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On Regularity Criteria for the 2D Generalized MHD System

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Abstract. This paper deals with the problem of regularity criteria for the 2D generalized MHD system with fractional dissipative terms $-\Lambda^{2\alpha}u$ for the velocity field and $-\Lambda^{2\beta}b$ for the magnetic field respectively. Various regularity criteria are established to guarantee smoothness of solutions. It turns out that our regularity criteria imply previous global existence results naturally.

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1. Introduction

We consider the following two-dimensional generalized MHD (GMHD, for simplicity) system:

$$\begin{cases} u_t + u\cdot\nabla u + \nabla\pi + \Lambda^{2\alpha}u - b\cdot\nabla b = 0, & (t,x)\in\mathbb{R}^+\times\mathbb{R}^2, \\ b_t + u\cdot\nabla b + \Lambda^{2\beta}b - b\cdot\nabla u = 0, & (t,x)\in\mathbb{R}^+\times\mathbb{R}^2, \\ \operatorname{div} u = 0, \operatorname{div} b = 0, & (t,x)\in\mathbb{R}^+\times\mathbb{R}^2, \\ (u,b)(x,0) = (u_0,b_0)(x), & x\in\mathbb{R}^2, \end{cases} \tag{1.1}$$

here $u = u(x,t)\in\mathbb{R}^2$, $b = b(x,t)\in\mathbb{R}^2$, and $\pi = \pi(x,t)\in\mathbb{R}$ represent the unknown velocity field, the magnetic field and the pressure respectively. $\alpha\geq 0, \beta\geq 0$ are real parameters. We identify the case $\alpha=\beta=0$ as the 2D GMHD system with zero velocity and zero magnetic diffusion respectively (so called idea MHD equations). $\Lambda = (-\Delta)^{\frac{1}{2}}$ is defined in terms of Fourier transform by $\widehat{\Lambda f}(\xi) = |\xi|\hat{f}(\xi)$.

First of all, local well-posedness and global existence results are established in [1–4]. Then, we mention some results about the global regularity theory for the 2D GMHD systems. In [5], Tran, Yu and Zhai proved that smooth solutions are global in the following three cases: $\alpha\geq 1, \beta\geq 1$; $0\leq\alpha<\frac{1}{2}, 2\alpha+\beta>2$; $\alpha\geq 2, \beta=0$. Recently, Jiu and Zhao got a global regular solution under the assumption that $0\leq\alpha<\frac{1}{2}$, $\beta\geq 1$, $3\alpha+2\beta>3$. In particular, it was proved the solution exists globally for the case $\alpha=0, \beta>\frac{3}{2}$, this result was also proved independently in [6,7]. Later, global regularity for the case $\alpha=0,\beta>1$ was established in [8] and [9] independently. Meanwhile, using the Fourier series analysis Ji proved the global regularity criterion when $\frac{1}{2}<\alpha\leq 1, \beta=1$ in [10]. Very recently, it was improved that the solution of the 2D GMHD system exists globally for the case $\alpha>0, \beta=1$ in [11].

Extensive studies on regularity criterion theory have also been made for the 2D GMHD systems with $\alpha=1, \beta=0$. The following regularity criterion on the magnetic field

$$b\in L^p(0,T;W^{2,q}(\mathbb{R}^2)), \quad \textit{with} \quad \frac{2}{p}+\frac{1}{q}\leq 2,\ 1\leq p\leq\frac{4}{3},\ 2<q\leq\infty,$$

is given in [12]. But it is not scaling invariant. Later, in [13], Fan and Ozawa proved a regularity criterion on the velocity field as $\nabla u\in L^1(0,T;L^\infty(\mathbb{R}^2))$. A regularity criterion in terms of $b\otimes b$ as

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$b \otimes b \in L^1(0,T;BMO(\mathbb{R}^2))$ is proved in [14], and another regularity criterion in terms of ∇b as $\nabla b \in L^1(0,T;BMO(\mathbb{R}^2))$ is proved in [15].

Now, we introduce some notations which will be used in this paper. Use $\|\cdot\|_p$ to denote the $L^p(\mathbb{R}^2)$ norm. Throughout this paper, C denotes a generic positive constant (generally large), it may be different from line to line. Use $\hat{f}$ to denote the Fourier transform of f. We introduce the norm $L^{p,q}$

$$\|f\|_{L^{p,q}} = \begin{cases} (\int_0^t \|f(\cdot,\tau)\|_{L^q}^p d\tau)^{\frac{1}{p}}, & if\ 1 \le p < \infty, \\ esssup_{\ 0<\tau<t} \|f\|_{L^q}, & if\ p = \infty. \end{cases}$$

From [16], we know that if $\alpha = \beta$ and $(u,b)(x,t)$ is a solution to (1.1), then $(u_\lambda, b_\lambda)(x,t)$ with any $\lambda > 0$ is also a solution, where $u_\lambda(x,t) = \lambda^{2\alpha-1}u(\lambda x, \lambda^{2\alpha}t)$ and $b_\lambda(x,t) = \lambda^{2\alpha-1}b(\lambda x, \lambda^{2\alpha}t)$. Direct calculation yields the norms $\|u\|_{L^{p,q}}$ and $\|\Lambda^\gamma u\|_{L^{p,q}}$ are scaling dimension zero for $\frac{2\alpha}{p} + \frac{2}{q} = 2\alpha - 1$ and $\frac{2\alpha}{p} + \frac{2}{q} = 2\alpha + \gamma - 1$ respectively. It should be noted that both equations have the same scaling property according to the dimensions 2 and 3. Another similarity from a scaling viewpoint can be found in the 2D dissipative quasi-geostrophic equation whose global well-posedness had been solved by Kiselev–Nazarov–Volberg [17], and whose further development of regularity of weak solutions was fully established by Caffarelli–Vasser [18]. In this respect, we may say that the topic of the present paper plays a central role in the research of regularity theorems on solutions arising in the equations of the fluid mechanics. This paper is devoted to obtaining some scaling invariant regularity criteria for the general system (1.1). Such a criterion on the 3D Euler equations was first obtained by Beale–Kato–Majda [19] in the case when $\omega \in L^1(0,T;L^\infty(\mathbb{R}^3))$. It should be emphasized that the present paper treats also the marginal case when $\omega \in L^1(0,T;BMO)$ like Kozono–Taniuchi [20] in the reference. Here, we also would like to call attention to a recent work on related generalized Hall-MHD system [21] and references therein.

