\nabla D_t^k V\|_\infty &\le c\big[\|D_t^k V\|_{2,q}^\alpha \|D_t^k V\|_q^{1-\alpha} + \|D_t^k V\|_q\big]\\
&\le c\big[\|D_t^k U\|_{W^{2,q}(\Omega^c)}^\alpha \|D_t^k U\|_\infty^{1-\alpha} + \|D_t^k U\|_\infty\big]\\
&\le c\big[\|\nabla^2 D_t^k U\|_\infty^\alpha \|D_t^k U\|_\infty^{1-\alpha} + \|D_t^k U\|_\infty\big], \text{ with } \alpha = \frac{2+q}{2q}.
\end{aligned}\tag{59}$$

In (58)–(59) the constant c is independent of U and of t.

3. Since Theorem 6 ensures that $U \in C([0,T);C(\mathbb{R}^2))$ and $u_0 = 0$ on $\partial\Omega$, we get

$$\lim_{t\to 0}\|U(t)\|_{C(\partial\Omega)} = 0.$$

Hence, via $(58)_1$ for $k = 0$ we also deduce

$$\lim_{t\to 0}\|V(t)\|_{C(\overline{\Omega})} = 0. \tag{60}$$

Recalling setting (24) we state

Lemma 12. *Let $q \geq 2$ and F be given by* (57)*. Then for $\eta \in (0,1)$ we get*

$$\|F(t)\|_q \leq \mathbb{F}(t, U_0) \quad \textit{and} \quad \|F_t(t)\|_q \leq \mathbb{F}_t(t, U_0), \tag{61}$$

where we have set $\mathbb{F}(t, U_0) := cQ(U_0, \nabla U_0)(t^{-\frac{1}{2}} + 1 + t^{2+\eta})$ *and* $\mathbb{F}_t(t, U_0) := cQ(U_0, \nabla U_0)(t^{-\frac{3}{2}} + 1 + t^{3+\eta})$

Proof. We limit ourselves to prove the estimate for $\|F\|_q$. The estimate for $\|F_t\|_q$ follows the same arguments. Applying Minkowski's inequality we deduce

$$\|F\|_q \leq \|V_t\|_q + \|\Delta V\|_q + \|V \cdot \nabla U\|_q + \|V \cdot \nabla V\|_q + \|U \cdot \nabla V\|_q. \tag{62}$$

We estimate $\|V_t\|_q$ and $\|\Delta V\|_q$. From (58) by the support compactness of V, V_t, we have

$$\|V_t\|_q + \|\Delta V\|_q \leq c\|V_t\|_\infty + \|\Delta V\|_q \leq c(\|U_t\|_{L^\infty(\partial\Omega)} + \|U\|_{W^{2,q}(\Omega^c)}).$$

Recalling (26), via estimate $(41)_1$ we estimate $\|U_t\|_{L^\infty(\partial\Omega)}$. Via estimates (25) by interpolation we estimate $\|U\|_{W^{2,q}(\Omega^c)}$. Hence we get

$$\|V_t\|_q + \|\Delta V\|_q \leq cQ(U_0, \omega_0)(t^{-\frac{1}{2}} + 1 + t^{2+\eta}).$$

Since V has a compact support, applying Hölder's inequality, we get

$$\|V \cdot \nabla U\|_q + \|V \cdot \nabla V\|_q + \|U \cdot \nabla V\|_q \leq c\|V\|_\infty \big[\|\nabla U\|_{L^q(\operatorname{supp} V)} + \|\nabla V\|_q\big] + \|U\|_\infty \|\nabla V\|_q. \tag{63}$$

By virtue of (48), we deduce that

$$\|V\|_\infty \leq c\|U\|_\infty \text{ and } \|\nabla V\|_q \leq c\|U\|_{1,\infty}.$$

Thus we get

$$\|V\|_\infty \big[\|\nabla U\|_{L^q(\operatorname{supp} V)} + \|\nabla V\|_q\big] + \|U\|_\infty \|\nabla V\|_q \leq c\|U\|_\infty \|U\|_{1,\infty}.$$

Hence, recalling estimates (18) and $(25)_2$, *a fortiori* we obtain

$$\|V\|_\infty \big[\|\nabla U\|_{L^q(\operatorname{supp} V)} + \|\nabla V\|_q\big] + \|U\|_\infty \|\nabla V\|_q \leq cQ(U_0, \nabla U_0)(t^{-\frac{1}{2}} + 1 + t^{2+\eta}).$$

In the case of $\|F_t\|_q$ we have

$$F_t = -V_{tt} + \Delta V_t - (V_t \cdot \nabla)(U + V) - (V \cdot \nabla)(U_t + V_t) - (U_t \cdot \nabla)V - (U \cdot \nabla)V_t.$$

Following the same arguments employed estimating (62), via estimates (58), then estimates (41) and (42), we get

$$\|V_{tt}\|_q + \|\Delta V_t\|_q \leq cQ(U_0, \nabla U_0)(t^{-\frac{3}{2}} + 1 + t^{3+\eta}).$$

As well as, since V, V_t have compact support we obtain

$$\begin{aligned} &\|(V_t \cdot \nabla)(U + V) - (V \cdot \nabla)(U_t + V_t) - (U_t \cdot \nabla)V - (U \cdot \nabla)V_t\|_q \\ &\quad \leq \|V_t\|_\infty \|\nabla(U + V)\|_{L^q(\operatorname{supp} V)} + \|V\|_\infty \|\nabla(U_t + V_t)\|_{L^q(\operatorname{supp} V)} \\ &\qquad + \|U_t\|_\infty \|V\|_q + \|U\|_\infty \|\nabla V_t\|_q. \end{aligned}$$

Hence via estimates (58), then estimates (25) for U and (41) for U_t, *a fortiori* we obtain

$$\begin{aligned} &\|(V_t \cdot \nabla)(U + V) - (V \cdot \nabla)(U_t + V_t) - (U_t \cdot \nabla)V - (U \cdot \nabla)V_t\|_q \\ &\quad \leq cQ(U_0, \nabla U_0)(t^{-\frac{3}{2}} + 1 + t^{3+\eta}), \end{aligned}$$

then estimate $(61)_2$ follows. □

Before discussing the result of successive sections, we make clear the goals. We develop the L^2-theory for problem (56) in order to solve the problem (1) in the exterior domain. This strategy is possible because by means of the solution U to the Cauchy problem we translate the initial data U_0 in a boundary data, that is the trace of U on the compact boundary $\partial\Omega$. This leads to problem (56) where the data is a suitable body force F with compact support. We do not only have to discuss the existence of w solution to problem (56), but also the behavior of the L^∞-norm of w in neighborhood of $t = 0$ and for large t. This requires precise estimates in order to obtain $\lim_{t\to 0} \|w(t)\|_\infty = 0$. Instead for large t we limit ourselves to prove an exponential growth of the solution.

5.2. L^2-Theory for Problem (56)

Lemma 13. *For all $T > 0$, assume that $w \in C([0,T); J^{1,2}(\Omega)) \cap L^2(0,T; W^{2,2}(\Omega))$ and $w_t \in L^2(0,T; L^2(\Omega))$. Further, assume also that w solves (56). Then, we get*

$$\|w(t)\|_2 \le E(t, U_0), \quad \int_0^t \|\nabla w(\tau)\|_2^2 d\tau \le \mathbb{E}(t, U_0), \text{ for all } t > 0, \tag{64}$$

where

$$E(t, U_0) := cQ(U_0, \nabla U_0)(t^{\frac{1}{2}} + t + t^{3+\eta})e^{c\int_0^t G_0^2(\tau,U_0)d\tau},$$
$$\mathbb{E}(t,U_0) := cE^2(t, U_0)\Big[\int_0^t G_0^2(\tau, U_0)d\tau + 1\Big],$$

c is independent of U and t (G_0 is defined in (26)). Finally, we get

$$\lim_{t\to 0} t^{-\frac{1}{2}}\|w(t)\|_2 \le cQ(U_0, \nabla U_0), \quad \text{and} \quad \lim_{t\to 0} t^{-1}\mathbb{E}(t, U_0) \le cQ^2(U_0, \nabla U_0). \tag{65}$$

Proof. Taking inner product (56) with w, we obtain

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|_2^2 + \|\nabla w(t)\|_2^2 = -(\nabla U(t)\cdot w(t), w(t)) + (\nabla w(t)\cdot V(t), w(t)) + (F(t), w(t)) = I_1 + I_2 + I_3. \tag{66}$$

Applying Hölder's inequality, recalling assumption ii., and estimate (58) finally applying the Cauchy inequality, we get

$$|I_1 + I_2| \le \|w\|_2\|\nabla w\|_2(\|V\|_\infty + \|U\|_\infty) \le c\|U\|_\infty^2\|w\|_2^2 + \tfrac{1}{2}\|\nabla w\|_2^2. \tag{67}$$

