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Global Solution to the Incompressible Inhomogeneous Navier–Stokes Equations with Some Large Initial Data

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Abstract. In this paper, we prove that the incompressible inhomogeneous Navier–Stokes equations have a unique global solution with initial data (a_0, u_0) in critical Besov spaces $\dot{B}^{\frac{n}{q}}_{q,1}(\mathbb{R}^n) \times \dot{B}^{\frac{n}{p}-1}_{p,1}(\mathbb{R}^n)$ satisfying a nonlinear smallness condition for all $(p, q) \in [1, 2n) \times [1, \infty)$, $-\frac{1}{n} \le \frac{1}{p} - \frac{1}{q} \le \frac{1}{n}$ and $\frac{1}{p} + \frac{1}{q} > \frac{1}{n}$. We also construct an initial data satisfying that nonlinear smallness condition, but the norm of each component of the initial velocity field can be arbitrarily large in $\dot{B}^{\frac{n}{p}-1}_{p,1}(\mathbb{R}^n)$ with $n < p < 2n$.

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1. Introduction

In this paper,we study the global well-posedness of the following incompressible inhomogeneous Navier–Stokes equations with initial data in critical Besov spaces

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) - \mu \Delta u + \nabla \Pi = 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n, \\ \operatorname{div} u = 0, \\ (\rho, u)|_{t=0} = (\rho_0, u_0)(x), \qquad\qquad x \in \mathbb{R}^n. \end{cases} \tag{1.1}$$

Here $\rho = \rho(t, x) \in \mathbb{R}^+$ and $u = u(t, x) \in \mathbb{R}^n$ stand for the density and velocity field respectively, and Π is a scalar pressure function. The viscosity coefficient $\mu > 0$ is a given positive real number. Throughout, we assume that the space dimensions $n \ge 2$.

Global weak solutions with finite energy to system (1.1) have been built up first by the Russian school [6]. We also refer to [18] for an overview of results on weak solutions. As is known, the key ingredient to construct weak solutions is the following conservation law:

$$\|\sqrt{\rho} u(t)\|_{L^2}^2 + 2\mu \int_0^t \|\nabla u(\tau)\|_{L^2}^2 d\tau = \|\sqrt{\rho_0} u_0\|_{L^2}^2.$$

However, the uniqueness of weak solutions is not known in general. Ladyzhenskaya and Solonnikov [17] initiated the studies for unique solvability of system (1.1) in a bounded domain Ω with homogeneous Dirichlet boundary condition for u. Similar results were established by Danchin [12] in $\mathbb{R}^n$ with initial data in the almost critical Sobolev spaces. On the other hand, from the viewpoint of physics, it is interesting to study the case for which density is discontinuous. Recently, Danchin and Mucha [14] proved

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by using a Lagrangian approach that the system (1.1) has a unique local solution with initial data $(\rho_0, u_0) \in L^\infty(\mathbb{R}^n) \times H^2(\mathbb{R}^n)$ if initial vacuum does not occur, see also some improvements in [15,21].

When the density ρ is away from zero, we can use the transform $a \stackrel{\text{def}}{=} \rho^{-1} - 1$ to turn (1.1) into:

$$\begin{cases} a_t + u \cdot \nabla a = 0, \\ u_t + u \cdot \nabla u + (1+a)\,(\nabla \Pi - \mu \Delta u) = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n, \\ \operatorname{div} u = 0, \\ (a,u)|_{t=0} = (a_0, u_0)(x), \qquad\qquad x \in \mathbb{R}^n. \end{cases} \tag{1.2}$$

Just as the homogeneous Navier–Stokes equations, the system (1.2) also has a scaling. Indeed, if (a,u) solves (1.2) with initial data (a_0, u_0), then for any $\lambda > 0$,

$$(a,u)_\lambda(t,x) \stackrel{\text{def}}{=} (a(\lambda^2 \cdot, \lambda \cdot), \lambda u(\lambda^2 \cdot, \lambda \cdot))$$

also solves (1.2) with initial data $(a_0(\lambda \cdot), \lambda u_0(\lambda \cdot))$. Moreover, the norm of $(a,u) \in \dot{B}^{\frac{n}{q}}_{q,1}(\mathbb{R}^n) \times \dot{B}^{\frac{n}{p}-1}_{p,1}(\mathbb{R}^n)$ is scaling invariant under this change of scale. In [11], Danchin derived the global well-poesdness of (1.2) under the assumptions that the initial velocity is small in critical homogeneous Besov spaces, and that the initial density is close to a positive constant. This result has been extended in [1,5], where the smallness condition for a_0 is still required. Then Abidi, Gui and Zhang removed the smallness condition for a_0 in [3,4]. Finally, we remark that in the very interesting paper [13], Danchin and Mucha proved the well-posedness of (1.1) provided that ρ_0 is close to a positive constant in the multiplier space of $\dot{B}^{\frac{n}{p}-1}_{p,1}(\mathbb{R}^n)$. More precisely, they proved:

Theorem 1.1 (See [13]). *Let $p \in [1, 2n)$, $u_0 \in \dot{B}^{\frac{n}{p}-1}_{p,1}(\mathbb{R}^n)$ with $\operatorname{div} u_0 = 0$. Assume that ρ_0 belongs to the multiplier space $\mathscr{M}(\dot{B}^{\frac{n}{p}-1}_{p,1}(\mathbb{R}^n))$. There exists a positive constant c_0 depending only on p and n such that if*

$$\|\rho_0 - 1\|_{\mathscr{M}(\dot{B}^{\frac{n}{p}-1}_{p,1})} \le c_0, \tag{1.3}$$

then there exists some $T > 0$ such that the system (1.1) has a unique local solution $(\rho, u, \nabla\Pi)$ with $\rho \in L^\infty_T(\mathscr{M}(\dot{B}^{\frac{n}{p}-1}_{p,1}(\mathbb{R}^n)))$, $u \in C_b([0,T]; \dot{B}^{\frac{n}{p}-1}_{p,1}(\mathbb{R}^n)) \cap L^1_T(\dot{B}^{\frac{n}{p}+1}_{p,1}(\mathbb{R}^n))$ and $\nabla\Pi \in L^1_T(\dot{B}^{\frac{n}{p}-1}_{p,1}(\mathbb{R}^n))$. Moreover, if $\|u_0\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}} \le c_0 \mu$, then $T = \infty$.

Here, $\mathscr{M}(\dot{B}^s_{p,1})$ is the multiplier spaces consisting of all the distributions f such that ψf is in $\dot{B}^s_{p,1}$ whenever ψ is in $\dot{B}^s_{p,1}$, endowed with the norm

$$\|f\|_{\mathscr{M}(\dot{B}^s_{p,1})} \stackrel{\text{def}}{=} \sup_{\psi \in \dot{B}^s_{p,1}:\, \|\psi\|_{\dot{B}^s_{p,1}}=1} \|\psi f\|_{\dot{B}^s_{p,1}}.$$

Motivated by [10,16,20] concerning the global well-posedness of (1.2) with the third component of the initial velocity field being large, we aim to relax the smallness condition in [20] so that (1.2) is still globally well-posed. We remark that the smallness condition we are going to present is somewhat similar to that in [9, Theorem 1.2]. Moreover, although the initial data satisfies the smallness condition we presented, the initial velocity could be large in every direction (see Theorem 1.3).

In the sequel, we denote by $S_\mu(t) \stackrel{\text{def}}{=} e^{\mu t \Delta}$ the heat flow, and by $S(t) \stackrel{\text{def}}{=} S_1(t)$. We look for the global solution u of the form $u_F + \bar{u}$, where $u_F(t) \stackrel{\text{def}}{=} S_\mu(t) u_0$ solves the free heat equation $\partial_t u_F - \mu \Delta u_F = 0$ with initial data u_0. Moreover, by classical estimate for heat flow (see Lemma 2.4 below), it is easy to observe that

$$\|u_F\|_{\widetilde{L}^\infty\left(\mathbb{R}^+; \dot{B}^{\frac{n}{p}-1}_{p,1}\right)} + \mu \|u_F\|_{L^1\left(\mathbb{R}^+; \dot{B}^{\frac{n}{p}+1}_{p,1}\right)} \le C \|u_0\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}, \tag{1.4}$$

whenever $u_0 \in \dot{B}^{\frac{n}{p}-1}_{p,1}(\mathbb{R}^n)$ for $1 \le p \le \infty$.

Let us state our main results.

