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A Multi-grid Decoupling Method for the Coupled Fluid Flow with the Porous Media Flow

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Abstract. In this paper, we propose a multi-grid decoupling method for the coupled Navier–Stokes–Darcy problem with the Beavers–Joseph–Saffman interface condition. The basic idea of the method is to first solve a much smaller global problem on a very coarse initial grid, then solve a linearized Newton problem and a Darcy problem in parallel on all the subsequently refined grids. Error bounds of the approximate solution for the proposed method are analyzed, and optimal error estimates are obtained. Numerical experiments are conducted to verify the theoretical analysis and indicate the effectiveness of the proposed method.

Keywords. Navier–Stokes equations, Darcy's law, decoupling, multi-grid technique, finite element method.

1. Introduction

The coupling of fluid flow and porous media flow has received more and more attention and has become a very active research area in recent years. The major reason lies in its wide spectrum of real world applications, including the environmental problem of groundwater contamination through rivers, the industrial manufacturing of filters, the biological modeling of the coupled circulatory system with the surrounding tissue, and so on.

The related numerical models are the coupled Stokes–Darcy model or the coupled Navier–Stokes–Darcy model, especially the interface conditions include Beavers–Joseph–Saffman or Beavers–Joseph interface conditions. Many workers pay attention to studying their mathematical analysis and numerical methods, the readers can refer to [1,3–20,22–33,35–40]. However, most of previous works related to the coupled Stokes–Darcy problem. In this paper, we focus on the nonlinear case, that is, the coupled Navier–Stokes–Darcy problem. In [6], a decoupled and linearized two-grid algorithm was proposed and investigated. To further improve the effectiveness of solving the coupled Navier–Stokes–Darcy problem, we now extend the algorithm in [6] and propose a multi-grid decoupling method. In this method, one only first solve a much smaller global problem on a very coarse initial grid, then solve a linearized Newton problem and a Darcy problem in parallel on all the subsequently refined grids. Moreover, we note that the numerical analysis in [6] only obtained the optimal order of convergence for the porous media flow and half order lower than the optimal one for the fluid flow. In our work, we analyze the error bounds of the approximate solution for the proposed method, and obtain the optimal error estimates for two flows. Numerical results well agree with the theoretical predictions, and also demonstrate the effectiveness of the proposed method.

The rest of the paper is organized as follows. In Sect. 2, the coupled Navier–Stokes–Darcy problem is given. In Sect. 3, the multi-grid decoupling method is presented. In Sect. 4, convergence of the proposed

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method is analyzed. Finally, numerical experiment is conducted to verify the accuracy of theoretical analysis and illustrate the effectiveness of the multi-grid method in Sect. 5.

2. Coupled Navier–Stokes–Darcy Problem

We consider a bounded region $\Omega \subset R^d\,(d = 2 \text{ or } 3)$ for the coupled Navier–Stokes–Darcy model, the domain Ω includes a fluid flow region Ω_f and a porous media region Ω_p. Two regions are separated by the interface $\Gamma = \partial\Omega_f \cap \partial\Omega_p$. Let n_f and n_p be the unit outward normal vectors on $\partial\Omega_f$ and $\partial\Omega_p$, respectively. τ_i, $i = 1, \ldots, d-1$, are the unit tangential vectors on the interface Γ. Obviously, $n_p = -n_f$ on the interface Γ.

The Navier–Stokes equations for the fluid velocity u and pressure p describe the flow in Ω_f:

$$-\nu\Delta u + (u\cdot\nabla)u + \nabla p = f_1 \quad \text{in } \Omega_f, \tag{2.1}$$

$$\nabla\cdot u = 0 \quad \text{in } \Omega_f. \tag{2.2}$$

Here, ν is the kinetic viscosity, f_1 is the external force acting on the fluid flow.

The Darcy equations for the fluid velocity u_p and the piezometric head φ govern the flow in Ω_p:

$$u_p = -\mathbf{K}\nabla\varphi \quad \text{in } \Omega_p, \tag{2.3}$$

$$\nabla\cdot u_p = f_2 \quad \text{in } \Omega_p. \tag{2.4}$$

Here $\mathbf{K} = \{K_{ij}\}_{d\times d}$ is the hydraulic conductivity tensor, symmetric and positive definite, denoting permeability of the rock. In this paper, we assume $\mathbf{K} = diag(K, \ldots, K)$ with $K \in L^\infty(\Omega_p)$, $K > 0$. f_2 describes the external forces acting on the porous media flow.

Eliminating u_p from (2.3)–(2.4), we have the following equation:

$$-\nabla\cdot(\mathbf{K}\nabla\varphi) = f_2 \quad \text{in } \Omega_p. \tag{2.5}$$

The mixed system presented above is coupled across the following interface boundary conditions on Γ.

Conservation of mass is displayed by

$$u\cdot n_f + u_p\cdot n_p = 0 \quad \text{on } \Gamma. \tag{2.6}$$

This condition on Γ is the continuity of the normal velocity, which is a consequence of the incompressibility.

The balance of the normal forces is stated by

$$p - \nu n_f\frac{\partial u}{\partial n_f} = g\varphi \quad \text{on } \Gamma, \tag{2.7}$$

where g is the gravitational acceleration.

For the third one, we consider the well known Beavers–Joseph–Saffman interface condition (see [34]),

$$-\nu\tau_i\frac{\partial u}{\partial n_f} = \alpha\sqrt{\frac{\nu g}{\mathrm{tr}(\mathbf{K})}}u\cdot\tau_i \;\; 1 \le i \le (d-1) \quad \text{on } \Gamma. \tag{2.8}$$

Here, α is a positive parameter and matters with the properties of the porous medium, it is experimentally determined. Beavers–Joseph–Saffman interface condition is the simplified Beavers–Joseph interface condition (see [2]) and is widely accepted and used.

For the sake of simplicity, we impose homogeneous boundary conditions on external boundary, i.e., $u = 0$ on $\partial\Omega_f \setminus \Gamma$, $\varphi = 0$ on $\partial\Omega_p \setminus \Gamma$.

