J. Math. Fluid Mech. 17 (2015), 599–626

1422-6928/15/040599-28
DOI 10.1007/s00021-015-0209-6

Journal of Mathematical Fluid Mechanics

Weighted Decay Results for the Nonstationary Stokes Flow and Navier–Stokes Equations in Half Spaces

Pigong Han

Communicated by D. Chae

Abstract. The weighted L^q-L^q ($q=1,\infty$) estimates for the Stokes flow are given in half spaces. Further large-time weighted decays for the second spatial derivatives of the Navier–Stokes equations are established, where the unboundedness of the projection operator $P:\ L^q(\mathbb{R}^n_+)\to L^q_\sigma(\mathbb{R}^n_+)$ ($q=1,\infty$) is overcome by employing a decomposition for the convection term. The main results in this article are motivated by the work in Bae (J Differ Equ 222:1–20, 2006; J Math Fluid Mech 10:503–530, 2008) and Bae and Jin (Proc R Soc Edinb Sect A 135:461–477, 2005).

Mathematics Subject Classification. 35Q35, 35B40, 75D05, 76D07.

Keywords. Navier–Stokes equations, Stokes solution formula, large-time behavior, strong solution.

1. Introduction and Main Results

The time behavior of strong solutions to the Navier–Stokes initial-value problem is considered in the half space

$$\begin{cases} \partial_t u-\Delta u+(u\cdot\nabla)u+\nabla p=0 & \text{in } \mathbb{R}^n_+\times(0,\infty),\\ \nabla\cdot u=0 & \text{in } \mathbb{R}^n_+\times(0,\infty),\\ u(x,t)=0 & \text{on } \partial\mathbb{R}^n_+\times(0,\infty),\\ u(x,0)=u_0 & \text{in } \mathbb{R}^n_+, \end{cases} \tag{1.1}$$

where $n\geq 2$, and $\mathbb{R}^n_+=\{x=(x',x_n)\in\mathbb{R}^n\mid x_n>0\}$ is the upper-half space of $\mathbb{R}^n$; $u=(u_1(x,t),u_2(x,t),\ldots,u_n(x,t))$ and $p=p(x,t)$ denote unknown velocity vector and the pressure respectively, while initial data $u_0(x)$ is assumed to satisfy a *compatibility condition*: $\nabla\cdot u_0=0$ in $\mathbb{R}^n_+$ and the normal component of u_0 equals to zero on the boundary $\partial\mathbb{R}^n_+$.

Now we introduce and explain some notations which are frequently used in this article: Let $\mathbb{N}$ denote the positive integer set; $C^\infty_{0,\sigma}(\mathbb{R}^n_+)=\{u\in C^\infty_0(\mathbb{R}^n_+);\ \nabla\cdot u=0 \text{ in } \mathbb{R}^n_+\}$; $L^q_\sigma(\mathbb{R}^n_+)$ $(1\leq q<\infty)$ denotes the closure of $C^\infty_{0,\sigma}(\mathbb{R}^n_+)$ with respect to $\|\cdot\|_{L^q(\mathbb{R}^n_+)}$, where $L^q(\mathbb{R}^n_+)$ represents the usual Lebesgue space of vector-valued functions. The norm of $L^\infty(\mathbb{R}^n_+)$ is given by $\|u\|_{L^\infty(\mathbb{R}^n_+)}=ess\ \sup_{x\in\mathbb{R}^n_+}|u(x)|$. By symbol C, it means a generic positive constant whose value may change from line to line.

u is called a weak solution of (1.1) if $u\in L^\infty(0,\infty;L^2_\sigma(\mathbb{R}^n_+))\bigcap L^2_{loc}([0,\infty);H^1_0(\mathbb{R}^n_+))$ satisfies problem (1.1) in the sense of distribution, with initial data $u_0\in L^2_\sigma(\mathbb{R}^n_+)$. Moreover, the energy inequality holds for almost all $t\in[0,\infty)$ including $t=0$: $\|u(t)\|^2_{L^2(\mathbb{R}^n_+)}+2\int_0^t\|\nabla u(s)\|^2_{L^2(\mathbb{R}^n_+)}ds\leq\|u_0\|^2_{L^2(\mathbb{R}^n_+)}$. Further we call u is a strong solution of (1.1) if u satisfies Serrin's condition: $u\in L^q(0,\infty;L^r(\mathbb{R}^n_+))$ with $\frac{2}{q}+\frac{n}{r}\leq 1$, $2\leq q<\infty$, $n<r\leq\infty$.

Let A denote the Stokes operator $-P\Delta$ in $\mathbb{R}^n_+$, where P is the projection: $L^r(\mathbb{R}^n_+)\longrightarrow L^r_\sigma(\mathbb{R}^n_+)$, $1<r<\infty$. Then the function $e^{-tA}u_0$ solves the Stokes system, that is, problem (1.1) with $u\cdot\nabla u$

Birkhäuser

deleted, with the initial data u_0. It is not difficult to find that one weak solution u of (1.1) with initial data $u_0 \in L^2_\sigma(\mathbb{R}^n_+)$ can be rewritten as

$$u(t) = e^{-tA}u_0 - \int_0^t e^{-(t-s)A}Pu(s)\cdot\nabla u(s)ds.$$

Observe that $e^{-(t-s)A}Pu(s)\cdot\nabla u(s)$, $t > s \geq 0$ is another Stokes flow with initial data $Pu(s)\cdot\nabla u(s)$. So in order to study properties of Navier–Stokes equations, it is necessary to analyze the structures of the Stokes flow.

Regularity-decay estimates like $L^q - L^r$ estimates are extensively studied especially for $1 < q, r < \infty$. If $\Omega = \mathbb{R}^n$, the Stokes flow $e^{-tA}u_0$ is reduced to a solution of the heat equation with initial data u_0 because there is no boundary condition. However few results are available for $q = r = 1, \infty$ in the cases where the boundary $\partial\Omega$ is nonempty. The main difficulty lies in the fact that the projection associated with the Helmholtz decomposition is not bounded in L^1 and L^∞ type spaces, because it involves singular integral operators such as Riestz operators. For examples, it is known that $\|\nabla e^{-tA}u_0\|_{L^q(\mathbb{R}^n_+)} \leq Ct^{-\frac{1}{2}}\|u_0\|_{L^q(\mathbb{R}^n_+)}$ with $1 < q < \infty$ (see [41]). Furthermore if $u_0 \in L^1_\sigma(\mathbb{R}^n_+)$ satisfies some additional conditions, Bae and Choe [3] showed that the decay rate of $\nabla e^{-tA}u_0$ in $L^q(\mathbb{R}^n_+)$ $(1 < q < \infty)$ could be controlled by t^{-1} times a constant. If the initial data u_0 lies in an appropriate weighted space, Bae [1,2] estimated the time decay rates of the gradient of Stokes solutions in $L^1(\mathbb{R}^n_+)$. Jin [29,30] obtained some weighted spatial and temporal decay estimates in a half space. Recently Han [23] established the weighted decay properties for the Stokes flow: Let $u_0 = (u_{01}, u_{02}, \cdots, u_{0n})$ satisfy $\nabla\cdot u_0 = 0$ in $\mathbb{R}^n_+$ $(n \geq 2)$, and $u_{0n}|_{\partial\mathbb{R}^n_+} = 0$. Then for $0 \leq \beta \leq 1$ and $t > 0$

$$\|\nabla^k e^{-tA}u_0\|_{L^q(\mathbb{R}^n_+)} \leq Ct^{-\frac{\beta}{2}-\frac{k}{2}-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\|x_n^\beta u_0\|_{L^r(\mathbb{R}^n_+)}, \qquad k = 0, 1, 2, \ldots,$$

provided either $1 \leq r < q \leq \infty$ or $1 < r \leq q < \infty$. Here it should be pointed out that the $L^q - L^r$ estimates with $q = r = 1, \infty$ for the Stokes flow $e^{-tA}u_0$ are not established by the author in [23], and the methods employed in [23] are not applicable directly to such cases.

Theorem 1.1. *Let $a = (a_1, a_2, \ldots, a_n)$ satisfy $\nabla\cdot a = 0$ in $\mathbb{R}^n_+$ $(n \geq 2)$, and $a_n|_{\partial\mathbb{R}^n_+} = 0$. Then for $k \in \mathbb{N}$, $0 \leq \beta \leq 1$ and $t > 0$*

$$\|\nabla^k e^{-tA}a\|_{L^1(\mathbb{R}^n_+)} \leq Ct^{-\frac{\beta}{2}-\frac{k}{2}}\||x|^\beta a\|_{L^1(\mathbb{R}^n_+)}; \tag{1.2}$$

and

$$\|\nabla^k e^{-tA}a\|_{L^\infty(\mathbb{R}^n_+)} \leq Ct^{-\frac{k+1}{2}}\||x|a\|_{L^\infty(\mathbb{R}^n_+)}. \tag{1.3}$$

Remark. Obviously the conclusions in Theorem 1.1 with $\beta = 0$ can be regarded as a complement to the results in [20]: Let a be given in Theorem 1.1. Then for $t > 0$

$$\|\nabla^k e^{-tA}a\|_{L^q(\mathbb{R}^n_+)} \leq Ct^{-\frac{k}{2}-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\|a\|_{L^r(\mathbb{R}^n_+)}, \qquad k \in \mathbb{N},$$

provided either $1 \leq r < q \leq \infty$ or $1 < r \leq q < \infty$.

Let a be given in Theorem 1.1. We are not sure whether the following estimate is true for $0 < \alpha < 1$ and $t > 0$

$$\|\nabla^k e^{-tA}a\|_{L^\infty(\mathbb{R}^n_+)} \leq Ct^{-\frac{k}{2}-\frac{\alpha}{2}}\||x|^\alpha a\|_{L^\infty(\mathbb{R}^n_+)}, \qquad k \in \mathbb{N}. \tag{1.4}$$

For example, let $k=1$, $0<\alpha<1$, and $\ell_1,\, m_1\geq 0$, $\ell_1+m_1\leq n+1+\alpha$. It follows from (2.33) below that for $x=(x',x_n)\in\mathbb{R}^n$ and $t>0$

$$\begin{aligned}
&\|\sup_{s>0}|G_s*k_0(\alpha,x,\cdot)(y)|\|_{L^1(\mathbb{R}^n_y)}\\
&\quad\leq\sum_{k=1}^{2}C_{\ell_1,m_1}t^{\frac{\ell_1+m_1-n-1}{2}-\frac{\alpha}{2}}\int_{\widetilde{\Omega}_k}|y'|^{-\ell_1}dy'\\
&\qquad\times\bigg(\int_{\mathbb{R}^1}|y_n|^{-m_1(1-\alpha)}|(1-\tau_0)x_n+\tau_0y_n|^{-m_1\alpha}dy_n\\
&\qquad+|x_n|^{-m_1(1-\alpha)}\tau_0^{-m_1\alpha}\int_{\mathbb{R}^1}|y_n|^{-m_1\alpha}dy_n\bigg)\\
&\quad=+\infty\ \text{ if }\ m_1=0;\ \text{ and }\ \longrightarrow\infty\quad\text{as}\quad x_n\longrightarrow 0\ \text{ if }\ m_1>0,
\end{aligned}$$

where $\tau_0\in(0,1)$, $\widetilde{\Omega}_k$ $(k=1,2)$ are given in the proof of (2.11) below, $k_0(\alpha,x,y)$ is from (2.32) below.
This shows for $t>0$

$$\sup_{x\in\mathbb{R}^n}\|k_0(\alpha,x,\cdot)\|_{\mathcal{H}^1(\mathbb{R}^n)}=\sup_{x\in\mathbb{R}^n}\|\sup_{s>0}|G_s*k_0(\alpha,x,\cdot)(y)|\|_{L^1(\mathbb{R}^n_y)}\leq\infty. \tag{1.5}$$

Similarly, let $\ell_2,\, m_2\geq 0$, $0\leq\ell_2+m_2\leq n+1+\alpha$, and $k_1(\alpha,x,y)$ is from (2.36) below. It follows from (2.37) below that for $x=(x',x_n)\in\mathbb{R}^n$ and $t>0$

$$\begin{aligned}
&\|\sup_{s>0}|G_s*k_1(\alpha,x,\cdot)|\|_{L^1(\mathbb{R}^n)}\\
&\quad\leq\sum_{k=1}^{2}C_{\ell_2,m_2}t^{\frac{\ell_2+m_2-n-1}{2}-\frac{\alpha}{2}}\int_{\widetilde{\Omega}_k}|z'|^{-\ell_2}dz'\\
&\qquad\times\bigg(\int_{\mathbb{R}^1}|z_n|^{-m_2(1-\alpha)}|(1-\tau_1)x_n+\tau_1z_n|^{-m_2\alpha}dz_n\\
&\qquad+|x_n|^{-m_2(1-\alpha)}\tau_1^{-m_2\alpha}\int_{\mathbb{R}^1}|z_n|^{-m_2\alpha}dz_n\bigg),\quad\tau_1\in(0,1)\\
&\quad=+\infty\ \text{ if }\ m_2=0;\ \text{ and }\ \longrightarrow\infty\quad\text{as}\quad x_n\longrightarrow 0\ \text{ if }\ m_2>0,
\end{aligned}$$

from which,

$$\sup_{x\in\mathbb{R}^n}\|k_1(\alpha,x,\cdot)\|_{\mathcal{H}^1(\mathbb{R}^n)}=\sup_{x\in\mathbb{R}^n}\|\sup_{s>0}|G_s*k_1(\alpha,x,\cdot)|\|_{L^1(\mathbb{R}^n)}\leq\infty. \tag{1.6}$$

In addition, from (2.42) below, we also have for $t>0$

$$\sup_{x\in\mathbb{R}^n}\|k_2(\alpha,x,\cdot)\|_{\mathcal{H}^1(\mathbb{R}^n)}\leq\infty, \tag{1.7}$$

where $k_2(\alpha,x,y)$ is from (2.41) below.

From (1.5)–(1.7), we can not find a finite upper bound to control $\sup_{x\in\mathbb{R}^n}\|k_j(\alpha,x,\cdot)\|_{\mathcal{H}^1(\mathbb{R}^n)}$, $j=0,1,2$. Then by checking the proof of Lemma 2.2, we are not sure whether the estimate of (1.4) is valid.

The second main result in this article deals with another class of weighted L^q-L^r estimates for the Stokes flow. That is,

Theorem 1.2. *Let $u_0=(u_{01},u_{02},\ldots,u_{0n})$ satisfy $\nabla\cdot u_0=0$ in $\mathbb{R}^n_+$ $(n\geq 2)$, and $u_{0n}|_{\partial\mathbb{R}^n_+}=0$. Assume $1\leq r<q\leq\infty$ or $1<r\leq q\leq\infty$. Then for $k\in\{0\}\bigcup\mathbb{N}$, $0\leq\alpha<k+n(\frac{1}{r}-\frac{1}{q})$ and $t>0$*

$$\begin{aligned}
&\|x_n^{\alpha}\nabla^k e^{-tA}u_0\|_{L^q(\mathbb{R}^n_+)}\leq C(t^{\frac{\alpha}{2}-\frac{k}{2}-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\|u_0\|_{L^r(\mathbb{R}^n_+)}+t^{-\frac{k}{2}-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\|y_n^{\alpha}u_0\|_{L^r(\mathbb{R}^n_+)});\\
&\||x'|^{\alpha}\nabla^k e^{-tA}u_0\|_{L^q(\mathbb{R}^n_+)}\leq C(t^{\frac{\alpha}{2}-\frac{k}{2}-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\|u_0\|_{L^r(\mathbb{R}^n_+)}+t^{-\frac{k}{2}-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\||y'|^{\alpha}u_0\|_{L^r(\mathbb{R}^n_+)}).
\end{aligned}$$

Remark. Let u_0, k, α, q, r be given in Theorem 1.2. It is not hard to find from Theorem 1.2 that for $t > 0$

$$\||x|^\alpha \nabla^k e^{-tA} u_0\|_{L^q(\mathbb{R}^n_+)} \le C(t^{\frac{\alpha}{2}-\frac{k}{2}-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\|u_0\|_{L^r(\mathbb{R}^n_+)} + t^{-\frac{k}{2}-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\||y|^\alpha u_0\|_{L^r(\mathbb{R}^n_+)}).$$

In recent years, much attention has been paid to the Navier–Stokes equations. Bae and Choe [4], Bae and Jin [5], Caffarelli et al. [14], Lin [28], Chae [15–19] considered the regularity and related topics on the Navier–Stokes equations, which brings about a deeper understanding on the regularity theory and narrowed the gap between what we can get from the global existence theory and what we need for the uniqueness and regularity of global weak solutions to the incompressible Navier–Stokes equations.

Bae and Choe [3], Bae and Jin [6–10], Brandolese [11,12], Brandolese and Vigneron [13], Fujigaki and Miyakawa [20], Schonbek [31–36] considered the asymptotic behavior for weak and strong solutions of (1.1) on the whole space $\mathbb{R}^n$, and many important and interesting results are obtained. Bae and Jin [9], He and Wang [27] showed the weighted energy inequality for solutions of the Cauchy problem under some conditions on the initial data. The situation changes in the case of domains with boundary; the difficulty comes from the lack of the weighted estimate with respect to pressure because of the appearance and the non-compactness of the boundary.

To our knowledge, few weighted decay results are available on the second derivatives of solutions of (1.1) in $L^r(\mathbb{R}^n_+)$ with $1 \le r \le \infty$. In this article, we use Solonnikov's solution formula and a decomposition for the convection term of (1.1), not the integral formula of solutions of (1.1), to establish the weighted decays.

Theorem 1.3. *Assume* $u_0 \in L^1(\mathbb{R}^n_+) \bigcap L^q_\sigma(\mathbb{R}^n_+)$ $(n \ge 2)$ *for all* $1 < q < \infty$*, satisfies* $\||x'|u_0\|_{L^1(\mathbb{R}^n_+)} + \||x'|u_0\|_{L^2(\mathbb{R}^n_+)} + \|(1+|x'|)\nabla u_0\|_{L^2(\mathbb{R}^n_+)} < \infty$. *There exists a number* $\eta_0 > 0$ *such that if* $\|u_0\|_{L^n(\mathbb{R}^n_+)} \le \eta_0$ (*smallness condition is unnecessary if* $n = 2$), *then problem* (1.1) *possesses a unique strong solution* u *which satisfies for* $t \ge t_0$ *with some number* $t_0 \ge 1$

$$\||x'|^\beta \nabla^2 u(t)\|_{L^r(\mathbb{R}^n_+)} \le C t^{-1-\frac{n}{2}(1-\frac{1}{r})+\frac{\beta}{2}} \quad \text{with } 1 \le r \le \infty,\ 0 < \beta < 1,\ n \ge 3.$$

Further if $\|x_n u_0\|_{L^1(\mathbb{R}^n_+)} < \infty$ *for* $n = 2$*, then for* $t \ge t_0$

$$\||x'|^\beta \nabla^2 u(t)\|_{L^r(\mathbb{R}^n_+)} \le C t^{-\frac{1}{2}-(1-\frac{1}{r})+\frac{\beta}{2}} \quad \text{with } n = 2.$$

Remark. Let $x = (x', x_n), y = (y', y_n) \in \mathbb{R}^n_+$ and r, β be given in Theorem 1.3. Recently, making full use of the structure of the half space and Solonnikov's solution formula, the author [26] could avoid the singularity caused by the weight x_n^β, and established the decay of $\|x_n^\beta \nabla^2 u(t)\|_{L^r(\mathbb{R}^n_+)}$. However, it becomes hard to handle another class of decays: $\||x'|^\beta \nabla^2 u(t)\|_{L^r(\mathbb{R}^n_+)}$, the main reason is that the weight $|y'|^\beta$ results in the strong singularity in Solonnikov's solution formula (this weight comes from the inequality: $|x'|^\beta \le |x'-y'|^\beta + |y'|^\beta$). In order to overcome this difficulty, we have to search new ideas and technical approaches, see Lemma 3.1 below for the details.