Theorem 1.2. *Let* $(p,q)\in[1,2n)\times[1,\infty)$, $-\frac{1}{n}\le\frac{1}{p}-\frac{1}{q}\le\frac{1}{n}$ *and* $\frac{1}{p}+\frac{1}{q}>\frac{1}{n}$. *Then there exist positive constants* c_1 *and* C_1 *such that, for any data* $(a_0,u_0)\in\dot B^{\frac{n}{q}}_{q,1}(\mathbb{R}^n)\times\dot B^{\frac{n}{p}-1}_{p,1}(\mathbb{R}^n)$ *satisfying* $\operatorname{div}u_0=0$ *and*

$$\delta\overset{\text{def}}{=}\left(\mu\|a_0\|_{\dot B^{\frac{n}{q}}_{q,1}}+C_F\right)\exp\left\{C_1\|u_0\|^2_{\dot B^{\frac{n}{p}-1}_{p,1}}/\mu^2\right\}\le\frac{c_1}{2C_1}\mu, \tag{1.5}$$

with $C_F=\|u_F\cdot\nabla u_F\|_{L^1(\mathbb{R}^+;\dot B^{\frac{n}{p}-1}_{p,1})}$, *the system* (1.2) *has a unique global solution* $(a,u,\nabla\Pi)$ *with* $a\in C_b([0,\infty);\dot B^{\frac{n}{q}}_{q,1}(\mathbb{R}^n))$, $u\in C_b([0,\infty);\dot B^{\frac{n}{p}-1}_{p,1}(\mathbb{R}^n))\cap L^1(\mathbb{R}^+;\dot B^{\frac{n}{p}+1}_{p,1}(\mathbb{R}^n))$ *and* $\nabla\Pi\in L^1(\mathbb{R}^+;\dot B^{\frac{n}{p}-1}_{p,1}(\mathbb{R}^n))$. *Moreover, there holds*

$$\|\bar u\|_{\widetilde L^\infty\left(\mathbb{R}^+;\dot B^{\frac{n}{p}-1}_{p,1}\right)}+\mu\|\bar u\|_{L^1\left(\mathbb{R}^+;\dot B^{\frac{n}{p}+1}_{p,1}\right)}+\|\nabla\Pi\|_{L^1\left(\mathbb{R}^+;\dot B^{\frac{n}{p}-1}_{p,1}\right)}+\mu\|a\|_{\widetilde L^\infty\left(\mathbb{R}^+;\dot B^{\frac{n}{q}}_{q,1}\right)}\le\frac{1}{2}c_1\mu. \tag{1.6}$$

Remark 1.1. (a) We should mention that if the initial data $a_0\in\dot B^{\frac{n}{q}}_{q,1}(\mathbb{R}^n)$ and $u_0=(u_0^h,u_0^n)\in\dot B^{\frac{n}{p}-1}_{p,1}(\mathbb{R}^n)$ satisfy the following smallness condition (see [20])

$$\left(\mu\|a_0\|_{\dot B^{\frac{n}{q}}_{q,1}}+\|u_0^h\|_{\dot B^{\frac{n}{p}-1}_{p,1}}\right)\exp\left\{C_0\|u_0^n\|^2_{\dot B^{\frac{n}{p}-1}_{p,1}}/\mu^2\right\}\le c_0\mu$$

with c_0 small enough and C_0 large enough, then (1.5) is fulfilled. The key observation is that $\operatorname{div}u_F=0$ implies $\partial_n u_F^n=-\operatorname{div}_h u_F^h$, so that

$$C_F\le C\int_0^\infty\|u_F\|_{\dot B^{\frac{n}{p}-1}_{p,1}}\|u_F^h\|_{\dot B^{\frac{n}{p}+1}_{p,1}}+\|u_F^h\|_{\dot B^{\frac{n}{p}-1}_{p,1}}\|u_F^n\|_{\dot B^{\frac{n}{p}+1}_{p,1}}dt\le\frac{C}{\mu}\|u_0\|_{\dot B^{\frac{n}{p}-1}_{p,1}}\|u_0^h\|_{\dot B^{\frac{n}{p}-1}_{p,1}}.$$

(b) If the viscosity coefficient depends on the density by a regular positive function $\mu=\mu(\rho)$, then the diffusion term in the momentum equation of (1.2) reads $(1+a)\operatorname{div}(\tilde\mu(a)(\nabla u+\nabla^\top u))$, where $\tilde\mu(a)=\mu(\frac{1}{1+a})$. In this case, our theorem remains true under the stronger condition $q\le p$.

Motivated by Chemin and Gallagher [9], we give the following theorem to ensure that Theorem 1.2 is relevant.

Theorem 1.3. *Let* $n\ge3$, $p\in(n,2n)$, $\alpha,\varepsilon\in(0,1)$ *and* ϕ,ψ *be in the Schwartz space* $\mathscr{S}(\mathbb{R}^n)$. *Define divergence free vector fields by*

$$\begin{aligned}v_{0,\varepsilon}(x)&\overset{\text{def}}{=}\frac{(-\log\varepsilon)^{\frac14}}{\varepsilon^{1-\frac{n}{p}}}(\partial_2\ldots\partial_{n-1}\phi_\varepsilon,\partial_1\partial_3\ldots\partial_{n-1}\phi_\varepsilon,\ldots,\partial_1\ldots\partial_{n-3}\partial_{n-1}\phi_\varepsilon,\\&\qquad-(n-2)\partial_1\ldots\partial_{n-2}\phi_\varepsilon,0),\\ \omega_{0,\varepsilon}(x)&\overset{\text{def}}{=}\frac{(-\log\varepsilon)^{\frac14}}{\varepsilon^{1-\frac{n}{p}+\frac{\alpha}{p}-\alpha}}(0,\ldots,0,\partial_2\ldots\partial_{n-2}\partial_n\psi^\varepsilon,-\partial_2\ldots\partial_{n-1}\psi^\varepsilon),\end{aligned}$$

where

$$\phi_\varepsilon(x)\overset{\text{def}}{=}\cos\left(\frac{x_n}{\varepsilon}\right)\phi(x),\quad\psi^\varepsilon(x)\overset{\text{def}}{=}\cos\left(\frac{x_1}{\varepsilon}\right)\psi\left(x_1,\ldots,x_{n-2},\frac{x_{n-1}}{\varepsilon^\alpha},x_n\right). \tag{1.7}$$

Then there exists a positive constant C *such that for any* ε *small enough, the divergence free vector field* $u_{0,\varepsilon}(x)\overset{\text{def}}{=}\mu(v_{0,\varepsilon}+\omega_{0,\varepsilon})(x)$ *satisfies*

$$C^{-1}\mu(-\log\varepsilon)^{\frac14}\le\|u^i_{0,\varepsilon}\|_{\dot B^{\frac{n}{p}-1}_{p,1}}\le C\mu(-\log\varepsilon)^{\frac14},\quad i=1,\ldots,n, \tag{1.8}$$

and

$$\|S_\mu(t)u_{0,\varepsilon}\cdot\nabla S_\mu(t)u_{0,\varepsilon}\|_{L^1\left(\mathbb{R}^+;\dot B^{\frac{n}{p}-1}_{p,1}\right)}\le C\mu\varepsilon^\gamma(-\log\varepsilon)^{\frac12}, \tag{1.9}$$

with $\gamma=(2n-\alpha)(\frac1p-\frac{1}{2n})\in(0,1)$.

Remark 1.2. Let a_0 be in the unit sphere of $\dot{B}^{\frac{n}{q}}_{q,1}(\mathbb{R}^n)$. Theorem 1.3 ensures that the initial data $(\varepsilon a_0, u_{0,\varepsilon})$ with ε small enough verifies the nonlinear smallness condition (1.5), thus, generates a unique global solution to (1.2). However, (1.8) ensures that $u_{0,\varepsilon}$ can be arbitrarily large in every direction.

The paper is organized as follows. In Sect. 2, we present some basic facts on Littlewood–Paley analysis. The next section is devoted to the proof of Theorem 1.2 by applying some appropriate weighted norms (see Propositions 3.1, 3.2). In the last section, we shall complete the proof of Theorem 1.3, which follows the lines of that of Chemin and Gallagher [9].

Notation For two operators A and B, we denote $[A, B] = AB - BA$, the commutator between A and B. The letter C stands for a generic constant whose meaning is clear from the context. We sometimes write $a \lesssim b$ instead of $a \leq Cb$. The Fourier transform of u is denoted either by $\hat{u}$ or $\mathscr{F}u$, the inverse by $\mathscr{F}^{-1}u$.

For X a Banach space and I an interval of $\mathbb{R}$, we denote by $C(I; X)$ the set of continuous functions on I with values in X, and by $C_b(I; X)$ the subset of bounded functions of $C(I; X)$. For $q \in [1, +\infty]$, $L^q(I; X)$ stands for the set of measurable functions on I with values in X, such that $t \mapsto \|f(t)\|_X$ belongs to $L^q(I)$. For short, we sometimes write $L^q_T(X)$ instead of $L^q((0, T); X)$.