To present the weak formulation of the coupled model, we define the following spaces:

$$H_f = \{v \in (H^1(\Omega_f))^d : v = 0 \text{ on } \partial\Omega_f \setminus \Gamma\},$$
$$H_p = \{\psi \in H^1(\Omega_p) : \psi = 0 \text{ on } \partial\Omega_p \setminus \Gamma\}.$$
$$W = H_f \times H_p,$$
$$Q = L^2(\Omega_f).$$

For the convenience of notation, we denote $L^2(\Omega_{f/p})$ inner product and $L^2(\Gamma)$ inner product by $(\cdot,\cdot)_{f/p}$ and $(\cdot,\cdot)_\Gamma$, respectively; the corresponding norms are denoted by $\|\cdot\|_{f/p}$ and $\|\cdot\|_\Gamma$.

The spaces H_f and H_p are equipped with the following norms: $\forall\, u \in H_f$ and $\forall\, \varphi \in H_p$,

$$\|u\|_{H_f} = \|\nabla u\|_f = \sqrt{(\nabla u, \nabla u)_f},$$
$$\|\varphi\|_{H_p} = \|\nabla \varphi\|_p = \sqrt{(\nabla \varphi, \nabla \varphi)_p}.$$

And the space W is equipped with the following norms: $\forall\, w \in W$,

$$\|w\|_0 = \sqrt{(u,u)_f + (\varphi,\varphi)_p},$$
$$\|w\|_W = \sqrt{\nu(\nabla u, \nabla u)_f + g(\mathbf{K}\nabla\varphi, \nabla\varphi)_p}.$$

In addition, we define the trilinear form

$$a_N(l,u,v) = ((l\cdot\nabla)u, v)_f \quad \forall\, l,u,v \in H_f. \tag{2.9}$$

We shall assume that physical parameters g, ν, α and K are positive constants, f_1 and f_2 are smooth enough. Then the weak formulation of the coupled Navier–Stokes–Darcy model is given by: find $w = (u,\varphi) \in W$ and $p \in Q$, such that

$$a(w;w,z) + b(z,p) = f(z) \quad \forall\, z = (v,\psi) \in W, \tag{2.10}$$
$$b(w,q) = 0 \quad \forall\, q \in Q, \tag{2.11}$$

where

$$a(w;w,z) = a_f(u,v) + a_p(\varphi,\psi) + a_\Gamma(w,z) + a_N(u;u,v),$$
$$a_f(u,v) = \nu(\nabla u, \nabla v)_f + \alpha\sqrt{\frac{\nu g}{\mathrm{tr}(\mathbf{K})}}\sum_{i=1}^{d-1}(u\cdot\tau_i, v\cdot\tau_i)_\Gamma,$$
$$a_p(\varphi,\psi) = g(\mathbf{K}\nabla\varphi, \nabla\psi)_p,$$
$$a_\Gamma(w,z) = g((\varphi, v\cdot n_f)_\Gamma - (\psi, u\cdot n_f)_\Gamma),$$
$$b(z,p) \equiv b_f(v,p) = -(p, \nabla\cdot v)_f,$$
$$f(z) = (f_1,v)_f + g(f_2,\psi)_p.$$

In [14], we can find that the coupled problem (2.10)–(2.11) is well-posed, provided the physical parameter ν is large enough. In this paper, we follow the assumptions in [14].

3. Numerical Algorithms

Let τ_h be a quasi-uniform triangulation of the global domain, as well as compatible and quasi-uniform on Γ as described in [6,16]. Denote $W_h = H_{fh} \times H_{ph} \subset W$ and $Q_h \subset Q$ are the finite element subspaces defined on the partition τ_h. Moreover, we assume that (H_{fh}, Q_h) satisfy the following discrete inf-sup condition, i.e., there exists a positive constant β independent of h, such that $\forall\, q_h \in Q_h, \exists\, v_h \in H_{fh}$,

$$b_f(v_h,q_h) \ge \beta\|v_h\|_{H_f}\|q_h\|_Q. \tag{3.1}$$

Then the standard finite element discrete scheme for (2.10)–(2.11) on (W_h, Q_h) reads as follows.

Algorithm 1. Standard finite element scheme.

Find $w_h = (u_h, \varphi_h) \in W_h$ and $p_h \in Q_h$, such that

$$a(w_h; w_h, z_h) + b(z_h, p_h) = f(z_h) \quad \forall\, z_h = (v_h, \psi_h) \in W_h, \tag{3.2}$$

$$b(w_h, q_h) = 0 \quad \forall\, q_h \in Q_h. \tag{3.3}$$

The problem (3.2)–(3.3) is a coupled and nonlinear problem, solving it directly would have many difficulties, especially in numerical implementation. These difficulties increase as the mesh size decreases. In [6], a decoupled and linearized two-grid algorithm was proposed. In the method, for the interface boundary term, a coarse grid approximation was used; for the trilinear term, a Newton type linearization was applied. To further improve the efficiency of solving the coupled Navier–Stokes–Darcy problem, we extend the method above to the following linearized multi-grid method.

Algorithm 2. Multi-grid decoupling scheme.

1. On a relative coarse grid with mesh size H, solve the coupled problem (3.2)–(3.3): find $w_H = (u_H, \varphi_H) \in W_H$ and $p_H \in Q_H$, such that

$$a(w_H; w_H, z_H) + b(z_H, p_H) = f(z_H) \quad \forall\, z_H = (v_H, \psi_H) \in W_H, \tag{3.4}$$

$$b(w_H, q_H) = 0 \quad \forall\, q_H \in Q_H. \tag{3.5}$$

2. On a fine grid with mesh size h_1, solve two independent subproblems in parallel: one is the linear Newton problem and the other is the Darcy problem.