2. Weighted Decays for the Stokes Flows

It is well known that the Hardy space $\mathcal{H}^q(\mathbb{R}^n)$ with $1 \le q < \infty$ (see the definition in [40] for example) is a Banach space, and $\mathcal{H}^q(\mathbb{R}^n) = L^q(\mathbb{R}^n)$ if $1 < q < \infty$; $\mathcal{H}^1(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$. The crucial fact for our purpose is the boundedness of the Riesz transforms R_j $(1 \le j \le n)$ on all of the spaces $\mathcal{H}^q(\mathbb{R}^n)$ with $1 \le q < \infty$. Furthermore, a function $f \in L^1(\mathbb{R}^n)$ belongs to the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$ if $\sup_{s>0} |G_s * f(x)| \in L^1(\mathbb{R}^n)$, where the symbol $*$ denotes the convolution with respect to the space variable $x \in \mathbb{R}^n$, G_s is the Gauss kernel $G_s(x) = (4\pi s)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4s}}$, $s > 0$. The norm of $f \in \mathcal{H}^1(\mathbb{R}^n)$ can be defined by

$$\|f\|_{\mathcal{H}^1(\mathbb{R}^n)} =: \|\sup_{s>0} |G_s * f|\|_{L^1(\mathbb{R}^n)}.$$

The Hardy space on the half space is denoted by $\mathcal{H}^q(\mathbb{R}^n_+)$ $(1 \le q < \infty)$, whose norm can be defined by

$$\|f\|_{\mathcal{H}^q(\mathbb{R}^n_+)} =: \inf\left\{\|\widetilde{f}\|_{\mathcal{H}^q(\mathbb{R}^n)} \mid \widetilde{f} \in \mathcal{H}^q(\mathbb{R}^n),\ \widetilde{f}|_{\mathbb{R}^n_+} = f\right\}.$$

Let $\mathcal{F}$ be the Fourier transform in $\mathbb{R}^n$:

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x)dx.$$

The Riesz operators R_j $(j = 1, 2, \ldots, n)$, S_j $(j = 1, 2, \ldots, n-1)$, and the operator Λ are defined by

$$\mathcal{F}(R_j f)(\xi) = \frac{i\xi_j}{|\xi|}\mathcal{F}f(\xi), \quad \mathcal{F}(S_j f)(\xi) = \frac{i\xi_j}{|\xi'|}\mathcal{F}f(\xi), \quad \mathcal{F}(\Lambda f)(\xi) = |\xi'|\mathcal{F}f(\xi),$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_{n-1}, \xi_n) = (\xi', \xi_n) \in \mathbb{R}^{n-1} \times \mathbb{R}^1$.

Let r be the restriction operator from $\mathbb{R}^n$ to $\mathbb{R}^n_+$, and e is the extension operator from $\mathbb{R}^n_+$ to $\mathbb{R}^n$, which is defined by

$$ef(x) = \begin{cases} f(x) & \text{if } x_n \ge 0, \\ 0 & \text{if } x_n < 0. \end{cases}$$

Define the operators $E(t)$ and $F(t)$ by

$$E(t)f(x) = \int_{\mathbb{R}^n_+} [G_t(x' - y', x_n - y_n) - G_t(x' - y', x_n + y_n)]f(y)dy,$$

and

$$F(t)f(x) = \int_{\mathbb{R}^n_+} [G_t(x' - y', x_n - y_n) + G_t(x' - y', x_n + y_n)]f(y)dy.$$

By the solution formula in [41], the Stokes flow $u = (u', u_n) = e^{-tA}u_0$ can be represented as

$$\begin{cases} u_n = UE(t)V_1u_0, \\ u' = E(t)V_2u_0 - SUE(t)V_1u_0, \end{cases} \tag{2.1}$$

where the operators are defined by $Uf = rR' \cdot S(R' \cdot S + R_n)ef$, $V_1u_0 = -S \cdot u_0' + u_{0n}$, $V_2u_0 = u_0' + Su_{0n}$, $R' = (R_1, R_2, \ldots, R_{n-1})$, $S = (S_1, S_2, \ldots, S_{n-1})$, $u_0 = (u_{01}, u_{02}, \ldots, u_{0n}) = (u_0', u_{0n})$.

Note that the Stokes flow $u = (u', u_n) = e^{-tA}u_0$ is given as a restriction $r\overline{u}$ of one vector field $\overline{u} = (\overline{u}', \overline{u}_n)$:

$$\overline{u}_n = R' \cdot S(R' \cdot S + R_n)eE(t)V_1u_0 = R' \cdot S(R' \cdot S + R_n)(-S \cdot eE(t)u_0' + eE(t)u_{0n}); \tag{2.2}$$

$$\overline{u}' = E(t)V_2u_0 - SR' \cdot S(R' \cdot S + R_n)eE(t)V_1u_0 = E(t)u_0' + SE(t)u_{0n} - S\overline{u}_n. \tag{2.3}$$

Lemma 2.1. *Let $a = (a', a_n) = (a_1, a_2, \ldots, a_n)$ satisfy $\nabla \cdot a = 0$ in $\mathbb{R}^n_+$ $(n \ge 2)$, and $a_n|_{\partial\mathbb{R}^n_+} = 0$. Then for $0 \le \beta \le 1$ and $t > 0$*

$$\|\nabla e^{-tA}a\|_{L^1(\mathbb{R}^n_+)} \le Ct^{-\frac{1}{2}-\frac{\beta}{2}}\||x|^\beta a\|_{L^1(\mathbb{R}^n_+)}.$$

Proof. Note that the Stokes flow $e^{-tA}a$ is given as a restriction $r\overline{u}$ of one vector field $\overline{u} = (\overline{u}', \overline{u}_n)$, where $\overline{u}_n, \overline{u}'$ are given in (2.2), (2.3) respectively, with initial data $u_0 = a$. Moreover it follows from Lemma 1.2 in [21] that for $1 \le j \le n$ and $t > 0$

$$\partial_j\overline{u}_n = -R_j[R' \cdot \Lambda eE(t)a' - R_n\nabla' \cdot eE(t)a' + R' \cdot \nabla' eE(t)a_n + R_n\Lambda eE(t)a_n]; \tag{2.4}$$

$$\begin{aligned}\partial_j\overline{u}' = \partial_j E(t)a' + \partial_j\nabla'\Lambda^{-1}E(t)a_n + R_j[R'\nabla' \cdot eE(t)a' - R_n\nabla'\nabla'\Lambda^{-1} \cdot eE(t)a' \\ -R'\Lambda eE(t)a_n + R_n\nabla' eE(t)a_n],\end{aligned} \tag{2.5}$$

where $\nabla' = (\partial_1, \partial_2, \ldots, \partial_{n-1})$.

Denote the odd and even extensions of a function f from $\mathbb{R}^n_+$ to $\mathbb{R}^n$ respectively by

$$f^*(x',x_n) = \begin{cases} f(x',x_n) & \text{if } x_n \geq 0, \\ -f(x',-x_n) & \text{if } x_n < 0, \end{cases} \quad \text{and} \quad f_*(x',x_n) = \begin{cases} f(x',x_n) & \text{if } x_n \geq 0, \\ f(x',-x_n) & \text{if } x_n < 0. \end{cases}$$

Note that $G_{s+t} = G_s * G_t$, $\forall s,t > 0$. Let $1 \leq j \leq n$, $0 < \alpha \leq 1$. Then for any $x = (x', x_n) \in \mathbb{R}^n$ and $s,t > 0$

$$\begin{aligned} |G_s * [\partial_j E(t) f](x)| &= |G_s * [\partial_j G_t * f^*](x)| = |\partial_j G_{s+t} * f^*(x)| \\ &= \left| \int_{\mathbb{R}^n} \big(\partial_{x_j} G_{s+t}(x'-y', x_n - y_n) - \partial_{x_j} G_{s+t}(x'-y', x_n)\big) f^*(y',y_n) dy' dy_n \right| \\ &\leq \int_{\mathbb{R}^n} \left|\partial_{x_j} G_{s+t}(x'-y', x_n - y_n) - \partial_{x_j} G_{s+t}(x'-y', x_n)\right|^{1-\alpha} \\ &\quad \times \left| \int_0^1 \partial_{x_j} \partial_{x_n} G_{s+t}(x'-y', x_n - \tau y_n) d\tau \right|^{\alpha} |y_n|^{\alpha} |f^*(y',y_n)| dy' dy_n. \end{aligned} \tag{2.6}$$

For $1 \leq j \leq n$, $\ell_1, m_1 \geq 0$ and $s,t > 0$,

$$\begin{aligned} &|\partial_{x_j} G_{s+t}(x'-y', x_n - y_n) - \partial_{x_j} G_{s+t}(x'-y', x_n)| \\ &\quad \leq C_{\ell_1,m_1} (s+t)^{\frac{\ell_1+m_1-n-1}{2}} |x'-y'|^{-\ell_1} \big(|x_n - y_n|^{-m_1} + |x_n|^{-m_1}\big); \end{aligned}$$

and

$$\begin{aligned} &\left| \int_0^1 \partial_{x_j} \partial_{x_n} G_{s+t}(x'-y', x_n - \tau y_n) d\tau \right| \\ &\quad \leq C(s+t)^{-1} \int_0^1 G_{s+t}(x'-y', x_n - \tau y_n) d\tau \\ &\quad \leq C_{\ell_1,m_1} (s+t)^{\frac{\ell_1+m_1-n-2}{2}} |x'-y'|^{-\ell_1} \int_0^1 |x_n - \tau y_n|^{-m_1} d\tau. \end{aligned}$$

Therefore we conclude from (2.6) that for $s,t > 0$

$$\begin{aligned} &\big|G_s * [\partial_j E(t) f](x)\big| \\ &\quad \leq \int_{\mathbb{R}^n} C_{\ell_1,m_1} t^{\frac{\ell_1+m_1-n-1}{2} - \frac{\alpha}{2}} |x'-y'|^{-\ell_1} (|x_n - y_n|^{-m_1} + |x_n|^{-m_1})^{1-\alpha} \\ &\quad\quad \times \left(\int_0^1 |x_n - \tau y_n|^{-m_1} d\tau \right)^{\alpha} |y_n|^{\alpha} |f^*(y',y_n)| dy' dy_n, \end{aligned} \tag{2.7}$$

where $1 \leq j \leq n$, ℓ_1, $m_1 \geq 0$, $\ell_1 + m_1 \leq n + 1 + \alpha$, $0 < \alpha \leq 1$.

Let $\alpha = 1$ in (2.7), we get for $t > 0$

$$\begin{aligned} &\| \sup_{s>0} |G_s * [\partial_j E(t) f]| \|_{L^1(\mathbb{R}^n)} \\ &\quad \leq \int_{\mathbb{R}^n} C_{\ell_1,m_1} t^{\frac{\ell_1+m_1-n-1}{2} - \frac{1}{2}} |x'|^{-\ell_1} |x_n|^{-m_1} dx' dx_n \int_{\mathbb{R}^n} |y_n| |f^*(y',y_n)| dy' dy_n \\ &\quad \leq \sum_{k=1}^{4} C_{\ell_1,m_1} t^{\frac{\ell_1+m_1-n-2}{2}} \int_{\Omega_k} |x'|^{-\ell_1} |x_n|^{-m_1} dx' dx_n \int_{\mathbb{R}^n} |y_n| |f^*(y',y_n)| dy' dy_n, \end{aligned} \tag{2.8}$$

where $1 \leq j \leq n$, $\ell_1, m_1 \geq 0$ satisfy $\ell_1 + m_1 \leq n + 2$; and

$$\begin{aligned} \Omega_1 &= \{(x',x_n) \in \mathbb{R}^n;\ \ |x'| \leq t^{\frac{1}{2}} \ \textit{and}\ \ |x_n| \leq t^{\frac{1}{2}}\}; \\ \Omega_2 &= \{(x',x_n) \in \mathbb{R}^n;\ \ |x'| > t^{\frac{1}{2}} \ \textit{and}\ \ |x_n| \leq t^{\frac{1}{2}}\}; \\ \Omega_3 &= \{(x',x_n) \in \mathbb{R}^n;\ \ |x'| \leq t^{\frac{1}{2}} \ \textit{and}\ \ |x_n| > t^{\frac{1}{2}}\}; \\ \Omega_4 &= \{(x',x_n) \in \mathbb{R}^n;\ \ |x'| > t^{\frac{1}{2}} \ \textit{and}\ \ |x_n| > t^{\frac{1}{2}}\}. \end{aligned}$$

For each integration in (2.8), we take suitable ℓ_1 and m_1 such that $\ell_1 = m_1 = 0$ in Ω_1; $\ell_1 = n, m_1 = 0$ in Ω_2; $\ell_1 = 0, m_1 = 2$ in Ω_3; $\ell_1 = n - \frac{1}{2}, m_1 = \frac{3}{2}$ in Ω_4. Then we obtain for $t > 0$

$$\sum_{k=1}^{4} C_{\ell_1,m_1} t^{\frac{\ell_1+m_1-n-2}{2}} \int_{\Omega_k} |x'|^{-\ell_1} |x_n|^{-m_1} dx' dx_n \leq Ct^{-1}. \tag{2.9}$$

Combining (2.8) and (2.9), we conclude for $1 \leq j \leq n$ and $t > 0$

$$\| \sup_{s>0} |G_s * [\partial_j E(t) f]| \|_{L^1(\mathbb{R}^n)} \leq Ct^{-1} \int_{\mathbb{R}^n_+} y_n |f(y', y_n)| dy' dy_n. \tag{2.10}$$

Let $0 < \alpha < 1$, and j, ℓ_1, m_1 be given in (2.7). It follows from (2.7) that for $t > 0$

$$\begin{aligned} &\| \sup_{s>0} |G_s * [\partial_j E(t) f]| \|_{L^1(\mathbb{R}^n)} \\ &\leq \sum_{k=1}^{2} C_{\ell_1,m_1} t^{\frac{\ell_1+m_1-n-1}{2} - \frac{\alpha}{2}} \int_{\widetilde{\Omega}_k} |x'|^{-\ell_1} dx' \\ &\quad \times \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^1} |x_n|^{-m_1(1-\alpha)} |x_n - (\tau_0 - 1) y_n|^{-m_1 \alpha} dx_n \right. \\ &\quad \left. + \int_{\mathbb{R}^1} |x_n|^{-m_1(1-\alpha)} |x_n - \tau_0 y_n|^{-m_1 \alpha} dx_n \right) |y_n|^{\alpha} |f^*(y', y_n)| dy' dy_n, \end{aligned} \tag{2.11}$$

where $\tau_0 \in (0, 1)$, $\widetilde{\Omega}_1 = \{x' \in \mathbb{R}^{n-1};\ |x'| \leq t^{\frac{1}{2}}\}$ and $\widetilde{\Omega}_2 = \{x' \in \mathbb{R}^{n-1};\ |x'| > t^{\frac{1}{2}}\}$.

To proceed, we recall a useful estimate (see Lemma 7.2, II.7, page 75 in [22]): Let $m \geq 1$ and $\lambda, \mu \in (0, m)$. Then, if $\lambda + \mu > m$, there exists a constant $C = C(\lambda, \mu, m)$ such that

$$\int_{\mathbb{R}^m} |x - y|^{-\lambda} |y|^{-\mu} dy \leq C|x|^{-(\lambda+\mu-m)}. \tag{2.12}$$

Set $m_1 = 1 + \delta_1$, where $0 < \delta_1 < \min\{\frac{1}{\alpha} - 1, \frac{1}{1-\alpha} - 1, \alpha\}$. Then $m_1 \alpha < 1$, $m_1(1 - \alpha) < 1$ and $m_1 \alpha + m_1(1 - \alpha) = m_1 = 1 + \delta_1 > 1$. Using (2.12) with $m = 1$, $\lambda = m_1 \alpha$, $\mu = m_1(1 - \alpha)$, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^1} |x_n|^{-m_1(1-\alpha)} |x_n - (\tau_0 - 1) y_n|^{-m_1 \alpha} dx_n \\ &\quad + \int_{\mathbb{R}^1} |x_n|^{-m_1(1-\alpha)} |x_n - \tau_0 y_n|^{-m_1 \alpha} dx_n \leq C |y_n|^{-\delta_1}. \end{aligned} \tag{2.13}$$

Let $m_1 = 1 + \delta_1$ be fixed as above. In (2.11), we take $\ell_1 = 0$ in $\widetilde{\Omega}_1$; $n - 1 < \ell_1 \leq n + 1 + \alpha - m_1 = n + \alpha - \delta_1$ in $\widetilde{\Omega}_2$, which is possible due to the choice of δ_1. Then for $0 < \alpha < 1$ and $t > 0$

$$\sum_{k=1}^{2} C_{\ell_1,m_1} t^{\frac{\ell_1+m_1-n-1}{2} - \frac{\alpha}{2}} \int_{\widetilde{\Omega}_k} |x'|^{-\ell_1} dx' \leq Ct^{-\frac{1}{2} - \frac{\alpha-\delta_1}{2}}. \tag{2.14}$$

Combining (2.11), (2.13) and (2.14), we conclude for $1 \leq j \leq n$, $0 < \alpha < 1$ and $t > 0$

$$\| \sup_{s>0} |G_s * [\partial_j E(t) f]| \|_{L^1(\mathbb{R}^n)} \leq Ct^{-\frac{1}{2} - \frac{\alpha-\delta_1}{2}} \int_{\mathbb{R}^n_+} y_n^{\alpha-\delta_1} |f(y', y_n)| dy' dy_n. \tag{2.15}$$

For any number $0 < \beta < 1$, we always can find $\alpha \in (\beta, 1)$ such that $\delta_1 = \alpha - \beta$ satisfies $0 < \delta_1 < \min\{\frac{1}{\alpha} - 1, \frac{1}{1-\alpha} - 1, \alpha\}$, and $m_1 \alpha < 1$, $m_1(1 - \alpha) < 1$ with $m_1 = 1 + \delta_1$. This is possible if we let α be close to β sufficiently. Whence from (2.15), we deduce for $0 < \beta < 1$ and $t > 0$

$$\| \sup_{s>0} |G_s * [\partial_j E(t) f]| \|_{L^1(\mathbb{R}^n)} \leq Ct^{-\frac{1}{2} - \frac{\beta}{2}} \int_{\mathbb{R}^n_+} y_n^{\beta} |f(y', y_n)| dy' dy_n. \tag{2.16}$$

Combining (2.10) and (2.16), we obtain for $1 \le j \le n$, $0 < \beta \le 1$ and $t > 0$

$$\|\partial_j E(t)f\|_{\mathcal{H}^1(\mathbb{R}^n)} = \|\sup_{s>0}|G_s * [\partial_j E(t)f]|\|_{L^1(\mathbb{R}^n)} \le Ct^{-\frac{1}{2}-\frac{\beta}{2}} \int_{\mathbb{R}^n_+} y_n^\beta |f(y', y_n)|dy'dy_n. \tag{2.17}$$

Let $G_t^{(n-1)}$, $G_t^{(1)}$ denote Gauss kernels in $\mathbb{R}^{n-1}$ and $\mathbb{R}^1$, respectively:

$$G_t^{(n-1)}(x') = (4\pi t)^{-\frac{n-1}{2}} e^{-\frac{|x'|^2}{4t}}, \quad G_t^{(1)}(x_n) = (4\pi t)^{-\frac{1}{2}} e^{-\frac{|x_n|^2}{4t}}, \quad x' \in \mathbb{R}^{n-1},\ x_n \in \mathbb{R}^1.$$

Then the Gauss kernel $G_t(x)$ in $\mathbb{R}^n$ can be written as:

$$G_t(x) = G_t^{(n-1)}(x')G_t^{(1)}(x_n), \quad x = (x', x_n) \in \mathbb{R}^n.$$

Let $1 \le j, k \le n-1$, $0 < \alpha \le 1$. Then for any $x = (x', x_n) \in \mathbb{R}^n$ and $s, t > 0$

$$\begin{aligned}
&|G_s * [\partial_j\partial_k\Lambda^{-1}eE(t)f](x)| \\
&\quad = |[G_s^{(n-1)}G_s^{(1)}] * [\partial_j\partial_k\Lambda^{-1}e(G_t^{(n-1)}G_t^{(1)}) * f^*](x)| \\
&\quad = \Bigg|\partial_j\partial_k\Lambda^{-1}\int_{\mathbb{R}^n} G_{s+t}^{(n-1)}(x'-z')\int_{-\infty}^{\infty}\theta(y_n)G_s^{(1)}(x_n-y_n) \\
&\qquad \times [G_t^{(1)}(y_n-z_n) - G_t^{(1)}(y_n)]f^*(z', z_n)dy_ndz'dz_n\Bigg| \\
&\quad \le \int_{\mathbb{R}^n}|\partial_{x_j}\partial_{x_k}\Lambda^{-1}G_{s+t}^{(n-1)}(x'-z')|\int_{-\infty}^{\infty}G_s^{(1)}(x_n-y_n) \\
&\qquad \times [G_t^{(1)}(y_n-z_n) + G_t^{(1)}(y_n)]^{1-\alpha} \\
&\qquad \times \left|\int_0^1 \partial_{y_n}G_t^{(1)}(y_n-\tau z_n)d\tau\right|^\alpha |z_n|^\alpha|f^*(z', z_n)|dy_ndz'dz_n \\
&\quad \le Ct^{-\frac{\alpha}{2}}\int_{\mathbb{R}^n}|\partial_{x_j}\partial_{x_k}\Lambda^{-1}G_{s+t}^{(n-1)}(x'-z')|[G_{s+t}^{(1)}(x_n-z_n) + G_{s+t}^{(1)}(x_n)]^{1-\alpha} \\
&\qquad \times |G_{2(s+t)}^{(1)}(x_n-\tau_1 z_n)|^\alpha|z_n|^\alpha|f^*(z', z_n)|dz'dz_n,
\end{aligned} \tag{2.18}$$

where $\tau_1 \in (0,1)$, and $\theta(s) = 1$ if $s \ge 0$; 0 if $s < 0$; and the estimate is used in the above proof of (2.18): $|\frac{d}{d\lambda}G_t^{(1)}(\lambda)| \le Ct^{-\frac{1}{2}}G_{2t}^{(1)}(\lambda)$, $\lambda > 0$.