2. Littlewood–Paley Theory

In this section, we recall some basic facts on Littlewood–Paley theory (see [7] for instance). Let χ, φ be two smooth radial functions valued in the interval [0,1], the support of χ be the ball $\mathscr{B} = \{\xi \in \mathbb{R}^n : |\xi| \leq \frac{4}{3}\}$ while the support of φ be the annulus $\mathscr{C} = \{\xi \in \mathbb{R}^n : \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$, and satisfy

$$\sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1 \quad \text{for} \quad \xi \in \mathbb{R}^n\backslash\{0\};$$
$$\chi(\xi) + \sum_{j\geq 0} \varphi(2^{-j}\xi) = 1 \quad \text{for} \quad \xi \in \mathbb{R}^n.$$

Denote by $h \overset{\text{def}}{=} \mathscr{F}^{-1}\varphi$ and $\widetilde{h} \overset{\text{def}}{=} \mathscr{F}^{-1}\chi$, the homogeneous dyadic blocks $\dot{\Delta}_j$ and the homogeneous low-frequency cutoff operators $\dot{S}_j$ are defined for all $j \in \mathbb{Z}$ by

$$\dot{\Delta}_j u = \varphi(2^{-j}D)u = 2^{nj}\int_{\mathbb{R}^n} h(2^j y)u(x-y)dy,$$
$$\dot{S}_j u = \chi(2^{-j}D)u = 2^{nj}\int_{\mathbb{R}^n} \widetilde{h}(2^j y)u(x-y)dy.$$

Denote by $\mathscr{S}'_h(\mathbb{R}^n)$ the space of tempered distributions u such that

$$\lim_{j\to-\infty} \dot{S}_j u = 0 \ \ in \ \ \mathscr{S}'.$$

Then we have the formal decomposition

$$u = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j u, \quad \forall u \in \mathscr{S}'_h(\mathbb{R}^n).$$

Moreover, the Littlewood–Paley decomposition satisfies the property of almost orthogonality:

$$\dot{\Delta}_k\dot{\Delta}_j u \equiv 0 \ \text{ if } \ |k-j| \geq 2, \quad \text{and} \quad \dot{\Delta}_k(\dot{S}_{j-1}u\dot{\Delta}_j v) \equiv 0 \ \text{ if } \ |k-j| \geq 5.$$

Definition 2.1. *Let $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$. The homogeneous Besov space $\dot{B}^s_{p,r}(\mathbb{R}^n)$ consists of all the distributions u in $\mathscr{S}'_h(\mathbb{R}^n)$ such that*

$$\|u\|_{\dot{B}^s_{p,r}} \overset{\text{def}}{=} \left\|\left(2^{js}\|\dot{\Delta}_j u\|_{L^p}\right)_{j\in\mathbb{Z}}\right\|_{l^r} < \infty.$$

Remark 2.1. Let $s \in \mathbb{R}$, $1 \leq p, r \leq \infty$ and $u \in \mathscr{S}_h'(\mathbb{R}^n)$. Then there exists a positive constant C such that u belongs to $\dot{B}^s_{p,r}(\mathbb{R}^n)$ if and only if there exists $\{c_{j,r}\}_{j\in\mathbb{Z}}$ such that $c_{j,r} \geq 0$, $\|c_{j,r}\|_{l^r} = 1$ and

$$\|\dot{\Delta}_j u\|_{L^p} \leq C c_{j,r} 2^{-js} \|u\|_{\dot{B}^s_{p,r}}, \quad \forall j \in \mathbb{Z}.$$

If $r = 1$, we denote by $d_j \stackrel{\text{def}}{=} c_{j,1}$.

To gain a better description of the regularization effect to the transport-diffusion equation, we should use the Chemin-Lerner type norms (see [7]):

Definition 2.2. *Let $s \in \mathbb{R}$ and $0 < T \leq +\infty$. We define*

$$\|u\|_{\widetilde{L}^\sigma_T(\dot{B}^s_{p,r})} \stackrel{\text{def}}{=} \left(\sum_{j\in\mathbb{Z}} 2^{jrs} \left(\int_0^T \|\dot{\Delta}_j u(t)\|^\sigma_{L^p} dt \right)^{\frac{r}{\sigma}} \right)^{\frac{1}{r}}$$

for $p \in [1, \infty]$, $r, \sigma \in [1, \infty)$, and with the standard modification for $r = \infty$ or $\sigma = \infty$.

Let us recall the fundamental properties for the Besov spaces.

Lemma 2.1. *Let $\mathscr{C} \subset \mathbb{R}^n$ be an annulus and $\mathscr{B} \subset \mathbb{R}^n$ be a ball. There exists a positive constant C such that for any $0 \leq k \in \mathbb{Z}$, any $\lambda > 0$, any $1 \leq p, q \leq \infty$ with $q \geq p$, and any function $u \in L^p$, we have*

$$\operatorname{supp} \widehat{u} \subset \lambda\mathscr{B} \Longrightarrow \|D^k u\|_{L^q} \stackrel{\text{def}}{=} \sum_{|\alpha|=k} \|\partial^\alpha u\|_{L^q} \leq C^{k+1} \lambda^{k+n(\frac{1}{p}-\frac{1}{q})} \|u\|_{L^p},$$
$$\operatorname{supp} \widehat{u} \subset \lambda\mathscr{C} \Longrightarrow C^{-k-1}\lambda^k \|u\|_{L^p} \leq \|D^k u\|_{L^p} \leq C^{k+1}\lambda^k \|u\|_{L^p}.$$

On the other hand, it has been demonstrated that the Bony's decomposition [7,8] is very effective to deal with nonlinear problems. Here, we recall the Bony's decomposition in the homogeneous context:

$$uv = \dot{T}_u v + \dot{T}_v u + \dot{R}(u, v),$$

where

$$\dot{T}_u v \stackrel{\text{def}}{=} \sum_{j\in\mathbb{Z}} \dot{S}_{j-1} u \dot{\Delta}_j v, \quad \dot{R}(u,v) \stackrel{\text{def}}{=} \sum_{j\in\mathbb{Z}} \dot{\Delta}_j u \widetilde{\dot{\Delta}}_j v, \quad \textit{and} \quad \widetilde{\dot{\Delta}}_j v \stackrel{\text{def}}{=} \sum_{|j'-j|\leq 1} \dot{\Delta}_{j'} v.$$

Lemma 2.2. *Let $1 \leq p, q \leq \infty$, $s_1 \leq \frac{n}{q}$, $s_2 \leq n \min\{\frac{1}{p}, \frac{1}{q}\}$ and $s_1 + s_2 > n \max\{0, \frac{1}{p} + \frac{1}{q} - 1\}$. Then*

$$\|ab\|_{\dot{B}^{s_1+s_2-\frac{n}{q}}_{p,1}} \lesssim \|a\|_{\dot{B}^{s_1}_{q,1}} \|b\|_{\dot{B}^{s_2}_{p,1}}, \quad \forall (a,b) \in \dot{B}^{s_1}_{q,1}(\mathbb{R}^n) \times \dot{B}^{s_2}_{p,1}(\mathbb{R}^n). \tag{2.1}$$

Proof. This lemma is proved in [20] in the case when $q \leq p$. We shall only prove (2.1) for $q > p$. Applying Bony's decomposition, we have

$$ab = \dot{T}_a b + \dot{T}_b a + \dot{R}(a, b).$$

Then applying Lemma 2.1, we get for $s_1 \leq \frac{n}{q}$

$$\|\dot{\Delta}_j(\dot{T}_a b)\|_{L^p} \lesssim \sum_{|j'-j|\leq 4} \|\dot{S}_{j'-1} a\|_{L^\infty} \|\dot{\Delta}_{j'} b\|_{L^p} \lesssim d_j 2^{-j(s_1+s_2-\frac{n}{q})} \|a\|_{\dot{B}^{s_1}_{q,1}} \|b\|_{\dot{B}^{s_2}_{p,1}},$$

and for $s_2 \leq \frac{n}{q}$

$$\|\dot{\Delta}_j(\dot{T}_b a)\|_{L^p} \lesssim \sum_{|j'-j|\leq 4} \|\dot{\Delta}_{j'} a\|_{L^q} \|\dot{S}_{j'-1} b\|_{L^{\frac{pq}{q-p}}} \lesssim d_j 2^{-j(s_1+s_2-\frac{n}{q})} \|a\|_{\dot{B}^{s_1}_{q,1}} \|b\|_{\dot{B}^{s_2}_{p,1}}.$$

If $\frac{1}{p}+\frac{1}{q}\geq 1=\frac{1}{p}+\frac{1}{p'}$, we infer

$$\begin{aligned}\|\dot\Delta_j\big(\dot R(a,b)\big)\|_{L^p}&\lesssim 2^{nj(1-\frac{1}{p})}\sum_{j'\geq j-3}\|\dot\Delta_{j'}a\|_{L^{p'}}\|\widetilde{\dot\Delta}_{j'}b\|_{L^p}\\&\lesssim 2^{nj(1-\frac{1}{p})}\|a\|_{\dot B^{s_1}_{q,1}}\|b\|_{\dot B^{s_2}_{p,1}}\sum_{j'\geq j-3}d_{j'}2^{-j'\big(s_1+s_2-n(\frac{1}{p}+\frac{1}{q}-1)\big)}\\&\lesssim d_j2^{-j(s_1+s_2-\frac{n}{q})}\|a\|_{\dot B^{s_1}_{q,1}}\|b\|_{\dot B^{s_2}_{p,1}},\end{aligned}$$

for $s_1+s_2>n(\frac{1}{p}+\frac{1}{q}-1)$. Finally, in the case when $\frac{1}{p}+\frac{1}{q}\stackrel{\text{def}}{=}\frac{1}{r}<1$, notice that $s_1+s_2>0$, one has

$$\begin{aligned}\|\dot\Delta_j\big(\dot R(a,b)\big)\|_{L^p}&\lesssim 2^{nj(\frac{1}{r}-\frac{1}{p})}\sum_{j'\geq j-3}\|\dot\Delta_{j'}a\|_{L^q}\|\widetilde{\dot\Delta}_{j'}b\|_{L^p}\\&\lesssim d_j2^{-j(s_1+s_2-\frac{n}{q})}\|a\|_{\dot B^{s_1}_{q,1}}\|b\|_{\dot B^{s_2}_{p,1}}.\end{aligned}$$

This completes the proof of the lemma. ☐

To prove our theorems, we shall also use the following lemmas, the proof of which could be found in [7].