For the term I_3 we get:

$$|I_3| \le \|F\|_2\|w\|_2.$$

Hence (66) becomes

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|_2^2 + \frac{1}{2}\|\nabla w(t)\|_2^2 \le c\|U\|_\infty^2\|w\|_2^2 + \|F\|_2\|w\|_2, \tag{68}$$

which implies

$$\frac{d}{dt}\|w(t)\|_2 \le c\|U\|_\infty^2\|w\|_2 + \|F\|_2.$$

Recalling estimates (18) for $\|U\|_\infty$ and (61) for $\|F\|_2$, the last differential inequality implies the estimate for the L^2-norm of w and after integrating (68) one deduce completely (64). □

Remark 3. By virtue of the inequality $1 + x \le \exp(x)$, $x \ge 0$, an estimate of $\mathbb{E}$ is the following:

$$\mathbb{E}(t, U_0) \le Q^2(U_0, \nabla U_0)(t^{\frac{1}{2}} + t^{3+\eta})^2 \exp(c\int_0^t G_0^2(\tau, U_0)d\tau),\ t > 0. \tag{69}$$

Lemma 14. *Let $U, V \in W^{1,\infty}(\Omega)$, $w \in J^{1,2}(\Omega) \cap W^{2,2}(\Omega)$ and $w_t \in L^2(\Omega)$. Then, for all $\varepsilon > 0$, there exists a constant $c(\varepsilon)$ independent of the functions such that*

$$\begin{aligned}
&|(w \cdot \nabla w, P\Delta w)| \le c\|w\|_2(1 + \|w\|_2\|\nabla w\|_2)\|\nabla w\|_2^3 + \varepsilon\|P\Delta w\|_2^2,\\
&|((U+V) \cdot \nabla w, P\Delta w)| \le c\|U+V\|_\infty^2\|\nabla w\|_2^2 + \varepsilon\|P\Delta w\|_2^2,\\
&|(w \cdot (\nabla U + \nabla V), P\Delta w)| \le c\|w\|_2^2\|\nabla U + \nabla V\|_\infty^2 + \varepsilon\|P\Delta w\|_2^2,\\
&|(F, P\Delta w)| \le c\|F\|_2^2 + \varepsilon\|P\Delta w\|_2^2,
\end{aligned} \tag{70}$$

and

$$\begin{aligned}
&|(w \cdot \nabla w, w_t)| \le c\|w\|_2(1 + \|w\|_2\|\nabla w\|_2)\|\nabla w\|_2^3 + \varepsilon\|P\Delta w\|_2^2 + \varepsilon\|w_t\|_2^2,\\
&|((U+V) \cdot \nabla w, w_t)| \le c\|U+V\|_\infty^2\|\nabla w\|_2^2 + \varepsilon\|w_t\|_2^2,\\
&|(w \cdot (\nabla U + \nabla V), w_t)| \le c\|w\|_2^2\|\nabla U + \nabla V\|_\infty^2 + \varepsilon\|w_t\|_2^2,\\
&|(F, w_t)| \le c\|F\|_2^2 + \varepsilon\|w_t\|_2^2.
\end{aligned} \tag{71}$$

Proof. Applying Hölder's inequality, we obtain

$$\begin{aligned}
&|(w \cdot \nabla w, P\Delta w)| \le \|w\|_\infty\|\nabla w\|_2\|P\Delta w\|_2,\\
&|((U+V) \cdot \nabla w, P\Delta w)| \le \|U+V\|_\infty\|\nabla w\|_2\|P\Delta w\|_2,\\
&|(w \times (\omega_U + \omega_V), P\Delta w)| \le \|w\|_2\|\nabla U + \nabla V\|_\infty\|P\Delta w\|_2,\\
&|(F, P\Delta w)| \le \|F\|_2\|P\Delta w\|_2,
\end{aligned} \tag{72}$$

and

$$\begin{aligned}
&|(w \cdot \nabla w, w_t)| \le \|w\|_\infty\|\nabla w\|_2\|w_t\|_2,\\
&|((U+V) \cdot \nabla w, w_t)| \le \|U+V\|_\infty\|\nabla w\|_2\|w_t\|_2,\\
&|(w \cdot (\nabla U + \nabla V), w_t)| \le \|w\|_2\|\nabla U + \nabla V\|_\infty\|w_t\|_2,\\
&|(F, w_t)| \le \|F\|_2\|w_t\|_2.
\end{aligned} \tag{73}$$

Moreover, estimate (53), estimate (55) and the Poincaré inequality furnish

$$\|w\|_\infty \le c\|w\|_2^{\frac12}\|D^2 w\|_2^{\frac12} \le c\|w\|_2^{\frac12}\|P\Delta w\|_2^{\frac12} + \|w\|_2^{\frac12}\|\nabla w\|_2^{\frac12}.$$

Substituting the last inequality on the right hand side of $(72)_1$ and $(73)_1$, the thesis is a consequence of the Cauchy inequality. □

Lemma 15. *For all $T > 0$, assume that $w \in C([0,T); J^{1,2}(\Omega)) \cap L^2(0,T; W^{2,2}(\Omega))$ and $w_t \in L^2(0,T; L^2(\Omega))$. Further, assume also that w solves (56). Then, for all $\eta \in (0,1]$ it holds that*

$$\begin{aligned}
&\|\nabla w(t)\|_2^2 \le \frac{c}{t} H(t,U_0)\mathbb{H}(t,U_0,1),\\
&\overline{\lim_{t\to 0}}\|\nabla w(t)\|_2^2 \le cQ^2(U_0,\nabla U_0),\\
&t^\eta\|\nabla w(t)\|_2^2 + 2\int_0^t \tau^\eta\big(\|P\Delta w(\tau)\|_2^2 + \|w_\tau(\tau)\|_2^2\big)d\tau \le H(t,U_0)\mathbb{H}(t,U_0,\eta),
\end{aligned} \tag{74}$$

where we have set

$$H(t,U_0) := ce^{cE^2\mathbb{E} + ct^{\frac12}E\mathbb{E}^{\frac12} + c\int_0^t G_0^2(\tau,U_0)d\tau},$$

$$\mathbb{H}(t,U_0,\eta) := \Big[E^2(t,U_0)\int_0^t \big[G_0^2 + G_0^{2-2\alpha}G_2^{2\alpha} + G_1^2\big]\tau^\eta d\tau + \int_0^t \mathbb{F}^2(\tau,U_0)\tau^\eta d\tau + t^\eta\mathbb{E}(t,U_0)\Big],$$

where $\alpha = \frac{2+q}{2q}, q > 2$, and the constant c is independent of U and t. Finally, it holds that

$$\lim_{t\to 0} t^{-\eta}\mathbb{H}(t,U_0) \le cQ^2(U_0,\nabla U_0). \tag{75}$$

Proof. Taking the inner product of the first equation of (56) with $P\Delta w$, one with w_t, and summing up the results, we obtain

$$\begin{aligned}
&\tfrac{d}{dt}\|\nabla w\|_2^2 + \|P\Delta w\|_2^2 + \|w_t\|_2^2 \\
&\quad = (w\cdot\nabla w, P\Delta w) + ((U+V)\cdot\nabla w, P\Delta w) + (w\cdot(\nabla U+\nabla V), P\Delta w) - (F, P\Delta w) \\
&\quad - (w\cdot\nabla w, w_t) - ((U+V)\cdot\nabla w, w_t) - (w\cdot(\nabla U+\nabla V), w_t) + (F, w_t).
\end{aligned}$$

By virtue of estimates (70)–(71) with $\varepsilon = \frac{1}{8}$ we deduce the differential inequality

$$\begin{aligned}
&\frac{d}{dt}\|\nabla w\|_2^2 + \frac{1}{8}\|P\Delta w\|_2^2 + \frac{1}{8}\|w_t\|_2^2 \le c\Big[\|w\|_2^2\|\nabla w\|_2^4 \\
&\quad + \|w\|_2\|\nabla w\|_2^3 + \|U+V\|_\infty^2\|\nabla w\|_2^2 + \|w\|_2^2\|\nabla U+\nabla V\|_\infty^2 + \|F\|_2^2\Big].
\end{aligned} \tag{76}$$

Hence multiplying by t, setting $h(t) := t\|\nabla w(t)\|_2^2$, we obtain by a simple handling

$$\begin{aligned}
h'(t) + \frac{1}{8}t\|P\Delta w\|_2^2 + \frac{1}{8}t\|w_t\|_2^2 &\le c(\|w\|_2^2\|\nabla w\|_2^2 + \|w\|_2\|\nabla w\|_2)h(t) + c\|U+V\|_\infty^2 h(t) \\
&+ ct\|w\|_2^2\|\nabla U+\nabla V\|_\infty^2 + ct\|F\|_2^2 + \|\nabla w\|_2^2.
\end{aligned}$$

Taking into account that the coefficients of the above differential inequality enjoys the following estimates:

- estimate (64) for w,
- estimate (18) for $\|U\|_\infty$ and $(58)_1$ for $\|V\|_\infty$,
- estimate $(25)_2$ for $\|\nabla U\|_\infty$ and estimate (59) for $\|\nabla V||_\infty$,
- estimate (61) for $\|F\|_2$.

