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On the Vanishing Dissipation Limit for the Full Navier–Stokes–Fourier System with Non-slip Condition

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Abstract. In this paper, we study the vanishing dissipation limit problem for the full Navier–Stokes–Fourier equations with non-slip boundary condition in a smooth bounded domain $\Omega \subseteq \mathbb{R}^3$. By using Kato's idea (Math Sci Res Inst Publ 2:85–98, 1984) of constructing an artificial boundary layer, we obtain a sufficient condition for the convergence of the solution of the full Navier–Stokes–Fourier equations to the solution of the compressible Euler equations in the energy space $L^2(\Omega)$ uniformly in time.

Mathematics Subject Classification. 35Q30, 76N20.

Keywords. Vanishing dissipation limit, full Navier–Stokes–Fourier system, non-slip condition.

Contents

1. Introduction

This paper is devoted to the issue of the vanishing dissipation limit for the full Navier–Stokes–Fourier system with non-slip boundary condition. The state of a general compressible, viscous, and heat conducting fluid can be characterized by three basic variables, the density $\rho = \rho(t, x)$, the velocity field $u = u(t, x)$, and the absolute temperature $\vartheta = \vartheta(t, x)$, where t is the time, x is the spatial variable. Moreover, the pressure p, as well as the specific energy e and the specific entropy s, is a typical thermostatic variable attributed to a system in thermodynamic equilibrium, and they can be represented as numerical functions of the density ρ and the absolute temperature ϑ. The motion of a general compressible, viscous, and heat conducting fluid with non-slip and adiabatic boundary conditions in a smooth bounded domain $\Omega \subseteq \mathbb{R}^3$, is governed by the following problem in $\{(t,x)|t>0, x\in\Omega\}$:

$$\begin{cases}\partial_t\rho+div(\rho u)=0,\\ \partial_t(\rho u)+div(\rho u\otimes u)+\nabla p(\rho,\vartheta)=div\mathbb{S}(\vartheta,\nabla u),\\ \partial_t(\rho s(\rho,\vartheta))+div(\rho s(\rho,\vartheta)u)+div\left(\frac{q(\vartheta,\nabla\vartheta)}{\vartheta}\right)=\varpi,\\ u=q\cdot\vec{n}=0,\quad \text{on }\partial\Omega\\ \rho|_{t=0}=\rho_0,(\rho u)|_{t=0}=\rho_0u_0,(\rho s(\rho,\vartheta))|_{t=0}=\rho_0 s(\rho_0,\vartheta_0),\end{cases}\tag{1.1}$$

where $\vec{n}$ denotes the unit outward normal vector to the boundary of Ω. The viscous stress $\mathbb{S}$ is described by Newton's law,

$$\mathbb{S}(\vartheta,\nabla u):=\bar{\mu}(\vartheta)\left(\nabla u+\nabla^T u-\frac{2}{3}(div\ u)Id\right)+\bar{\eta}(\vartheta)(div\ u)Id,$$

q is the heat flux satisfying Fourier's law,

$$q=-\bar{\kappa}(\vartheta)\nabla\vartheta,$$

and ϖ stands for entropy production rate satisfying

$$\varpi\geq\frac{1}{\vartheta}\left(\mathbb{S}(\vartheta,\nabla u):\nabla u-\frac{q(\vartheta,\nabla\vartheta)\cdot\nabla\vartheta}{\vartheta}\right).$$

The pressure $p=p(\rho,\vartheta)$ contains two parts:

$$p(\rho,\vartheta)=p_M(\rho,\vartheta)+p_R(\rho,\vartheta),$$

where $p_M(\rho,\vartheta)$ and $p_R(\rho,\vartheta)$ denote the molecular pressure and radiation component respectively. Accordingly, the internal energy and specific entropy read as

$$e(\rho,\vartheta)=e_M(\rho,\vartheta)+e_R(\rho,\vartheta),\ \ s(\rho,\vartheta)=s_M(\rho,\vartheta)+s_R(\rho,\vartheta)$$

respectively. At low viscosity, thermal conductivity and radiation, it is natural to think that the dissipative terms in (1.1) will be much smaller than the nonlinear terms and the motion of the fluid is assumed to be governed by the following compressible Euler system in $\{(t,x)|t>0, x\in\Omega\}$:

$$\begin{cases}\partial_t\rho^E+div(\rho^E u^E)=0,\\ \partial_t(\rho^E u^E)+div(\rho^E u^E\otimes u^E)+\nabla p_M(\rho^E,\vartheta^E)=0,\\ \partial_t(\rho^E s_M(\rho^E,\vartheta^E))+div(\rho^E s_M(\rho^E,\vartheta^E)u^E)=0,\\ (u^E\cdot\vec{n})|_{x\in\partial\Omega}=0,\\ \rho^E|_{t=0}=\rho_0^E,(\rho^E u^E)|_{t=0}=\rho_0^E u_0^E,(\rho^E s(\rho^E,\vartheta^E))|_{t=0}=\rho_0^E s(\rho_0^E,\vartheta_0^E),\end{cases}\tag{1.2}$$

However this approximation does not hold in general near the boundary $\partial\Omega$, since the full Navier–Stokes–Fourier system and the Euler system admit different boundary conditions in (1.1) and (1.2) respectively, leading to a fast change of the flow in a neighborhood of boundary $\partial\Omega$. The vanishing dissipation limit problem for the full Navier–Stokes–Fourier equations is a physically significant but challenging problem, due to this mismatch on the boundary.

The study of vanishing viscosity limit for solutions of the Navier–Stokes equations is one of the classical problems in the mathematical analysis of fluid mechanics. For the incompressible Navier–Stokes equations with non-slip boundary condition, Prandtl [10] introduced a concept of viscosity-dependent layer, that is, boundary layer of the flow near the boundary. He also derived a simplified set of equations which is now called the Prandtl equations to describe the motion of such layers. According to Prandtl's theory, the motion of the fluid in the boundary layer can be described by the Prandtl equations, and out of the boundary layer, the fluid can be approximated as inviscid one. Till now, there have been many interesting mathematical results on the well-posedness of the Prandtl equations, while the rigorous justification of Prandtl's boundary layer theory is known only for some special cases. One can see the recent survey article [9], and the references therein.

There is another way to analyze this vanishing viscosity problem without use of the Prandtl equations, which was introduced by Kato in [5]. By constructing an artificial boundary layer with thickness proportional to the viscosity, Kato studied the small viscosity limit of the incompressible viscous fluid with the non-slip boundary condition, and concluded that the viscous fluid can be approximated by the inviscid fluid in the energy space under a dissipation condition of energy in the boundary layer. Later, Wang [12] relaxed Kato's condition to the case only containing the tangential derivatives of the tangential or normal velocity at the expense of increasing the thickness of the layer slightly. For the compressible Navier–Stokes system, Sueur [11] gave a sufficient condition for the convergence to hold for the compressible viscous flow, which can be regarded as an extension of Kato's result in the compressible fluid. Recently, Kato's idea was also used to investigate the vanishing α limit for the Euler-α system[7], and the relation between that and the vanishing viscosity limit for the Navier–Stokes equations[8]. More results and references can be found in the survey article [9].

For the full Navier–Stokes–Fourier system, Feireisl [4] considered the vanishing dissipation limit problem with a complete slip boundary condition, and obtained a convergence result. This result is path dependent, that means, the vanishing rates of the singular parameters are interrelated in a special way. Our main goal in this paper is to develop the idea of [12] to study the vanishing dissipation limit of the full Navier–Stokes–Fourier system with non-slip boundary condition. We shall obtain a sufficient condition, which contains only the tangential or normal component of velocity and the integrability of temperature near the boundary, for the convergence to take place in the energy space $L^2(\Omega)$ uniformly in time. A key lemma what we shall use is the relative energy inequality, derived from the definition of weak solutions to the problem (1.1). To get our main result, we construct an artificial boundary layer v in Kato's way and take $U = u^E - v$ as a test function into the relative energy inequality, where u^E is a smooth solution to the problem (1.2). The existence theory of weak solutions to the full Navier–Stokes–Fourier system was given in [2]. Since the temperature can not be controlled by the relative entropy under the structural restrictions, which are required for the well-posedness of (1.1) when radiation vanishes, our result is also path dependent.

This paper is organized as follows. In Sect. 2, we present some preliminary results on the well-posedness of the problems of the full Navier–Stokes–Fourier system with non-slip condition and the Euler system, then state the main result of this paper. In Sect. 3, we introduce the relative energy inequality given in [3] and a vital lemma. The proof of the main result will be given in Sect. 4.

2. Preliminaries and the Main Result

In the following calculation, we shall use the notation C to denote a generic constant that may change from line to line.

At first, in accordance with the second law of thermodynamics, the continuously differentiable thermodynamic functions p, e and s are interrelated through the Gibbs' function:

$$\vartheta Ds(\rho,\vartheta) = De(\rho,\vartheta) + p(\rho,\vartheta) D\left(\frac{1}{\rho}\right), \tag{2.1}$$

where the symbol D stands for the differential with respect to the variables ρ, ϑ.

Let us first recall a definition from [2], of weak solutions to the full Naiver–Stokes–Fourier system with non-slip conditions.

Definition 2.1. Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. For any $T>0$, we say that a trio $\{\rho, \vartheta, u\}$ is a weak solution of the problem (1.1) for the Naiver–Stokes–Fourier system with non-slip boundary condition on $[0, T]$, if

$$\rho, \text{div } u, \rho|u|^2, p, \rho e, \rho s, \frac{1}{\vartheta}\mathbb{S} : \nabla u, \frac{1}{\vartheta^2} q \cdot \nabla\vartheta \in L^1((0,T)\times\Omega),$$
$$\rho u, \rho s u, \frac{q}{\vartheta} \in L^1((0,T)\times\Omega), \mathbb{S} \in L^1((0,T)\times\Omega),$$
$$\nabla u \in L^1((0,T); L^\alpha(\Omega)), \vartheta \in L^\alpha((0,T)\times\Omega), \nabla\vartheta \in L^\alpha((0,T)\times\Omega),$$
$$\rho(t,x) \geq 0,\ \vartheta(t,x) > 0 \text{ for } a.a.(t,x) \in (0,T)\times\Omega,\ u|_{\partial\Omega} = 0,$$

for certain $\alpha > 1$, and
(i) the identity

$$\int_\Omega \rho(\sigma,\cdot)\phi(\sigma,\cdot)dx - \int_\Omega \rho_0\phi(0,\cdot)dx = \int_0^\sigma \int_\Omega (\rho\partial_t\phi + \rho u \cdot \nabla\phi)dxdt$$

holds for any $\phi \in C^1([0,T]\times\bar{\Omega})$ and any $\sigma \in [0,T]$;
(ii) the identity

$$\int_\Omega \rho(\sigma,\cdot)u(\sigma,\cdot)\phi(\sigma,\cdot)dx - \int_\Omega \rho_0 u_0\phi(0,\cdot)dx$$
$$= \int_0^\sigma \int_\Omega (\rho u \cdot \partial_t\phi + \rho u \otimes u : \nabla\phi + p(\rho,\vartheta)div\phi - \varepsilon\mathbb{S}(\vartheta,\nabla u) : \nabla\phi)dxdt$$

holds for any $\phi \in C^1([0,T]\times\bar{\Omega};\mathbb{R}^3)$, $\phi|_{\partial\Omega} = 0$, and any $\sigma \in [0,T]$;
(iii) the identity

$$\int_\Omega \rho_0 s(\rho_0,\vartheta_0)\phi(0,\cdot)dx - \int_\Omega \rho s(\rho,\vartheta)(\sigma,\cdot)\phi(\sigma,\cdot)dx + \int_0^\sigma \int_\Omega \phi\varpi dxdt$$
$$= -\int_0^\sigma \int_\Omega \left(\rho s(\rho,\vartheta)\partial_t\phi + \rho s(\rho,\vartheta)u\cdot\nabla\phi + \frac{q(\vartheta,\nabla\vartheta)\cdot\nabla\phi}{\vartheta}\right)dxdt$$

holds for any $\phi \in C^1([0,T]\times\bar{\Omega})$, and any $\sigma \in [0,T]$;
(iv) the total energy is conserved:

$$\int_\Omega \left(\frac{1}{2}\rho|u|^2 + \rho e(\rho,\vartheta)\right)(\sigma,\cdot)dx = \int_\Omega \left(\frac{1}{2}\rho_0|u_0|^2 + \rho_0 e(\rho_0,\vartheta_0)\right)dx \tag{2.2}$$

for any $\sigma \in [0,T]$.

To quote an existence result of weak solutions given in [2], we give several assumptions on the state of functions in the Navier–Stokes–Fourier equations. Suppose that transport coefficients in (1.1) take the form

$$\bar{\mu}(\vartheta) \sim \mu(1+\vartheta), \quad 0 \leq \bar{\eta}(\vartheta) \lesssim \mu(1+\vartheta),\ \mu > 0, \tag{2.3}$$

and

$$\bar{\kappa}(\vartheta) \sim \kappa(1+\vartheta) + a\vartheta^3, \kappa, a > 0, \tag{2.4}$$

where "$a \lesssim b$" means $a \leq Cb$ for a positive constant C, "$a \sim b$" means $a \lesssim b$ and $b \lesssim a$. Assume that the pressure $p = p(\rho,\vartheta)$ can be written in the form

$$p(\rho,\vartheta) = p_M(\rho,\vartheta) + p_R(\rho,\vartheta), \tag{2.5}$$

where

$$p_M(\rho,\vartheta) = \vartheta^{\frac{5}{2}} P\left(\frac{\rho}{\vartheta^{\frac{3}{2}}}\right) \text{ and } p_R(\rho,\vartheta) = \frac{a}{3}\vartheta^4, \tag{2.6}$$

with

$$P \in C^1[0,\infty) \cap C^2(0,\infty), P(0) = 0, P'(Z) > 0 \text{ for all } Z \geq 0. \tag{2.7}$$

In addition, the internal energy has the from

$$e(\rho,\vartheta) = e_M(\rho,\vartheta) + e_R(\rho,\vartheta), \tag{2.8}$$

where, in agreement with (2.1)

$$e_M(\rho,\vartheta) = \frac{3}{2}\frac{\vartheta^{\frac{5}{2}}}{\rho}P(\frac{\rho}{\vartheta^{\frac{3}{2}}}),\ e_R(\rho,\vartheta) = a\frac{\vartheta^4}{\rho}. \tag{2.9}$$

The entropy s has the expression,

$$s(\rho,\vartheta) = s_M(\rho,\vartheta) + s_R(\rho,\vartheta), \quad \text{where } s_R(\rho,\vartheta) = \frac{4a}{3}\frac{\vartheta^3}{\rho}. \tag{2.10}$$

In accordance with the thermodynamic stability of the fluid system, we suppose

$$0 < c \le \partial_\vartheta e_M(\rho,\vartheta) \le C,\ \partial_\rho p(\rho,\vartheta) > 0,\ \text{for all } \rho,\vartheta \ge 0. \tag{2.11}$$

By virtue of the first inequality in (2.11), we have

$$0 < c \le \frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z} \le C \text{ for all } Z > 0. \tag{2.12}$$

This implies that the function $Z \to \frac{P(Z)}{Z^{\frac{5}{3}}}$ is decreasing, and we suppose that

$$\lim_{Z\to\infty} \frac{P(Z)}{Z^{\frac{5}{3}}} = P_\infty > 0. \tag{2.13}$$

Finally, in agreement with the third law of thermodynamics, we suppose that

$$s_M(\rho,\vartheta) = S\left(\frac{\rho}{\vartheta^{\frac{3}{2}}}\right), S'(Z) = -\frac{3}{2}\frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z^2} < 0, \lim_{Z\to\infty} S(Z) = 0. \tag{2.14}$$

Under the above assumptions, the existence of weak solutions to the problem (1.1) has been obtained in [2]:

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of calss $C^{2,\nu}, \nu \in (0,1)$. Assume that the initial data $(\rho_0, \vartheta_0, u_0)$ satisfy*

$$\begin{gathered} \rho_0 \ge 0,\ \vartheta_0 > 0,\ \rho_0 \in L^{\frac{5}{3}}(\Omega),\ \rho_0 s(\rho_0,\vartheta_0) \in L^1(\Omega), \\ \rho_0 u_0 = 0 \ a.a.\ on\ the\ set\ \{x \in \Omega | \rho_0(x) = 0\}, \\ \int_\Omega \rho_0 dx > 0,\ \int_\Omega \frac{|\rho_0 u_0|^2}{\rho_0} dx < \infty,\ E_0 = \int_\Omega \left(\frac{1}{2}\rho_0|u_0|^2 + \rho_0 e(\rho_0,\vartheta_0)\right) dx < \infty, \end{gathered} \tag{2.15}$$

the thermodynamic functions p, e, s and the transport coefficients μ, η, κ obey the structural hypotheses (2.3)–(2.14), then for any $T > 0$, there exists a weak solution $\{\rho, \vartheta, u\}$ of the problem (1.1) for the Navier–Stokes–Fourier equations on $[0,T]$. Moreover,

$$\begin{aligned} \vartheta \in L^\infty((0,T); L^4(\Omega)) \cap L^2(0,T; W^{1,2}(\Omega)),\ u \in L^2(0,T; W_0^{1,2}(\Omega)), \\ \rho \in C_w([0,T]; L^{\frac{5}{3}}(\Omega)) \cap C([0,T]; L^1(\Omega)),\ \rho u \in C_w([0,T]; L^{\frac{5}{4}}(\Omega)), \\ \rho(t,x) \ge 0,\ \vartheta(t,x) > 0 \ for\ a.a.(t,x) \in (0,T) \times \Omega, \end{aligned}$$

We say that $(\rho^E, \vartheta^E, u^E)$ is a classical solution of the Euler system (1.2) in $(0,T) \times \Omega$ if

$$\rho^E, \vartheta^E, u^E \in C^1([0,T] \times \bar{\Omega}),\ \rho^E(t,x) \ge \underline{\rho} > 0,\ \vartheta^E \ge \underline{\vartheta} > 0,$$

for all $(t,x) \in [0,T] \times \bar{\Omega}$, and $(\rho^E, \vartheta^E, u^E)$ satisfies (1.2). The following existence of a smooth solution to the problem (1.2) for the Euler equations can be found in many works, cf.[1,6]:

Proposition 2.2. *Let $s \ge 3$. Suppose that $(\rho_0^E, \vartheta_0^E, u_0^E) \in H^s(\Omega)$, satisfies the compatibility conditions up to order $s-1$. Then there exist a positive T_0 such that the problem (1.2) has a unique solution $(\rho^E, \vartheta^E, u^E)$ on $[0, T_0]$ such that*

$$(\rho^E, \vartheta^E, u^E) \in \cap_{k=0}^{s-1}([0,T_0]; H^{s-k}(\Omega)).$$

Denote by

$$\Gamma_\epsilon := \{x \in \Omega | d_\Omega(x) < \epsilon\}.$$

with $d_\Omega(x) := dist(x, \partial\Omega)$. The main result of this paper is as follows:

Theorem 2.1. *Assume $T > 0$ and $(\rho^E, \vartheta^E, u^E)$ is the classical solution of the problem (1.2) for the Euler equations corresponding to an initial data $(\rho_0^E, \vartheta_0^E, u_0^E)$ on $[0,T]$ as given in Proposition 2.2 such that $\rho^E \in [\underline{\rho}, \overline{\rho}], \vartheta^E \in [\underline{\vartheta}, \overline{\vartheta}]$, where $\underline{\rho} \le \overline{\rho}$, $\underline{\vartheta} \le \overline{\vartheta}$ are positive constants. For the viscosity and heat conduction coefficients $\mu, \kappa, a \in (0,1)$ given in (2.4)–(2.5), with the relations $a \sim \mu^2$ and $\kappa = O(a^{\frac{1}{4}})$, let $(\rho_0^\mu, \vartheta_0^\mu, u_0^\mu)$ be an initial data such that the problem (1.1) for the full Navier–Stokes–Fourier system has a weak solution $(\rho^\mu, \vartheta^\mu, u^\mu)$ on $[0,T]$ as given in Proposition 2.1. Assume that ρ_0^μ and ϑ_0^μ are bounded with positive lower and upper bounds, moreover the initial datum satisfy*

$$\lim_{\mu\to 0}\left(\| \rho_0^\mu - \rho_0^E \|_{L^2(\Omega)} + \| \vartheta_0^\mu - \vartheta_0^E \|_{L^2(\Omega)} + \int_\Omega \rho_0^\mu |u_0^\mu - u_0^E|^2 dx\right) = 0, \tag{2.16}$$

then we have

$$\sup_{t\in[0,T]}\left(\| \rho^\mu - \rho^E \|_{L^2(\Omega)} + \| \vartheta^\mu - \vartheta^E \|_{L^2(\Omega)} + \int_\Omega \rho^\mu |u^\mu - u^E|^2 dx\right)(t) \to 0, \tag{2.17}$$

when $\mu \to 0$, if for some $1 < \lambda \le \infty$,

$$\|\vartheta^\mu\|_{L^2(0,T;L^{2\lambda}(\Gamma_\delta))} \le C \tag{2.18}$$

and one of the following two conditions

$$\mu^{\frac{\lambda-1}{\lambda+1}} \int_{[0,T]\times\Gamma_\delta} \left(\frac{\rho^\mu (u^\mu \cdot n)}{d_\Omega}\right)^2 dxdt \to 0 \text{ when } \mu \to 0, \tag{2.19}$$

and

$$\mu^{\frac{\lambda-1}{\lambda+1}} \int_{[0,T]\times\Gamma_\delta} \left(\frac{\rho^\mu (u^\mu \cdot \tau)}{d_\Omega}\right)^2 dxdt \to 0 \text{ when } \mu \to 0, \tag{2.20}$$

holds, where $u^\mu \cdot n$ and $u^\mu \cdot \tau$ denote the normal and the tangential components of u^μ respectively, and $\delta \to 0$ when $\mu \to 0$, with $\mu = o(\delta^{1+\frac{1}{\lambda}})$.

3. Relative Energy

As in [3], we introduce the following relative energy $\mathcal{E}([\rho,\vartheta]|[\tilde\rho,\tilde\vartheta])$ of (ρ,ϑ) with respect to $(\tilde\rho,\tilde\vartheta)$:

$$\mathcal{E}([\rho,\vartheta]|[\tilde\rho,\tilde\vartheta]) = \mathcal{H}_{\tilde\vartheta}(\rho,\vartheta) - \partial_\rho \mathcal{H}_{\tilde\vartheta}(\tilde\rho,\tilde\vartheta)(\rho - \tilde\rho) - \mathcal{H}_{\tilde\vartheta}(\tilde\rho,\tilde\vartheta), \tag{3.1}$$

where $\mathcal{H}_{\tilde\vartheta}(\rho,\vartheta)$ is a thermodynamic potential termed ballistic free energy

$$\mathcal{H}_{\tilde\vartheta}(\rho,\vartheta) = \rho e(\rho,\vartheta) - \tilde\vartheta \rho s(\rho,\vartheta).$$

Next we introduce a relative energy inequality which will be used to prove our main result:

Proposition 3.1. *Let $T > 0$ and $(\rho^\mu, \vartheta^\mu, u^\mu)$ be a finite energy weak solution of the problem (1.1) for the full Naiver–Stokes–Fourier system on [0,T] associated to an initial data $(\rho_0^\mu, \vartheta_0^\mu, u_0^\mu)$. Then, for any smooth test functions (r, Θ, U), r and Θ bounded below away from zero in $[0,T]\times\Omega$ and $U|_{\partial\Omega} = 0$, we* *have the following relative energy inequality:*

$$\begin{aligned}
&\int_\Omega \left(\frac{1}{2}\rho^\mu|u^\mu - U|^2 + \mathcal{E}([\rho^\mu,\vartheta^\mu]|[r,\Theta])\right)(\sigma)dx \\
&\quad + \int_0^\sigma\int_\Omega \frac{\Theta}{\vartheta^\mu}\left(\mathbb{S}(\vartheta^\mu,\nabla u^\mu):\nabla u^\mu - \frac{q(\vartheta^\mu,\nabla\vartheta^\mu)\cdot\nabla\vartheta^\mu}{\vartheta^\mu}\right)dxdt \\
&\le \int_\Omega \left(\frac{1}{2}\rho_0^\mu|u_0^\mu - U_0|^2 + \mathcal{E}([\rho_0^\mu,\vartheta_0^\mu]|[r_0,\Theta_0])\right)dx \\
&\quad + \int_0^\sigma\int_\Omega \rho^\mu(\partial_t U + (u^\mu\cdot\nabla)U)\cdot(U-u^\mu)dxdt - \int_0^\sigma\int_\Omega p(\rho^\mu,\vartheta^\mu)divU dxdt \\
&\quad + \int_0^\sigma\int_\Omega \mathbb{S}(\vartheta^\mu,\nabla u^\mu):\nabla U dxdt - \int_0^\sigma\int_\Omega \rho^\mu(s(\rho^\mu,\vartheta^\mu)-s(r,\Theta))(\partial_t\Theta + u^\mu\cdot\nabla\Theta)dxdt \\
&\quad - \int_0^\sigma\int_\Omega \frac{q(\vartheta^\mu,\nabla\vartheta^\mu)}{\vartheta^\mu}\cdot\nabla\Theta dxdt + \int_0^\sigma\int_\Omega \left(\left(1-\frac{\rho^\mu}{r}\right)\partial_t p(r,\Theta) - \frac{\rho^\mu}{r}u^\mu\cdot\nabla p(r,\Theta)\right)dxdt \qquad (3.2)
\end{aligned}$$

for any $\sigma\in[0,T]$, where $(r_0,\Theta_0,U_0)(x) = (r,\Theta,U)|_{t=0}$.

The proof of this proposition can be found in [3].

At the end of this section, we introduce two important inequalities, which will be used in the next section.

Lemma 3.1. *For any constants*

$$0 < \underline{\underline{\rho}} < \underline{\rho} < \overline{\rho} < \overline{\overline{\rho}},\ 0 < \underline{\underline{\vartheta}} < \underline{\vartheta} < \overline{\vartheta} < \overline{\overline{\vartheta}},$$

define

$$K := \left\{(\rho,\vartheta)\in\mathbb{R}^2 | \underline{\underline{\rho}} \le \rho \le \overline{\overline{\rho}}, \underline{\underline{\vartheta}} \le \vartheta \le \overline{\overline{\vartheta}}\right\}.$$

Then there exists a constant C_K depending only on K such that for any given $\tilde\rho\in[\underline{\rho},\overline{\rho}], \tilde\vartheta\in[\underline{\vartheta},\overline{\vartheta}]$,
(i) when $(\rho,\vartheta)\in K$,

$$\mathcal{E}([\rho,\vartheta]|[\tilde\rho,\tilde\vartheta]) \sim (|\rho-\tilde\rho|^2 + |\vartheta-\tilde\vartheta|^2),$$

(ii) when $(\rho,\vartheta)\in[0,\infty)\times(0,\infty)\backslash K$,

$$\mathcal{E}([\rho,\vartheta]|[\tilde\rho,\tilde\vartheta]) \ge C_K(1+\rho+\rho e(\rho,\vartheta)+|\rho s(\rho,\vartheta)|).$$

Proof. The rough idea of the proof was given in [2]. For completeness, we give the detail calculation here.

Decompose $\mathcal{E}([\rho,\vartheta]|[\tilde\rho,\tilde\vartheta])$ into

$$\mathcal{E}([\rho,\vartheta]|[\tilde\rho,\tilde\vartheta]) = F[\rho] + G[\rho,\vartheta],$$

where

$$F[\rho] = H_{\tilde\vartheta}(\rho,\tilde\vartheta) - \partial_\rho H_{\tilde\vartheta}(\tilde\rho,\tilde\vartheta)(\rho-\tilde\rho) - H_{\tilde\vartheta}(\tilde\rho,\tilde\vartheta),\ G[\rho,\vartheta] = H_{\tilde\vartheta}(\rho,\vartheta) - H_{\tilde\vartheta}(\rho,\tilde\vartheta).$$

Using (2.1) and (2.11) we can see that

$$F'[\rho] = \partial_\rho H_{\tilde\vartheta}(\rho,\tilde\vartheta) - \partial_\rho H_{\tilde\vartheta}(\tilde\rho,\tilde\vartheta),\ F''[\rho] = \partial^2_{\rho,\rho}H_{\tilde\vartheta}(\rho,\tilde\vartheta) = \frac{1}{\rho}\partial_\rho p(\rho,\tilde\vartheta) > 0.$$

This implies

$$\rho\to F[\rho] \text{ is strictly convex,}$$

and $F[\rho]$ attains its global minimum 0 at $\rho=\tilde\rho$. On the other hand, using (2.1) again we deduce that

$$\partial_\vartheta G[\rho,\vartheta] = \partial_\vartheta H_{\tilde\vartheta}(\rho,\vartheta) = \rho\partial_\vartheta s(\rho,\vartheta)(\vartheta-\tilde\vartheta) = \frac{\rho}{\vartheta}\partial_\vartheta e(\rho,\vartheta)(\vartheta-\tilde\vartheta),$$

this means that $G[\rho,\cdot]$ is strictly decreasing for $\vartheta<\tilde\vartheta$ and strictly increasing for $\vartheta>\tilde\vartheta$ for any $\rho>0$.

If $(\rho,\vartheta)\in K$, by using the Taylor-Lagrange formula we get

$$\left[\inf_{\underline{\underline{\rho}}\le\rho\le\overline{\overline{\rho}}}\partial^2_{\rho,\rho}H_{\tilde{\vartheta}}(\rho,\tilde{\vartheta})\right](\rho-\tilde{\rho})^2\le F[\rho]\le\left[\sup_{\underline{\underline{\rho}}\le\rho\le\overline{\overline{\rho}}}\partial^2_{\rho,\rho}H_{\tilde{\vartheta}}(\rho,\tilde{\vartheta})\right](\rho-\tilde{\rho})^2.$$

Moreover, we have

$$G[\rho,\vartheta]=H_{\tilde{\vartheta}}(\rho,\vartheta)-H_{\tilde{\vartheta}}(\rho,\tilde{\vartheta})=\int_{\tilde{\vartheta}}^{\vartheta}\rho\partial_\vartheta s(\rho,t)(t-\tilde{\vartheta})dt,$$

which implies,

$$\frac{1}{2}\left[\inf_{(\rho,\vartheta)\in K}\rho\partial_\vartheta s(\rho,\vartheta)\right](\vartheta-\tilde{\vartheta})^2\le G[\rho,\vartheta]\le\frac{1}{2}\left[\inf_{(\rho,\vartheta)\in K}\rho\partial_\vartheta s(\rho,\vartheta)\right](\vartheta-\tilde{\vartheta})^2.$$

Thus, we get the assertion given in (i).

If $(\rho,\vartheta)\in[0,\infty)\times(0,\infty)\backslash K$, since

$$\partial_\rho\mathcal{E}([\rho,\vartheta]|[\tilde{\rho},\tilde{\vartheta}])=\partial_\rho H_{\tilde{\vartheta}}(\rho,\vartheta)-\partial_\rho H_{\tilde{\vartheta}}(\tilde{\rho},\tilde{\vartheta}),$$

for any fixed $\vartheta>0$, there exists a $\varrho[\vartheta]$ such that $\mathcal{E}([\rho,\vartheta]|[\tilde{\rho},\tilde{\vartheta}])$ is strictly decreasing for $\rho<\varrho[\vartheta]$ and strictly increasing for $\rho>\varrho[\vartheta]$, and attains its global minimum at $\varrho=\varrho[\vartheta]$. Notice that

$$\partial_\vartheta\mathcal{E}([\rho,\vartheta]|[\tilde{\rho},\tilde{\vartheta}])=\frac{\rho}{\vartheta}\partial_\vartheta e(\rho,\vartheta)(\vartheta-\tilde{\vartheta}),$$

we know that $\mathcal{E}([\rho,\cdot]|[\tilde{\rho},\tilde{\vartheta}])$ is strictly decreasing for $\vartheta<\tilde{\vartheta}$ and strictly increasing for $\vartheta>\tilde{\vartheta}$ when $\rho>0$. Since $\mathcal{E}([\rho,\vartheta]|[\tilde{\rho},\tilde{\vartheta}])$ attains its global minimum only at the point $(\tilde{\rho},\tilde{\vartheta})$ and ∂K is compact, we conclude that

$$\mathcal{E}([\rho,\vartheta]|[\tilde{\rho},\tilde{\vartheta}])\ge\inf_{(\rho,\vartheta)\in\partial K}\mathcal{E}([\rho,\vartheta]|[\tilde{\rho},\tilde{\vartheta}])>0,\tag{3.3}$$

for all $(\rho,\vartheta)\in[0,\infty)\times(0,\infty)\backslash K$.