   In the fluid region Ω_f, find $u^{h_1} \in H_{fh_1} \supset H_{fH}$, $p^{h_1} \in Q_{h_1} \supset Q_H$ such that

$$\begin{aligned} &a_f(u^{h_1}, v_{h_1}) + a_N(u_H; u^{h_1}, v_{h_1}) + a_N(u^{h_1}; u_H, v_{h_1}) + b_f(v_{h_1}, p^{h_1}) \\ &\quad = (f_1, v_{h_1})_f + a_N(u_H; u_H, v_{h_1}) - g(\varphi_H, v_{h_1} \cdot n_f)_\Gamma \qquad \forall\, v_{h_1} \in H_{fh_1}, \end{aligned} \tag{3.6}$$

$$b_f(u^{h_1}, q_{h_1}) = 0 \qquad \forall\, q_{h_1} \in Q_{h_1}. \tag{3.7}$$

   In the porous media region Ω_p, find $\varphi^{h_1} \in H_{ph_1} \supset H_{pH}$ such that

$$a_p(\varphi^{h_1}, \psi_{h_1}) = g(f_2, \psi_{h_1})_p + g(\psi_{h_1}, u_H \cdot n_f)_\Gamma \quad \forall\, \psi_{h_1} \in H_{ph_1}. \tag{3.8}$$

3. For $i = 1, 2, \ldots, I$, on the fine grid with mesh size (h_{i+1}), solve the following two subproblems in parallel.

   In the fluid region Ω_f, find $u^{h_{i+1}} \in H_{fh_{i+1}} \supset H_{fh_i}$, $p^{h_{i+1}} \in Q_{h_{i+1}} \supset Q_{h_i}$ such that for $\forall\, v_{h_{i+1}} \in H_{fh_{i+1}}$ and $\forall q_{h_{i+1}} \in Q_{h_{i+1}}$

$$\begin{aligned} &a_f(u^{h_{i+1}}, v_{h_{i+1}}) + a_N(u^{h_i}; u^{h_{i+1}}, v_{h_{i+1}}) + a_N(u^{h_{i+1}}; u^{h_i}, v_{h_{i+1}}) + b_f(v_{h_{i+1}}, p^{h_{i+1}}) \\ &\quad = (f_1, v_{h_{i+1}})_f + a_N(u^{h_i}; u^{h_i}, v_{h_{i+1}}) - g(\varphi^{h_i}, v_{h_{i+1}} \cdot n_f)_\Gamma, \end{aligned} \tag{3.9}$$

$$b_f(u^{h_{i+1}}, q_{h_{i+1}}) = 0. \tag{3.10}$$

   In the porous media region Ω_p, find $\varphi^{h_{i+1}} \in H_{ph_{i+1}} \supset H_{ph_i}$ such that

$$a_p(\varphi^{h_{i+1}}, \psi_{h_{i+1}}) = g(f_2, \psi_{h_{i+1}})_p + g(\psi_{h_{i+1}}, u^{h_i} \cdot n_f)_\Gamma \quad \forall\, \psi_{h_{i+1}} \in H_{ph_{i+1}}. \tag{3.11}$$

It is easily seen that one only requires to solve a small global problem on a very coarse initial grid, and then on a series of refined grids solve a linearized Navier–Stokes subproblem and a Darcy subproblem, which are independent of each other on the same fine grid and hence can be solved in parallel. Moreover, in the next section, we will show that the proposed multi-grid method has the same accuracy as the standard finite element method, when the mesh sizes h_{i+1} and h_i are taken properly.

4. Convergence Analysis

In this section, we shall analyze the errors between $(u^{h_{i+1}}, p^{h_{i+1}}, \varphi^{h_{i+1}})$ obtained by the problem (3.9)–(3.11) and (u, p, φ) obtained by the problem (2.10)–(2.11).

Before further analysis, we assume (u, p, φ) satisfies:

$$u \in (H^2(\Omega_f))^d, \quad p \in H^1(\Omega_f), \quad \varphi \in H^2(\Omega_p), \tag{4.1}$$

and thus finite element spaces of first order approximation $O(\tilde{h})$, $\tilde{h} = H, h_i$ are available in following analysis for the fluid and porous media regions.

For the sake of simplicity, we denote by C a generic positive constant which may depend on physical parameters but is independent of mesh size. Moreover, it may have different values at different occasions.

For the trilinear term $a_N(l; u, v)$ defined by (2.9), we have the following estimate:

$$|a_N(l, u, v)| \leq C\|\nabla l\|_f \|\nabla u\|_f \|\nabla v\|_f. \tag{4.2}$$

The following lemma [7,38] gives the convergence of the standard finite element method, which is useful for our analysis.

Lemma 4.1. *For the coupled problem* (3.2)–(3.3)*, we have the following error estimates*

$$\|w - w_h\|_0 + h(\|w - w_h\|_W + \|p - p_h\|_f) \leq Ch^2. \tag{4.3}$$

For the second step solution $(u^{h_1}, p^{h_1}, \varphi^{h_1})$, we have the following result.

Theorem 4.1. *For $(u^{h_1}, p^{h_1}, \varphi^{h_1})$, the following error estimate holds:*

$$\|\nabla(\varphi_{h_1} - \varphi^{h_1})\|_p \leq CH^2, \tag{4.4}$$

$$\|\nabla(u_{h_1} - u^{h_1})\|_f + \|p_{h_1} - p^{h_1}\|_f \leq CH^2, \tag{4.5}$$

where $(u_{h_1}, p_{h_1}, \varphi_{h_1})$ is the solution of (3.2)–(3.3) *with a mesh of size h_1.*

Proof. It follows from [6], we have the estimate (4.4).

Now, we will state H^1 error for the velocity and L^2 error for the pressure in energy norms.

Taking $z_h = (v_{h_1}, 0)$ in (3.2) and $q_h = q_{h_1}$ in (3.3), and combining with the problem (3.6)–(3.7), for $\forall v_{h_1} \in H_{fh_1}$ and $\forall q_{h_1} \in Q_{h_1}$, we have

$$\begin{aligned} &a_f(u_{h_1} - u^{h_1}, v_{h_1}) + b_f(v_{h_1}, p_{h_1} - p^{h_1}) + [a_N(u_{h_1}; u_{h_1}, v_{h_1}) - a_N(u_H; u^{h_1}, v_{h_1}) \\ &\quad - a_N(u^{h_1}; u_H, v_{h_1}) + a_N(u_H; u_H, v_{h_1})] = -g(\varphi_{h_1} - \varphi_H, v_{h_1} \cdot n_f)_\Gamma, \end{aligned} \tag{4.6}$$

$$b_f(u_{h_1} - u^{h_1}, q_{h_1}) = 0. \tag{4.7}$$

Note that,

$$\begin{aligned} &a_N(u_{h_1}; u_{h_1}, v_{h_1}) - a_N(u_H; u^{h_1}, v_{h_1}) - a_N(u^{h_1}; u_H, v_{h_1}) + a_N(u_H; u_H, v_{h_1}) \\ &= a_N(u_H; u_{h_1} - u^{h_1}, v_{h_1}) + a_N(u_{h_1} - u^{h_1}; u_H, v_{h_1}) + a_N(u_{h_1} - u_H; u_{h_1} - u_H, v_{h_1}). \end{aligned} \tag{4.8}$$