Integrating the differential inequality and using (59) with $\alpha = \frac{2+q}{2q}$, one deduces

$$t\|\nabla w(t)\|_2^2 \le cH(t,U_0)E^2(t,U_0)\int_0^t \big[G_0^2 + G_0^{2-2\alpha}G_2^{2\alpha} + G_1^2\big]\tau d\tau + cH\int_0^t \mathbb{F}^2(\tau,U_0)\tau d\tau + H\mathbb{E}(t,U_0),$$

which implies $(74)_1$. In particular, taking into account the behavior of H, E in $t = 0$ and estimate (65), one easily deduces $(74)_2$. Now, we consider the case of $\eta \in (0,1)$. Multiplying differential inequality (76) by t^η, and setting $h(t) := t^\eta\|\nabla w(t)\|_2^2$, we get the differential inequality:

$$\begin{aligned}
h'(t) + \frac{1}{8}t^\eta\|P\Delta w\|_2^2 + \frac{1}{8}t^\eta\|w_t\|_2^2 &\le c(\|w\|_2^2\|\nabla w\|_2^2 + \|w\|_2\|\nabla w\|_2)h(t) + c\|U+V\|_\infty^2 h(t) \\
&+ ct^\eta\|w\|_2^2\|\nabla U+\nabla V\|_\infty^2 + ct^\eta\|F\|_2^2 + \eta t^{\eta-1}\|\nabla w(t)\|_2^2.
\end{aligned}$$

Since the estimate with $\eta = 1$ ensures also $\|\nabla w(t)\|_2$ bounded in any neighborhood of $t = 0$, then all the terms on the right hand side of the last differential inequality are integrable. Therefore, by integration we have proved estimate $(74)_3$. Finally, the limit property (75) is an immediate consequence of the definition of $\mathbb{H}$ and of $(74)_2$. □

Lemma 16. *Let $U, V \in W^{1,\infty}(\Omega)$,, $w \in J^{1,2}(\Omega) \cap W^{2,2}(\Omega)$ and $w_t \in L^2(\Omega)$. Then, for all $\varepsilon > 0$, there exists a constant $c(\varepsilon)$ independent of the functions such that*

$$\begin{aligned}
&I_1 := |(w_t\cdot\nabla w, w_t)| \le c\|w_t\|_2^2\|\nabla w\|_2^2 + \varepsilon\|\nabla w_t\|_2^2, \\
&I_2 := |((U_t+V_t)\cdot\nabla w, w_t)| \le \|U_t+V_t\|_\infty\|\nabla w\|_2\|w_t\|_2, \\
&I_3 := |(w_t\cdot\nabla w_t, (U+V))| \le c\|w_t\|_2^2\|\nabla U+\nabla V\|_\infty^2 + \varepsilon\|\nabla w_t\|_2^2 \\
&I_4 := |(w\cdot\nabla w_t, (U_t+V_t))| \le \|w\|_2^2\|U_t+V_t\|_\infty^2 + \varepsilon\|\nabla w_t\|_2^2 \\
&I_5 := |(F_t, w_t)| \le \|F_t\|_2\|w_t\|_2.
\end{aligned} \tag{77}$$

Proof. Applying Hölder's inequality, the Gagliardo–Nirenberg inequality (53) and the Cauchy inequality we get the estimates (77). □

Lemma 17. *For all $T > 0$, assume that $w, w_t \in C([0,T); J^{1,2}(\Omega)) \cap L^2(0,T; W^{2,2}(\Omega))$ and $w_{tt} \in L^2(0,T; L^2(\Omega))$. Assume that w solves (56) and estimate $(64)_1$ and $(74)_3$ hold, then*

$$t^2\|w_t\|_2^2 \le ct^{1-\frac{\eta}{2}} K(t,U_0)\mathbb{K}(t,U_0), \tag{78}$$

where we set $K(t,U_0) := e^{\mathbb{E}+\int_0^t G_0^2 d\tau}$ and

$$\mathbb{K}(t,U_0,\eta) := \mathbb{H}^{\frac{1}{2}}(t,U_0,\eta)\Big[t^{-\frac{\eta}{2}}H(t,U_0)\mathbb{H}^{\frac{1}{2}}(t,U_0,\eta) + \big[t^{\frac{1}{2}}\mathbb{E}^{\frac{1}{2}}(t)\sup_{(0,t)}\tau^{\frac{1}{2}}D_1(\tau,U_0) + \sup_{(0,t)}\tau^{\frac{3}{2}}\mathbb{F}_\tau(\tau,U_0)\big]H^{\frac{1}{2}}(t,U_0)\Big] + t^{1+\frac{\eta}{2}}E^2(t,U_0)\sup_{(0,t)}\tau D_1^2(\tau,U_0)$$

with c independent of U and t. Finally, for a suitable $c(U_0,\nabla U_0)$ it holds that

$$\begin{aligned}&\overline{\lim_{t\to 0}}\, t^{\frac{1}{2}}\|w_t(t)\|_2 \le cQ(U_0,\nabla U_0),\\ &\|w_t(t)\|_2 \le cQ(U_0,\nabla U_0)t^{-\frac{1}{2}}e^{A(\mathbb{E}(t)+\int_0^t G_0^2(\tau,U_0)d\tau)},\end{aligned} \tag{79}$$

where A is a suitable constant.

Proof. After differentiating with respect to t the first equation of (56), taking inner product (56) with w_t, and integrating by parts we obtain

$$\tfrac{1}{2}\tfrac{d}{dt}\|w_t(t)\|_2^2 + \|\nabla w_t(t)\|_2^2 \le \sum_{i=1}^5 I_i, \tag{80}$$

where I_i, for $i = 1,\dots,5$ are defined in (77). Applying estimates (77) to the right hand side of (80), we deduce the differential inequality

$$\begin{aligned}\frac{d}{dt}\|w_t\|_2^2 \le\, & c\|w_t\|_2^2(\|\nabla w\|_2^2 + \|U+V\|_\infty^2)\\ & + \|w_t\|_2(\|\nabla w\|_2\|U_t+V_t\|_\infty + \|F_t\|_2) + \|w\|_2^2\|U_t+V_t\|_\infty^2,\end{aligned}$$

which is equivalent to the following

$$\begin{aligned}\frac{d}{dt}(t^2\|w_t\|_2^2) \le\, & ct^2\|w_t\|_2^2(\|\nabla w\|_2^2 + \|U+V\|_\infty^2) + 2t\|w_t\|_2^2\\ & + t^2\|w_t\|_2(\|\nabla w\|_2\|U_t+V_t\|_\infty + \|F_t\|_2) + t^2\|w\|_2^2\|U_t+V_t\|_\infty^2.\end{aligned} \tag{81}$$

By virtue of the assumption (64) for $\|\nabla w\|_2$, via estimate $(25)_1$ for $\|U\|_\infty$ and via estimate $(58)_1$ for $\|V\|_\infty$, we get

$$\int_0^t \big[\|\nabla w\|_2^2 + \|U\|_\infty^2 + \|V\|_\infty^2\big]d\tau \le \mathbb{E}(t) + \int_0^t G_0^2 d\tau.$$

From inequality $(74)_3$ easily follows

$$2\int_0^t \tau\|w_\tau\|_2^2 d\tau \le t^{1-\eta}H(t,U_0)\mathbb{H}(t,U_0,\eta).$$

Moreover, by virtue of the assumptions (74) on $\|w_t\|_2$ and (64) for $\|\nabla w\|_2$, estimate $(41)_1$ for the L^∞-norm of U_t and $(58)_1$ for V_t, we have

$$\int_0^t \tau^2\|w_\tau\|_2\|\nabla w\|_2\|U_\tau+V_\tau\|_\infty d\tau \le ct^{\frac{3}{2}-\frac{\eta}{2}}\mathbb{E}^{\frac{1}{2}}(t)H^{\frac{1}{2}}(t)\mathbb{H}^{\frac{1}{2}}(t)\sup_{(0,t)}\tau^{\frac{1}{2}}D_1(\tau).$$

Since $\|w\|_2$ satisfies estimate (64), $\|U_t\|_\infty$ estimate $(41)_1$ and $\|V_t\|_\infty$ estimate $(58)_1$, we obtain

$$\int_0^t \tau^2\|w\|_2^2\|U_\tau+V_\tau\|_\infty^2 d\tau \le E^2(t)\int_0^t \tau^2 D_1^2 d\tau \le t^2E^2(t)\sup_{(0,t)}\tau D_1^2(\tau).$$

Finally, recalling (61), and employing $(74)_3$, we have

$$\int_0^t \tau^2\|w_\tau\|_2\mathbb{F}_\tau d\tau \le t^{1-\frac{\eta}{2}}H^{\frac12}(t,U_0)\mathbb{H}^{\frac12}(t,U_0,\eta)\sup_{(0,t)}\tau^{\frac32}\mathbb{F}_\tau.$$