Note that for $m_2 \ge 0$ and $s, t > 0$

$$\begin{aligned}
G_{s+t}^{(1)}(x_n-z_n) + G_{s+t}^{(1)}(x_n) &\le C_{m_2}(s+t)^{\frac{m_2-1}{2}}(|x_n-z_n|^{-m_2} + |x_n|^{-m_2}); \\
|G_{2(s+t)}^{(1)}(x_n-\tau_1 z_n)| &\le C_{m_2}(s+t)^{\frac{m_2-1}{2}}|x_n-\tau_1 z_n|^{-m_2}.
\end{aligned}$$

Moreover for $1 \le j, k \le n-1$ (see Lemma 2.2 in [23])

$$|\partial_{x_j}\partial_{x_k}\Lambda^{-1}G_{s+t}^{(n-1)}(x'-z')| \le C_{\ell_2}(s+t)^{\frac{\ell_2-n}{2}}|x'-z'|^{-\ell_2}$$

for any $0 \le \ell_2 \le n$.

Therefore, from (2.9) and (2.18), we conclude for $s, t > 0$

$$\begin{aligned}
&|G_s * [\partial_j\partial_k\Lambda^{-1}eE(t)f](x)| \\
&\quad \le \int_{\mathbb{R}^n} C_{\ell_2,m_2}t^{\frac{\ell_2+m_2-n-1}{2}-\frac{\alpha}{2}}|x'-z'|^{-\ell_2}\big(|x_n-z_n|^{-m_2} + |x_n|^{-m_2}\big)^{1-\alpha} \\
&\qquad \times|x_n-\tau_1 z_n|^{-m_2})^\alpha|z_n|^\alpha|f^*(z', z_n)|dz'dz_n.
\end{aligned} \tag{2.19}$$

where $1 \le j, k \le n-1,\ 0 \le \ell_2 \le n,\ m_2 \ge 0$, $0 \le \ell_2 + m_2 \le n + 1 + \alpha$, $0 < \alpha \le 1$.

Let $\alpha = 1$. Then from (2.19), and using (2.9) with $\ell_2 = \ell_1 \in [0, n]$, $m_2 = m_1 \geq 0$, $\ell_2 + m_2 \leq n+1$, we get for $1 \leq j, k \leq n-1$ and $t > 0$

$$\begin{aligned}
&\|\sup_{s>0} |G_s * [\partial_j \partial_k \Lambda^{-1} eE(t)f]|\|_{L^1(\mathbb{R}^n)} \\
&\leq \sum_{k=1}^{4} C_{\ell_2, m_2} t^{\frac{\ell_2+m_2-n-2}{2}} \int_{\Omega_k} |x'|^{-\ell_2} |x_n|^{-m_2} dx' dx_n \int_{\mathbb{R}^n} |z_n| |f^*(z', z_n)| dz' dz_n \\
&\leq Ct^{-1} \int_{\mathbb{R}^n_+} z_n |f(z', z_n)| dz' dz_n,
\end{aligned} \tag{2.20}$$

where Ω_i $(i = 1, 2, 3, 4)$ are given in the proof of (2.8).

Let $0 < \beta < 1$, and take $\ell_2 = \ell_1 \in [0, n], m_2 = m_1$ in (2.19) satisfying $\ell_1 + m_1 \leq n+1$, and $\alpha \in (\beta, 1)$ such that $\delta_1 = \alpha - \beta$, where δ_1 is the selected number given in the proof of (2.13). From (2.19), and following the proofs of (2.11)–(2.16), we obtain for $t > 0$

$$\begin{aligned}
&\|\sup_{s>0} |G_s * [\partial_j \partial_k \Lambda^{-1} eE(t)f]|\|_{L^1(\mathbb{R}^n)} \\
&\leq \sum_{k=1}^{2} C_{\ell_1, m_1} t^{\frac{\ell_1+m_1-n-1}{2} - \frac{\alpha}{2}} \int_{\widetilde{\Omega}_k} |x'|^{-\ell_1} dx' \\
&\quad \times \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^1} |x_n|^{-m_1(1-\alpha)} |x_n - (\tau_1 - 1) z_n|^{-m_1 \alpha} dx_n \\
&\quad + \int_{\mathbb{R}^1} |x_n|^{-m_1(1-\alpha)} |x_n - \tau_1 z_n|^{-m_1 \alpha} dx_n \Big) |z_n|^{\alpha} |f^*(z', z_n)| dz' dz_n \\
&\leq Ct^{-\frac{1}{2} - \frac{\beta}{2}} \int_{\mathbb{R}^n_+} z_n^{\beta} |f(z', z_n)| dz' dz_n, \quad 1 \leq j, k \leq n-1,
\end{aligned} \tag{2.21}$$

where $\widetilde{\Omega}_i$ $(i = 1, 2)$ are from the proof of (2.11).

Combining (2.20) and (2.21), we obtain for $1 \leq j, k \leq n-1$, $0 < \beta \leq 1$ and $t > 0$

$$\begin{aligned}
\|\partial_j \partial_k \Lambda^{-1} eE(t)f\|_{\mathcal{H}^1(\mathbb{R}^n)} &= \|\sup_{s>0} |G_s * [\partial_j \partial_k \Lambda^{-1} eE(t)f]|\|_{L^1(\mathbb{R}^n)} \\
&\leq Ct^{-\frac{1}{2} - \frac{\beta}{2}} \int_{\mathbb{R}^n_+} y_n^{\beta} |f(y', y_n)| dy' dy_n.
\end{aligned} \tag{2.22}$$

Since $\sum_{j=1}^{n-1} \partial_j \partial_j \Lambda^{-1} = -\Lambda$. Using (2.22), we get for $0 < \beta \leq 1$ and $t > 0$

$$\|\Lambda eE(t)f\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq Ct^{-\frac{1}{2} - \frac{\beta}{2}} \int_{\mathbb{R}^n_+} y_n^{\beta} |f(y', y_n)| dy' dy_n. \tag{2.23}$$

To proceed, we have to deal with a special term in (2.5) with $j = 0$: $\partial_n \nabla' \Lambda^{-1} E(t) a_n$, whose estimate in $\mathcal{H}^1$ space is different from (2.22) and (2.23).

Note that $\nabla \cdot a = 0$ in $\mathbb{R}^n_+$ and $a_n(y', y_n)|_{\partial \mathbb{R}^n_+} = a_n(y', 0) = 0$. We have for almost all $y_n > 0$

$$\begin{aligned}
\int_{\mathbb{R}^{n-1}} a_n(y', y_n) dy' &= \int_{\mathbb{R}^{n-1}} (a_n(y', y_n) - a_n(y', 0)) dy' \\
&= \int_{\mathbb{R}^{n-1}} \int_0^1 y_n \partial_n a_n(y', \lambda y_n) d\lambda dy' \\
&= -\int_0^1 y_n \sum_{k=1}^{n-1} \int_{\mathbb{R}^{n-1}} \partial_k a_k(y', \lambda y_n) dy' d\lambda = 0,
\end{aligned}$$

from which, we find $\int_{\mathbb{R}^{n-1}} a_n^*(y',y_n)dy' = 0$ for almost all $y_n > 0$. Whence for $1 \le j \le n-1$ and $s,t > 0$

$$\begin{aligned}
&|\{G_s * [\partial_n\partial_j\Lambda^{-1}E(t)a_n]\}(x)| \\
&\quad = |\{G_s * [\partial_n\partial_j\Lambda^{-1}G_t * a_n^*]\}(x)| = |[\partial_n\partial_j\Lambda^{-1}G_{s+t} * a_n^*](x)| \\
&\quad = \left|\int_{\mathbb{R}^n} \partial_{x_j}\Lambda^{-1}[G_{s+t}^{(n-1)}(x'-y') - G_{s+t}^{(n-1)}(x')]\partial_{x_n}G_{s+t}^{(1)}(x_n-y_n)a_n^*(y',y_n)dy'dy_n\right| \\
&\quad \le \int_{\mathbb{R}^n} [|\partial_{x_j}\Lambda^{-1}G_{s+t}^{(n-1)}(x'-y')| + |\partial_{x_j}\Lambda^{-1}G_{s+t}^{(n-1)}(x')|]^{1-\alpha} \\
&\quad\quad \times \int_0^1 |\nabla'\partial_{x_j}\Lambda^{-1}G_{s+t}^{(n-1)}(x'-\tau y')|^{\alpha}|\partial_{x_n}G_{s+t}^{(1)}(x_n-y_n)||y'|^{\alpha}|a_n^*(y',y_n)|d\tau dy'dy_n \\
&\quad \le \int_{\mathbb{R}^n} C_{\ell_3,\ell_4} t^{\frac{\ell_3(1-\alpha)+\ell_4\alpha+m_3-n-1-\alpha}{2}} [|x'-y'|^{-\ell_3} + |x'|^{-\ell_3}]^{1-\alpha} \\
&\quad\quad \times |x'-\tau_2 y'|^{-\ell_4\alpha}|x_n-y_n|^{-m_3}|y'|^{\alpha}|a_n^*(y',y_n)|dy'dy_n, \quad \tau_2 \in (0,1),
\end{aligned} \tag{2.24}$$

where $0 \le \ell_3 \le n-1$, $0 \le \ell_4 \le n$, $m_3 \ge 0$, and $\ell_3(1-\alpha) + \ell_4\alpha + m_3 \le n+1+\alpha$. In the proof of (2.24), we employed the following estimates (see Lemma 2.2 in [23]):

$$\begin{aligned}
&|\partial_{x_j}\Lambda^{-1}G_{s+t}^{(n-1)}(x'-y')| + |\partial_{x_j}\Lambda^{-1}G_{s+t}^{(n-1)}(x')| \\
&\quad \le C_{\ell_3}(s+t)^{\frac{\ell_3+1-n}{2}}(|x'-y'|^{-\ell_3} + |x'|^{-\ell_3}), \quad 0 \le \ell_3 \le n-1; \\
&|\nabla'\partial_{x_j}\Lambda^{-1}G_{s+t}^{(n-1)}(x'-\tau_2 y')| \le C_{\ell_4}(s+t)^{\frac{\ell_4-n}{2}}|x'-\tau_2 y'|^{-\ell_4}, \quad 0 \le \ell_4 \le n; \\
&|\partial_{x_n}G_{s+t}^{(1)}(x_n-y_n)| \le C_{m_3}(s+t)^{\frac{m_3-2}{2}}|x_n-y_n|^{-m_3}, \quad m_3 \ge 0.
\end{aligned}$$

If $\alpha = 1$. Then from (2.24), and using (2.9) with $\ell_3 = 0$, $\ell_4 = \ell_1 \in [0,n]$, $m_3 = m_1 \ge 0$, $\ell_4 + m_3 \le n+2$, we get for $t > 0$

$$\begin{aligned}
&\|\sup_{s>0}|G_s * [\partial_n\partial_j\Lambda^{-1}eE(t)f]|\|_{L^1(\mathbb{R}^n)} \\
&\quad \le \sum_{k=1}^{4} C_{\ell_4,m_3} t^{\frac{\ell_4+m_3-n-2}{2}} \int_{\Omega_k} |x'|^{-\ell_4}|x_n|^{-m_3}dx'dx_n \int_{\mathbb{R}^n} |y'||a_n^*(y',y_n)|dy'dy_n \\
&\quad \le Ct^{-1}\int_{\mathbb{R}^n_+} |y'||a_n(y',y_n)|dy'dy_n,
\end{aligned} \tag{2.25}$$

where $1 \le j \le n-1$, and Ω_i $(i = 1,2,3,4)$ arise from the proof of (2.8).

Let $0 < \alpha < 1$, $0 \le \ell_3 \le n-1$, $0 \le \ell_4 \le n$, $m_3 \ge 0$, and $\ell_3(1-\alpha) + \ell_4\alpha + m_3 \le n+1+\alpha$. Then for $t > 0$

$$\begin{aligned}
&\|\sup_{s>0}|G_s * [\partial_n\partial_j\Lambda^{-1}eE(t)a_n]|\|_{L^1(\mathbb{R}^n)} \\
&\quad \le \int_{\mathbb{R}^n} C_{\ell_3,\ell_4} t^{\frac{\ell_3(1-\alpha)+\ell_4\alpha-n-1}{2}-\frac{\alpha}{2}} \bigg(\int_{\mathbb{R}^{n-1}} |x'|^{-\ell_3(1-\alpha)}|x'-(\tau_2-1)y'|^{-\ell_4\alpha}dx' \\
&\quad\quad + \int_{\mathbb{R}^{n-1}} |x'|^{-\ell_3(1-\alpha)}|x'-\tau_2 y'|^{-\ell_4\alpha}dx'\bigg) \\
&\quad\quad \times \bigg(\sum_{k=1}^{2} C_{m_3} t^{\frac{m_3}{2}} \int_{Q_k} |x_n|^{-m_3}dx_n\bigg)|y'|^{\alpha}|a_n^*(y',y_n)|dy'dy_n,
\end{aligned} \tag{2.26}$$

where $Q_1 = \{x_n \in \mathbb{R}^1;\ |x_n| \le t^{\frac{1}{2}}\}$ and $Q_2 = \{x_n \in \mathbb{R}^1;\ |x_n| > t^{\frac{1}{2}}\}$.

Take $\ell_3 = n-1$, $\ell_4 = (n-1+\delta_2) \le n$ in (2.26), where $0 < \delta_2 < \min\{1, (n-1)(\frac{1}{\alpha}-1)\}$. Then $\ell_3(1-\alpha) < n-1$ and $\ell_4\alpha = (n-1+\delta_2)\alpha < n-1$. Moreover, $\ell_3(1-\alpha) + \ell_4\alpha = n-1+\delta_2\alpha > n-1$. It follows from (2.12) with $m = n-1$, $\lambda = \ell_4\alpha$, $\mu = \ell_3(1-\alpha)$, that for $t > 0$

$$C_{\ell_3,\ell_4} t^{\frac{\ell_3(1-\alpha)+\ell_4\alpha-n-1}{2}-\frac{\alpha}{2}} \left(\int_{\mathbb{R}^{n-1}} |x'|^{-\ell_3(1-\alpha)} |x'-(\tau_2-1)y'|^{-\ell_4\alpha} dx' + \int_{\mathbb{R}^{n-1}} |x'|^{-\ell_3(1-\alpha)} |x'-\tau_2 y'|^{-\ell_4\alpha} dx' \right) \le C t^{-1-\frac{(1-\delta_2)\alpha}{2}} |y'|^{-\delta_2\alpha}. \tag{2.27}$$

Let ℓ_3, ℓ_4 be fixed in the proof of (2.27). Take $m_3 = 0$ in Q_2; $1 < m_3 \le n+1+\alpha-\ell_3(1-\alpha)+\ell_4\alpha = 2+(1-\delta_2)\alpha$ in Q_2. Then we obtain for $t > 0$

$$\sum_{k=1}^{2} C_{m_3} t^{\frac{m_3}{2}} \int_{Q_k} |x_n|^{-m_3} dx_n \le C t^{\frac{1}{2}}. \tag{2.28}$$

By the choices of ℓ_3, ℓ_4, m_3 in (2.27) and (2.28), we always have $\ell_3(1-\alpha)+\ell_4\alpha+m_4 \le n+1+\alpha$.

Combining (2.26), (2.27) and (2.28), we conclude for $1 \le j \le n-1$ and $t > 0$

$$\| \sup_{s>0} |G_s * [\partial_n \partial_j \Lambda^{-1} eE(t) a_n]| \|_{L^1(\mathbb{R}^n)} \le C t^{-\frac{1}{2}-\frac{(1-\delta_2)\alpha}{2}} \int_{\mathbb{R}^n_+} |y'|^{(1-\delta_2)\alpha} |a_n(y', y_n)| dy' dy_n, \tag{2.29}$$

where $0 < \alpha < 1$ and $0 < \delta_2 < \min\{1, (n-1)(\frac{1}{\alpha}-1)\}$.

For any number $0 < \beta < 1$, we can find $\alpha \in (\beta, 1)$ such that $\delta_2 = 1 - \frac{\beta}{\alpha}$ or $(1-\delta_2)\alpha = \beta$ satisfies $0 < \delta_2 < \min\{1, (n-1)(\frac{1}{\alpha}-1)\}$, and $\ell_3(1-\alpha) < n-1$, $\ell_4\alpha = (n-1+\delta_2)\alpha < n-1$; moreover, $\ell_3(1-\alpha)+\ell_4\alpha = n-1+\delta_2\alpha > n-1$, where $\ell_3 = n-1$, $\ell_4 = (n-1+\delta_2) \le n$. This is possible because δ_2 becomes sufficiently small if we let α be close to β enough.

