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Global Regularity and Long-time Behavior of the Solutions to the 2D Boussinesq Equations without Diffusivity in a Bounded Domain

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Abstract. New results are obtained for global regularity and long-time behavior of the solutions to the 2D Boussinesq equations for the flow of an incompressible fluid with positive viscosity and zero diffusivity in a smooth bounded domain. Our first result for global boundedness of the solution (u, θ) in $D(A) \times H^1$ improves considerably the main result of the recent article (Hu et al. in J Math Phys 54(8):081507, 2013). Our second result on global boundedness of the solution (u, θ) in $V \times H^1$ for both bounded domain and the whole space $\mathbb{R}^2$ is a new one. It has been open and also seems much more challenging than the first result. Global regularity of the solution (u, θ) in $D(A) \times H^2$ is also proved.

Keywords. Two dimensional dissipative Boussinesq equations, zero diffusivity, global regularity, long time behavior.

1. Introduction

Suppose that $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$. Consider the following Boussinesq equations for the flow of an incompressible fluid with positive viscosity and zero diffusivity:

$$\partial_t u - \Delta u + u \cdot \nabla u + \nabla p = \theta e_2, \quad x \in \Omega,\ t > 0 \tag{1.1}$$

$$\nabla \cdot u = 0, \quad x \in \Omega,\ t > 0 \tag{1.2}$$

$$\partial_t \theta + u \cdot \nabla \theta = 0, \quad x \in \Omega,\ t > 0, \tag{1.3}$$

where u is the vector field of fluid velocity, p and θ are the fluid pressure and density/temperature field and $e_2 = (0, 1) \in \mathbb{R}^2$. For simplicity of presentation, the viscosity constant has been set as 1. The standard no-slip Dirichlet boundary condition is imposed:

$$u\big|_{\partial\Omega} = 0, \quad t > 0. \tag{1.4}$$

The above equations are also equipped with initial conditions:

$$u(x,t)\big|_{t=0} = u_0(x), \quad \theta(x,t)\Big|_{t=0} = \theta_0(x), \quad x \in \Omega, \tag{1.5}$$

with proper u_0 and θ_0 to be specified later.

The 2D Boussinesq equations with *positive* viscosity and *positive* diffusivity is well known to be globally well-posed for both weak and strong solutions. This can be obtained by following the classic theory for the 2D Navier-Stokes equations, cf. e.g. [3,12]. See also a detailed proof of the global existence of the strong solutions as given in [11]. If viscosity and diffusivity *both* vanish, then the global existence of strong solutions is still an open problem.

Motivated by the works of [2,8], there have been extensive research activities on the global regularity of the solutions to the 2D Boussinesq equations for many cases sitting between the easiest one with full viscosity and diffusivity and the hardest one with zero viscosity and diffusivity. For simplicity of presentation, we refrain from discussing or citing extensive references for all those varying cases. In the following, we briefly discuss existing results just for the case to be studied in this article, i.e. the one

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with *positive and full* viscosity and *zero* diffusivity. The notations to be used in this article are basically standard and will be defined and explained in details in Sect. 2.

First of all, existence and uniqueness of global weak solutions have been established in [4] for $\Omega = \mathbb{R}^2$ and in [6] for Ω being a smooth bounded domain. Global regularity for the solution $(u, \theta) \in H^1 \times L^2$ can be easily proved. See e.g. [2,5,8] for $\Omega = \mathbb{R}^2$, [10] for $\Omega = \mathbb{T}^2$ and [7] for Ω being a smooth bounded domain. See also Theorems 2.1 and 2.2.

For the case $\Omega = \mathbb{R}^2$, global in time regularity is proved in [2] for $(u, \theta) \in H^m \times H^m$, with the integer $m \geqslant 3$ and in [8] for $(u, \theta) \in H^m \times H^{m-1}$, with the integer $m \geqslant 3$. Global regularity is also proved in [4] for $(u, \theta) \in H^s \times H^{s-1}$, with real $s \geqslant 3$.

For the case Ω being a bounded domain with a physical boundary, the particular difficulty in addition to complication of nonlinearity is the intensive vorticity along the boundary, which requires more careful treatment. Mathematically, it is convenient to use vorticity equation when $\Omega = \mathbb{R}^2$, as vortex stretching does not exist. Indeed, previous works for the case $\Omega = \mathbb{R}^2$ all take advantage of this feature. However, for a bounded domain Ω, except the case of $\Omega = \mathbb{T}^2$, this approach is not as feasible since it is difficult to deal with vorticity along the boundary of Ω. The case of Ω being a smooth bounded domain with no-slip boundary condition on u has been studied in [9], the main result of which is the global regularity for solutions $(u, \theta) \in H^3 \times H^3$ under some *extra compatibility conditions* for initial datum.

As pointed out by [7], due to the special feature of the equations, global in time regularity in the Sobolev spaces of intermediate orders is in fact much more difficult to be established than in the higher order Sobolve spaces $H^m \times H^m$ and $H^m \times H^{m-1}$ for $m \geqslant 3$ as studied in the above mentioned literature. The main result of [7] is global in time regularity of the solutions $(u, \theta) \in D(A) \times H^1$ for the case of a smooth bounded domain Ω with no-slip boundary condition on u. Here, A is the Stokes operator of u with no-slip boundary condition on $\partial\Omega$. See Sect. 2 for notations. This result is proved in [7] using a new Gronwall lemma for two coupled nonlinear differential inequalities with one involving a logarithmic function. No additional compatibility conditions is required for u_0 beyond $u_0 \in D(A)$. The analysis of [7] also applies to the case of $\Omega = \mathbb{R}^2$ or $\mathbb{T}^2$ with minor modifications to obtain similar results. It might be worthy of mentioning that for the case of $\Omega = \mathbb{R}^2$, global regularity in $H^2 \times H^1$ is also obtained in [8] using a boot-strapping argument, though much more complicated than the approach of [7]. Notice also that vorticity equation is used in [8]. Thus, it seems that the analysis of [8] can not cover the case of a bounded domain with non-periodic boundary conditions.

In this article, we study further the global regularity of solutions of this system and especially the long-time behavior of the solutions of this system. We will focus on the case of a *bounded* domain $\Omega \subset \mathbb{R}^2$. All the conclusions obtained in this article can be extended without adjustment to the case of Ω for which Poincaré inequality is valid. All the results except for those related to uniform upper bounds are also valid for the case $\Omega = \mathbb{R}^2$.

First, a new proof of the $D(A) \times H^1$ global regularity will be presented using a method quite different from those of [7,8]. The new proof improves considerably the result of [7] in the following sense: the best upper bounds of $\|Au(t)\|_2$ and $\|\nabla\theta(t)\|_2$ for $t \in [0, +\infty)$ that can be derived from the analysis of [7] both grow *double exponentially* with respect to t as $t \to \infty$; while under the same conditions as those of [7] and somewhat surprisingly, our analysis provides a *uniform* upper bound for $\|A(t)u\|_2$ with respect to $t \in [0, \infty)$ (and indeed also a *bounded absorbing set* for $u(t)$ in $D(A)$ as $t \to \infty$) and an upper bound for $\|\nabla\theta(t)\|_2$ which is only *single exponential* with repsect to t^2 as $t \to \infty$. See Theorems 3.1 and 3.2 for detailed statements of the result. Our new analysis seems also more transparent than previous ones. Notice that due to the inductive feature of the method of [8], it seems quite difficult to get upper bounds for $\|Au(t)\|_2$ and $\|\nabla\theta(t)\|_2$ in *explicit form of functions* of t from the boot-strapping argument of [8]. In this sense, the method of [8] seems to be less sharp than that of [7] and, as already mentioned above, it does not apply to the case of a bounded Ω with non-periodic boundary conditions.

