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An Eulerian–Lagrangian Form for the Euler Equations in Sobolev Spaces

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Abstract. In 2000 Constantin showed that the incompressible Euler equations can be written in an "Eulerian–Lagrangian" form which involves the back-to-labels map (the inverse of the trajectory map for each fixed time). In the same paper a local existence result is proved in certain Hölder spaces $C^{1,\mu}$. We review the Eulerian–Lagrangian formulation of the equations and prove that given initial data in H^s for $n \geq 2$ and $s > \frac{n}{2} + 1$, a unique local-in-time solution exists on the n-torus that is continuous into H^s and C^1 into H^{s-1}. These solutions automatically have C^1 trajectories. The proof here is direct and does not appeal to results already known about the classical formulation. Moreover, these solutions are regular enough that the classical and Eulerian–Lagrangian formulations are equivalent, therefore what we present amounts to an alternative approach to some of the standard theory.

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1. Introduction

We study a reformulation (following Constantin [2]) of the incompressible Euler equations on a domain $\mathbb{T}^n := \mathbb{R}^n/2\pi\mathbb{Z}^n$ in the absence of external forcing. The Euler equations model the flow of an incompressible inviscid fluid and are (classically) formulated in terms of a divergence-free vector field u (i.e. $\nabla \cdot u = 0$) as follows:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = 0 \tag{1}$$

where p is a scalar potential representing internal pressure (as opposed to physical pressure at a boundary). The divergence-free condition reflects the incompressibility constraint.

In two and particularly in three dimensions, these equations continue to be of great interest; some recent surveys include [5,8,18]. As an illustration of the challenge posed by these equations we note that unlike the Navier–Stokes equations where global weak solutions have been known to exist since 1934 due to Leray [12], existence of global weak solutions of the Euler equations (on periodic domains) was not proved until 2011 by Wiedemann [17], following the work of DeLellis and Székelyhidi [7]. On the spatial domain $\mathbb{R}^3$, more regular local solutions ($u \in C^0([0,T]; H^s) \cap C^1([0,T]; H^{s-1})$ with $s > 5/2$) have been known to exist since the 1970s due to Kato et al. see for example [10,11].

In the study of the Navier–Stokes equations, results such as those found in [15] motivate us to approach the classical equations of fluid mechanics from a more Lagrangian viewpoint. In that paper, Robinson and Sadowski show that if u is a suitable weak solution of the 3D Navier–Stokes equations in the sense of Caffarelli et al. [1] and in addition $u \in L^{6/5}(0,T;L^\infty)$, then almost every particle trajectory is unique and C^1 in time. The arguments there are based on the fact that almost all trajectories avoid the set of points (x,t) where singularities could develop using the fact that the set of such points has box-counting dimension at most 5/3.

Constantin has studied a form for the Euler equations that involves both the classical velocity field and the so called back-to-labels map A which is defined to be the inverse of the trajectory map X at each time t. More precisely, for an evolving vector field u defined on $\mathbb{T}^n \times [0,T]$, the trajectory map solves

$$\begin{cases} \frac{\mathrm{d}X}{\mathrm{d}t}(y,t) = u(X(y,t),t), \\ X(y,0) = y \end{cases} \tag{2}$$

for each $y \in \mathbb{T}^n$. If u is divergence-free and sufficiently regular then X is well defined and $X(\cdot,t)$ is bijective for each t. In this case we can define the back-to-labels map A by setting

$$A(\cdot,t) \coloneqq X^{-1}(\cdot,t), \tag{3}$$

where we consider X as a map $X(\cdot,t) : \mathbb{T}^n \to \mathbb{T}^n$ for each $t \in [0,T]$. For the *Eulerian–Lagrangian* form, as we shall continue to call it, Constantin [2] proved local existence and uniqueness results in certain Hölder spaces on $\mathbb{R}^3$ for solutions that are periodic, or satisfy suitable decay conditions.

As Yudovich [18] has noted, a similar combination of Eulerian and Lagrangian approaches was used to investigate the Euler equations in Hölder spaces, by Günther and Lichtenstein independently, as early as the 1920s [9,13].

First we will review the Eulerian–Lagrangian formulation and discuss how it is formally equivalent to the usual Euler equations. We then turn to the main topic of this paper which is the proof of an existence and uniqueness result for the Eulerian–Lagrangian formulation in $C^0([0,T];H^s(\mathbb{T}^n))$ with $s > \frac{n}{2}+1$ in dimension $n \geq 2$. The proof is self contained, in the sense that it neither appeals to results about the classical Euler equations, nor to the problem in Hölder spaces.

2. The Eulerian–Lagrangian form of the equations

The Eulerian–Lagrangian form of the Euler equations comprises the following system:

$$\partial_t A + (u \cdot \nabla)A = 0, \tag{4}$$

$$u = \mathbb{P}((\nabla A)^* v), \tag{5}$$

$$\partial_t v + (u \cdot \nabla)v = 0. \tag{6}$$

Given an initial divergence-free velocity u_0 for the classical equations, we choose initial conditions for the above system as follows:

$$A(x,0) = x, \tag{7}$$

$$u(x,0) = v(x,0) = u_0(x). \tag{8}$$

We use the notation $\mathbb{P}$ for the Leray projector onto the space of divergence-free functions. For a matrix M, M^* denotes the transposed matrix. The vector field v is called the *virtual velocity* and represents the initial velocity transported by the flow.

It will often be convenient to treat A as a perturbation of the identity map on $\mathbb{T}^n$. In this case we use the notation $\eta(x,t) \coloneqq A(x,t) - x$ and replace (4) and (7) with the equations

$$\partial_t \eta + (u \cdot \nabla)\eta + u = 0, \quad \eta(x,0) = 0 \tag{9}$$

respectively. We do this because the identity map (hence A) does not have sufficient Sobolev regularity when considered as a function on the torus with values in $\mathbb{R}^n$ (i.e. without accounting for the topology of the target torus).

The following proposition encapsulates the derivation of (5) (sometimes called the Weber formula) which can be found in [2].