By letting $v_{h_1} = u_{h_1} - u^{h_1}$, $q_{h_1} = p_{h_1} - p^{h_1}$ in the problem (4.6)–(4.7), we can get

$$\begin{aligned} \nu\|\nabla(u_{h_1} - u^{h_1})\|_f^2 &\leq |a_N(u_H; u_{h_1} - u^{h_1}, u_{h_1} - u^{h_1}) + a_N(u_{h_1} - u^{h_1}; u_H, u_{h_1} - u^{h_1})| \\ &\quad + |a_N(u_{h_1} - u_H; u_{h_1} - u_H, u_{h_1} - u^{h_1})| + |g(\varphi_{h_1} - \varphi_H, (u_{h_1} - u^{h_1}) \cdot n_f)_\Gamma|. \end{aligned} \tag{4.9}$$

For the trilinear terms in the right side of (4.9), we can obtain that

$$|a_N(u_H; u_{h_1} - u^{h_1}, u_{h_1} - u^{h_1}) + a_N(u_{h_1} - u^{h_1}; u_H, u_{h_1} - u^{h_1})| \leq \frac{C}{\sqrt{2\nu}}\|\nabla(u_{h_1} - u^{h_1})\|_f^2, \tag{4.10}$$

and

$$|a_N(u_{h_1} - u_H; u_{h_1} - u_H, u_{h_1} - u^{h_1})| \leq CH^2\|\nabla(u_{h_1} - u^{h_1})\|_f. \tag{4.11}$$

To estimate the interface term in the right side of (4.9), we introduce an auxiliary problem in Ω_p: find $\xi \in H^1(\Omega_p)$ such that,

$$\begin{cases} -\nabla\cdot(\mathbf{K}\nabla\xi)=0 & \text{in } \Omega_p,\\ \mathbf{K}\nabla\xi\cdot n_p=(u_{h_1}-u^{h_1})\cdot n_p & \text{on } \Gamma,\\ \xi=0 & \text{on } \partial\Omega_p\backslash\Gamma. \end{cases} \tag{4.12}$$

Then the weak formulation of problem (4.12) reads as follows: find $\xi \in H_p$ such that for $\forall\,\eta\in H_p$,

$$(\mathbf{K}\nabla\xi,\nabla\eta)_p=(\eta,(u_{h_1}-u^{h_1})\cdot n_p)_\Gamma.$$

By Lax–Milgram theorem [21], the solution ξ of above problem is existent and unique. Then taking $\eta=\xi$ in the above equation, we have

$$\left\|\mathbf{K}^{\frac12}\nabla\xi\right\|_p^2\le C\|\xi\|_{L^2(\Gamma)}\|u_{h_1}-u^{h_1}\|_{L^2(\Gamma)}\le C\|\mathbf{K}^{\frac12}\nabla\xi\|_p\|\nabla(u_{h_1}-u^{h_1})\|_f,$$

which implies

$$\|\mathbf{K}^{\frac12}\nabla\xi\|_p\le C\|\nabla(u_{h_1}-u^{h_1})\|_f. \tag{4.13}$$

Moreover, when Γ is smooth enough, and $(u_{h_1}-u^{h_1})\cdot n_p\in H^{\frac12}(\Gamma)$, $\xi\in H^2(\Omega_p)$ and

$$\|\xi\|_{H^2(\Omega_p)}\le C\|\nabla(u_{h_1}-u^{h_1})\|_f. \tag{4.14}$$

Therefore, we have

$$\begin{aligned} &|g(\varphi_{h_1}-\varphi_H,(u_{h_1}-u^{h_1})\cdot n_f)_\Gamma|\\ &\quad=|g(\varphi_{h_1}-\varphi_H,\mathbf{K}\nabla\xi\cdot n_p)_{\partial\Omega_p}|\\ &\quad=|g(\varphi_{h_1}-\varphi_H,\nabla\cdot(\mathbf{K}\nabla\xi))_p+g(\mathbf{K}\nabla(\varphi_{h_1}-\varphi_H),\nabla\xi)_p|\\ &\quad=|g(\mathbf{K}\nabla(\varphi_{h_1}-\varphi_H),\nabla\xi)_p|\\ &\quad\le|a_p(\varphi_{h_1}-\varphi^{h_1},\xi)|+|a_p(\varphi^{h_1}-\varphi_H,\xi)|. \end{aligned} \tag{4.15}$$

Using (4.4) and (4.13), we get

$$|a_p(\varphi_{h_1}-\varphi^{h_1},\xi)|\le C\|\nabla(\varphi_{h_1}-\varphi^{h_1})\|_p\left\|\mathbf{K}^{\frac12}\nabla\xi\right\|_p\le CH^2\|\nabla(u_{h_1}-u^{h_1})\|_f. \tag{4.16}$$

For the last term in the right side of (4.15), we can obtain that,

$$a_p(\varphi^{h_1}-\varphi_H,\xi)=a_p(\varphi^{h_1}-\varphi_H,\xi-\psi_H)\quad\forall\,\psi_H\in H_{pH},$$

where, we used the fact that

$$(\mathbf{K}\nabla(\varphi^{h_1}-\varphi_H),\nabla(\psi_H))_p=0\quad\forall\,\psi_H\in H_{pH}.$$

