J. Math. Fluid Mech. 17 (2015), 287–293

1422-6928/15/020287-7
DOI 10.1007/s00021-015-0202-0

Journal of Mathematical
Fluid Mechanics

The Liouville Theorem for the Steady-State Navier–Stokes Problem for Axially Symmetric 3D Solutions in Absence of Swirl

Mikhail Korobkov, Konstantin Pileckas and Remigio Russo

Abstract. We study the Navier–Stokes equations of steady motion of a viscous incompressible fluid in $\mathbb{R}^3$. We prove that there are no nontrivial solution of these equations defined in the whole space $\mathbb{R}^3$ for axially symmetric case with no swirl (the Liouville theorem). Also we prove the conditional Liouville type theorem for axial symmetric solutions to the Euler system.

Mathematics Subject Classification. 35Q30 · 76D03 · 76D05.

Keywords. Three dimensional exterior axially symmetric domains, stationary Navier–Stokes equations, boundary–value problem.

1. Introduction

The historical Liouville theorem (1847) states that a bounded entire holomorphic function is constant. Nowadays, if $\mathscr{K}[\mathbf{u}]$ denotes an elliptic differential operator (in general, nonlinear), among other issues the name of Liouville is associated with the search of the conditions on $\mathbf{u}$ assuring that the system

$$\begin{aligned} &\mathscr{K}[\mathbf{u}] = \mathbf{0} \quad \text{in } \mathbb{R}^m, \\ &\lim_{|x|\to+\infty} \mathbf{u}(x) = \mathbf{u}_0 \end{aligned} \tag{1.1}$$

where $\mathbf{u}_0$ is an assigned constant vector, has the only solution $\mathbf{u} = \mathbf{u}_0$. In this sense the Liouville theorem for (1.1) is the first step in the study of the uniqueness questions concerning the operator $\mathscr{K}$. We recall, that the Liouville theorem holds for homogeneous elliptic systems and is strictly linked with the regularity of weak solutions for nonlinear elliptic systems with nonconstant coefficients (see [9]).

In this paper we are concerned with the case where $\mathscr{K}$ is the *stationary Navier–Stokes* operator. To be precise, we consider in $\mathbb{R}^3$ the following problem

$$\begin{cases} -\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{0}, \\ \operatorname{div}\mathbf{u} = 0, \\ \lim\limits_{|x|\to+\infty} \mathbf{u}(x) = \mathbf{u}_0, \end{cases} \tag{1.2}$$

where ν is the kinematical viscosity, $\mathbf{u} : \mathbb{R}^3 \to \mathbb{R}^3$ and $p : \mathbb{R}^3 \to \mathbb{R}$ are the unknown velocity and pressure fields.

A D-solution to (1.2) is a pair $(\mathbf{u}, p)$ such that

$$\int\limits_{\mathbb{R}^3} |\nabla\mathbf{u}|^2\, \mathrm{d}x < +\infty. \tag{1.3}$$

Among the several open problems in the theory of the steady–state Navier–Stokes equation the following Liouville's type one is of some interest (see, e.g., [7], p. 729):

(LT) *is a D-solution to* (1.2) in $\mathbb{R}^3$, vanishing at infinity, identically zero?

Birkhäuser

In addition to its theoretical meaning, (LT) can be viewed as the first step in the study of the very hard problem of uniqueness of solutions to nonhomogeneous problem for the Navier–Stokes equations and the asymptotics of such solutions as $|x| \to +\infty$ (see [1,3,5,7,10,17,18,20]). As far as we know, the only results concerning (LT) are due to Galdi [7] and, more recently to Chae and Yoneda [2]. In [2,7] (LT) is respectively proved under the additional conditions that $\mathbf{u}$ belongs to $L^{9/2}(\mathbb{R}^3)$ or has a suitable behavior at infinity, respectively. For $n = 2$ (LT) is proved in [8], while for $n = 4$ is simply obtained by a standard integration and Sobolev's inequality [7]. For $n > 4$ it is not known whether a D-solution is regular so the technique used to prove the theorem for $n = 4$ are no longer applicable.

In this paper we consider (LT) in the class of axially symmetric solutions. Here and henceforth we use the following usual notation. Let $O_{x_1}, O_{x_2}, O_{x_3}$ be coordinate axis in $\mathbb{R}^3$ and $\theta = \operatorname{arctg}(x_2/x_1)$, $r = (x_1^2 + x_2^2)^{1/2}$, $z = x_3$ be cylindrical coordinates. Denote by v_θ, v_r, v_z the projections of the vector $\mathbf{v}$ on the axes θ, r, z.