As our next main results, we prove the global regularity for $(u, \theta) \in V \times H^1$ and for $(u, \theta) \in D(A) \times H^2$. Here V is the standard space of divergence-free vector fields in L^2 with their gradients also in L^2. See Sect. 2 for notations. Especially, the global regularity for $(u, \theta) \in V \times H^1$ seems to be quite unexpected complement of the existing results for global and local regularity. Notice that there are obvious essential reasons for previous analysis of regularity results to focus on the case $(u, \theta) \in H^2 \times H^1$. For example, due to lack of diffusivity in the θ equation (1.3), even though $\|\theta(t)\|_p$ is conserved for all $t \geqslant 0$, there is no direct way to get global or even local in time regularity $\theta \in H^1$ just from θ equation (1.3) alone. Moreover, it seems still extremely difficult to obtain regularity of θ in H^1 by just coupling it with the enstropy equation of u, even just local in time. One key obstacle is that, in dealing with the involved nonlinearity, $\|u\|_{H^2}$ can not dominate $\|\nabla u\|_\infty$ in 2D domain via Sobolev imbedding. Even with one order higher regularity of u coupled, it is still technically complicated to combine the tool of Brezis–Gallouet type inequalities to deal with regularity of θ in H^1, as can be seen in [7,8] and in the proof of the first main result of this article as given in Sect. 3. Now, with only H^1 regularity for u coupling with H^1 regularity for θ, proving global regularity of $(u, \theta) \in V \times H^1$ appears *much* more challenging than proving global regularity of $(u, \theta) \in D(A) \times H^1$, since Brezis–Gallouet type inequalities would *not* help any more. Indeed, to the knowledge of the author, existence of $V \times H^1$ solutions has been *open* for any kind of domains, even though the essential difficulty here is indeed only a *local* property. In this article, some new ideas from spectral decomposition analysis will be used to resolve completely the essential difficulty. The main result on global regularity of $(u, \theta) \in V \times H^1$ thus obtained will be summarized in Theorem 4.1. As the second application of spectral decomposition and for completeness, we prove our next main result, Theorem 5.1, for the global regularity of $(u, \theta) \in D(A) \times H^2$. Since we will not use Brezis–Gallouet type inequalities here, the proof is a lot simpler than otherwise.

We comment that our proofs of the above mentioned global regularity results are self-contained and global regularity results in higher order Sobolev spaces can thus be obtained as consequence of our global regularity results.

The rest of this article is organized as following: In Sect. 2, we give the notations, briefly review the background results and recall some important facts crucial to later analysis. In Sect. 3, we state and prove Theorems 3.1 and 3.2 for the global regularity of the solutions $(u, \theta) \in D(A) \times H$. In Sect. 4, we state and prove Theorem 4.1 for the global regularity of the solutions $(u, \theta) \in V \times H^1$. In Sect. 5, we state and prove Theorem 5.1 for the global regularity of the solutions $(u, \theta) \in D(A) \times H^2$.

2. Preliminaries

2.1. Notations and Some Basic Results

Throughout this article, we use the notations that, for real numbers A and B,

$$A \preccurlyeq B \quad \text{iff } A \leqslant C \cdot B,$$

and

$$A \approx B \quad \text{iff } c \cdot A \leqslant B \leqslant C \cdot A,$$

for some positive constants c and C independent of A and B.

Recall that Ω is a bounded domain in $\mathbb{R}^2$ with sufficiently smooth boundary $\partial\Omega$. By a domain, we always mean an open and connected open subset of $\mathbb{R}^2$. Denote by $L^p(\Omega)$ $(1 \leqslant p < +\infty)$ the classic Lebesgue L^p space with the norm:

$$\|f\|_p := \left(\int_\Omega |f(x)|^p \, dx \right)^{\frac{1}{p}}, \quad \forall f \in L^p(\Omega).$$

The Lebsgue space $L^\infty(\Omega)$ is defined as the Banach space of (classes of) real functions on Ω which are measurable and essentially bounded with the norm

$$\|f\|_\infty := \operatorname{ess\,sup}_{x \in \Omega} |f(x)|.$$

Denote by $H^m(\Omega)$ $(m \geqslant 1)$ the Sobolev space for square-integrable functions with square-integrable weak derivatives up to order m with the norm

$$\|f\|_{H^m} := \left(\sum_{|\alpha| \leqslant m} \|D^\alpha f\|_2^2 \right)^{\frac{1}{2}}, \quad \forall f \in H^m(\Omega).$$

We use the standard notations of the following functional spaces for the initial value problem with no-slip boundary condition:

$$H := \left\{ v \in (L^2(\Omega))^2 \;\middle|\; \nabla \cdot v = 0 \ \text{ in } \Omega, \quad v \cdot n = 0 \ \text{ on } \partial\Omega \right\},$$
$$V := \left\{ v \in (H_0^1(\Omega))^2 \;\middle|\; \nabla \cdot v = 0 \ \text{ in } \Omega \right\},$$

where the equalities are in distribution sense, n is the unit vector normal to $\partial\Omega$ and outward with respect to Ω and $H_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$, i.e. the space of the functions in H^1 with zero-trace on $\partial\Omega$. Later on, we do not distinguish the notations for vector and scalar function spaces, which are self-evident from the context. Therefore, we will use e.g. $L^p(\Omega)$ and $H^m(\Omega)$ or simply L^p and H^m to denote $(L^p(\Omega))^2$ and $(H^m(\Omega))^2$ respectively provided there is no confusion.

Denote by $\mathbb{P}$ the Helmholtz orthogonal projector in $(L^2(\Omega))^2$ onto H. The Stoke operator $A : D(A) \subset H \mapsto H$ is defined as

$$A := -\mathbb{P}\Delta, \quad D(A) = (H^2(\Omega))^2 \cap V.$$

Recall that

$$\|\mathbb{P}v\|_H \preccurlyeq \|v\|_2, \ \forall v \in L^2(\Omega)^2, \quad \|\nabla \mathbb{P}v\|_2 \preccurlyeq \|v\|_{H^1}, \ \forall v \in H^1(\Omega)^2. \tag{2.1}$$

By the spectral theorem for A, there exists a sequence $\{\lambda_j\}_1^\infty$

$$0 < \lambda_1 \leqslant \cdots \leqslant \lambda_j \leqslant \lambda_{j+1} \leqslant \cdots, \quad \lim_{j \to \infty} \lambda_j = \infty,$$

and a family $\{w_j\}_1^\infty \subset D(A)$ which is orthonormal in H such that

$$Aw_j = \lambda_j w_j, \quad \forall j = 1, 2, \ldots$$

Denote, for $\alpha \in [0, \infty)$,

$$A^\alpha v := \sum_{j=1}^\infty \lambda_j^\alpha (v, w_j) w_j, \quad \forall v \in D(A^\alpha),$$

where $(\cdot, \cdot)$ is the inner product in H and

$$D(A^\alpha) := \left\{ v \in H \;\middle|\; \sum_{j=1}^\infty \lambda_j^{2\alpha} |(v, w_j)|^2 < \infty \right\}.$$

Notice that, $D(A^{\frac{1}{2}}) = V$ and

$$\|A^{\frac{1}{2}} v\|_2 = \|\nabla v\|_2, \quad \forall v \in V. \tag{2.2}$$

Moreover, $\|A \cdot\|_2$ is equivalent to $\|\cdot\|_{H^2}$ on $D(A)$, i.e.

$$\|Av\|_2 \approx \|v\|_{H^2} \quad \forall v \in D(A).$$

For $0 \leqslant r \leqslant s \leqslant 1$, there is a Poincaré inequality:

$$\lambda_1^{s-r} \|A^r u\|_2 \leqslant \|A^s u\|_2.$$

Define the family of spectral projection operators $\{P_m\}_1^\infty$ as

$$P_1 v := \sum_{\lambda_j < 2} (u, w_j) w_j, \quad P_m v := \sum_{2^{m-1} \leqslant \lambda_j < 2^m} (u, w_j) w_j, \quad \text{for } m \geqslant 2.$$

Then,

$$\|P_m A^s v\|_2 = \|A^s P_m v\|_2 \approx 2^{ms} \|P_m v\|_2, \quad \text{for } m \geqslant 1, \ s \geqslant 0. \tag{2.3}$$

Introducing the bi-linear operator,

$$B(u,v) := \mathbb{P}(u\cdot\nabla v), \quad \forall u,v\in V,$$

which maps $V\times V$ into V' (the dual space of V), one can also reformulate equations (1.1)–(1.2) as

$$u_t + Au + B(u,u) = \mathbb{P}(\theta e_2). \tag{2.4}$$

It can be shown that (see e.g. [7])

$$\|\nabla B(u,v)\|_2 \preccurlyeq \|u\|_2^{\frac{1}{4}}\|Au\|_2^{\frac{3}{4}}\|v\|_2^{\frac{1}{4}}\|Av\|_2^{\frac{3}{4}} + \|u\|_2^{\frac{1}{2}}\|Au\|_2^{\frac{1}{2}}\|Av\|_2. \tag{2.5}$$

Definition 2.1. Given $(u_0,\theta_0)\in H\times L^2$, $(u(t),\theta(t))$ is a *(global) weak solution* of the initial boundary value problem (1.1)–(1.5) if for any $T>0$, the following statements are valid:

1. $\theta\in C([0,T];H)$, $v\in C([0,T])\cap L^2(0,T;V)$.
2. For any $\psi\in(C^1([0,T]\times\Omega))^2$ such that $\nabla\cdot\psi=0$,

$$\int_0^T \Big[(u,\partial_t\psi) - (\nabla u,\nabla\psi) - (u\cdot\nabla u - \theta e_2,\psi)\Big]\, dt = (u(T),\psi(T)) - (u_0,\psi(0)).$$

3. For any $\phi\in C^1([0,T]\times\Omega)$,

$$\int_0^T \Big[(\theta,\partial_t\phi) - (u\cdot\phi)\Big]\, dt = (\theta(T),\phi(T)) - (\theta_0,\phi(0)).$$

Notice that one could define a weak solution even somewhat *weaker* than the above one. We adopt this one since existence and especially uniqueness of the weak solution to the problem (1.1)–(1.5) has been proved in [4] for the case when $\Omega=\mathbb{R}^2$, and in [6] for the case of a smooth (C^2) bounded domain.