Proposition 1. *Let $n \geq 2$, consider $u \in C^1((0,T) \times \mathbb{T}^n)$, with $u(0) \in C^1(\mathbb{T}^n)$. If u is divergence-free and satisfies (1) for some p, with spatially periodic boundary conditions then $A \in C^1((0,T) \times \mathbb{T}^n; \mathbb{T}^n)$ and u satisfies (5) with $v(x,t) = u_0(A(x,t))$.*

Proof. From the regularity assumptions on u and periodicity of the domain we deduce that the trajectories $X(y,\cdot) \in C^2(0,T)$ and $\nabla X(y,\cdot) \in C^1(0,T)$ for all $y \in \mathbb{T}^n$, we also have $X, \frac{\partial X}{\partial t} \in C^1((0,T)\times\mathbb{T}^n)$. It follows from the divergence-free condition that $\det \nabla X \equiv 1$, so X is volume preserving and locally injective, hence bijective, given that $\mathbb{T}^n$ has finite volume. By the inverse function theorem we see that A exists and is an element of $C^1((0,T)\times\mathbb{T}^n)$. We now have enough regularity to make the following calculations rigorous.

From (1) and (2) we obtain

$$\frac{\partial^2 X}{\partial t^2}(y,t) = -\nabla p(X(y,t),t),$$

which is of course just a Lagrangian interpretation of the Euler equations. Setting $\tilde{p}(y,t) = p(X(y,t),t)$ this becomes

$$\frac{\partial^2 X}{\partial t^2} = -((\nabla X)^*)^{-1}\nabla \tilde{p}(y,t).$$

Multiplying through by $(\nabla X)^*$ and changing the order of differentiation yields

$$\frac{\partial}{\partial t}\left[\frac{\partial X_j}{\partial t}\frac{\partial X_j}{\partial y_i}\right] = \frac{\partial}{\partial y_i}\left[-\tilde{p} + \frac{1}{2}\left|\frac{\partial X}{\partial t}\right|^2\right] \tag{10}$$

for $i = 1,\ldots,n$, where there is an implicit sum over $j = 1,\ldots,n$ and X_j, y_i denote the components in $\mathbb{R}^n$ of X, y respectively. Integrating (10) in time, multiplying the corresponding vector equation by $(\nabla A)^*$ and evaluating at $A(x,t)$ gives

$$u(x,t) = \frac{\partial X}{\partial t}(A(x,t),t) = (\nabla A)^* u_0(A(x,t)) - \nabla q \tag{11}$$

where

$$q(x,t) = \int_0^t \tilde{p}(A(x,t),s) - \frac{1}{2}\left|\frac{\partial X}{\partial t}(A(x,t),s)\right|^2 \mathrm{d}s.$$

As gradients lie in the kernel of the Leray projector, applying $\mathbb{P}$ to (11) shows that u satisfies (5) as required. Note that $v(x,t) = u_0(A(x,t))$ satisfies (6), hence solutions to the Euler equations indeed solve the Eulerian–Lagrangian form. □

The converse is a little more technical.

Proposition 2. *Let $s > \frac{n}{2}+1$ and u, v, $\eta \in C^0([0,T];H^s)\cap C^1([0,T];H^{s-1})$ satisfy (5), (6), (8) and (9). Then for some $p \in C^0([0,T];H^s)$ u solves (1).*

Proof. Since $H^{s-1}(\mathbb{T}^n) \hookrightarrow L^\infty(\mathbb{T}^n)$ is an algebra, we have that if $f,g \in H^{s-1}$ (scalar valued) then

$$\partial_{x_i}(fg) = (\partial_{x_i} f)g + f(\partial_{x_i} g)$$

as an equlity of L^2 functions, for $i = 1,2,\ldots,n$. Therefore, denoting the material derivative by $\mathrm{D}_t := \partial_t + (u\cdot\nabla)$, for $f,g \in C^0([0,T];H^{s-1})\cap C^1([0,T];H^{s-2})$ we have

$$\mathrm{D}_t(fg) = (\mathrm{D}_t f)g + f(\mathrm{D}_t g). \tag{12}$$

Moreover, if $f \in H^s$,

$$(u\cdot\nabla)\nabla f = \nabla((u\cdot\nabla)f) - (\nabla u)^*\nabla f.$$

Hence the classical commutation relation

$$\mathrm{D}_t\nabla f = \nabla \mathrm{D}_t f - (\nabla u)^*\nabla f \tag{13}$$

holds as an equality in L^2, when $f \in C^0([0,T];H^s)\cap C^1([0,T];H^{s-1})$.

Since u satisfies (5), we may write

$$u(x,t) = v + (\nabla\eta)^* v - \nabla q \tag{14}$$

for some real-valued q. Then by (12) and (13) the following calculations are justified:

$$\begin{aligned} \mathrm{D}_t u &= \mathrm{D}_t v + (\mathrm{D}_t \nabla\eta)^* v + (\nabla\eta)^* \mathrm{D}_t v - \mathrm{D}_t \nabla q \\ &= (\nabla \mathrm{D}_t \eta)^* v - (\nabla u)^* (\nabla\eta)^* v - \nabla \mathrm{D}_t q + (\nabla u)^* \nabla q \\ &= -(\nabla u)^* [v + (\nabla\eta)^* v - \nabla q] - \nabla \mathrm{D}_t q \\ &= -(\nabla u)^* u - \nabla \mathrm{D}_t q \\ &= -\nabla p \end{aligned} \tag{15}$$

where $p = \frac{1}{2}|u|^2 + \mathrm{D}_t q$. □

3. An Existence and Uniqueness Theorem

For $r \geq 0$, we will use the notation H^r variously for scalar or vector valued functions in $H^r(\mathbb{T}^n)$ (componentwise), where this does not cause ambiguity. We will often consider functions in spaces of the form $C^0([0,T];(H^s(\mathbb{T}^n))^n)$. To simplify notation we define $\Sigma_s(T)$ (usually denoted Σ_s) for $T \geq 0$ and $s \geq 0$ by

$$\Sigma_s(T) := C^0([0,T];(H^s(\mathbb{T}^n))^n).$$

We consider the natural norm on Σ_s:

$$\|u\|_{\Sigma_s} = \sup_{t\in[0,T]} \|u(t)\|_{H^s}.$$

The aim of the rest of this paper is to prove the following theorem.