Thanks to the above estimates, by integrating the differential inequality (81), we get

$$t^2\|w_t\|_2^2 \le ct^{1-\frac{\eta}{2}}K(t,U_0)\mathbb{K}(t,U_0),$$

which furnishes (78), where we have set $K(t,U_0) := e^{\mathbb{E}(t)+\int_0^t G_0^2 d\tau}$ and

$$\mathbb{K}(t,U_0,\eta) := \mathbb{H}^{\frac12}(t,U_0,\eta)\Big[t^{-\frac{\eta}{2}}H(t,U_0)\mathbb{H}^{\frac12}(t,U_0,\eta) + \big[t^{\frac12}\mathbb{E}^{\frac12}(t)\sup_{(0,t)}\tau^{\frac12}D_1(\tau,U_0) + \sup_{(0,t)}\tau^{\frac32}\mathbb{F}_\tau(\tau,U_0)\big]H^{\frac12}(t,U_0)\Big] + t^{1+\frac{\eta}{2}}E^2(t,U_0)\sup_{(0,t)}\tau D_1^2(\tau,U_0).$$

Hence recalling (75), we arrive at $(79)_1$. Finally, taking into account that the function

$$e^{\mathbb{E}(t)+\int_0^t G_0^2(\tau,U_0)d\tau}$$

has the most grow for large t, so that a suitable constant A allows to include all the divergent terms of the right hand side of (78), we obtain $(79)_2$. □

Corollary 2. *Under the same hypotheses of Lemma 17, we have*

$$\begin{aligned}&\overline{\lim_{t\to0}}\, t^{\frac12}\|D^2w(t)\|_2 \le cQ(U_0,\nabla U_0),\\ &\|D^2w(t)\|_2 \le cQ(U_0,\nabla U_0)t^{-\frac12}e^{A(\mathbb{E}(t)+\int_0^t G_0^2(\tau,U_0)d\tau)},\end{aligned} \tag{82}$$

where c is independent of U and t.

Proof. Since it holds that

$$P\Delta w = w_t + P((w+U+V)\cdot\nabla)w + P((w+V)\cdot\nabla)(U+V) + (U\cdot\nabla)V + P(V_t-\Delta V),$$

then, via estimate $(79)_2$ for w_t, estimate $(74)_{1,2}$ for ∇w, estimate $(64)_1$ for w, estimates $(25)_{1,2}$ for U and ∇U respectively, and estimate (58) with $k=0$ for V, by means of a suitable $Q(U_0,\nabla U_0)$ we get

$$\|P\Delta w\|_2 \le cQ(U_0,\nabla U_0)t^{-\frac12}e^{A(\mathbb{E}(t)+\int_0^t G_0^2(\tau,U_0)d\tau)}.$$

Now, via the last estimate, to prove (82) it is enough to apply inequality (55) of Lemma 11, and to employ (64). □

Lemma 18. *Under the Assumption 1 there exists a solution (w,π_w) to problem (56) such that, for all $T>0$ and $\delta>0$,*

$$\begin{aligned}&w\in C(0,T;L^2(\Omega))\cap C(\delta,T;J^{1,2}(\Omega)),\ q\in[2,\infty),\ \lim_{t\to0}\|w(t)\|_q = 0,\\ &t^{\frac12}D^2w,\ t^{\frac12}w_t \in L^\infty(0,T;L^2(\Omega)),\ t^{\frac12}\nabla\pi_w \in L^\infty(0,T;L^2(\Omega)),\\ &\|w(t)\|_\infty \le E^{\frac12}(t,U_0)Q^{\frac12}(U_0,\nabla U_0)t^{-\frac14}e^{A(\mathbb{E}(t)+\int_0^t G_0^2(\tau,U_0)d\tau)}\end{aligned} \tag{83}$$

for all $t>0$.

Proof. We employ the Faedo-Galerkin method, which in 2-D Navier–Stokes equations furnishes the existence of a global regular solution, regular in the sense of estimates (83). Employing the Galerkin method in the way suggested e.g. in [14], first we prove all the estimates assuming that F is mollified with respect to t. Of course the mollified shares the same properties of F. For both the approximations, that is Galerkin and mollified F, we omit index. Actually, it is enough to take care that the estimates are independent of the particular approximation. Since we assume that F is mollified, then the Galerkin approximation enjoys the hypotheses of Lemma 13 and Lemma 15. Hence for the Galerkin approximation estimates (64) and (74) hold. As well as, via Lemma 17 and Corollary 2 for the Galerkin approximation estimates (64), (74) and (82) hold. As consequence there exists a limit w such that, for all $T>0$, w is a solution to problem (56), and, for all $\delta>0$, $w\in C([0,T);J^2(\Omega))\cap C([\delta,T);J^{1,2}(\Omega))\cap L^2(\delta,T;W^{2,2}(\Omega))$ with $w_t,\,\nabla\pi_w\in L^2(\delta,T;L^2(\Omega))$. Of course, $w\in C([\delta,T);J^{1,2}(\Omega))$ implies $w\in C([\delta,T);J^q(\Omega))$ for all $q\in[2,\infty)$. Finally, since the limit w also verifies $(82)_2$ and $(65)_1$ for all $t>0$, via the Gagliardo–Nirenberg estimate (53), we obtain $w\in L^\infty(0,T;L^\infty(\Omega))$, and $\lim_{t\to0}\|w(t)\|_q=0$ for $q\in[2,\infty)$. □

5.3. L^q-Estimates for Solution to Problem (56)

Lemma 19. *Let $h(t)$ be a non negative continuous function such that*

$$h(t)\le H(t)+\int_0^t a(t,s)h(s)ds,\ \text{for all}\ t\in[0,T),\tag{84}$$

where $H(t)\ge0$ is a continuous function on $[0,T]$, and, for all $t\in[0,T)$, $a(t,s)\ge0$ is continuous in $[0,t)$ and belongs to $L^1(0,t)$ in such a way that for some $\delta\in(0,1)$ there exists $\eta>0$:

$$\int_{t-\eta}^t a(t,\tau)d\tau<\delta,\ \text{for all}\ t\in[0,T).$$

Then we have

$$h(t)\le\frac{H(t)}{1-\delta}\exp\frac{\gamma}{1-\delta}t,\ \text{for all}\ t\in[0,T],\ \gamma=\sup_{t\in[0,T)}\left\{\sup_{s\in[0,t-\eta)}a(t,s)\right\}.\tag{85}$$

Proof. The proof is a trivial generalization of [12, Lemma 4]. □

Lemma 20. *In the same hypotheses of Lemma 17, for all $s\in[2,\infty)$, the following estimate holds*

$$\|w(t)\|_\infty\le ct^{\frac12\frac{s-2}{s+2}}c(U_0,\nabla U_0),\ \text{for all}\ t\in(0,1).\tag{86}$$

Proof. We start recalling that $w\in L^\infty(0,T;L^\infty(\Omega))$ with estimate $(83)_3$. In order to prove property (86) we begin proving

$$\text{for}\ s>2,\ \|w(t)\|_s\le ct^{\frac12},\tag{87}$$

which is an improvement of the one previously given. For this aim, we employ a duality argument. We set $\widehat\varphi(s,x):=\varphi(t-s,x)$, where $\varphi(\tau,x)$ is the solution to the Stokes problem (49) furnished by Theorem 8 and corresponding to the initial data $\varphi_0\in\mathscr{C}_0(\Omega)$. Multiplying equation $(56)_1$ by φ and integrating by parts on $(0,t)\times\Omega$, we derive

$$(w(t),\varphi_0)=\int_0^t\Big[((w+U+V)\cdot\nabla\varphi,w)+(w\cdot\nabla\varphi,(U+V))+(F,\varphi)\Big]d\tau.$$

Applying Hölder's inequality, we get

$$|(w(t),\varphi_0)|\le\int_0^t\Big[\|w+U+V\|_\infty\|\nabla\varphi\|_{s'}\|w\|_s+\|U+v\|_\infty\|\nabla\varphi\|_{s'}\|w\|_s+\|F\|_s\|\varphi\|_{s'}\Big]d\tau.$$

Via estimates $(25)_1$ for U, $(58)_1$ for V and $(83)_3$ for w, semigroup property for φ and estimate (61) for F, we get with obvious meaning of the symbols

$$|(w(t),\varphi_0)|\le\|\varphi_0\|_{s'}\int_0^t b(\tau)(t-\tau)^{-\frac12}\|w\|_sd\tau+c\|\varphi_0\|_{s'}\int_0^t\mathbb{F}(\tau)d\tau.$$

The last inequality furnishes the following estimate

$$\|w(t)\|_s\le H(t)+\int_0^t a(t,\tau)\|w(s)\|_sd\tau,\ \text{ for all}\ t>0,$$

with a obvious meaning of function a. Hence by virtue of (84)–(85), we obtain that (87) holds in a neighborhood of $t = 0$. Employing the Gagliardo–Nirenberg inequality, we have

$$\|w(t)\|_{C(\overline{\Omega})} \leq c\|D^2 w(t)\|_2^a \|w(t)\|_s^{1-a} \quad \text{with } a = \tfrac{2}{s+2}.$$

Now, estimate (86) is a consequence of (82) and (87). □

The following lemma is close to the ones proved by Solonnikov [24] for the linearized Navier–Stokes problem in $\Omega \subseteq \mathbb{R}^3$.