A function f is said to be *axially symmetric* if it does not depend on θ. A vector-valued function $\mathbf{h} = (h_\theta, h_r, h_z)$ is called *axially symmetric* if h_θ, h_r and h_z do not depend on θ. A vector-valued function $\mathbf{h}' = (h_\theta, h_r, h_z)$ is called *axially symmetric with no swirl* if $h_\theta = 0$ while h_r and h_z do not depend on θ.

Our main result is the following

Theorem 1.1. *Let* $(\mathbf{u}, p)$ *be an axially symmetric D-solution to* (1.2) *in* $\mathbb{R}^3$. *If* $\mathbf{u}_0 = 0$ *and* $\mathbf{u}$ *has no swirl, then* $\mathbf{u} \equiv \mathbf{0}$.

For $\mathbf{u}_0 \neq \mathbf{0}$ it is well–known, by classical results of Babenko [1], that $\mathbf{u} - \mathbf{u}_0 \in L^q(\mathbb{R}^3)$ for all $q > 2$ (see also [7] p. 698 (last edition)), and the Liouville Theorem follows by a simple integration by parts (see, e.g., [7, Theorem X.7.2]). Accordingly, Theorem 1.1 can be strengthened in the following way.

Theorem 1.2. *Let* $\mathbf{u}$ *be a D-solution to the Navier–Stokes equations* (1.2) *in* $\mathbb{R}^3$. *Assume that at least one of the following two conditions is fulfilled:*

(1) $\mathbf{u}_0 \neq \mathbf{0}$;
(2) $\mathbf{u}$ *is axially symmetric with no swirl. Then* $\mathbf{u} \equiv \text{const}$.

In addition, we prove some conditional Liouville type theorem for axial-symmetric solutions to the Euler system (more precisely, under the assumption that the total head pressure is nonpositive, see Theorem 2.1).

The paper is organized as follows: in Sect. 2.1 we discuss some integral identities for solutions to the Euler equations, and in Sect. 2.2 we apply these results to prove Theorem 1.1.

2. Proofs of the Main Results

2.1. Some Identities for Solutions to the Euler System

For $q \ge 1$ denote by $D^{k,q}(\Omega)$ the set of functions $f \in W^{k,q}_{\mathrm{loc}}(\Omega)$ such that $\|f\|_{D^{k,q}}(\Omega) = \|\nabla^k f\|_{L^q(\Omega)} < \infty$. Further, $D_0^{1,2}(\Omega)$ is the closure of the set $C_0^\infty(\Omega)$ of all smooth functions having compact supports in Ω with respect to the norm $\|\cdot\|_{D^{1,2}(\Omega)}$, and $H(\Omega) = \{\mathbf{v} \in D_0^{1,2}(\Omega) : \operatorname{div} \mathbf{v} = 0\}$. Note that for $\Omega = \mathbb{R}^3, \Omega = \mathbb{R}^3_+$ or an exterior domain $\Omega \subset \mathbb{R}^3$ with Lipschits boundary, elements from $H(\Omega)$ can be approximated in the norm $\|\cdot\|_{D^{1,2}(\Omega)}$ by smooth divergence free vector fields with compact supports (see [11,15]).

Assume that the following conditions are fulfilled:

(E) The axially symmetric functions $\mathbf{v} \in H(\mathbb{R}^3)$, $p \in D^{1,3/2}(\Omega)$ satisfy the Euler system

$$\begin{cases} (\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla p = 0 & \text{in } \mathbb{R}^3, \\ \operatorname{div} \mathbf{v} = 0 & \text{in } \mathbb{R}^3, \\ \lim\limits_{|x| \to +\infty} \mathbf{v}(x) = \mathbf{0}. \end{cases} \tag{2.1}$$

By Sobolev Imbedding Theorem,

$$\|\mathbf{v}\|_{L^6(\mathbb{R}^3)} < \infty. \tag{2.2}$$

Adding some constant to p (if necessary), by virtue of the Sobolev inequality (see, e.g., [7] II.6), we may assume without loss of generality that

$$\|p\|_{L^3(\mathbb{R}^3)} < \infty. \tag{2.3}$$

We need also the following fact, which is a simple consequence of well-known results.