As mentioned in Sect. 1, for $\Omega=\mathbb{R}^2$, global regularity of the solutions in $H^m\times H^m$ with integer $m\geqslant 3$ is proved in [2] and global regularity in $H^m\times H^{m-1}$ with integer $m\geqslant 3$ is proved in [8], which is extended in [4] for $(u,\theta)\in H^s\times H^{s-1}$, with real $s\geqslant 3$. In [9] global regularity in $H^3\times H^3$ is proved when Ω is a smooth *bounded* domain, under *extra* compatibility conditions for initial data. In the more recent work [7], global regularity in $D(A)\times H^1$ is obtained where Ω can be either a bounded domain, $\mathbb{T}^2$ or $\mathbb{R}^2$. No *extra* compatibility conditions for initial data other than $u_0\in D(A)$ is needed.

The main goal of this article is to study global regularity and long-time behavior of the solutions for the case of a bounded domain. The main results of this article, as discussed in Sect. 1, are Theorems 3.1, 3.2, 4.1 and 5.1. The first two improve considerably the main result of [7]. The last two are new results which seem to have been open.

For convenience of reading, we recall the following formulation of the Uniform Growall Lemma, which will be used frequently in later discussion. The detailed proof of the lemma can be found e.g. in [13].

Lemma 2.1 (Uniform Gronwall Lemma). *Let g, h and y be three non-negative locally integrable functions on $(t_0,+\infty)$ such that*

$$\frac{dy}{dt}\leqslant gy+h, \quad \forall t\geqslant t_0,$$

and

$$\int_t^{t+r} g(s)\, ds\leqslant a_1, \quad \int_t^{t+r} h(s)\, ds\leqslant a_2, \quad \int_t^{t+r} y(s)\, ds\leqslant a_3, \quad \forall t\geqslant t_0,$$

where r, a_1, a_2 and a_3 are positive constants. Then

$$y(t+r)\leqslant\left(\frac{a_3}{r}+a_2\right)e^{a_1}, \quad \forall t\geqslant t_0.$$

2.2. Global Regularity in $H \times L^2$

For convenience of later discussion, we recall the following global regularity result:

Theorem 2.1. *Suppose* $(u_0, \theta_0) \in H \times L^2$. *Then,* (1.1)–(1.5) *has a unique weak solution* (u, θ). *Moreover, the following are valid:*

$$\|\theta(t)\|_2 = \|\theta_0\|_2, \quad \forall t > 0, \tag{2.6}$$

$$\|u(t)\|_2^2 \leqslant \|u_0\|_2^2 e^{-\lambda_1 t} + \frac{\|\theta_0\|_2^2}{\lambda_1^2}\left(1 - e^{-\lambda_1 t}\right), \quad \forall t > 0. \tag{2.7}$$

$$\int_t^{t+r} \|A^{\frac{1}{2}} u(\tau)\|_2^2 \, d\tau \leqslant \|u_0\|_2^2 e^{-\lambda_1 t} + \frac{\|\theta_0\|_2^2}{\lambda_1^2}\left(1 - e^{-\lambda_1 t}\right) + \frac{r}{\lambda_1}\|\theta_0\|_2^2, \quad \forall t, r > 0. \tag{2.8}$$

Proof. Existence and uniqueness of the weak solution have already been established in [6]. For completeness, we provide the brief formal a priori estimates yielding (2.6)–(2.8). Due to existence and uniqueness of the weak solution, these a priori estimates can be justified rigorously via standard argument.

First of all, (2.6) is an immediate consequence of (1.2)–(1.4) via the following simple computations:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\theta\|_2^2 &= \int_\Omega \theta_t \theta \, dx = -\int_\Omega (u \cdot \nabla\theta)\theta \, dx = -\frac{1}{2}\int_\Omega u \cdot \nabla(\theta^2) \, dx \\ &= \frac{1}{2}\int_{\partial\Omega} (u \cdot n)\theta^2 \, dS - \frac{1}{2}\int_\Omega (\nabla \cdot u)\theta^2 \, dx = 0.\end{aligned}$$

Next, taking inner product of (1.1) with u and using (1.2) and (1.4), we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \|A^{\frac{1}{2}} u\|_2^2 \leqslant \|\theta\|_2 \|u\| \leqslant \frac{\lambda_1}{2}\|u\|_2^2 + \frac{1}{2\lambda_1}\|\theta\|_2^2 \leqslant \frac{1}{2}\|A^{\frac{1}{2}} u\|_2^2 + \frac{1}{2\lambda_1}\|\theta\|_2^2.$$

Therefore,

$$\frac{d}{dt}\|u\|_2^2 + \|A^{\frac{1}{2}} u\|_2^2 \leqslant \frac{1}{\lambda_1}\|\theta_0\|_2^2, \tag{2.9}$$

implying that

$$\frac{d}{dt}\|u\|_2^2 + \lambda_1\|u\|_2^2 \leqslant \frac{1}{\lambda_1}\|\theta_0\|_2^2,$$

from which (2.7) follows by a direct integration.

Finally, (2.8) can be obtained from (2.9) and (2.7). □

Remark 2.1. We see from (2.7) that, for any fixed $\|\theta_0\|_2$, there is a *bounded* absorbing set for u in H with the radius being $\|\theta_0\|_2/\lambda_1$ and not depending on $\|u_0\|_2$. It is also easy to see from (2.8) that there is a *bounded* absorbing set for $\int_t^{t+r} \|A^{\frac{1}{2}} u\|_2^2 d\tau$ in $\mathbb{R}_+$ with fixed $\|\theta_0\|_2$ and $r(>0)$.

2.3. Global Regularity in $V \times L^2$

The following global regularity will also be used in later discussion:

Theorem 2.2. *Suppose that* $(u_0, \theta_0) \in V \times L^2$. *Then, there is a* $t_0 = t_0(u_0, \theta_0) > 0$, *such that the unique weak solution* (u, θ) *of* (1.1)–(1.5) *satisfies*

$$\|A^{\frac{1}{2}} u(t)\|_2^2 \leqslant 1 + 2\|A^{\frac{1}{2}} u_0\|_2^2, \quad \forall t \in [0, t_0]. \tag{2.10}$$

Moreover, for $t \geqslant t_0$, *the following uniform bound of* $\|A^{\frac{1}{2}} u(t)\|_2$ *is valid:*

$$\|A^{\frac{1}{2}} u(t)\|_2^2 \leqslant \left[\|u_0\|_2^2 e^{-\lambda_1 t} t_0^{-1} + (\lambda_1^{-1} + t_0)\|\theta_0\|_2^2 + \frac{\|\theta_0\|_2^2}{\lambda_1^2 t_0}\left(1 - e^{-\lambda_1 t}\right)\right] e^{a_1}, \tag{2.11}$$

where, there is a generic constant $c_0 > 0$, such that

$$a_1 \leqslant c_0 \left(\|u_0\|_2^2 e^{-\lambda_1 t} + \frac{\|\theta_0\|_2^2}{\lambda_1^2} \right) \left[\|u_0\|_2^2 e^{-\lambda_1 t} + \frac{\|\theta_0\|_2^2}{\lambda_1^2} \left(1 - e^{-\lambda_1 t}\right) + \frac{t_0}{\lambda_1} \|\theta_0\|_2^2 \right]. \tag{2.12}$$

Furthermore, for any $t, r > 0$,

$$\int_t^{t+r} \|Au(\tau)\|_2^2 \, d\tau \preccurlyeq \|A^{\frac{1}{2}} u(t)\|_2^2 + r\|\theta_0\|_2^2 + \int_t^{t+r} \left[\|u_0\|_2^2 e^{-\lambda_1 \tau} + \left(\frac{\|\theta_0\|_2}{\lambda_1} \right)^2 \right] \|A^{\frac{1}{2}} u(\tau)\|_2^4 \, d\tau. \tag{2.13}$$

Proof. Due to Theorem 2.1, all the following a priori estimates can be justified rigorously. Taking the inner product of (2.4) with Au, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|A^{\frac{1}{2}}u\|_2^2 + \|Au\|_2^2 &\leqslant \|\theta\|_2 \|Au\|_2 + |\langle B(u,u), Au\rangle| \\ &\leqslant \|\theta_0\|_2\|Au\|_2 + C\|u\cdot\nabla u\|_2\|Au\|_2 \\ &\leqslant \|\theta_0\|_2\|Au\|_2 + C\|u\|_2^{\frac{1}{2}}\|\nabla u\|_2\|Au\|_2^{\frac{3}{2}} \\ &\leqslant \frac{1}{2}\|Au\|_2^2 + C\left(\|\theta_0\|_2^2 + \|u\|_2^2\|A^{\frac{1}{2}}u\|_2^4\right), \end{aligned}$$

where we have used Gagliardo–Nirenberg inequality. Therefore, we obtain

$$\frac{d}{dt}\|A^{\frac{1}{2}}u\|_2^2 + \|Au\|_2^2 \preccurlyeq \|\theta_0\|_2^2 + \|u\|_2^2\|A^{\frac{1}{2}}u\|_2^4. \tag{2.14}$$

Step 1 Proof of regularity local in time.