Lemma 21. *For all $T > 0$, let $w \in C([0,T); J^{1,2}(\Omega)) \cap L^2(0,T; W^{2,2}(\Omega))$ and $w_t \in L^2(0,T; L^2(\Omega))$. Then, for all $T > 0$ it holds that*

$$r \in (1,2),\ w \in C([0,T); J^r(\Omega)) \cap L^r(0,T; W^{2,r}(\Omega)),\ w_t, \nabla\pi_w \in L^r(0,T; L^r(\Omega)). \tag{88}$$

Proof. We introduce the following auxiliary problem:

$$\begin{aligned}&\psi_t - \Delta\psi + \nabla\pi_\psi = -h_R(U+V+w)\cdot\nabla\psi - h_R\psi\cdot\nabla(U+V) + F \text{ in } (0,T)\times\Omega,\\ &\nabla\cdot\psi = 0 \text{ in } (0,T)\times\Omega,\ \psi = 0 \text{ on } (0,T)\times\partial\Omega,\ \psi = \psi_0 \text{ on } \{0\}\times\Omega,\end{aligned} \tag{89}$$

here h_R is a cut-off function. We assume that $\psi_0 \in J^{2-\frac{2}{r},r}(\Omega) \cap J^{1,2}(\Omega)$. We recall that $U, V, w \in L^\infty(0,T; L^\infty(\Omega))$, $\nabla U \in L^\infty(0,T; L^\infty(\Omega))$ and for all $q \in (1,\infty)$, $\nabla V, t^{\frac{1}{2}}F \in L^\infty(0,T; L^q(\Omega))$. Applying the Galerkin method, as derived for the proof of Lemma 18, we can prove the existence of a solution (ψ, π_ψ) enjoying, uniformly with respect to R, the following property:

$$\begin{aligned}&\psi \in C([0,T); J^2(\Omega)) \cap L^\infty(0,T; J^{1,2}(\Omega))\\ &t^{\frac{\eta}{2}}\psi_t, t^{\frac{\eta}{2}}D^2\psi, t^{\frac{\eta}{2}}\nabla\pi_\psi \in L^2(0,T; L^2(\Omega)).\end{aligned} \tag{90}$$

Hence, thanks to the cutoff function h_R, the right hand side of $(89)_1$, denoted shortly by $\overline{F}$, belongs to $L^r(0,T; L^r(\Omega))$. By virtue of Theorem 7 we deduce the existence of a unique $(\overline{\psi}, \pi_{\overline{\psi}})$ such that

$$\begin{aligned}&\text{for all } R > 0,\ r \in (1,2),\ \overline{\psi} \in C([0,T); J^r(\Omega))\\ &\int_0^t \Big[\|\overline{\psi}_\tau\|_r^r + \|D^2\overline{\psi}\|_r^r + \|\nabla\pi_{\overline{\psi}}\|_r^r\Big] d\tau \leq c\|\psi_0\|^r_{J^{2-\frac{2}{r},r}} + c\int_0^t \|\overline{F}\|_r^r d\tau\end{aligned}$$

and solving the problem

$$\begin{aligned}&\overline{\psi}_t - \Delta\overline{\psi} + \nabla\pi_{\overline{\psi}} = \overline{F} \text{ in } (0,T)\times\Omega,\\ &\nabla\cdot\overline{\psi} = 0 \text{ in } (0,T)\times\Omega,\ \overline{\psi} = 0 \text{ on } (0,T)\times\partial\Omega,\ \overline{\psi} = \psi_0 \text{ on } \{0\}\times\Omega.\end{aligned}$$

The difference $\Psi := \overline{\psi} - \psi$ and $\pi_\Psi := \pi_{\overline{\psi}} - \pi_\psi$ solves an homogeneous Stokes problem, for which, by the duality, trivially follows that $\Psi = 0$ and $\pi_\Psi = 0$ up to a function of t. Therefore, we get, as $(\overline{\psi}, \pi_{\overline{\psi}})$,

$$\begin{aligned}&\text{for all } R > 0,\ r \in (1,2),\ \psi \in C([0,T); J^r(\Omega)),\\ &\int_0^t \Big[\|\psi_\tau\|_r^r + \|D^2\psi\|_r^r + \|\nabla\pi_\psi\|_r^r\Big] d\tau \leq c\|\psi_0\|^r_{J^{2-\frac{2}{r},r}} + c\int_0^t \|\overline{F}\|_r^r d\tau,\end{aligned}$$

and, we recall,

$$\begin{aligned}\|\overline{F}\|_r &= \|-h_R(U+V+w)\cdot\nabla\psi - h_R\psi\cdot\nabla(U+V) + F\|_r\\ &\leq \|U+V+w\|_\infty\|\nabla\psi\|_r + \|\nabla U\|_\infty\|\psi\|_r + \|\nabla V\|_r\|\psi\|_\infty + \|F\|_r.\end{aligned} \tag{91}$$

Now, we prove that, there exists a $T > 0$ such that, uniform in R,

$$\begin{aligned}&r \in (1,2),\ \psi \in C([0,T; J^r(\Omega)),\\ &\int_0^t \Big[\|\psi\|_r^r + \|\psi_\tau\|_r^r + \|D^2\psi\|_r^r + \|\nabla\pi_\psi\|_r^r\Big] d\tau \leq c(T)\|\psi_0\|^r_{J^{2-\frac{2}{r},r}}.\end{aligned} \tag{92}$$

We recall:

$$\|\psi(t)\|_r \le \|\psi_0\|_r + \int_0^t \|\psi(\tau)\|_r d\tau,\ t>0,$$

and, via the Gagliardo–Nirenberg inequality,

$$\begin{aligned}\|\psi\|_\infty \le \|D^2\psi\|_r^{\frac{1}{r}}\|\psi\|_r^{1-\frac{1}{r}} \le c\|D^2\psi\|_r^{\frac{1}{r}}\left[\int_0^t \|\psi_\tau\|_r^r d\tau\right]^{\frac{r-1}{r^2}} t^{\frac{1}{r'^2}},\\ \|\nabla\psi\|_r \le c\|D^2\psi\|_s^{\frac12}\|\psi\|_s^{\frac12} \le c\|D^2\psi\|_r^{\frac12}\left[\int_0^t \|\psi_\tau\|_r^r d\tau\right]^{\frac{1}{2r}} t^{\frac{1}{2r'}}, \quad t>0.\end{aligned} \tag{93}$$

Taking estimate (91) into account, we obtain the following estimate uniform in $R>0$

$$\begin{aligned}\|\overline{F}\|_r^r \le A(t)\|D^2\psi\|_r^{\frac{r}{2}}\left[\int_0^t \|\psi_\tau\|_r^r d\tau\right]^{\frac12} t^{\frac{r-1}{2}} + B(t)\left[\int_0^t \|\psi_\tau\|_r^r d\tau\right] t^{r-1}\\ +C(t)\|D^2\psi\|_r\left[\int_0^t \|\psi_\tau\|_r^r d\tau\right]^{\frac{1}{r'}} t^{\frac{(1-r)^2}{r}} + \|F\|_r^r, \quad t>0,\end{aligned}$$

with obvius meaning of the symbols. Integrating on $(0,t)$, applying Hölder's inequality and, subsequently, Cauchy's inequality, we obtain

$$\begin{aligned}\int_0^t \|\overline{F}\|_r^r d\tau \le A(t)t^{\frac{r}{2}}\left[\int_0^t \|D^2\psi\|_r^r d\tau + \int_0^t \|\psi_\tau\|_r^r d\tau\right] + B(t)t^{r-1}\left[\int_0^t \|\psi_\tau\|_r^r d\tau\right]\\ +C(t)t^{\frac{2-r}{r'}}\left[\int_0^t \|D^2\psi\|_r^r d\tau + \int_0^t \|\psi_\tau\|_r^r d\tau\right] + \int_0^t \|F\|_r^r d\tau\, t>0,\end{aligned}$$

uniform in R. Fixing $T>0$ such that the coefficients satisfy

$$A(T)T^{\frac{r}{2}} + B(T)T^{r-1} + C(T)T^{\frac{2-r}{r'}} < 1,$$

we deduce (92). Since problem (89) is linear, $C(T)$ is independent of ψ_0, and $\psi(T) \in J^{2-\frac{2}{r},r}(\Omega) \cap J^{1,2}(\Omega)$, $r \in (1,2)$, we can extend the validity of (92) to all $T>0$. Thus we can consider the limit function, denoted by $(\overline{w}, \pi_{\overline{w}})$ of the family $\{(\psi, \pi_\psi)\}$ of solutions to problem (89). This limit enjoys of the properties (90) and (92). For $\psi_0 = 0$, the pair $(\overline{w}, \pi_{\overline{w}})$ just proved, solves problem

$$\begin{aligned}\overline{w}_t - \Delta\overline{w} + \nabla\pi_{\overline{w}} = -(U+V+w)\cdot\nabla\overline{w} - \overline{w}\cdot\nabla(U+V) + F \text{ in } (0,T)\times\Omega,\\ \nabla\cdot\overline{w} = 0 \text{ in } (0,T)\times\Omega,\ \overline{w}=0 \text{ on } (0,T)\times\partial\Omega,\ \overline{w}=0 \text{ on } \{0\}\times\Omega.\end{aligned} \tag{94}$$