Lemma 2.1. (see, e.g., Lemma 4.2 from [12]) *Let the conditions* (E) *be fulfilled. Then*

$$p \in D^{2,1}(\mathbb{R}^3) \cap D^{1,3/2}(\mathbb{R}^3). \tag{2.4}$$

For reader's convenience we reproduce the proof of this fact.

Proof. Clearly, $p \in D^{1,3/2}(\mathbb{R}^3)$ is the weak solution to the Poisson equation

$$\Delta p = -\nabla \mathbf{v} \cdot \nabla \mathbf{v}^\top \quad \text{in } \mathbb{R}^3. \tag{2.5}$$

Let

$$G(x) = \frac{1}{4\pi} \int\limits_{\Omega} \frac{(\nabla \mathbf{v} \cdot \nabla \mathbf{v}^\top)(y)}{|x-y|} dy.$$

By the results of [4], $\nabla \mathbf{v} \cdot \nabla \mathbf{v}^\top$ belongs to the Hardy space $\mathcal{H}^1(\mathbb{R}^3)$. Hence by Calderón–Zygmund theorem for Hardy's spaces [19] $G \in D^{2,1}(\mathbb{R}^3) \cap D^{1,3/2}(\mathbb{R}^3)$. Consider the function $p_* = p - G$. By construction $p_* \in D^{1,3/2}(\mathbb{R}^3)$ and $\Delta p_* = 0$ in $\mathbb{R}^3$. In particular, $\nabla p_* \in L^{3/2}(\mathbb{R}^3)$ is a harmonic (in the sense of distributions) function. From the mean-value property it follows that $p_* \equiv \text{const}$. Consequently, $p \in D^{2,1}(\mathbb{R}^3)$. □

Let $\Phi(x) = p(x) + \frac{1}{2}|\mathbf{v}(x)|^2$ be the total head pressure corresponding to the solution $(\mathbf{v}, p)$ of the Euler equations (2.1). Suppose that Φ satisfies the inequality

$$\Phi(x) \le 0. \tag{2.6}$$

Because of axial symmetry condition, it is natural to consider the restriction of functions on the half-plane $P_+ = \{(0, x_2, x_3) : x_2 > 0,\ x_3 \in \mathbb{R}\}$. Of course, on P_+ the coordinates x_2, x_3 coincides with coordinates r, z.

In cylindrical coordinates for the axially symmetric solution the Euler system (2.1) can be written as

$$\begin{cases} \dfrac{\partial p}{\partial z} + v_r \dfrac{\partial v_z}{\partial r} + v_z \dfrac{\partial v_z}{\partial z} = 0, \\ \dfrac{\partial p}{\partial r} - \dfrac{(v_\theta)^2}{r} + v_r \dfrac{\partial v_r}{\partial r} + v_z \dfrac{\partial v_r}{\partial z} = 0, \\ \dfrac{v_\theta v_r}{r} + v_r \dfrac{\partial v_\theta}{\partial r} + v_z \dfrac{\partial v_\theta}{\partial z} = 0, \\ \dfrac{1}{r} \dfrac{\partial (r v_r)}{\partial r} + \dfrac{\partial v_z}{\partial z} = 0, \\ \lim\limits_{|x| \to +\infty} \mathbf{v}(x) = 0 \end{cases} \tag{2.7}$$

(these equations are fulfilled for almost all $x \in P_+$).

First, we discuss the integrability properties of $\mathbf{v}, p$ on the half-plane P_+. The standard formula

$$\int\limits_{\mathbb{R}^3} f \, \mathrm{d}x = \iint\limits_{P_+} r f \, \mathrm{d}z \, \mathrm{d}r$$

holds for any axially symmetric function $f \in L^1(\mathbb{R}^3)$. Further, for every axially symmetric vector function $\mathbf{g} = (g_\theta, g_r, g_z)$ we have

$$|\nabla \mathbf{g}|^2 = \frac{|g_\theta|^2}{r^2} + \frac{|g_r|^2}{r^2} + |\partial_r g_r|^2 + |\partial_z g_r|^2 + |\partial_r g_\theta|^2 + |\partial_z g_\theta|^2 + |\partial_r g_z|^2 + |\partial_z g_z|^2. \tag{2.8}$$