Lemma 2.3. *Let $1\leq p,q\leq\infty$, $s\leq 1+n\min\{\frac{1}{p},\frac{1}{q}\}$, $a\in\dot B^s_{q,1}(\mathbb{R}^n)$ and $u\in\dot B^{\frac{n}{p}+1}_{p,1}(\mathbb{R}^n)$. Assume that*

$$s>-n\min\left\{\frac{1}{p},1-\frac{1}{q}\right\},\quad or\quad s>-1-n\min\left\{\frac{1}{p},1-\frac{1}{q}\right\}\quad if\quad \operatorname{div}u=0.$$

Then there holds

$$\|[u\cdot\nabla,\dot\Delta_j]a\|_{L^q}\lesssim d_j2^{-js}\|u\|_{\dot B^{\frac{n}{p}+1}_{p,1}}\|a\|_{\dot B^s_{q,1}}.$$

Lemma 2.4. *Let $\mathscr{C}\subset\mathbb{R}^n$ be an annulus. Then there exists a positive constant c, such that for any $1\leq p\leq\infty$ and $\lambda>0$, we have*

$$\operatorname{supp}\hat u\subset\lambda\mathscr{C}\Rightarrow\|S(t)u\|_{L^p}\lesssim e^{-ct\lambda^2}\|u\|_{L^p}.$$

Lemma 2.5. *Let $s>0$ and $1\leq p,r\leq\infty$. There exists a positive constant C such that*

$$C^{-1}\mu^{-s}\|u\|_{\dot B^{-2s}_{p,r}}\leq\Big\|t^s\|S_\mu(t)u\|_{L^p}\Big\|_{L^r(\mathbb{R}^+,\frac{dt}{t})}\leq C\mu^{-s}\|u\|_{\dot B^{-2s}_{p,r}},$$

for all $u\in\dot B^{-2s}_{p,r}(\mathbb{R}^n)$.

3. Proof of Theorem 1.2

This section is devoted to complete the proof of Theorem 1.2. The key ingredient is to estimate some linear equations by applying the following weighted norms [20]:

$$\|a\|_{L^1_{T,f}(X)}\stackrel{\text{def}}{=}\int_0^T f(t)\|a(t)\|_X\,dt,$$

where $f(t)\in L^1_{loc}(\mathbb{R}^+)$, $f(t)\geq 0$ and $(X,\|\cdot\|_X)$ is a normed space.

In the sequel, we take the weighted function $f(t)=\|u_F(t)\|_{\dot B^{\frac{n}{p}+1}_{p,1}}$ and define

$$a_\lambda(t,x)\stackrel{\text{def}}{=}a(t,x)\exp\left\{-\lambda\int_0^t f(t')dt'\right\}\tag{3.1}$$

with $\lambda\geq 0$.

The first linear equation we are concerned is the free transport equation.

Proposition 3.1. *Let* $a_0 \in \dot{B}^{\frac{n}{q}}_{q,1}(\mathbb{R}^n)$ *and* $u \in L^\infty_T(\dot{B}^{\frac{n}{p}-1}_{p,1}(\mathbb{R}^n)) \cap L^1_T(\dot{B}^{\frac{n}{p}+1}_{p,1}(\mathbb{R}^n))$ *with* $1 \le p \le \infty$, $1 \le q < \infty$ *and* $\frac{1}{q} - \frac{1}{p} \le \frac{1}{n}$. *Then the transport equation*

$$\partial_t a + u \cdot \nabla a = 0, \quad (t,x) \in (0,T] \times \mathbb{R}^n, \tag{3.2}$$

has a unique solution $a \in C([0,T]; \dot{B}^{\frac{n}{q}}_{q,1}(\mathbb{R}^n))$ *with initial data* a_0. *Moreover, there exists a constant* $C > 0$ *such that*

$$\begin{aligned} &\|a_\lambda\|_{\widetilde{L}^\infty_t\left(\dot{B}^{\frac{n}{q}}_{q,1}\right)} + (\lambda - C)\|a_\lambda\|_{L^1_{t,f}\left(\dot{B}^{\frac{n}{q}}_{q,1}\right)} \\ &\quad \le C\left(\|a_0\|_{\dot{B}^{\frac{n}{q}}_{q,1}} + \|a\|_{L^\infty_t\left(\dot{B}^{\frac{n}{q}}_{q,1}\right)}\|\bar{u}_\lambda\|_{L^1_t\left(\dot{B}^{\frac{n}{p}+1}_{p,1}\right)}\right), \quad \forall t \in [0,T]. \end{aligned} \tag{3.3}$$

Proof. Both the existence and uniqueness of a solution to (3.2) essentially follow from estimate (3.3). For simplicity, we just present the a priori estimate for smooth enough solution of (3.2). With the notation of a_λ, (3.2) is reduced to

$$\partial_t a_\lambda + \lambda f(t) a_\lambda + u \cdot \nabla a_\lambda = 0, \quad (t,x) \in [0,T] \times \mathbb{R}^n.$$

Applying $\dot{\Delta}_j$ to the above equation and taking the L^2 inner product of the resulting equation with $|\dot{\Delta}_j a_\lambda|^{q-2}\dot{\Delta}_j a_\lambda$, we obtain

$$\frac{1}{q}\frac{d}{dt}\|\dot{\Delta}_j a_\lambda\|^q_{L^q} + \lambda f(t)\|\dot{\Delta}_j a_\lambda\|^q_{L^q} = \frac{1}{q}\int_{\mathbb{R}^n} \operatorname{div} u\, |\dot{\Delta}_j a_\lambda|^q dx + \int_{\mathbb{R}^n} [u\cdot\nabla, \dot{\Delta}_j] a_\lambda\, |\dot{\Delta}_j a_\lambda|^{q-2}\dot{\Delta}_j a_\lambda dx.$$

From this, using Remark 2.1 and Lemma 2.3, we get

$$\frac{d}{dt}\|\dot{\Delta}_j a_\lambda\|_{L^q} + \lambda f(t)\|\dot{\Delta}_j a_\lambda\|_{L^q} \lesssim d_j(t) 2^{-\frac{n}{q}j}\|u\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}}\|a_\lambda\|_{\dot{B}^{\frac{n}{q}}_{q,1}}.$$

Integrating the above inequality over $[0,t]$ and using $u = u_F + \bar{u}$ lead to

$$\begin{aligned} &\|\dot{\Delta}_j a_\lambda\|_{L^\infty_t(L^q)} + \lambda\int_0^t f(\tau)\|\dot{\Delta}_j a_\lambda(\tau)\|_{L^q} d\tau \\ &\quad \lesssim \|\dot{\Delta}_j a_0\|_{L^q} + \int_0^t d_j(\tau) 2^{-\frac{n}{q}j}\|u(\tau)\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}}\|a_\lambda(\tau)\|_{\dot{B}^{\frac{n}{q}}_{q,1}} d\tau \\ &\quad \lesssim \|\dot{\Delta}_j a_0\|_{L^q} + \int_0^t d_j(\tau) 2^{-\frac{n}{q}j}\|\bar{u}_\lambda(\tau)\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}}\|a(\tau)\|_{\dot{B}^{\frac{n}{q}}_{q,1}} d\tau \\ &\qquad + \int_0^t d_j(\tau) 2^{-\frac{n}{q}j} f(\tau)\|a_\lambda(\tau)\|_{\dot{B}^{\frac{n}{q}}_{q,1}} d\tau. \end{aligned}$$

Multiplying the above inequality by $2^{\frac{n}{q}j}$ and taking summation for $j \in \mathbb{Z}$, we arrive at

$$\begin{aligned} &\|a_\lambda\|_{\widetilde{L}^\infty_t\left(\dot{B}^{\frac{n}{q}}_{q,1}\right)} + \lambda\|a_\lambda\|_{L^1_{t,f}\left(\dot{B}^{\frac{n}{q}}_{q,1}\right)} \\ &\quad \lesssim \|a_0\|_{\dot{B}^{\frac{n}{q}}_{q,1}} + \|a\|_{L^\infty_t\left(\dot{B}^{\frac{n}{q}}_{q,1}\right)}\|\bar{u}_\lambda\|_{L^1_t\left(\dot{B}^{\frac{n}{p}+1}_{p,1}\right)} + \|a_\lambda\|_{L^1_{t,f}\left(\dot{B}^{\frac{n}{q}}_{q,1}\right)}. \end{aligned}$$

This completes the proof of the proposition. □

We next give the estimate for Stokes system.