By (2.7) and (2.14), we have

$$\frac{d}{dt}\|A^{\frac{1}{2}}u\|_2^2 + \|Au\|_2^2 \preccurlyeq \|\theta_0\|_2^2 + \left[\|u_0\|_2^2 + \left(\frac{\|\theta_0\|_2}{\lambda_1}\right)^2\right]\|A^{\frac{1}{2}}u\|_2^4.$$

Hence,

$$\frac{d}{dt}\|A^{\frac{1}{2}}u\|_2^2 \leqslant C_0 + C_1\|A^{\frac{1}{2}}u\|_2^4, \tag{2.15}$$

where

$$C_0 = C\|\theta_0\|_2^2, \quad C_1 = C\left(\|u_0\|_2^2 + \frac{\|\theta_0\|_2^2}{\lambda_1^2}\right).$$

Let $y(t) = 1 + \|A^{\frac{1}{2}}u(t)\|_2^2$ and $C_2 = \max\{C_0, C_1\}$. Then, (2.15) implies that

$$y'(t) \leqslant C_2 y^2.$$

Integrating this inequality with respect to t yields

$$y(t) \leqslant \frac{y(0)}{(1 - C_2 y(0) t)}.$$

Choose $t_0 > 0$ such that

$$t_0 \leqslant \frac{1}{2C_2 y(0)} = \frac{1}{2C_2\left(1 + \|A^{\frac{1}{2}}u_0\|_2^2\right)}.$$

Then, for $t \in [0, t_0]$,

$$1 + \|A^{\frac{1}{2}}u(t)\|_2^2 \leqslant \frac{1 + \|A^{\frac{1}{2}}u_0\|_2^2}{1 - C_2\left(1 + \|A^{\frac{1}{2}}u_0\|_2^2\right)t} \leqslant 2\left(1 + \|A^{\frac{1}{2}}u_0\|_2^2\right).$$

This proves the local estimate (2.10).

Step 2 Proof of regularity global in time.

Recall that (2.8) implies

$$\int_t^{t+r} \|A^{\frac{1}{2}}u(\tau)\|_2^2 \, d\tau \leqslant \|u_0\|_2^2 + \frac{\|\theta_0\|_2^2}{\lambda_1^2} + \frac{r}{\lambda_1}\|\theta_0\|_2^2. \tag{2.16}$$

Apply Lemma 2.1, (2.15) with

$$g(s) = C_1 y(s), \quad h(s) = C_0, \quad y(s) = \|A^{\frac{1}{2}}u(s)\|_2^2$$
$$a_1 = C_1 a_3, \quad a_2 = C_0 r, \quad a_3 = \|u_0\|_2^2 + \frac{\|\theta_0\|_2^2}{\lambda_1^2} + \frac{r}{\lambda_1}\|\theta_0\|_2^2.$$

Then, due to (2.16), Lemma 2.1 confirms that

$$\|A^{\frac{1}{2}}u(t+r)\|_2^2 \leqslant \left(\frac{a_3}{r} + C_0 r\right) e^{C_1 a_3}, \quad \forall\, t \geqslant 0. \tag{2.17}$$

Therefore, global in time regularity $u \in L^\infty([0,\infty); V)$ follows from (2.17) (with $r = t_0$) and (2.10).

Notice that, even though the combination of (2.10) and (2.17) gives an upper bound for $\|A^{\frac{1}{2}}u(t)\|_2$ which is uniform with respect to $t \in [0,\infty)$, the upper bound depends on u_0 and θ_0. In the following, we prove an improved upper bound for $\|A^{\frac{1}{2}}u(t)\|_2$ which is uniform with respect to $u_0 \in V$ for large t (depending on u_0). When $\theta_0 \in L^2$ is fixed, this upper bound implies an upper bound for the radius of an absorbing set for u in V. To do so, we use the sharper estimate that

$$y(t+r) \leqslant \left(\frac{1}{r}\int_t^{t+r} y(s)ds + \int_t^{t+r} h(s)ds\right) e^{\int_t^{t+r} g(s)ds}, \quad t \geqslant 0, r > 0, \tag{2.18}$$

under the same assumption of Lemma 2.1, except that a_1, a_2, a_3 and r are constants. Validity of (2.18) is obvious from the proof of Lemma 2.1. See e.g. [13]. Applying (2.18) to (2.14) (ignoring the term $\|Au\|_2^2$) with $r = t_0$ and

$$g(s) = C\|u(s)\|_2^2\|A^{\frac{1}{2}}u(s)\|_2^2, \quad h(s) = C\|\theta_0\|_2^2, \quad y(s) = \|A^{\frac{1}{2}}u(s)\|_2^2,$$

yields the global estimate (2.11), (2.12) with the help of (2.7) and (2.8).

Finally, (2.13) follows from integrating (2.14) with respect to t from t to $t + r$. □

Remark 2.2. We see from (2.11) and (2.12) that, for any fixed $\|\theta_0\|_2$, there is a *bounded* absorbing set for u in space V with the radius depending on $\|\theta_0\|_2$ but not u_0. It is also easy to see from (2.13) that there is a *bounded* absorbing set of $\int_t^{t+r} \|Au\|_2^2 d\tau$ in $\mathbb{R}_+$ for fixed $\|\theta_0\|_2$ and $r(> 0)$ which is also independent of u_0.

3. Global Regularity in $D(A) \times H^1$

The first main result of this article is achieved via an alternative proof of the global regularity of the solution $(u,\theta) \in D(A) \times H^1$. Proving this global regularity is the main result of the recent work [7]. The improvement of our new result over that of [7][1] is in the following sense: the *best* upper bounds for $\|Au(t)\|_2$ and $\|\nabla\theta(t)\|_2$ that can be obtained from the proof of [7] both increase *double exponentially* with respect to t as $t \to \infty$; while under the same conditions, our new proof provides a *uniform* upper bound for $\|Au(t)\|_2$ with respect to $t \in [0,\infty)$ and an upper bound for $\|\nabla\theta(t)\|_2$ which grows only *single exponentially* with respect to t^2. Indeed, we also obtain a bounded absorbing set for $\|Au(t)\|_2$ in $\mathbb{R}_+$, which depends only on θ_0 but not on u_0. To achieve the improvement, our method has to be quite different from the previous ones. We split the statement of our first main result into Theorems 3.1 and 3.2. We also split the proof of our first main result into the following three subsections.

[1] The approach of [8] applies only to the case of Ω with no boundary and yields less sharp estimate than that of [7].

3.1. Local Regularity in $D(A) \times H^1$

As the preparation for the analysis in the next two subsections, we prove the following local in time boundedness of $\|Au(t)\|_2$ and $\|\nabla\theta(t)\|_2$:

Proposition 3.1. *Suppose $(u_0, \theta_0) \in D(A) \times H^1$. Then, there exists a $t_1 \in (0, t_0]$, such that*

$$\|Au\|_2^2 + \|\nabla\theta\|_2^2 \leqslant 2\left(\|Au_0\|_2^2 + \|\nabla\theta_0\|_2^2\right), \quad \forall t \in [0, t_1], \tag{3.1}$$

where $t_0 > 0$ is given in (2.10).

