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Nonlinear Stability of Convection in a Porous Layer with Solid Partitions

B. Straughan

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Abstract. We show that for many classes of convection problem involving a porous layer, or layers, interleaved with finite but non-deformable solid layers, the global nonlinear stability threshold is exactly the same as the linear instability one. The layer(s) of porous material may be of Darcy type, Brinkman type, possess an anisotropic permeability, or even be such that they are of local thermal non-equilibrium type where the fluid and solid matrix constituting the porous material may have different temperatures. The key to the global stability result lies in proving the linear operator attached to the convection problem is a symmetric operator while the nonlinear terms must satisfy appropriate conditions.

1. Introduction

Interesting recent work of Rees and Genc [28] and Patil and Rees [25] has derived linear instability thresholds for the problem of thermal convection in a horizontal layer heated below where the layer is composed of identical finite thickness horizontal layers of a Darcy porous material saturated by an incompressible fluid interleaved with identical infinitesimal, Rees and Genc [28], and identical finite thickness, Patil and Rees [25], horizontal solid layers which are themselves heat conducting. This work is likely to have much application in modern heat transfer systems especially since such devices are now prevalent in the microfluidic industry. In fact, Egorov [9] provides a brief study of such a convection problem and he draws attention to applications in the thermal insulation of tanks of cryogenic liquids. Such tanks are widely used for fuel storage involving space research and travel, see e.g. Jordan and Puri [16].

The work of Rees and Genc [28] and Patil and Rees [25] concentrates on determining Rayleigh number thresholds for the linear instability problem. The object of this work is to show that the linear instability problem yields exactly the same results for the global nonlinear stability one. This is, of course, a very powerful result since it shows that the linear theory has completely captured the physics of the onset of thermal convection, and no large amplitude sub-critical instabilities can arise. To achieve our results we appeal to work of Galdi and Straughan [13], see also Galdi and Rionero [12], Galdi and Padula [11], and we show the linear operator associated to the multi-layer convection problem is symmetric, in a precise sense, and the nonlinear terms are compatible with the theory of Galdi and Straughan [13]. It is worth noting that highly non-trivial and technical ramifications and generalizations of the symmetry ideas put forward by Galdi and Straughan [13] are still occupying much attention in energy stability studies in fluid mechanics, see e.g. Capone et al. [6], Capone and Rionero [7], Falsaperla et al. [10], Georgescu and Palese [14], Hill and Malashetty [15], Lombardo et al. [18], Rionero [30].

We establish the coincidence of the linear instability and nonlinear stability boundaries firstly when the porous material is one of Darcy type but may possess an anisotropic permeability. Egorov [9] and Böttger et al. [3] point out that in practical applications the porous layer will often be anisotropic. Also, much recent attention has been devoted to experiments, and to nonlinear stability in thermal convection in a

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single porous layer, when the properties of the layer are anisotropic, see e.g. Altawallbeh et al. [1], Böttger et al. [3], Capone et al. [4,5], Kumar and Bhadauria [17], Nield [21], Rionero [29], Saravan and Brindha [31], Shiina and Hishida [35], Shivakumara et al. [36], Straughan [37,39,42], Straughan and Walker [43], and Tiwari et al. [44], and the references therein. We also prove the coincidence of the analogous problem when the porous medium is of Brinkman type.

Finally, we consider the situation where the porous layer(s) may be of local thermal non-equilibrium (LTNE) type, i.e. where the fluid and the solid skeleton may have different temperatures. This is a non-trivial problem when the layer also contains solid heat conducting layers, due to the prescription of appropriate conditions at the interface between the layers, cf. Nield [22], Nield and Kuznetsov [23]. We should point out that LTNE convection problems in a single porous or fluid layer are the topic of much current research, see e.g. Banu and Rees [2], Rees et al. [27], Rees [26], Malashetty et al. [19], Malashetty et al. [20], Saravanan and Sivakumar [33], Saravanan and Brindha [32], Vadasz [45], Straughan [38,41], Scott and Straughan [34], for the porous case, and Nield and Kuznetsov [24] and Straughan [40], for the case of a nanofluid.