Proposition 3.2. *Let* $u_0 \in \dot{B}^s_{p,1}(\mathbb{R}^n)$ *with* $\operatorname{div} u_0 = 0$ *and* $F \in L^1_T(\dot{B}^s_{p,1}(\mathbb{R}^n))$ *with* $p \in [1, \infty]$ *and* $s \in \mathbb{R}$. *Then the system*

$$\begin{cases} u_t - \mu\Delta u + \nabla\Pi = F, \quad (t,x) \in (0,T] \times \mathbb{R}^n, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0 \end{cases} \tag{3.4}$$

has a unique solution $(u, \nabla\Pi)$ *with* $u \in C([0,T]; \dot{B}^s_{p,1}(\mathbb{R}^n))$ *and* $\nabla\Pi \in L^1_T(\dot{B}^s_{p,1}(\mathbb{R}^n))$. *Moreover, the following estimate is valid:*

$$\|u_\lambda\|_{\widetilde{L}^\infty_T(\dot{B}^s_{p,1})} + \lambda\|u_\lambda\|_{L^1_{T,f}(\dot{B}^s_{p,1})} + \mu\|u_\lambda\|_{L^1_T(\dot{B}^{s+2}_{p,1})} + \|\nabla\Pi_\lambda\|_{L^1_T(\dot{B}^s_{p,1})} \lesssim \|u_0\|_{\dot{B}^s_{p,1}} + \|F_\lambda\|_{L^1_T(\dot{B}^s_{p,1})}. \tag{3.5}$$

Proof. We just need to prove (3.5). Let $\mathbb{P}$ be the Leray projection operator and $\mathbb{Q} \overset{\text{def}}{=} \mathbb{I}_{\text{d}} - \mathbb{P} = -\nabla(-\Delta)^{-1}\operatorname{div}$. Applying $\mathbb{Q}$ to the first equation of (3.4) and using the divergence free condition for u, we have

$$\nabla\Pi = \mathbb{Q}F,$$

which gives rise to

$$\|\nabla\Pi_\lambda\|_{L^1_T\left(\dot{B}^{\frac{n}{p}-1}_{p,1}\right)} \lesssim \|F_\lambda\|_{L^1_T\left(\dot{B}^{\frac{n}{p}-1}_{p,1}\right)}. \tag{3.6}$$

Applying $\mathbb{P}$ to the first equation of (3.4), then we get by Duhamel's formula that

$$u(t) = S_\mu(t)u_0 + \int_0^t S_\mu(t-\tau)\mathbb{P}F(\tau)d\tau.$$

Applying Lemma 2.4 gives rise to

$$\|\dot{\Delta}_j u(t)\|_{L^p} \lesssim e^{-c\mu 2^{2j}t}\|\dot{\Delta}_j u_0\|_{L^p} + \int_0^t e^{-c\mu 2^{2j}(t-\tau)}\|\dot{\Delta}_j F(\tau)\|_{L^p}d\tau.$$

Multiplying the above inequality by $\exp\left\{-\lambda\int_0^t f(t')dt'\right\}$, we obtain

$$\begin{aligned} \|\dot{\Delta}_j u_\lambda(t)\|_{L^p} \lesssim\, & e^{-c\mu 2^{2j}t}\exp\left\{-\lambda\int_0^t f(t')dt'\right\}\|\dot{\Delta}_j u_0\|_{L^p} \\ & + \int_0^t e^{-c\mu 2^{2j}(t-\tau)}\exp\left\{-\lambda\int_\tau^t f(t')dt'\right\}\|\dot{\Delta}_j F_\lambda(\tau)\|_{L^p}d\tau. \end{aligned} \tag{3.7}$$

Then it is easy to observe that

$$\|\dot{\Delta}_j u_\lambda\|_{L^\infty_T(L^p)} \lesssim \|\dot{\Delta}_j u_0\|_{L^p} + \int_0^T \|\dot{\Delta}_j F_\lambda(\tau)\|_{L^p}d\tau,$$

from which, we deduce that

$$\|u_\lambda\|_{\widetilde{L}^\infty_T(\dot{B}^s_{p,1})} \lesssim \|u_0\|_{\dot{B}^s_{p,1}} + \|F_\lambda\|_{L^1_T(\dot{B}^s_{p,1})}. \tag{3.8}$$

On the other hand, integrating (3.7) over $[0,T]$, we arrive at

$$\begin{aligned} \|\dot{\Delta}_j u_\lambda\|_{L^1_T(L^p)} &\lesssim \|\dot{\Delta}_j u_0\|_{L^p}\int_0^T e^{-c\mu 2^{2j}t}dt + \int_0^T\int_0^t e^{-c\mu 2^{2j}(t-\tau)}\|\dot{\Delta}_j F_\lambda(\tau)\|_{L^p}d\tau dt \\ &\lesssim \mu^{-1}2^{-2j}\|\dot{\Delta}_j u_0\|_{L^p} + \int_0^T \|\dot{\Delta}_j F_\lambda(\tau)\|_{L^p}d\tau\int_\tau^T e^{-c\mu 2^{2j}(t-\tau)}dt \\ &\lesssim \mu^{-1}2^{-2j}\left(\|\dot{\Delta}_j u_0\|_{L^p} + \int_0^T \|\dot{\Delta}_j F_\lambda(\tau)\|_{L^p}d\tau\right), \end{aligned}$$

which implies

$$\mu\|u_\lambda\|_{L^1_T(\dot B^{s+2}_{p,1})} \lesssim \|u_0\|_{\dot B^s_{p,1}} + \|F_\lambda\|_{L^1_T(\dot B^s_{p,1})}. \tag{3.9}$$

To control the weighted norm in (3.5), we integrate (3.7) $\times\lambda f(t)$ over $[0,T]$ to obtain

$$\begin{aligned}
&\lambda\int_0^T \|\dot\Delta_j u_\lambda(t)\|_{L^p} f(t)dt\\
&\lesssim \|\dot\Delta_j u_0\|_{L^p}\int_0^T \lambda f(t)\exp\left\{-\lambda\int_0^t f(t')dt'\right\}dt\\
&\quad+\int_0^T\int_0^t \lambda f(t)\exp\left\{-\lambda\int_\tau^t f(t')dt'\right\}\|\dot\Delta_j F_\lambda(\tau)\|_{L^p}d\tau dt\\
&= -\|\dot\Delta_j u_0\|_{L^p}\int_0^T \frac{d}{dt}\exp\left\{-\lambda\int_0^t f(t')dt'\right\}dt\\
&\quad-\int_0^T\|\dot\Delta_j F_\lambda(\tau)\|_{L^p}d\tau\int_\tau^T\frac{d}{dt}\exp\left\{-\lambda\int_\tau^t f(t')dt'\right\}dt\\
&\lesssim \|\dot\Delta_j u_0\|_{L^p}+\int_0^T\|\dot\Delta_j F_\lambda(\tau)\|_{L^p}d\tau.
\end{aligned}$$

As a consequence, we have

$$\lambda\|u_\lambda\|_{L^1_{T,f}(\dot B^s_{p,1})} \lesssim \|u_0\|_{\dot B^s_{p,1}} + \|F_\lambda\|_{L^1_T(\dot B^s_{p,1})}. \tag{3.10}$$

Combining (3.6) and (3.8)–(3.10) completes the proof of the proposition. □

Now let us come back to the proof of Theorem 1.2. We first infer from (1.5) that $\|a_0\|_{\dot B^{\frac nq}_{q,1}} \le \frac{c_1}{2C_1}$. On the other hand, we can get by applying Lemma 2.2 that $\dot B^{\frac nq}_{q,1}(\mathbb R^n)\hookrightarrow \mathscr M(\dot B^{\frac np-1}_{p,1}(\mathbb R^n))$ with p,q satisfying the conditions listed in Theorem 1.2. Thus, thanks to Theorem 2.61 in [7], we have

$$\|\rho_0-1\|_{\mathscr M(\dot B^{\frac np-1}_{p,1})} \lesssim \left\|\frac{a_0}{1+a_0}\right\|_{\dot B^{\frac nq}_{q,1}} \lesssim \|a_0\|_{\dot B^{\frac nq}_{q,1}} \lesssim \frac{c_1}{C_1}.$$

Taking c_1 small enough or C_1 large enough gives rise to (1.3). Then Theorem 1.1 ensures that there exists a positive time T so that (1.2) has a unique solution $(a,u,\nabla\Pi)$ with

$$\begin{aligned}
a\in C([0,T];\dot B^{\frac nq}_{q,1}(\mathbb R^n)),\ u\in C([0,T];\dot B^{\frac np-1}_{p,1}(\mathbb R^n))\cap L^1_T(\dot B^{\frac np+1}_{p,1}(\mathbb R^n)),\\
\text{and}\ \ \nabla\Pi\in L^1_T(\dot B^{\frac np-1}_{p,1}(\mathbb R^n)).
\end{aligned} \tag{3.11}$$

We denote T^* to be the largest time so that there holds (3.11). Hence to prove Theorem 1.2, we only need to prove that $T^*=\infty$ and there holds (1.6).