Our main results are the following four theorems. In what follows, we set $\rho = \max\{\frac{2}{\alpha}, 2\}$, $\varrho = \max\{\frac{2}{\beta}, 2\}$. The first theorem is for large α and β.

Theorem 1.1. *Let $\alpha, \beta \ge \frac{1}{2}$. Suppose $u_0(x), b_0(x) \in H^2(\mathbb{R}^2)$ and $(u,b)(x,t)$ is a local smooth solution of the system (1.1). If $\omega(x,t), j(x,t)$ satisfy*

$$\int_0^T \|\omega(\cdot,t)\|_p^{\frac{1}{1-\theta_\beta}} dt \le C(T), \tag{1.2}$$

or

$$\int_0^T \|\omega(\cdot,t)\|_\varrho^2 dt \le C(T), \tag{1.3}$$

or

$$\int_0^T \|j(\cdot,t)\|_\rho^2 dt \le C(T), \tag{1.4}$$

or

$$\int_0^T \|j(\cdot,t)\|_\varrho^2 dt \le C(T), \tag{1.5}$$

then $(u,b)(x,t)$ is a regular solution in $(0,T']$ for some $T' > T$. Here $\omega = -\partial_2 u_1 + \partial_1 u_2$, $j = -\partial_2 b_1 + \partial_1 b_2$, $\theta_\beta = \frac{1}{p\beta}$, $p \ge \frac{1}{\beta}$.

Remark 1.1. The regularity criteria in Theorem 1.1 are given in terms of ω or j. A natural question is wether the regularity criteria can be given in terms of $\Lambda^\alpha u$ or $\Lambda^\beta b$. Here, we can prove that for $1 > \alpha, \beta \ge \frac{1}{2}$, if $\Lambda^\beta b$ satisfies

$$\int_0^T \|\Lambda^\beta b(\cdot,t)\|_s^{\frac{2}{1-\rho\alpha}} dt \le C(T), \tag{1.6}$$

then the solution remains smooth on $(0,T]$, where $\rho_\alpha = \frac{\frac{2}{s}-(2\beta-1)}{\alpha}$, $\frac{2\beta-1}{2} \le \frac{1}{s} \le \frac{2\beta-1+\alpha}{2}$. It seems that the regularity criteria in terms of $\Lambda^\alpha u\,(\alpha < 1)$ is much more difficult. We hope we can investigate it in the future.

In comparison with the 3D Navier-Stokes equations, the most important exponents α and β in the 2D MHD equations are the case when $\alpha = \beta = \frac{1}{2}$. It is well-known that the 2D generalized Navier-Stokes equation possesses a global classical solution with $\alpha \ge 1$ (see [22] for details). Thanks to Theorem 1.1, we have the following interesting improvement.

Corollary 1.1. Let $\alpha \ge \frac{1}{2}$ and $u_0(x) \in H^2(\mathbb{R}^2)$. Then, the 2D generalized Navier-Stokes equation

$$\begin{cases} u_t + u\cdot\nabla u + \nabla\pi + \Lambda^{2\alpha}u = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^2, \\ divu = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^2, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^2, \end{cases} \tag{1.7}$$

has a unique global classical solution.

The following theorems are established for the cases with small α or β.

Theorem 1.2. *Let $0 < \alpha < \frac{1}{2}$, $\beta > 0$ or $\alpha > 0$, $0 < \beta < \frac{1}{2}$. Suppose $u_0(x), b_0(x) \in H^2(\mathbb{R}^2)$ and $(u,b)(x,t)$ is a local smooth solution of the system (1.1). If*

$$\int_0^T (\|\omega(\cdot,t)\|_p^{\frac{1}{1-\sigma}} + \|j(\cdot,t)\|_p^{\frac{1}{1-\sigma}})dt \le C(T). \tag{1.8}$$

Then $(u,b)(x,t)$ is a regular solution in $(0,T']$ for some $T' > T$. Here $\sigma = \max\{\theta_\alpha, \theta_\beta\}$, $\theta_\alpha = \frac{1}{p\alpha}$, $p\alpha \ge 1$.

Theorem 1.3. *Let $\alpha > 0$, $\beta = 0$. Suppose $u_0(x), b_0(x) \in H^2(\mathbb{R}^2)$ and $(u,b)(x,t)$ is a local smooth solution of the system (1.1). if $(u,b)(x,t)$ satisfies*

$$\int_0^T (\|\nabla u(\cdot,t)\|_\infty + \|j(\cdot,t)\|_p^{\frac{2}{2-\theta}})dt \le C(T), \tag{1.9}$$

or

$$\int_0^T (\|\nabla u(\cdot,t)\|_\infty + \|j(\cdot,t)\|_\rho^2)dt \le C(T), \tag{1.10}$$

then $(u,b)(x,t)$ is a regular solution in $(0,T']$ for some $T' > T$, where $\theta = \frac{2}{p\alpha}$, $p\alpha \ge 2$.

For the case $\alpha = 0$, $\beta > 0$ we can give the following regularity criteria.

Theorem 1.4. *Let $\alpha = 0$, $\beta > 0$. Suppose $u_0(x), b_0(x) \in H^2(\mathbb{R}^2)$ and $(u,b)(x,t)$ is a local smooth solution of the system (1.1). Suppose the corresponding solution satisfies*

$$\int_0^T (\|\nabla u(\cdot,t)\|_\infty + \|j(\cdot,t)\|_\varrho^2)dt \le C(T), \tag{1.11}$$

or

$$\int_0^T \|\omega(\cdot,t)\|_{BMO}dt \le C(T), \int_0^T \|\omega(\cdot,t)\|_\varrho^2 dt \le C(T),$$
$$and \quad \int_0^T \|j(\cdot,t)\|_\varrho^2 dt \le C(T), \tag{1.12}$$

then $(u,b)(x,t)$ is a regular solution in $(0,T']$ for some $T' > T$.

Before giving the proofs of our main theorems, we would like to give some remarks about our results.

Remark 1.2. If $\alpha = \beta$, the regularity criteria given by (1.2), (1.4), (1.5), (1.6), (1.8) satisfy $2\beta(1-\theta_\beta)+\frac{2}{p} = 2\beta$, $\frac{2\alpha}{2} + \frac{2}{\rho} = 2\alpha \, for \, \alpha \le 1$, $\frac{2\beta}{2} + \frac{2}{\varrho} = 2\beta \, for \, \beta \le 1$, $\beta(1-\rho_\beta) + \frac{2}{s} = 3\beta - 1$ and $2\beta(1-\sigma) + \frac{2}{p} = 2\beta$ respectively, they are all scaling invariant.