2. The Darcy Equations

The problem of Patil and Rees [25] consists of N identical horizontal porous layers of the same thickness interleaved with $N-1$ identical layers of a rigid heat conducting solid each of the same thickness, where $N \in \mathbb{N}$ with $N \geq 2$. The temperatures of the upper surface of the top layer, at $z = d$, and lower surface of the bottom layer, at $z = 0$, are fixed at constant values T_U and T_L, respectively, with $T_L > T_U$. The object of Patil and Rees [25] is to obtain the linear instability critical Rayleigh and wave numbers for various porous and solid properties and various depths of the solid and porous layers.

To study the nonlinear stability of the above problem it is sufficient to restrict attention to a two layer case of a porous layer above a rigid solid layer. Once the mathematical problem for this configuration is analysed, the extension of the equivalence between the linear instability problem and the global nonlinear stability problem is easily extended to the Patil–Rees problem. Hence, we firstly consider the problem of an infinite horizontal layer occupying the region $\{(x, y) \in \mathbb{R}^2\} \times \{z \in (0, d)\}$ with gravity acting in the negative z-direction (downward). The layer is divided into two horizontal layers namely those involving $z \in (0, h_s)$ and $z \in (h_s, d)$ where $0 < h_s < d$. A porous medium occupies the upper part of the layer whereas the lower part, $z \in (0, h_s)$, is occupied by a rigid heat conducting solid. The porous material is taken to be of Darcy type although the permeability is allowed to be anisotropic.

The basic equations for thermal convection in the porous medium are given by (1.27), (4.6) of Straughan [39], and are

$$\begin{aligned} &\frac{\mu}{C}\, M_{ij} v_j = -p_{,i} - k_i g \rho_0 \big(1 - \tilde{\alpha}[T - T_0]\big), \\ &v_{i,i} = 0, \\ &(\rho c)_m T_{,t} + (\rho_0 c)_f v_i T_{,i} = k_m \Delta T, \end{aligned} \tag{1}$$

where $\mathbf{v}, p, T$ are the pore averaged velocity, pressure, and temperature in the porous medium. The quantities $\mu, C, g, \rho_0, \tilde{\alpha}, T_0$ and k_m are dynamic viscosity of the fluid saturating the porous medium, an appropriate constant, gravity, a reference density, the expansion coefficient of the fluid, a reference temperature, and the averaged thermal conductivity in the porous medium. In fact, $k_m = k_s(1-\epsilon) + k_f \epsilon$ where ϵ is the porosity, k_f is the thermal conductivity of the fluid and k_s is the thermal conductivity of the material forming the solid skeleton of the porous medium. The quantities $c_f, (\rho c)_m$ are the specific heat at constant pressure of the fluid and the averaged product of the density and specific heat of the porous medium. Precisely,

$$(\rho c)_m = \rho_0 c_f \epsilon + (\rho c)_s (1 - \epsilon)$$

where ρ_s and c_s denote the density and specific heat of the solid skeleton. The tensor M_{ij} is symmetric, dimensionless and positive-definite, i.e.

$$M_{ij} \xi_i \xi_j \geq \mu_0 \xi_i \xi_i, \tag{2}$$

for some $\mu_0 > 0$. A specific example of such an anisotropic tensor is given on p. 149 of Straughan [39]. In Eq. (1) and throughout we employ standard indicial notation with a repeated Roman index denoting summation from 1 to 3. The symbol Δ denotes the Laplace operator, and a subscript $,i$ denotes $\partial/\partial x_i$. Equations (1) hold in the domain $\mathbb{R}^2 \times \{z \in (h_s, d)\} \times \{t > 0\}$.

In the domain $\mathbb{R}^2 \times \{z \in (0, h_s)\} \times \{t > 0\}$ the temperature, $T^s \equiv T_s$, of the solid satisfies the equation

$$(\rho c)_S T^s_{,t} = k_S \Delta T_s, \tag{3}$$

where ρ_S, c_S, k_S denote the density, specific heat, and the thermal conductivity of the rigid solid.