Proof. Taking inner product of (2.4) with A^2u and using (2.1) and (2.5), we have

$$\begin{aligned}\frac{d}{dt}\|Au\|_2^2 + \|A^{\frac{3}{2}}u\|_2^2 &\preccurlyeq \|\theta\|_{H^1}^2 + \|u\|_2\|Au\|_2^3 \\ &\preccurlyeq \|\theta_0\|_2^2 + \|\nabla\theta\|_2^2 + \|u\|_2\|Au\|_2^3.\end{aligned} \tag{3.2}$$

Applying ∇ to (1.3) and taking the inner product of the derived equation with $\nabla\theta$, we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla\theta\|_2^2 = -\int_\Omega \sum_{i,j=1}^{2} \partial_j\theta\partial_j u_i\partial_i\theta \, dx \leqslant \|\nabla u\|_\infty\|\nabla\theta\|_2^2, \tag{3.3}$$

where we have used an integration by parts, no-slip boundary condition and the fact that $\nabla \cdot u = 0$. Therefore,

$$\frac{d}{dt}\|\nabla\theta\|_2^2 \leqslant \frac{1}{2}\|A^{\frac{3}{2}}u\|_2^2 + C\|\nabla\theta\|_2^4. \tag{3.4}$$

Adding (3.2) and (3.4) yields

$$\frac{d}{dt}\left(\|Au\|_2^2 + \|\nabla\theta\|_2^2\right) + \|A^{\frac{3}{2}}u\|_2^2 \leqslant C(1 + \|\theta_0\|_2^2) + C\left(\|Au\|_2^2 + \|\nabla\theta\|_2^2\right)^2.$$

Therefore, there exists a $t_1 \in (0, t_0]$ such that (3.1) is valid. This finishes the proof of local regularity of $(u, \theta) \in L^\infty(0, t_1; D(A) \times H^1)$. □

3.2. Global Uniform Estimates of $||Au||_2$ and $||u_t(t)||_2$

In this subsection, we prove the global in time uniform boundedness of $\|Au(t)\|_2$. Our approach is completely different from previous methods used in [7], which gives an upper bound of $\|Au(t)\|_2$ that is double exponential with respect to t as $t \to \infty$.

Theorem 3.1. *Suppose $(u_0, \theta_0) \in (D(A), H^1)$. Then,*

$$u \in L^\infty(0, +\infty; D(A)), \quad u_t \in L^\infty(0, +\infty; H) \cap L^2_{loc}(0, +\infty; V).$$

Moreover, for fixed $\|\theta_0\|_{H^1}$, there exists a bounded absorbing set in $\mathbb{R}_+$ for $\|Au(t)\|_2$, $\|u_t(t)\|_2$ and $\int_t^{t+1}\|u_t(\tau)\|_V^2 d\tau$, which is independent of u_0.

Proof. To prove uniform boundedness of $u \in L^\infty(0, \infty; D(A))$, it is enough to prove uniform boundedness of $u_t \in L^\infty(t_1, \infty; H)$ for sufficiently small $t_1 > 0$. However, we will prove in the following slightly stronger uniform boundedness of $u_t \in L^\infty(0, \infty; H)$, *without* imposing the condition that $u_t(0) \in H$.

Step 1 Local regularity of $u_t \in L^\infty(0, t_1; H)$.

By (2.4), we have

$$\|u_t\|_2^2 = \langle -Au - B(u, u) + \mathbb{P}\theta e_2, u_t\rangle \leqslant (\|Au\|_2 + \|u \cdot \nabla u\|_2 + \|\theta\|_2)\|u_t\|_2.$$

Therefore,

$$\|u_t\|_2 \preccurlyeq \|Au\|_2(1 + \|\nabla u\|_2) + \|\theta_0\|_2. \tag{3.5}$$

Hence, by (2.10), (3.1) and (3.5), we have

$$\|u_t\|_2 \preccurlyeq (\|Au_0\|_2 + \|\nabla\theta_0\|)(1 + \|\nabla u_0\|_2) + \|\theta_0\|_2, \quad \forall t \in (0, t_1],$$

which proves local regularity of $u_t \in L^\infty(0, t_1; H)$.

Step 2 Global regularity of $u_t \in L^\infty(0,\infty;H)\cap L^2(0,\infty;V)$.

First, square (3.5) and integrate with respect to t to get

$$\int_t^{t+r} \|u_t(\tau)\|_2^2\, d\tau \preccurlyeq r\|\theta_0\|_2^2 + \int_t^{t+r} \|Au(\tau)\|_2^2(1+\|\nabla u(\tau)\|_2^2)\, d\tau. \tag{3.6}$$

By (3.6), Theorems 2.1 and 2.2, we see that $\int_t^{t+r} \|u_t(\tau)\|_2^2\, d\tau$ is uniformly bounded. By Remarks 2.1 and 2.2, it is also easy to see that there is a bounded absorbing set in $\mathbb{R}_+$ for $\int_t^{t+r} \|u_t(\tau)\|_2^2\, d\tau$ when θ_0 and $r>0$ are fixed. Next, apply ∂_t to (1.1) to get

$$\partial_t u_t - \Delta u_t + u_t\cdot\nabla u + u\cdot\nabla u_t + \nabla p_t = \theta_t e_2. \tag{3.7}$$

Taking inner product of (3.7) with respect u_t yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_t\|_2^2 + \|\nabla u_t\|_2^2 &= -\langle u_t\cdot\nabla u, u_t\rangle - \langle u\cdot\nabla u_t, u_t\rangle - \langle\nabla p_t, u_t\rangle + \langle\theta_t e_2, u_t\rangle\\ &= -\langle u_t\cdot\nabla u, u_t\rangle - \langle u\cdot\nabla\theta e_2, u_t\rangle\\ &= -\langle u_t\cdot\nabla u, u_t\rangle + \langle\theta u, \nabla u_{2,t}\rangle\\ &\leqslant \|\nabla u\|_2\|u_t\|_4^2 + \|u\theta\|_2^2 + \frac{1}{4}\|\nabla u_t\|_2^2\\ &\leqslant C\|\nabla u\|_2^2\|u_t\|_2^2 + C\|Au\|_2^2\|\theta_0\|_2^2 + \frac{1}{2}\|\nabla u_t\|_2^2.\end{aligned}$$

Therefore,

$$\frac{d}{dt}\|u_t\|_2^2 + \|\nabla u_t\|_2^2 \preccurlyeq \|\nabla u\|_2^2\|u_t\|_2^2 + \|Au\|_2^2\|\theta_0\|_2^2 \tag{3.8}$$

Recall (3.6), Theorems 2.1 and 2.2 to conclude that $\int_t^{t+r} \|u_t(\tau)\|_2^2\, d\tau$, $\|\nabla u(t)\|_2$ and $\int_t^{t+r} \|Au(\tau)\|_2^2\, d\tau$ are all uniformly bounded. Moreover, there is a *bounded* absorbing set for all the three quantities. Therefore, we can apply Lemma 2.1 to (3.8) to get the existence of a bounded set in $\mathbb{R}_+$ which absorbs $\|u_t(t)\|_2$. Moreover, with the help of the local regularity in *Step 1*, we get the uniform boundedness of $\|u_t(t)\|_2$ for all $t\in(0,+\infty)$. Then, integrate (3.8) with respect to t and use the fact $u_t \in L^\infty(0,\infty;H)$ and Therorem 2.2 to get $u_t \in L^2_{loc}(0,\infty;V)$.

Step 3 Global regularity $u\in L^\infty(0,\infty;D(A))$.

By (2.4), we have

$$\begin{aligned}\|Au(t)\|_2^2 &= \langle\mathbb{P}\theta e_2 - u_t - B(u,u), Au\rangle\\ &\leqslant (\|\theta\|_2 + \|u_t\|_2)\|Au\|_2 + C\|u\|_2^{\frac{1}{2}}\|A^{\frac{1}{2}}u\|_2\|Au\|_2^{\frac{3}{2}}\\ &\leqslant C(\|\theta\|_2^2 + \|u_t\|_2^2 + \|u\|_2^2\|A^{\frac{1}{2}}u\|_2^4) + \frac{1}{2}\|Au(t)\|_2^2.\end{aligned}$$

Thus,

$$\|Au(t)\|_2^2 \preccurlyeq \|\theta_0\|_2^2 + \|u_t\|_2^2 + \|u\|_2^2\|A^{\frac{1}{2}}u\|_2^4.$$

Therefore, $\|Au(t)\|_2$ is also uniformly bounded for $t\in[0,\infty)$ and there exists a bounded absorbing set for $\|Au(t)\|_2$ in $\mathbb{R}_+$ as well. □

3.3. Global Single Exponential Estimate of $||\nabla\theta||_2$

Now, we prove global boundedness of $\|\nabla\theta\|_2$. Our approach is still quite different from previous methods used in [7]. Notice that, *even if assuming that the uniform boundedness of* $\|Au(t)\|_2$ *were already known*, following previous approache would still results in *double exponential growth* of the upper bound of $\|\nabla\theta(t)\|_2$ with respect to t.