Remark 1.3. If $\alpha = 0$, $\beta > 1$,one can prove $\omega \in L^\infty(0,T;L^p(\mathbb{R}^2)), 2 \leq p \leq \infty$, $j \in L^2(0,T;L^p(\mathbb{R}^2)), 2 \leq p < \infty$ (refer [9] for details) the regularity criteria in Theorem 1.4 is satisfied naturally. So we recover the result in [8] and [9].

Remark 1.4. If $\alpha > 0$, $\beta = 1$, combining the result in [11] and the equations (2.1), (2.7) and (2.11) we can obtain $u, b \in L^\infty(0,T;H^2(\mathbb{R}^2))$, the regularity criteria in (1.8) is satisfied naturally. So the corresponding solution exists globally.

Remark 1.5. If $\alpha \geq 2$, $\beta = 0$, one can prove $\omega \in L^2(0,T;H^2(\mathbb{R}^2))$, $j \in L^\infty(0,T;H^1(\mathbb{R}^2)), 2 \leq p < \infty$ (refer [5] for details), the regularity criteria in (1.9) is satisfied naturally. So the corresponding solution exists globally.

Remark 1.6. Let $0 < \beta < \frac{1}{2} \leq \alpha$ and $\alpha + \beta \geq 1$ or $0 < \alpha < \frac{1}{2} \leq \beta$. The regularity criteria can also be given in terms of ω.

For $0 < \beta < \frac{1}{2} \leq \alpha$ and $\alpha + \beta \geq 1$ the regularity criteria can be given by (1.3) or

$$\int_0^T \|\omega(\cdot,t)\|_p^{\frac{1}{1-\theta_\beta}} dt \leq C(T). \tag{1.13}$$

For $0 < \alpha < \frac{1}{2} \leq \beta$ the regularity criteria can be given by

$$\int_0^T \|\omega(\cdot,t)\|_p^{\frac{1}{1-\theta_\alpha}} dt \leq C(T). \tag{1.14}$$

2. Proof of Theorem 1.1

In this section, we devote to prove Theorem 1.1. Under the assumption in Theorem 1.1, if $u \in L^\infty(0,T;H^1(\mathbb{R}^2))$ and $b \in L^\infty(0,T;H^1(\mathbb{R}^2))$, we can deduce $u \in L^\infty(0,T;H^2(\mathbb{R}^2))$ and $b \in L^\infty(0,T;H^2(\mathbb{R}^2))$. So it is sufficient to give a priori estimates to bound H^1 norms of u and b.

Proof. Multiplying both sides of the equations of u, b in (1.1) by u, b respectively, integrating over $\mathbb{R}^2$ and adding the resulting equations together we obtain

$$\|u\|_2^2(t) + \|b\|_2^2(t) + 2\int_0^t \|\Lambda^\alpha u\|_2^2 + \|\Lambda^\beta b\|_2^2 ds = \|u_0\|_2^2 + \|b_0\|_2^2. \tag{2.1}$$

Let $\omega = -\partial_2 u_1 + \partial_1 u_2$, $j = -\partial_2 b_1 + \partial_1 b_2$, then we obtain the following equations for ω and j:

$$\begin{cases} \omega_t + u \cdot \nabla\omega + \Lambda^{2\alpha}\omega - b \cdot \nabla j = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^2, \\ j_t + u \cdot \nabla j + \Lambda^{2\beta} j - b \cdot \nabla\omega = Q(\nabla u, \nabla b), & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^2, \\ \operatorname{div} u = 0, \operatorname{div} b = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^2, \\ (u,b)(x,0) = (u_0,b_0)(x), & x \in \mathbb{R}^2, \end{cases} \tag{2.2}$$

where $Q(\nabla u, \nabla b) = 2\partial_1 b_1(\partial_1 u_2 + \partial_2 u_1) + 2\partial_2 u_2(\partial_1 b_2 + \partial_2 b_1)$.

Now, we are ready to give the H^1 estimation for (u,b), multiplying the equations of ω, j in (2.2) by ω, j respectively, integrating over $\mathbb{R}^2$ and adding the resulting equations together we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}(\|\omega\|_2^2 + \|j\|_2^2) + \|\Lambda^\alpha\omega\|_2^2 + \|\Lambda^\beta j\|_2^2 &= \int_{R^2} Q(\nabla u, \nabla b) j dx \\ &\leq C\|\omega\|_p\|j\|_{2q}^2 \\ &\leq C\|j\|_2^{2-2\theta_\beta}\|\Lambda^\beta j\|_2^{2\theta_\beta}\|\omega\|_p \\ &\leq C\|j\|_2^2\|\omega\|_p^{\frac{1}{1-\theta_\beta}} + \epsilon\|\Lambda^\beta j\|_2^2, \end{aligned} \tag{2.3}$$

where $p > \frac{1}{\beta}$, $\frac{1}{p} + \frac{1}{q} = 1$. Here we have used the Galiardo–Nirenberg inequality

$$\|j\|_{2q} \le C\|j\|_2^{1-\theta_\beta}\|\Lambda^\beta j\|_2^{\theta_\beta}, \quad \frac{1}{2q} = \left(\frac{1}{2} - \frac{\beta}{2}\right)\theta_\beta + \frac{1-\theta_\beta}{2}, \quad 0 \le \theta_\beta < 1, \tag{2.4}$$

then it yields

$$\theta_\beta = \frac{\frac{1}{2} - \frac{1}{2q}}{\frac{\beta}{2}} = \frac{\frac{1}{2p}}{\frac{\beta}{2}} = \frac{1}{p\beta}.$$

Thanks to regularity criteria (1.2), we obtain

$$(\|\omega\|_2^2 + \|j\|_2^2) + \int_0^t \|\Lambda^\alpha\omega\|_2^2 + \|\Lambda^\beta j\|_2^2 dt \le C(T). \tag{2.5}$$