Whence from (2.29), we deduce for $0 < \beta < 1$, $1 \le j \le n-1$ and $t > 0$

$$\| \sup_{s>0} |G_s * [\partial_n \partial_j \Lambda^{-1} eE(t) a_n]| \|_{L^1(\mathbb{R}^n)} \le C t^{-\frac{1}{2}-\frac{\beta}{2}} \int_{\mathbb{R}^n_+} |y'|^{\beta} |a_n(y', y_n)| dy' dy_n. \tag{2.30}$$

Combining (2.25) and (2.30), we obtain for $0 < \beta \le 1$ and $t > 0$

$$\begin{aligned} \|\partial_n \nabla' \Lambda^{-1} eE(t) a_n\|_{\mathcal{H}^1(\mathbb{R}^n)} &= \| \sup_{s>0} |G_s * [\partial_n \nabla' \Lambda^{-1} eE(t) a_n]| \|_{L^1(\mathbb{R}^n)} \\ &\le C t^{-\frac{1}{2}-\frac{\beta}{2}} \int_{\mathbb{R}^n_+} |y'|^{\beta} |a_n(y', y_n)| dy' dy_n. \end{aligned} \tag{2.31}$$

For the convenience, we denote the norm of the Riesz operator R_j $(1 \le j \le n)$ in $\mathcal{H}^1(\mathbb{R}^n)$ by $|||R_j|||_{\mathcal{H}^1}$. From (2.4), (2.5), (2.17), (2.22), (2.23) and (2.31), we conclude for $1 \le j \le n$, $0 < \beta \le 1$ and $t > 0$

$$\begin{aligned} \|\partial_j \overline{u}_n\|_{\mathcal{H}^1} &\le |||R_j|||_{\mathcal{H}^1} \Big\{ \sum_{k=1}^{n-1} |||R_k|||_{\mathcal{H}^1} \|\Lambda eE(t) a_k\|_{\mathcal{H}^1} + |||R_n|||_{\mathcal{H}^1} \sum_{k=1}^{n-1} \|\partial_k eE(t) a_k\|_{\mathcal{H}^1} \\ &\quad + \sum_{k=1}^{n-1} |||R_k|||_{\mathcal{H}^1} \|\partial_k eE(t) a_n\|_{\mathcal{H}^1} + |||R_n|||_{\mathcal{H}^1} \|\Lambda eE(t) a_n\|_{\mathcal{H}^1} \Big\} \\ &\le C t^{-\frac{1}{2}-\frac{\beta}{2}} \int_{\mathbb{R}^n_+} |y|^{\beta} |a(y)| dy; \end{aligned}$$

$$\begin{aligned} \|\partial_j \overline{u}'\|_{\mathcal{H}^1} &\le \|\partial_j E(t) a'\|_{\mathcal{H}^1} + \|\partial_j \nabla' \Lambda^{-1} E(t) a_n\|_{\mathcal{H}^1} + |||R_j|||_{\mathcal{H}^1} \Big\{ |||R'|||_{\mathcal{H}^1} \sum_{k=1}^{n-1} \|\partial_k eE(t) a_k\|_{\mathcal{H}^1} \\ &\quad + |||R_n|||_{\mathcal{H}^1} \sum_{k=1}^{n-1} \|\nabla' \partial_k \Lambda^{-1} eE(t) a_k\|_{\mathcal{H}^1} + |||R'|||_{\mathcal{H}^1} \|\Lambda eE(t) a_n\|_{\mathcal{H}^1} + |||R_n|||_{\mathcal{H}^1} \|\nabla' eE(t) a_n\|_{\mathcal{H}^1} \Big\} \\ &\le C t^{-\frac{1}{2}-\frac{\beta}{2}} \int_{\mathbb{R}^n_+} |y|^{\beta} |a(y)| dy. \end{aligned}$$

Since $e^{-tA}a = \overline{u}(t)\,|_{\mathbb{R}^n_+}$, we get for $t > 0$

$$\|\nabla e^{-tA}a\|_{L^1(\mathbb{R}^n_+)} \leq \|\nabla \overline{u}(t)\|_{L^1(\mathbb{R}^n)} \leq \|\nabla \overline{u}(t)\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq Ct^{-\frac{1}{2}-\frac{\beta}{2}} \int_{\mathbb{R}^n_+} |y|^\beta |a(y)| dy.$$

□

Lemma 2.2. *Let $a = (a_1, a_2, \dots, a_n)$ satisfy $\nabla \cdot a = 0$ in $\mathbb{R}^n_+$ $(n \geq 2)$, and $a_n\,|_{\partial\mathbb{R}^n_+} = 0$. Then for $t > 0$*

$$\|\nabla e^{-tA}a\|_{L^\infty(\mathbb{R}^n_+)} \leq Ct^{-1}\||x|a\|_{L^\infty(\mathbb{R}^n_+)}.$$

Proof. Let $1 \leq j \leq n$, $0 < \alpha \leq 1$. Following the proof of (2.6), we find for $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}^1$ and $t > 0$

$$\begin{aligned}
|\partial_j E(t) f(x)| &= |\partial_j G_t * f^*(x)| \\
&= \Big| \int_{\mathbb{R}^n} \big(\partial_{x_j} G_t(x'-y', x_n - y_n) - \partial_{x_j} G_t(x'-y', x_n)\big) f^*(y', y_n) dy' dy_n \Big| \\
&\leq \int_{\mathbb{R}^n} \big[|\partial_{x_j} G_t(x'-y', x_n - y_n)| + |\partial_{x_j} G_t(x'-y', x_n)|\big]^{1-\alpha} \\
&\quad \times \big|\partial_{x_j}\partial_{x_n} G_t(x'-y', x_n - \tau_0 y_n)\big|^\alpha |y_n|^\alpha |f^*(y', y_n)| dy' dy_n, \quad \tau_0 \in (0,1) \\
&= \int_{\mathbb{R}^n} k_0(\alpha, x, y) |y_n|^\alpha |f^*(y', y_n)| dy' dy_n.
\end{aligned} \tag{2.32}$$

Let $0 < \alpha \leq 1$ and $1 \leq j \leq n$. Checking the proof of (2.7), we get for $x = (x', x_n),\ y = (y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R}^1$ and $s, t > 0$

$$\begin{aligned}
&\big|G_s * k_0(\alpha, x, \cdot)(y)\big| \\
&\quad \leq \big[|\partial_{x_j} G_{s+t}(x'-y', x_n - y_n)| + |\partial_{x_j} G_{s+t}(x'-y', x_n)|\big]^{1-\alpha} \\
&\qquad \times \big|\partial_{x_j}\partial_{x_n} G_{s+t}(x'-y', x_n - \tau_0 y_n)\big|^\alpha \\
&\quad \leq C_{\ell_1, m_1} t^{\frac{\ell_1 + m_1 - n - 1}{2} - \frac{\alpha}{2}} |x'-y'|^{-\ell_1} \big(|x_n - y_n|^{-m_1} + |x_n|^{-m_1}\big)^{1-\alpha} \\
&\qquad \times |x_n - \tau_0 y_n|^{-m_1 \alpha},
\end{aligned} \tag{2.33}$$

where $\ell_1,\, m_1 \geq 0,\, \ell_1 + m_1 \leq n + 1 + \alpha$.

Let $\alpha = 1$. Using (2.9), and from (2.33), we have for $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}^1$ and $t > 0$

$$\begin{aligned}
\|k_0(1, x, \cdot)\|_{\mathcal{H}^1(\mathbb{R}^n)} &= \|\sup_{s>0} |G_s * k_0(1, x, \cdot)(y)|\|_{L^1(\mathbb{R}^n_y)} \\
&\leq \int_{\mathbb{R}^n} C_{\ell_1, m_1} t^{\frac{\ell_1 + m_1 - n - 1}{2} - \frac{1}{2}} |x'-y'|^{-\ell_1} |x_n - \tau_0 y_n|^{-m_1} dy' dy_n \\
&\leq \sum_{k=1}^4 \int_{\Omega_k} \widetilde{C}_{\ell_1, m_1} t^{\frac{\ell_1 + m_1 - n - 2}{2}} |y'|^{-\ell_1} |y_n|^{-m_1} dy' dy_n \leq Ct^{-1},
\end{aligned} \tag{2.34}$$

where $1 \leq j \leq n$, $\ell_1, m_1 \geq 0$ satisfy $\ell_1 + m_1 \leq n + 2$, and $\Omega_k (k = 1, 2, 3, 4)$ are given in the proof of (2.8).

Combining (2.32) and (2.34), we obtain for $j, k, \ell = 1, 2, \dots, n$, and $t > 0$

$$\begin{aligned}
&\|\partial_j E(t) f\|_{L^\infty(\mathbb{R}^n)} + \|R_k R_\ell \partial_j E(t) f\|_{L^\infty(\mathbb{R}^n)} \\
&\leq \big(\sup_{x \in \mathbb{R}^n} \|k_0(1, x, \cdot)\|_{L^1(\mathbb{R}^n)} + \sup_{x \in \mathbb{R}^n} \|R_k R_\ell k_0(1, x, \cdot)\|_{L^1(\mathbb{R}^n)}\big) \|y_n f^*\|_{L^\infty(\mathbb{R}^n)} \\
&\leq \big(\sup_{x \in \mathbb{R}^n} \|k_0(1, x, \cdot)\|_{\mathcal{H}^1(\mathbb{R}^n)} + \sup_{x \in \mathbb{R}^n} \|R_k R_\ell k_0(1, x, \cdot)\|_{\mathcal{H}^1(\mathbb{R}^n)}\big) \|y_n f\|_{L^\infty(\mathbb{R}^n_+)} \\
&\leq (1 + \||R_k\||_{\mathcal{H}^1} \||R_\ell\||_{\mathcal{H}^1}) \sup_{x \in \mathbb{R}^n} \|k_0(1, x, \cdot)\|_{\mathcal{H}^1(\mathbb{R}^n)} \|y_n f\|_{L^\infty(\mathbb{R}^n_+)} \\
&\leq Ct^{-1} \|y_n f\|_{L^\infty(\mathbb{R}^n_+)}.
\end{aligned} \tag{2.35}$$

Let $1 \le j, k \le n-1$, $0 < \alpha \le 1$. Checking the proof of (2.18), we have for $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}^1$ and $t > 0$

$$\begin{aligned}
&|\partial_j \partial_k \Lambda^{-1} eE(t) f(x)| \\
&\quad \le \int_{\mathbb{R}^n} |\partial_{x_j} \partial_{x_k} \Lambda^{-1} G_t^{(n-1)}(x'-z')| [G_t^{(1)}(x_n - z_n) + G_t^{(1)}(x_n)]^{1-\alpha} \\
&\qquad \times |\partial_{x_n} G_t^{(1)}(x_n - \tau_1 z_n)|^{\alpha} |z_n|^{\alpha} |f^*(z', z_n)| dz' dz_n, \quad \textit{where} \;\; \tau_1 \in (0,1) \\
&\quad = \int_{\mathbb{R}^n} k_1(\alpha, x, z) ||z_n|^{\alpha} |f^*(z', z_n)| dz' dz_n.
\end{aligned} \tag{2.36}$$

From (2.36), we get for $x = (x', x_n),\ z = (z', z_n) \in \mathbb{R}^{n-1} \times \mathbb{R}^1$ and $s, t > 0$

$$\begin{aligned}
&\big| G_s * [k_1(\alpha, x, z)] \big| \\
&\quad \le |\partial_{x_j} \partial_{x_k} \Lambda^{-1} G_{s+t}^{(n-1)}(x'-z')| [G_{s+t}^{(1)}(x_n - z_n) + G_{s+t}^{(1)}(x_n)]^{1-\alpha} \\
&\qquad \times |\partial_{x_n} G_{s+t}^{(1)}(x_n - \tau_1 z_n)|^{\alpha} \\
&\quad \le C_{\ell_2, m_2} t^{\frac{\ell_2 + m_2 - n - 1}{2} - \frac{\alpha}{2}} |x' - z'|^{-\ell_2} \big(|x_n - z_n|^{-m_2} + |x_n|^{-m_2} \big)^{1-\alpha} \\
&\qquad \times |x_n - \tau_1 z_n|^{-m_2})^{\alpha}.
\end{aligned} \tag{2.37}$$

where $1 \le j, k \le n-1,\ \ell_2,\ m_2 \ge 0,\ 0 \le \ell_2 + m_2 \le n + 1 + \alpha,\ 0 < \alpha \le 1$.

Let $\alpha = 1$ in (2.37), and from (2.9), we get for $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}^1$ and $t > 0$

$$\begin{aligned}
&\|k_1(1, x, \cdot)\|_{\mathcal{H}^1(\mathbb{R}^n)} = \| \sup_{s>0} |G_s * k_1(1, x, \cdot)(z)| \|_{L^1(\mathbb{R}^n_z)} \\
&\quad \le \int_{\mathbb{R}^n} C_{\ell_2, m_2} t^{\frac{\ell_2 + m_2 - n - 1}{2} - \frac{1}{2}} |x' - z'|^{-\ell_2} |x_n - \tau_1 z_n|^{-m_2} dz' dz_n \\
&\quad \le \sum_{k=1}^{4} \int_{\Omega_k} \widetilde{C}_{\ell_2, m_2} t^{\frac{\ell_2 + m_2 - n - 2}{2}} |y'|^{-\ell_2} |y_n|^{-m_2} dy' dy_n \le C t^{-1},
\end{aligned} \tag{2.38}$$

where Ω_k $(k = 1, 2, 3, 4)$ are given in the proof of (2.8).

Let $\alpha = 1$ in (2.36), together with (2.38), we get for $1 \le j, k \le n-1$, $1 \le m, \ell \le n$ and $t > 0$

$$\begin{aligned}
&\|\partial_j \partial_k \Lambda^{-1} eE(t) f|\|_{L^\infty(\mathbb{R}^n)} + \|R_m R_\ell \partial_j \partial_k \Lambda^{-1} eE(t) f|\|_{L^\infty(\mathbb{R}^n)} \\
&\quad \le (\sup_{x \in \mathbb{R}^n} \|k_1(1, x, \cdot)\|_{L^1(\mathbb{R}^n)} + \sup_{x \in \mathbb{R}^n} \|R_m R_\ell k_1(1, x, \cdot)\|_{L^1(\mathbb{R}^n)}) \|z_n f^*\|_{L^\infty(\mathbb{R}^n)} \\
&\quad \le C(\sup_{x \in \mathbb{R}^n} \|k_1(1, x, \cdot)\|_{\mathcal{H}^1(\mathbb{R}^n)} + \sup_{x \in \mathbb{R}^n} \|R_m R_\ell k_1(1, x, \cdot)\|_{\mathcal{H}^1(\mathbb{R}^n)}) \|z_n f\|_{L^\infty(\mathbb{R}^n_+)} \\
&\quad \le C(1 + \||R_m\||_{\mathcal{H}^1} \||R_\ell\||_{\mathcal{H}^1}) \sup_{x \in \mathbb{R}^n} \|k_1(1, x, \cdot)\|_{\mathcal{H}^1(\mathbb{R}^n)} \|z_n f\|_{L^\infty(\mathbb{R}^n_+)} \\
&\quad \le C t^{-1} \|z_n f\|_{L^\infty(\mathbb{R}^n_+)}.
\end{aligned} \tag{2.39}$$

Since $\sum_{j=1}^{n-1} \partial_j \partial_j \Lambda^{-1} = -\Lambda$. Using (2.39), we get for $m, \ell = 1, 2, \ldots, n$ and $t > 0$

$$\|\Lambda eE(t) f\|_{L^\infty(\mathbb{R}^n)} + \|R_m R_\ell \Lambda eE(t) f\|_{L^\infty(\mathbb{R}^n)} \le C t^{-1} \|z_n f\|_{L^\infty(\mathbb{R}^n_+)}. \tag{2.40}$$

Now we pay attention to $\partial_n \nabla' \Lambda^{-1} E(t) a_n$ in (2.5) with $j = 0$. Checking the proof of (2.24), we get for $1 \le j \le n-1$, $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}^1$ and $t > 0$

$$\begin{aligned}
&|\partial_n \partial_j \Lambda^{-1} E(t) a_n(x)| \\
&\quad \le \int_{\mathbb{R}^n} [|\partial_{x_j} \Lambda^{-1} G_t^{(n-1)}(x' - y')| + |\partial_{x_j} \Lambda^{-1} G_t^{(n-1)}(x')|]^{1-\alpha} \\
&\quad \times \int_0^1 |\nabla' \partial_{x_j} \Lambda^{-1} G_t^{(n-1)}(x' - \tau y')|^{\alpha} |\partial_{x_n} G_t^{(1)}(x_n - y_n)| |y'|^{\alpha} |a_n^*(y', y_n)| d\tau dy' dy_n \\
&\quad = \int_{\mathbb{R}^n} k_2(\alpha, x, y) |y'|^{\alpha} |a_n^*(y', y_n)| d\tau dy' dy_n.
\end{aligned} \tag{2.41}$$

Whence we infer for $1 \le j \le n-1$, $x=(x', x_n), y=(y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R}^1$ and $s, t > 0$

$$\begin{aligned}|\{G_s * k_2(\alpha, x, y)| &\le [|\partial_{x_j}\Lambda^{-1}G^{(n-1)}_{s+t}(x'-y')| + |\partial_{x_j}\Lambda^{-1}G^{(n-1)}_{s+t}(x')|]^{1-\alpha} && (2.42)\\ &\quad \times \int_0^1 |\nabla'\partial_{x_j}\Lambda^{-1}G^{(n-1)}_{s+t}(x'-\tau y')|^{\alpha} d\tau |\partial_{x_n} G^{(1)}_{s+t}(x_n - y_n)| && (2.43)\\ &\le C_{\ell_3,\ell_4} t^{\frac{\ell_3(1-\alpha)+\ell_4\alpha+m_3-n-1-\alpha}{2}} [|x'-y'|^{-\ell_3} + |x'|^{-\ell_3}]^{1-\alpha} && (2.44)\\ &\quad \times |x'-\tau_2 y'|^{-\ell_4\alpha} |x_n - y_n|^{-m_3}, \quad \tau_2 \in (0,1), && (2.45)\end{aligned}$$

where $0 \le \ell_3 \le n-1$, $0 \le \ell_4 \le n$, $m_3 \ge 0$, and $\ell_3(1-\alpha) + \ell_4\alpha + m_3 \le n+1+\alpha$.

Let $\alpha = 1$ in (2.42). Using (2.9) with $\ell_4 = \ell_1 \in [0, n]$, $m_3 = m_1 \ge 0$, $\ell_4 + m_3 \le n+2$, we get for $x \in \mathbb{R}^n$ and $t > 0$

$$\begin{aligned}\|k_2(1, x, \cdot)\|_{\mathcal{H}^1(\mathbb{R}^n)} &= \|\sup_{s>0} |G_s * k_2(1, x, \cdot)(y)|\|_{L^1(\mathbb{R}^n_y)}\\ &\le \sum_{k=1}^{4} C_{\ell_4, m_3} t^{\frac{\ell_4+m_3-n-2}{2}} \int_{\Omega_k} |y'|^{-\ell_4} |y_n|^{-m_3} dy' dy_n \qquad (2.46)\\ &\le Ct^{-1},\end{aligned}$$

where $\Omega_k (k = 1, 2, 3, 4)$ are given in (2.8).

Combining (2.41) and (2.46), we obtain for $t > 0$

$$\begin{aligned}\|\partial_n \nabla' \Lambda^{-1} eE(t) a_n\|_{L^\infty(\mathbb{R}^n)} &\le \sup_{x \in \mathbb{R}^n} \|k_2(1, x, \cdot)\|_{L^1(\mathbb{R}^n)} \||y'| a_n^*\|_{L^\infty(\mathbb{R}^n)}\\ &\le Ct^{-1} \||y'| a_n\|_{L^\infty(\mathbb{R}^n_+)}.\end{aligned} \qquad (2.47)$$

From (2.4), (2.5), (2.35), (2.39), (2.40) and (2.47), we conclude for $1 \le j \le n$ and $t > 0$

$$\begin{aligned}\|\partial_j \overline{u}_n\|_{L^\infty(\mathbb{R}^n)} &\le \sum_{k=1}^{n-1} \|R_j R_k \Lambda eE(t) a_k\|_{L^\infty(\mathbb{R}^n)} + \sum_{k=1}^{n-1} \|R_j R_n \partial_k eE(t) a_k\|_{L^\infty(\mathbb{R}^n)}\\ &\quad + \sum_{k=1}^{n-1} \|R_j R_k \partial_k eE(t) a_n\|_{L^\infty(\mathbb{R}^n)} + \|R_j R_n \Lambda eE(t) a_n\|_{L^\infty(\mathbb{R}^n)}\\ &\le Ct^{-1} \|y_n a\|_{L^\infty(\mathbb{R}^n_+)};\\ \|\partial_j \overline{u}'\|_{L^\infty(\mathbb{R}^n)} &\le \|\partial_j E(t) a'\|_{L^\infty(\mathbb{R}^n)} + \|\partial_j \nabla' \Lambda^{-1} E(t) a_n\|_{L^\infty(\mathbb{R}^n)}\\ &\quad + \sum_{k=1}^{n-1} \|R_j R' \partial_k eE(t) a_k\|_{L^\infty(\mathbb{R}^n)} + \sum_{k=1}^{n-1} \|R_n \nabla' \partial_k \Lambda^{-1} eE(t) a_k\|_{L^\infty(\mathbb{R}^n)}\\ &\quad + \|R' \Lambda eE(t) a_n\|_{L^\infty(\mathbb{R}^n)} + \|R_n \nabla' eE(t) a_n\|_{L^\infty(\mathbb{R}^n)}\\ &\le Ct^{-1} (\|y_n a\|_{L^\infty(\mathbb{R}^n_+)} + \||y'| a\|_{L^\infty(\mathbb{R}^n_+)})\\ &\le Ct^{-1} \||y| a\|_{L^\infty(\mathbb{R}^n_+)}.\end{aligned}$$

Note that $e^{-tA} a = \overline{u}(t)\,|_{\mathbb{R}^n_+}$, we have for $t > 0$

$$\|\nabla e^{-tA} a\|_{L^\infty(\mathbb{R}^n_+)} \le \|\nabla \overline{u}(t)\|_{L^\infty(\mathbb{R}^n)} \le Ct^{-1} \||y| a\|_{L^\infty(\mathbb{R}^n_+)}.$$

□

Lemma 2.3. *Let $b = (b_1, b_2, \dots, b_n)$ satisfy $\nabla \cdot b = 0$ in $\mathbb{R}^n_+$ $(n \ge 2)$, and $b\,|_{\partial\mathbb{R}^n_+} = 0$. Then for $q = 1, \infty$ and $t > 0$*

$$\|\nabla^k e^{-tA} b\|_{L^q(\mathbb{R}^n_+)} \le Ct^{-\frac{k-1}{2}} \|\nabla b\|_{L^q(\mathbb{R}^n_+)}, \qquad k = 2, 3, \dots$$

Proof. Note that the Stokes flow $e^{-tA}b$ is given as a restriction $r\overline{u}$ of one vector field $\overline{u} = (\overline{u}', \overline{u}_n)$, where $\overline{u}_n, \overline{u}'$ are given in (2.2), (2.3) respectively, with initial data $u_0 = b$. Moreover the estimates hold for any $1 \le k, j \le n$ and $t > 0$ (see (3.34)–(3.36) in [23])

$$\begin{aligned}\partial_k\partial_j\overline{u}_n = & -R_j\big\{R'\cdot\Lambda eE(t)(1-\delta_{kn})\partial_k b' + \delta_{kn}R_n\Lambda eE(t)\nabla'\cdot b' \\ & - R_n\nabla'\cdot eE(t)(1-\delta_{kn})\partial_k b' - \delta_{kn}R_n\partial_n eE(t)\nabla'\cdot b' \\ & + R'\cdot\nabla' eE(t)(1-\delta_{kn})\partial_k b_n + \delta_{kn}R'\cdot\partial_n eE(t)\nabla' b_n \\ & + R_n\Lambda eE(t)(1-\delta_{kn})\partial_k b_n + \delta_{kn}R_n\Lambda eF(t)\partial_n b_n\big\},\end{aligned} \tag{2.48}$$

where $\delta_{kn} = 0$ if $1 \le k \le n-1$; $\delta_{kn} = 1$ if $k = n$.