Hence,

$$\begin{aligned} |a_p(\varphi^{h_1}-\varphi_H,\xi)|&=\inf_{\forall\,\psi_H\in H_{pH}}|a_p(\varphi^{h_1}-\varphi_H,\xi-\psi_H)|\\ &\le C\|\nabla(\varphi^{h_1}-\varphi_H)\|_p\inf_{\forall\,\psi_H\in H_{pH}}\|\nabla(\xi-\psi_H)\|_p\\ &\le C(\|\nabla(\varphi^{h_1}-\varphi_{h_1})\|_p+\|\nabla(\varphi_{h_1}-\varphi_H)\|_p)\cdot H\|\xi\|_{H^2(\Omega_p)}\\ &\le CH^2\|\nabla(u_{h_1}-u^{h_1})\|_f. \end{aligned} \tag{4.17}$$

Combining (4.15) with (4.16) and (4.17) leads to

$$|g(\varphi_{h_1}-\varphi_H,(u_{h_1}-u^{h_1})\cdot n_f)_\Gamma|\le CH^2\|\nabla(u_{h_1}-u^{h_1})\|_f. \tag{4.18}$$

By applying (4.10), (4.11) and (4.18) into (4.9), we get

$$\|\nabla(u_{h_1}-u^{h_1})\|_f\le CH^2. \tag{4.19}$$

Next, from (4.6), (4.4) and (4.19) , we can derive that

$$\begin{aligned}
&|b(v_{h_1}, p_{h_1} - p^{h_1})| \\
&\quad \le |a_f(u_{h_1} - u^{h_1}, v_{h_1})| + |g(\varphi_{h_1} - \varphi_H, v_{h_1} \cdot n_f)_\Gamma| \\
&\qquad + |a_N(u_H; u_{h_1} - u^{h_1}, v_{h_1}) + a_N(u_{h_1} - u^{h_1}; u_H, v_{h_1}) + a_N(u_{h_1} - u_H; u_{h_1} - u_H, v_{h_1})| \\
&\quad \le C(\|\nabla(u_{h_1} - u^{h_1})\|_f + \|\nabla(u_{h_1} - u_H)\|_f^2 + \|\nabla(\varphi^{h_1} - \varphi_{h_1})\|_p)\|\nabla v_{h_1}\|_f \\
&\quad \le CH^2\|\nabla v_{h_1}\|_f. \qquad (4.20)
\end{aligned}$$

In view of the discrete inf-sup condition (3.1), we have

$$\|\nabla(p_{h_1} - p^{h_1})\|_f \le CH^2. \qquad (4.21)$$

Then (4.5) follows from (4.19) and (4.21). □

Together Lemma 4.1, Theorem 4.1 and triangle inequalities, we can easily obtain the following theorem.

Theorem 4.2. *The solution* $(u^{h_1}, p^{h_1}, \varphi^{h_1})$ *defined by the problem* (3.6)–(3.8) *satisfies*

$$\|\nabla(\varphi - \varphi^{h_1})\|_p \le C(h_1 + H^2), \qquad (4.22)$$

$$\|\nabla(u - u^{h_1})\|_f + \|p - p^{h_1}\|_f \le C(h_1 + H^2). \qquad (4.23)$$

Now, we will give the convergence results for the final step solution $(u^{h_{i+1}}, p^{h_{i+1}}, \varphi^{h_{i+1}})$ of the proposed multi-grid method.

Theorem 4.3. *The solution* $(u^{h_{i+1}}, p^{h_{i+1}}, \varphi^{h_{i+1}})$ *defined by the solution* (3.9)–(3.11) *satisfies*

$$\|\nabla(\varphi_{h_{i+1}} - \varphi^{h_{i+1}})\|_p \le C(h_i\|\nabla(u - u^{h_i})\|_f + \|u - u^{h_i}\|_f), \qquad (4.24)$$

and,

$$\begin{aligned}
&\|\nabla(u_{h_{i+1}} - u^{h_{i+1}})\|_f + \|p_{h_{i+1}} - p^{h_{i+1}}\|_f \\
&\quad \le C(h_i\|\nabla(\varphi - \varphi^{h_i})\|_p + h_i\|\nabla(u - u^{h_i})\|_f + \|\nabla(u - u^{h_i})\|_f^2 + \|u - u^{h_i}\|_f). \qquad (4.25)
\end{aligned}$$

Proof. First, we will show (4.24).

Setting $h = h_{i+1}$ in the problem (3.2)–(3.3), then we have the following two coupled problems: for $\forall\, \psi_{h_{i+1}} \in H_{ph_{i+1}}$,

$$a_p(\varphi_{h_{i+1}}, \psi_{h_{i+1}}) = g(f_2, \psi_{h_{i+1}})_p + g(\psi_{h_{i+1}}, u_{h_{i+1}} \cdot n_f)_\Gamma, \qquad (4.26)$$

and for $\forall v_{h_{i+1}} \in H_{fh_{i+1}}$, $\forall q_{h_{i+1}} \in Q_{h_{i+1}}$,

$$\begin{aligned}
&a_f(u_{h_{i+1}}, v_{h_{i+1}}) + a_N(u_{h_{i+1}}; u_{h_{i+1}}, v_{h_{i+1}}) + b_f(v_{h_{i+1}}, p_{h_{i+1}}) \\
&\quad = (f_1, v_{h_{i+1}})_f - g(\varphi_{h_{i+1}}, v_{h_{i+1}} \cdot n_f)_\Gamma, \qquad (4.27) \\
&b_f(u_{h_{i+1}}, q_{h_{i+1}}) = 0. \qquad (4.28)
\end{aligned}$$

Comparing (4.26) with (3.11), and letting $\psi_{h_{i+1}} = \varphi_{h_{i+1}} - \varphi^{h_{i+1}}$, we have

$$a_p(\varphi_{h_{i+1}} - \varphi^{h_{i+1}}, \varphi_{h_{i+1}} - \varphi^{h_{i+1}}) = g(\varphi_{h_{i+1}} - \varphi^{h_{i+1}}, (u_{h_{i+1}} - u^{h_i}) \cdot n_f)_\Gamma. \qquad (4.29)$$

To further estimate, we introduce an auxiliary problem in fluid region Ω_f: find $\chi \in H^1(\Omega_f)$, such that

$$\begin{cases}
-\triangle\chi = 0 & \text{in } \Omega_f, \\
\chi = \varphi_{h_{i+1}} - \varphi^{h_{i+1}} & \text{on } \Gamma, \\
\chi = 0 & \text{on } \partial\Omega_f \backslash \Gamma.
\end{cases}$$