On the other hand, if $\beta > 0$, we have

$$\frac{1}{2}\frac{d}{dt}(\|\omega\|_2^2 + \|j\|_2^2) + \|\Lambda^\alpha\omega\|_2^2 + \|\Lambda^\beta j\|_2^2 \le \begin{cases} C\|\omega\|_{\frac{2}{\beta}}^2\|j\|_2^2 + \frac{1}{2}\|\Lambda^\beta j\|_2^2, & if\ 0 < \beta < 1, \\ C\|\omega\|_2^2\|j\|_2^2 + \frac{1}{2}\|\Lambda j\|_2^2, & if\ \beta \ge 1, \end{cases}$$
$$\le C\|\omega\|_\varrho^2\|j\|_2^2 + \frac{1}{2}\|\Lambda^\beta j\|_2^2. \tag{2.6}$$

Or

$$\frac{1}{2}\frac{d}{dt}(\|\omega\|_2^2 + \|j\|_2^2) + \|\Lambda^\alpha\omega\|_2^2 + \|\Lambda^\beta j\|_2^2 \le \begin{cases} C\|\omega\|_2^2\|j\|_{\frac{2}{\beta}}^2 + \frac{1}{2}\|\Lambda^\beta j\|_2^2, & if\ 0 < \beta < 1, \\ C\|\omega\|_2^2\|j\|_2^2 + \frac{1}{2}\|\Lambda j\|_2^2, & if\ \beta \ge 1, \end{cases}$$
$$\le C\|\omega\|_2^2\|j\|_\varrho^2 + \frac{1}{2}\|\Lambda^\beta j\|_2^2. \tag{2.7}$$

Here we have used the Galiardo–Nirenberg inequality

$$\|j\|_4^2 \le C\|\Lambda^\beta j\|_2\|j\|_{\frac{2}{\beta}},$$

and the estimate given in [20]

$$\|\partial^\alpha f \cdot \partial^\beta g\|_r \le C(\|f\|_{BMO}\|\Lambda^{|\alpha|+|\beta|}g\|_r + \|g\|_{BMO}\|\Lambda^{|\alpha|+|\beta|}f\|_r).$$

We need the condition (1.3) or (1.5) to guarantee the H^1 estimate (2.5).

If $\alpha > 0$, using the Hölder's inequality and Young's inequality we have

$$\frac{1}{2}\frac{d}{dt}(\|\omega\|_2^2 + \|j\|_2^2) + \|\Lambda^\alpha\omega\|_2^2 + \|\Lambda^\beta j\|_2^2 \le \begin{cases} C\|\omega\|_{\frac{2}{1-\alpha}}\|j\|_{\frac{4}{1+\alpha}}^2 \le \frac{1}{2}\|\Lambda^\alpha\omega\|_2^2 + C\|j\|_2^2\|j\|_{\frac{2}{\alpha}}^2, & 0 < \alpha < 1, \\ \frac{1}{2}\|\nabla\omega\|_2^2 + C\|j\|_2^4, & \alpha \ge 1, \end{cases}$$
$$\le C\|j\|_2^2\|j\|_\rho^2 + \frac{1}{2}\|\Lambda^\alpha\omega\|_2^2. \tag{2.8}$$

Here we have used the embeddings $H^s(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ for $s < \frac{n}{2}$ and $\frac{1}{q} = \frac{1}{2} - \frac{s}{n}$ and $H^1 \hookrightarrow BMO$ in 2D. Finally, use (1.4) we obtain (2.5). Now we complete the H^1 estimation.

As the MHD system stays regular beyond T if and only if $\int_0^T \|\omega\|_{BMO} + \|j\|_{BMO}dt < \infty$. Using the embedding $H^1(\mathbb{R}^2) \hookrightarrow BMO(\mathbb{R}^2)$ in 2D, if $\omega, j \in L^\infty(0,T;H^1(\mathbb{R}^2))$, we can deduce that $\omega, j \in L^\infty(0,T;BMO(\mathbb{R}^2))$. Now, we only have to prove $\omega, j \in L^\infty(0,T;H^1(\mathbb{R}^2))$. Form (2.2), we have

$$\begin{cases} (\partial_i\omega)_t + u\cdot\nabla(\partial_i\omega) + \Lambda^{2\alpha}(\partial_i\omega) = -\partial_i u\cdot\nabla\omega + \partial_i(b\cdot\nabla j), \\ (\partial_i j)_t + u\cdot\nabla(\partial_i j) + \Lambda^{2\beta}(\partial_i j) = -\partial_i u\cdot\nabla j + \partial_i(b\cdot\nabla\omega) + \partial_i Q(\nabla u, \nabla b). \end{cases} \tag{2.9}$$

Multiplying the equation of $\partial_i\omega$, $\partial_i j$ in (2.9) by $\partial_i\omega$, $\partial_i j$ respectively, integrating over $\mathbb{R}^2$ and adding the resulting equations together we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\partial_i\omega\|_2^2+\|\partial_i j\|_2^2)+\|\Lambda^\alpha\partial_i\omega\|_2^2+\|\Lambda^\beta\partial_i j\|_2^2\\
&=-\int_{\mathbb{R}^2}(\partial_i u\cdot\nabla\omega)\partial_i\omega dx+\int_{\mathbb{R}^2}(\partial_i b\cdot\nabla j)\partial_i\omega dx-\int_{\mathbb{R}^2}(\partial_i u\cdot\nabla j)\partial_i j dx\\
&\quad+\int_{\mathbb{R}^2}(\partial_i b\cdot\nabla\omega)\partial_i j dx+\int_{\mathbb{R}^2}\partial_i Q(\nabla u,\nabla b)\partial_i j dx\\
&:=I_1+I_2+I_3+I_4+I_5.
\end{aligned}\tag{2.10}$$

If $\alpha\geq\frac{1}{2}$, $\beta\geq\frac{1}{2}$, the following Galiardo–Nirenberg inequalities will be used in our estimation:

$$\begin{aligned}
&\|\nabla f\|_{L^3}\leq C\|\Lambda^{\frac{1}{2}}f\|_2^{\frac{1}{6}}\|\Lambda^{\frac{1}{2}}\nabla f\|_2^{\frac{5}{6}},\\
&\|\nabla f\|_{L^3}\leq C\|\nabla f\|_2^{\frac{1}{3}}\|\Lambda^{\frac{1}{2}}\nabla f\|_2^{\frac{2}{3}},\\
&\|\nabla f\|_{L^3}=\|\nabla f\|_{L^3}^{\frac{2}{3}}\|\nabla f\|_{L^3}^{\frac{1}{3}}\leq C\|\Lambda^{\frac{1}{2}}f\|_2^{\frac{1}{9}}\|\nabla f\|_2^{\frac{1}{9}}\|\Lambda^{\frac{1}{2}}\nabla f\|_2^{\frac{7}{9}},\\
&\|f\|_{L^3}\leq C\|f\|_2^{\frac{7}{9}}\|\Lambda^{\frac{1}{2}}\nabla f\|_2^{\frac{2}{9}}.
\end{aligned}$$