In particular, $\frac{|g_r|}{r} \le |\nabla \mathbf{g}|$. Thus, by virtue of $\nabla^2 p \in L^1(\mathbb{R}^3)$, we can apply the above formula to the function $\mathbf{g} = \nabla p = (\partial_r p, 0, \partial_z p)$. Then we obtain that

$$\frac{|\partial_r p|}{r} \le |\nabla^2 p|,$$

consequently, $\frac{\partial_r p}{r} \in L^1(\mathbb{R}^3)$, i.e.,

$$\partial_r p \in L^1(P_+). \tag{2.9}$$

Since for each $z \in \mathbb{R}$ the convergence $p(r,z) \to 0$ as $r \to \infty$ holds,[1] we have $p(r,z) = -\int_r^\infty \partial_r p(\rho, z)\,d\rho$. Therefore, by (2.9), we conclude that

$$p(r,\cdot) \in L^1(\mathbb{R}) \tag{2.10}$$

for each $r > 0$. Moreover,

$$\int\limits_{\mathbb{R}} p(t,z)\,\mathrm{d}z = -\int\limits_t^\infty \int\limits_{\mathbb{R}} \partial_r p(r,z)\,\mathrm{d}z\,\mathrm{d}r \to 0 \qquad \text{as } t \to \infty. \tag{2.11}$$

From the inequality $\Phi(x) \le 0$ [see the Assumption (2.6)] it follows that

$$|\mathbf{v}|^2(r,\cdot) \in L^1(\mathbb{R}) \tag{2.12}$$

for each $r > 0$. Further, (2.8) and $\nabla \mathbf{v} \in L^2(\mathbb{R}^3)$ imply

$$\frac{|v_\theta|^2}{r} + \frac{|v_r|^2}{r} \in L^1(P_+). \tag{2.13}$$

From the Euler system (2.1) it follows by direct calculation that for any smooth vector function $\mathbf{g}$ we have

$$\operatorname{div}[p\,\mathbf{g} + (\mathbf{v}\cdot\mathbf{g})\mathbf{v}] = p\,\operatorname{div}\mathbf{g} + \big[(\mathbf{v}\cdot\nabla)\mathbf{g}\big]\cdot\mathbf{v}. \tag{2.14}$$

We apply this formula two times for $\mathbf{g} = r\mathbf{e}_r$ and $\mathbf{g} = \frac{1}{r}\mathbf{e}_r$, where $\mathbf{e}_r$ is the unit vector parallel to the r-axis.

(I) Let $\mathbf{g} = r\mathbf{e}_r$. Then for axially symmetric $\mathbf{v}$ and p we get

$$\operatorname{div}[p\,\mathbf{g} + (\mathbf{v}\cdot\mathbf{g})\mathbf{v}] = 2p + v_\theta^2 + v_r^2.$$

Integrating this identity over the three dimensional infinite cylinder $C_t = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : r = \sqrt{x_1^2 + x_2^2} < t,\ x_3 \in \mathbb{R}\}$, we obtain

$$t^2 \int\limits_{\mathbb{R}} [p(t,z) + v_r^2(t,z)]\,\mathrm{d}z = \iint\limits_{P_t} r[2p + v_\theta^2 + v_r^2]\,\mathrm{d}z\,\mathrm{d}r, \tag{2.15}$$

where $P_t = \{(r,z) \in P_+ : r < t\}$.

(II) Let $\mathbf{g} = \frac{1}{r}\mathbf{e}_r$. Then for axially symmetric functions the identity (2.14) takes the form

$$\operatorname{div}[p\,\mathbf{g} + (\mathbf{v}\cdot\mathbf{g})\mathbf{v}] = \frac{1}{r^2}(v_\theta^2 - v_r^2).$$

[1] It follows from the formula $p \in L^3(\mathbb{R}^3) \cap D^{2,1}(\mathbb{R}^3) \cap D^{1,3/2}(\mathbb{R}^3)$.

Since we have an essential singularity at $r = 0$, we need to integrate this identity over the cylindrical annulus $C_{t_0t} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : r = \sqrt{x_1^2 + x_2^2} \in (t_0, t),\, x_3 \in \mathbb{R}\}$ to obtain

$$\int_{\mathbb{R}} [p(t,z) + v_r^2(t,z)]\,\mathrm{d}z - \int_{\mathbb{R}} [p(t_0,z) + v_r^2(t_0,z)]\,\mathrm{d}z = \iint_{P_{t_0t}} \left[\frac{1}{r}(v_\theta^2 - v_r^2)\right] \mathrm{d}z\,\mathrm{d}r, \tag{2.16}$$

where $P_{t_0t} = \{(r,z) \in P_+ : r \in (t_0,t), z \in \mathbb{R}\}$.