Easily, we have

$$\|\chi\|_{H^1(\Omega_f)} \le C\|\varphi_{h_{i+1}} - \varphi^{h_{i+1}}\|_{H_{00}^{1/2}(\Gamma)} \le C\|\nabla(\varphi_{h_{i+1}} - \varphi^{h_{i+1}})\|_p. \qquad (4.30)$$

Note that $u_{h_{i+1}}$ and u^{h_i} satisfy the divergence-free property, we have,

$$(q_{h_i}, \nabla \cdot (u_{h_{i+1}} - u^{h_i}))_f = 0 \quad \forall\, q_{h_i} \in Q_{h_i}. \tag{4.31}$$

Then, for $\forall\, q_{h_i} \in Q_{h_i}$, we get

$$\begin{aligned}(\varphi_{h_{i+1}} - \varphi^{h_{i+1}}, (u_{h_{i+1}} - u^{h_i}) \cdot n_f)_\Gamma \\ &= (\varphi_{h_{i+1}} - \varphi^{h_{i+1}}, (u_{h_{i+1}} - u^{h_i}) \cdot n_f)_{\partial\Omega_f} \\ &= (\chi, \nabla \cdot (u_{h_{i+1}} - u^{h_i}))_f + (u_{h_{i+1}} - u^{h_i}, \nabla\chi)_f \\ &= (\chi - q_{h_i}, \nabla \cdot (u_{h_{i+1}} - u^{h_i}))_f + (u_{h_{i+1}} - u^{h_i}, \nabla\chi)_f.\end{aligned} \tag{4.32}$$

It follows from (4.29) and (4.32) that

$$\begin{aligned}&\|\nabla(\varphi_{h_{i+1}} - \varphi^{h_{i+1}})\|_p^2 \\ &\leq C \inf_{\forall\, q_{h_i} \in Q_{h_i}} |(\chi - q_{h_i}, \nabla \cdot (u_{h_{i+1}} - u^{h_i}))_f| + C|(u_{h_{i+1}} - u^{h_i}, \nabla\chi)_f| \\ &\leq C(h_i\|\nabla(u_{h_{i+1}} - u^{h_i})\|_f + \|u_{h_{i+1}} - u^{h_i}\|_f)\|\chi\|_{H^1(\Omega_f)} \\ &\leq C(h_i\|\nabla(u - u^{h_i})\|_f + \|u - u^{h_i}\|_f)\|\nabla(\varphi_{h_{i+1}} - \varphi^{h_{i+1}})\|_p,\end{aligned} \tag{4.33}$$

which yields (4.4).

Next, we will show

$$\|\nabla(u_{h_{i+1}} - u^{h_{i+1}})\|_f \leq C\left(\|\nabla(\varphi_{h_{i+1}} - \varphi^{h_{i+1}})\|_p + h_i\|\nabla(\varphi - \varphi^{h_i})\|_p + \|\nabla(u - u^{h_i})\|_f^2\right). \tag{4.34}$$

Subtracting (3.9)–(3.10) from (4.27)–(4.28), for $\forall\, v_{h_{i+1}} \in H_{fh_{i+1}}$, $\forall\, q_{h_{i+1}} \in Q_{h_{i+1}}$, gives

$$\begin{aligned}&a_f(u_{h_{i+1}} - u^{h_{i+1}}, v_{h_{i+1}}) + b_f(v_{h_{i+1}}, p_{h_{i+1}} - p^{h_{i+1}}) \\ &\quad = -a_N(u^{h_i}; u_{h_{i+1}} - u^{h_{i+1}}, v_{h_{i+1}}) - a_N(u_{h_{i+1}} - u^{h_{i+1}}; u^{h_i}, v_{h_{i+1}}) \\ &\qquad - a_N(u_{h_{i+1}} - u^{h_i}; u_{h_{i+1}} - u^{h_i}, v_{h_{i+1}}) - g(\varphi_{h_{i+1}} - \varphi^{h_i}, v_{h_{i+1}} \cdot n_f)_\Gamma,\end{aligned} \tag{4.35}$$

$$b_f(u_{h_{i+1}} - u^{h_{i+1}}, q_{h_{i+1}}) = 0. \tag{4.36}$$

Taking $v_{h_{i+1}} = u_{h_{i+1}} - u^{h_{i+1}}$ in (4.35) and $q_{h_{i+1}} = u_{h_{i+1}} - u^{h_{i+1}}$ in (4.36) leads to

$$\begin{aligned}&\nu\|\nabla(u_{h_{i+1}} - u^{h_{i+1}})\|_f^2 \\ &\leq |a_N(u^{h_i}; u_{h_{i+1}} - u^{h_{i+1}}, u_{h_{i+1}} - u^{h_{i+1}}) + a_N(u_{h_{i+1}} - u^{h_{i+1}}; u^{h_i}, u_{h_{i+1}} - u^{h_{i+1}})| \\ &\quad + |a_N(u_{h_{i+1}} - u^{h_i}; u_{h_{i+1}} - u^{h_i}, u_{h_{i+1}} - u^{h_{i+1}})| + |g(\varphi_{h_{i+1}} - \varphi^{h_i}, u_{h_{i+1}} - u^{h_{i+1}}) \cdot n_f)_\Gamma| \\ &\leq C\left(\frac{1}{\sqrt{2\nu}}\|\nabla(u_{h_{i+1}} - u^{h_{i+1}})\|_f^2 + \|\nabla(u_{h_{i+1}} - u^{h_i})\|_f^2\|\nabla(u_{h_{i+1}} - u^{h_{i+1}})\|_f\right) \\ &\quad + |g(\varphi_{h_{i+1}} - \varphi^{h_i}, (u_{h_{i+1}} - u^{h_{i+1}}) \cdot n_f)_\Gamma|,\end{aligned} \tag{4.37}$$

which yields

$$\|\nabla(u_{h_{i+1}} - u^{h_{i+1}})\|_f^2 \leq C(\|\nabla(u - u^{h_i})\|_f^2\|\nabla(u_{h_{i+1}} - u^{h_{i+1}})\|_f + |g(\varphi_{h_{i+1}} - \varphi^{h_i}, (u_{h_{i+1}} - u^{h_{i+1}}) \cdot n_f)_\Gamma|). \tag{4.38}$$