Now, we we are ready to give the estimate for the right hand of (2.10).

$$\begin{aligned}
I_1&=-\int_{\mathbb{R}^2}(\partial_i u\cdot\nabla\omega)\partial_i\omega dx\leq C\|\omega\|_{L^3}\|\nabla\omega\|_{L^3}^2\\
&\leq C\|\omega\|_{L^3}\|\nabla\omega\|_{L^3}^{\frac{4}{3}}\|\nabla\omega\|_{L^3}^{\frac{2}{3}}\\
&\leq C\|\omega\|_2^{\frac{7}{9}}\|\Lambda^{\frac{1}{2}}\nabla\omega\|_2^{\frac{2}{9}}\|\Lambda^{\frac{1}{2}}\omega\|_2^{\frac{2}{9}}\|\nabla\omega\|_2^{\frac{2}{9}}\|\Lambda^{\frac{1}{2}}\nabla\omega\|_2^{\frac{14}{9}}\\
&\leq C\|\omega\|_2^{\frac{7}{9}}\|\Lambda^{\frac{1}{2}}\omega\|_2^{\frac{2}{9}}\|\nabla\omega\|_2^{\frac{2}{9}}\|\Lambda^{\frac{1}{2}}\nabla\omega\|_2^{\frac{16}{9}}\\
&\leq C(\epsilon)\|\omega\|_2^7\|\Lambda^{\frac{1}{2}}\omega\|_2^2\|\nabla\omega\|_2^2+\epsilon\|\Lambda^{\frac{1}{2}}\nabla\omega\|_2^2.
\end{aligned}$$

$$\begin{aligned}
I_2&=\int_{\mathbb{R}^2}(\partial_i b\cdot\nabla j)\partial_i\omega dx\leq C\|j\|_{L^3}\|\nabla j\|_{L^3}\|\nabla\omega\|_{L^3}\\
&\leq C\|j\|_2^{\frac{7}{9}}\|\Lambda^{\frac{1}{2}}\nabla j\|_2^{\frac{2}{9}}\|\Lambda^{\frac{1}{2}}j\|_2^{\frac{1}{9}}\|\nabla j\|_2^{\frac{1}{9}}\|\Lambda^{\frac{1}{2}}\nabla j\|_2^{\frac{7}{9}}\|\Lambda^{\frac{1}{2}}\omega\|_2^{\frac{1}{9}}\|\nabla\omega\|_2^{\frac{1}{9}}\|\Lambda^{\frac{1}{2}}\nabla\omega\|_2^{\frac{7}{9}}\\
&\leq C\|j\|_2^{\frac{7}{9}}\|\Lambda^{\frac{1}{2}}j\|_2^{\frac{1}{9}}\|\nabla j\|_2^{\frac{1}{9}}\|\Lambda^{\frac{1}{2}}\nabla j\|_2\|\Lambda^{\frac{1}{2}}\omega\|_2^{\frac{1}{9}}\|\nabla\omega\|_2^{\frac{1}{9}}\|\Lambda^{\frac{1}{2}}\nabla\omega\|_2^{\frac{7}{9}}\\
&\leq C(\epsilon)\|j\|_2^7(\|\Lambda^{\frac{1}{2}}j\|_2^2\|\nabla j\|_2^2+\|\Lambda^{\frac{1}{2}}\omega\|_2^2\|\nabla\omega\|_2^2)+\epsilon(\|\Lambda^{\frac{1}{2}}\nabla j\|_2^2+\|\Lambda^{\frac{1}{2}}\nabla\omega\|_2^2).
\end{aligned}$$

$$\begin{aligned}
I_3&=-\int_{\mathbb{R}^2}(\partial_i u\cdot\nabla j)\partial_i j dx\leq C\|\omega\|_{L^3}\|\nabla j\|_{L^3}^2\\
&\leq C\|\omega\|_{L^3}\|\nabla j\|_{L^3}^{\frac{4}{3}}\|\nabla j\|_{L^3}^{\frac{2}{3}}\\
&\leq C(\epsilon)\|\omega\|_2^7\|\Lambda^{\frac{1}{2}}j\|_2^2\|\nabla j\|_2^2+\epsilon(\|\Lambda^{\frac{1}{2}}\nabla j\|_2^2+\|\Lambda^{\frac{1}{2}}\nabla\omega\|_2^2).
\end{aligned}$$

As

$$I_4=\int_{\mathbb{R}^2}(\partial_i b\cdot\nabla\omega)\partial_i j dx\leq C\|j\|_{L^3}\|\nabla j\|_{L^3}\|\nabla\omega\|_{L^3},$$

we can use the same method as I_2 to cope with it.

$$\begin{aligned}
I_5&=\int_{\mathbb{R}^2}\partial_i Q(\nabla u,\nabla b)\partial_i j dx\leq C\|j\|_3\|\nabla j\|_3\|\nabla\omega\|_3+C\|\omega\|_3\|\nabla j\|_3^2\\
&\leq C(\epsilon)(\|j\|_2^7+\|\omega\|_2^7)(\|\Lambda^{\frac{1}{2}}j\|_2^2\|\nabla j\|_2^2+\|\Lambda^{\frac{1}{2}}\omega\|_2^2\|\nabla\omega\|_2^2)+\epsilon(\|\Lambda^{\frac{1}{2}}\nabla j\|_2^2+\|\Lambda^{\frac{1}{2}}\nabla\omega\|_2^2).
\end{aligned}$$

In the above estimation we have also used Hölder's inequality and Young's inequality.

Finally, putting the above results together and setting ϵ suitably small, we deduce

$$\begin{aligned}&\frac{d}{dt}(\|\nabla\omega\|_2^2+\|\nabla j\|_2^2)+\|\Lambda^\alpha\nabla\omega\|_2^2+\|\Lambda^\beta\nabla j\|_2^2\\&\quad\leq C(\|\omega\|_2^7+\|j\|_2^7)(\|\Lambda^{\frac{1}{2}}j\|_2^2\|\nabla j\|_2^2+\|\Lambda^{\frac{1}{2}}\omega\|_2^2\|\nabla\omega\|_2^2).\end{aligned} \tag{2.11}$$

Then, by Gronwall's inequality and (2.5) we have

$$(\|\nabla\omega\|_2^2+\|\nabla j\|_2^2)+\int_0^t\|\Lambda^\alpha\nabla\omega\|_2^2+\|\Lambda^\beta\nabla j\|_2^2dt\leq C(T). \tag{2.12}$$

That is to say $\omega, j\in L^\infty(0,T;H^1(\mathbb{R}^2))$.