Let $1 \le k \le n$. Then for any $t > 0$

$$\begin{aligned}\partial_k\partial_n\overline{u}' = & \,\partial_n E(t)(1-\delta_{kn})\partial_k b' + \delta_{kn}\partial_n F(t)\partial_n b' \\ & - \nabla'\big(\nabla'\Lambda^{-1}\cdot F(t)(1-\delta_{kn})\partial_k a'\big) - \nabla'(\nabla'\Lambda^{-1}\cdot E(t)\delta_{kn}\partial_n b') \\ & + R_n\big\{R'\nabla'\cdot eE(t)(1-\delta_{kn})\partial_k b' + R'\delta_{kn}\partial_n eE(t)\nabla'\cdot b' \\ & - R_n\nabla'\big(\nabla'\Lambda^{-1}\cdot eE(t)(1-\delta_{kn})\partial_k b'\big) - R_n\nabla'\big(\nabla'\Lambda^{-1}\cdot\delta_{kn}\partial_n eE(t)b'\big) \\ & - R'\Lambda eE(t)(1-\delta_{kn})\partial_k b_n - R'\Lambda\delta_{kn}eF(t)\partial_n b_n \\ & + R_n\nabla' eE(t)(1-\delta_{kn})\partial_k b_n + R_n\delta_{kn}\nabla' eF(t)\partial_n b_n\big\}.\end{aligned} \tag{2.49}$$

Let $1 \le k \le n,\ 1 \le j \le n-1$. Then for any $t > 0$

$$\begin{aligned}\partial_k\partial_j\overline{u}' = & \,\partial_k E(t)\partial_j b' + \partial_j\nabla'\Lambda^{-1}E(t)(1-\delta_{kn})\partial_k b_n + \partial_j\nabla'\Lambda^{-1}F(t)\delta_{kn}\partial_n b_n \\ & + R_k\big\{R'\nabla'\cdot eE(t)\partial_j b' - R_n\nabla'(\nabla'\Lambda^{-1}\cdot eE(t)\partial_j b') \\ & - R'\Lambda eE(t)\partial_j b_n + R_n\nabla' eE(t)\partial_j b_n\big\}.\end{aligned} \tag{2.50}$$

Note that $\partial_n eE(t)b'$ appears from the term $R_n\nabla'\big(\nabla'\Lambda^{-1}\cdot\delta_{kn}\partial_n eE(t)b'$ in (2.49). Since $b\,|_{\partial\mathbb{R}^n_+} = 0$, using the explicit formulas of $E(t), F(t)$, we find for $t > 0$

$$\begin{aligned}&\langle\partial_n eE(t)b(x), \varphi(x)\rangle \\ &= \int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+}\varphi(x)b(y)\partial_{x_n}[G_t(x'-y', x_n-y_n) - G_t(x'-y', x_n+y_n)]dxdy \\ &= \int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+}\varphi(x)b(y)\partial_{y_n}[-G_t(x'-y', x_n-y_n) - G_t(x'-y', x_n+y_n)]dxdy \\ &= \int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+}\varphi(x)[G_t(x'-y', x_n-y_n) + G_t(x'-y', x_n+y_n)]\partial_n b(y)dxdy \\ &= \langle eF(t)\partial_n b(x), \varphi(x)\rangle,\end{aligned}$$

for any $\varphi \in C_0^\infty(\mathbb{R}^n)$, from which we get

$$\partial_n eE(t)b = eF(t)\partial_n b. \tag{2.51}$$

To proceed, we need the following variants of the estimates, which are from Lemma 2.4 in [21]: Assume $a \in L^1(\mathbb{R}^n_+)$. Then for $t > 0$

$$\begin{aligned}&\|\nabla^m\partial_\ell E(t)a\|_{\mathcal{H}^1(\mathbb{R}^n)} + \|\nabla^m\partial_\ell F(t)a\|_{\mathcal{H}^1(\mathbb{R}^n)} + \|\nabla^m\partial_j\partial_k\Lambda^{-1}eE(t)a\|_{\mathcal{H}^1(\mathbb{R}^n)} \\ &\quad + \|\nabla^m\partial_j\partial_k\Lambda^{-1}F(t)a\|_{\mathcal{H}^1(\mathbb{R}^n)} + \|\nabla^m\Lambda eE(t)a\|_{\mathcal{H}^1(\mathbb{R}^n)} + \|\nabla^m\Lambda eF(t)a\|_{\mathcal{H}^1(\mathbb{R}^n)} \\ &\le Ct^{-\frac{1}{2}-\frac{m}{2}}\|a\|_{L^1(\mathbb{R}^n_+)} \quad \text{for } 0 \le \ell \le n,\ 1 \le j, k \le n-1,\ m = 0, 1, \ldots\end{aligned} \tag{2.52}$$

Note that the Riesz operators R_j $(1 \le j \le n)$ are bounded in the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$. Therefore from (2.48)–(2.52), we obtain for $1 \le j, k \le n$ and $t > 0$

$$\|\nabla^m\partial_j\partial_k\overline{u}\|_{\mathcal{H}^1(\mathbb{R}^n)} \le Ct^{-\frac{1}{2}-\frac{m}{2}}\|\nabla b\|_{L^1(\mathbb{R}^n_+)}.$$

In addition, since $\overline{u}$ [given in (2.2), (2.3)] is the extension of the Stokes flow $e^{-tA}b$ from $\mathbb{R}^n_+$ to $\mathbb{R}^n$, and $e^{-tA}b = \overline{u}\,|_{\mathbb{R}^n_+}$. We find for $m = 0, 1, \ldots$ and $t > 0$

$$\|\nabla^{m+2}e^{-tA}b\|_{L^1(\mathbb{R}^n_+)} \le \|\nabla^{m+2}\overline{u}\|_{L^1(\mathbb{R}^n)} \le \|\nabla^{m+2}\overline{u}\|_{\mathcal{H}^1(\mathbb{R}^n)} \le Ct^{-\frac{1}{2}-\frac{m}{2}}\|\nabla b\|_{\mathcal{H}^1(\mathbb{R}^n_+)}. \tag{2.53}$$

The proof of Lemma 2.3 with $q = 1$ follows from (2.53).

Recall that $G_t(x, y) = G_t(y, x)$, and for $1 \le j, k \le n$, $1 \le \ell, m \le n-1$

$$R_{x_j}R_{x_k}\partial_{x_\ell}\partial_{x_m}\Lambda_x^{-1}G_t(x, y) = R_{y_j}R_{y_k}\partial_{y_\ell}\partial_{y_m}\Lambda_y^{-1}G_t(x, y). \tag{2.54}$$

In addition, it holds for $t > 0$ (see Lemma 4.2 in [37])

$$\sup_{x\in\mathbb{R}^n}\|\partial_{y_j}\partial_{y_k}\Lambda_y^{-1}G_t(x,\cdot)\|_{\mathcal{H}^1(\mathbb{R}^n)} \le Ct^{-\frac{1}{2}}. \tag{2.55}$$

Since $\sum_{j=1}^{n-1}\partial_j\partial_j\Lambda^{-1} = -\Lambda$. Using (2.55), we get for $t > 0$

$$\sup_{x\in\mathbb{R}^n}\|\Lambda_y G_t(x,\cdot)\|_{\mathcal{H}^1(\mathbb{R}^n)} \le Ct^{-\frac{1}{2}}. \tag{2.56}$$

Note that $E(t)f = G_t * f^*$, $F(t)f = G_t * f_*$. It follows from (2.49)–(2.51) and (2.54)–(2.56) that for $t > 0$

$$\begin{aligned}
\|\partial_k\partial_j\overline{u}_n\|_{L^\infty(\mathbb{R}^n)} \le \sum_{m=1}^{n-1}\sup_{x\in\mathbb{R}^n}\big\{&\|R_{y_j}R_{y_m}\Lambda_y eG_t(x,\cdot)\|_{L^1(\mathbb{R}^n)}\|(\partial_k b_m)^*\|_{L^\infty(\mathbb{R}^n)}\\
&+\|R_{y_j}R_{y_n}\Lambda_y eG_t(x,\cdot)\|_{L^1(\mathbb{R}^n)}\|(\partial_m b_m)^*\|_{L^\infty(\mathbb{R}^n)}\\
&+\|R_{y_j}R_{y_n}\partial_m eG_t(x,\cdot)\|_{L^1(\mathbb{R}^n)}\|(\partial_k b_m)^*\|_{L^\infty(\mathbb{R}^n)}\\
&+\|R_{y_j}R_{y_n}\partial_n eG_t(x,\cdot)\|_{L^1(\mathbb{R}^n)}\|(\partial_m b_m)^*\|_{L^\infty(\mathbb{R}^n)}\\
&+\|R_{y_j}R_{y_m}\partial_m eG_t(x,\cdot)\|_{L^1(\mathbb{R}^n)}\|(\partial_k b_n)^*\|_{L^\infty(\mathbb{R}^n)}\\
&+\|R_{y_j}R_{y_m}\partial_n eG_t(x,\cdot)\|_{L^1(\mathbb{R}^n)}\|(\partial_m b_n)^*\|_{L^\infty(\mathbb{R}^n)}\big\}\\
&+\sup_{x\in\mathbb{R}^n}\big\{\|R_{y_j}R_{y_n}\Lambda_y eG_t(x,\cdot)\|_{L^1(\mathbb{R}^n)}\|(\partial_k b_n)^*\|_{L^\infty(\mathbb{R}^n)}\\
&+\|R_{y_j}R_{y_n}\Lambda_y eG_t(x,\cdot)\|_{L^1(\mathbb{R}^n)}\|(\partial_n b_n)_*\|_{L^\infty(\mathbb{R}^n)}\big\}\\
\le C\big(\sup_{x\in\mathbb{R}^n}&\|\Lambda_y eG_t(x,\cdot)\|_{\mathcal{H}^1(\mathbb{R}^n)} + \sup_{x\in\mathbb{R}^n}\|\nabla eG_t(x,\cdot)\|_{\mathcal{H}^1(\mathbb{R}^n)}\big)\|\nabla b\|_{L^\infty(\mathbb{R}^n)}\\
\le Ct^{-\frac{1}{2}}&\|\nabla b\|_{L^\infty(\mathbb{R}^n)} \quad \text{for} \quad 1 \le k, j \le n;
\end{aligned} \tag{2.57}$$

$$\begin{aligned}
\|\partial_k\partial_n\overline{u}'\|_{L^\infty(\mathbb{R}^n)} \le C\sup_{x\in\mathbb{R}^n}\big\{&\|\partial_n G_t\|_{L^1(\mathbb{R}^n)}\|\partial_k b'\|_{L^\infty(\mathbb{R}^n_+)} + \|\partial_n G_t\|_{L^1(\mathbb{R}^n)}\|\partial_n b'\|_{L^\infty(\mathbb{R}^n_+)}\\
&+\|\nabla'_y\nabla'_y\Lambda_y^{-1}G_t(x,\cdot)\|_{\mathcal{H}^1(\mathbb{R}^n)}\|\partial_k b'\|_{L^\infty(\mathbb{R}^n_+)} + \|\nabla'_y\nabla'_y\Lambda_y^{-1}G_t(x,\cdot)\|_{\mathcal{H}^1(\mathbb{R}^n)}\|\partial_n b'\|_{L^\infty(\mathbb{R}^n_+)}\\
&+\||R_n\||_{\mathcal{H}^1}\||R'\||_{\mathcal{H}^1}\|\nabla'_y G_t(x,\cdot)\|_{\mathcal{H}^1(\mathbb{R}^n)}\|\partial_k b'\|_{L^\infty(\mathbb{R}^n_+)}\\
&+\||R_n\||_{\mathcal{H}^1}\||R'\||_{\mathcal{H}^1}\|\partial_{y_n}G_t(x,\cdot)\|_{\mathcal{H}^1(\mathbb{R}^n)}\|\nabla'\cdot b'\|_{L^\infty(\mathbb{R}^n_+)}\\
&+\||R_n\||^2_{\mathcal{H}^1}\|\nabla'_y\nabla'_y\Lambda_y^{-1}G_t(x,\cdot)\|_{\mathcal{H}^1(\mathbb{R}^n)}\|\partial_k b'\|_{L^\infty(\mathbb{R}^n_+)}\\
&+\||R_n\||^2_{\mathcal{H}^1}\|\nabla'_y\nabla'_y\Lambda_y^{-1}G_t(x,\cdot)\|_{\mathcal{H}^1(\mathbb{R}^n)}\|\partial_n b'\|_{L^\infty(\mathbb{R}^n_+)}\\
&+\||R_n\||_{\mathcal{H}^1}\||R'\||_{\mathcal{H}^1}\|\Lambda_y G_t(x,\cdot)\|_{\mathcal{H}^1(\mathbb{R}^n)}\|\partial_k b_n\|_{L^\infty(\mathbb{R}^n_+)}\\
&+\||R_n\||_{\mathcal{H}^1}\||R'\||_{\mathcal{H}^1}\|\Lambda_y G_t(x,\cdot)\|_{\mathcal{H}^1(\mathbb{R}^n)}\|\partial_n b_n\|_{L^\infty(\mathbb{R}^n_+)}\\
&+\||R_n\||^2_{\mathcal{H}^1}\|\nabla'_y G_t(x,\cdot)\|_{\mathcal{H}^1(\mathbb{R}^n)}\|\partial_k b_n\|_{L^\infty(\mathbb{R}^n_+)} + \||R_n\||^2_{\mathcal{H}^1}\|\nabla'_y G_t(x,\cdot)\|_{\mathcal{H}^1(\mathbb{R}^n)}\|\partial_n b_n\|_{L^\infty(\mathbb{R}^n_+)}\big\}\\
\le Ct^{-\frac{1}{2}}&\|\nabla b\|_{L^\infty(\mathbb{R}^n)} \quad \text{for} \quad 1 \le k \le n;
\end{aligned} \tag{2.58}$$

and

$$\begin{aligned}\|\partial_k\partial_j\overline{u}'\|_{L^\infty(\mathbb{R}^n)} \le C \sup_{x\in\mathbb{R}^n} \{&\|\partial_k G_t\|_{L^1(\mathbb{R}^n)}\|\partial_j b'\|_{L^\infty(\mathbb{R}^n_+)}\\ &+\|\partial_{y_j}\nabla'_y\Lambda_y^{-1}G_t(x,\cdot)\|_{\mathcal{H}^1(\mathbb{R}^n)}\|\partial_k b_n\|_{L^\infty(\mathbb{R}^n_+)} + \|\partial_{y_j}\nabla'_y\Lambda_y^{-1}G_t(x,\cdot)\|_{\mathcal{H}^1(\mathbb{R}^n)}\|\partial_n b_n\|_{L^\infty(\mathbb{R}^n_+)}\\ &+\||R_k\||_{\mathcal{H}^1}\||R'\||_{\mathcal{H}^1}\||\nabla'_y G_t(x,\cdot)\|_{\mathcal{H}^1(\mathbb{R}^n)}\|\partial_j b'\|_{L^\infty(\mathbb{R}^n_+)}\\ &+\||R_k\||_{\mathcal{H}^1}\|\||R_n\||_{\mathcal{H}^1}\||\nabla'_y\nabla'_y\Lambda_y^{-1}G_t(x,\cdot)\|_{\mathcal{H}^1(\mathbb{R}^n)}\|\partial_j b'\|_{L^\infty(\mathbb{R}^n_+)}\\ &+\||R_k\||_{\mathcal{H}^1}\|\||R'\||_{\mathcal{H}^1}\||\Lambda_y G_t(x,\cdot)\|_{\mathcal{H}^1(\mathbb{R}^n)}\|\partial_j b_n\|_{L^\infty(\mathbb{R}^n_+)}\\ &+\||R_k\||_{\mathcal{H}^1}\|\||R_n\||_{\mathcal{H}^1}\||\nabla'_y G_t(x,\cdot)\|_{\mathcal{H}^1(\mathbb{R}^n)}\|\partial_j b_n\|_{L^\infty(\mathbb{R}^n_+)}\}\\ \le Ct^{-\frac{1}{2}}\|\nabla b\|_{L^\infty(\mathbb{R}^n)} \quad &\text{for} \quad 1\le k\le n, \ \ 1\le j\le n-1.\end{aligned} \tag{2.59}$$

Whence from (2.57)–(2.59), we derive for $1\le j,k\le n$ and $t>0$

$$\|\partial_j\partial_k\overline{u}\|_{L^\infty(\mathbb{R}^n)} \le Ct^{-\frac{1}{2}}\|\nabla b\|_{L^\infty(\mathbb{R}^n_+)}.$$

Note that $\overline{u}$ [given in (2.2), (2.3) with initial data b] is the extension of the Stokes flow $e^{-tA}b$ from $\mathbb{R}^n_+$ to $\mathbb{R}^n$, and $e^{-tA}b=\overline{u}\,|_{\mathbb{R}^n_+}$. We conclude for $t>0$

$$\|\nabla^2 e^{-tA}b\|_{L^\infty(\mathbb{R}^n_+)} \le \|\nabla^2\overline{u}\|_{L^\infty(\mathbb{R}^n)} \le Ct^{-\frac{1}{2}}\|\nabla b\|_{L^\infty(\mathbb{R}^n_+)}. \tag{2.60}$$

Repeating the proof of (2.60), it is not hard to find that for $m=1,2,\cdots$ and $t>0$

$$\|\nabla^{m+2}e^{-tA}b\|_{L^\infty(\mathbb{R}^n_+)} \le \|\nabla^2\overline{u}\|_{L^\infty(\mathbb{R}^n)} \le Ct^{-\frac{1}{2}-\frac{m}{2}}\|\nabla b\|_{L^\infty(\mathbb{R}^n_+)},$$

from which and (2.60), we complete the proof of Lemma 2.3 with $q=\infty$. □

Proof of Theorem 1.1. From Lemmata 2.1, 2.3, we obtain for $k=2,\ldots$, $0<\beta\le 1$ and $t>0$

$$\begin{aligned}\|\nabla^k e^{-tA}a\|_{L^1(\mathbb{R}^n_+)} &= \|\nabla^k e^{-\frac{t}{2}A}e^{-\frac{t}{2}A}a\|_{L^1(\mathbb{R}^n_+)}\\ &\le Ct^{-\frac{k-1}{2}}\|\nabla e^{-\frac{t}{2}A}a\|_{L^1(\mathbb{R}^n_+)}\\ &\le Ct^{-\frac{\beta}{2}-\frac{k}{2}}\||x|^\beta a\|_{L^1(\mathbb{R}^n_+)},\end{aligned}$$

which, together with Lemma 2.1 implies that (1.2) is valid for $k\in\mathbb{N}$.