Notice that $G[\rho,\vartheta]\ge0$ for all $\rho\ge0,\vartheta>0$, we have

$$\mathcal{E}([\rho,\vartheta]|[\tilde{\rho},\tilde{\vartheta}])\ge F[\rho].$$

If $\rho<\overline{\overline{\rho}}$, we get from (3.3) that

$$\mathcal{E}([\rho,\vartheta]|[\tilde{\rho},\tilde{\vartheta}])\ge C\rho.\tag{3.4}$$

If $\rho\ge\overline{\overline{\rho}}$, we define a function of ρ:

$$\mathcal{F}(\rho)=F[\rho]-\frac{F[\overline{\overline{\rho}}]}{\overline{\overline{\rho}}}\rho.$$

Since

$$\begin{aligned}\mathcal{F}'(\rho)=F'[\rho]-\frac{F[\overline{\overline{\rho}}]}{\overline{\overline{\rho}}}&=\frac{1}{\overline{\overline{\rho}}}(\overline{\overline{\rho}}\partial_\rho H_{\tilde{\vartheta}}(\rho,\tilde{\vartheta})-\tilde{\rho}\partial_\rho H_{\tilde{\vartheta}}(\tilde{\rho},\tilde{\vartheta})-H_{\tilde{\vartheta}}(\overline{\overline{\rho}},\tilde{\vartheta})+H_{\tilde{\vartheta}}(\tilde{\rho},\tilde{\vartheta}))\\&>\frac{\overline{\overline{\rho}}-\tilde{\rho}}{\overline{\overline{\rho}}}(\partial_\rho H_{\tilde{\vartheta}}(\overline{\rho},\tilde{\vartheta})-\partial_\rho H_{\tilde{\vartheta}}(\overline{\rho},\tilde{\vartheta}))=0,\end{aligned}$$

and $\mathcal{F}(\overline{\overline{\rho}})=0$, we can obtain

$$\mathcal{F}(\rho)\ge0\text{ for all }\rho\ge\overline{\overline{\rho}},$$

this implies

$$\mathcal{E}([\rho,\vartheta]|[\tilde{\rho},\tilde{\vartheta}])\ge F[\rho]\ge\frac{F[\overline{\overline{\rho}}]}{\overline{\overline{\rho}}}\rho,\quad\text{for all }\rho\ge\overline{\overline{\rho}}.\tag{3.5}$$

Combining (3.4) with (3.5), we have

$$\mathcal{E}([\rho,\vartheta]|[\tilde{\rho},\tilde{\vartheta}]) \geq C\rho, \quad \text{for all } (\rho,\vartheta) \in [0,\infty)\times(0,\infty)\backslash K. \tag{3.6}$$

If $\rho s(\rho,\vartheta) < 0$, we can easily see that

$$H_{\tilde{\vartheta}}(\rho,\vartheta) = \rho e(\rho,\vartheta) + \tilde{\vartheta}|\rho s(\rho,\vartheta)|,$$

by using (3.3) and (3.6) we can get

$$\begin{aligned}\mathcal{E}([\rho,\vartheta]|[\tilde{\rho},\tilde{\vartheta}]) \geq& \rho e(\rho,\vartheta) + \tilde{\vartheta}|\rho s(\rho,\vartheta)| - |\partial_\rho H_{\tilde{\vartheta}}(\tilde{\rho},\tilde{\vartheta})(\rho-\tilde{\rho}) + H_{\tilde{\vartheta}}(\tilde{\rho},\tilde{\vartheta})|\\ \geq& \rho e(\rho,\vartheta) + \tilde{\vartheta}|\rho s(\rho,\vartheta)| - C\mathcal{E}([\rho,\vartheta]|[\tilde{\rho},\tilde{\vartheta}]),\end{aligned}$$

which implies

$$\mathcal{E}([\rho,\vartheta]|[\tilde{\rho},\tilde{\vartheta}]) \geq C\rho e(\rho,\vartheta) + \tilde{\vartheta}|\rho s(\rho,\vartheta)|. \tag{3.7}$$

If $\rho s(\rho,\vartheta) \geq 0$, from $\mathcal{E}([\rho,\vartheta]|[\tilde{\rho},\tilde{\vartheta}]) \geq 0$ we know

$$H_{2\tilde{\vartheta}}(\rho,\vartheta) \geq \partial_\rho H_{2\tilde{\vartheta}}(\tilde{\rho},\tilde{\vartheta})(\rho-\tilde{\rho}) + H_{2\tilde{\vartheta}}(\tilde{\rho},\tilde{\vartheta}),$$

this implies

$$\begin{aligned}H_{\tilde{\vartheta}}(\rho,\vartheta) &= \frac{1}{2}\rho e(\rho,\vartheta) + \frac{1}{2}H_{2\tilde{\vartheta}}(\rho,\vartheta) \geq \frac{1}{2}\rho e(\rho,\vartheta) + \frac{1}{2}(\partial_\rho H_{2\tilde{\vartheta}}(\tilde{\rho},\tilde{\vartheta})(\rho-\tilde{\rho}) + H_{2\tilde{\vartheta}}(\tilde{\rho},\tilde{\vartheta})),\\ H_{\tilde{\vartheta}}(\rho,\vartheta) &= \tilde{\vartheta}\rho s(\rho,\vartheta) + H_{2\tilde{\vartheta}}(\rho,\vartheta) \geq \tilde{\vartheta}\rho s(\rho,\vartheta) + \partial_\rho H_{2\tilde{\vartheta}}(\tilde{\rho},\tilde{\vartheta})(\rho-\tilde{\rho}) + H_{2\tilde{\vartheta}}(\tilde{\rho},\tilde{\vartheta}),\end{aligned}$$

thus

$$H_{\tilde{\vartheta}}(\rho,\vartheta) \geq \frac{1}{4}(\rho e(\rho,\vartheta) + \tilde{\vartheta}\rho s(\rho,\vartheta)) - |\partial_\rho H_{2\tilde{\vartheta}}(\tilde{\rho},\tilde{\vartheta})(\rho-\tilde{\rho}) + H_{2\tilde{\vartheta}}(\tilde{\rho},\tilde{\vartheta})|.$$

With the help of (3.3) and (3.6) we get

$$\mathcal{E}([\rho,\vartheta]|[\tilde{\rho},\tilde{\vartheta}]) \geq H_{\tilde{\vartheta}}(\rho,\vartheta) - C\mathcal{E}([\rho,\vartheta]|[\tilde{\rho},\tilde{\vartheta}]) \geq \frac{1}{4}(\rho e(\rho,\vartheta) + \tilde{\vartheta}\rho s(\rho,\vartheta)) - C\mathcal{E}([\rho,\vartheta]|[\tilde{\rho},\tilde{\vartheta}]),$$

which implies

$$\mathcal{E}([\rho,\vartheta]|[\tilde{\rho},\tilde{\vartheta}]) \geq C(\rho e(\rho,\vartheta) + \tilde{\vartheta}\rho s(\rho,\vartheta)). \tag{3.8}$$

Summing up (3.3), (3.6), (3.7) and (3.8) we have (ii). □

Remark 3.1. If the initial datum ρ_0^μ and ϑ_0^μ are bounded with positive lower and upper bounds, then by using Lemma 3.1 we can deduce from (2.16) that

$$\mathcal{E}([\rho_0,\vartheta_0]|[\rho_0^E,\vartheta_0^E]) \to 0 \text{ as } \mu \to 0.$$

4. Proof of the Main Result

In this section we will prove our main result, Theorem 2.1.

First from (2.9) and (2.13), we know

$$\lim_{\vartheta\to 0^+} e_M(\rho,\vartheta) = \frac{3}{2}\lim_{Z\to\infty}\rho^{\frac{2}{3}}\frac{P(Z)}{Z^{\frac{5}{3}}} = \frac{3}{2}\rho^{\frac{2}{3}}P_\infty. \tag{4.1}$$

Combining (2.8), (2.9), (2.11) with (4.1), we deduce

$$\rho e(\rho,\vartheta) \geq C\left(\rho^{\frac{5}{3}} + \rho\vartheta + a\vartheta^4\right). \tag{4.2}$$

Let $(\rho^\mu,\vartheta^\mu,u^\mu)$ and (ρ^E,ϑ^E,u^E) be weak and classical solutions to problems (1.1) and (1.2), respectively, as given in Propositions 2.1 and 2.2. Let

$$\mathbb{E}(t) = \int_\Omega \left(\frac{1}{2}\rho^\mu|u^\mu - u^E|^2 + \mathcal{E}([\rho^\mu,\vartheta^\mu]|[\rho^E,\vartheta^E])\right)dx.$$

It is easy to deduce that

$$\mathbb{E}(t) \in L^\infty(0,T). \tag{4.3}$$

For simplicity of notations, we drop the index μ in the following calculation.

We introduce a Kato type "fake" boundary layer, by defining

$$v := \xi\left(\frac{d_\Omega(x)}{\delta}\right) u^E|_{\partial\Omega},$$

with

$$\xi \in C^\infty[0,\infty), \quad \xi(0)=1, \quad \|\, \xi \,\|_{L^\infty} < \infty, \quad \|\, \xi' \,\|_{L^\infty} < \infty, \quad \operatorname{supp} \xi \subseteq [0,1)$$

and $\delta = \delta(\mu)$ tending to zero as $\mu \to 0$, which will be determined later.

It is obvious to see that v has the following properties:

$$\begin{aligned} v_n = 0,\ \|v\|_{L^\infty([0,T]\times\Omega)} &= O(1), \\ \|\partial_t v\|_{L^\infty([0,T]\times\Omega)} = O(1),\ \|div\ v\|_{L^\infty([0,T]\times\Omega)} &= O(1), \\ \|\partial_\tau v_\tau\|_{L^\infty([0,T]\times\Omega)} = O(1),\ \|\partial_n v_\tau\|_{L^\infty([0,T]\times\Omega)} &= O(\delta^{-1}), \end{aligned} \tag{4.4}$$

where v_n and v_τ denote the normal and the tangential components of v, ∂_n and ∂_τ denote the normal and the tangential derivatives respectively.

Set $(r,\Theta,U) = (\rho^E, \vartheta^E, u^E - v)$. Since $U|_{\partial\Omega} = 0$, by applying Proposition 3.1 we obtain

$$\begin{aligned} &\int_\Omega \left(\frac{1}{2}\rho|u-U|^2 + \mathcal{E}([\rho,\vartheta]|[\rho^E,\vartheta^E])\right)dx + \int_0^\sigma\!\!\int_\Omega \frac{\vartheta^E}{\vartheta}\left(\mathbb{S}(\vartheta,\nabla u):\nabla u - \frac{q(\vartheta,\nabla\vartheta)\cdot\nabla\vartheta}{\vartheta}\right)dxdt \\ &\quad \le \int_\Omega \left(\frac{1}{2}\rho_0|u_0 - U(0,\cdot)|^2 + \mathcal{E}([\rho_0,\vartheta_0]|[\rho^E(0,\cdot),\vartheta^E(0,\cdot)])\right)dx \\ &\qquad + \int_0^\sigma\!\!\int_\Omega \rho(\partial_t U + (u\cdot\nabla)U)\cdot(U-u)dxdt - \int_0^\sigma\!\!\int_\Omega p(\rho,\vartheta) div\ U dxdt \\ &\qquad + \int_0^\sigma\!\!\int_\Omega \mathbb{S}(\vartheta,\nabla u):\nabla U dxdt - \int_0^\sigma\!\!\int_\Omega \rho(s(\rho,\vartheta) - s(\rho^E,\vartheta^E))(\partial_t\vartheta^E + u\cdot\nabla\vartheta^E)dxdt \\ &\qquad - \int_0^\sigma\!\!\int_\Omega \frac{q(\vartheta,\nabla\vartheta)}{\vartheta}\cdot\nabla\vartheta^E dxdt + \int_0^\sigma\!\!\int_\Omega \left(\left(1-\frac{\rho}{\rho^E}\right)\partial_t p(\rho^E,\vartheta^E) - \frac{\rho}{\rho^E}u\cdot\nabla p(\rho^E,\vartheta^E)\right)dxdt. \end{aligned} \tag{4.5}$$

We divide the terms on the right hand side of (4.5) into four parts. The first part comes from the initial data

$$E_0 = \int_\Omega \left(\frac{1}{2}\rho_0|u_0 - U(0,\cdot)|^2 + \mathcal{E}([\rho_0,\vartheta_0]|[\rho^E(0,\cdot),\vartheta^E(0,\cdot)])\right)dx,$$

the second one is the mobility part,

$$\begin{aligned} \mathcal{R}_1 = &\int_0^\sigma\!\!\int_\Omega \rho(\partial_t U + (u\cdot\nabla)U)\cdot(U-u)dxdt - \int_0^\sigma\!\!\int_\Omega p_M(\rho,\vartheta) div\ U dxdt \\ &- \int_0^\sigma\!\!\int_\Omega \rho(s_M(\rho,\vartheta) - s_M(\rho^E,\vartheta^E))(\partial_t\vartheta^E + u\cdot\nabla\vartheta^E)dxdt \\ &+ \int_0^\sigma\!\!\int_\Omega \left(\left(1-\frac{\rho}{\rho^E}\right)\partial_t p_M(\rho^E,\vartheta^E) - \frac{\rho}{\rho^E}u\cdot\nabla p_M(\rho^E,\vartheta^E)\right)dxdt, \end{aligned}$$

the third part is the radiation component,

$$\begin{aligned} \mathcal{R}_2 = &-\int_0^\sigma\!\!\int_\Omega p_R(\rho,\vartheta) div\ U dxdt - \int_0^\sigma\!\!\int_\Omega \rho(s_R(\rho,\vartheta) - s_R(\rho^E,\vartheta^E))(\partial_t\vartheta^E + u\cdot\nabla\vartheta^E)dxdt \\ &+ \int_0^\sigma\!\!\int_\Omega \left(\left(1-\frac{\rho}{\rho^E}\right)\partial_t p_R(\rho^E,\vartheta^E) - \frac{\rho}{\rho^E}u\cdot\nabla p_R(\rho^E,\vartheta^E)\right)dxdt, \end{aligned}$$

and the last one is the dissipation component,

$$\mathcal{R}_3 = \int_0^\sigma \int_\Omega \mathbb{S}(\vartheta, \nabla u) : \nabla U dxdt - \int_0^\sigma \int_\Omega \frac{q(\vartheta, \nabla \vartheta)}{\vartheta} \cdot \nabla \vartheta^E dxdt.$$

In next subsections, we shall estimate each part and then get the main result finally by using the Gronwall inequality.