Theorem 1. *If $n \geq 2$, $s > \frac{n}{2} + 1$ and $u_0 \in H^s$ is divergence free then there exists $T > 0$, such that the system (4–6) with initial conditions (7) and (8) has a unique solution A, u, v such that $\eta, u, v \in \Sigma_s(T) \cap C^1([0,T]; H^{s-1})$ where $\eta(x,t) = A(x,t) - x$. Moreover $A \in C^1([0,T] \times \mathbb{T}^n)$ as a map into the torus.*

We will prove this by constructing a contracting iteration scheme using the Eqs. (5), (6) and (9). More precisely, given $u \in \Sigma_s(T)$ we find $v, \eta \in \Sigma_s \cap C^1([0,T] \times \mathbb{T}^n)$, solutions of

$$\partial_t \eta + (u \cdot \nabla)\eta = -u,\ \eta(0,x) = 0$$

and

$$\partial_t v + (u \cdot \nabla) v = 0,\ v(0,x) = u_0(x).$$

We then construct the next iterate of u, using

$$u' = \mathbb{P}[(\nabla A)^* v]$$

and show that $u \mapsto u'$ is a contraction on a certain subset of Σ_s.

In the case of Hölder spaces, Constantin constructed an iteration scheme that was instead a contraction with respect to A. This involves controlling differences between candidate virtual velocities (v_1 and v_2, say) in terms of the difference between the respective back-to-labels maps (A_1 and A_2). This can be achieved, using the fact that $v_i = u_0(A_i)$ is a solution to (6). In the Hölder setting this is a natural way to proceed, however, relying on this a posteriori knowledge about the solution introduces an extra technicality when we work in Sobolev spaces. For this reason we will proceed as described above, relying only on a priori estimates. Following the proof, we shall see how the argument differs if the contraction is with respect to A, in particular we get an alternative proof under the additional assumption that $s \in \mathbb{Z}$.

We begin the proof of Theorem 1 by stating two inequalities concerning the advection term $(u \cdot \nabla)v$, using the notation $B(u,v) := (u \cdot \nabla)v$. Both of these results can be proved following the steps in [6,16] (the only difference being that B here does not include a Leray projection).

Lemma 1. *For $s > \frac{n}{2}$ there exists $C_1 > 0$ such that if $u \in H^s$ and $v \in H^{s+1}$ then $B(u,v) \in H^s$ and*

$$\|B(u,v)\|_{H^s} \leq C_1 \|u\|_{H^s} \|v\|_{H^{s+1}}. \tag{16}$$

This is really just the fact that H^s is a Banach algebra. For the second lemma the assumption that u is divergence-free allows us to "save a derivative" by means of the identities

$$(B(u,(-\Delta)^{r/2}v),(-\Delta)^{r/2}v)_{L^2}=0$$

for $r\in[0,s]$.

Lemma 2. *If $s>\frac{n}{2}+1$ there exists $C_2>0$ such that for $u\in H^s$, $v\in H^{s+1}$ with u divergence-free we have*

$$|(B(u,v),v)_{H^s}|\le C_2\|u\|_{H^s}\|v\|^2_{H^s}. \tag{17}$$

We use the following shorthand for closed balls in Σ_s:

$$B_M=\overline{B_{\|\cdot\|_{\Sigma_s}}(0,M)},$$

i.e. B_M is the closed unit ball centred at the origin of radius $M>0$ with respect to the norm $\|\cdot\|_{\Sigma_s}$. Where ambiguity could arise we write $B_M(T)$ for the closed ball in $\Sigma_s(T)$.

Lemma 3. *If $s>\frac{n}{2}+1$ and $\eta,v\in\Sigma_s(T)$ then $\mathbb{P}[(\nabla\eta)^*v]\in\Sigma_s$ and there exists a constant $C_3>0$ (independent of η, v, t and T) such that for fixed t,*

$$\|\mathbb{P}[(\nabla\eta)^*v]\|_{H^r}\le C_3\|\eta\|_{H^s}\|v\|_{H^r}, \tag{18}$$

where $r=s$ or $r=s-1$. Furthermore, there exists $C_3'>0$ such that for any $M>0$ and $T>0$, the following bounds hold uniformly with respect to $t\in[0,T]$ for any $\eta_1,\eta_2,v_1,v_2\in B_M(T)$:

$$\|\mathbb{P}[(\nabla\eta_1)^*v_1-(\nabla\eta_2)^*v_2]\|_X\le C_3'M(\|\eta_1-\eta_2\|_X+\|v_1-v_2\|_X). \tag{19}$$

where X is $L^2(\mathbb{T}^n)$ or H^{s-1}.

Proof. For continuity into H^{s-1} we use the fact that H^{s-1} is a Banach algebra. More precisely, we see that

$$\begin{aligned}\|\mathbb{P}[(\nabla\eta_1)^*v_1-(\nabla\eta_2)^*v_2]\|_{H^{s-1}}\le{}& C\|\eta_1-\eta_2\|_{H^s}\|v_1+v_2\|_{H^{s-1}}\\ &+C\|\nabla\eta_1+\nabla\eta_2\|_{H^{s-1}}\|v_1-v_2\|_{H^{s-1}},\end{aligned} \tag{20}$$

where $C>0$ is independent of the η_i and v_i. The key step in the proof of (18) when $r=s$ is that if $\eta,v\in C^2$ then for some $q\in H^s$,

$$\begin{aligned}\partial_{x_i}\mathbb{P}[(\nabla\eta)^*v]&=\partial_{x_i}(\partial_{x_j}\eta_k v_k)-\partial_{x_i}\partial_{x_j}q\\ &=\partial_{x_j}(\partial_{x_i}\eta_k v_k)-\partial_{x_i}\eta_k\partial_{x_j}v_k+\partial_{x_j}\eta_k\partial_{x_i}v_k-\partial_{x_i}\partial_{x_j}q\end{aligned}$$

where sums are taken implicitly over k. The left-hand side is already divergence-free so projecting again removes the gradient terms and yields

$$\partial_{x_i}\mathbb{P}[(\nabla\eta)^*v]=\mathbb{P}[(\nabla\eta)^*\partial_{x_i}v-(\nabla v)^*\partial_{x_i}\eta]. \tag{21}$$

By continuity, this still holds if we only have $\eta,v\in H^s$. A calculation similar to (20) applied to (21) yields continuity with respect to the H^s norm as claimed.