The proof of Theorem 1.1 is complete. □

Now we give the proof of Remark 1.1.

Proof. If in addition, $1>\alpha,\beta\geq\frac{1}{2}$, the estimates (2.1) and (2.12) is the same as that in Theorem 1.1. Now we only have to give the estimate (2.5). Using the following inequalities

$$\|\omega\|_p\leq\|\omega\|_2^{1-\rho_\alpha}\|\Lambda^\alpha\omega\|_2^{\rho_\alpha},\quad \frac{1}{p}=\left(\frac{1}{2}-\frac{\alpha}{2}\right)\rho_\alpha+\frac{1-\rho_\alpha}{2},\quad 0\leq\rho_\alpha\leq 1,$$

$$\|j\|_{2q}\leq C\|\Lambda^\beta b\|_s^{1-\theta_s}\|\Lambda^{1+\beta}b\|_2^{\theta_s},\quad \frac{1}{2q}=\frac{1-\beta}{2}+\frac{1-\theta_s}{s},\quad 1-\beta\leq\theta_s\leq 1,$$

and letting $\theta_s=\frac{1}{2}$, we can deduce

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|\omega\|_2^2+\|j\|_2^2)+\|\Lambda^\alpha\omega\|_2^2+\|\Lambda^\beta j\|_2^2&\leq C\|\omega\|_p\|j\|_{2q}^2.\\&\leq C\|\omega\|_2^{1-\rho_\alpha}\|\Lambda^\alpha\omega\|_2^{\rho_\alpha}\|\Lambda^\beta b\|_s^{2-2\theta_s}\|\Lambda^\beta j\|_2^{2\theta_s}\\&\leq C\|\omega\|_2^2\|\Lambda^\beta b\|_s^{\frac{2}{1-\rho_\alpha}}+\epsilon\|\Lambda^\alpha\omega\|_2^2+\epsilon\|\Lambda^\beta j\|_2^2,\end{aligned}$$

where

$$\rho_\alpha=\frac{\frac{1}{2}-\frac{1}{p}}{\frac{\alpha}{2}}=\frac{p-2}{p\alpha},\quad 2\leq p\leq\frac{2}{1-\alpha},$$

$$\theta_s=1-s\left(\frac{\beta}{2}-\frac{1}{2p}\right),\quad s=\frac{p}{p\beta-1}.$$

The regularity criterion (1.6) also guarantees the H^1 estimate (2.5). □

3. Proof of Theorem 1.2

First, we use (1.8) and (2.3) to obtain the H^1 estimation (2.5).

Under the assumption of Theorem 1.2, we can't establish the estimation (2.12) as that in Theorem 1.1. Now, we should give the estimate of $I_i, i=1,2,3,4,5$ which are defined in Sect. 2. If $0<\alpha<\frac{1}{2}$, $\beta>0$ or $\alpha>0$, $0<\beta<\frac{1}{2}$, use (2.4), Hölder's inequality and Young's inequality we obtain

$$\begin{aligned}I_1&=-\int_{R^2}(\partial_i u\cdot\nabla\omega)\partial_i\omega dx\leq C\|\omega\|_p\|\nabla\omega\|_{2q}^2\\&\leq C\|\omega\|_p\|\nabla\omega\|_2^{2(1-\theta_\alpha)}\|\Lambda^{1+\alpha}\omega\|_2^{2\theta_\alpha}\\&\leq C(\epsilon)\|\omega\|_p^{\frac{1}{1-\theta_\alpha}}\|\nabla\omega\|_2^2+\epsilon\|\Lambda^{1+\alpha}\omega\|_2^2,\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$, and $\theta_\alpha=\frac{1-\frac{1}{q}}{\alpha}=\frac{1}{p\alpha}$.

$$\begin{aligned} I_3 &= -\int_{R^2}(\partial_i u\cdot\nabla j)\partial_i j dx \le C\|\omega\|_p\|\nabla j\|_{2q}^2 \\ &\le C\|\omega\|_p\|\nabla j\|_2^{2(1-\theta_\beta)}\|\Lambda^{1+\beta}j\|_2^{2\theta_\beta} \\ &\le C(\epsilon)\|\omega\|_p^{\frac{1}{1-\theta_\beta}}\|\nabla j\|_2^2+\epsilon\|\Lambda^{1+\beta}j\|_2^2. \end{aligned} \tag{3.1}$$

Similarly, $I_i, i=2,4,5$, can be bounded as follows.

$$\begin{aligned} I_2, I_4 &\le C\|j\|_p\|\nabla j\|_{2q}\|\nabla\omega\|_{2q} \\ &\le C(\epsilon)\left(\|j\|_p^{\frac{1}{1-\theta_\beta}}\|\nabla j\|_2^2+\|j\|_p^{\frac{1}{1-\theta_\alpha}}\|\nabla\omega\|_2^2\right)+\epsilon(\|\Lambda^{1+\alpha}\omega\|_2^2+\|\Lambda^{1+\beta}j\|_2^2). \\ I_5 &\le C\|j\|_p\|\nabla j\|_{2q}\|\nabla\omega\|_{2q}+C\|\omega\|_p\|\nabla j\|_{2q}^2 \\ &\le C(\epsilon)[\left(\|j\|_p^{\frac{1}{1-\theta_\beta}}+\|\omega\|_p^{\frac{1}{1-\theta_\beta}}\right)\|\nabla j\|_2^2+\|j\|_p^{\frac{1}{1-\theta_\alpha}}\|\nabla\omega\|_2^2]+\epsilon(\|\Lambda^{1+\alpha}\omega\|_2^2+\|\Lambda^{1+\beta}j\|_2^2). \end{aligned}$$

Finally, Let ϵ suitably small and put the above results together, we deduce

$$\begin{aligned} &\frac{d}{dt}(\|\nabla\omega\|_2^2+\|\nabla j\|_2^2)+\|\Lambda^\alpha\nabla\omega\|_2^2+\|\Lambda^\beta\nabla j\|_2^2 \\ &\quad\le C[(\|j\|_p^{\frac{1}{1-\theta_\beta}}+\|\omega\|_p^{\frac{1}{1-\theta_\beta}})\|\nabla j\|_2^2+(\|j\|_p^{\frac{1}{1-\theta_\alpha}}+\|\omega\|_p^{\frac{1}{1-\theta_\alpha}})\|\nabla\omega\|_2^2]. \end{aligned} \tag{3.2}$$

If we have

$$\int_0^T\|\omega\|_p^{\frac{1}{1-\theta_\alpha}}dt, \int_0^T\|\omega\|_p^{\frac{1}{1-\theta_\beta}}dt, \int_0^T\|j\|_p^{\frac{1}{1-\theta_\alpha}}dt, \int_0^T\|j\|_p^{\frac{1}{1-\theta_\beta}}dt \le C(T). \tag{3.3}$$

By the Gronwall's inequality we can deduce (2.12). As the regularity criteria in Theorem 1.2 covered (3.3), so we finish our proof.