Theorem 3.2. *Suppose* $(u_0, \theta_0) \in (D(A), H^1)$*. Then, for any* $T > 0$*, we have* $\theta \in L^\infty(0, T; H^1)$*. Moreover, there is a constant* $C > 0$ *depending only on* $t_1 (> 0)$*,* u_0 *and* θ_0*, such that*

$$\|\nabla\theta(t)\| \leqslant Ce^{Ct^2}, \quad \forall t \geqslant t_1 > 0. \tag{3.9}$$

Moreover, there exists a $C > 0$ *depending only* θ_0*, such that*

$$\lim_{t\to\infty} \|\nabla\theta(t)\| e^{-Ct^2} \leqslant C. \tag{3.10}$$

Proof. By the local regularity result Proposition 3.1, we only need to prove global boundedness of $\theta \in L^\infty_{\text{loc}}(t_1, \infty; H^1)$, or simply $\nabla\theta \in L^\infty_{\text{loc}}(t_1, \infty; L^2)$, and it is enough to prove (3.9).

Since $\|Au(t)\|_2$ is uniformly bounded for $t \in [0, +\infty)$ by Theorem 3.1, we can use Brezis–Gallouet inequality (see [1]) to obtain

$$\|\nabla u(t)\|_\infty \leqslant C\left[1 + \log^{\frac{1}{2}}(1 + \|A^{\frac{3}{2}}u(t)\|_2)\right], \tag{3.11}$$

where C is *independent* of t and u and *depends only on* $\|\theta_0\|_2$. Therefore, by (3.3), we get

$$\frac{d}{dt}\|\nabla\theta\|_2^2 \leqslant C\left[1 + \log^{\frac{1}{2}}(1 + \|A^{\frac{3}{2}}u(t)\|_2)\right]\|\nabla\theta\|_2^2,$$

that is

$$\|\nabla\theta(t)\|_2^2 \leqslant \|\nabla\theta_0\|_2^2 \exp\left\{C\int_0^t \left[1 + \log^{\frac{1}{2}}(1 + \|A^{\frac{3}{2}}u(\tau)\|_2)\right] d\tau\right\}.$$

Due to concavity of the function $f(x) := \log^{\frac{1}{2}}(1 + x)$ for $x \in [0, +\infty)$, Jensen's inequality implies

$$\frac{1}{t}\int_0^t \left[1 + \log^{\frac{1}{2}}(1 + \|A^{\frac{3}{2}}u(\tau)\|_2)\right]\, d\tau \leqslant 1 + \log^{\frac{1}{2}}\left(1 + \frac{1}{t}\int_0^t \|A^{\frac{3}{2}}u(\tau)\|_2\, d\tau\right).$$

Hence, the above two inequalities yield

$$\begin{aligned}\|\nabla\theta(t)\|_2^2 &\leqslant \|\nabla\theta_0\|_2^2 \exp\left\{Ct + Ct\log^{\frac{1}{2}}\left[1 + \frac{1}{t}\int_0^t \|A^{\frac{3}{2}}u(\tau)\|_2\, d\tau\right]\right\} \\ &\leqslant \|\nabla\theta_0\|_2^2\left[1 + \frac{1}{t}\int_0^t \|A^{\frac{3}{2}}u(\tau)\|_2\, d\tau\right] e^{C(t+t^2)}.\end{aligned}$$

Notice that in the last step of the above inequality, we have used Cauchy-Schwartz inequality.

Let $t \geqslant t_1 > 0$. Then, we obtain form the above estimate

$$\begin{aligned}\|\nabla\theta(t)\|_2^2 &\leqslant \|\nabla\theta_0\|_2^2 e^{C(t+t^2)} + \|\nabla\theta_0\|_2^2 \frac{e^{C(t+t^2)}}{t}\int_0^t \|A^{\frac{3}{2}}u(\tau)\|_2\, d\tau \\ &\leqslant \|\nabla\theta_0\|_2^2 e^{C(t+t^2)} + \|\nabla\theta_0\|_2^2 \frac{e^{C(t+t^2)}}{\sqrt{t}}\left(\int_0^t \|A^{\frac{3}{2}}u(\tau)\|_2^2\, d\tau\right)^{\frac{1}{2}} \\ &\leqslant \left[\|\nabla\theta_0\|_2^2 + \frac{\|\nabla\theta_0\|_2^4}{t_1}\right] e^{C(t+t^2)} + \int_0^t \|A^{\frac{3}{2}}u(\tau)\|_2^2\, d\tau.\end{aligned}$$

Integrating (3.2), we have

$$\int_0^t \|A^{\frac{3}{2}}u(\tau)\|_2^2\, d\tau \leqslant \|Au_0\|_2^2 + C\int_0^t \left[\|\theta_0\|_2^2 + \|u\|_2\|Au\|_2^3 + \|\nabla\theta\|_2^2\right] d\tau.$$

Therefore, the above two inequalities give us, for $t \geqslant t_1$,

$$\|\nabla\theta(t)\|_2^2 \leqslant \left[\|\nabla\theta_0\|_2^2 + \frac{\|\nabla\theta_0\|_2^4}{t_1}\right] e^{C(t+t^2)} + C(1+t) + C\int_0^t \|\nabla\theta(\tau)\|_2^2\, d\tau, \tag{3.12}$$

where we have used uniform boundedness of $u \in L^\infty(0, \infty, D(A))$ as given in Theorem 3.1. Denote

$$y(t) := \int_0^t \|\nabla\theta(\tau)\|_2^2\, d\tau.$$

Then, (3.12) becomes

$$y'(\tau) = \|\nabla\theta(\tau)\|_2^2 \leqslant C(\tau+1) + Ce^{C(\tau+\tau^2)} + Cy(\tau),$$

that is

$$y'(\tau) - Cy(\tau) \leqslant C(\tau+1) + Ce^{C(\tau+\tau^2)}.$$

Integrating the above inequality for τ from t_1 to $t > t_1$, we get

$$\begin{aligned}\int_0^t \|\nabla\theta(\tau)\|_2^2 \, d\tau &= y(t) \\ &\leqslant e^{C(t-t_1)}\left[y(t_1) + C(t+1)(t-t_1) + C\int_{t_1}^t e^{C(\tau+\tau^2)} \, d\tau\right].\end{aligned} \tag{3.13}$$

Noticing that

$$\int_{t_1}^t e^{C(\tau+\tau^2)} \, d\tau \leqslant \int_{t_1}^t e^{C(\tau+1)^2} \, d\tau \leqslant \frac{1}{Ct_1}\left(e^{C(t+1)^2} - e^{C(t_1+1)^2}\right) \leqslant Ce^{Ct^2},$$

and recalling local regularity result from Proposition 3.1:

$$\|\nabla\theta(t)\|_2^2 \leqslant 2(\|Au_0\|_2^2 + \|u_0\|_2^2), \quad \forall t \in [0, t_1],$$

we obtain from (3.13) that, for $t \geqslant t_1 > 0$,

$$\int_0^t \|\nabla\theta(\tau)\|_2^2 \, d\tau \leqslant Ce^{Ct^2}, \tag{3.14}$$

where the C's in the right-hand side of (3.14) may depend only on $\|\theta_0\|_2$, $\|u_0\|$, $\|Au\|_0$ and t_1. Now, (3.9) follows immediately from (3.12) and (3.14). It is easy to see from our proof that (3.10) is also valid. □

4. Global Regularity in $V \times H^1$

In this section, we prove global regularity of the solution $(u, \theta) \in V \times H^1$. As can be seen from Sect. 3, the Brezis–Gallouet inequality (3.11) is one of the key steps in obtaining estimate on $\|\nabla\theta(t)\|_2$. However, to use (3.11), one has to deal with $\|A^{\frac{3}{2}}u\|_2$. Therefore, it is natrual to introduce the estimate (3.2). This is the main reason for studying local and global regularity of $(u, \theta) \in D(A) \times H^1$. Notice that, it is possible to use a less demanding variant of Brezis–Gallouet inequality involving only $\|A^{1+\varepsilon}u\|_2$, with $\varepsilon > 0$, which however would be quite complicated. If we now study local and global regularity of $(u, \theta) \in V \times H^1$, the estimate (3.2) will not be available and more importantly the Brezis–Gallouet type inequality (3.11) (and its variants) can not help. Therefore, *essential new difficulties appear here*. It is quite remarkable that our following analysis using the idea of spectral decompostion can help to resolve the problem rather conveniently. Our next main result of this article is presented the following:

Theorem 4.1. *Suppose* $(u_0, \theta_0) \in V \times H^1$ *and* (u, θ) *is the* (unique) *weak solution to the Eqs.* (1.1)–(1.3) *with conditions* (1.4) *and* (1.5). *Then,*

$$u \in L^\infty(0, \infty; V), \quad \theta \in L^\infty_{loc}(0, \infty; H^1).$$

Moreover, for fixed θ_0, *there is a bounded absorbing set in* $\mathbb{R}_+$ *for* $\|\nabla u(t)\|_2$ *and* $\|Au(t)\|_2$ *as* $t \to \infty$, *which is independent of* u_0. *Furthermore, there is a constant* $C > 0$ *depending only on* $t_1(> 0)$, u_0 *and* θ_0, *such that*

$$\|\nabla\theta(t)\|_2 \leqslant Ce^{Ct^2}, \quad \forall t \geqslant t_1 > 0. \tag{4.1}$$

Proof. Step 1 We first prove local in time existence of the solution (u, θ) in $V \times H^1$, that is, there exists a $t_0 = t_0(u_0, \theta_0) > 0$, such that $(u, \theta) \in L^\infty(0, t_0; V \times H^1)$.