On the other hand, it follows from Lemmata 2.2, 2.3 that for $k=2,\ldots$ and $t>0$

$$\begin{aligned}\|\nabla^k e^{-tA}a\|_{L^\infty(\mathbb{R}^n_+)} &= \|\nabla^k e^{-\frac{t}{2}A}e^{-\frac{t}{2}A}a\|_{L^\infty(\mathbb{R}^n_+)}\\ &\le Ct^{-\frac{k-1}{2}}\|\nabla e^{-\frac{t}{2}A}a\|_{L^\infty(\mathbb{R}^n_+)}\\ &\le Ct^{-\frac{k+1}{2}}\||x|a\|_{L^\infty(\mathbb{R}^n_+)},\end{aligned}$$

which, together with Lemma 2.2 implies that (1.3) is true for $k\in\mathbb{N}$. □

Proof of Theorem 1.2. Note that the Stokes flows with initial data u_0 can be expressed by $e^{-tA}u_0$. Moreover, $e^{-tA}u_0$ can also be given as follows (see [38,39]):

$$e^{-tA}u_0(x) = \int_{\mathbb{R}^n_+}\mathcal{M}(x,y,t)u_0(y)dy, \tag{2.61}$$

where $\mathcal{M}=(M_{ij})_{i,j=1,2,\ldots,n}$ is defined by

$$\begin{aligned}M_{ij}(x,y,t) &= \delta_{ij}(G_t(x-y)-G_t(x-y^*))\\ &\quad +4(1-\delta_{jn})\frac{\partial}{\partial x_j}\int_0^{x_n}\int_{\mathbb{R}^{n-1}}\frac{\partial E(x-z)}{\partial x_i}G_t(z-y^*)dz\\ &\equiv \delta_{ij}G_t(x-y)+M^*_{ij}(x,y,t)\\ &\equiv \delta_{ij}(G_t(x-y)-G_t(x'-y',x_n+y_n))+N^*_{ij}(x,y,t),\end{aligned} \tag{2.62}$$

$y^* = (y_1, y_2, \dots, -y_n)$, $G_t(x) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$ is the Gauss kernel, and

$$E(z) = \begin{cases} -\dfrac{\Gamma(\frac{n}{2})}{2(n-2)\pi^{\frac{n}{2}}} \dfrac{1}{|z|^{n-2}} & \text{if } n > 2, \\ \dfrac{1}{2\pi} \log |z| & \text{if } n = 2, \end{cases}$$

is the fundamental solution of the Laplace equation. Moreover, the following estimate for M^*_{ij}, N^*_{ij} in (2.62) holds

$$\begin{aligned} &|\partial_t^s \partial_x^k \partial_y^m M^*_{ij}(x,y,t)| + |\partial_t^s \partial_x^k \partial_y^m N^*_{ij}(x,y,t)| \\ &\quad \le C t^{-s-\frac{m_n}{2}} (t + x_n^2)^{-\frac{k_n}{2}} (|x - y^*|^2 + t)^{-\frac{n+|k'|+|m'|}{2}} e^{-\frac{c y_n^2}{t}}, \end{aligned} \tag{2.63}$$

where $m = (m_1, m_2, \dots, m_{n-1}, m_n) = (m', m_n)$, $k = (k_1, k_2, \cdots, k_{n-1}, k_n) = (k', k_n)$.

Let $s = |k| = |m| = 0$ in (2.63), we find for any $\alpha \ge 0$,

$$(|x'|^2 + x_n^2 + t)^{-\frac{n}{2}} f_\alpha(x) \notin L^1(\mathbb{R}^n_+), \quad \text{where} \quad f_\alpha(x) = x_n^\alpha, \quad |x'|^\alpha, \quad |x|^\alpha.$$

Note that the results in Theorem 1.2 have been shown to be true in [20] for the case: $(k, \alpha) = (0, 0)$ and $1 < r = q \le \infty$. Let $1 < r = q \le \infty$. Then $k \ge 1$, which follows from the assumptions in Theorem 1.2:

$$k = 0, 1, 2, \dots \quad \text{and} \quad 0 \le \alpha < k + n\left(\frac{1}{r} - \frac{1}{q}\right), \quad \text{here } 1 \le r < q \le \infty, \quad \text{or} \quad 1 < r \le q < \infty.$$

Using (2.61) and (2.63), we get for $t > 0$

$$\begin{aligned} &\|x_n^\alpha \nabla^k e^{-tA} u_0\|_{L^q(\mathbb{R}^n_+)} \\ &\quad \le \| \int_{\mathbb{R}^n_+} |x_n - y_n|^\alpha |\nabla_x^k G_t(x-y)| |u_0(y)| dy \|_{L^q(\mathbb{R}^n_+)} \\ &\qquad + \| \int_{\mathbb{R}^n_+} |\nabla_x^k G_t(x-y)| y_n^\alpha |u_0(y)| dy \|_{L^q(\mathbb{R}^n_+)} \\ &\qquad + C \| \int_{\mathbb{R}^n_+} (t + x_n^2)^{-\frac{k_n}{2}} (x_n + y_n)^\alpha (|x - y^*|^2 + t)^{-\frac{n+|k'|}{2}} |u_0(y)| dy \|_{L^q(\mathbb{R}^n_+)} \\ &\quad \le \|x_n^\alpha \nabla^k G_t\|_{L^{(1+\frac{1}{q}-\frac{1}{r})^{-1}}(\mathbb{R}^n_+)} \|u_0\|_{L^r(\mathbb{R}^n_+)} + \|\nabla^k G_t\|_{L^{(1+\frac{1}{q}-\frac{1}{r})^{-1}}(\mathbb{R}^n_+)} \|y_n^\alpha u_0\|_{L^r(\mathbb{R}^n_+)} \\ &\qquad + \|(t + x_n^2)^{-\frac{k_n}{2}} x_n^\alpha (|x'|^2 + x_n^2 + t)^{-\frac{n+|k'|}{2}}\|_{L^{(1+\frac{1}{q}-\frac{1}{r})^{-1}}(\mathbb{R}^n_+)} \|u_0\|_{L^r(\mathbb{R}^n_+)} \\ &\quad \le C(t^{\frac{\alpha}{2}-\frac{k}{2}-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})} \|u_0\|_{L^r(\mathbb{R}^n_+)} + t^{-\frac{k}{2}-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})} \|y_n^\alpha u_0\|_{L^r(\mathbb{R}^n_+)}); \end{aligned}$$

and

$$\begin{aligned} &\||x'|^\alpha \nabla^k e^{-tA} u_0\|_{L^q(\mathbb{R}^n_+)} \\ &\quad \le \| \int_{\mathbb{R}^n_+} |x' - y'|^\alpha |\nabla_x^k G_t(x-y)| |u_0(y)| dy \|_{L^q(\mathbb{R}^n_+)} \\ &\qquad + \| \int_{\mathbb{R}^n_+} |\nabla_x^k G_t(x-y)| |y'|^\alpha |u_0(y)| dy \|_{L^q(\mathbb{R}^n_+)} \\ &\qquad + C \| \int_{\mathbb{R}^n_+} (t + x_n^2)^{-\frac{k_n}{2}} |x' - y'|^\alpha (|x - y^*|^2 + t)^{-\frac{n+|k'|}{2}} |u_0(y)| dy \|_{L^q(\mathbb{R}^n_+)} \\ &\qquad + C \| \int_{\mathbb{R}^n_+} (t + x_n^2)^{-\frac{k_n}{2}} (|x - y^*|^2 + t)^{-\frac{n+|k'|}{2}} |y'|^\alpha |u_0(y)| dy \|_{L^q(\mathbb{R}^n_+)} \end{aligned}$$

$$
\begin{aligned}
&\le \||x'|^\alpha \nabla^k G_t\|_{L^{(1+\frac{1}{q}-\frac{1}{r})^{-1}}(\mathbb{R}^n_+)} \|u_0\|_{L^r(\mathbb{R}^n_+)} + \|\nabla^k G_t\|_{L^{(1+\frac{1}{q}-\frac{1}{r})^{-1}}(\mathbb{R}^n_+)} \||y'|^\alpha u_0\|_{L^r(\mathbb{R}^n_+)} \\
&\quad + \|(t+x_n^2)^{-\frac{k_n}{2}} |x'|^\alpha (|x'|^2 + x_n^2 + t)^{-\frac{n+|k'|}{2}}\|_{L^{(1+\frac{1}{q}-\frac{1}{r})^{-1}}(\mathbb{R}^n_+)} \|u_0\|_{L^r(\mathbb{R}^n_+)} \\
&\quad + \|(t+x_n^2)^{-\frac{k_n}{2}} (|x'|^2 + x_n^2 + t)^{-\frac{n+|k'|}{2}}\|_{L^{(1+\frac{1}{q}-\frac{1}{r})^{-1}}(\mathbb{R}^n_+)} \||y'|^\alpha u_0\|_{L^r(\mathbb{R}^n_+)} \\
&\le C(t^{\frac{\alpha}{2}-\frac{k}{2}-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})} \|u_0\|_{L^r(\mathbb{R}^n_+)} + t^{-\frac{k}{2}-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})} \||y'|^\alpha u_0\|_{L^r(\mathbb{R}^n_+)}).
\end{aligned}
$$

From the above arguments, we complete the proof of Theorem 1.2. □

3. Weighted Decay Properties for the Navier–Stokes Flows

Let $g = \mathcal{N} f$ denote the solution of the Neumann problem

$$
\begin{cases} -\Delta g = f & \text{in } \mathbb{R}^n_+, \\ \partial_\nu g = 0 & \text{on } \partial\mathbb{R}^n_+. \end{cases}
$$

Then (see [24])

$$
\mathcal{N} = \int_0^\infty F(\tau) d\tau. \tag{3.1}
$$

Moreover the following decomposition holds for $u \in L^2_\sigma(\mathbb{R}^n_+) \bigcap H^1_0(\mathbb{R}^n_+)$

$$
P(u \cdot \nabla u) = u \cdot \nabla u + \sum_{i,j=1}^n \nabla \mathcal{N} \partial_i \partial_j (u_i u_j). \tag{3.2}
$$

The following lemma plays a crucial role in the proof of Theorem 1.3, from which the strong singularity can be avoided. It should be pointed out that such a study is of independent interest.

Lemma 3.1. *Let $0 < \theta < 1$, $0 < \alpha < 1$, $1 \le k \le n$, $n \ge 2$ and $1 \le q \le \infty$. Then for any $u \in C^\infty_{0,\sigma}(\mathbb{R}^n_+)$*

$$
\begin{aligned}
&\left\| \sum_{i,j=1}^n x_n^{-\theta} |x'|^\alpha \partial_k \mathcal{N} \partial_i \partial_j (u_i u_j) \right\|_{L^q(\mathbb{R}^n_+)} \\
&\quad \le C(\|u\|^2_{L^{2q}(\mathbb{R}^n_+)} + \|\nabla u\|^2_{L^{2q}(\mathbb{R}^n_+)} + \||y'|^{\frac{\alpha}{2}} u\|^2_{L^{2q}(\mathbb{R}^n_+)} + \||y'|^{\frac{\alpha}{2}} \nabla u\|^2_{L^{2q}(\mathbb{R}^n_+)}).
\end{aligned} \tag{3.3}
$$

Proof. Denote the odd and even extensions of a function f from $\mathbb{R}^n_+$ to $\mathbb{R}^n$ by f^*, f_*, respectively, see their definitions in the proof of Lemma 2.1 in Sect. 2. Recall the following inequality (see [23]): Let $0 \le \gamma < 1$, $1 \le k \le n$ and $1 \le q \le \infty$, then for any $u \in C^\infty_{0,\sigma}(\mathbb{R}^n_+)$

$$
\begin{aligned}
&\left\| \sum_{i,j=1}^n |x'|^\gamma \partial_k \mathcal{N} \partial_i \partial_j (u_i u_j) \right\|_{L^q(\mathbb{R}^n_+)} \\
&\quad \le C(\|u\|^2_{L^{2q}(\mathbb{R}^n_+)} + \|\nabla u\|^2_{L^{2q}(\mathbb{R}^n_+)} + \||y'|^{\frac{\gamma}{2}} u\|^2_{L^{2q}(\mathbb{R}^n_+)} + \||y'|^{\frac{\gamma}{2}} \nabla u\|^2_{L^{2q}(\mathbb{R}^n_+)}).
\end{aligned} \tag{3.4}
$$

Let $0 < \theta < 1$, $0 < \alpha < 1$ and $1 \le q \le \infty$. From (3.1), (3.4), one has for any $1 \le k \le n$

$$
\begin{aligned}
&\left\| \sum_{i,j=1}^n x_n^{-\theta} |x'|^\alpha \partial_k \mathcal{N} \partial_i \partial_j (u_i u_j) \right\|_{L^q(\mathbb{R}^n_+)} \\
&\le \left\| \sum_{i,j=1}^n x_n^{-\theta} |x'|^\alpha \partial_k \int_0^1 G_\tau * [\partial_i \partial_j (u_i u_j)]_* d\tau \right\|_{L^q(\mathbb{R}^{n-1} \times (0,1))}
\end{aligned}
$$

$$
\begin{aligned}
&+\left\|\sum_{i,j=1}^{n} x_n^{-\theta}|x'|^{\alpha}\partial_k\int_1^{\infty}(\partial_i\partial_j G_\tau)*w_{ij}d\tau\right\|_{L^q(\mathbb{R}^{n-1}\times(0,1))}\\
&+\left\|\sum_{i,j=1}^{n}|x'|^{\alpha}\partial_k\mathcal{N}\partial_i\partial_j(u_iu_j)\right\|_{L^q(\mathbb{R}^{n-1}\times[1,\infty))}\\
\le\ & C\sup_{y\in\mathbb{R}^n}\left\|\int_0^1 x_n^{-\theta}|x'-y'|^{\alpha}|\partial_k G_\tau(x-y)|d\tau\right\|_{L^1(\mathbb{R}^{n-1}\times(0,1))}\left\|\sum_{i,j=1}^{n}\partial_i\partial_j(u_iu_j)\right\|_{L^q(\mathbb{R}^n_+)}\\
&+C\sup_{y\in\mathbb{R}^n}\left\|\int_0^1 x_n^{-\theta}|\partial_k G_\tau(x-y)|d\tau\right\|_{L^1(\mathbb{R}^{n-1}\times(0,1))}\left\|\sum_{i,j=1}^{n}|y'|^{\alpha}\partial_i\partial_j(u_iu_j)\right\|_{L^q(\mathbb{R}^n_+)}\\
&+C\sup_{y\in\mathbb{R}^n}\sum_{i,j=1}^{n}\left\|\int_1^{\infty}x_n^{-\theta}|x'-y'|^{\alpha}|\partial_k\partial_i\partial_j G_\tau(x-y)|d\tau\right\|_{L^1(\mathbb{R}^{n-1}\times(0,1))}\|w_{ij}\|_{L^q(\mathbb{R}^n_+)}\\
&+C\sup_{y\in\mathbb{R}^n}\sum_{i,j=1}^{n}\left\|\int_1^{\infty}x_n^{-\theta}|\partial_k\partial_i\partial_j G_\tau(x-y)|d\tau\|_{L^1(\mathbb{R}^{n-1}\times(0,1))}\||y'|^{\alpha}w_{ij}\right\|_{L^q(\mathbb{R}^n_+)}\\
&+\left\|\sum_{i,j=1}^{n}|x'|^{\alpha}\partial_k\mathcal{N}\partial_i\partial_j(u_iu_j)\right\|_{L^q(\mathbb{R}^n_+)},
\end{aligned}
\tag{3.5}
$$

where $w_{ij}=(u_iu_j)_*$ if $1\le i,j\le n-1$ or $i=j=n$; $w_{in}=(u_iu_n)^*$ if $1\le i\le n-1$; $w_{nj}=(u_nu_j)^*$ if $1\le j\le n-1$.

In the following arguments, we take $1\le i,j,k\le n$, $0<\alpha<1$ and $0<\theta<1$. Take $q_1,q_2\in(1,\infty)$ such that $\frac{1}{q_1}+\frac{1}{q_2}=1$ and $\theta q_1<1$. Then for any $y=(y',y_n)\in\mathbb{R}^{n-1}\times\mathbb{R}^1$

$$
\begin{aligned}
&\left\|\int_0^1 x_n^{-\theta}|x'-y'|^{\alpha}|\partial_k G_\tau(x-y)|d\tau\right\|_{L^1(\mathbb{R}^{n-1}\times(0,1))}\\
&\quad\le C\int_0^1\int_{\mathbb{R}^{n-1}}\int_0^1\tau^{-\frac12+\frac{\alpha}{2}}x_n^{-\theta}\left(\frac{|x'-y'|}{2\sqrt{\tau}}\right)^{\alpha}\frac{|x_k-y_k|}{2\sqrt{\tau}}G_\tau(x-y)dxd\tau\\
&\quad\le C\int_0^1\int_{\mathbb{R}^{n-1}}\int_0^1\tau^{-1+\frac{\alpha}{2}}x_n^{-\theta}(4\pi\tau)^{-\frac{n-1}{2}}e^{-\frac{|x'-y'|^2}{8\tau}}e^{-\frac{(x_n-y_n)^2}{8\tau}}dx'dx_nd\tau\\
&\quad\le C\int_0^1\tau^{-1+\frac{\alpha}{2}}\left(\int_0^1 x_n^{-\theta q_1}dx_n\right)^{\frac{1}{q_1}}\left(\int_0^1 e^{-\frac{(x_n-y_n)^2q_2}{8\tau}}dx_n\right)^{\frac{1}{q_2}}d\tau\\
&\quad\le C\int_0^1\tau^{-1+\frac{\alpha}{2}+\frac{1}{2q_2}}d\tau\le C;
\end{aligned}
\tag{3.6}
$$

$$
\begin{aligned}
&\left\|\int_1^{\infty}x_n^{-\theta}|x'-y'|^{\alpha}|\partial_k\partial_i\partial_j G_\tau(x-y)|d\tau\right\|_{L^1(\mathbb{R}^{n-1}\times(0,1))}\\
&\quad\le C\int_1^{\infty}\tau^{-\frac32+\frac{\alpha}{2}}\int_{\mathbb{R}^{n-1}}\int_0^1 x_n^{-\theta}\left(\frac{|x'-y'|}{2\sqrt{\tau}}\right)^{\alpha}\left(\frac{|x_i-y_i|}{2\sqrt{\tau}}\frac{|x_j-y_j|}{2\sqrt{\tau}}\frac{|x_k-y_k|}{2\sqrt{\tau}}\right.\\
&\qquad\left.+\frac{|x_k-y_k|\delta_{ij}+|x_i-y_i|\delta_{kj}+|x_j-y_j|\delta_{ki}}{2\sqrt{\tau}}\right)\\
&\qquad\times(4\pi\tau)^{-\frac{n}{2}}e^{-\frac{|x'-y'|^2}{4\tau}}e^{-\frac{|x_n-y_n|^2}{4\tau}}dx'dx_nd\tau
\end{aligned}
$$

$$\leq C\int_1^\infty\int_0^1 \tau^{-2+\frac{\alpha}{2}}x_n^{-\theta}e^{-\frac{(x_n-y_n)^2}{8\tau}}dx_nd\tau$$
$$\leq C\int_1^\infty \tau^{-2+\frac{\alpha}{2}}d\tau\int_0^1 x_n^{-\theta}dx_n \leq C. \tag{3.7}$$

It is not difficult to find that the estimates (3.6), (3.7) are still valid for $\alpha = 0$. That is,

$$\left\|\int_0^1 x_n^{-\theta}|\partial_k G_\tau(x-y)|d\tau\right\|_{L^1(\mathbb{R}^{n-1}\times(0,1))} \leq C\int_0^1 \tau^{-1+\frac{1}{2q_2}}d\tau \leq C, \tag{3.8}$$

and

$$\left\|\int_1^\infty x_n^{-\theta}|\partial_k\partial_i\partial_j G_\tau(x-y)|d\tau\right\|_{L^1(\mathbb{R}^{n-1}\times(0,1))} \leq C\int_1^\infty \tau^{-2}d\tau\int_0^1 x_n^{-\theta}dx_n \leq C. \tag{3.9}$$

Note that for $0 \leq \gamma < 1$ and $1 \leq q \leq \infty$,

$$\begin{aligned}&\left\|\sum_{i,j=1}^n |y'|^\gamma\partial_i\partial_j(u_iu_j)\right\|_{L^q(\mathbb{R}^n_+)} + \sum_{i,j=1}^n \||y'|^\gamma w_{ij}\|_{L^q(\mathbb{R}^n_+)}\\ &\quad\leq C(\||y'|^{\frac{\gamma}{2}}u\|^2_{L^{2q}(\mathbb{R}^n_+)} + \||y'|^{\frac{\gamma}{2}}\nabla u\|^2_{L^{2q}(\mathbb{R}^n_+)}).\end{aligned} \tag{3.10}$$

From (3.4)–(3.10), we conclude that for $0 < \alpha < 1$, $0 < \theta < 1$, $1 \leq k \leq n$ and $1 \leq q \leq \infty$

$$\begin{aligned}&\left\|\sum_{i,j=1}^n x_n^{-\theta}|x'|^\alpha\partial_k\mathcal{N}\partial_i\partial_j(u_iu_j)\right\|_{L^q(\mathbb{R}^n_+)}\\ &\quad\leq C(\|u\|^2_{L^{2q}(\mathbb{R}^n_+)} + \|\nabla u\|^2_{L^{2q}(\mathbb{R}^n_+)} + \||y'|^{\frac{\alpha}{2}}u\|^2_{L^{2q}(\mathbb{R}^n_+)} + \||y'|^{\frac{\alpha}{2}}\nabla u\|^2_{L^{2q}(\mathbb{R}^n_+)}),\end{aligned}$$

which is (3.3). □

Lemma 3.2. [20,25] *Assume $u_0 \in L^1(\mathbb{R}^n_+)\bigcap L^q_\sigma(\mathbb{R}^n_+)$ $(n \geq 2)$ for all $1 < q < \infty$. Then there exists a number $\eta_0 > 0$ such that if $\|u_0\|_{L^n(\mathbb{R}^n_+)} \leq \eta_0$ (smallness condition is unnecessary if $n = 2$), problem (1.1) possesses a unique strong solution u which satisfies for all $1 < r \leq \infty$, $k = 0, 1, 2$ and $t \geq 1$*

$$\|\nabla^k u(t)\|_{L^r(\mathbb{R}^n_+)} \leq Ct^{-\frac{k}{2}-\frac{n}{2}(1-\frac{1}{r})}.$$

Further if u_0 satisfies $\|x_nu_0\|_{L^1(\mathbb{R}^n_+)} < \infty$, then

$$\|\nabla^k u(t)\|_{L^r(\mathbb{R}^n_+)} \leq Ct^{-\frac{1}{2}-\frac{k}{2}-\frac{n}{2}(1-\frac{1}{r})}.$$

In the above two estimates, we require $1 < r < \infty$ if $k = 2$.