We first get from (1.2) that $(\bar u,\nabla\Pi)$ solves the Stokes system

$$\begin{cases}\bar u_t-\mu\Delta\bar u+\nabla\Pi=G, \quad (t,x)\in(0,T^*)\times\mathbb R^n,\\ \operatorname{div}\bar u=0,\\ \bar u|_{t=0}=0\end{cases}$$

with

$$G=a(\mu\Delta\bar u-\nabla\Pi)+\mu a\Delta u_F-(\bar u\cdot\nabla\bar u+\operatorname{div}(u_F\otimes\bar u+\bar u\otimes u_F)+u_F\cdot\nabla u_F).$$

Applying Lemma 2.2, we have for any $t < T^*$ that

$$\|a(\mu\Delta\bar{u}_\lambda - \nabla\Pi_\lambda)\|_{L^1_t\left(\dot{B}^{\frac{n}{p}-1}_{p,1}\right)} \lesssim \|a\|_{L^\infty_t\left(\dot{B}^{\frac{n}{q}}_{q,1}\right)}\|\mu\Delta\bar{u}_\lambda, \nabla\Pi_\lambda\|_{L^1_t\left(\dot{B}^{\frac{n}{p}-1}_{p,1}\right)},$$

$$\|\mu a_\lambda\Delta u_F\|_{L^1_t\left(\dot{B}^{\frac{n}{p}-1}_{p,1}\right)} \lesssim \mu\|a_\lambda\|_{L^1_{t,f}\left(\dot{B}^{\frac{n}{q}}_{q,1}\right)},$$

$$\|\bar{u}\cdot\nabla\bar{u}_\lambda + (u_F\cdot\nabla u_F)_\lambda\|_{L^1_t\left(\dot{B}^{\frac{n}{p}-1}_{p,1}\right)} \lesssim \|\bar{u}\|_{L^\infty_t\left(\dot{B}^{\frac{n}{p}-1}_{p,1}\right)}\|\bar{u}_\lambda\|_{L^1_t\left(\dot{B}^{\frac{n}{p}+1}_{p,1}\right)} + C_F,$$

Using Lemma 2.2 , (1.4) and interpolation inequality, we have for any $\eta > 0$ that

$$\begin{aligned}\|\operatorname{div}(u_F\otimes\bar{u}_\lambda + \bar{u}_\lambda\otimes u_F)\|_{L^1_t\left(\dot{B}^{\frac{n}{p}-1}_{p,1}\right)} &\lesssim \int_0^t \|\bar{u}_\lambda\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\|u_F\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\,d\tau\\ &\lesssim \int_0^t \|\bar{u}_\lambda\|^{\frac{1}{2}}_{\dot{B}^{\frac{n}{p}+1}_{p,1}}\|\bar{u}_\lambda\|^{\frac{1}{2}}_{\dot{B}^{\frac{n}{p}-1}_{p,1}}\|u_F\|^{\frac{1}{2}}_{\dot{B}^{\frac{n}{p}+1}_{p,1}}\|u_F\|^{\frac{1}{2}}_{\dot{B}^{\frac{n}{p}-1}_{p,1}}\,d\tau\\ &\lesssim \eta\|\bar{u}_\lambda\|_{L^1_t\left(\dot{B}^{\frac{n}{p}+1}_{p,1}\right)}\frac{1}{\eta}\|u_0\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}\|\bar{u}_\lambda\|_{L^1_{t,f}\left(\dot{B}^{\frac{n}{p}-1}_{p,1}\right)},\quad 0\le t<T^*.\end{aligned}$$

Finally, we conclude that

$$\begin{aligned}\|G_\lambda\|_{L^1_t\left(\dot{B}^{\frac{n}{p}-1}_{p,1}\right)} \lesssim{}& \left(\eta + \mu\|a\|_{L^\infty_t\left(\dot{B}^{\frac{n}{q}}_{q,1}\right)} + \|\bar{u}\|_{L^\infty_t\left(\dot{B}^{\frac{n}{p}-1}_{p,1}\right)}\right)\|\bar{u}_\lambda\|_{L^1_t\left(\dot{B}^{\frac{n}{p}+1}_{p,1}\right)}\\ &+ \frac{1}{\eta}\|u_0\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}\|\bar{u}_\lambda\|_{L^1_{t,f}\left(\dot{B}^{\frac{n}{p}-1}_{p,1}\right)} + \|a\|_{L^\infty_t\left(\dot{B}^{\frac{n}{q}}_{q,1}\right)}\|\nabla\Pi_\lambda\|_{L^1_t\left(\dot{B}^{\frac{n}{p}-1}_{p,1}\right)}\\ &+ \mu\|a_\lambda\|_{L^1_{t,f}\left(\dot{B}^{\frac{n}{q}}_{q,1}\right)} + C_F,\qquad 0\le t<T^*.\end{aligned}$$

Applying Proposition 3.2 and taking $\eta \ll \mu$ in the above inequality, we arrive at

$$\begin{aligned}&\|\bar{u}_\lambda\|_{\widetilde{L}^\infty_t\left(\dot{B}^{\frac{n}{p}-1}_{p,1}\right)} + \lambda\|\bar{u}_\lambda\|_{L^1_{t,f}\left(\dot{B}^{\frac{n}{p}-1}_{p,1}\right)} + \mu\|\bar{u}_\lambda\|_{L^1_t\left(\dot{B}^{\frac{n}{p}+1}_{p,1}\right)} + \|\nabla\Pi_\lambda\|_{L^1_t\left(\dot{B}^{\frac{n}{p}-1}_{p,1}\right)}\\ &\lesssim \left(\mu\|a\|_{L^\infty_t\left(\dot{B}^{\frac{n}{q}}_{q,1}\right)} + \|\bar{u}\|_{L^\infty_t\left(\dot{B}^{\frac{n}{p}-1}_{p,1}\right)}\right)\|\bar{u}_\lambda\|_{L^1_t\left(\dot{B}^{\frac{n}{p}+1}_{p,1}\right)}\\ &\quad+ \mu^{-1}\|u_0\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}\|\bar{u}_\lambda\|_{L^1_{t,f}\left(\dot{B}^{\frac{n}{p}-1}_{p,1}\right)} + \|a\|_{L^\infty_t\left(\dot{B}^{\frac{n}{q}}_{q,1}\right)}\|\nabla\Pi_\lambda\|_{L^1_t\left(\dot{B}^{\frac{n}{p}-1}_{p,1}\right)}\\ &\quad+ \mu\|a_\lambda\|_{L^1_{t,f}\left(\dot{B}^{\frac{n}{q}}_{q,1}\right)} + C_F,\qquad 0\le t<T^*.\end{aligned}\tag{3.12}$$

Summing up (3.12) and (3.3)$\times\mu$, we obtain

$$\begin{aligned}&\|\bar{u}_\lambda\|_{\widetilde{L}^\infty_t\left(\dot{B}^{\frac{n}{p}-1}_{p,1}\right)} + \mu\|\bar{u}_\lambda\|_{L^1_t\left(\dot{B}^{\frac{n}{p}+1}_{p,1}\right)} + \|\nabla\Pi_\lambda\|_{L^1_t\left(\dot{B}^{\frac{n}{p}-1}_{p,1}\right)} + \mu\|a_\lambda\|_{\widetilde{L}^\infty_t\left(\dot{B}^{\frac{n}{q}}_{q,1}\right)}\\ &\quad+ \left(\lambda - C\mu^{-1}\|u_0\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}\right)\|\bar{u}_\lambda\|_{L^1_{t,f}\left(\dot{B}^{\frac{n}{p}-1}_{p,1}\right)} + \mu(\lambda - C)\|a_\lambda\|_{L^1_{t,f}\left(\dot{B}^{\frac{n}{q}}_{q,1}\right)}\\ &\le C\left\{\left(\mu\|a\|_{L^\infty_t\left(\dot{B}^{\frac{n}{q}}_{q,1}\right)} + \|\bar{u}\|_{L^\infty_t\left(\dot{B}^{\frac{n}{p}-1}_{p,1}\right)}\right)\|\bar{u}_\lambda\|_{L^1_t\left(\dot{B}^{\frac{n}{p}+1}_{p,1}\right)}\right.\\ &\quad\left.+ \|a\|_{L^\infty_t\left(\dot{B}^{\frac{n}{q}}_{q,1}\right)}\|\nabla\Pi_\lambda\|_{L^1_t\left(\dot{B}^{\frac{n}{p}-1}_{p,1}\right)} + \mu\|a_0\|_{\dot{B}^{\frac{n}{q}}_{q,1}} + C_F\right\},\qquad 0\le t<T^*.\end{aligned}\tag{3.13}$$