Since $(u, \theta) \in L^\infty(0, \infty; V \times L^2)$, we only need to prove the local regularity of $\theta \in L^\infty(0, t_0; H^1)$ for some $t_0 > 0$. Due to existence and uniqueness of the weak solution, we only need to carry out some *a priori* estimates formally in the following proof. These *a priori* estimates can be rigorously justified with a standard approximation argument.

By (3.3), we have

$$\|\nabla\theta(t)\|_2 \leqslant \|\nabla\theta_0\|_2 \exp\left\{\int_0^t \|\nabla u(\tau)\|_\infty \, d\tau\right\} \tag{4.2}$$

Apply P_m to (2.4) and then take inner product with $P_m u$. We get

$$\frac{1}{2}\frac{d}{dt}\|P_m u(t)\|_2^2 + \langle P_m A u, P_m u\rangle = \langle P_m\left[\mathbb{P}(\theta e_2) - B(u,u)\right], P_m u\rangle.$$

Noticing that

$$\langle P_m A u, P_m u\rangle = \sum_{2^{m-1} \leqslant \lambda_j < 2^m} \lambda_j |(u, w_j)|^2 \geqslant 2^{m-1}\|P_m u\|_2^2, \quad \forall m \geqslant 1,$$

we get

$$\frac{d}{dt}\|P_m u(t)\|_2 + 2^{m-1}\|P_m u\|_2 \leqslant \|P_m \mathbb{P}(\theta e_2)\|_2 + \|P_m B(u,u)\|_2.$$

Thus,

$$\begin{aligned} &2\|P_m u(t)\|_2 + \int_0^t 2^m \|P_m u(\tau)\|_2 \, d\tau \\ &\quad \leqslant 2\|P_m u_0\|_2 + 2\int_0^t \left(\|P_m \mathbb{P}(\theta(\tau) e_2)\|_2 + \|P_m B(u(\tau), u(\tau))\|_2\right) d\tau. \end{aligned} \tag{4.3}$$

Multiplying (4.3) by $2^{\frac{m}{2}}$ and using (2.3) yields

$$\begin{aligned} \int_0^t 2^{m+\frac{m}{2}} \|P_m u(\tau)\|_2 d\tau \preccurlyeq \|A^{\frac{1}{2}} P_m u_0\|_2 &+ \int_0^t \|A^{\frac{1}{2}} P_m \mathbb{P}(\theta(\tau) e_2)\|_2 \, d\tau \\ &+ \int_0^t \|A^{\frac{1}{2}} P_m B(u(\tau), u(\tau))\|_2 \, d\tau. \end{aligned} \tag{4.4}$$

Squaring both sides of (4.4) and summing up with respect to m yields

$$\begin{aligned} &\sum_{m=1}^\infty \left(\int_0^t 2^{m+\frac{m}{2}} \|P_m u\|_2 \, d\tau\right)^2 \\ &\quad \preccurlyeq \sum_{m=1}^\infty \|A^{\frac{1}{2}} P_m u_0\|_2^2 + \sum_{m=1}^\infty \left(\int_0^t \|A^{\frac{1}{2}} P_m \mathbb{P}(\theta(\tau) e_2)\|_2 \, d\tau\right)^2 \\ &\qquad + \sum_{m=1}^\infty \left(\int_0^t \|A^{\frac{1}{2}} P_m B(u(\tau), u(\tau))\|_2 \, d\tau\right)^2 \\ &\quad = \|A^{\frac{1}{2}} u_0\|_2^2 + \sum_{m=1}^\infty \left(\int_0^t \|A^{\frac{1}{2}} P_m \mathbb{P}(\theta(\tau) e_2)\|_2 \, d\tau\right)^2 \\ &\qquad + \sum_{m=1}^\infty \left(\int_0^t \|A^{\frac{1}{2}} P_m B(u(\tau), u(\tau))\|_2 \, d\tau\right)^2. \end{aligned}$$

Then, take square root of the above inequality to get

$$\begin{aligned}
&\left[\sum_{m=1}^{\infty}\left(\int_0^t 2^{m+\frac{m}{2}}\|P_m u\|_2\,d\tau\right)^2\right]^{\frac12}\\
&\preccurlyeq \|A^{\frac12}u_0\|_2+\left[\sum_{m=1}^{\infty}\left(\int_0^t \|A^{\frac12}P_m\mathbb{P}(\theta(\tau)e_2)\|_2\,d\tau\right)^2\right]^{\frac12}\\
&\quad+\left[\sum_{m=1}^{\infty}\left(\int_0^t \|A^{\frac12}P_m B(u(\tau),u(\tau))\|_2\,d\tau\right)^2\right]^{\frac12}\\
&\leqslant \|A^{\frac12}u_0\|_2+\int_0^t\left(\sum_{m=1}^{\infty}\|A^{\frac12}P_m\mathbb{P}(\theta(\tau)e_2)\|_2^2\right)^{\frac12}d\tau\\
&\quad+\int_0^t\left(\sum_{m=1}^{\infty}\|A^{\frac12}P_m B(u(\tau),u(\tau))\|_2^2\right)^{\frac12}d\tau\\
&=\|A^{\frac12}u_0\|_2+\int_0^t\|A^{\frac12}\mathbb{P}(\theta(\tau)e_2)\|_2\,d\tau+\int_0^t\|A^{\frac12}B(u(\tau),u(\tau))\|_2\,d\tau,
\end{aligned}\tag{4.5}$$

where Minkowski's inequality is used twice for the second inequality of (4.5).

By Agmon's inequality, (2.2) and (2.3), it is easy to see that

$$\begin{aligned}
\|\nabla P_m u(\tau)\|_\infty &\preccurlyeq \|\nabla P_m u(\tau)\|_2^{\frac12}\|A\nabla P_m u(\tau)\|_2^{\frac12}\\
&\preccurlyeq 2^{\frac{m}{2}}\|\nabla P_m u(\tau)\|_2=2^{\frac{m}{2}}\|A^{\frac12}P_m u(\tau)\|_2\preccurlyeq 2^m\|P_m u(\tau)\|_2.
\end{aligned}\tag{4.6}$$

Therefore, by (4.6) and (4.5), we obtain

$$\begin{aligned}
\int_0^t\|\nabla u(\tau)\|_\infty\,d\tau &\leqslant \int_0^t\sum_{m=1}^{\infty}\|\nabla P_m u(\tau)\|_\infty\,d\tau\\
&=\sum_{m=1}^{\infty}2^{-\frac{m}{2}}\int_0^t 2^{m+\frac{m}{2}}\|P_m u(\tau)\|_2\,d\tau\\
&\leqslant\left[\sum_{m=1}^{\infty}\left(\int_0^t 2^{m+\frac{m}{2}}\|P_m u\|_2\,d\tau\right)^2\right]^{\frac12}\\
&\preccurlyeq\|\nabla u_0\|_2+\int_0^t\|\nabla\mathbb{P}(\theta(\tau)e_2)\|_2\,d\tau+\int_0^t\|\nabla B(u(\tau),u(\tau))\|_2\,d\tau.
\end{aligned}\tag{4.7}$$

The last term of (4.7) can be estimated using (2.5) as

$$\begin{aligned}
\int_0^t\|\nabla B(u(\tau),u(\tau))\|_2\,d\tau &\preccurlyeq \int_0^t\|u(\tau)\|_2^{\frac12}\|Au(\tau)\|_2^{\frac32}\,d\tau\\
&\preccurlyeq\left(\int_0^t\|u(\tau)\|_2^2\,d\tau\right)^{\frac14}\left(\int_0^t\|Au(\tau)\|_2^2\,d\tau\right)^{\frac34}.
\end{aligned}\tag{4.8}$$