The inequalities (18) for $r=s-1$ and $r=s$ are obtained by taking the H^{s-1} norms of $\mathbb{P}[(\nabla\eta)^*v]$ and (21) respectively.

To prove (19), we again use the fact that $\mathbb{P}$ removes gradients. Indeed for f, $g\in H^s$, we have

$$\mathbb{P}((\nabla f)^*g)=\mathbb{P}(\nabla(f\cdot g)-(\nabla g)^*f)=-\mathbb{P}((\nabla g)^*f). \tag{22}$$

Setting $f=\eta_1-\eta_2$, $g=v_1+v_2$, we see that the calculations in (20) can be modified to give the required result. Note that for the L^2 bound we use the fact that (20) holds if we replace H^s with L^∞ and H^{s-1} with L^2. □

The next lemma gives uniform bounds on the H^s norms of solutions to the transport Eqs. (4) and (6). We will consider the following system:

$$\begin{cases}\partial_t f+(u\cdot\nabla)f=g\\ f(0)=f_0\end{cases} \tag{23}$$

where $f,g:[0,T]\times\mathbb{T}^n\to\mathbb{R}^n$ and u is divergence free.

Lemma 4. *Let $s > \frac{n}{2} + 1$ and fix $f_0 \in H^s$, $g \in \Sigma_s$. If $u \in \Sigma_s$ is non-zero and divergence free then there exists a unique solution f to (23). Furthermore, the solution $f \in \Sigma_s \cap C^1([0,T]; H^{s-1}) \cap C^1([0,T] \times \mathbb{T}^n)$ and there exists $C_4 > 0$ (from Lemma 2) such that if $r, t \in [0,T]$ we have:*

$$\|f(t)\|_{H^s} \leq \left(\|f(r)\|_{H^s} + \frac{\|g\|_{\Sigma_s}}{C_4\|u\|_{\Sigma_s}}\right)\exp(C_4|t-r|\|u\|_{\Sigma_s}) - \frac{\|g\|_{\Sigma_s}}{C_4\|u\|_{\Sigma_s}}. \tag{24}$$

Proof. By the method of characteristics we obtain a solution $f \in C^1([0,T] \times \mathbb{T}^n)$. The formal argument that follows motivates our consideration of the regularity of f. Taking the H^s product of (23) with f yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f\|_{H^s}^2 = -(B(u,f),f)_{H^s} + (f,g)_{H^s}.$$

By Lemma 2, there exists $C > 0$ such that for all $t \in [0,T]$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f(t)\|_{H^s}^2 \leq C\|u(t)\|_{H^s}\|f(t)\|_{H^s}^2 + \|g(t)\|_{H^s}\|f(t)\|_{H^s}. \tag{25}$$

Now (24) follows from Gronwall's inequality. In the case $r > t$, this argument is applied to the time-reversed equation, that is, using the fact that for fixed r, $-f(r-t)$ is transported by $-u(r-t)$ with forcing $g(r-t)$.

To properly justify this we can proceed by a Galerkin method. For each $N \in \mathbb{N}$ we find a solution to the system

$$\begin{cases} \partial_t f_N + P_N B(u_N, f_N) = g_N \\ f_N(r) = P_N f(r), \end{cases} \tag{26}$$

on $[r,T]$, where P_N denotes truncation up to Fourier modes of order N (in space), $u_N := P_N u$ and $g_N := P_N g$. The estimate (24) applies to f_N so by a standard argument using the Aubin–Lions lemma we obtain a weak solution $h \in L^\infty(r,T;H^s)$ such that $\partial_t h \in L^\infty(r,T;H^{s-1})$, hence $h \in C^0([0,T];H^{s-1})$. Using the divergence free property we obtain uniqueness of solutions $h \in L^2(r,T;H^1)$ with time derivative $\partial_t h \in L^2(r,T;L^2)$. Indeed, if h and $\tilde{h}$ are two such solutions it follows from (23) that

$$\frac{\mathrm{d}}{\mathrm{d}s}\|h - \tilde{h}\|_{L^2}^2 = 0.$$

Therefore $f = h$, i.e. this weak solution agrees with our C^1 classical solution on $[r,T]$.

We now prove (24) in the case $r \leq t$. Since $f_N \to f$ in $L^2(r,T;H^{s-1})$, we may choose a dense countable subset $\{t_k\}_{k=1}^\infty \subset [r,T]$ such that $f_N(t_k) \to f(t_k)$ in H^{s-1} as $N \to \infty$ for each k. The formal argument above is valid on the truncated system, thus

$$\|f_N(t_k)\|_{H^s} \leq \left(\|P_N f(r)\|_{H^s} + \frac{\|g\|_{\Sigma_s}}{C\|u_N\|_{\Sigma_s}}\right)\exp(C|t_k - r|\|u\|_{\Sigma_s}) - \frac{\|g_N\|_{\Sigma_s}}{C\|u\|_{\Sigma_s}}. \tag{27}$$

Hence, passing to a subsequence of f_N for each k with a diagonalisation argument, we may assume that for all k, $f_N(t_k)$ converges weakly in H^s as $N \to \infty$. Moreover, by the choice of the points t_k and uniqueness of weak limits, we must have $f_N(t_k) \rightharpoonup f(t_k)$ in H^s. Taking the lim inf of (27) with respect to $N \to \infty$ yields

$$\|f(t_k)\|_{H^s} \leq \left(\|f(r)\|_{H^s} + \frac{\|g\|_{\Sigma_s}}{C\|u\|_{\Sigma_s}}\right)\exp(C|t_k - r|\|u\|_{\Sigma_s}) - \frac{\|g\|_{\Sigma_s}}{C\|u\|_{\Sigma_s}}. \tag{28}$$

To prove (24) and the weak continuity of f into H^s we will use the fact that a weakly convergent sequence in H^{s-1} that is also bounded in H^s must converge weakly in H^s to the same limit by the Banach–Alaoglu theorem. Indeed if $x_k \rightharpoonup x$ in H^{s-1} is bounded in H^s then any subsequence admits a further subsequence converging weakly in H^s to x by the uniqueness of weak limits.