The following Lemma is a variant of Theorem 1.2 in [23,26], and we omit the details of its proof here.

Lemma 3.3. *Assume $u_0 \in L^1(\mathbb{R}^n_+)\bigcap L^q_\sigma(\mathbb{R}^n_+)(n \geq 2)$ for all $1 < q < \infty$, satisfies $\||x'|u_0\|_{L^1(\mathbb{R}^n_+)} + \||x'|u_0\|_{L^2(\mathbb{R}^n_+)} + \|(1 + |x'|)\nabla u_0\|_{L^2(\mathbb{R}^n_+)} < \infty$. Let u be the strong solution of (1.1) given in Lemma 3.2. Then for $n \geq 3$ and $t \geq 1$*

$$\||x'|^\beta u(t)\|_{L^r(\mathbb{R}^n_+)} \leq Ct^{-\frac{n}{2}(1-\frac{1}{r})+\frac{\beta}{2}} \tag{3.11}$$

with $1 < r \leq \infty$ and $0 < \beta < \min\{1,\ n(1-\frac{1}{r})\}$;

$$\||x'|^\beta\nabla u(t)\|_{L^r(\mathbb{R}^n_+)} \leq Ct^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{r})+\frac{\beta}{2}} \tag{3.12}$$

provided that $1 \leq r \leq \infty$ and $0 < \beta < 1$. Further if $\|x_nu_0\|_{L^1(\mathbb{R}^n_+)} < \infty$, then the estimates (3.11), (3.12) hold true for $n \geq 2$.

Proof of Theorem 1.3. Step 1. Without the assumption: $\|x_n u_0\|_{L^1(\mathbb{R}^n_+)} < \infty$. Let $1 \le r \le \infty$ and $0 < \beta < 1$, $n \ge 3$. Assume that u is the strong solution of (1.1) given in Lemma 3.2. Then it can be expressed as (see [38,39])

$$u(x,t) = \int_{\mathbb{R}^n_+} \mathcal{M}(x,y,t)u_0(y)dy - \int_0^t \int_{\mathbb{R}^n_+} \mathcal{M}(x,y,t-s)Pu(y,s)\cdot\nabla u(y,s)dyds,$$

Let $1 \le k, j \le n$ and $t \ge 1$. Then

$$\begin{aligned}
&\Big\| \int_{\mathbb{R}^n_+} |x'|^\beta \partial_{x_k}\partial_{x_j}\mathcal{M}(x,y,t)u_0(y)dy\Big\|_{L^r(\mathbb{R}^n_+)} \\
&\le C\int_{\mathbb{R}^n_+} \big(\||x'-y'|^\beta \partial_k\partial_j G_t(x-y)\|_{L^r(\mathbb{R}^n_+)} + \|\partial_k\partial_j G_t\|_{L^r(\mathbb{R}^n_+)}|y'|^\beta\big)|u_0(y)|dy \\
&\quad + C\|u_0\|_{L^1(\mathbb{R}^n_+)} \times \sup_{y=(y',y_n)\in\mathbb{R}^n_+} \|(x_n+\sqrt{t})^{-\ell_n}(|x'-y'|+x_n+y_n+\sqrt{t})^{\beta-n-|\ell'|}\|_{L^r(\mathbb{R}^n_+)} \\
&\quad + C\||y'|^\beta u_0\|_{L^1(\mathbb{R}^n_+)} \\
&\quad \times \sup_{y=(y',y_n)\in\mathbb{R}^n_+} \big\|(x_n+\sqrt{t})^{-\ell_n}(|x'-y'|+x_n+y_n+\sqrt{t}\big)^{-n-|\ell'|}\|_{L^r(\mathbb{R}^n_+)} \\
&\le C(t^{-1-\frac{n}{2}(1-\frac{1}{r})+\frac{\beta}{2}}\|u_0\|_{L^1(\mathbb{R}^n_+)} + t^{-1-\frac{n}{2}(1-\frac{1}{r})}\||y'|^\beta u_0\|_{L^1(\mathbb{R}^n_+)}) \\
&\le Ct^{-1-\frac{n}{2}(1-\frac{1}{r})+\frac{\beta}{2}}, \quad \text{where } 2 = |(\ell',\, \ell_n)|.
\end{aligned} \tag{3.13}$$

Let $1 \le k, j \le n$. By (3.2), (3.4) and Lemmata 3.2, 3.3, one has for any $t \ge 1$

$$\begin{aligned}
&\Big\||x'|^\beta \int_0^{\frac{t}{2}}\int_{\mathbb{R}^n_+} \partial_{x_k}\partial_{x_j}\mathcal{M}(x,y,t-s)Pu(y,s)\cdot\nabla u(y,s)dyds\Big\|_{L^r(\mathbb{R}^n_+)} \\
&\le C\int_0^{\frac{t}{2}}\int_{\mathbb{R}^n_+} \||x'-y'|^\beta \partial_{x_k}\partial_{x_j}G_{t-s}(x-y)\|_{L^r(\mathbb{R}^n_{+,x})}|Pu(y,s)\cdot\nabla u(y,s)|dyds \\
&\quad + C\int_0^{\frac{t}{2}}\int_{\mathbb{R}^n_+} \|\partial_k\partial_j G_{t-s}\|_{L^r(\mathbb{R}^n_+)}|y'|^\beta|Pu(y,s)\cdot\nabla u(y,s)|dyds \\
&\quad + C\int_0^{\frac{t}{2}} \sup_{y\in\mathbb{R}^n_+} \||x'-y'|^\beta\partial_{x_k}\partial_{x_j}\mathcal{M}^*(x,y,t-s)\|_{L^r(\mathbb{R}^n_+)} \\
&\quad \times \|Pu(s)\cdot\nabla u(s)\|_{L^1(\mathbb{R}^n_+)}ds \\
&\quad + C\int_0^{\frac{t}{2}} \sup_{y\in\mathbb{R}^n_+} \|\partial_{x_k}\partial_{x_j}\mathcal{M}^*(x,y,t-s)\|_{L^r(\mathbb{R}^n_+)}\||y'|^\beta Pu(s)\cdot\nabla u(s)\|_{L^1(\mathbb{R}^n_+)}ds \\
&\le Ct^{-1-\frac{n}{2}(1-\frac{1}{r})}\int_0^{\frac{t}{2}} \||y'|^\beta Pu(s)\cdot\nabla u(s)\|_{L^1(\mathbb{R}^n_+)}ds \\
&\quad + Ct^{-1-\frac{n}{2}(1-\frac{1}{r})+\frac{\beta}{2}}\int_0^{\frac{t}{2}} \|Pu(s)\cdot\nabla u(s)\|_{L^1(\mathbb{R}^n_+)}ds \\
&\le Ct^{-1-\frac{n}{2}(1-\frac{1}{r})}\int_0^{\frac{t}{2}} (\|u(s)\|^2_{L^2(\mathbb{R}^n_+)} + \|\nabla u(s)\|^2_{L^2(\mathbb{R}^n_+)} \\
&\quad + \||y'|^{\frac{\beta}{2}}u(s)\|^2_{L^2(\mathbb{R}^n_+)} + \||y'|^{\frac{\beta}{2}}\nabla u(s)\|^2_{L^2(\mathbb{R}^n_+)})ds \\
&\quad + Ct^{-1-\frac{n}{2}(1-\frac{1}{r})+\frac{\beta}{2}}\int_0^{\frac{t}{2}} (\|u(s)\|^2_{L^2(\mathbb{R}^n_+)} + \|\nabla u(s)\|^2_{L^2(\mathbb{R}^n_+)})ds
\end{aligned} \tag{3.14}$$

$$
\begin{aligned}
&\le Ct^{-1-\frac{n}{2}(1-\frac{1}{r})}\int_0^{\frac{t}{2}}(1+s)^{-\frac{n}{2}+\frac{\beta}{2}}ds + Ct^{-1-\frac{n}{2}(1-\frac{1}{r})+\frac{\beta}{2}}\int_0^{\frac{t}{2}}(1+s)^{-\frac{n}{2}}ds\\
&\le Ct^{-1-\frac{n}{2}(1-\frac{1}{r})+\frac{\beta}{2}}.
\end{aligned}
$$

Note that for $0 < s < t$

$$
\begin{aligned}
&\int_{\mathbb{R}^n_+}\partial_{x_n}(G_{t-s}(x-y)-G_{t-s}(x-y^*))Pu(y,s)\cdot\nabla u(y,s)dy\\
&\quad = -\int_{\mathbb{R}^n_+}\partial_{y_n}(G_{t-s}(x-y)-G_{t-s}(x-y^*))Pu(y,s)\cdot\nabla u(y,s)dy\\
&\qquad -2\int_{\mathbb{R}^n_+}\partial_{x_n}G_{t-s}(x-y^*)Pu(y,s)\cdot\nabla u(y,s)dy\\
&\quad = \int_{\mathbb{R}^n_+}(G_{t-s}(x-y)-G_{t-s}(x-y^*))\partial_n Pu(y,s)\cdot\nabla u(y,s)dy\\
&\qquad -2\int_{\mathbb{R}^n_+}\partial_{x_n}G_{t-s}(x-y^*)Pu(y,s)\cdot\nabla u(y,s)dy,
\end{aligned}
$$

which yields for $1 \le k \le n$ and $0 < s < t$

$$
\begin{aligned}
&\int_{\mathbb{R}^n_+}\partial_{x_k}\partial_{x_n}G_{t-s}(x-y)Pu(y,s)\cdot\nabla u(y,s)dy\\
&\quad = \int_{\mathbb{R}^n_+}\partial_{x_k}(G_{t-s}(x-y)-G_{t-s}(x-y^*))\partial_n Pu(y,s)\cdot\nabla u(y,s)dy\\
&\qquad -\int_{\mathbb{R}^n_+}\partial_{x_k}\partial_{x_n}G_{t-s}(x-y^*)Pu(y,s)\cdot\nabla u(y,s)dy.
\end{aligned}
\tag{3.15}
$$

By (3.2), (3.15) and Lemmata 3.1–3.3, we obtain for any $t \ge 1$

$$
\begin{aligned}
&\left\| |x'|^\beta \int_{\frac{t}{2}}^t\int_{\mathbb{R}^n_+}\partial_{x_n}\partial_{x_n}\mathcal{M}(x,y,t-s)Pu(y,s)\cdot\nabla u(y,s)dyds\right\|_{L^r(\mathbb{R}^n_+)}\\
&\le C\int_{\frac{t}{2}}^t \sup_{y\in\mathbb{R}^n_+}\||x'-y'|^\beta\partial_{x_n}G_{t-s}(x-y)\|_{L^1(\mathbb{R}^n_+)}\|\partial_n Pu(s)\cdot\nabla u(s)\|_{L^r(\mathbb{R}^n_+)}ds\\
&\quad +C\int_{\frac{t}{2}}^t \sup_{y\in\mathbb{R}^n_+}\|\partial_{x_n}G_{t-s}(x-y)\|_{L^1(\mathbb{R}^n_+)}\||y'|^\beta\partial_n Pu(s)\cdot\nabla u(s)\|_{L^r(\mathbb{R}^n_+)}ds\\
&\quad +C\int_{\frac{t}{2}}^t \sup_{y\in\mathbb{R}^n_+}\||x'-y'|^\beta\partial_{x_n}\partial_{x_n}G_{t-s}(x-y^*)\|_{L^1(\mathbb{R}^n_+)}\|Pu(s)\cdot\nabla u(s)\|_{L^r(\mathbb{R}^n_+)}ds\\
&\quad +C\int_{\frac{t}{2}}^t \sup_{y\in\mathbb{R}^n_+}\|(x_n+y_n)^{2\epsilon}\partial_{x_n}\partial_{x_n}G_{t-s}(x-y^*)\|_{L^1(\mathbb{R}^n_+)}\|y_n^{-2\epsilon}|y'|^\beta Pu(s)\cdot\nabla u(s)\|_{L^r(\mathbb{R}^n_+)}ds\\
&\quad +C\int_{\frac{t}{2}}^t \sup_{y\in\mathbb{R}^n_+}\||x'-y'|^\beta\partial_{x_n}\partial_{x_n}\mathcal{M}^*(x,y,t-s)\|_{L^1(\mathbb{R}^n_+)}\|Pu(s)\cdot\nabla u(s)\|_{L^r(\mathbb{R}^n_+)}ds\\
&\quad +C\int_{\frac{t}{2}}^t \sup_{y\in\mathbb{R}^n_+}\|(x_n+y_n)^{2\epsilon}\partial_{x_n}\partial_{x_n}\mathcal{M}^*(x,y,t-s)\|_{L^1(\mathbb{R}^n_+)}\|y_n^{-2\epsilon}|y'|^\beta Pu(s)\cdot\nabla u(s)\|_{L^r(\mathbb{R}^n_+)}ds\\
&\le C\int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2}+\frac{\beta}{2}}\|\partial_n Pu(s)\cdot\nabla u(s)\|_{L^r(\mathbb{R}^n_+)}ds
\end{aligned}
$$

$$
\begin{aligned}
&+C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{2}}\||y'|^{\beta}\partial_n Pu(s)\cdot\nabla u(s)\|_{L^r(\mathbb{R}^n_+)}ds\\
&+C\int_{\frac{t}{2}}^{t}(t-s)^{-1+\frac{\beta}{2}}\|Pu(s)\cdot\nabla u(s)\|_{L^r(\mathbb{R}^n_+)}ds\\
&+C\int_{\frac{t}{2}}^{t}(t-s)^{-1+\epsilon}\|y_n^{-2\epsilon}|y'|^{\beta}Pu(s)\cdot\nabla u(s)\|_{L^r(\mathbb{R}^n_+)}ds\\
\le{}& C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{2}+\frac{\beta}{2}}\big(\|u(s)\|^2_{L^{2r}(\mathbb{R}^n_+)}+\|\nabla u(s)\|^2_{L^{2r}(\mathbb{R}^n_+)}+\|\nabla^2 u(s)\|^2_{L^{2r}(\mathbb{R}^n_+)}\big)ds\\
&+C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{2}}(\||y'|^{\frac{\beta}{2}}u(s)\|^2_{L^{2r}(\mathbb{R}^n_+)}+\||y'|^{\frac{\beta}{2}}\nabla u(s)\|^2_{L^{2r}(\mathbb{R}^n_+)}\\
&+\|\nabla u(s)\|_{L^\infty(\mathbb{R}^n_+)}\||y'|^{\beta}\nabla^2 u(s)\|_{L^r(\mathbb{R}^n_+)}\\
&+\|u(s)\|^2_{L^{2r}(\mathbb{R}^n_+)}+\|\nabla u(s)\|^2_{L^{2r}(\mathbb{R}^n_+)}+\|\nabla^2 u(s)\|^2_{L^{2r}(\mathbb{R}^n_+)})ds\\
&+C\int_{\frac{t}{2}}^{t}(t-s)^{-1+\frac{\beta}{2}}(\|u(s)\|^2_{L^{2r}(\mathbb{R}^n_+)}+\|\nabla u(s)\|^2_{L^{2r}(\mathbb{R}^n_+)})ds\\
&+C\int_{\frac{t}{2}}^{t}(t-s)^{-1+\epsilon}(\|u(s)\|^2_{L^{2r}(\mathbb{R}^n_+)}+\|\nabla u(s)\|^2_{L^{2r}(\mathbb{R}^n_+)}\\
&+\||y'|^{\frac{\beta}{2}}u(s)\|^2_{L^{2r}(\mathbb{R}^n_+)}+\||y'|^{\frac{\beta}{2}}\nabla u(s)\|^2_{L^{2r}(\mathbb{R}^n_+)})ds\\
\le{}& C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{2}+\frac{\beta}{2}}s^{-n(1-\frac{1}{2r})}ds\\
&+C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{2}}(s^{-n(1-\frac{1}{2r})}+s^{-n(1-\frac{1}{2r})+\frac{\beta}{2}})ds\\
&+C\int_{\frac{t}{2}}^{t}(t-s)^{-1+\frac{\beta}{2}}s^{-n(1-\frac{1}{2r})}ds\\
&+C\int_{\frac{t}{2}}^{t}(t-s)^{-1+\epsilon}(s^{-n(1-\frac{1}{2r})}+s^{-n(1-\frac{1}{2r})+\frac{\beta}{2}})ds\\
&+Cf(t)\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{2}}s^{-\frac{1}{2}-\frac{n}{2}+\frac{\beta}{2}-1-\frac{n}{2}(1-\frac{1}{r})}ds\\
&\le Ct^{-\frac{n-1}{2}-\frac{n}{2}(1-\frac{1}{r})+\frac{\beta}{2}}+Ct^{-\frac{n}{2}+\frac{\beta}{2}-1-\frac{n}{2}(1-\frac{1}{r})}f(t),
\end{aligned}\tag{3.16}
$$

where $\epsilon\in(0,\frac{1}{2})$, $f(t)=\sup_{0<s\le t}[s^{1+\frac{n}{2}(1-\frac{1}{r})-\frac{\beta}{2}}\||y'|^{\beta}\nabla^2 u(s)\|_{L^r(\mathbb{R}^n_+)}]$. Here we used the estimate (see Lemma 3.1 in [26]): Let $1\le k,\ell\le n$ and $1\le q\le\infty$, $\frac{1}{q}=\frac{1}{q_1}+\frac{1}{q_2}$, $q\le q_i\le\infty$ $(i=1,2)$. Then for any $u\in C^\infty_{0,\sigma}(\mathbb{R}^n_+)$

$$
\begin{aligned}
&\left\|\sum_{i,j=1}^{n}|x'|^{\gamma}\partial_k\partial_\ell\mathcal{N}\partial_i\partial_j(u_iu_j)\right\|_{L^q(\mathbb{R}^n_+)}\\
&\quad\le C(\|u\|^2_{L^{2q}(\mathbb{R}^n_+)}+\|\nabla u\|^2_{L^{2q}(\mathbb{R}^n_+)}+\|\nabla^2 u\|^2_{L^{2q}(\mathbb{R}^n_+)}\\
&\qquad+\||y'|^{\frac{\gamma}{2}}u\|^2_{L^{2q}(\mathbb{R}^n_+)}+\||y'|^{\frac{\gamma}{2}-\theta}\nabla u\|_{L^{q_1}(\mathbb{R}^n_+)}\||y'|^{\frac{\gamma}{2}+\theta}\nabla^2 u\|_{L^{q_2}(\mathbb{R}^n_+)})
\end{aligned}
$$

where $0\le\gamma<2$, $0\le\theta\le\frac{\gamma}{2}$.