Since $\int_{\mathbb{R}} [p(t,z) + v_r^2(t,z)]\,\mathrm{d}z \to 0$ as $t \to +\infty$ and

$$\iint_{P_+} \left|\frac{1}{r}(v_\theta^2 - v_r^2)\right| \mathrm{d}z\,\mathrm{d}r < \infty,$$

we obtain immediately from the above formulas that

$$\int_{\mathbb{R}} [p(t,z) + v_r^2(t,z)]\,\mathrm{d}z = \iint_{P_{t\infty}} \left[\frac{1}{r}(v_r^2 - v_\theta^2)\right] \mathrm{d}z\,\mathrm{d}r, \tag{2.17}$$

where $P_{t\infty} = \{(r,z) \in P_+ : r \in (t, +\infty),\, z \in \mathbb{R}\}$.

If $\mathbf{v}$ is axially symmetric with no swirl, then under the condition (2.6) the formulas (2.15) and (2.17) imply $\mathbf{v} \equiv \mathbf{0}$. Indeed, from (2.15) and the condition $\Phi(x) \le 0$ it follows that

$$\int_{\mathbb{R}} [p(t,z) + v_r^2(t,z)]\,\mathrm{d}z \le 0, \quad \forall t > 0,$$

while, if $v_\theta(x) = 0$, from (2.17) we get

$$\int_{\mathbb{R}} [p(t,z) + v_r^2(t,z)]\,\mathrm{d}z \ge 0, \quad \forall t > 0.$$

From the last two inequalities it follows that $\int_{\mathbb{R}} [p(t,z) + v_r^2(t,z)]\,\mathrm{d}z \equiv 0$ for all $t > 0$, and thus, by virtue of (2.17) and $v_\theta(x) = 0$, we obtain $v_r \equiv 0$. Therefore, by $(2.7)_4$ we conclude that $v_z \equiv 0$.

Thus, we have proved the following conditional theorem.

Theorem 2.1. *Let $(\mathbf{v}, p)$ be an axially symmetric D-solution to (2.1) in $\mathbb{R}^3$. If $\mathbf{v}$ has no swirl and the corresponding head pressure $\Phi \in L^3(\mathbb{R}^3)$ satisfies the condition (2.6) ($\Phi(x) \le 0$), then $\mathbf{v} \equiv \mathbf{0}$.*

Remark 2.1. This theorem is only conditional because we cannot prove the assumption (2.6) for an arbitrary D-solution of the Euler equations vanishing at infinity. However, as we will see below, this condition is valid for solutions to the Navier–Stokes equations.

2.2. Proof of the Liouville Theorem for the Navier–Stokes System

Let $\mathbf{u} \in H(\mathbb{R}^3)$ be a D-solution to the Navier–Stokes system in the whole space $\mathbb{R}^3$:

$$\begin{cases} -\nu\Delta\mathbf{u} + \big(\mathbf{u}\cdot\nabla\big)\mathbf{u} + \nabla p = \mathbf{0} & \text{in } \mathbb{R}^3, \\ \operatorname{div}\,\mathbf{u} = 0 & \text{in } \mathbb{R}^3, \\ \lim\limits_{|x| \to +\infty} \mathbf{u}(x) = \mathbf{0}. \end{cases} \tag{2.18}$$

By Sobolev Imbedding theorem $\mathbf{u} \in L^6(\mathbb{R}^3)$ and by the Hölder inequality $\big(\mathbf{u}\cdot\nabla\big)\mathbf{u} \in L^{3/2}(\mathbb{R}^3)$. Then, using the standard estimates for the solutions of the Stokes system (see, e.g., the Lemma 2.2 and Corollary 2.1 from [12]), we have

$$\|\nabla p\|_{L^{3/2}(\mathbb{R}^3)} < \infty. \tag{2.19}$$

Adding a constant to p (if necessary), by Sobolev inequality, we may assume without loss of generality that

$$\|p\|_{L^3(\mathbb{R}^3)} < \infty. \tag{2.20}$$

Denote by Φ the total head pressure $\Phi = p + \frac{1}{2}|\mathbf{u}|^2$ corresponding to the Navier–Stokes equations (2.18). It is well-known that Φ satisfies the classical identity