From this, (24) follows by the density of $\{t_k\}$ and the continuity of f into H^{s-1}. Indeed, in the case $t \geq r$, for any subsequence $(t_{k_\ell})_{\ell=1}^\infty \subset (t_k)_{k=1}^\infty$ such that $t_{k_\ell} \to t$ we have $f(t_{k_\ell}) \rightharpoonup f(t)$ in H^s. Applying (28) at t_{k_ℓ} and taking the lim inf as $\ell \to \infty$ yeilds (24) at time t. For $t < r$ the required bounds are obtained in the same way from the time-reversed version of (26).

We have shown that $\|f(t)\|_{H^s}$ in bounded uniformly, not merely almost everywhere. Therefore for any fixed $\tau \in [0,T]$ and any sequence $\{\tau_k\} \subset [0,T]$ such that $\tau_k \to \tau$ we deduce, by the continuity into H^{s-1}, that $f(\tau_k) \rightharpoonup f(\tau)$ in H^s. This says that f is weakly continuous into H^s.

To see that $f \in \Sigma_s$ it is therefore enough to show that $\|f(t)\|_{H^s}$ is continuous. This is the case since for all $r, t \in [0,T]$, (24) gives bounds of the form

$$(\|f(r)\|_{H^s} + \alpha)\mathrm{e}^{-\beta|t-r|} - \alpha \le \|f(t)\|_{H^s} \le (\|f(r)\|_{H^s} + \alpha)\mathrm{e}^{\beta|t-r|} - \alpha$$

for time independent constants $\alpha, \beta > 0$, where the first inequality comes from (24) with r and t interchanged.

The fact that $f \in C^1([0,T]; H^{s-1})$ follows from the fact that $\partial_t f \in \Sigma_{s-1}$ which can be seen from the regularity of the other terms in (23). □

Lemma 5. *For $s > n/2 + 1$ fix u_1, $u_2 \in \Sigma_s$ and $f_0 \in H^s$. Let $g_1 = g_2 = 0$ or $g_i = -u_i$ for $i = 1, 2$. If f_1, f_2 are the solutions of (23) corresponding to u_1, u_2, g_1, g_2 respectively, then in the case that $g_1 = g_2 = 0$, there exists $C_5 > 0$ depending only on s such that*

$$\|f_1(t) - f_2(t)\|_{L^2} \le C_5 \|f_1 + f_2\|_{\Sigma_s} \|u_1 - u_2\|_{\Sigma_0} t \tag{29}$$

for all $t \in [0,T]$. In the case that $g_i = -u_i$ for $i = 1, 2$ we instead have

$$\|f_1(t) - f_2(t)\|_{L^2} \le (C_5 \|f_1 + f_2\|_{\Sigma_s} + 1) \|u_1 - u_2\|_{\Sigma_0} t \tag{30}$$

Proof. Using the anti-symmetry of $(B(u_1 - u_2, \cdot), \cdot)_{L^2}$ we have, for $t \in [0,T]$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|f_1 - f_2\|_{L^2}^2 &\le |(B(u_1 - u_2, f_1 + f_2), f_1 - f_2)_{L^2}| + 2|(g_1 - g_2, f_1 - f_2)| \\ &\le C\|f_1 + f_2\|_{H^s}\|u_1 - u_2\|_{L^2}\|f_1 - f_2\|_{L^2} + 2\|g_1 - g_2\|_{\Sigma_0}\|f_1 - f_2\|_{L^2} \\ &\le C\|f_1 + f_2\|_{\Sigma_s}\|u_1 - u_2\|_{\Sigma_0}\|f_1 - f_2\|_{L^2} + 2\|g_1 - g_2\|_{\Sigma_0}\|f_1 - f_2\|_{L^2}\end{aligned}$$

Where C depends on the embedding $H^{s-1} \hookrightarrow L^\infty$. Formally dividing by $\|f_1 - f_2\|_{L^2}$ and integrating the resulting inequality gives (29) or (30) depending on the choice of g_1 and g_2. Justifying this last step is straightforward. □

We are now in a position to prove the main result.

Proof of Theorem 1. Fix $s > n/2 + 1$ and let C_3, C_4 be the constants in (18), (24) (from Lemmas 3 and 4) respectively. Fix $M > \|u_0\|_{H^s}$ and $T > 0$ so that

$$\exp(C_4 TM)\|u_0\|_{H^s}\left(\frac{C_3}{C_4}[\exp(C_4 TM) - 1] + 1\right) \le M.$$

Let $u \in B_M(T)$ be a divergence free function and let η be the solution of (23) for the flow u with initial data $\eta_0 = 0$ and forcing $g = u$. Let v be the solution for initial data $v_0 = u_0$ with $g = 0$. Define $Su := \mathbb{P}[(\nabla\eta)^* v + v]$, then by Lemmas 3 and 4,

$$\|Su(t)\|_{H^s} \le \exp(C_4 tM)\|u_0\|_{H^s}\left(\frac{C_3}{C_4}[\exp(C_4 tM) - 1] + 1\right) \le M \tag{31}$$

for all $t \in [0,T]$. Hence $S : B_M(T) \to B_M(T)$. Note that $Su(\cdot, 0) = u_0$ even if $u(\cdot, 0) \ne u_0$.