Combining (4.7) and (4.8), we get

$$\begin{aligned}
\int_0^t\|\nabla u(\tau)\|_\infty\,d\tau &\preccurlyeq\|\nabla u_0\|_2+t+\int_0^t\|\nabla\theta(\tau)\|_2\,d\tau\\
&\preccurlyeq\|\nabla u_0\|_2+\int_0^t(1+\|\nabla\theta(\tau)\|_2)\,d\tau,
\end{aligned}\tag{4.9}$$

where we have used Theorems 2.1, 2.2 and (2.15). By (4.2) and (4.9), we obtain

$$\|\nabla\theta(t)\|_2 \preccurlyeq \exp\left\{c_0\int_0^t (1+\|\nabla\theta(t)\|_2)\,d\tau\right\}, \tag{4.10}$$

where $c_0>0$ is a constant depending on initial data. Denote:

$$y(t) := c_0\int_0^t (1+\|\nabla\theta(t)\|_2)\,d\tau.$$

By (4.10), we have

$$y'(t) \leqslant C_1 e^{y(t)},$$

where $C_1>0$ is a constant depending on u_0 and θ_0. Thus, there exists a $t_0>0$, such that

$$c_0\int_0^t (1+\|\nabla\theta(t)\|_2)\,d\tau = y(t) \leqslant \ln\left(\frac{1}{1-C_1 t}\right) < +\infty, \quad \text{for } t\in[0,t_0].$$

Therefore, by (4.10),

$$\|\nabla\theta(t)\|_2 \leqslant \frac{C_2}{1-C_1 t} < +\infty, \quad t\in[0,t_0],$$

where $C_2>0$ is also a constant depending on u_0 and θ_0. This finishes the proof of $\theta\in L^\infty(0,t_0;H^1)$.
Step 2 We prove global in time existence of θ in H^1.

Suppose that $(u_0,\theta_0)\in V\times H^1$. Then, by *Step 1*, there is $t_0>0$, such that $(u,\theta)\in L^\infty(0,t_0;V\times H^1)$. Therefore, by (2.14),

$$\begin{aligned}\|A^{\frac12}u(t_0)\|_2^2 + \int_0^{t_0}\|Au(s)\|_2^2 ds &\leqslant \|A^{\frac12}u_0\|_2^2 + Ct_0\|\theta_0\|_2^2\\ &+ C\int_0^{t_0}\|u(s)\|_2^2\|A^{\frac12}u(s)\|_2^4 ds < \infty.\end{aligned}$$

Therefore, there exists a $t_*\in(0,t_0)$, such that $u(t_*)\in D(A)$. Since $(u(t_*),\theta(t_*))\in D(A)\times H^1$, by the global existence results of Theorems 3.1 and 3.2, we have

$$u\in L^\infty(t_*,+\infty;D(A)), \quad \theta\in L^\infty_{\text{loc}}(t_*,\infty;H^1),$$

thus proving the global regularity $(u,\theta)\in L^\infty(0,T;V\times H^1)$ for any $T>0$. The rest part of the theorem follows from Theorems 2.2, 3.1 and 3.2. □

Remark 4.1. Theorem 4.1 is obviously also valid for $\Omega=\mathbb{T}^2$. By Littlewood-Paley frequence decomposition, the method of our proof of global regularity in $V\times H^1$ can be extended immediately to the case of $\Omega=\mathbb{R}^2$.

5. Global Regularity in $D(A)\times H^2$

The following theorem is our last main result for global regularity, which we include here for completeness. Since we will not use Brezis–Gallouet type inequalities in our proof of Theorem 5.1, the proof is a lot simpler than otherwise.

Theorem 5.1. *Suppose* $(u_0,\theta_0)\in D(A)\times H^2$ *and* (u,θ) *is the* (unique) *weak solution to the Eqs.* (1.1)–(1.3) *with conditions* (1.4) *and* (1.5)*. Then,*

$$u\in L^\infty(0,\infty;D(A)), \quad \textit{and} \quad \theta\in L^\infty([0,T];H^2), \quad \forall T>0.$$

Moreover, for fixed θ_0*, there is a bounded absorbing set in* $\mathbb{R}_+$ *for* $\|\nabla u(t)\|_2$ *and* $\|Au(t)\|_2$ *as* $t\to\infty$*, which is independent of* u_0*.*

Proof. It is easy to see that we just need to prove that for any $T > 0$

$$\theta \in L^\infty(0, T; H^2).$$

All the other statements of the theorem are already confirmed by the previous theorems.

Applying ∂_x^2 to (1.3) and then taking inner product of the derived equation with $\partial_x^2\theta$, we obtain

$$\begin{aligned}\frac{d}{dt}\|\partial_x^2\theta\|_2^2 &= -2\langle(\partial_x^2 u)\cdot\nabla\theta + 2(\partial_x u)\cdot\nabla(\partial_x\theta), \partial_x^2\theta\rangle \\ &\preccurlyeq \|\nabla\theta\|_4\|Au\|_4\|\nabla^2\theta\| + \|\nabla u\|_\infty\|\nabla^2\theta\|_2^2 \\ &\preccurlyeq \|\nabla\theta\|_2^{\frac12}\|\nabla^2\theta\|_2^{\frac32}\|Au\|_2^{\frac12}\|A^{\frac32}u\|_2^{\frac12} + \|\nabla u\|_\infty\|\nabla^2\theta\|_2^2 \\ &\preccurlyeq \|\nabla\theta\|_2^2\|Au\|_2^2\|A^{\frac32}u\|_2^2 + (1+\|\nabla u\|_\infty)\|\nabla^2\theta\|_2^2.\end{aligned}$$

Similar estimates can be obtained for $\partial_{xy}^2\theta$ and $\partial_y^2\theta$. Therefore, we have

$$\frac{d}{dt}\|\nabla^2\theta\|_2^2 \preccurlyeq \|\nabla\theta\|_2^2\|Au\|_2^2\|A^{\frac32}u\|_2^2 + (1+\|\nabla u\|_\infty)\|\nabla^2\theta\|_2^2.$$

Applying Gronwall lemma to the above inequality, we obtain

$$\begin{aligned}\|\nabla^2\theta(t)\|_2^2 &\preccurlyeq \|\nabla^2\theta_0\|_2^2 e^{\int_0^t(1+\|\nabla u(\tau)\|_\infty)\ d\tau} \\ &\quad + \int_0^t \|\nabla\theta(\tau)\|_2^2\|Au(\tau)\|_2^2\|A^{\frac32}u(\tau)\|_2^2 e^{\int_\tau^t(1+\|\nabla u(s)\|_\infty)\ ds}\ d\tau.\end{aligned} \tag{5.1}$$

By (4.7) and (4.8), we have

$$\begin{aligned}\int_0^t \|\nabla u(\tau)\|_\infty\ d\tau &\preccurlyeq \|\nabla u_0\|_2 + \int_0^t \|\nabla\theta(\tau)\|_2\ d\tau \\ &\quad + \left(\int_0^t \|u(\tau)\|_2^2\ d\tau\right)^{\frac14}\left(\int_0^t \|Au(\tau)\|_2^2\ d\tau\right)^{\frac34}\end{aligned} \tag{5.2}$$

Case 1. Suppose that $t \in [0, t_1]$. Then, by (5.2)

$$\int_0^t \|\nabla u(\tau)\|_\infty\ d\tau \leqslant \int_0^{t_1} \|\nabla u(\tau)\|_\infty\ d\tau \leqslant C < \infty,$$

where we have applied Theorems 2.1, 2.2 (or Theorem 3.1) and Theorem 3.2 (or Theorem 4.1) to (5.2) with t replaced by t_0. Then, apply these theorems and the above inequality to (5.1) to get

$$\|\nabla^2\theta(t)\|_2^2 \preccurlyeq \|\nabla^2\theta_0\|_2^2 + \int_0^{t_1} \|A^{\frac32}u(\tau)\|_2^2\ d\tau, \quad \forall t \in [0, t_1].$$

Furthermore, by (3.2), we get that for $t \in [0, t_1]$

$$\|\nabla^2\theta(t)\|_2^2 \preccurlyeq \|\nabla^2\theta_0\|_2^2 + \|\theta_0\|_2^2 t + \int_0^t (\|\nabla\theta(\tau)\|_2^2 + \|u(\tau)\|_2^2\|Au(\tau)\|_2^2)\ d\tau,$$

from which we derive regularity of local boundedness:

$$\theta \in L^\infty(0, t_1, H^2).$$

Case 2. Suppose $t \in [t_1, T]$. This can be treated the same way as *Case 1* and we can obtain a upper bound on $\|\nabla^2\theta(t)\|_2^2$ for $t \in [t_1, T]$. □

Remark 5.1. Since (4.7) and (4.8) are all still valid for $\Omega = \mathbb{R}^2$. Global regularity of $(u, \theta) \in D(A) \times H^2$ is also still valid when $\Omega = \mathbb{R}^2$.

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