4.1. Estimates of the Mobility Terms

With the Euler equation (1.2) and the Gibbs' relation (2.1), we consider $\mathcal{R}_1$ in this subsection. Decompose the first term of $\mathcal{R}_1$ as follows

$$\begin{aligned}
&\int_0^\sigma \int_\Omega \rho(\partial_t U + (u \cdot \nabla)U) \cdot (U-u) dxdt \\
&\quad = \int_0^\sigma \int_\Omega \rho(\partial_t u^E + (u^E \cdot \nabla)u^E) \cdot (U-u)dxdt + \int_0^\sigma \int_\Omega \rho(u-u^E) \cdot \nabla u^E \cdot (U-u)dxdt \\
&\quad\quad - \int_0^\sigma \int_\Omega \rho(\partial_t v + (u \cdot \nabla)v) \cdot (U-u)dxdt. \qquad (4.6)
\end{aligned}$$

From (1.2) we have that

$$\partial_t u^E + (u^E \cdot \nabla)u^E = -\frac{1}{\rho^E} \nabla p_M(\rho^E, \vartheta^E),$$

which together with (4.6) gives that

$$\begin{aligned}
&\int_0^\sigma \int_\Omega \rho(\partial_t U + (u \cdot \nabla)U) \cdot (U-u)dxdt \\
&\quad = \int_0^\sigma \int_\Omega \frac{\rho}{\rho^E} \nabla p_M(\rho^E, \vartheta^E) \cdot (u-U)dxdt + \int_0^\sigma \int_\Omega \rho(u-u^E) \cdot \nabla u^E \cdot (U-u)dxdt \\
&\quad\quad - \int_0^\sigma \int_\Omega \rho(\partial_t v + (u \cdot \nabla)v) \cdot (U-u)dxdt \\
&\quad = \int_0^\sigma \int_\Omega \frac{\rho - \rho^E}{\rho^E} \nabla p_M(\rho^E, \vartheta^E) \cdot (u-U)dxdt - \int_0^\sigma \int_\Omega div(u-U)p_M(\rho^E, \vartheta^E)dxdt \\
&\quad\quad + \int_0^\sigma \int_\Omega \rho(u-u^E) \cdot \nabla u^E \cdot (U-u)dxdt - \int_0^\sigma \int_\Omega \rho(\partial_t v + (u \cdot \nabla)v) \cdot (U-u)dxdt. \qquad (4.7)
\end{aligned}$$

Next we decompose the last term of $\mathcal{R}_1$ into two parts,

$$\begin{aligned}
&\int_0^\sigma \int_\Omega \left(\left(1 - \frac{\rho}{\rho^E}\right) \partial_t p_M(\rho^E, \vartheta^E) - \frac{\rho}{\rho^E} u \cdot \nabla p_M(\rho^E, \vartheta^E) \right) dxdt \\
&= \int_0^\sigma \int_\Omega \left(1 - \frac{\rho}{\rho^E}\right) \left(\partial_t p_M(\rho^E, \vartheta^E) + u \cdot \nabla p_M(\rho^E, \vartheta^E)\right) dxdt + \int_0^\sigma \int_\Omega p_M(\rho^E, \vartheta^E) div\ udxdt. \qquad (4.8)
\end{aligned}$$

Combining (4.7) with (4.8), it follows

$$\begin{aligned}
&\int_0^\sigma \int_\Omega \rho(\partial_t U + (u \cdot \nabla)U) \cdot (U-u)dxdt \\
&\quad + \int_0^\sigma \int_\Omega \left(\left(1 - \frac{\rho}{\rho^E}\right) \partial_t p(\rho^E, \vartheta^E) - \frac{\rho}{\rho^E} u \cdot \nabla p(\rho^E, \vartheta^E) \right) dxdt \\
&= \int_0^\sigma \int_\Omega \left(1 - \frac{\rho}{\rho^E}\right) \left(\partial_t p_M(\rho^E, \vartheta^E) + U \cdot \nabla p_M(\rho^E, \vartheta^E)\right) dxdt + \int_0^\sigma \int_\Omega p_M(\rho^E, \vartheta^E) div\ Udxdt \\
&\quad + \int_0^\sigma \int_\Omega \rho((u-u^E) \cdot \nabla)u^E \cdot (U-u)dxdt - \int_0^\sigma \int_\Omega \rho(\partial_t v + (u \cdot \nabla)v) \cdot (U-u)dxdt. \qquad (4.9)
\end{aligned}$$

We observe that

$$\partial_t p_M(\rho^E,\vartheta^E)+U\cdot\nabla p_M(\rho^E,\vartheta^E)=\partial_\rho p_M(\rho^E,\vartheta^E)\left(\partial_t\rho^E+U\cdot\nabla\rho^E\right)\\ +\partial_\vartheta p_M(\rho^E,\vartheta^E)\left(\partial_t\vartheta^E+U\cdot\nabla\vartheta^E\right).$$

Moreover, (2.1) implies that

$$\rho^E\partial_\rho s_M(\rho^E,\vartheta^E)=-\frac{1}{\rho^E}\partial_\vartheta p_M(\rho^E,\vartheta^E). \tag{4.10}$$

Using again (1.2) we get

$$\partial_t\rho^E+u^E\cdot\nabla\rho^E=-(div\ u^E)\rho^E. \tag{4.11}$$

From (4.10) and (4.11) we can obtain that

$$\begin{aligned}
&\int_0^\sigma\int_\Omega\left(1-\frac{\rho}{\rho^E}\right)\left(\partial_t p_M(\rho^E,\vartheta^E)+U\cdot\nabla p_M(\rho^E,\vartheta^E)\right)dxdt\\
&\quad=\int_0^\sigma\int_\Omega\rho^E(\rho-\rho^E)\partial_\rho s_M(\rho^E,\vartheta^E)\left(\partial_t\vartheta^E+U\cdot\nabla\vartheta^E\right)dxdt\\
&\qquad-\int_0^\sigma\int_\Omega(\rho^E-\rho)\partial_\rho p_M(\rho^E,\vartheta^E)div\ u^E dxdt-\int_0^\sigma\int_\Omega\frac{\rho^E-\rho}{\rho^E}\partial_\rho p_M(\rho^E,\vartheta^E)v\cdot\nabla\rho^E dxdt\\
&\quad=\int_0^\sigma\int_\Omega\rho^E(\rho-\rho^E)\partial_\rho s_M(\rho^E,\vartheta^E)\left(\partial_t\vartheta^E+U\cdot\nabla\vartheta^E\right)dxdt\\
&\qquad-\int_0^\sigma\int_\Omega(\rho^E-\rho)\partial_\rho p_M(\rho^E,\vartheta^E)div\ U dxdt-\int_0^\sigma\int_\Omega\frac{\rho^E-\rho}{\rho^E}\partial_\rho p_M(\rho^E,\vartheta^E)v\cdot\nabla\rho^E dxdt\\
&\qquad-\int_0^\sigma\int_\Omega(\rho^E-\rho)\partial_\rho p_M(\rho^E,\vartheta^E)div\ v dxdt.
\end{aligned}\tag{4.12}$$

Combining (4.5), (4.9) with (4.12), we conclude that

$$\begin{aligned}
\mathcal{R}_1=&-\int_0^\sigma\int_\Omega\left[\rho\left(s_M(\rho,\vartheta)-s_M(\rho^E,\vartheta^E)\right)-\rho^E\partial_\rho s_M(\rho^E,\vartheta^E)(\rho-\rho^E)\right](\partial_t\vartheta^E+U\cdot\nabla\vartheta^E)dxdt\\
&+\int_0^\sigma\int_\Omega\left(p_M(\rho^E,\vartheta^E)-\partial_\rho p_M(\rho^E,\vartheta^E)(\rho^E-\rho)-p_M(\rho,\vartheta)\right)div\ U dxdt\\
&-\int_0^\sigma\int_\Omega(\rho^E-\rho)\partial_\rho p_M(\rho^E,\vartheta^E)div\ v dxdt-\int_0^\sigma\int_\Omega\frac{\rho^E-\rho}{\rho^E}v\cdot\partial_\rho p_M(\rho^E,\vartheta^E)\nabla\rho^E\\
&+\int_0^\sigma\int_\Omega\rho((u-u^E)\cdot\nabla)u^E\cdot(U-u)dxdt+\int_0^\sigma\int_\Omega\rho(s_M(\rho,\vartheta)-s_M(\rho^E,\vartheta^E))(U-u)\cdot\nabla\vartheta^E dxdt\\
&-\int_0^\sigma\int_\Omega\rho(\partial_t v+(u\cdot\nabla)v)\cdot(U-u)dxdt\\
=&\sum_{j=1}^7\mathcal{R}_1^j,
\end{aligned}\tag{4.13}$$

with obvious notions $\mathcal{R}_1^j(1\le j\le 7)$. We shall estimate each term step by step.

Denote by

$$\Omega_{ess}=\{x\in\Omega|(\rho,\vartheta)\in K\},\ \Omega_{res}=\Omega\backslash\Omega_{ess},$$

with

$$K=\left\{(\rho,\vartheta)\in\mathbb{R}^2|\frac{\rho}{2}\le\rho\le 2\overline{\rho},\frac{\vartheta}{2}\le\vartheta\le 2\overline{\vartheta}\right\}.$$

Let $\mathcal{R}_1^j = \mathcal{R}_{1ess}^j + \mathcal{R}_{1res}^j (1 \le j \le 7)$, where $\mathcal{R}_{1ess}^j$, $\mathcal{R}_{1res}^j$ denote the integrals in the domains Ω_{ess} and Ω_{res} respectively. We will estimate $\mathcal{R}_1^1$ together with $\mathcal{R}_1^2$ by using Lemma 3.1.

First we consider the essential part $\mathcal{R}_{1ess}^1 + \mathcal{R}_{1ess}^2$. Using (4.10) and (4.11) again, we deduce

$$\begin{aligned}\partial_\vartheta p_M(\rho^E,\vartheta^E) div\ u^E &= -\frac{1}{\rho^E}\partial_\vartheta p_M(\rho^E,\vartheta^E)(\partial_t\rho^E + u^E\cdot\nabla\rho^E)\\ &= \rho^E(\partial_t s_M(\rho^E,\vartheta^E) + u^E\cdot s_M(\rho^E,\vartheta^E)) - \rho^E\partial_\vartheta s_M(\rho^E,\vartheta^E)(\partial_t\vartheta^E + u^E\cdot\nabla\vartheta^E),\end{aligned}$$

which together with (1.2) implies

$$\partial_\vartheta p_M(\rho^E,\vartheta^E) div\ u^E = -\rho^E\partial_\vartheta s_M(\rho^E,\vartheta^E)(\partial_t\vartheta^E + u^E\cdot\nabla\vartheta^E). \tag{4.14}$$

Combining (4.13) with (4.14), we conclude that

$$\begin{aligned}\mathcal{R}_{1ess}^1 + \mathcal{R}_{1ess}^2 = &- \int_0^\sigma\int_{\Omega_{ess}} \big[\rho\left(s_M(\rho,\vartheta) - s_M(\rho^E,\vartheta^E)\right) - \rho^E\partial_\rho s_M(\rho^E,\vartheta^E)(\rho-\rho^E)\\ &-\rho^E\partial_\vartheta s_M(\rho^E,\vartheta^E)(\vartheta-\vartheta^E)\big]\left(\partial_t\vartheta^E + U\cdot\nabla\vartheta^E\right)dxdt\\ &+ \int_0^\sigma\int_{\Omega_{ess}} \big(p_M(\rho^E,\vartheta^E) - \partial_\rho p_M(\rho^E,\vartheta^E)(\rho^E-\rho)\\ &-\partial_\vartheta p_M(\rho^E,\vartheta^E)(\vartheta^E-\vartheta) - p_M(\rho,\vartheta)\big)\, div\ U dxdt\\ &- \int_0^\sigma\int_{\Omega_{ess}} (\vartheta^E-\vartheta)\partial_\vartheta p_M(\rho^E,\vartheta^E) div\ v dxdt\\ &- \int_0^\sigma\int_{\Omega_{ess}} \rho^E(\vartheta^E-\vartheta)v\cdot\partial_\vartheta s_M(\rho^E,\vartheta^E)\nabla\vartheta^E dxdt\\ = &\ \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4,\end{aligned} \tag{4.15}$$

with the obvious notations $\mathcal{I}_j$, $(1 \le j \le 4)$.

With the help of the Taylor-Lagrange formula and Lemma 3.1, we have

$$\begin{aligned}\mathcal{I}_1 = &- \int_0^\sigma\int_{\Omega_{ess}} (\rho-\rho^E)(s_M(\rho,\vartheta) - s_M(\rho^E,\vartheta^E))(\partial_t\vartheta^E + U\cdot\nabla\vartheta^E)dxdt\\ &- \int_0^\sigma\int_{\Omega_{ess}} \rho^E\left(s_M(\rho,\vartheta) - \partial_\rho s_M(\rho^E,\vartheta^E)(\rho-\rho^E) - \partial_\vartheta s_M(\rho^E,\vartheta^E)(\vartheta-\vartheta^E) - s_M(\rho^E,\vartheta^E)\right)\\ &\cdot(\partial_t\vartheta^E + U\cdot\nabla\vartheta^E)dxdt\\ \le &\ C\int_0^\sigma\int_{\Omega_{ess}} |\rho-\rho^E|\left(\sup_{(\rho,\vartheta)\in K}|\partial_\rho s_M(\rho,\vartheta)||\rho-\rho^E| + \sup_{(\rho,\vartheta)\in K}|\partial_\vartheta s_M(\rho,\vartheta)||\vartheta-\vartheta^E|\right)dxdt\\ &+ C\int_0^\sigma\int_{\Omega_{ess}} \sup_{(\rho,\vartheta)\in K}|\partial^2_{\rho,\rho} s_M(\rho,\vartheta)||\rho-\rho^E|^2 + \sup_{(\rho,\vartheta)\in K}|\partial^2_{\rho,\vartheta} s_M(\rho,\vartheta)||\rho-\rho^E||\vartheta-\vartheta^E|\\ &+ \sup_{(\rho,\vartheta)\in K}|\partial^2_{\vartheta,\vartheta} s_M(\rho,\vartheta)||\vartheta-\vartheta^E|^2 dxdt\\ \le &\ C\int_0^\sigma\int_{\Omega_{ess}} |\rho-\rho^E|^2 + |\vartheta-\vartheta^E|^2 dxdt \le C\int_0^\sigma\int_\Omega \mathcal{E}([\rho,\vartheta]|[\rho^E,\vartheta^E])dxdt.\end{aligned} \tag{4.16}$$

Similarly, we can estimate $\mathcal{I}_2$ on the right-hand side of (4.15) by using (4.4) to get

$$\begin{aligned}\mathcal{I}_2 &\le C\int_0^\sigma\int_{\Omega_{ess}} \left|p_M(\rho^E,\vartheta^E) - \partial_\rho p_M(\rho^E,\vartheta^E)(\rho^E-\rho) - \partial_\vartheta p_M(\rho^E,\vartheta^E)(\vartheta^E-\vartheta) - p_M(\rho,\vartheta)\right| dxdt\\ &\le C\int_0^\sigma\int_{\Omega_{ess}} (|\rho-\rho^E|^2 + |\vartheta-\vartheta^E|^2)dxdt\\ &\le C\int_0^\sigma\int_\Omega \mathcal{E}([\rho,\vartheta]|[\rho^E,\vartheta^E])dxdt.\end{aligned} \tag{4.17}$$

Using the Young inequality, (4.4) and Lemma 3.1, we obtain

$$\mathcal{I}_3 \le C\int_0^\sigma\int_{\Gamma_\delta\cap\Omega_{ess}} |\vartheta-\vartheta^E|^2 dxdt + C\int_0^\sigma\int_{\Gamma_\delta} |div\ v|^2 dxdt$$

$$\le C\int_0^\sigma\int_\Omega \mathcal{E}([\rho,\vartheta]|[\rho^E,\vartheta^E])dxdt + C\delta. \tag{4.18}$$

Similarly, we have

$$\mathcal{I}_4 \le C\int_0^\sigma\int_{\Gamma_\delta\cap\Omega_{ess}} |\vartheta-\vartheta^E|^2 dxdt + C\int_0^\sigma\int_{\Gamma_\delta} |v|^2 dxdt$$

$$\le C\int_0^\sigma\int_\Omega \mathcal{E}([\rho,\vartheta]|[\rho^E,\vartheta^E])dxdt + C\delta. \tag{4.19}$$

Thus we get from (4.15)–(4.19) that

$$\mathcal{R}^1_{1ess} + \mathcal{R}^2_{1ess} \le C\int_0^\sigma\int_\Omega \mathcal{E}([\rho,\vartheta]|[\rho^E,\vartheta^E])dxdt + C\delta. \tag{4.20}$$

Now we consider the residual parts $\mathcal{R}^1_{1res} + \mathcal{R}^2_{1res}$. Recall that

$$\begin{aligned}\mathcal{R}^1_{1res} &+ \mathcal{R}^2_{1res}\\ &= -\int_0^\sigma\int_{\Omega_{res}} \rho\left(s_M(\rho,\vartheta) - s_M(\rho^E,\vartheta^E)\right) - \rho^E\partial_\rho s_M(\rho^E,\vartheta^E)(\rho-\rho^E)(\partial_t\vartheta^E + U\cdot\nabla\vartheta^E)dxdt\\ &\quad + \int_0^\sigma\int_{\Omega_{res}} \left(p_M(\rho^E,\vartheta^E) - \partial_\rho p_M(\rho^E,\vartheta^E)(\rho^E-\rho) - p_M(\rho,\vartheta)\right) div\ U dxdt.\end{aligned} \tag{4.21}$$

Since $p_M(\rho,\vartheta) = \frac{2}{3}\rho e_M(\rho,\vartheta)$, by using the Young inequality, (4.2), (4.4) and Lemma 3.1, we deduce that

$$\begin{aligned}\mathcal{R}^1_{1res} + \mathcal{R}^2_{1res} &\le C\int_0^\sigma\int_{\Omega_{res}} (\rho s_M(\rho,\vartheta) + \rho e_M + \rho + 1)dxdt\\ &\le C\int_0^\sigma\int_{\Omega_{res}} (\rho s(\rho,\vartheta) + \rho e(\rho,\vartheta))dxdt\\ &\le C\int_0^\sigma\int_\Omega \mathcal{E}([\rho,\vartheta]|[\rho^E,\vartheta^E])dxdt.\end{aligned} \tag{4.22}$$

Combining (4.20) with (4.22), we get

Lemma 4.1. *For the terms $\mathcal{R}^1_1$ and $\mathcal{R}^2_1$ given in (4.13), we have the estimate*

$$\mathcal{R}^1_1 + \mathcal{R}^2_1 \le C\int_0^\sigma\int_\Omega \mathcal{E}([\rho,\vartheta]|[\rho^E,\vartheta^E])dxdt + C\delta. \tag{4.23}$$

where $\mathcal{E}([\rho,\vartheta]|[\rho^E,\vartheta^E])$ is the relative energy defined in (3.1).