We next show that S is a contraction on $B_M(T)$ in the L^2 norm if T is sufficiently small. For u_1, $u_2 \in B_M(T)$ we construct v_i and η_i from u_i as above for $i = 1, 2$ with $v_1(\cdot, 0) = v_2(\cdot, 0) = u_0$. Now

$$\begin{aligned}\|Su_1 - Su_2\|_{L^2} &\le C_a\|\eta_1 - \eta_2\|_{L^2} + C_b\|v_1 - v_2\|_{L^2} \\ &\le (C_c\|v_1 + v_2\|_{\Sigma_s} + C_d\|\eta_1 + \eta_2\|_{\Sigma_s} + C_e)T\|u_1 - u_2\|_{\Sigma_0} \\ &\le C(u_0, M, T)\|u_1 - u_2\|_{\Sigma_0},\end{aligned} \tag{32}$$

where $C_a, \ldots, C_e$ denote various constants arising from the application of Lemmas 3, 4 and 5. Keeping careful track of the constants shows that $C(u_0, M, T)$ is given by the formula

$$C(u_0, M, T) := 2T\left[\left(C_5(C_3'M+1)\|u_0\|_{H^s} + \frac{C_3'C_5M}{C_4}\right)\exp(C_4TM) \quad + C_3'M\left(\frac{1}{2} - \frac{C_5}{C_4}\right)\right] \tag{33}$$

Where C_3', C_4, C_5 are the constants from Lemmas 3, 4 and 5 respectively. Taking the supremum of (32) with respect to t and choosing $T > 0$ small enough, we see that S is a contraction in the required sense.

We conclude that S has a unique accumulation point u, in the closure of B_M with respect to $\|\cdot\|_{\Sigma_0}$. Since $B_M(T)$ is convex and closed in Σ_s it is weakly closed, hence $u \in B_M(T)$ is a fixed point of S. A fixed point of S, along with associated back-to-labels map and virtual velocity, clearly give a solution to the Eulerian–Lagrangian formulation of the Euler equations with the required regularity. The contraction argument gives uniqueness in $B_M(T)$ and it remains to prove that we have uniqueness in $\Sigma_s(T)$.

Since S is a contraction on $B_M(\widetilde{T})$ for any $\widetilde{T} \in (0, T]$, we have by continuity of $\|u(t)\|_{H^s}$, that if u', A' and v' also satisfy (4–6) with $u' \in \Sigma_s(T)$, then $u(t) = u'(t)$ when $0 \le t \le \min(T, \inf\{r : \|u'(r)\|_{H^s} = M\})$.

Now we know that for all $k \in \mathbb{N}$ there exists $T_k \le T$ such that S is a contraction on $B_{M+1/k}(T_k)$ and we may assume $T_k \to T$ as $k \to \infty$. By the previous observation, this means that u is the unique solution in $\Sigma_s(T - \varepsilon)$ for all $\varepsilon > 0$, hence by continuity u is the unique solution in Σ_s as required.

The proof that $u \in C^1([0, T]; H^{s-1})$ uses the same trick as Lemma 3 to save a spatial derivative (we have only shown that $\nabla\eta_t \in H^{s-2}$, which might otherwise limit the regularity of u). More precisely, using (22) and the fact that $u = \mathbb{P}[(\nabla\eta)^* v + v]$, it can be shown that

$$\begin{aligned}
&\frac{1}{h}\left\|u(t+h) - u(t) - h\mathbb{P}[(\nabla\eta(t))^*\partial_t v(t) + \partial_t v(t) - (\nabla v(t))^*\partial_t\eta(t)]\right\|_{H^{s-1}} \\
&\quad\le \frac{1}{2h}\left\|\mathbb{P}[(\nabla\eta(t+h) + \nabla\eta(t))^*(v(t+h) - v(t) - h\partial_t v)]\right\|_{H^{s-1}} \\
&\qquad + \frac{1}{2h}\left\|\mathbb{P}[(\nabla v(t+h) + \nabla v(t))^*(\eta(t+h) - \eta(t) - h\partial_t\eta)]\right\|_{H^{s-1}} \\
&\qquad + \frac{1}{2}\|\mathbb{P}[(\nabla\eta(t+h) - \nabla\eta(t))^*\partial_t v(t)]\|_{H^{s-1}} \\
&\qquad + \frac{1}{2}\|\mathbb{P}[(\nabla v(t+h) - \nabla v(t))^*\partial_t\eta(t)]\|_{H^{s-1}} \\
&\qquad + \frac{1}{h}\|v(t+h) - v(t) - h\partial_t v(t)\|_{H^{s-1}}.
\end{aligned}$$

Since H^{s-1} is an algebra and $\eta, v \in C^0([0, T]; H^s) \cap C^1([0, T]; H^{s-1})$, the right-hand side vanishes as $h \to 0$. Therefore $u \in C^1([0, T]; H^{s-1})$ and

$$\partial_t u = \mathbb{P}[(\nabla\eta(t))^*\partial_t v(t) + \partial_t v(t) - (\nabla v(t))^*\partial_t\eta(t)].$$

□

4. An Alternative Iteration

Here we exhibit an alternative proof of existence and uniqueness for (4–6), which is based on contractions with respect to A rather than u. The extra technicality in this approach is contained in the following lemma, which is proved in an appendix. We will denote the identity map on $\mathbb{T}^n$ by ι and use the correspondence between maps $\mathbb{T}^n \to \mathbb{R}^n$ and $\mathbb{T}^n \to \mathbb{T}^n$ without comment.

Lemma 6. *Let $s \in \mathbb{Z}$ with $s > \frac{n}{2} + 1$ and fix $f, g \in H^s$. If $g + \iota$ is a volume preserving map then $f \circ (g + \iota) \in H^s$ and*

$$\|f \circ (g + \iota)\|_{H^s} \le C_6\|f\|_{H^s}(\|g\|_{H^s} + (2\pi)^n)^s \tag{34}$$

for some $C_6 > 0$ depending only on s and the constants from some Sobolev embeddings.

This allows us to write a second proof of existence and uniqueness of solutions in Σ_s for $s > n/2 + 1$ in the case $s \in \mathbb{Z}$.