To complete the proof, we shall use the method of continuity. For this, we define

$$T^{**} \stackrel{\text{def}}{=} \sup\left\{t\in[0,T^*): \|\bar u\|_{\widetilde L^\infty_t\left(\dot B^{\frac np-1}_{p,1}\right)}+\mu\|\bar u\|_{L^1_t\left(\dot B^{\frac np+1}_{p,1}\right)}+\|\nabla\Pi\|_{L^1_t\left(\dot B^{\frac np-1}_{p,1}\right)}+\mu\|a\|_{\widetilde L^\infty_t\left(\dot B^{\frac nq}_{q,1}\right)}\le c_1\mu\right\}. \tag{3.14}$$

Fixing the constant C in (3.13) and taking $c_1=\frac{1}{2C}$, $\lambda=C+C\mu^{-1}\|u_0\|_{\dot B^{\frac np-1}_{p,1}}$, we deduce from (3.13) and (3.14) that

$$\begin{aligned}&\|\bar u_\lambda\|_{\widetilde L^\infty_t\left(\dot B^{\frac np-1}_{p,1}\right)}+\mu\|\bar u_\lambda\|_{L^1_t\left(\dot B^{\frac np+1}_{p,1}\right)}+\|\nabla\Pi_\lambda\|_{L^1_t\left(\dot B^{\frac np-1}_{p,1}\right)}+\mu\|a_\lambda\|_{\widetilde L^\infty_t\left(\dot B^{\frac nq}_{q,1}\right)}\\&\quad\le 2C\left(\mu\|a_0\|_{\dot B^{\frac nq}_{q,1}}+C_F\right),\qquad \text{for } t\le T^{**}.\end{aligned}$$

As a consequence,

$$\begin{aligned}&\|\bar u\|_{\widetilde L^\infty_t\left(\dot B^{\frac np-1}_{p,1}\right)}+\mu\|\bar u\|_{L^1_t\left(\dot B^{\frac np+1}_{p,1}\right)}+\|\nabla\Pi\|_{L^1_t\left(\dot B^{\frac np-1}_{p,1}\right)}+\mu\|a\|_{\widetilde L^\infty_t\left(\dot B^{\frac nq}_{q,1}\right)}\\&\quad\le 2C\left(\mu\|a_0\|_{\dot B^{\frac nq}_{q,1}}+C_F\right)\exp\left(\lambda\int_0^t f(t')dt'\right)\\&\quad\le C_1\left(\mu\|a_0\|_{\dot B^{\frac nq}_{q,1}}+C_F\right)\exp\left(C_1\|u_0\|^2_{\dot B^{\frac np-1}_{p,1}}/\mu^2\right),\qquad\text{for } t\le T^{**},\end{aligned}$$

with some constant $C_1>C$. The above estimate along with the smallness condition (1.5) implies that

$$\|\bar u\|_{\widetilde L^\infty_t\left(\dot B^{\frac np-1}_{p,1}\right)}+\mu\|\bar u\|_{L^1_t\left(\dot B^{\frac np+1}_{p,1}\right)}+\|\nabla\Pi\|_{L^1_t\left(\dot B^{\frac np-1}_{p,1}\right)}+\mu\|a\|_{\widetilde L^\infty_t\left(\dot B^{\frac nq}_{q,1}\right)}\le\frac12 c_1\mu \tag{3.15}$$

for $t\le T^{**}$.

Now we can conclude that $T^*=\infty$ by using the standard method of continuity. In fact, it follows from (3.14) and (3.15) that $T^{**}=T^*$. If $T^*<\infty$, in view of (3.15), we can use Theorem 1.1 to extend the local existence time of solution satisfying (3.11) beyond T^*. But this is a contradiction with our assumption that T^* is maximal. Whence we conclude that $T^*=\infty$ and the conclusion of Theorem 1.2 follows.

4. Proof of Theorem 1.3

In this section, we shall verify that the divergence free vector field $u_{0,\varepsilon}$ introduced in Theorem 1.3 can be large in every direction, namely, we shall check (1.8). We shall also show that the nonlinear smallness assumption (1.9) is valid. Let us start by proving the following lemma:

Lemma 4.1. *Let $n\ge3$, $f\in\mathscr S(\mathbb R^n)$ be given, $p\in(1,\infty]$, $\sigma\in(0,n(1-\frac1p))$ and $\alpha\in[0,1)$. Then there exists a constant $C>0$ such that for any $\varepsilon\in(0,1)$ small enough, the function*

$$f^\varepsilon(x)\stackrel{\text{def}}{=} e^{i\frac{x_1}{\varepsilon}}f\left(x_1,\dots,x_{n-2},\frac{x_{n-1}}{\varepsilon^\alpha},x_n\right)$$

satisfies

$$C^{-1}\varepsilon^{\sigma+\frac\alpha p}\le\|f^\varepsilon\|_{\dot B^{-\sigma}_{p,1}}\le C\varepsilon^{\sigma+\frac\alpha p}.$$

Proof. Applying Lemma 3.1 in [9], we can obtain the upper bound of $\|f^\varepsilon\|_{\dot B^{-\sigma}_{p,1}}$. In order to bound from below $\|f^\varepsilon\|_{\dot B^{-\sigma}_{p,1}}$, we first approximate f by g in the following sense

$$\sup_{x\in\mathbb R^n}(1+|x|^2)^k|D^\alpha(f-g)(x)|\le\eta\ \ \textit{and}\ \ \hat g\in\mathscr D(\mathbb R^n)$$

with k large enough and η small enough. In particular, we have

$$\|f^\varepsilon - g^\varepsilon\|_{L^p} \le \eta\varepsilon^{\frac{\alpha}{p}}, \quad \text{and} \quad \|f^\varepsilon - g^\varepsilon\|_{\dot B^{-\sigma}_{p,1}} \le \eta\varepsilon^{\sigma+\frac{\alpha}{p}}. \tag{4.1}$$

Suppose the support of $\hat g$ is included in the ball $B(0,R)$ for some $R>0$, then the support of $\hat{g^\varepsilon}$ is included in the ball $B((\varepsilon^{-1},0,\ldots,0),\varepsilon^{-\alpha}R)$. This ball is included in $\varepsilon^{-1}\mathscr{C}$ for some annulus $\mathscr{C}$. Choose a smooth function $\zeta \in \mathscr{D}(\mathbb{R}^n)$ with value 1 in the neighborhood of $\mathscr{C}$ and vanishing identically near 0. Then we have

$$g^\varepsilon = \mathcal{F}^{-1}(\zeta(\varepsilon\xi)\hat{g^\varepsilon}) = \mathcal{F}^{-1}\left(\zeta(\varepsilon\xi)e^{\varepsilon^2|\xi|^2}\widehat{e^{\varepsilon^2\Delta}g^\varepsilon}\right).$$

In particular, for $0<t<\varepsilon^2$, we infer from the above equality that

$$\|g^\varepsilon\|_{L^p} \lesssim \|e^{\varepsilon^2\Delta}g^\varepsilon\|_{L^p} \lesssim \|e^{t\Delta}g^\varepsilon\|_{L^p},$$

which along with (4.1) and Lemma 2.5 implies

$$\begin{aligned}\|g^\varepsilon\|_{\dot B^{-\sigma}_{p,1}} &\approx \int_0^\infty t^{\frac{\sigma}{2}-1}\|e^{t\Delta}g^\varepsilon\|_{L^p}dt \ge C^{-1}\|g^\varepsilon\|_{L^p}\int_0^{\varepsilon^2} t^{\frac{\sigma}{2}-1}dt\\ &\ge C^{-1}\varepsilon^\sigma(\|f^\varepsilon\|_{L^p} - \eta\varepsilon^{\frac{\alpha}{p}}) \ge C^{-1}\varepsilon^{\sigma+\frac{\alpha}{p}}(\|f\|_{L^p}-\eta).\end{aligned}$$

Again thanks to (4.1), we arrive at

$$\|f^\varepsilon\|_{\dot B^{-\sigma}_{p,1}} \ge \|g^\varepsilon\|_{\dot B^{-\sigma}_{p,1}} - \eta\varepsilon^{\sigma+\frac{\alpha}{p}} \ge C^{-1}\varepsilon^{\sigma+\frac{\alpha}{p}}(\|f\|_{L^p}-C\eta).$$

Taking $\eta \ll \|f\|_{L^p}$, we complete the proof of the lemma. □

Remark 4.1. With some slight modifications of the above proof, the lemma remains true for real valued functions

$$\begin{aligned} f^\varepsilon(x) &= \cos\left(\frac{x_1}{\varepsilon}\right) f\left(x_1,\ldots,x_{n-2},\frac{x_{n-1}}{\varepsilon^\alpha},x_n\right),\\ \text{or}\quad f^\varepsilon(x) &= \sin\left(\frac{x_1}{\varepsilon}\right) f\left(x_1,\ldots,x_{n-2},\frac{x_{n-1}}{\varepsilon^\alpha},x_n\right). \end{aligned} \tag{4.2}$$

We note that (1.8) is an immediate consequence of Lemma 4.1. Indeed, applying Lemma 4.1 for $\alpha=0$, we infer that the $\dot B^{\frac{n}{p}-1}_{p,1}$ norm of the j-th ($j<n$) component of $v_{0,\varepsilon}$ is equivalent to $(-\log\varepsilon)^{\frac14}$. While $\|\omega^{n-1}_{0,\varepsilon}\|_{\dot B^{\frac{n}{p}-1}_{p,1}}$ and $\|\omega^{n}_{0,\varepsilon}\|_{\dot B^{\frac{n}{p}-1}_{p,1}}$ are equivalent respectively to $\varepsilon^\alpha(-\log\varepsilon)^{\frac14}$ and $(-\log\varepsilon)^{\frac14}$. Thus, we require $\alpha>0$ in Theorem 1.3 so that there holds (1.8).