The estimates of the essential parts of $\mathcal{R}^3_1$ and $\mathcal{R}^4_1$ are similar to that in (4.18) and (4.19), and we can get

$$\mathcal{R}^3_{1ess} + \mathcal{R}^4_{1ess} \le C\int_0^\sigma\int_\Omega \mathcal{E}([\rho,\vartheta]|[\rho^E,\vartheta^E])dxdt + C\delta. \tag{4.24}$$

For the residual parts of $\mathcal{R}^3_1$ and $\mathcal{R}^4_1$, using Lemma 3.1 and (4.4) we have

$$\begin{aligned}\mathcal{R}^3_{1res} &= -\int_0^\sigma\int_{\Omega_{res}} (\rho^E-\rho)\partial_\rho p_M(\rho^E,\vartheta^E) div\ v dxdt\\ &\le C\int_0^\sigma\int_{\Omega_{res}} (\rho+1)dxdt\\ &\le C\int_0^\sigma\int_\Omega \mathcal{E}([\rho,\vartheta]|[\rho^E,\vartheta^E])dxdt\end{aligned} \tag{4.25}$$

and

$$\begin{aligned}\mathcal{R}_{1res}^4 &= -\int_0^\sigma\int_{\Omega_{res}} \frac{\rho^E-\rho}{\rho^E} v\cdot\partial_\rho p_M(\rho^E,\vartheta^E)\nabla\rho^E dxdt\\ &\le C\int_0^\sigma\int_{\Omega_{res}}(\rho+1)dxdt\\ &\le C\int_0^\sigma\int_\Omega \mathcal{E}([\rho,\vartheta]|[\rho^E,\vartheta^E])dxdt \end{aligned}\tag{4.26}$$

Combining (4.24) (4.25) with (4.26), we get

Lemma 4.2. *For the terms $\mathcal{R}_1^3$ and $\mathcal{R}_1^4$ given in* (4.13)*, we have the estimate*

$$\mathcal{R}_1^3+\mathcal{R}_1^4 \le C\int_0^\sigma\int_\Omega \mathcal{E}([\rho,\vartheta]|[\rho^E,\vartheta^E])dxdt + C\delta.\tag{4.27}$$

Now we estimate $\mathcal{R}_1^5$. Decompose it into two parts:

$$\begin{aligned}\mathcal{R}_1^5 &= \int_0^\sigma\int_\Omega \rho((u-u^E)\cdot\nabla)u^E\cdot(U-u)dxdt\\ &= -\int_0^\sigma\int_\Omega \rho((u-U)\cdot\nabla)u^E\cdot(u-U)dxdt - \int_0^\sigma\int_\Omega \rho(v\cdot\nabla)u^E\cdot(U-u)dxdt\\ &= \mathcal{J}_1+\mathcal{J}_2.\end{aligned}$$

It is clearly that

$$\mathcal{J}_1 \le C\int_0^\sigma\int_\Omega \rho|u-U|^2dxdt.\tag{4.28}$$

For $\mathcal{J}_2$, we can easily deduce from (4.4) and Lemma 3.1 that

$$\begin{aligned}\mathcal{J}_2 &\le C\int_0^\sigma\int_{\Omega_{ess}}\rho|v||u-U|dxdt + C\int_0^\sigma\int_{\Omega_{res}}\rho|u-U|dxdt\\ &\le C\int_0^\sigma\int_\Omega \rho|u-U|^2dxdt + \int_0^\sigma\int_{\Gamma_\delta}|v|^2dxdt + C\int_0^\sigma\int_{\Omega_{res}}\rho dxdt\\ &\le C\int_0^\sigma\int_\Omega \rho|u-U|^2dxdt + C\int_0^\sigma\int_\Omega \mathcal{E}([\rho,\vartheta]|[\rho^E,\vartheta^E])dxdt + C\delta.\end{aligned}\tag{4.29}$$

Combining (4.28) with (4.29), we have

Lemma 4.3. *For the term $\mathcal{R}_1^5$ given in* (4.13)*, we have*

$$\mathcal{R}_1^5 \le C\int_0^\sigma\int_\Omega \rho|u-U|^2dxdt + C\int_0^\sigma\int_\Omega \mathcal{E}([\rho,\vartheta]|[\rho^E,\vartheta^E])dxdt + C\delta.\tag{4.30}$$

where $\mathcal{E}([\rho,\vartheta]|[\rho^E,\vartheta^E])$ is the relative energy defined in (3.1)*.*

Now we study $\mathcal{R}_1^6$. It is easy to see that

$$\mathcal{R}_{1ess}^6 \le C\int_0^\sigma\int_{\Omega_{ess}} |\rho(s_M(\rho,\vartheta)-s_M(\rho^E,\vartheta^E))||u-U|dxdt.$$

Using the Young inequality, the Taylor-Lagrange formula and Lemma 3.1, we obtain

$$\begin{aligned}
&\int_0^\sigma\int_{\Omega_{ess}} |\rho(s_M(\rho,\vartheta)-s_M(\rho^E,\vartheta^E))||u-U|dxdt\\
&\quad\le C\int_0^\sigma\Big(\int_{\Omega_{ess}}(s_M(\rho,\vartheta)-s_M(\rho^E,\vartheta^E))^2dx\Big)^{\frac12}\left(\int_\Omega \rho|u-U|^2dx\right)^{\frac12}dt\\
&\quad\le C\int_0^\sigma \mathcal{G}(t)\left(\int_\Omega \rho|u-U|^2dx\right)^{\frac12}dt\\
&\quad\le C\int_0^\sigma\int_\Omega \mathcal{E}([\rho,\vartheta]|[\rho^E,\vartheta^E])dxdt + C\int_0^\sigma\int_\Omega \rho|u-U|^2dxdt,
\end{aligned}\tag{4.31}$$

where

$$\mathcal{G}(t)=\left(\int_{\Omega_{ess}}\sup_{(\rho,\vartheta)\in K}|\partial_\rho s_M(\rho,\vartheta)|^2|\rho-\rho^E|^2+\sup_{(\rho,\vartheta)\in K}|\partial_\vartheta s_M(\rho,\vartheta)|^2|\vartheta-\vartheta^E|^2dx\right)^{\frac12}.$$

By using (2.1), (2.5), (2.6), (2.11) and (2.12) we can deduce that

$$s_M(\rho,\vartheta)\le C\left(1+[\log\vartheta]^+ + |\log\rho|\right).$$

Thus with (4.2) and Lemma 3.1, we have

$$\begin{aligned}
\mathcal{R}^6_{1res}&\le C\int_0^\sigma\int_{\Omega_{res}}\rho(s_M(\rho,\vartheta)-s_M(\rho^E,\vartheta^E))^2dxdt + C\int_0^\sigma\int_\Omega \rho|u-U|^2dxdt\\
&\le C\int_0^\sigma\int_{\Omega_{res}}\left(\rho+\rho\vartheta+\rho^{\frac53}+C\right)dxdt + C\int_0^\sigma\int_\Omega \rho|u-U|^2dxdt\\
&\le C\int_0^\sigma\int_\Omega \mathcal{E}([\rho,\vartheta]|[\rho^E,\vartheta^E])dxdt + C\int_0^\sigma\int_\Omega \rho|u-U|^2dxdt.
\end{aligned}\tag{4.32}$$

From (4.31) and (4.32) we obtain

Lemma 4.4. *For the term $\mathcal{R}^6_1$ given in (4.13), we have the estimate*

$$\mathcal{R}^6_1\le C\int_0^\sigma\int_\Omega \mathcal{E}([\rho,\vartheta]|[\rho^E,\vartheta^E])dxdt + C\int_0^\sigma\int_\Omega \rho|u-U|^2dxdt.\tag{4.33}$$

Finally we turn to study $\mathcal{R}^7_1$. For simplicity of presentation, we consider the case of boundary being flat. As usual, one can treat the problem with a general smooth boundary, by using the technique of localization and transforming the curved boundary into a flat one. Without loss of generality we assume that the domain lies in the upper half plane, $\Omega=\{(x_1,x_2,x_3)|(x_1,x_2)\in\mathbb{R}^2, x_3>0\}$, with $\{x_3=0\}$ being the boundary.

We decompose $\mathcal{R}^7_1$ into two parts,

$$\mathcal{R}^7_1=\mathcal{A}_1+\mathcal{A}_2,\tag{4.34}$$

with

$$\mathcal{A}_1=\int_0^\sigma\int_\Omega \rho\partial_t v\cdot(u-U)dxdt,\ \mathcal{A}_2=\int_0^\sigma\int_\Omega \rho(u\cdot\nabla)v\cdot(u-U)dxdt.$$

Using (4.4) and the Hölder inequality, we estimate $\mathcal{A}_1$ as follows,

$$\begin{aligned}
\mathcal{A}_1&\le C\int_0^\sigma\int_{\Omega_{ess}}\rho|\partial_t v||u-U|dxdt + C\int_0^\sigma\int_{\Omega_{res}}\rho|u-U|dxdt\\
&\le C\int_0^\sigma\int_\Omega \rho|u-U|^2dxdt + C\int_0^\sigma\int_{\Gamma_\delta}|\partial_t v|^2dxdt + C\int_0^\sigma\int_{\Omega_{res}}\rho dxdt\\
&\le C\int_0^\sigma\int_\Omega \rho|u-U|^2dxdt + C\int_0^\sigma\int_\Omega \mathcal{E}([\rho,\vartheta]|[\rho^E,\vartheta^E])dxdt + C\delta.
\end{aligned}\tag{4.35}$$

For $\mathcal{A}_2$, with the construction of v we can decompose it into

$$\mathcal{A}_2 = \int_0^\sigma\int_\Omega \rho \sum_{i=1}^{2}\left[\left(\sum_{j=1}^{3} u_j\partial_j\right) v_i w_i\right] dxdt, \tag{4.36}$$

by denoting $w = u - U$. With (4.4), the Hölder inequality and Lemma 3.1, we get

$$\begin{aligned}\int_0^\sigma\int_\Omega \rho u_j w_i \partial_j v_i dxdt &= \int_0^\sigma\int_\Omega \rho w_j w_i \partial_j v_i dxdt + \int_0^\sigma\int_\Omega \rho U_j w_i \partial_j v_i dxdt,\\ &\le C\int_0^\sigma\int_\Omega \rho|w|^2 dxdt + \int_0^\sigma\left(\int_\Omega \rho|w_i|^2dx\right)^{\frac12}\left(\int_\Omega \rho|U_j\partial_j v_i|^2dx\right)^{\frac12} dt\\ &\le C\int_0^\sigma\int_\Omega \rho|w|^2dxdt + C\int_0^\sigma\int_\Omega \mathcal{E}([\rho,\vartheta]|[\rho^E,\vartheta^E])dxdt + C\delta\end{aligned} \tag{4.37}$$

for $i, j \in \{1, 2\}$, by noticing that

$$\begin{aligned}\int_\Omega \rho|U_j\partial_j v_i|^2dx &= \int_{\Omega_{ess}} \rho|U_j\partial_j v_i|^2dx + \int_{\Omega_{res}} \rho|U_j\partial_j v_i|^2dx\\ &\le C\int_{\Gamma_\delta\cap\Omega_{ess}} 1dx + C\int_{\Gamma_\delta\cap\Omega_{res}} \rho dx\\ &\le C\delta + C\int_\Omega \mathcal{E}([\rho,\vartheta]|[\rho^E,\vartheta^E])dx.\end{aligned}$$

Now let us study another two terms of $\mathcal{A}_2$ given in (4.36), that is,

$$\mathcal{B}_1 = \int_0^\sigma\int_\Omega \rho u_3 w_1 \partial_3 v_1 dxdt,\ \mathcal{B}_2 = \int_0^\sigma\int_\Omega \rho u_3 w_2 \partial_3 v_2 dxdt$$

The first way to estimate $\mathcal{B}_1$ is as follows. By using (4.4), the Young inequality and the Poincaré inequality, we have

$$\begin{aligned}\mathcal{B}_1 &= \int_0^\sigma\int_\Omega \frac{\rho u_3}{d_\Omega}\cdot w_1 d_\Omega \partial_3 v_1 dxdt\\ &\le C\delta\int_0^\sigma \|\frac{\rho u_3}{d_\Omega}\|_{L^2(\Gamma_\delta)}\|\partial_3 w_1\|_{L^2(\Omega)}dt\\ &\le C(\eta)\frac{\delta^2}{\mu}\|\frac{\rho u_3}{d_\Omega}\|^2_{L^2([0,T]\times\Gamma_\delta)} + \frac{\eta\mu}{8}\int_0^\sigma \|\partial_3 w_1\|^2_{L^2(\Omega)}dt,\end{aligned}$$

for any fixed $\eta > 0$. Obviously,

$$\begin{aligned}\mu\int_0^\sigma\int_\Omega |\nabla w|^2dxdt &\le \mu\int_0^\sigma\int_\Omega |\nabla u|^2dxdt + \mu\int_0^\sigma\int_{\Gamma_\delta} |\nabla v|^2dxdt + \mu\int_0^\sigma\int_\Omega |\nabla u^E|^2dxdt\\ &\le \mu\int_0^\sigma\int_\Omega |\nabla u|^2dxdt + C\frac{\mu}{\delta} + C\mu,\end{aligned}$$

thus we have

$$\mathcal{B}_1 \le \frac{\eta\mu}{8}\int_0^\sigma\int_\Omega |\nabla u|^2dxdt + C\frac{\mu}{\delta} + C\mu. \tag{4.38}$$

On the other hand, to estimate $\mathcal{B}_1$ we can decompose it into two parts,

$$\mathcal{B}_1 = \int_0^\sigma\int_\Omega \rho u_3 u_1 \partial_3 v_1 dxdt - \int_0^\sigma\int_\Omega \rho u_3 U_1 \partial_3 v_1 dxdt. \tag{4.39}$$

For the first term on the right side of (4.39), by using (4.4), the Young inequality and the Poincaré inequality we have

$$\begin{aligned}\int_0^\sigma\int_\Omega \rho u_3 u_1\partial_3 v_1 dxdt &= \int_0^\sigma\int_\Omega \frac{\rho u_1}{d_\Omega}\cdot u_3 d_\Omega \partial_3 v_1 dxdt\\ &\le C\delta\int_0^\sigma \|\frac{\rho u_1}{d_\Omega}\|_{L^2(\Gamma_\delta)}\|\partial_3 u_3\|_{L^2(\Omega)}dt\\ &\le C(\eta)\frac{\delta^2}{\mu}\|\frac{\rho u_1}{d_\Omega}\|^2_{L^2([0,T]\times\Gamma_\delta)}+\frac{\eta\mu}{8}\int_0^\sigma \|\partial_3 u_3\|^2_{L^2(\Omega)}dt,\end{aligned} \tag{4.40}$$

Next we study the second term on the right hand side of (4.39). First from the definition of weak solution of (1.1), we know that the identity

$$\int_0^\sigma\int_\Omega (\rho\partial_t\phi+\rho u\cdot\nabla\phi)dxdt=\int_\Omega \rho(\sigma,\cdot)\phi(\sigma,\cdot)dx-\int_\Omega \rho_0\phi(0,\cdot)dx \tag{4.41}$$

holds for any $\phi\in C^1([0,T]\times\bar\Omega;\mathbb{R})$ and any $\sigma\in[0,T]$. Noticing that $v_1u_1^E\in C^1([0,T]\times\bar\Omega;\mathbb{R})$, we take $\phi=v_1u_1^E$ in (4.41) and get

$$\int_\Omega \rho(\sigma,\cdot)(v_1u_1^E)(\sigma,\cdot)dx-\int_\Omega \rho_0(v_1u_1^E)(0,\cdot)dx=\int_0^\sigma\int_\Omega (\rho\partial_t(v_1u_1^E)+\rho u\cdot\nabla(v_1u_1^E))dxdt,$$

which implies

$$\begin{aligned}\int_0^\sigma\int_\Omega \rho u_3u_1^E\partial_3v_1dxdt &= \int_\Omega \rho(\sigma,\cdot)(v_1u_1^E)(\sigma,\cdot)dx-\int_\Omega \rho_0(v_1u_1^E)(0,\cdot)dx-\int_0^\sigma\int_\Omega \rho\partial_t(v_1u_1^E)dxdt\\ &\quad -\int_0^\sigma\int_\Omega \rho u_2\partial_2(v_1u_1^E)dxdt-\int_0^\sigma\int_\Omega \rho u_1\partial_1(v_1u_1^E)dxdt\\ &\quad -\int_0^\sigma\int_\Omega \rho u_3v_1\partial_3u_1^Edxdt.\end{aligned} \tag{4.42}$$