4. Proof of Theorem 1.3

Firstly we give the H^1 estimation for (u,b):

$$\frac{1}{2}\frac{d}{dt}(\|\omega\|_2^2+\|j\|_2^2)+\|\Lambda^\alpha\omega\|_2^2+\|\Lambda^\beta j\|_2^2\le C\|\nabla u\|_\infty\|j\|_2^2.$$

Then use the Gronwall's inequality and (1.11) we obtain (2.5).

In order to give the H^2 estimation for (u,b), we should estimate $I_i, i=1,2,3,4,5$ which are defined in Sect. 2.

$$I_1=-\int_{\mathbb{R}^2}(\partial_i u\cdot\nabla\omega)\partial_i\omega dx\le C\|\nabla u\|_\infty\|\nabla\omega\|_2^2. \tag{4.1}$$

$$I_3=-\int_{\mathbb{R}^2}(\partial_i u\cdot\nabla j)\partial_i j dx\le C\|\nabla u\|_\infty\|\nabla j\|_2^2. \tag{4.2}$$

$$\begin{aligned} I_2, I_4 &\le C\|\nabla j\|_2\|j\|_p\|\nabla\omega\|_q\le\|\nabla j\|_2\|j\|_p\|\nabla\omega\|_2^{1-\theta}\|\Lambda^\alpha\nabla\omega\|_2^\theta \\ &\le C\|j\|_p^{\frac{2}{2-\theta}}(\|\nabla\omega\|_2^2+\|\nabla j\|_2^2)+\epsilon\|\Lambda^\alpha\nabla\omega\|_2^2. \end{aligned}$$

Here $\theta=\frac{2}{p\alpha}$, $p\alpha\ge 2$, $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$. Here we have used the Galiardo–Nirenberg inequality

$$\|\nabla\omega\|_q\le C\|\nabla\omega\|_2^{1-\theta}\|\Lambda^\alpha\nabla\omega\|_2^\theta,\quad \frac{1}{q}=\left(\frac{1}{2}-\frac{\alpha}{2}\right)\theta+\frac{1-\theta}{2},\quad 0\le\theta\le 1,$$

with

$$\theta = \frac{\frac{1}{2} - \frac{1}{q}}{\frac{\alpha}{2}} = \frac{\frac{1}{p}}{\frac{\alpha}{2}} = \frac{2}{p\alpha}.$$

On the other hand we can estimate I_2, I_4 as

$$\begin{aligned} I_2, I_4 &\le \begin{cases} C\|j\|_{\frac{2}{\alpha}} \|\Lambda^\alpha \nabla \omega\|_2 \|\nabla j\|_2 \le C\|j\|^2_{\frac{2}{\alpha}} \|\nabla j\|_2^2 + \epsilon \|\Lambda^{\alpha+1}\omega\|_2^2, & 0 < \alpha < 1, \\ C\|j\|_2^2 \|\nabla j\|_2^2 + \epsilon \|\nabla^2 \omega\|_2^2, & \alpha \ge 1, \end{cases} \\ &\le C\|j\|_\rho^2 \|\nabla j\|_2^2 + \epsilon \|\Lambda^{\alpha+1}\omega\|_2^2. \end{aligned} \tag{4.3}$$

$$I_5 \le C\|\nabla u\|_\infty \|\nabla j\|_2^2 + C\|j\|_p^{\frac{2}{2-\theta}} (\|\nabla j\|_2^2 + \|\nabla \omega\|_2^2) + \epsilon \|\Lambda^{1+\alpha}\omega\|_2^2,$$

or

$$I_5 \le C\|\nabla u\|_\infty \|\nabla j\|_2^2 + C\|j\|_\rho^2 \|\nabla j\|_2^2 + \epsilon \|\Lambda^{\alpha+1}\omega\|_2^2.$$

Finally, let ϵ suitably small and put the above results, we deduce

$$\frac{d}{dt}(\|\nabla \omega\|_2^2 + \|\nabla j\|_2^2) + \|\Lambda^\alpha \nabla \omega\|_2^2 + \|\Lambda^\beta \nabla j\|_2^2 \le C\left(\|\nabla u\|_\infty + \|j\|_p^{\frac{2}{2-\theta}}\right)(\|\nabla j\|_2^2 + \|\nabla \omega\|_2^2),$$

or

$$\frac{d}{dt}(\|\nabla \omega\|_2^2 + \|\nabla j\|_2^2) + \|\Lambda^\alpha \nabla \omega\|_2^2 + \|\Lambda^\beta \nabla j\|_2^2 \le C(\|\nabla u\|_\infty + \|j\|_\rho^2)(\|\nabla j\|_2^2 + \|\nabla \omega\|_2^2),$$

Then, by the Gronwall's inequality, (1.9) or (1.10) we obtain (2.12).

Now we finish our proof.

5. Proof of Theorem 1.4

The H^1 estimation for (u, b) is the same as that in the proof of Theorem 1.3.