In order to control the nonlinearity $S_\mu(t)u_{0,\varepsilon}\cdot\nabla S_\mu(t)u_{0,\varepsilon}$, we need to prove the following lemma, which is a slight improvement of Lemma 3.2 in [9].

Lemma 4.2. *Let $p\in(1,\infty]$, $\sigma\in(0,n(1-\frac1p))$ and $\theta=\frac{\sigma}{n(1-\frac1p)}\in(0,1)$. Then for any $f,g\in\dot H^{-1}(\mathbb{R}^n)\cap\dot B^{-1}_{2p,2}(\mathbb{R}^n)$, we have*

$$\|S_\mu(t)fS_\mu(t)g\|_{L^1(\mathbb{R}^+;\dot B^{-\sigma}_{p,1})} \lesssim \mu^{-1}(\|f\|_{\dot H^{-1}}\|g\|_{\dot H^{-1}})^\theta(\|f\|_{\dot B^{-1}_{2p,2}}\|g\|_{\dot B^{-1}_{2p,2}})^{1-\theta}.$$

Proof. Denote by $E_j \overset{\text{def}}{=} \|\dot\Delta_j(S_\mu(t)fS_\mu(t)g)\|_{L^1(\mathbb{R}^+;L^p)}$. Then for any integer j_0, we have

$$\|S_\mu(t)fS_\mu(t)g\|_{L^1(\mathbb{R}^+;\dot B^{-\sigma}_{p,1})} = \sum_{j\le j_0}2^{-j\sigma}E_j + \sum_{j>j_0}2^{-j\sigma}E_j. \tag{4.3}$$

For the low frequencies, using Lemma 2.1 and the Cauchy–Schwarz inequality, we get

$$\begin{aligned}E_j &\lesssim 2^{nj(1-\frac1p)}\|S_\mu(t)fS_\mu(t)g\|_{L^1(\mathbb{R}^+;L^1)}\\ &\lesssim 2^{nj(1-\frac1p)}\|S_\mu(t)f\|_{L^2(\mathbb{R}^+;L^2)}\|S_\mu(t)g\|_{L^2(\mathbb{R}^+;L^2)}.\end{aligned}$$

Whence we deduce by using Lemma 2.5 that

$$E_j \lesssim 2^{nj(1-\frac{1}{p})}\mu^{-1}\|f\|_{\dot{H}^{-1}}\|g\|_{\dot{H}^{-1}}. \tag{4.4}$$

For the high frequencies, we have

$$E_j \lesssim \|S_\mu(t)f\|_{L^2(\mathbb{R}^+;L^{2p})}\|S_\mu(t)g\|_{L^2(\mathbb{R}^+;L^{2p})} \lesssim \mu^{-1}\|f\|_{\dot{B}^{-1}_{2p,2}}\|g\|_{\dot{B}^{-1}_{2p,2}}. \tag{4.5}$$

Substituting (4.4) and (4.5) into (4.3), we infer

$$\|S_\mu(t)fS_\mu(t)g\|_{L^1(\mathbb{R}^+;\dot{B}^{-\sigma}_{p,1})} \lesssim \mu^{-1}\Big(2^{(n-\frac{n}{p}-\sigma)j_0}\|f\|_{\dot{H}^{-1}}\|g\|_{\dot{H}^{-1}} + 2^{-\sigma j_0}\|f\|_{\dot{B}^{-1}_{2p,2}}\|g\|_{\dot{B}^{-1}_{2p,2}}\Big).$$

Choosing j_0 to be the integer part of

$$\frac{\theta}{\sigma}\log_2\left(\frac{\|f\|_{\dot{B}^{-1}_{2p,2}}\|g\|_{\dot{B}^{-1}_{2p,2}}}{\|f\|_{\dot{H}^{-1}}\|g\|_{\dot{H}^{-1}}}\right)$$

gives the desired result. □

Finally, we are ready to check (1.9). Using the notations in (1.7) and (4.2), we get by easy but tedious computations,

$$\|S_\mu(t)v_{0,\varepsilon}\cdot\nabla S_\mu(t)v_{0,\varepsilon}\|_{L^1\left(\mathbb{R}^+;\dot{B}^{\frac{n}{p}-1}_{p,1}\right)} \lesssim \frac{(-\log\varepsilon)^{\frac{1}{2}}}{\varepsilon^{2(1-\frac{n}{p})}}\|S_\mu(t)\tilde{f}_\varepsilon S_\mu(t)\tilde{g}_\varepsilon\|_{L^1\left(\mathbb{R}^+;\dot{B}^{\frac{n}{p}-1}_{p,1}\right)},$$

$$\|S_\mu(t)\omega_{0,\varepsilon}\cdot\nabla S_\mu(t)\omega_{0,\varepsilon}\|_{L^1\left(\mathbb{R}^+;\dot{B}^{\frac{n}{p}-1}_{p,1}\right)} \lesssim \frac{(-\log\varepsilon)^{\frac{1}{2}}}{\varepsilon^{2(1-\frac{n}{p}+\frac{\alpha}{p})}}\|S_\mu(t)\tilde{f}^\varepsilon S_\mu(t)\tilde{g}^\varepsilon\|_{L^1\left(\mathbb{R}^+;\dot{B}^{\frac{n}{p}-1}_{p,1}\right)},$$

and

$$\|S_\mu(t)v_{0,\varepsilon}\cdot\nabla S_\mu(t)\omega_{0,\varepsilon}\|_{L^1\left(\mathbb{R}^+;\dot{B}^{\frac{n}{p}-1}_{p,1}\right)} + \|S_\mu(t)\omega_{0,\varepsilon}\cdot\nabla S_\mu(t)v_{0,\varepsilon}\|_{L^1\left(\mathbb{R}^+;\dot{B}^{\frac{n}{p}-1}_{p,1}\right)}$$
$$\lesssim \frac{(-\log\varepsilon)^{\frac{1}{2}}}{\varepsilon^{3-\frac{2n}{p}+\frac{\alpha}{p}}}\|S_\mu(t)\tilde{f}_\varepsilon S_\mu(t)\tilde{g}^\varepsilon\|_{L^1\left(\mathbb{R}^+;\dot{B}^{\frac{n}{p}-1}_{p,1}\right)},$$

where $\tilde{f}$ and $\tilde{g}$ are various Schwartz functions and may be different on different lines. On the other hand, applying Lemmas 4.1 and 4.2 gives rise to

$$\|S_\mu(t)\tilde{f}_\varepsilon S_\mu(t)\tilde{g}_\varepsilon\|_{L^1\left(\mathbb{R}^+;\dot{B}^{\frac{n}{p}-1}_{p,1}\right)} \lesssim \mu^{-1}\left(\|\tilde{f}_\varepsilon\|_{\dot{H}^{-1}}\|\tilde{g}_\varepsilon\|_{\dot{H}^{-1}}\right)^\theta\left(\|\tilde{f}_\varepsilon\|_{\dot{B}^{-1}_{2p,2}}\|\tilde{g}_\varepsilon\|_{\dot{B}^{-1}_{2p,2}}\right)^{1-\theta} \lesssim \mu^{-1}\varepsilon^2.$$

Similarly, we have

$$\|S_\mu(t)\tilde{f}^\varepsilon S_\mu(t)\tilde{g}^\varepsilon\|_{L^1\left(\mathbb{R}^+;\dot{B}^{\frac{n}{p}-1}_{p,1}\right)} \lesssim \mu^{-1}\varepsilon^{2+\frac{\alpha}{n}},$$
$$\|S_\mu(t)\tilde{f}_\varepsilon S_\mu(t)\tilde{g}^\varepsilon\|_{L^1\left(\mathbb{R}^+;\dot{B}^{\frac{n}{p}-1}_{p,1}\right)} \lesssim \mu^{-1}\varepsilon^{2+\frac{\alpha}{2n}}.$$

As a consequence, we conclude that (1.9) is valid and the proof of Theorem 1.3 is completed.

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