Similarly, we have the identity

$$\begin{aligned}\int_0^\sigma\int_\Omega \rho u_3v_1\partial_3v_1dxdt &= \frac12\int_\Omega \rho(\sigma,\cdot)v_1^2(\sigma,\cdot)dx-\frac12\int_\Omega \rho_0v_1^2(0,\cdot)dx-\frac12\int_0^\sigma\int_\Omega \rho\partial_t(v_1^2)dxdt\\ &\quad -\int_0^\sigma\int_\Omega \rho u_2v_1\partial_2v_1dxdt-\int_0^\sigma\int_\Omega \rho u_1v_1\partial_1v_1dxdt\end{aligned} \tag{4.43}$$

by choosing $\phi=v_1^2$ in (4.41). Thus, from (4.42) and (4.43) we have by noticing $U_1=u_1^E-v_1$,

$$\int_0^\sigma\int_\Omega \rho u_3U_1\partial_3v_1dxdt=\int_{\Gamma_\delta}\rho(\sigma,\cdot)(v_1u_1^E)(\sigma,\cdot)dx-\int_0^\sigma\int_{\Gamma_\delta}\rho w_2\partial_2(v_1u_1^E)dxdt+\mathcal{R} \tag{4.44}$$

with

$$\begin{aligned}\mathcal{R}=&-\int_{\Gamma_\delta}\rho_0(v_1u_1^E)(0,\cdot)dx \quad -\int_0^\sigma\int_{\Gamma_\delta}\rho\partial_t(v_1u_1^E)dxdt\\ &-\frac12\int_{\Gamma_\delta}\rho(\sigma,\cdot)v_1^2(\sigma,\cdot)dx \quad +\frac12\int_{\Gamma_\delta}\rho_0v_1^2(0,\cdot)dx+\frac12\int_0^\sigma\int_{\Gamma_\delta}\rho\partial_t(v_1^2)dxdt\\ &-\int_0^\sigma\int_{\Gamma_\delta}\rho w_1\partial_1(v_1u_1^E)dxdt-\int_0^\sigma\int_{\Gamma_\delta}\rho w_3v_1\partial_3u_1^Edxdt\\ &+\int_0^\sigma\int_{\Gamma_\delta}\rho w_2v_1\partial_2v_1dxdt \quad +\int_0^\sigma\int_{\Gamma_\delta}\rho w_1v_1\partial_1v_1dxdt\\ &-\int_0^\sigma\int_{\Gamma_\delta}\rho U_2\partial_2(v_1u_1^E)dxdt-\int_0^\sigma\int_{\Gamma_\delta}\rho U_1\partial_1(v_1u_1^E)dxdt-\int_0^\sigma\int_{\Gamma_\delta}\rho U_3v_1\partial_3u_1^Edxdt\end{aligned}$$

$$+\int_0^\sigma\int_{\Gamma_\delta}\rho U_2 v_1\partial_2 v_1 dxdt \quad +\int_0^\sigma\int_{\Gamma_\delta}\rho U_1 v_1\partial_1 v_1 dxdt.$$

From (4.4), we can deduce that

$$\begin{aligned}\int_{\Gamma_\delta}\rho(\sigma,\cdot)(v_1u_1^E)(\sigma,\cdot)dx &= \int_{\Gamma_\delta\cap\Omega_{ess}}\rho(\sigma,\cdot)(v_1u_1^E)(\sigma,\cdot)dx+\int_{\Gamma_\delta\cap\Omega_{res}}\rho(\sigma,\cdot)(v_1u_1^E)(\sigma,\cdot)dx\\ &\le C\int_{\Gamma_\delta}1dx+C\left(\int_{\Gamma_\delta\cap\Omega_{res}}\rho^{\frac{5}{3}}dx\right)^{\frac{3}{5}}\left(\int_{\Gamma_\delta}1dx\right)^{\frac{2}{5}}\\ &\le \frac{\eta}{16}\int_\Omega\mathcal{E}([\rho,\vartheta]|[\rho^E,\vartheta^E])dx+C(\eta)\delta. \end{aligned}\tag{4.45}$$

and

$$\begin{aligned}\int_0^\sigma\int_{\Gamma_\delta}\rho w_2\partial_2(v_1u_1^E)dxdt &\le C\int_0^\sigma\left(\int_\Omega\rho|w|^2dx\right)^{\frac{1}{2}}\left(\int_{\Gamma_\delta}\rho dx\right)^{\frac{1}{2}}dt\\ &\le C\int_0^\sigma\int_\Omega\rho|w|^2dxdt+C\int_0^\sigma\int_\Omega\mathcal{E}([\rho,\vartheta]|[\rho^E,\vartheta^E])dxdt+C\delta.\end{aligned}\tag{4.46}$$

We can estimate each term of $\mathcal{R}$ given in (4.44) in a way similar to (4.45) and (4.46) and get

$$\begin{aligned}\int_0^\sigma\int_\Omega\rho u_3U_1\partial_3v_1dxdt &\le C\int_0^\sigma\int_\Omega\rho|w|^2dxdt+C\int_0^\sigma\int_\Omega\mathcal{E}([\rho,\vartheta]|[\rho^E,\vartheta^E])dxdt\\ &\quad+\frac{3}{8}\eta\int_\Omega\mathcal{E}([\rho,\vartheta]|[\rho^E,\vartheta^E])dx+C(\eta)\delta.\end{aligned}\tag{4.47}$$

Plugging (4.40) and (4.47) into (4.39), we get

$$\begin{aligned}\mathcal{B}_1 &\le C(\eta)\frac{\delta^2}{\mu}\|\frac{\rho u_1}{d_\Omega}\|^2_{L^2([0,T]\times\Gamma_\delta)}+\frac{\eta\mu}{8}\int_0^\sigma\|\partial_3u_3\|^2_{L^2(\Omega)}dt+C\int_0^\sigma\int_\Omega\rho|w|^2dxdt\\ &\quad+\frac{3}{8}\eta\int_\Omega\mathcal{E}([\rho,\vartheta]|[\rho^E,\vartheta^E])dx+C\int_0^\sigma\int_\Omega\mathcal{E}([\rho,\vartheta]|[\rho^E,\vartheta^E])dxdt+C(\eta)\delta.\end{aligned}\tag{4.48}$$

Similar estimates can be obtained for $\mathcal{B}_2$.

Combining (4.35)–(4.38), (4.48) with (4.34) it follows

Lemma 4.5. *For the term $\mathcal{R}_1^7$ given in (4.13), we have the estimates*

$$\begin{aligned}\mathcal{R}_1^7 &\le C\int_0^\sigma\int_\Omega\rho|u-U|^2dxdt+ C\int_0^\sigma\int_\Omega\mathcal{E}([\rho,\vartheta]|[\rho^E,\vartheta^E])dxdt+\frac{\eta}{4}\mu\int_0^\sigma\int_\Omega|\nabla u|^2dxdt\\ &\quad+C(\eta)\frac{\delta^2}{\mu}\|\frac{\rho u\cdot n}{d_\Omega}\|^2_{L^2([0,T]\times\Gamma_\delta)}+C(\eta)\frac{\mu}{\delta}+C\delta+C\mu\end{aligned}\tag{4.49}$$

and

$$\begin{aligned}\mathcal{R}_1^7 &\le C\int_0^\sigma\int_\Omega\rho|u-U|^2dxdt+C\int_0^\sigma\int_\Omega\mathcal{E}([\rho,\vartheta]|[\rho^E,\vartheta^E])dxdt+\frac{\eta}{2}\mu\int_0^\sigma\int_\Omega|\nabla u|^2dxdt\\ &\quad+C(\eta)\frac{\delta^2}{\mu}\|\frac{\rho u\cdot\tau}{d_\Omega}\|^2_{L^2([0,T]\times\Gamma_\delta)}+\frac{3}{4}\eta\int_\Omega\mathcal{E}([\rho,\vartheta]|[\rho^E,\vartheta^E])dx+C(\eta)\frac{\mu}{\delta}+C\delta.\end{aligned}\tag{4.49'}$$

where $\mathcal{E}([\rho,\vartheta]|[\rho^E,\vartheta^E])$ is the relative energy defined in (3.1), and η is an arbitrary positive constant.

Summarizing the results from Lemma 4.1 to Lemma 4.5, we obtain

Proposition 4.1. *For the term $\mathcal{R}_1$ given in (4.5), we have*

$$\mathcal{R}_1 \le C\int_0^\sigma\int_\Omega \rho|u-U|^2dxdt + C\int_0^\sigma\int_\Omega \mathcal{E}([\rho,\vartheta]|[\rho^E,\vartheta^E])dxdt + \frac{\eta}{4}\mu\int_0^\sigma\int_\Omega |\nabla u|^2dxdt + C(\eta)\frac{\delta^2}{\mu}\|\frac{\rho u\cdot n}{d_\Omega}\|^2_{L^2([0,T]\times\Gamma_\delta)} + C(\eta)\frac{\mu}{\delta} + C\delta + C\mu \tag{4.50}$$

and

$$\mathcal{R}_1 \le C\int_0^\sigma\int_\Omega \rho|u-U|^2dxdt + C\int_0^\sigma\int_\Omega \mathcal{E}([\rho,\vartheta]|[\rho^E,\vartheta^E])dxdt + \frac{\eta}{2}\mu\int_0^\sigma\int_\Omega |\nabla u|^2dxdt + C(\eta)\frac{\delta^2}{\mu}\|\frac{\rho u\cdot \tau}{d_\Omega}\|^2_{L^2([0,T]\times\Gamma_\delta)} + \frac{3}{4}\eta\int_\Omega \mathcal{E}([\rho,\vartheta]|[\rho^E,\vartheta^E])dx + C(\eta)\frac{\mu}{\delta} + C\delta. \tag{4.50'}$$

where $\mathcal{E}([\rho,\vartheta]|[\rho^E,\vartheta^E])$ is the relative energy defined in (3.1), and η is an arbitrary positive constant.

4.2. Estimates of the Radiation Terms

We will estimate the radiation component $\mathcal{R}_2$ in this subsection. Recall that

$$\mathcal{R}_2 = -\int_0^\sigma\int_\Omega p_R(\rho,\vartheta)div\ Udxdt - \int_0^\sigma\int_\Omega \rho(s_R(\rho,\vartheta)-s_R(\rho^E,\vartheta^E))(\partial_t\vartheta^E + u\cdot\nabla\vartheta^E)dxdt + \int_0^\sigma\int_\Omega\left(\left(1-\frac{\rho}{\rho^E}\right)\partial_t p_R(\rho^E,\vartheta^E) - \frac{\rho}{\rho^E}u\cdot\nabla p_R(\rho^E,\vartheta^E)\right)dxdt, \tag{4.51}$$

From (2.6) and (2.10), we know $\partial_\vartheta p_R(\rho^E,\vartheta^E) = \rho^E s_R(\rho^E,\vartheta^E)$, thus we obtain that

$$\begin{aligned}
&\int_0^\sigma\int_\Omega\left(\left(1-\frac{\rho}{\rho^E}\right)\partial_t p_R(\rho^E,\vartheta^E) - \frac{\rho}{\rho^E}u\cdot\nabla p_R(\rho^E,\vartheta^E)\right)dxdt\\
&= \int_0^\sigma\int_\Omega \partial_t p_R(\rho^E,\vartheta^E)dxdt - \int_0^\sigma\int_\Omega \frac{\rho}{\rho^E}\left(\partial_t p_R(\rho^E,\vartheta^E) + u\cdot\nabla p_R(\rho^E,\vartheta^E)\right)dxdt\\
&= \int_0^\sigma\int_\Omega \partial_t p_R(\rho^E,\vartheta^E)dxdt - \int_0^\sigma\int_\Omega \frac{\rho}{\rho^E}\partial_\vartheta p_R(\rho^E,\vartheta^E)\left(\partial_t\vartheta^E + u\cdot\nabla\vartheta^E\right)dxdt\\
&= \int_0^\sigma\int_\Omega \partial_t p_R(\rho^E,\vartheta^E)dxdt - \int_0^\sigma\int_\Omega \rho s_R(\rho^E,\vartheta^E)\left(\partial_t\vartheta^E + u\cdot\nabla\vartheta^E\right)dxdt.
\end{aligned} \tag{4.52}$$

By (2.6), (2.10), from (4.51) and (4.52) we obtain

$$\mathcal{R}_2 = -\int_0^\sigma\int_\Omega p_R(\rho,\vartheta)div\ Udxdt - \int_0^\sigma\int_\Omega \rho s_R(\rho,\vartheta)(\partial_t\vartheta^E + u\cdot\nabla\vartheta^E)dxdt + \int_0^\sigma\int_\Omega \partial_t p_R(\rho^E,\vartheta^E)dxdt = \mathcal{R}_2^1 + \mathcal{R}_2^2, \tag{4.53}$$

where

$$\mathcal{R}_2^1 = -\int_0^\sigma\int_\Omega \frac{a}{3}\vartheta^4 div\ Udxdt - \int_0^\sigma\int_\Omega \frac{4a}{3}\vartheta^3\partial_t\vartheta^E dxdt + \int_0^\sigma\int_\Omega \frac{a}{3}\partial_t\left({\vartheta^E}^4\right)dxdt,$$

and

$$\mathcal{R}_2^2 = -\int_0^\sigma\int_\Omega \frac{4a}{3}\vartheta^3 u\cdot\nabla\vartheta^E dxdt.$$

From (4.2), (4.4) and Lemma 3.1, it is easy to deduce that

$$\begin{aligned}\mathcal{R}_2^1 &\leq C\int_0^\sigma\int_{\Omega_{res}} a\vartheta^4 dxdt + Ca\\ &\leq C\int_0^\sigma\int_\Omega \mathcal{E}([\rho,\vartheta]|[\rho^E,\vartheta^E])dxdt + Ca. \end{aligned}\tag{4.54}$$

We decompose $\mathcal{R}_2^2$ into

$$\mathcal{R}_2^2 = \mathcal{R}_{2ess}^2 + \mathcal{R}_{2res}^2,\tag{4.55}$$

with

$$\mathcal{R}_{2ess}^2 = -\int_0^\sigma\int_{\Omega_{ess}} \frac{4a}{3}\vartheta^3 u\cdot\nabla\vartheta^E dxdt,\ \mathcal{R}_{2res}^2 = -\int_0^\sigma\int_{\Omega_{res}} \frac{4a}{3}\vartheta^3 u\cdot\nabla\vartheta^E dxdt.$$

By (2.2) we have

$$\begin{aligned}\mathcal{R}_{2ess}^2 &\leq C\int_0^\sigma\int_{\Omega_{ess}} \frac{4a}{3}\frac{\vartheta^3}{\rho^{\frac12}}\rho^{\frac12}|u|dxdt\\ &\leq Ca\int_0^\sigma\left(\int_{\Omega_{ess}}\frac{\vartheta^6}{\rho}dx\right)^{\frac12}\left(\int_\Omega \rho|u|^2dx\right)^{\frac12}dt \leq Ca,\end{aligned}\tag{4.56}$$

and by the Hölder inequality we obtain

$$\begin{aligned}\mathcal{R}_{2res}^2 &\leq C\int_0^\sigma\int_{\Omega_{res}} a\vartheta^3|u|dxdt\\ &\leq C\int_0^\sigma\left(\int_{\Omega_{res}} a^{\frac43}\vartheta^4dx\right)^{\frac34}\left(\int_\Omega |u|^4dx\right)^{\frac14}dt.\end{aligned}\tag{4.57}$$

Since the Sobolev imbedding and the Poincaré inequality imply

$$\|u\|_{L^4(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)},$$

we deduce from (4.57) that

$$\begin{aligned}\mathcal{R}_{2ess}^2 &\leq C(\eta)a^{\frac12}\mu^{-1}\int_0^\sigma\left(\int_\Omega \mathcal{E}([\rho,\vartheta]|[\rho^E,\vartheta^E])dx\right)^{\frac32}dt + \frac{\eta}{4}\mu\int_0^\sigma\int_\Omega|\nabla u|^2dxdt\\ &\leq C(\eta)a^{\frac12}\mu^{-1}\int_0^\sigma\int_\Omega \mathcal{E}([\rho,\vartheta]|[\rho^E,\vartheta^E])dxdt + \frac{\eta}{4}\mu\int_0^\sigma\int_\Omega|\nabla u|^2dxdt,\end{aligned}\tag{4.58}$$

by using the Cauchy inequality, (4.2), (4.3) and Lemma 3.1.