To estimate the interface term in the above inequality, we introduce the following problem: find $\zeta \in H^1(\Omega_p)$ such that,

$$\begin{cases}-\nabla \cdot (\mathbf{K}\nabla\zeta) = 0 & \text{in } \Omega_p, \\ \mathbf{K}\nabla\zeta \cdot n_p = (u_{h_{i+1}} - u^{h_{i+1}}) \cdot n_p & \text{on } \Gamma, \\ \zeta = 0 & \text{on } \partial\Omega_p\backslash\Gamma.\end{cases} \tag{4.39}$$

This problem is similar to the problem (4.12), therefore,

$$\left\|\mathbf{K}^{\frac{1}{2}}\nabla\zeta\right\|_p \leq C\|\nabla(u_{h_{i+1}} - u^{h_{i+1}})\|_f. \tag{4.40}$$

When Γ is smooth enough, we have

$$\|\zeta\|_{H^2(\Omega_p)} \leq C\|\nabla(u_{h_{i+1}} - u^{h_{i+1}})\|_f. \tag{4.41}$$

Then, for $\forall\, \psi_{h_i} \in H_{ph_i}$, we can obtain from (4.40) and (4.41) that

$$\begin{aligned}
&|g(\varphi_{h_{i+1}} - \varphi^{h_i}, (u_{h_{i+1}} - u^{h_{i+1}}) \cdot n_f)_\Gamma| \\
&\quad = |g(\varphi_{h_{i+1}} - \varphi^{h_i}, (\mathbf{K}\nabla\zeta) \cdot n_p)_{\partial\Omega_p}| \\
&\quad = |g(\varphi_{h_{i+1}} - \varphi^{h_i}, \nabla \cdot (\mathbf{K}\nabla\zeta))_p + g(\mathbf{K}\nabla(\varphi_{h_{i+1}} - \varphi^{h_i}), \nabla\zeta)_p| \\
&\quad = g|(\mathbf{K}\nabla(\varphi_{h_{i+1}} - \varphi^{h_i}), \nabla\zeta)_p| \\
&\quad = g|(\mathbf{K}\nabla(\varphi_{h_{i+1}} - \varphi^{h_{i+1}}), \nabla\zeta)_p| + g|(\mathbf{K}\nabla(\varphi^{h_{i+1}} - \varphi^{h_i}), \nabla(\zeta - \psi_{h_i}))_p| \\
&\quad \leq C(\|\nabla(\varphi_{h_{i+1}} - \varphi^{h_{i+1}})\|_p\|\nabla\zeta\|_p + \|\nabla(\varphi_{h_{i+1}} - \varphi^{h_i})\|_p \cdot h_i\|\zeta\|_{H^2(\Omega_p)}) \\
&\quad \leq C(\|\nabla(\varphi_{h_{i+1}} - \varphi^{h_{i+1}})\|_p + h_i\|\nabla(\varphi - \varphi^{h_i})\|_p)\|\nabla(u_{h_{i+1}} - u^{h_{i+1}})\|_f.
\end{aligned} \tag{4.42}$$

By using (4.38) and (4.42), we get (4.34).

Finally, thanks to (4.35) and the above estimations, we get

$$\begin{aligned}
&|b_f(v_{h_{i+1}}, p_{h_{i+1}} - p^{h_{i+1}})| \\
&\quad \leq |a_N(u^{h_i}; u_{h_{i+1}} - u^{h_{i+1}}, v_{h_{i+1}}) + a_N(u_{h_{i+1}} - u^{h_{i+1}}; u^{h_i}, v_{h_{i+1}})| \\
&\qquad + |a_N(u_{h_{i+1}} - u^{h_i}; u_{h_{i+1}} - u^{h_i}, u_{h_{i+1}} - u^{h_{i+1}})| + g|(\varphi_{h_{i+1}} - \varphi^{h_{i+1}}, v_{h_{i+1}} \cdot n_f)_\Gamma| \\
&\quad \leq C(\|\nabla(u_{h_{i+1}} - u^{h_{i+1}})\|_f + \|\nabla(u_{h_{i+1}} - u^{h_i})\|_f^2 \\
&\qquad + \|\nabla(\varphi_{h_{i+1}} - \varphi^{h_{i+1}})\|_p + h_i\|\nabla(\varphi_{h_{i+1}} - \varphi^{h_i})\|_p)\|\nabla v_{h_{i+1}}\|_f,
\end{aligned}$$

which, together with the discrete inf-sup condition and the inequality (4.34), gives

$$\begin{aligned}
&\|p_{h_{i+1}} - p^{h_{i+1}}\|_f \\
&\quad \leq C(|\nabla(u_{h_{i+1}} - u^{h_{i+1}})\|_f + \|\nabla(\varphi_{h_{i+1}} - \varphi^{h_{i+1}})\|_p + h_i\|\nabla(\varphi - \varphi^{h_i})\|_p + \|\nabla(u - u^{h_i})\|_f^2).
\end{aligned} \tag{4.43}$$

Then (4.25) follows from (4.34), (4.43) and (4.24). □

Applying Lemma 4.1, Theorem 4.3 and the triangle inequalities, we can immediately get the following theorem.

Theorem 4.4. *For $i = 1, 2, \ldots, I$, the following error estimates hold:*

$$\|\nabla(\varphi - \varphi^{h_{i+1}})\|_p \leq C(h_{i+1} + h_i\|\nabla(u - u^{h_i})\|_f + \|u - u^{h_i}\|_f), \tag{4.44}$$

and

$$\begin{aligned}
&\|\nabla(u - u^{h_{i+1}})\|_f + \|p - p^{h_{i+1}}\|_f \\
&\quad \leq C(h_{i+1} + h_i\|\nabla(\varphi - \varphi^{h_i})\|_p + h_i\|\nabla(u - u^{h_i})\|_f + \|\nabla(u - u^{h_i})\|_f^2 + \|u - u^{h_i}\|_f).
\end{aligned} \tag{4.45}$$