$$\Delta\Phi = \omega^2 + \frac{1}{\nu}\,\mathrm{div}(\Phi\mathbf{u}), \tag{2.21}$$

where $\omega(x) = |\,\mathrm{curl}\,\mathbf{u}|$. Suppose $\mathbf{u} \neq \mathbf{0}$. Since $\mathbf{u}(x) \to \mathbf{0}$, $p(x) \to 0$ *uniformly* as $|x| \to \infty$ (see, e.g., the works of Finn [6] and Ladyzhenskaya [13,14]), by (2.21) and Hopf maximum principle (see, e.g., [16]) we have the inequality

$$\Phi(x) < 0 \quad \forall x \in \mathbb{R}^3. \tag{2.22}$$

For the solution $(\mathbf{u}, p)$ of the Navier–Stokes system (2.18) we can repeat "word by word" the proof of Lemma 2.1 and we again conclude that

$$p \in D^{2,1}(\mathbb{R}^3) \cap D^{1,3/2}(\mathbb{R}^3). \tag{2.23}$$

In particular, $\partial_r p \in L^1(P_+)$ and, consequently,

$$p(r,\cdot) \in L^1(\mathbb{R}), \qquad |\mathbf{u}(r,\cdot)|^2 \in L^1(\mathbb{R}) \tag{2.24}$$

for each $r > 0$ (see the previous subsection).

From (2.18) it follows by direct calculation that for any smooth function $\mathbf{g}$ we have

$$\mathrm{div}[p\,\mathbf{g} + (\mathbf{u}\cdot\mathbf{g})\mathbf{u}] = p\,\mathrm{div}\,\mathbf{g} + \big[(\mathbf{u}\cdot\nabla)\mathbf{g}\big]\cdot\mathbf{u} - \nu\mathbf{g}\cdot\mathrm{curl}\,\boldsymbol{\omega}, \tag{2.25}$$

where $\boldsymbol{\omega} = (\omega_\theta, \omega_r, \omega_z) = \mathrm{curl}\,\mathbf{u}$.

If a function $\mathbf{g}$ is of type $\mathbf{g} = g(r)\mathbf{e}_r$, where $\mathbf{e}_r$ is the unit vector parallel to the r-axis, then by direct calculation

$$-\nu\mathbf{g}\cdot\mathrm{curl}\,\boldsymbol{\omega} = \nu g(r)\frac{\partial\omega_\theta}{\partial z}.$$

Consequently, for any cylindrical annulus $C_{t_0 t} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : r = \sqrt{x_1^2 + x_2^2} \in (t_0, t)\}$ we have

$$\int\limits_{C_{t_0 t}} \mathbf{g}\cdot\mathrm{curl}\,\boldsymbol{\omega}\,\mathrm{d}x = 0.$$

Therefore, identities (2.15) and (2.17) will be still fulfilled if we replace the solution $\mathbf{v}$ of the Euler equations (2.1) by the solution $\mathbf{u}$ of the Navier–Stokes equations (2.18). Now we can obtain a contradiction in the same way as in the previous subsection: from (2.22), (2.15) it follows for $\mathbf{u} \not\equiv 0$ that

$$\int\limits_{\mathbb{R}} [p(t,z) + u_r^2(t,z)]\,\mathrm{d}z < 0 \qquad \forall t > 0,$$

However, if $u_\theta \equiv 0$, then the equality (2.17) implies

$$\int\limits_{\mathbb{R}} [p(t,z) + u_r^2(t,z)]\,\mathrm{d}z \geq 0.$$

Thus, $\mathbf{u}(x) \equiv 0$ and Theorem 1.1 is proved. □

Acknowledgements. The research of M. Korobkov was partially supported by the Russian Foundation for Basic Research (project No. 14-01-00768-a) and by Dynasty Foundation. The research of K. Pileckas was funded by the Lithuanian-Swiss cooperation programme to reduce economic and social disparities within the enlarged European Union under project agreement No. CH-3-ŠMM-01/01. The research of R. Russo was supported by the Research Council of Lithuania (Grant No. VIZ-TYR-130).