Moreover, the pair (w, π_w) solves problem (94), and by the regularity of both the solutions, it is easy to deduce that they coincide up to function of t for the pressure field. The lemma is completely proved. □

6. Proof of Theorem 4

We start with the following auxiliary result:

Theorem 9. *Under the Assumption 1 there exists a solution (w, π_w) to problem (56) such that, for all $T>0$ and $\delta>0$,*

$$\begin{aligned} q\in[2,\infty],\, w\in C(0,T;L^q(\Omega))\cap C^2((\delta,T)\times\Omega),\ \pi_w\in C^2((\delta,T)\times\Omega_R),\\ \lim_{t\to0}\|w(t)\|_q = 0,\ \textit{and for } s>2,\ \|w(t)\|_\infty \le ct^{\frac12\frac{s-2}{s+2}}Q(u_0,\nabla u_0) \textit{ for all } t\in(0,1),\\ \|w(t)\|_\infty \le E^{\frac12}(t,u_0)Q^{\frac12}(U_0,\nabla U_0)t^{-\frac14}e^{A(\mathbb{E}(t)+\int_0^t G_0^2(\tau,U_0)d\tau)} < \infty,\\ \textit{for all } r\in(1,2),\ \nabla\pi_w\in L^r(0,T;L^r(\Omega)) \textit{ and } t^{\frac12}\nabla\pi_w\in L^2(0,T;L^2(\Omega)).\end{aligned} \tag{95}$$

Proof. For the proof of the existence, of the limit properties, L^∞-estimate and for the integrability $(95)_4$ we avail ourselves of Lemmas 18, 20 and 21. Now, recalling the regularity of U, V, F, classical arguments concerning the regularity of 2-D weak solutions can be applied in order to prove the regularity expressed by $(95)_1$. Hence the proof is achieved. □

Proof of Theorem 4. Now, we are in a position to prove the existence claimed in Theorem 4. We look for a solution u to problem (1) in the following form:

$$u := U + V + w, \text{ and } \pi_u := \pi_U + \pi_w, \tag{96}$$

where (U, π_U) solves the Navier–Stokes Cauchy problem with initial data u_0, V is the extension exhibited in Sect. 4.1 and (w, π_w) is a solution to problem (56) furnished by Theorem 9. By construction we determine that the pair (u, π_u) defined by (96) is a solution to system $(1)_1$ and satisfies the homogeneous boundary condition. Moreover, we have achieved that $U \in C([0,\infty); BUC(\mathbb{R}^2))$, the limit property (60) for V and estimate $(95)_2$ for $\|w\|_\infty$. Thus we obtain

$$\lim_{t\to 0} \|U(t) - u_0\|_{BUC(\mathbb{R}^2)}, \;\; \lim_{t\to 0} \|V(t)\|_{C(\overline{\Omega})} = 0, \quad \lim_{t\to 0} \|w(t)\|_{C(\overline{\Omega})} = 0,$$

so that u assumes the initial data u_0 in strong form in $C(\overline{\Omega})$. Regarding estimate $(13)_2$ we point out that it is a consequence of estimate $(25)_1$ for U, estimate $(58)_1$ for V and finally of estimates $(95)_2$-$(95)_3$ for w. Obviously, we mean that $c(t, u_0, \nabla u_0)$ is suitable function which sums the contributes not exponential of the estimate related to w, and of estimate (69). If we take into account the results related to the pressure field π_U obtained in Sect. 3 and related to the pressure field π_w obtained in Theorem 9, the existence is completely achieved. For the uniqueness we limit ourselves to remark that it is a consequence of the uniqueness result proved in [1]. Actually the solution (u, π_u) just constructed is also a mild solution to the integral equation (8). Hence the uniqueness follows from the one proved for Theorem 2. □

7. Proof of Theorem 5

Let us consider a smooth non-negative cutoff function h such that $h(\xi) = 1$ for $\xi \in [0, 1]$ and $h(\xi) = 0$ for $\xi \geq 2$. We define the integral operator

$$A_k(x) := \int_{\mathbb{R}^2} \nabla\mathscr{E}(x - y)(u \cdot \nabla)(h_k u)dy, \tag{97}$$

where $\{h_k(y)\}$ is a sequence of smooth not negative cutoff functions, with $h_k(x) := h(|x|/k)$, and $\mathscr{E}(z)$ is the fundamental solution of 2-D Laplacian operator. We assume that

$$\text{for all } \lambda \in (0,1), \begin{cases} u \in C^{0,\lambda}(\mathbb{R}^2), \text{ with } u = 0 \text{ in } \mathbb{R}^2 - \Omega, \\ \nabla u \in C^{0,\lambda}(\overline{\Omega}), \text{ with } \nabla \cdot u = 0 \text{ in } \mathbb{R}^2. \end{cases} \tag{98}$$

Lemma 22. *The sequences $\{A_k\}$ admits the following pointwise limits of continuous functions in $\mathbb{R}^2$:*

$$\begin{aligned} &\{\nabla \cdot A_k\} \to \Pi, \quad \{\text{rot} A_k\} \to a, \quad \{\nabla\nabla \cdot A_k\} \to \nabla\Pi, \\ &\{\Delta\, A_k\} \to -u \cdot \nabla u, \quad \{\Delta\nabla \cdot A_k\} \to -\nabla \cdot (u \cdot \nabla)u. \end{aligned} \tag{99}$$

In particular we get

$$\begin{aligned} \textit{for all } \varepsilon \in (0, \lambda], \quad & |\Pi(x)| + |a(x)| \leq c|x|^\varepsilon \|u\|_\infty [u]^{(\varepsilon)}, \\ & |\nabla\Pi|_\infty + |\nabla a|_\infty \leq c\big[\|(u \cdot \nabla)u\|_{1,\lambda} + \|u\|_\infty^2\big], \\ \textit{for all } \lambda' \in (0, \lambda),\; & [\nabla^i \Pi]^{(\lambda')} \leq c[\nabla^{i-1}(u \cdot \nabla)u]^{(\lambda)}, \; \textit{for } i = 1, 2. \end{aligned} \tag{100}$$

Moreover the following relation holds:

$$n \cdot \nabla\Pi = n \cdot \text{rot}\, a \quad \textit{on} \quad \partial\Omega, \tag{101}$$

where n is the unit outer normal to $\partial\Omega$.

Proof. The proof of this lemma is analogous to the one given for Lemma 3. For this claim it is enough to verify that the kernel which is defined in the operator (97) and the one that is defined in the operator (32) are formally the same. So we omit the proof of (99) and (100). To prove the boundary relation (101) we remark that from (99) and assumption for u on $\partial\Omega$ it follows

$$n\cdot\nabla\Pi = n\cdot\big[\nabla\Pi+(u\cdot\nabla)u\big] = n\lim_k\big[\nabla\nabla\cdot A_k-\Delta A_k\big] = n\cdot\lim_k \mathrm{rot}(\mathrm{rot}A_k) = n\cdot\mathrm{rot}a.$$

□

Let us consider the Neumann problem

$$\Delta p = 0 \text{ in } \Omega,\ p\to c_\infty \text{ for } |x|\to 0,\ \frac{dp}{dn} = (\mathrm{rot}\, b)\cdot n \text{ on } \partial\Omega. \tag{102}$$

The following result holds for problem (102):

Theorem 10. *Assume that in (102) $b\in W^{1-\frac1q,q}(\partial\Omega)$. Then, there exist a solution p to problem (102) and a constant c independent of b such that*

$$\|p\|_{L^q(\Omega_R)}+\|\nabla p\|_q \le c < b >_q^{1-\frac1q}, \tag{103}$$

where $< \cdot >_q^{1-\frac1q}$ is the seminorm on $\partial\Omega$. The solution p to problem (102) is unique up to a constant.

Let us consider the Neumann problem

$$\Delta\pi_u = -\nabla\cdot\big[(u\cdot\nabla)u\big] \text{ in } \Omega,\quad \frac{dp_u}{dn} = \mathrm{rot}\,\mathrm{rot}\, u\cdot n \text{ on } \partial\Omega. \tag{104}$$

The following result holds for problem (104):

Theorem 11. *Assume that in problem (104) u satisfies (98). Then, there exist a solution to problem (104) such that $\pi_u\in C^2\Omega)$, and, for $\lambda>\frac12$, we get*

$$|\pi_u(x)|\le c\Big[|x|^\varepsilon\|u\|_\infty[u]^{(\varepsilon)}+\|(u\cdot\nabla)u\|_{1,\lambda}+\|u\|_\infty^2+[\nabla u]^{(\lambda)}\Big]. \tag{105}$$

The solution π_u is unique up to a constant.