5. Numerical Experiment

Let the computational domain $\Omega_f = (0,1)\times(1,2)$ and $\Omega_p = (0,1)\times(0,1)$ with the interface $\Gamma = (0,1)\times\{1\}$. The exact solution satisfying the coupled Navier–Stokes–Darcy problem is

$$\begin{cases}
u = \left[x^2(y-1)^2 + y, -\frac{2}{3}x(y-1)^3 + 2 - \pi\sin(\pi x)\right]^T, \\
p = [2 - \pi\sin(\pi x)]\sin\left(\frac{\pi}{2}y\right), \\
\varphi = [2 - \pi\sin(\pi x)][1 - y - \cos(\pi y)].
\end{cases}$$

The external force f_1 and the source term f_2 follow the exact solution. For the sake of simplicity, the physical parameters ν, g, K and α are set to 1. The finite element spaces we chosen are the well-known

TABLE 1. *The convergence performance of the standard finite element method*

h	$\|\nabla(\varphi-\varphi_h)\|_p$	Rate	$\|\nabla(u-u_h)\|_f$	Rate	$\|p-p_h\|_f$	Rate	CPU
$\frac{1}{4}$	1.59477	–	1.94324	–	1.69231	–	0.312
$\frac{1}{16}$	0.375153	1.04	0.360627	1.21	0.150438	1.74	16.583
$\frac{1}{64}$	0.092468	1.01	0.091558	0.99	0.039472	0.96	1395.01

TABLE 2. *The convergence performance of the multi-grid method*

h_i	$\|\nabla(\varphi-\varphi^{h_i})\|_p$	$\|\nabla(u-u^{h_i})\|_f$	$\|p-p^{h_i}\|_f$	CPU
$H=\frac{1}{2}$	2.54989	3.07155	2.12104	0.093
$h_1=\frac{1}{4}$	1.57518	1.5484	1.26509	0.032
$h_2=\frac{1}{16}$	0.377258	0.370801	0.164897	0.483
$h_3=\frac{1}{256}$	0.023431	0.022552	0.009187	211.926

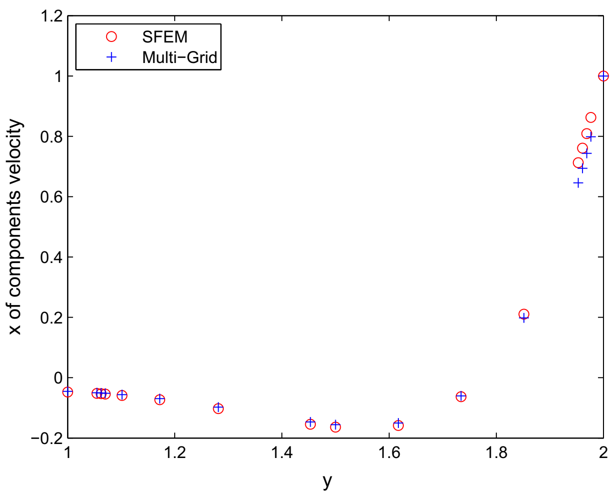

FIG. 1. Flow velocity profiles along the vertical centerline

Mini elements ($P1b-P1$) for fluid region and the linear Lagrangian elements ($P1$) for porous media region.

For comparison of the approximation accuracy, we first consider the standard finite element method, the errors between the exact solution and the numerical solutions of the standard finite element method are listed in Table 1. Observed that, the errors are of the order of $O(h)$.

Next, we compute the errors between the exact solution and the third step solutions of the multi-grid decoupling method. $h_0=H=1/2$ are set, h_i and h_{i-1} $(i=1,2,3)$ satisfy $h_i=h_{i-1}^2$. The corresponding numerical results are showed in Table 2. By a comparison, we can clearly see that the accuracy of the solutions computed by the multi-grid method is comparable to that obtained by the standard finite element method. To further illustrate the effectiveness of the proposed method, we also list the computational time. Compared with the standard finite element method, our method could save a large amount of computational time.

The second experiment, we consider a modified driven cavity flow with the Dirichlet boundary conditions for the Navier–Stokes region:

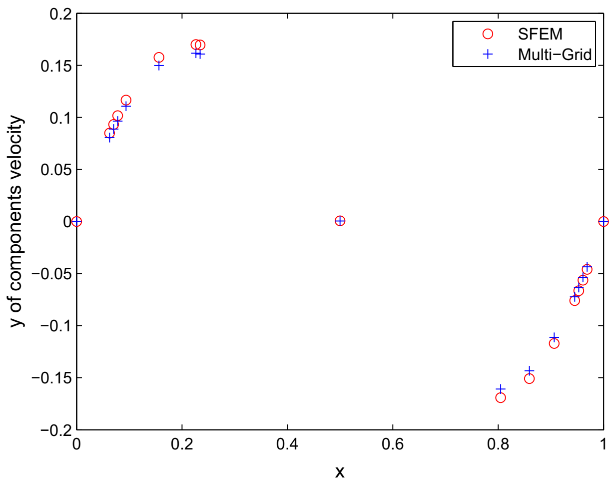

FIG. 2. Flow velocity profiles along the horizontal centerline

$$\begin{cases} u = [sin(\pi x), 0]^T & on\ (0,1) \times 2, \\ u = [0,0] & on\ 0 \times (1,2) \cup 1 \times (1,2). \end{cases}$$

Take the physical parameters g, ν, α equal to 1, and $\mathbf{K} = \mathbf{I}$. Let $f_1 = 0$ in Ω_f, $f_2 = 0$ in Ω_p, and $\varphi = 0$ on $\partial\Omega_p\ \Gamma$. In Figs. 1 and 2, we show the velocity profiles along the vertical and horizontal centerlines obtained from Algorithm 1 (SFEM), and Algorithm 2 (Multi-Grid). As seen from figures, they are almost identical each other.

In a word, we propose a multi-grid decoupling method for the coupled Navier–Stokes–Darcy problem. Numerical results suggest that our proposed method does not degrade the accuracy of the solutions. Moreover, using our method, only a much smaller global problem is solved on a very coarse initial grid, then one only need to solve a linearized Newton problem and a Darcy problem in parallel on all the subsequently refined grids. Therefore, the multi-grid decoupling method can save much more time than the standard finite element method.

Compliance with ethical standards
Conflict of interest The authors declare that they have no conflict of interest.

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