References

[1] Babenko, K.I.: On stationary solutions of the problem of flow past a body. Mat. Sb. **91**, 3–27 (1973). English translation: Math. SSSR Sbornik **20**, 1–25 (1973)

[2] Chae, D., Yoneda, T.: On the Liouville theorem for the stationary Navier–Stokes equations in a critical case. J. Math. Anal. Appl. **405**, 706–710 (2013)

[3] Choe, H.J., Jin, B.J.: Asymptotic properties of axis–symmetric D-solutions of the Navier Stokes equations. J. Math. Fluid Mech. **11**, 208–232 (2009)

[4] Coifman, R.R., Lions, J.L., Meier, Y., Semmes, S.: Compensated compactness and Hardy spaces. J. Math. Pures App. IX Sér. **72**, 247–286 (1993)

[5] Farwig, R.: The stationary Navier–Stokes equations in a 3D-exterior domain, recent topics on mathematical theory of viscous incompressible fluid. Lect. Notes Numer. App. Anal. **16**, 53–1961 (1998)

[6] Finn, R.: On the steady-state solutions of the Navier–Stokes equations. III. Acta Math. **105**, 197–244 (1961)

[7] Galdi, G.P.: An introduction to the mathematical theory of the Navier–Stokes equation. In: Steady-state problems, 2nd ed. Springer, Berlin (2011)

[8] Gilbarg, D., Weinberger, H.: Asymptotic properties of steady plane solutions of the Navier–Stokes equations with bounded Dirichlet integral. Ann. Scuola Norm. Sup. Pisa (4) **5**, 301–404 (1978)

[9] Giaquinta, M.: Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems, Annals of Mathematical Studies, vol. 105. Princeton University Press, Princeton (1983)

[10] Hillairet, H., Wittwer, P.: On the existence of solutions to the planar exterior Navier–Stokes system. J. Differ. Equ. **255**, 2996–3019 (2013)

[11] Heywood, J.G.: On uniqueness questions in the theory of viscous flow. Acta. Math. **136**, 61–102 (1976)

[12] Korobkov, M.V., Pileckas, K., Russo, R.: The existence theorem for the steady Navier–Stokes problem in exterior axially symmetric 3D domains. Submitted to Acta Math. arXiv:1403.6921

[13] Ladyzhenskaya, O.A.: Investigation of the Navier–Stokes equations in the case of stationary motion of an incompressible fluid. Uspech Mat. Nauk **3**, 75–97 (1959, in Russian)

[14] Ladyzhenskaya, O.A.: The Mathematical Theory of Viscous Incompressible Fluid. Gordon and Breach, UK (1969)

[15] Ladyzhenskaya, O.A., Solonnikov, V.A.: On some problems of vector analysis and generalized formulations of boundary value problems for the Navier–Stokes equations. Zapiski Nauchn. Sem. LOMI **59**, 81–116 (1976, in Russian). English translation in J. Soviet Math. **10**(2), 257–286 (1978)

[16] Miranda, C.: Partial Differential Equations of Elliptic Type. Springer, Berlin (1970)

[17] Nazarov, S.A., Pileckas, K.: On steady Stokes and Navier–Stokes problems with zero velocity at infinity in a three–dimensional exterior domains. J. Math. Kyoto Univ. **40**, 475–492 (2000)

[18] Novotny, A., Padula, M.: Note on decay of solutions of steady Navier–Stokes equations in $3-D$ exterior domains. Differ. Integral Equ. **8**(7), 1833–1842 (1995)

[19] Stein, E.: Harmonic Analysis: Real-Variables Methods, Orthogonality and Oscillatory Integrals. Princeton University Press, Princeton (1993)

[20] Šverák, V., Tsai, T-P.: On the spatial decay of 3-D steady-state Navier–Stokes flows. Commun. Partial Differ. Equ. **25**, 2107–2117 (2000)

Mikhail Korobkov
Sobolev Institute of Mathematics
Novosibirsk State University
Koptyuga pr. 4, Pirogova Str. 2
630090 Novosibirsk
Russia
e-mail: korob@math.nsc.ru

Remigio Russo
Seconda Università di Napoli
Via Vivaldi 43
81100 Caserta
Italy
e-mail: remigio.russo@unina2.it

Konstantin Pileckas
Faculty of Mathematics and Informatics
Vilnius University
Naugarduko Str., 24
Vilnius 03225
Lithuania
e-mail: pileckas@ktl.mii.lt

(accepted: February 5, 2015; published online: April 17, 2015)