Combining all the estimates above, we can deduce that

Proposition 4.2. *For the radiation term $\mathcal{R}_2$ given in (4.5), we have the estimate*

$$\mathcal{R}_2 \leq C(\eta)\left(1 + a^{\frac12}\mu^{-1}\right)\int_0^\sigma\int_\Omega \mathcal{E}([\rho,\vartheta]|[\rho^E,\vartheta^E])dxdt + \frac{\eta}{4}\mu\int_0^\sigma\int_\Omega |\nabla u|^2dxdt + Ca.\tag{4.59}$$

for an arbitrary $\eta > 0$, where $\mathcal{E}([\rho,\vartheta]|[\rho^E,\vartheta^E])$ is the relative energy defined in (3.1).

4.3. Estimates of the Dissipation Terms

With the form of the transport coefficients (2.3) and (2.4), we consider $\mathcal{R}_3$.

$$\mathcal{R}_3 = \mathcal{R}_3^1 + \mathcal{R}_3^2$$

where

$$\mathcal{R}_3^1 = \int_0^\sigma\int_\Omega \mathbb{S}(\vartheta,\nabla u):\nabla U dxdt,\ \mathcal{R}_3^2 = -\int_0^\sigma\int_\Omega \frac{q(\vartheta,\nabla\vartheta)}{\vartheta}\cdot\nabla\vartheta^E dxdt$$

It is easy to have

$$\begin{aligned}\mathcal{R}_3^1 &\le C\mu\int_0^\sigma\int_\Omega(1+\vartheta)|\nabla u||\nabla U|dxdt\\ &\le C\mu\int_0^\sigma\int_\Omega(1+\vartheta)|\nabla u||\nabla u^E|dxdt + C\mu\int_0^\sigma\int_\Omega(1+\vartheta)|\nabla u||\nabla v|dxdt\\ &= \mathcal{C}_1+\mathcal{C}_2,\end{aligned}\tag{4.60}$$

with the obvious notations $\mathcal{C}_j,\ (1\le j\le 2)$. For $\mathcal{C}_1$, we use the Young inequality to get

$$\begin{aligned}\mathcal{C}_1 &\le \frac{\eta}{8}\mu\int_0^\sigma\int_\Omega|\nabla u|^2dxdt + C(\eta)\mu\int_0^\sigma\int_\Omega(1+\vartheta)^2dxdt\\ &\le \frac{\eta}{8}\mu\int_0^\sigma\int_\Omega|\nabla u|^2dxdt + C(\eta)\mu\int_0^\sigma\int_\Omega\vartheta^2dxdt + C(\eta)\mu.\end{aligned}\tag{4.61}$$

Thanks to Lemma 3.1, we get

$$\begin{aligned}\mu\int_0^\sigma\int_\Omega\vartheta^2dxdt &= \mu\int_0^\sigma\int_{\Omega_{ess}}\vartheta^2dxdt + \mu\int_0^\sigma\int_{\Omega_{res}}\vartheta^2dxdt\\ &\le C\mu + \mu a^{-\frac12}\int_0^\sigma\int_{\Omega_{res}}(a\vartheta^4+1)dxdt\\ &\le \mu a^{-\frac12}\int_0^\sigma\int_\Omega\mathcal{E}([\rho,\vartheta]|[\rho^E,\vartheta^E])dxdt + C\mu.\end{aligned}$$

Thus we deduce from (4.61) that

$$\mathcal{C}_1 \le \frac{\eta}{8}\mu\int_0^\sigma\int_\Omega|\nabla u|^2dxdt + C(\eta)\mu a^{-\frac12}\int_0^\sigma\int_\Omega\mathcal{E}([\rho,\vartheta]|[\rho^E,\vartheta^E])dxdt + C(\eta)\mu.\tag{4.62}$$

For $\mathcal{C}_2$, by using (4.4) we know that

$$\begin{aligned}\mathcal{C}_2 &\le \frac{\eta}{8}\mu\int_0^\sigma\int_\Omega|\nabla u|^2dxdt + C(\eta)\mu\delta^{-2}\int_0^\sigma\int_{\Gamma_\delta}(1+\vartheta)^2dxdt\\ &\le \frac{\eta}{8}\mu\int_0^\sigma\int_\Omega|\nabla u|^2dxdt + C(\eta)\mu\delta^{-2}\int_0^\sigma\int_{\Gamma_\delta}\vartheta^2dxdt + C(\eta)\mu\delta^{-1}.\end{aligned}\tag{4.63}$$

By using the Hölder inequality, we have

$$\begin{aligned}\mu\delta^{-2}\int_0^\sigma\int_{\Gamma_\delta}\vartheta^2dxdt &= \mu\delta^{-2}\int_0^\sigma\int_{\Gamma_\delta\cap\Omega_{ess}}\vartheta^2dxdt + \mu\delta^{-2}\int_0^\sigma\int_{\Gamma_\delta\cap\Omega_{res}}\vartheta^2dxdt\\ &\le C\mu\delta^{-1} + C\mu\delta^{-1-\frac{1}{\lambda}}\|\vartheta\|^2_{L^2(0,T;L^{2\lambda}(\Gamma_\delta))},\end{aligned}$$

for any $1<\lambda\le\infty$. Thus from (4.63) we can deduce that

$$\begin{aligned}\mathcal{C}_2 &\le \frac{\eta}{8}\mu\int_0^\sigma\int_\Omega|\nabla u|^2dxdt + C(\eta)\mu\delta^{-1-\frac{1}{\lambda}}\|\vartheta\|^2_{L^2(0,T;L^{2\lambda}(\Gamma_\delta))}\\ &\quad + C(\eta)\mu\delta^{-1}.\end{aligned}\tag{4.64}$$

Combining (4.62), (4.64) with (4.60), we get

$$\begin{aligned}\mathcal{R}_3^1 &\le \frac{\eta}{4}\mu\int_0^\sigma\int_\Omega|\nabla u|^2dxdt + C(\eta)\mu a^{-\frac12}\int_0^\sigma\int_\Omega\mathcal{E}([\rho,\vartheta]|[\rho^E,\vartheta^E])dxdt\\ &\quad + C(\eta)\mu\delta^{-1-\frac{1}{\lambda}}\|\vartheta\|^2_{L^2(0,T;L^{2\lambda}(\Gamma_\delta))} + C(\eta)\mu\delta^{-1} + C(\eta)\mu.\end{aligned}\tag{4.65}$$

Now we consider $\mathcal{R}_3^2$. With (2.4) we know

$$\mathcal{R}_3^2 \leq C\kappa \int_0^\sigma\int_\Omega \frac{1}{\vartheta}|\nabla\vartheta||\nabla\vartheta^E|dxdt + C\kappa\int_0^\sigma\int_\Omega |\nabla\vartheta||\nabla\vartheta^E|dxdt \\ + Ca\int_0^\sigma\int_\Omega \vartheta^2|\nabla\vartheta||\nabla\vartheta^E|dxdt = \mathcal{D}_1+\mathcal{D}_2+\mathcal{D}_3, \tag{4.66}$$

with the obvious notations $\mathcal{D}_j,\ (1\leq j\leq 3)$. Using the Cauchy inequality, we estimate $\mathcal{D}_1$ as follows:

$$\mathcal{D}_1 \leq C\kappa\int_0^\sigma\int_\Omega \left|\frac{\nabla\vartheta}{\vartheta}\right|dxdt \leq \eta\kappa\int_0^\sigma\int_\Omega\left|\frac{\nabla\vartheta}{\vartheta}\right|^2dxdt + C(\eta)\kappa. \tag{4.67}$$

For $\mathcal{D}_2$, by using the Hölder inequality and Lemma 3.1 we have

$$\begin{aligned}\mathcal{D}_2 &\leq \eta\kappa\int_0^\sigma\int_\Omega\frac{|\nabla\vartheta|^2}{\vartheta}dxdt + C(\eta)\kappa\int_0^\sigma\int_\Omega\vartheta dxdt\\ &\leq \eta\kappa\int_0^\sigma\int_\Omega\frac{|\nabla\vartheta|^2}{\vartheta}dxdt + C(\eta)\kappa\int_0^\sigma\int_{\Omega_{ess}}\vartheta dxdt + C(\eta)\kappa\int_0^\sigma\int_{\Omega_{res}}\vartheta dxdt\\ &\leq \eta\kappa\int_0^\sigma\int_\Omega\frac{|\nabla\vartheta|^2}{\vartheta}dxdt + C(\eta)\kappa a^{-\frac14}\int_0^\sigma\int_{\Omega_{res}}(a\vartheta^4+1)dxdt + C(\eta)\kappa\\ &\leq \eta\kappa\int_0^\sigma\int_\Omega\frac{|\nabla\vartheta|^2}{\vartheta}dxdt + C(\eta)\kappa a^{-\frac14}\int_0^\sigma\int_\Omega\mathcal{E}([\rho,\vartheta]|[\rho^E,\vartheta^E])dxdt + C(\eta)\kappa.\end{aligned} \tag{4.68}$$

Similarly, we have

$$\begin{aligned}\mathcal{D}_3 &\leq \eta a\int_0^\sigma\int_\Omega\vartheta\left|\nabla\vartheta\right|^2dxdt + C(\eta)a\int_0^\sigma\int_\Omega\vartheta^3dxdt\\ &\leq \eta a\int_0^\sigma\int_\Omega\vartheta\left|\nabla\vartheta\right|^2dxdt + C(\eta)a^{\frac14}\int_0^\sigma\int_\Omega\mathcal{E}([\rho,\vartheta]|[\rho^E,\vartheta^E])dxdt + C(\eta)a.\end{aligned} \tag{4.69}$$

Summing up (4.66)–(4.69) we get

$$\mathcal{R}_3^2 \leq \eta\kappa\int_0^\sigma\int_\Omega\left|\frac{\nabla\vartheta}{\vartheta}\right|^2dxdt + \eta\kappa\int_0^\sigma\int_\Omega\frac{|\nabla\vartheta|^2}{\vartheta}dxdt + \eta a\int_0^\sigma\int_\Omega\vartheta\left|\nabla\vartheta\right|^2dxdt \\ + C(\eta)\left(\kappa a^{-\frac14}+a^{\frac14}\right)\int_0^\sigma\int_\Omega\mathcal{E}([\rho,\vartheta]|[\rho^E,\vartheta^E])dxdt + C(\eta)\kappa + C(\eta)a. \tag{4.70}$$

Combining (4.65) with (4.70) we get

Proposition 4.3. *For the dissipation term $\mathcal{R}_3$ given in (4.5), we have the estimate*

$$\mathcal{R}_3 \leq \frac{\eta}{4}\mu\int_0^\sigma\int_\Omega|\nabla u|^2dxdt + \eta\kappa\int_0^\sigma\int_\Omega\left|\frac{\nabla\vartheta}{\vartheta}\right|^2dxdt + \eta\kappa\int_0^\sigma\int_\Omega\frac{|\nabla\vartheta|^2}{\vartheta}dxdt \\ + \eta a\int_0^\sigma\int_\Omega\vartheta\left|\nabla\vartheta\right|^2dxdt + C(\eta)\left(\mu a^{-\frac12}+\kappa a^{-\frac14}+a^{\frac14}\right)\int_0^\sigma\int_\Omega\mathcal{E}([\rho,\vartheta]|[\rho^E,\vartheta^E])dxdt \\ + C(\eta)\mu\delta^{-1-\frac{1}{\lambda}}\|\vartheta\|^2_{L^2(0,T;L^{2\lambda}(\Gamma_\delta))} + C(\eta)\mu\delta^{-1} + C(\eta)\mu + C(\eta)\kappa + C(\eta)a. \tag{4.71}$$

for any fixed $\lambda>1$ and $\eta>0$, where $\mathcal{E}([\rho,\vartheta]|[\rho^E,\vartheta^E])$ is the relative energy defined in (3.1).

4.4. Convergence

We'll prove the main result in this subsection, by using the estimates we have obtained above.

From (4.14), by using Proposition 4.1–4.3, we obtain

$$\begin{aligned}
&\int_\Omega \left(\frac{1}{2}\rho|u-U|^2+\mathcal{E}([\rho,\vartheta]|[\rho^E,\vartheta^E])\right)dx+\int_0^\sigma\!\!\int_\Omega \frac{\vartheta^E}{\vartheta}\left(\mathbb{S}(\vartheta,\nabla u):\nabla u-\frac{q(\vartheta,\nabla\vartheta)\cdot\nabla\vartheta}{\vartheta}\right)dxdt\\
&\le \int_\Omega \left(\frac{1}{2}\rho_0|u_0-U(0,\cdot)|^2+\mathcal{E}([\rho_0,\vartheta_0]|[\rho^E(0,\cdot),\vartheta^E(0,\cdot)])\right)dx\\
&\quad + C(\eta)\left(1+\mu^{-1}a^{\frac12}+\mu a^{-\frac12}+\kappa a^{-\frac14}\right)\int_0^\sigma\!\!\left(\int_\Omega \frac{1}{2}\rho|u-U|^2dx+\int_\Omega \mathcal{E}([\rho,\vartheta]|[\rho^E,\vartheta^E])dx\right)dt\\
&\quad + \eta\kappa\int_0^\sigma\!\!\int_\Omega \left|\frac{\nabla\vartheta}{\vartheta}\right|^2dxdt+\eta\kappa\int_0^\sigma\!\!\int_\Omega \frac{|\nabla\vartheta|^2}{\vartheta}dxdt+\eta a\int_0^\sigma\!\!\int_\Omega \vartheta\,|\nabla\vartheta|^2\,dxdt\\
&\quad + \eta\mu\int_0^\sigma\!\!\int_\Omega |\nabla u|^2dxdt+C(\eta)\frac{\delta^2}{\mu}\|\frac{\rho u\cdot n}{d_\Omega}\|^2_{L^2([0,T]\times\Gamma_\delta)}+C(\eta)\mu\delta^{-1-\frac{1}{\lambda}}\|\vartheta\|^2_{L^2(0,T;L^{2\lambda}(\Gamma_\delta))}\\
&\quad + C(\eta)\left(\frac{\mu}{\delta}+\delta+\mu+\kappa+a\right),
\end{aligned}\tag{4.72}$$

and

$$\begin{aligned}
&\int_\Omega \left(\frac{1}{2}\rho|u-U|^2+\mathcal{E}([\rho,\vartheta]|[\rho^E,\vartheta^E])\right)dx+\int_0^\sigma\!\!\int_\Omega \frac{\vartheta^E}{\vartheta}\left(\mathbb{S}(\vartheta,\nabla u):\nabla u-\frac{q(\vartheta,\nabla\vartheta)\cdot\nabla\vartheta}{\vartheta}\right)dxdt\\
&\le \int_\Omega \left(\frac{1}{2}\rho_0|u_0-U(0,\cdot)|^2+\mathcal{E}([\rho_0,\vartheta_0]|[\rho^E(0,\cdot),\vartheta^E(0,\cdot)])\right)dx\\
&\quad + C(\eta)\left(1+\mu^{-1}a^{\frac12}+\mu a^{-\frac12}+\kappa a^{-\frac14}\right)\int_0^\sigma\!\!\left(\int_\Omega \frac{1}{2}\rho|u-U|^2dx+\int_\Omega \mathcal{E}([\rho,\vartheta]|[\rho^E,\vartheta^E])dx\right)dt\\
&\quad + \eta\kappa\int_0^\sigma\!\!\int_\Omega \left|\frac{\nabla\vartheta}{\vartheta}\right|^2dxdt+\eta\kappa\int_0^\sigma\!\!\int_\Omega \frac{|\nabla\vartheta|^2}{\vartheta}dxdt+\eta a\int_0^\sigma\!\!\int_\Omega \vartheta\,|\nabla\vartheta|^2\,dxdt+\eta\mu\int_0^\sigma\!\!\int_\Omega |\nabla u|^2dxdt\\
&\quad + C(\eta)\frac{\delta^2}{\mu}\|\frac{\rho\cdot\tau}{d_\Omega}\|^2_{L^2([0,T]\times\Gamma_\delta)}+\frac{3}{4}\eta\int_\Omega \mathcal{E}([\rho,\vartheta]|[\rho^E,\vartheta^E])dx+C(\eta)\mu\delta^{-1-\frac{1}{\lambda}}\|\vartheta\|^2_{L^2(0,T;L^{2\lambda}(\Gamma_\delta))}\\
&\quad + C(\eta)\left(\frac{\mu}{\delta}+\delta+\mu+\kappa+a\right).
\end{aligned}\tag{4.72'}$$