Observe that for $x, y \in \mathbb{R}^n_+$ and $t > 0$

$$\partial_{x_k}\mathcal{M}(x,y,t) = -\partial_{y_k}\mathcal{M}(x,y,t), \quad 1 \le k \le n-1.$$

Consequently, we have for $1 \le \ell \le n$, $1 \le k \le n-1$ and $t > 0$

$$\begin{aligned}\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+} &\partial_{x_k}\partial_{x_\ell}\mathcal{M}(x,y,t-s)Pu(y,s)\cdot\nabla u(y,s)dyds\\ &= \int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\partial_{x_\ell}\mathcal{M}(x,y,t-s)\partial_{y_k}Pu(y,s)\cdot\nabla u(y,s)dyds.\end{aligned} \tag{3.17}$$

Following the proof of (3.16), and using (3.17), we get for $1 \le k \le n-1$, $1 \le \ell \le n$ and $t \ge 1$

$$\begin{aligned}&\left\||x'|^\beta\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\partial_{x_k}\partial_{x_\ell}\mathcal{M}(x,y,t-s)Pu(y,s)\cdot\nabla u(y,s)dyds\right\|_{L^r(\mathbb{R}^n_+)}\\
&\le C\int_{\frac{t}{2}}^{t}\sup_{y\in\mathbb{R}^n_+}\||x'-y'|^\beta\partial_{x_\ell}G_{t-s}(x-y)\|_{L^1(\mathbb{R}^n_+)}\|\partial_k Pu(s)\cdot\nabla u(s)\|_{L^r(\mathbb{R}^n_+)}ds\\
&\quad + C\int_{\frac{t}{2}}^{t}\sup_{y\in\mathbb{R}^n_+}\|\partial_{x_\ell}G_{t-s}(x-y)\|_{L^1(\mathbb{R}^n_+)}\||y'|^\beta\partial_k Pu(s)\cdot\nabla u(s)\|_{L^r(\mathbb{R}^n_+)}ds\\
&\quad + C\int_{\frac{t}{2}}^{t}\sup_{y\in\mathbb{R}^n_+}\||x'-y'|^\beta\partial_{x_\ell}\mathcal{M}^*(x,y,t-s)\|_{L^1(\mathbb{R}^n_+)}\|\partial_k Pu(s)\cdot\nabla u(s)\|_{L^r(\mathbb{R}^n_+)}ds\\
&\quad + C\int_{\frac{t}{2}}^{t}\sup_{y\in\mathbb{R}^n_+}\|\partial_{x_\ell}\mathcal{M}^*(x,y,t-s)\|_{L^1(\mathbb{R}^n_+)}\||y'|^\beta\partial_k Pu(s)\cdot\nabla u(s)\|_{L^r(\mathbb{R}^n_+)}ds\\
&\le C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{2}+\frac{\beta}{2}}\|\partial_k Pu(s)\cdot\nabla u(s)\|_{L^r(\mathbb{R}^n_+)}ds\\
&\quad + C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{2}}\||y'|^\beta\partial_k Pu(s)\cdot\nabla u(s)\|_{L^r(\mathbb{R}^n_+)}ds\\
&\le C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{2}+\frac{\beta}{2}}\left(\|u(s)\|^2_{L^{2r}(\mathbb{R}^n_+)} + \|\nabla u(s)\|^2_{L^{2r}(\mathbb{R}^n_+)} + \|\nabla^2 u(s)\|^2_{L^{2r}(\mathbb{R}^n_+)}\right)ds\\
&\quad + C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{2}}(\||y'|^{\frac{\beta}{2}}u(s)\|^2_{L^{2r}(\mathbb{R}^n_+)} + \||y'|^{\frac{\beta}{2}}\nabla u(s)\|^2_{L^{2r}(\mathbb{R}^n_+)}\\
&\quad + \|\nabla u(s)\|_{L^\infty(\mathbb{R}^n_+)}\||y'|^\beta\nabla^2 u(s)\|_{L^r(\mathbb{R}^n_+)}\\
&\quad + \|u(s)\|^2_{L^{2r}(\mathbb{R}^n_+)} + \|\nabla u(s)\|^2_{L^{2r}(\mathbb{R}^n_+)} + \|\nabla^2 u(s)\|^2_{L^{2r}(\mathbb{R}^n_+)})ds\\
&\le C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{2}+\frac{\beta}{2}}s^{-n(1-\frac{1}{2r})}ds\\
&\quad + C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{2}}(s^{-n(1-\frac{1}{2r})} + s^{-n(1-\frac{1}{2r})+\frac{\beta}{2}})ds\\
&\quad + Cf(t)\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{2}}s^{-\frac{1}{2}-\frac{n}{2}+\frac{\beta}{2}-1-\frac{n}{2}(1-\frac{1}{r})}ds\\
&\le Ct^{-\frac{n-1}{2}-\frac{n}{2}(1-\frac{1}{r})+\frac{\beta}{2}} + Ct^{-\frac{n}{2}+\frac{\beta}{2}-1-\frac{n}{2}(1-\frac{1}{r})}f(t),\end{aligned} \tag{3.18}$$

where the definition of $f(t)$ is given in the proof of (3.16).

It follows from (3.13), (3.14), (3.16) and (3.18) that for any $t \ge 1$

$$\||y'|^\beta\nabla^2 u(t)\|_{L^r(\mathbb{R}^n_+)} \le Ct^{-\frac{n}{2}+\frac{\beta}{2}-1-\frac{n}{2}(1-\frac{1}{r})}f(t) + Ct^{-1-\frac{n}{2}(1-\frac{1}{r})+\frac{\beta}{2}}.$$

and then

$$t^{1+\frac{n}{2}(1-\frac{1}{r})-\frac{\beta}{2}}\||y'|^\beta\nabla^2 u(t)\|_{L^r(\mathbb{R}^n_+)} \le C_0 t^{-\frac{n}{2}} f(t) + C.$$

Whence there exists $t_1 \ge 1$ satisfying $C_0 t_1^{-\frac{n}{2}} \le \frac{1}{2}$, such that for any $t \ge t_1$

$$f(t) \le \frac{1}{2} f(t) + C, \quad \text{and so} \quad f(t) \le 2C,$$

which implies for $t \ge t_1$

$$\||y'|^\beta\nabla^2 u(t)\|_{L^r(\mathbb{R}^n_+)} \le Ct^{-1-\frac{n}{2}(1-\frac{1}{r})+\frac{\beta}{2}}. \tag{3.19}$$

Step 2. Without loss of generalization, we suppose $\|x_n u_0\|_{L^1(\mathbb{R}^n_+)} < \infty$ for $n \ge 2$. Let $1 \le k, j \le n$, $0 < \beta < 1$, $n \ge 2$, $1 \le r \le \infty$. Using Lemmata 3.2, 3.3, and checking the proof of (3.14), we get for any $t \ge 1$

$$\begin{aligned}
&\left\||x'|^\beta \int_0^{\frac{t}{2}}\int_{\mathbb{R}^n_+} \partial_{x_k}\partial_{x_j}\mathcal{M}(x,y,t-s)Pu(y,s)\cdot\nabla u(y,s)dyds\right\|_{L^r(\mathbb{R}^n_+)}\\
&\le Ct^{-1-\frac{n}{2}(1-\frac{1}{r})}\int_0^{\frac{t}{2}}((1+s)^{-1-\frac{n}{2}} + (1+s)^{-\frac{n}{2}+\frac{\beta}{2}})ds\\
&\quad + Ct^{-1-\frac{n}{2}(1-\frac{1}{r})+\frac{\beta}{2}}\int_0^{\frac{t}{2}}(1+s)^{-1-\frac{n}{2}}ds\\
&\le Ct^{-1-\frac{n}{2}(1-\frac{1}{r})+\frac{\beta}{2}}.
\end{aligned} \tag{3.20}$$

Repeating the proofs of (3.16) and (3.18) respectively, we infer for any $t \ge 1$

$$\begin{aligned}
&\left\||x'|^\beta \int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+} \partial_{x_n}\partial_{x_n}\mathcal{M}(x,y,t-s)Pu(y,s)\cdot\nabla u(y,s)dyds\right\|_{L^r(\mathbb{R}^n_+)}\\
&\le C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{2}+\frac{\beta}{2}}s^{-1-n(1-\frac{1}{2r})}ds\\
&\quad + C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{2}}\big(s^{-1-n(1-\frac{1}{2r})} + s^{-n(1-\frac{1}{2r})+\frac{\beta}{2}}\big)ds\\
&\quad + C\int_{\frac{t}{2}}^{t}(t-s)^{-1+\frac{\beta}{2}}s^{-1-n(1-\frac{1}{2r})}ds\\
&\quad + C\int_{\frac{t}{2}}^{t}(t-s)^{-1+\epsilon}(s^{-1-n(1-\frac{1}{2r})} + s^{-n(1-\frac{1}{2r})+\frac{\beta}{2}})ds\\
&\quad + Cg(t)\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{2}}s^{-1-\frac{n}{2}+\frac{\beta}{2}-b(n)-\frac{n}{2}(1-\frac{1}{r})}ds\\
&\le Ct^{-\frac{n+1}{2}-\frac{n}{2}(1-\frac{1}{r})+\frac{\beta}{2}} + Ct^{-\frac{n-1}{2}-\frac{n}{2}(1-\frac{1}{r})+\frac{\beta}{2}} + Ct^{\epsilon-\frac{n}{2}-\frac{n}{2}(1-\frac{1}{r})+\frac{\beta}{2}}\\
&\quad + Ct^{-\frac{n+2}{2}-\frac{n}{2}(1-\frac{1}{r})+\frac{\beta}{2}} + Ct^{-\frac{n+1}{2}+\frac{\beta}{2}-b(n)-\frac{n}{2}(1-\frac{1}{r})}g(t), \quad \epsilon \in \left(0, \frac{1}{2}\right)\\
&\le Ct^{-\frac{n-1}{2}-\frac{n}{2}(1-\frac{1}{r})+\frac{\beta}{2}} + Ct^{-\frac{n+1}{2}+\frac{\beta}{2}-b(n)-\frac{n}{2}(1-\frac{1}{r})}g(t);
\end{aligned} \tag{3.21}$$

and for $1 \le k \le n-1$, $1 \le \ell \le n$

$$\begin{aligned}
&\left\||x'|^\beta \int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+} \partial_{x_k}\partial_{x_\ell}\mathcal{M}(x,y,t-s)Pu(y,s)\cdot\nabla u(y,s)dyds\right\|_{L^r(\mathbb{R}^n_+)}\\
&\quad \le Ct^{-\frac{n-1}{2}-\frac{n}{2}(1-\frac{1}{r})+\frac{\beta}{2}} + Ct^{-\frac{n+1}{2}+\frac{\beta}{2}-b(n)-\frac{n}{2}(1-\frac{1}{r})}g(t),
\end{aligned} \tag{3.22}$$

where $b(n) = \begin{cases} 1 & \text{if } n \geq 3, \\ \frac{1}{2} & \text{if } n = 2, \end{cases}$ and

$$g(t) = \sup_{0<s\leq t} [s^{b(n)+\frac{n}{2}(1-\frac{1}{r})-\frac{\beta}{2}} \||y'|^{\beta}\nabla^2 u(s)\|_{L^r(\mathbb{R}^n_+)}].$$

So from (3.13) and (3.20)–(3.22), we conclude for $t \geq 1$

$$t^{b(n)+\frac{n}{2}(1-\frac{1}{r})-\frac{\beta}{2}} \||y'|^{\beta}\nabla^2 u(t)\|_{L^r(\mathbb{R}^n_+)} \leq C_1 t^{-\frac{n+1}{2}} f(t) + C.$$

Whence there exists $t_2 \geq 1$ satisfying $C_1 t_2^{-\frac{n+1}{2}} \leq \frac{1}{2}$, such that for any $t \geq t_2$

$$g(t) \leq \frac{1}{2} g(t) + C, \text{ and so } g(t) \leq 2C,$$

which implies for $t \geq t_2$

$$\||y'|^{\beta}\nabla^2 u(t)\|_{L^r(\mathbb{R}^n_+)} \leq \begin{cases} Ct^{-1-\frac{n}{2}(1-\frac{1}{r})+\frac{\beta}{2}} & \text{if } n \geq 3, \\ Ct^{-\frac{1}{2}-(1-\frac{1}{r})+\frac{\beta}{2}} & \text{if } n = 2. \end{cases} \tag{3.23}$$

Combining (3.19) and (3.23), we establish the decays of $\||x'|^{\beta}\nabla^2 u(t)\|_{L^r(\mathbb{R}^n_+)}$ for any $t \geq t_0 = \max\{t_1, t_2\}$, with or without the assumption: $\|x_n u_0\|_{L^1(\mathbb{R}^n_+)} < \infty$. □

References

[1] Bae, H.: Temporal decays in L^1 and L^∞ for the Stokes flow. J. Differ. Equ. **222**, 1–20 (2006)
[2] Bae, H.: Temporal and spatial decays for the Stokes flow. J. Math. Fluid Mech. **10**, 503–530 (2008)
[3] Bae, H., Choe, H.: Decay rate for the incompressible flows in half spaces. Math. Z. **238**, 799–816 (2001)
[4] Bae, H., Choe, H.: A regularity criterion for the Navier–Stokes equations. Commun. Partial Differ. Equ. **32**, 1173–1187 (2007)
[5] Bae, H., Jin, J.: Regularity for the Navier–Stokes equations with slip boundary condition. Proc. Am. Math. Soc. **136**, 2439–2443 (2008)
[6] Bae, H., Jin, J.: Asymptotic behavior for the Navier–Stokes equations in 2D exterior domains. J. Funct. Anal. **240**, 508–529 (2006)
[7] Bae, H., Jin, J.: Temporal and spatial decay rates of Navier–Stokes solutions in exterior domains. Bull. Korean Math. Soc. **44**, 547–567 (2007)
[8] Bae, H., Jin, J.: Upper and lower bounds of temporal and spatial decays for the Navier–Stokes equations. J. Differ. Equ. **209**, 365–391 (2005)
[9] Bae, H., Jin, J.: Temporal and spatial decays for the Navier–Stokes equations. Proc. R. Soc. Edinb. Sect. A. **135**, 461–477 (2005)
[10] Bae, H., Jin, J.: Existence of strong mild solution of the Navier–Stokes equations in the half space with nondecaying initial data. J. Korean Math. Soc. **49**, 113–138 (2012)
[11] Brandolese, L.: On the localization of symmetric and asymmetric solutions of the Navier-Stokes equations in R^n. C.R. Acad. Sci. Paris S'Er. I Math. **332**, 125–130 (2001)
[12] Brandolese, L.: Space-time decay of Navier–Stokes flows invariant under rotations. Math. Ann. **329**, 685–706 (2004)
[13] Brandolese, L., Vigneron, F.: New asymptotic profiles of nonstationary solutions of the Navier–Stokes system. J. Math. Pures Appl. **88**, 64–86 (2007)
[14] Caffarelli, L., Kohn, R., Nirenberg, L.: Partial regularity of suitable weak solutions of the Navier–Stokes equations. Commun. Pure. Appl. Math. **35**, 771–831 (1982)
[15] Chae, D.: Conditions on the pressure for vanishing velocity in the incompressible fluid flows in R^N. Commun. Partial Differ. Equ. **37**, 1445–1455 (2012)
[16] Chae, D.: On the Liouville type theorems with weights for the Navier–Stokes equations and Euler equations. Differ. Integral Equ. **25**, 403–416 (2012)
[17] Chae, D.: Liouville type theorems for the Euler and the Navier–Stokes equations. Adv. Math. **228**, 2855–2868 (2011)
[18] Chae, D.: On the regularity conditions of suitable weak solutions of the 3D Navier–Stokes equations. J. Math. Fluid Mech. **12**, 171–180 (2010)
[19] Chae, D.: On the a priori estimates for the Euler, the Navier–Stokes and the quasi-geostrophic equations. Adv. Math. **221**, 1678–1702 (2009)
[20] Fujigaki, Y., Miyakawa, T.: Asymptotic profiles of non stationary incompressible Navier–Stokes flows in the half-space. Methods Appl. Anal. **8**, 121–158 (2001)
[21] Giga, Y., Matsui, S., Shimizu, Y.: On estimates in Hardy spaces for the Stokes flow in a half space. Math. Z. **231**, 383–396 (1999)

[22] Galdi, G.: An introduction to the mathematical theory of the Navier–Stokes equations, vol. I. Linearized Steady Problems, 38 edn . Springer, New York (1994)
[23] Han, P.: Weighted decay properties for the incompressible Stokes flow and Navier–Stokes equations in a half space. J. Differ. Equ. **253**, 1744–1778 (2012)
[24] Han, P.: Asymptotic behavior for the Stokes flow and Navier–Stokes equations in half spaces. J. Differ. Equ. **249**, 1817–1852 (2010)
[25] Han, P.: Decay results of solutions to the incompressible Navier–Stokes flows in a half space. J. Differ. Equ. **250**, 3937–3959 (2011)
[26] Han, P.: Weighted spatial decay rates for the Navier–Stokes flows in a half space. Proc. R. Soc. Edinb. Sect. A. **144**, 491–510 (2014)
[27] He, C., Wang, L.: Moment estimates for weak solutions to the Navier–Stokes equations in half-space. Math. Methods Appl. Sci. **32**, 1878–1892 (2009)
[28] Lin, F.: A new proof of the Caffarelli–Kohn–Nirenberg theorem. Commun. Pure Appl. Math. **51**, 241–257 (1998)
[29] Jin, B.: Weighted $L^q - L^1$ estimate of the Stokes flow in the half space. Nonlinear Anal. **72**, 1031–1043 (2010)
[30] Jin, B.: Spatial and temporal decay estimate of the Stokes flow of weighted L^1 initial data in the half space. Nonlinear Anal. **73**, 1394–1407 (2010)
[31] Schonbek, M.E.: L^2 decay for weak solutions of the Navier–Stokes equations. Arch. Rational Mech. Anal. **88**, 209–222 (1985)
[32] Schonbek, M.E.: Lower bounds of rates of decay for solutions to the Navier–Stokes equations. J. Am. Math. Soc. **4**, 423–449 (1991)
[33] Schonbek, M.E.: Asymptotic behavior of solutions to the three-dimensional Navier–Stokes equations. Indiana Univ. Math. J. **41**, 809–823 (1992)
[34] Schonbek, M.E.: Large time behaviour of solutions to the Navier–Stokes equations in H^m spaces. Commun. Partial Differ. Equ. **20**, 103–117 (1995)
[35] Schonbek, M.E.: The Fourier splitting method. Advances in Geometric Analysis and Continuum Mechanics. International Press, Cambridge (1995)
[36] Schonbek, M.E.: Total variation decay of solutions to the Navier–Stokes equations. Methods Appl. Anal. **7**, 555–564 (2000)
[37] Shimizu, Y.: L^∞-estimate of first-order space derivatives of Stokes flow in a half space. Funk. Ekvac. **42**, 291–309 (1999)
[38] Solonnikov, V.A.: Estimates for solutions of the nonstationary Stokes problem in anisotropic Sobolev spaces and estimates for the resolvent of the Stokes operator. Usp. Mat. Nauk. **58**, 123–156 (2003)
[39] Solonnikov, V.A.: On nonstationary Stokes problem and Navier–Stokes problem in a half-space with initial data nondecreasing at infinity. J. Math. Sci. **114**, 1726–1740 (2003)
[40] Stein, E.M.: Harmonic Analysis. Princeton University Press, Princeton (1993)
[41] Ukai, S.: A solution formula for the Stokes equation in $\mathbb{R}^N$. Commun. Pure Appl. Math. **XL**, 611–621 (1987)

Pigong Han
Academy of Mathematics and Systems Science
Chinese Academy of Sciences
Beijing 100190
China
e-mail: pghan@amss.ac.cn

(accepted: May 28, 2015; published online: June 26, 2015)