Proof. The proof of this lemma is a slight modification of the ones given in [24]. Actually, we look for π_u a pointwise estimate of the kind (105). For this task we employ Lemma 22. We set

$$\pi_u := \Pi + p,$$

where Π is given in Lemma 22 and p is the solution to problem (102) with $b=-a+\mathrm{rot}u$. Via formulas (99) and (101), and since p is harmonic it is immediate to verify that π_u is a solution. By virtue of estimate $(100)_1$ for π_u, we can relax our estimate to function p. Since p is a solution to a harmonic problem it is smooth far to the boundary, near to the boundary we employ estimate (103) for $q>2$, in such a way by embedding theorem we obtain that $p\in C(\Omega_R)$. Since $b=-a+\mathrm{rot}\,u$ we examine independently the seminorm of a and $\mathrm{rot}\,u$. By virtue of $(100)_2$ for the term a we get that

$$|a(x)-a(y)|\le|\nabla a|_\infty|x-y|\le c\big[\|(u\cdot\nabla)u\|_{1,\lambda}+\|u\|_\infty^2\big]|x-y|.$$

Analogously, by the Hölder assumption on ∇u, we get

$$|\mathrm{rot}\,u(x)-\mathrm{rot}\,u(y)|\le c|x-y|^\lambda[\nabla u]^{(\lambda)}.$$

These last estimates and the definition of $< \cdot >_q^{1-\frac1q}$ $(q>2)$ ensure the result provided that $\lambda>\frac12$. □

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

8. Appendix

We furnish lines of argument that lead to establish estimates (25) which in particular imply estimate $(19)_2$. We recall the following result related to Hölder's spaces

Theorem 12. *There exists a constant M such that for all $g \in C^{m,\mu}(\Omega)$ it holds that*[4]

$$\begin{aligned}&\textit{for } r\in\{0,\dots,m\}, \nu\in[0,\mu],\ [\nabla^r g]^{(\nu)} \le 2^{\frac{\mu-\nu}{\mu}} \|\nabla^r g\|_\infty^{\frac{\mu-\nu}{\mu}} [\nabla^r g]^{(\mu)\,\frac{\nu}{\mu}},\\ &\textit{for } r\in\{1,\dots,m\}, \nu\in[0,1],\ \|\nabla^r g\|_\infty \le M\Big[[\nabla^m g]^{(\mu)\,\frac{r-\nu}{m+\mu-\nu}} [g]^{(\nu)\,\frac{m-r+\mu}{r+\mu-\nu}} + [g]^{(\nu)}\Big].\end{aligned} \tag{106}$$

Of course we consider estimates (18) and $(19)_1$ as achieved from Theorem 6. Hence we look for estimates $(25)_{2,3}$.

The starting points are two. The former is formula (23), that for the convenience of the reader we reproduce:

$$\nabla U = \nabla e^{t\Delta} U_0 + \int_0^t (\nabla E(t-\tau), (U\times\omega)(\tau))d\tau. \tag{107}$$

The latter is the formula of the vorticity solution to Eq. (22):

$$\omega(t,x) = e^{t\Delta}\omega_0 + \int_0^t (e^{(t-\tau)\Delta}, U\cdot\nabla\omega)d\tau. \tag{108}$$

With obvious meaning of the symbols we also write

$$\nabla U := \nabla U_L + \nabla U_N,$$

as well as

$$\omega := \omega_L + \omega_N.$$

The following estimates hold: ($\lambda \in (0,1)$)

$$\begin{aligned}&|\nabla^r\Gamma(s,z)| + |\nabla^r E(s,z)| \le c(|z| + s^{\frac12})^{-2-r},\ \text{for all } (s,z)\in(0,\infty)\times\mathbb{R}^2,\\ &|\nabla^r\Gamma(s,\overline{z}) - \nabla^r\Gamma(s,\overline{\overline{z}})| \le c|\overline{z}-\overline{\overline{z}}|^\lambda \max\{(|\overline{z}|+s^{\frac12})^{-2-r-\lambda}, (|\overline{\overline{z}}|+s^{\frac12})^{-2-r-\lambda}\},\\ &|\nabla^r E(s,\overline{z}) - \nabla^r E(s,\overline{\overline{z}})| \le c|\overline{z}-\overline{\overline{z}}|^\lambda \max\{(|\overline{z}|+s^{\frac12})^{-2-r-\lambda}, (|\overline{\overline{z}}|+s^{\frac12})^{-2-r-\lambda}\},\\ &\qquad\qquad\qquad\text{for all } s>0, \text{ and } \overline{z},\overline{\overline{z}}\in\mathbb{R}^2,\end{aligned} \tag{109}$$

in the above inequality Γ denotes the kernel of the heat equation.

In the following lemma, the symbol $\mathbb{K}$ indicates indifferntly the kernels E or Γ.

Lemma 23. *Let us consider the integral transform*

$$\overline{U}(t,x) := \int_0^t (\nabla\mathbb{K}(t-\tau), g(\tau))d\tau,$$

with $g \in C(0,T;C^{0,\nu}(\mathbb{R}^2)), \nu\in(0,1)$, and

$$\textit{for } \alpha_0\in[0,1), \beta_0>0,\ \|g(\tau)\|_\infty \le A_0\tau^{-\alpha_0} + B_0\tau^{\beta_0},$$

$$\textit{for } \alpha_1\in[0,1), \beta_1>0,\ [g(\tau)]^{(\nu)} \le A_1\tau^{-\alpha_1} + B_1\tau^{\beta_1}, \tau>0.$$

Then, for all $\mu>0$ such that $\frac{\mu}{1-\mu}<\nu$, we get

$$[\nabla\overline{U}(t)]^{(\mu)} \le c\sum_{i=1}^4 M_i t^{-m_i} t^{\frac{\nu}{2}(1-\mu)-\frac{\mu}{2}},\ \textit{for all } t>0, \tag{110}$$

where we set $M_1 := A_0^\mu A_1^{1-\mu}$, $M_2 := A_0^\mu B_1^{1-\mu}$, $M_3 := B_0^\mu A_1^{1-\mu}$, $M_4 := B_0^\mu B_1^{1-\mu}$, and $m_1 := \alpha_0\mu + \alpha_1(1-\mu)$, $m_2 := \alpha_0\mu - \beta_1(1-\mu)$, $m_3 := \alpha_1(1-\mu) - \beta_0\mu$, $m_4 := -\beta_0\mu - \beta_1(1-\mu)$.

[4] For $\nu = 0$ we mean $[g]^{(0)} = \|g\|_\infty$.

Proof. This is a classical result, so that the proof is omitted. □

Taking into account (18) and $(19)_1$, via estimates $(109)_{1,3}$ for all $\lambda \in (0,1)$ we easily deduce

$$[\nabla U_N]^{(\lambda)} \le c\int_0^t (t-\tau)^{-\frac{1}{2}-\frac{\lambda}{2}}\|U\|_\infty\|\omega\|_\infty d\tau \le cG_0(t,U_0)\|\omega_0\|_\infty t^{\eta_1}, \tag{111}$$

where we set $\eta_1 = \frac{1}{2}-\frac{\lambda}{2}$. From the interpolation inequality (106), applying the Cauchy inequality and (18), we get

$$\begin{aligned}\|\nabla U_N(t)\|_\infty &\le c\Big[[\nabla U_N(t)]^{(\lambda)\,\frac{1}{1+\lambda}}\|U_N\|_\infty^{\frac{\lambda}{1+\lambda}} + \|U_N\|_\infty\Big]\\ &\le c[\nabla U_N(t)]^{(\lambda)\,\frac{1}{1+\lambda}}\Big[\|U\|_\infty^{\frac{\lambda}{1+\lambda}} + \|U_L\|_\infty^{\frac{\lambda}{1+\lambda}}\Big] + \|U\|_\infty + \|U_L\|_\infty\\ &\le c(1+t\|U_0\|_\infty^2)\|U_0\|_\infty + c[\nabla U_N(t)]^{(\lambda)} \le cG_0(t,U_0)(1+\|\omega_0\|_\infty t^{\eta_1}).\end{aligned} \tag{112}$$

Since $\|\nabla U_L\|_\infty \le c\|\nabla U_0\|_\infty$, via (112) we arrive at $(25)_2$. We recall the symbols G_i defined in (26), that from now up we make use. Since $[\nabla U_L]^{(\lambda_1)} \le c\|\nabla U_0\|_\infty t^{-\frac{\lambda_1}{2}}$ again via (111), we also deduce

$$[\nabla U]^{(\lambda_1)} \le c(\|\nabla U_0\|_\infty t^{-\frac{\lambda_1}{2}} + G_0(t,U_0)\|\omega_0\|_\infty t^{\eta_1}). \tag{113}$$

Of course the following holds:

$$[\omega]^{(\lambda_1)} \le c(\|\nabla U_0\|_\infty t^{-\frac{\lambda_1}{2}} + G_0(t,U_0)\|\omega_0\|_\infty t^{\eta_1}). \tag{114}$$