Fix $u_0 \in H^s$ and $M > 0$ and suppose $\eta \in B_M(T)$ for some $T > 0$ such that $\eta(t)+\iota$ is volume-preserving for all $t \in [0, T]$. Define u and v via $v = u_0 \circ (\eta + \iota)$ and $u = \mathbb{P}[(\nabla\eta)^* v + v]$. Construct η', the iterate of η by solving

$$\partial_t \eta' + (u \cdot \nabla)\eta' = -u,\ \eta'(x, 0) = 0.$$

By Lemmas 3, 4 and 6 we have

$$\|\eta'\|_{\Sigma_s} \le \frac{1}{C_4}\left[\exp(C_4 C_6 (C_3 M + 1)(M + (2\pi)^n)^s \|u_0\|_{H^s} T) - 1\right].$$

Hence for T small enough, we may assume $\eta' \in B_M(T)$ and since $\nabla \cdot u = 0$ we also have that $\eta' + \iota$ is volume preserving.

Now suppose that η_1, $\eta_2 \in B_M(T)$ and let η_1', η_2' be the respective iterates then

$$\|\eta_1' - \eta_2'\|_{\Sigma_0} \le 2(C_5 M + 1)(C_3' M + (C_3' M + 1)C_{\mathrm{Lip}})T\|\eta_1 - \eta_2\|_{\Sigma_0},$$

by Lemmas 3 and 5. Here C_{Lip} is the Lipschitz constant of u_0. It follows that, for small enough T, this iteration procedure is a contraction on $B_M(T)$ in the L^2 norm. Existence and uniqueness of solutions now follows using the same steps as in the previous method.

5. Conclusions

We have seen that Constantin's proof of local well-posedness for the Eulerian–Lagrangian formulation of the Euler equations in $C^{1,\mu}$ can be adapted to prove analogous results in the corresponding Sobolev spaces, H^s for $s > n/2 + 1$, directly. This involved different estimates, which may seem more familiar to some readers. We have given two different iteration schemes to deduce well-posedness using these estimates; iterating with respect to u is natural in this setting and leads to a fairly clean proof, whereas iterating with respect to the Lagrangian coordinate A involves estimates on the compositions of Sobolev functions, which are proved in Appendix A. It would be interesting to investigate these composition estimates further and extend them to non-integer Sobolev spaces, for example.

Robinson and Sadowski [15] have shown that in the case of the 3D Navier–Stokes equations, almost every Lagrangian trajectory is well-defined and C^1, for any suitable weak solution u, with $u \in L^{6/5}(0, T; L^\infty)$. This suggests it may be reasonable to study Eulerian–Lagrangian formulations for diffusive systems. For example, calculations analogous to the derivation above suggest that the Navier–Stokes equations can be formulated as

$$\partial_t A + (u \cdot \nabla)A = 0,\ u = \mathbb{P}((\nabla A)^* v),$$

with

$$\partial_t v + (u \cdot \nabla)v - ((\nabla A)^*)^{-1}\Delta(\nabla A)^* v = 0,$$

however obtaining results using such formulations has proved difficult, so far.

Constantin [3,4] has put forward an Eulerian–Lagrangian form in the viscous case, where diffusive terms appear in the equations for the back-to-labels map and the virtual velocity. Ideally we would be able to make a meaningful study of formulations where the back-to-labels map retains its physical meaning.

Alternatively, if one formally considers the equation satisfied by $(\nabla A)^* v$, one arrives at a formulation in magnetization variables. We recently showed that this leads to an interesting model system for Navier–Stokes, which is globally well-posed in $H^{1/2}$ in 3D [14].

Appendix A. Compositions in H^s

In this appendix we prove Lemma 6, which gives bounds on the compositions H^s functions with certain volume-preserving locally H^s functions where $s \in \mathbb{Z}$ with $s > \frac{n}{2}$.

To begin with we consider $g_i \in H^s$ and multi indices β_i with $|\beta_i| \in [1, s]$ for $i = 1, \ldots, \ell$. We call $p \in [1, \infty]$ *admissible* for $(\beta_i)_{1 \le i \le \ell}$ if there exists a constant $C > 0$ independent of $(g_i)_{1 \le i \le \ell}$ such that

$$\left\| \prod_{i=1}^{\ell} \mathrm{D}^{\beta_i} g_i \right\|_{L^p} \le C \prod_{i=1}^{\ell} \|g_i\|_{H^s}. \tag{35}$$

Of course p is admissible if there exist $q_1, \ldots, q_\ell \in [1, \infty)$ such that $H^{s-|\beta_i|} \hookrightarrow L^{q_i}$ for each i and

$$\sum_{i=1}^{\ell} \frac{1}{q_i} = \frac{1}{p},$$

or $p = \infty$ and $q_i = \infty$ for all i. We may assume, without loss of generality that there are constants k_1 and k_2 with $0 \le k_1 \le k_2 \le \ell$ such that

$$\begin{cases} s - |\beta_i| \in [0, n/2) & \text{for } \ 1 \le i \le k_1 \\ s - |\beta_i| = n/2 & \text{for } \ k_1 + 1 \le i \le k_2 \\ s - |\beta_i| > n/2 & \text{for } \ k_2 + 1 \le i \le \ell \end{cases}$$

So we have

$$\left\| \prod_{i=1}^{k_1} D^{\beta_i} g_i \right\|_{L^p} \le C \prod_{i=1}^{k_1} \|g_i\|_{H^s}$$

for

$$\frac{1}{p} \in \left[\sum_{i=1}^{k_1} \frac{n - 2(s - |\beta_i|)}{2n}, \frac{k_1}{2} \right].$$

Moreover

$$\left\| \prod_{i=k_1+1}^{k_2} D^{\beta_i} g_i \right\|_{L^p} \le C \prod_{i=k_1+1}^{k_2} \|g_i\|_{H^s}$$

for $p \in [2, \infty)$. Lastly,

$$\left\| \prod_{i=k_2+1}^{\ell} D^{\beta_i} g_i \right\|_{L^\infty} \le C \prod_{i=k_2+1}^{\ell} \|g_i\|_{H^s}.$$

Combining these observations we see that p is admissible if

$$\frac{1}{p} \in \left(\sum_{i=1}^{k_1} \frac{n - 2(s - |\beta_i|)}{2n}, \frac{\ell}{2} \right]. \tag{36}$$

or if $k_1 = k_2$ then p is still admissible if

$$\frac{1}{p} = \sum_{i=1}^{k_1} \frac{n - 2(s - |\beta_i|)}{2n}, \tag{37}$$

furthermore $p = \infty$ is admissible if $k_1 = k_2 = 0$.