For $\alpha = 0$, $\beta > 0$, in order to give the H^2 estimation for (u, b), we should estimate $I_i, i = 1, 2, 3, 4, 5$ which are defined in Sect. 2.

$$\begin{aligned} I_2, I_4 &\le \begin{cases} C\|j\|_{\frac{2}{\beta}} \|\Lambda^\beta \nabla j\|_2 \|\nabla \omega\|_2 \le C\|j\|^2_{\frac{2}{\beta}} \|\nabla \omega\|_2^2 + \epsilon \|\Lambda^\beta \nabla j\|_2^2, & 0 < \beta < 1, \\ C\|j\|_2^2 \|\nabla \omega\|_2^2 + \epsilon \|\nabla^2 j\|_2^2, & \beta \ge 1, \end{cases} \\ &\le C\|j\|_\varrho^2 \|\nabla \omega\|_2^2 + \epsilon \|\Lambda^\beta \nabla j\|_2^2. \end{aligned} \tag{5.1}$$

$$I_5 \le C\|j\|_\varrho^2 \|\nabla \omega\|_2^2 + C\|\nabla u\|_\infty \|\nabla j\|_2^2 + \epsilon \|\Lambda^\beta \nabla j\|_2^2.$$

Finally, let ϵ suitably small and put the above results and (4.1), (4.2) together, we deduce

$$\frac{d}{dt}(\|\nabla \omega\|_2^2 + \|\nabla j\|_2^2) + \|\Lambda^\alpha \nabla \omega\|_2^2 + \|\Lambda^\beta \nabla j\|_2^2 \le C(\|j\|_\varrho^2 + \|\nabla u\|_\infty)(\|\nabla \omega\|_2^2 + \|\nabla j\|_2^2). \tag{5.2}$$

Then, by using the Gronwall's inequality and (1.11) we obtain (2.12).

On the other hand, for $\alpha = 0$, $\beta > 0$, we use (2.7) to give the H^1 estimation for (u, b).

$$\frac{1}{2}\frac{d}{dt}(\|\omega\|_2^2 + \|j\|_2^2) + \|\Lambda^\beta j\|_2^2 \le C\|j\|_\varrho^2 \|\omega\|_2^2 + \frac{1}{2}\|\Lambda^\beta j\|_2^2.$$

Then use the Gronwall's inequality and (1.12) we obtain (2.5).

Now, we give the H^1 estimation for (ω, j). We should give the estimation of $I_i, i = 1, 2, 3, 4, 5$ which are defined in Sect. 2.

$$
\begin{aligned}
I_1 &= -\int_{\mathbb{R}^2} (\partial_i u \cdot \nabla\omega)\partial_i \omega dx \le C\|\omega\|_{BMO}\|\nabla\omega\|_2^2. \\
I_3 &\le \begin{cases} C\|\omega\|_{\frac{2}{\beta}}\|\Lambda^\beta \nabla j\|_2\|\nabla j\|_2 \le C\|\omega\|_{\frac{2}{\beta}}^2\|\nabla j\|_2^2 + \epsilon\|\Lambda^\beta\nabla j\|_2^2, & 0 < \beta < 1, \\ C\|\omega\|_2^2\|\nabla j\|_2^2 + \epsilon\|\nabla^2 j\|_2^2, & \beta \ge 1, \end{cases} \\
&\le \|\omega\|_\varrho^2\|\nabla j\|_2^2 + \epsilon\|\Lambda^{\beta+1} j\|_2^2.
\end{aligned} \tag{5.3}
$$

We use (4.3) to estimate I_2 and I_4.

$$
I_5 \le C\|j\|_\varrho^2\|\nabla\omega\|_2^2 + C\|\omega\|_\varrho^2\|\nabla j\|_2^2 + \epsilon\|\Lambda^\beta\nabla j\|_2^2.
$$

Finally, let ϵ suitably small and put the above results together, we deduce

$$
\frac{d}{dt}(\|\nabla\omega\|_2^2 + \|\nabla j\|_2^2) + \|\Lambda^\alpha\nabla\omega\|_2^2 + \|\Lambda^\beta\nabla j\|_2^2 \le C(\|\omega\|_{BMO} + \|j\|_\varrho^2 + \|\omega\|_\varrho^2)(\|\nabla j\|_2^2 + \|\nabla\omega\|_2^2).
$$

By the Gronwall's inequality and (1.12), we obtain (2.12).

Now we complete our proof.

From the proof of our Theorems we can also give the following regularity criterion

Theorem 5.1. *Let $\alpha, \beta > 0$. Suppose $u_0(x), b_0(x) \in H^2(\mathbb{R}^2)$ and $(u, b)(x, t)$ is a local smooth solution of the system (1.1). If*

$$
\int_0^T (\|\omega(\cdot, t)\|_\rho^2 + \|\omega(\cdot, t)\|_\varrho^2 + \|j(\cdot, t)\|_\rho^2)dt \le C(T), \tag{5.4}
$$

or

$$
\int_0^T (\|\omega(\cdot, t)\|_\rho^2 + \|\omega(\cdot, t)\|_\varrho^2 + \|j(\cdot, t)\|_\varrho^2)dt \le C(T). \tag{5.5}
$$

Then the solution remains smooth on $(0, T]$. Use this regularity criteria we can prove the existence of global regularity when $\alpha + \beta \ge 2$, $0 < \beta < 1$.

Proof. The H^1 estimation for (u, b) is given by (2.7), (2.8) or (5.3). In order to give the H^2 estimation for (u, b), we should estimate $I_i, i = 1, 2, 3, 4, 5$ which are defined in Sect. 2.

$$
I_1 = -\int_{\mathbb{R}^2} (\partial_i u \cdot \nabla\omega)\partial_i\omega dx \le C\|\omega\|_\rho^2\|\nabla\omega\|_2^2 + \epsilon\|\Lambda^{1+\alpha}\omega\|_2^2.
$$

Finally, let ϵ suitably small and put the above result, (4.3), (5.1) and (5.3) together, we deduce the H^2 estimation for (u, b).

In order to prove the global regularity we only have to prove $(u, b)(x, t)$ satisfies (5.4) or (5.5).

If $0 < \beta < 1$, we have $\alpha > 1$. By using the Galiardo–Nirenberg inequality

$$
\|\omega\|_{\frac{2}{\beta}} \le \|u\|_2^{1-\theta}\|\Lambda^\alpha u\|_2^\theta, \quad 0 \le \theta = \frac{2-\beta}{\alpha} \le 1, \tag{5.6}
$$

we have

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}(\|\omega\|_2^2 + \|j\|_2^2) + \|\Lambda^\alpha\omega\|_2^2 + \|\Lambda^\beta j\|_2^2 &= \int_{\mathbb{R}^2} Q(\nabla u, \nabla b) j dx \\
&\le C\|\Lambda^\alpha u\|_2^{2\theta}\|j\|_2^2 + \|\Lambda^\beta j\|_2^2.
\end{aligned}
$$

Then we deduce (5.4). □

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