By virtue of Korn's inequality and (2.3), there exist a constant $C_1>0$ such that

$$\int_0^\sigma\!\!\int_\Omega \frac{\vartheta^E}{\vartheta}\mathbb{S}(\vartheta,\nabla u):\nabla u dxdt\ge C_1\mu\int_0^\sigma\!\!\int_\Omega |\nabla u|^2dxdt.\tag{4.73}$$

With (2.4), it is easy to deduce that

$$\begin{aligned}
&-\int_0^\sigma\!\!\int_\Omega \frac{\vartheta^E}{\vartheta}\cdot\frac{q(\vartheta,\nabla\vartheta)\cdot\nabla\vartheta}{\vartheta}dxdt\\
&\quad\ge C_2\left(\kappa\int_0^\sigma\!\!\int_\Omega \left|\frac{\nabla\vartheta}{\vartheta}\right|^2dxdt+\kappa\int_0^\sigma\!\!\int_\Omega \frac{|\nabla\vartheta|^2}{\vartheta}dxdt+a\int_0^\sigma\!\!\int_\Omega \vartheta|\nabla\vartheta|^2dxdt\right),
\end{aligned}\tag{4.74}$$

for some constant $C_2>0$. By Lemma 3.1 and (4.2) we obtain that

$$\begin{aligned}
\int_{\Gamma_\delta}\rho|v|^2dx&=\int_{\Gamma_\delta\cap\Omega_{res}}\rho|v|^2dx+\int_{\Gamma_\delta\cap\Omega_{ess}}\rho|v|^2dx\\
&\le C\left(\int_{\Gamma_\delta\cap\Omega_{res}}\rho^{\frac53}dx\right)^{\frac35}\left(\int_{\Gamma_\delta\cap\Omega_{res}}1dx\right)^{\frac25}+C\delta\\
&\le \frac{\eta}{8}\int_\Omega \mathcal{E}([\rho,\vartheta]|[\rho^E,\vartheta^E])dx+C(\eta)\delta.
\end{aligned}\tag{4.75}$$

Similarly, we deduce that

$$\begin{aligned}
&\int_\Omega \left(\frac{1}{2}\rho_0|u_0-U(0,\cdot)|^2+\mathcal{E}([\rho_0,\vartheta_0]|[\rho^E(0,\cdot),\vartheta^E(0,\cdot)])\right)dx\\
&\le \mathbb{E}(0)+C\int_{\Gamma_\delta}\rho_0|v_0|^2dx\\
&\le \mathbb{E}(0)+C\int_{\Gamma_\delta}\rho_0dx\\
&\le \mathbb{E}(0)+\frac{\eta}{8}\int_\Omega \mathcal{E}([\rho,\vartheta]|[\rho^E,\vartheta^E])dx+C(\eta)\delta. \qquad (4.76)
\end{aligned}$$

We choose $\eta<\frac{1}{2}\min\{C_1,C_2,1\}$ in (4.72) or (4.72'), finally find with (4.73)–(4.76) that

$$\begin{aligned}
&\mathbb{E}(\sigma)+\int_0^\sigma\int_\Omega \frac{\vartheta^E}{\vartheta}\left(\mathbb{S}(\vartheta,\nabla u):\nabla u-\frac{q(\vartheta,\nabla\vartheta)\cdot\nabla\vartheta}{\vartheta}\right)dxdt\\
&\quad\le C\left(1+\mu^{-1}a^{\frac{1}{2}}+\mu a^{-\frac{1}{2}}+\kappa a^{-\frac{1}{4}}\right)\int_0^\sigma(\mathbb{E}(t)+\delta)\,dt\\
&\qquad+C\left(\frac{\delta^2}{\mu}\|\frac{\rho u\cdot n}{d_\Omega}\|^2_{L^2([0,T]\times\Gamma_\delta)}+\mu\delta^{-1-\frac{1}{\lambda}}\|\vartheta\|^2_{L^2(0,T;L^{2\lambda}(\Gamma_\delta))}+\frac{\mu}{\delta}+\delta+\mu+\kappa+a+\mathbb{E}(0)\right), \qquad (4.77)
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}(\sigma)+\int_0^\sigma\int_\Omega \frac{\vartheta^E}{\vartheta}\left(\mathbb{S}(\vartheta,\nabla u):\nabla u-\frac{q(\vartheta,\nabla\vartheta)\cdot\nabla\vartheta}{\vartheta}\right)dxdt \qquad (4.77')\\
&\quad\le C\left(1+\mu^{-1}a^{\frac{1}{2}}+\mu a^{-\frac{1}{2}}+\kappa a^{-\frac{1}{4}}\right)\int_0^\sigma(\mathbb{E}(t)+\delta)\,dt\\
&\qquad+C\left(\frac{\delta^2}{\mu}\|\frac{\rho u\cdot \tau}{d_\Omega}\|^2_{L^2([0,T]\times\Gamma_\delta)}+\mu\delta^{-1-\frac{1}{\lambda}}\|\vartheta\|^2_{L^2(0,T;L^{2\lambda}(\Gamma_\delta))}+\frac{\mu}{\delta}+\delta+\mu+\kappa+a+\mathbb{E}(0)\right),
\end{aligned}$$

for any $\sigma\in[0,T]$.

Therefore, we conclude

Proposition 4.4. *Assume that $T>0$, (ρ^E,ϑ^E,u^E) is the smooth solution to the problem (1.2) of the Euler equations on $[0,T]$ as given in Proposition 2.2, and $(\rho^\mu,\vartheta^\mu,u^\mu)$ is a weak solution to the problem (1.1) for the full Naiver–Stokes–Fourier equations (1.1) on $[0,T]$ as given in Proposition 2.1. Let $\mathbb{E}$ be defined as in (4.3). Then for any $\sigma\in[0,T]$, $1<\lambda\le\infty$ and small $\delta>0$, we have*

$$\begin{aligned}
&\mathbb{E}(\sigma)+\int_0^\sigma\int_\Omega \frac{\vartheta^E}{\vartheta^\mu}\left(\mathbb{S}(\vartheta^\mu,\nabla u^\mu):\nabla u^\mu-\frac{q(\vartheta^\mu,\nabla\vartheta^\mu)\cdot\nabla\vartheta^\mu}{\vartheta^\mu}\right)dxdt\\
&\quad\le C\left(1+\mu^{-1}a^{\frac{1}{2}}+\mu a^{-\frac{1}{2}}+\kappa a^{-\frac{1}{4}}\right)\int_0^\sigma(\mathbb{E}(t)+\delta)\,dt\\
&\qquad+C\left(\frac{\delta^2}{\mu}\|\frac{\rho^\mu u^\mu\cdot n}{d_\Omega}\|^2_{L^2([0,T]\times\Gamma_\delta)}+\mu\delta^{-1-\frac{1}{\lambda}}\|\vartheta^\mu\|^2_{L^2(0,T;L^{2\lambda}(\Gamma_\delta))}+\frac{\mu}{\delta}+\delta+\mu+\kappa+a+\mathbb{E}(0)\right), \qquad (4.78)
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}(\sigma)+\int_0^\sigma\int_\Omega \frac{\vartheta^E}{\vartheta^\mu}\left(\mathbb{S}(\vartheta^\mu,\nabla u^\mu):\nabla u^\mu-\frac{q(\vartheta^\mu,\nabla\vartheta^\mu)\cdot\nabla\vartheta^\mu}{\vartheta^\mu}\right)dxdt \qquad (4.78')\\
&\quad\le C\left(1+\mu^{-1}a^{\frac{1}{2}}+\mu a^{-\frac{1}{2}}+\kappa a^{-\frac{1}{4}}\right)\int_0^\sigma(\mathbb{E}(t)+\delta)\,dt\\
&\qquad+C\left(\frac{\delta^2}{\mu}\|\frac{\rho^\mu u^\mu\cdot \tau}{d_\Omega}\|^2_{L^2([0,T]\times\Gamma_\delta)}+\mu\delta^{-1-\frac{1}{\lambda}}\|\vartheta^\mu\|^2_{L^2(0,T;L^{2\lambda}(\Gamma_\delta))}+\frac{\mu}{\delta}+\delta+\mu+\kappa+a+\mathbb{E}(0)\right).
\end{aligned}$$

Now, let us prove the result given in Theorem 2.1 by using the idea developed in [12].

Proof of Theorem 2.1. We shall only prove the sufficiency of conditions (2.18) and (2.19) for the convergence (2.17) by using the inequality (4.78), while the sufficiency of (2.18) and (2.20) can be derived similarly by using the inequality (4.78').

Now if $a \sim \mu^2,\ \kappa = O(a^{\frac14})$ and (2.18) holds, we can deduce from (4.78) that

$$\mathbb{E}(\sigma) \le C\int_0^\sigma \mathbb{E}(t)dt + C\left(\frac{\delta^2}{\mu}\|\frac{\rho^\mu u^\mu\cdot n}{d_\Omega}\|^2_{L^2([0,T]\times\Gamma_\delta)} + \frac{\mu}{\delta^{1+\frac1\lambda}} + \delta+\mu+\kappa+a+\mathbb{E}(0)\right). \tag{4.79}$$

Denote by $\alpha := \frac{\mu}{\delta^{1+\frac1\lambda}}$, then $\frac{\delta^2}{\mu} = \frac{\mu^{\frac{\lambda-1}{\lambda+1}}}{\alpha^{\frac{2\lambda}{\lambda+1}}}$. For any $s>0$, define

$$F(s) = s + \frac{\mu^{\frac{\lambda-1}{\lambda+1}}}{s^{\frac{2\lambda}{\lambda+1}}}\|\frac{\rho^\mu u^\mu\cdot n}{d_\Omega}\|^2_{L^2([0,T]\times\Gamma_\delta)}.$$

Obviously, $F(s)$ attains its minimum at

$$s = \alpha_{ct} = \left(\frac{2\lambda}{\lambda+1}\mu^{\frac{\lambda-1}{\lambda+1}}\|\frac{\rho^\mu u^\mu\cdot n}{d_\Omega}\|^2_{L^2([0,T]\times\Gamma_\delta)}\right)^{\frac{\lambda+1}{3\lambda+1}}.$$

Now we choose a δ such that $\mu = o(\delta^{1+\frac1\lambda})$. If $\alpha_{ct}\ge\alpha$, then $\delta_{ct} = \left(\frac{\mu}{\alpha_{ct}}\right)^{\frac{\lambda}{\lambda+1}} \le \left(\frac{\mu}{\alpha}\right)^{\frac{\lambda}{\lambda+1}} = \delta$, and

$$\begin{aligned}\alpha_{ct} + \frac{\mu^{\frac{\lambda-1}{\lambda+1}}}{a_{ct}^{\frac{2\lambda}{\lambda+1}}}\|\frac{\rho^\mu u^\mu\cdot n}{d_\Omega}\|^2_{L^2([0,T]\times\Gamma_{\delta_{ct}})} &\le \alpha_{ct} + \frac{\mu^{\frac{\lambda-1}{\lambda+1}}}{a_{ct}^{\frac{2\lambda}{\lambda+1}}}\|\frac{\rho^\mu u^\mu\cdot n}{d_\Omega}\|^2_{L^2([0,T]\times\Gamma_\delta)}\\ &= \min_{s>0}F(s) = \frac{3\lambda+1}{2\lambda}\left(\frac{2\lambda}{\lambda+1}\mu^{\frac{\lambda-1}{\lambda+1}}\|\frac{\rho^\mu u^\mu\cdot n}{d_\Omega}\|^2_{L^2([0,T]\times\Gamma_\delta)}\right)^{\frac{\lambda+1}{3\lambda+1}}.\end{aligned} \tag{4.80}$$

Since

$$\alpha_{ct} = \left(\frac{2\lambda}{\lambda+1}\mu^{\frac{\lambda-1}{\lambda+1}}\|\frac{\rho^\mu u^\mu\cdot n}{d_\Omega}\|^2_{L^2([0,T]\times\Gamma_\delta)}\right)^{\frac{\lambda+1}{3\lambda+1}} \longrightarrow 0$$

as $\mu,\kappa,a\to0$ under the assumption (2.19), we know $\mu = \alpha_{ct}\delta_{ct}^{1+\frac1\lambda} = o(\delta_{ct}^{1+\frac1\lambda})$. Moreover, from $\delta\to0$ as $\mu\to0$ we get that $\delta_{ct}\le\delta$ gives rise to $\delta_{ct}\to0$ when $\mu\to0$. Thus, the inequality (4.79) holds for $\delta=\delta_{ct}$. Together with (4.77), we obtain

$$\mathbb{E}(\sigma) \le C\int_0^\sigma \mathbb{E}(t)dt + C\left(\left(\mu^{\frac{\lambda-1}{\lambda+1}}\|\frac{\rho^\mu u^\mu\cdot n}{d_\Omega}\|^2_{L^2([0,T]\times\Gamma_\delta)}\right)^{\frac{\lambda+1}{3\lambda+1}} + \delta+\mu+\kappa+a+\mathbb{E}(0)\right). \tag{4.81}$$

If $\alpha_{ct}<\alpha$, that is

$$\left(\frac{2\lambda}{\lambda+1}\mu^{\frac{\lambda-1}{\lambda+1}}\|\frac{\rho^\mu u^\mu\cdot n}{d_\Omega}\|^2_{L^2([0,T]\times\Gamma_\delta)}\right)^{\frac{\lambda+1}{3\lambda+1}} < \alpha,$$

we know

$$\mu^{\frac{\lambda-1}{\lambda+1}}\|\frac{\rho^\mu u^\mu\cdot n}{d_\Omega}\|^2_{L^2([0,T]\times\Gamma_\delta)} < \frac{\lambda+1}{2\lambda}\alpha^{\frac{3\lambda+1}{\lambda+1}},$$

which implies

$$\alpha + \frac{\mu^{\frac{\lambda-1}{\lambda+1}}}{\alpha^{\frac{2\lambda}{\lambda+1}}}\|\frac{\rho^{\mu,\kappa}u_3^{\mu,\kappa}}{d_\Omega}\|^2_{L^2((0,T)\times\Gamma_\delta)} < \frac{3\lambda+1}{2\lambda}\alpha.$$

Thus we have

$$\mathbb{E}(\sigma) \le C\int_0^\sigma \mathbb{E}(t)dt + C\left(\frac{\mu}{\delta^{1+\frac1\lambda}} + \delta+\mu+\kappa+a+\mathbb{E}(0)\right). \tag{4.82}$$

Combining (4.81) with (4.82), we deduce that

$$\mathbb{E}(\sigma) \leq C\int_0^\sigma \mathbb{E}(t)dt + C\left(\left(\mu^{\frac{\lambda-1}{\lambda+1}}\|\frac{\rho^\mu u^\mu\cdot n}{d_\Omega}\|^2_{L^2([0,T]\times\Gamma_\delta)}\right)^{\frac{\lambda+1}{3\lambda+1}} + \frac{\mu}{\delta^{1+\frac{1}{\lambda}}} + \delta + \mu + \kappa + a + \mathbb{E}(0)\right). \quad (4.83)$$

With Remark 3.1, we can deduce from (2.16) that $\mathbb{E}(0) \to 0$ as $\mu \to 0$. By using the classical Gronwall inequality, from Lemma 3.1 and (4.83) we get the conclusion given in Theorem 2.1 under the conditions (2.18) and (2.19), as δ is chosen such that $\mu = o(\delta^{1+\frac{1}{\lambda}})$, and $\delta \to 0$ as $\mu \to 0$. □

Remark 4.1. The boundedness assumption of the initial datum ρ_0^μ and ϑ_0^μ given in Theorem 2.1 is used to ensure that $\mathbb{E}(0) \to 0$ as $\mu \to 0$. If we remove this boundedness assumption of the initial datum ρ_0^μ and ϑ_0^μ, then it is easy to see that one has the same convergence result as given in (2.17), as long as we impose the condition $\mathcal{E}([\rho_0, \vartheta_0]|[\rho_0^E, \vartheta_0^E]) \to 0$ as $\mu \to 0$ in addition to the convergence assumption (2.16) on the initial datum.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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