We set

$$G^{\lambda_1}(t,U_0) := \text{the right hand side of (113) and (114)}.$$

So that employing interpolation $(106)_1$, we get

$$\begin{aligned}\nu\in[0,\lambda_1],\ [\omega]^{(\nu)} &\le 2^{\frac{\lambda_1-\nu}{\lambda_1}}\|\omega\|_\infty^{\frac{\lambda_1-\nu}{\lambda_1}}[\omega]^{(\lambda_1)\,\frac{\nu}{\lambda_1}}\\ &\le c\|\omega_0\|_\infty^{\frac{\lambda_1-\nu}{\lambda_1}}(\|\nabla U_0\|_\infty t^{-\frac{\lambda_1}{2}} + G_0(t,U_0)\|\omega_0\|_\infty t^{\eta_1})^{\frac{\nu}{\lambda_1}}\\ &\le c\|\nabla U_0\|_\infty(t^{-\frac{\nu}{2}} + G_0^{\frac{\nu}{\lambda_1}}(t,U_0)t^{\eta_2}),\end{aligned} \tag{115}$$

where we set $\eta_2 := \frac{\nu}{\lambda_1}\eta_1$. We give the estimate of $(19)_2$. Employing the interpolation inequality $(106)_2$, recalling that $U = U_L + U_N$, via estimate (112), we deduce also

$$\begin{aligned}[U_N]^{(\nu)} &\le c\|U_N\|_\infty^{1-\nu}[U_N]^{(1)^\nu} \le c(\|U\|_\infty + \|U_L\|_\infty)^{1-\nu}[U_N]^{(1)^\nu}\\ &\le c(\|U\|_\infty + \|U_L\|_\infty)^{1-\nu}\|\nabla U_N\|_\infty^\nu \le cG_0(t,U_0)(1+\|\omega_0\|_\infty t^{\eta_1})^\nu.\end{aligned}$$

Since $[U_L]^{(\nu)} \le c\|U_0\|_\infty t^{-\frac{\nu}{2}}$, we deduce $(19)_2$. We set

$$G^\lambda(t,U_0) := \text{the right hand side of } (19)_2.$$

Employing estimates (19) for ω and U, (115) for Hölder's seminorm of ω, we obtain an estimate for Hölder's seminorm of $U\times\omega$. Hence by virtue of (110) of Lemma 23, for $0 < \frac{\lambda_2}{1-\lambda_2} < \min\{\lambda,\lambda_1\}$, via representation formula (108) we arrive at an estimate of $[\nabla\omega_N(t)]^{(\lambda_2)}$:

$$[\nabla\omega_N(t)]^{(\lambda_2)} \le c\int_0^t \Big[(t-\tau)^{-1+\eta_2}G^\lambda\|\omega\|_\infty + (t-\tau)^{-1+\overline{\eta}_2}G_0G^{\lambda_1}\Big]d\tau,$$

where $\eta_2 := \frac{\lambda}{2}(1-\lambda_2) - \frac{\lambda_2}{2}$ and $\overline{\eta}_2 := \frac{\lambda_1}{2}(1-\lambda_2) - \frac{\lambda_2}{2}$. The last gives

$$\begin{aligned}[\nabla\omega_N(t)]^{(\lambda_2)} &\le c\|U_0\|_\infty\|\omega_0\|_\infty t^{\eta_2-\frac{\lambda}{2}} + G_0(t,U_0)(1+\|\omega_0\|_\infty t^{\eta_1})^{\lambda_2}\|\omega_0\|_\infty t^{\eta_2}\\ &\quad + \|\nabla U_0\|_\infty G_0(t,U_0)t^{\overline{\eta}_2-\frac{\lambda_1}{2}} + G_0^2(t,U_0)\|\omega_0\|_\infty t^{\eta_1+\overline{\eta}_2}\\ &\le Q(U_0,\nabla U_0)(t^{\min\{\eta_2-\frac{\lambda}{2},\overline{\eta}_2-\frac{\lambda_1}{2}\}} + 1 + t^{\overline{\lambda}_2}),\end{aligned}$$

where $\overline{\lambda}_2 := \max\{\eta_1\lambda_2+\eta_2, \eta_1+\overline{\eta}_2\}$. Hence taking into account the contribute of the $\nabla\omega_L$ we obtain the estimate

$$[\nabla\omega(t)]^{(\lambda_2)} \leq [\nabla\omega_L(t)]^{(\lambda_2)}+[\nabla\omega_N(t)]^{(\lambda_2)} \leq c\|\nabla U_0\|_\infty t^{-\frac{1}{2}-\frac{\lambda_2}{2}} \\ +Q(u_0,\nabla U_0)(1+t)^{1+\overline{\lambda}_2} =: G^{\lambda_2}(t,U_0), \text{ for all } t>0. \tag{116}$$

By virtue of Theorem 12, employing (116) we also deduce

$$\|\nabla\omega(t)\|_\infty \leq c\big([\nabla\omega]^{(\lambda_2)^a}\|\omega_0\|_\infty^{1-a} + \|\omega_0\|_\infty\big), \text{ for all } t>0,\ a=\tfrac{1}{1+\lambda_2}. \tag{117}$$

Making use of the representation (107) we deduce

$$\nabla^2 U = \nabla e^{t\Delta}\nabla U_0 - \int_0^t (\nabla E(t-\tau), \nabla(U\times\omega)(\tau)d\tau. \tag{118}$$

Analogously to the case of the estimate (116), employing the bound for Hölder's seminorm of $U\times\omega$, for $\frac{\lambda_2}{1-\lambda_2} < \{\lambda,\lambda_1\}$, we obtain

$$[\nabla^2 U_N(t)]^{(\lambda_2)} \leq c\|U_0\|_\infty\|\omega_0\|_\infty t^{\eta_2} + G_0(t,U_0)(1+\|\omega_0\|_\infty t^{\eta_1})^{\lambda_2}\|\omega_0\|_\infty t^{\eta_2} \\ +\|\nabla U_0\|_\infty G_0(t,U_0)t^{\overline{\eta}_2} + G_0(t,U_0)^{1+\frac{\lambda_2}{\lambda_1}} t^{\overline{\eta}_2} =: Q(U_0,\nabla U_0)(1+t)^{1+\overline{\lambda}_2}.$$

Hence taking into account the contribute of the linear part, we get

$$[\nabla^2 U(t)]^{(\lambda_2)} \leq [\nabla^2 U_L(t)]^{(\lambda_2)}+[\nabla^2 U_N(t)]^{(\lambda_2)} \leq c\|\nabla U_0\|_\infty t^{-\frac{1}{2}-\frac{\lambda_2}{2}} \\ +Q(u_0,\nabla U_0)(1+t)^{1+\overline{\lambda}_2} =: G^{\lambda_2}(t,U_0), \text{ for all } t>0. \tag{119}$$

Taking into account the estimates of Hölder's semminorms: $(19)_2$ for U, (116)–(117) for the $\nabla\omega$ and $(25)_2$ for ∇U and $(19)_1$ for ω, we have an estimate for the Hölder's seminorm of $\nabla(U\times\omega)$. Hence applying Lemma 23 to (118) for $0<\frac{\lambda_3}{1-\lambda_3}<\min\{\lambda,\lambda_1,\lambda_2\}$ we arrive at an estimate of

$$[\nabla^3 U]^{(\lambda_3)} \text{ which also implies the existence of } [\nabla^2\omega]^{(\lambda_3)}. \tag{120}$$

However we estimate the semi norm $[\nabla^2\omega(t)]^{(\lambda_3)}$ by means of the representation formula (108). We get

$$[\nabla^2\omega_N(t)]^{(\lambda_3)} \leq c\int_0^t \Big[(t-\tau)^{-1+\eta_3}G^\lambda\|\nabla\omega\|_\infty + (t-\tau)^{-1+\overline{\eta}_3}G_0 G^{\lambda_2}\Big]d\tau,$$

where $\eta_3 := \frac{\lambda}{2}(1-\lambda_3)-\frac{\lambda_3}{2}$ and $\overline{\eta}_2 := \frac{\lambda_2}{2}(1-\lambda_3)-\frac{\lambda_3}{2}$. We can deduce

$$\nabla^3 U = \nabla^2 e^{t\Delta}\nabla U_0 - \int_0^{\frac{t}{2}} (\nabla^2 E(t-\tau), \nabla(U\times\omega(\tau))d\tau \\ + \int_{\frac{t}{2}}^t (\nabla E(t-\tau), \nabla^2(U\times\omega(\tau))d\tau = Y_1+Y_2+Y_3.$$

Now we are in a position to deduce estimate $(25)_4$. Actually, for the the term ∇Y_1 is trivial to prove the estimate $c\|\nabla U_0\|_\infty t^{-\frac{3}{2}}$. As well as for the first integral term we easily get $\|\nabla Y_2(t)\|_\infty \leq ct^{-\frac{3}{2}}(1+ct\|U_0\|_\infty)\|U_0\|_\infty\|\omega_0\|_\infty$. Hence applying the Cauchy inequality we prove for ∇Y_2 the estimate $(25)_4$. Finally, taking into account the above Hölder's estimates related to U and ω, applying Lemma 23, by handling suitably the Cauchy inequality for the last term, we can deduce estimate $(25)_4$. Finally, we deduce $(25)_3$. To this end we apply $(106)_2$ with $g:=\nabla U$, for $\nu=\mu=0$, $r=1$ and $m=3$. Then a simple manipulation leads to $(25)_3$.

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