Note that if $p \in [1, \infty]$ is admissable and $f_i : \mathbb{T}^n \to \mathbb{R}^n$ are linear maps then we have (rather crudely)

$$\left\| \prod_{i=1}^{\ell} D^{\beta_i} (g_i + f_i) \right\|_{L^p} \le C \prod_{i=1}^{\ell} \|g_i\|_{H^s} + \|f_i\|_{\mathrm{op}} (2\pi)^{n/q_i}. \tag{38}$$

In the proof of the lemma below, we will need the fact that if $s > \frac{n}{2}$ and $\sum_{i=1}^{\ell} |\beta_i| \leq s$ then $p = 2$ is admissible for $(\beta_i)_{1 \leq i \leq \ell}$. Furthermore, we will need to show that if $s > n/2 + 1$ then there exists an admissible $p > \frac{n}{s-\ell}$ and that $p = \infty$ is admissible if $s = \ell > n/2 + 1$.

For the first claim, note that if $k_1 = 0$ or $k_1 = 1$ then $p = 2$ is clearly admissible. Otherwise, if $1 < k_1 \leq \ell$ and $s > n/2$, we have the following calculation:

$$\sum_{i=1}^{k_1} n - 2(s - |\beta_i|) \leq k_1 n - 2k_1 s + 2s = (k_1 - 1)(n - 2s) + n < n \tag{39}$$

so $p = 2$ is admissible. For the second claim, observe that if $s > n/2 + 1$ then

$$\sum_{i=1}^{k_1} n - 2(s - |\beta_i|) < 2\sum_{i=1}^{k_1} |\beta_i| - 2k_1 \leq 2(s - k_1) - 2\sum_{i=k_1+1}^{\ell} |\beta_i| \leq 2(s - \ell), \tag{40}$$

where the middle inequality uses the assumption that $\sum_{i=1}^{\ell} |\beta_i| \leq s$. Hence there exists an admissible value $p > \frac{n}{s-\ell}$, if $s - \ell > 0$. If $s = \ell$ then necessarily, $|\beta_i| = 1$ for $i = 1, \ldots, \ell$ hence $p = \infty$ is admissible by (37).

Lemma 6. *Let $s \in \mathbb{Z}$ with $s > \frac{n}{2} + 1$ and fix $f, g \in H^s$. Denote the identity map on $\mathbb{T}^n$ by ι. If $g + \iota$ is a volume preserving map then $f \circ (g + \iota) \in H^s(\mathbb{T}^n)$ and*

$$\|f \circ (g + \iota)\|_{H^s} \leq C\|f\|_{H^s}(\|g\|_{H^s} + (2\pi)^n)^s \tag{41}$$

for some $C > 0$ depending only on s and the constants from some Sobolev embeddings.

Proof. For each $k \in \mathbb{N}$, consider functions $f_k, g_k \in C^\infty(\mathbb{T}^n; \mathbb{R}^n)$ such that $f_k \to f$ in H^s and $g_k \to g$ in H^s. Without loss of generality we assume that $||\det \nabla(g_k(x) + x)| - 1| < \frac{1}{k+1}$ holds uniformly in x.

Now by the chain and Leibniz rules, we see that for a multi-index γ with $|\gamma| \leq s$, $D^\gamma(f_k \circ (g_k + \iota))$ is a (weighted) sum with summands of the form

$$((D^\alpha f_k) \circ (g_k + \iota)) \prod_{i=1}^{\ell} D^{\beta_i}(g_k^{r_i} + x_{r_i}), \tag{42}$$

where $\ell = |\alpha| \leq |\gamma|$ and $\sum_{i=1}^{\ell} |\beta_i| = |\gamma|$. Here g_k^i denotes the ith vector component of g_k. We seek to bound terms of the form (42) in L^2 using the preceding observations.

Since $D^\alpha f_k \in H^{s-\ell}$ and $g_k + \iota$ is "almost volume preserving" it can be seen that $(D^\alpha f_k) \circ (g_k + \iota) \in L^q$ if

$$\frac{1}{q} \in \left(\frac{1}{2} - \frac{s-\ell}{n}, \frac{1}{2}\right]$$

with $s - \ell \in (0, n/2]$ or

$$\frac{1}{q} = \frac{1}{2} - \frac{s-\ell}{n}$$

when $s - \ell \in (0, n/2)$. Of course, if $s - \ell > n/2$ then $D^\alpha f_k \in L^\infty$.

To bound (42) in L^2 therefore, we need to check that there is an admissible p such that,

$$\frac{1}{p} \in \left[0, \frac{s-\ell}{n}\right).$$

and that $p = \infty$ is admissible if $s = \ell$. This follows from the claims we proved before the statement of the lemma.

Now we see that

$$\|f_k \circ (g_k + \iota)\|_{H^s} \leq C\sqrt{1 + 1/k}\,\|f_k\|_{H^s}(\|g_k\|_{H^s} + (2\pi)^n)^s$$

where C depends only on Sobolev embeddings and some combinatorics. Since f_k and g_k converge we may assume that $f_k \circ (g_k + \iota)$ converges weakly in H^s. Thus the lemma is proved if we can show that $f_k \circ (g_k + \iota) \to f \circ (g + \iota)$ in L^2 for example. This is indeed the case:

$$\begin{aligned}&\|f \circ (g + \iota) - f_k \circ (g_k + \iota)\|_{L^2} \\ &\quad \le \|f \circ (g + \iota) - f \circ (g_k + \iota)\|_{L^2} + \|f \circ (g_k + \iota) - f_k \circ (g_k + \iota)\|_{L^2} \\ &\quad \le C_{\text{Lip}} \|g - g_k\|_{L^2} + \sqrt{1 + 1/k}\, \|f - f_k\|_{L^2},\end{aligned}$$

where we make use of the fact that $f \in H^s$ is Lipschitz since $s > n/2+1$ and denote by C_{Lip} the Lipschitz constant of f. □

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