The boundary conditions which must be satisfied involve no flow out of the boundaries $z = h_s, d$, fixed constant temperatures $T = T_U$ at $z = d$, $T = T_L$ at $z = 0$, where $T_L > T_U$, together with continuity of the temperature and of the heat flux across the interface $z = h_s$. Thus, if $\mathbf{v} = (u, v, w)$ then the boundary conditions are

$$\begin{aligned} &w = 0, \quad z = h_s, d; \quad T = T_U, \quad z = d; \quad T = T_L, \quad z = 0; \\ &T = T^s, \quad z = h_s; \quad k_m \frac{\partial T}{\partial z} = k_S \frac{\partial T^s}{\partial z}, \quad z = h_s, \end{aligned} \tag{4}$$

where the last condition recalls that the unit normal in the definition of the heat flux is the unit outward normal.

The basic steady state solution to Eqs. (1)–(4) for which $\bar{v}_i \equiv 0$, $\bar{T} = \bar{T}(z)$, $\bar{T}^s = \bar{T}^s(z)$, is found to be

$$\bar{T} = \alpha z + \beta, \quad \bar{T}^s = \alpha_s z + T_L. \tag{5}$$

To define the coefficients in Eqs. (5) we let h be such that $h + h_s = d$, $\tilde{k} = k_S h + k_m h_s$. Then,

$$\begin{aligned} &\alpha_s = -\frac{k_S}{\tilde{k}} \Delta T, \quad \alpha = -\frac{k_m}{\tilde{k}} \Delta T, \\ &\beta = T_U + \frac{k_S}{\tilde{k}} d \, \Delta T, \\ &\Delta T = T_L - T_U > 0. \end{aligned}$$

To study stability of this basic solution we introduce perturbations $u_i, \theta, \theta^s, \pi$ to $\bar{v}_i, \bar{T}, \bar{T}^s, \bar{p}$ as

$$v_i = \bar{v}_i + u_i, \quad T = \bar{T} + \theta, \quad T^s = \bar{T}^s + \theta^s, \quad p = \bar{p} + \pi.$$

Equations are derived for u_i, θ, θ^s from (1) to (4) and we non-dimensionalize these with the transformations

$$\begin{aligned} &x_i = x_i^* d, \quad t = \mathcal{T} t^*, \quad u_i = U u_i^*, \quad U = \frac{\kappa}{d}, \quad \mathcal{T} = \frac{d^2}{\kappa}, \quad \pi = \pi^* P, \\ &P = \frac{\mu U d}{C}, \quad K = \frac{k_S}{k_m C_1}, \quad C_1 = \frac{(\rho c)_S}{(\rho c)_m}, \quad \theta = T^\sharp \theta^*, \quad \theta^s = T^\sharp \theta^{s*}, \\ &T^\sharp = U d \sqrt{\frac{(\rho c)_f \mu \Delta T}{\tilde{k} \tilde{\alpha} \rho_0 g C}}, \end{aligned}$$

and introduce the Rayleigh number $R^2 = Ra$ as

$$R = d \sqrt{\frac{\tilde{\alpha} g \rho_0 C \, \Delta T (\rho c)_f}{\mu \tilde{k}}}.$$

Then the nonlinear non-dimensional perturbation equations may be written as, dropping all stars,

$$\begin{aligned} &M_{ij} u_j = -\pi_{,i} + R k_i \theta, \\ &u_{i,i} = 0, \\ &\theta_{,t} + u_i \theta_{,i} = R w + \Delta \theta, \end{aligned} \tag{6}$$

in $\mathbb{R}^2 \times \{z \in (h_s/d, 1)\} \times \{t > 0\}$, and

$$\theta^s_{,t} = K \Delta \theta^s, \tag{7}$$

in $\mathbb{R}^2 \times \{z \in (0, h_s/d)\} \times \{t > 0\}$. The non-dimensional boundary conditions are

$$
\begin{aligned}
&w = 0, \quad \text{on} \quad z = h_s/d, 1; \quad \theta = 0, \quad \text{on} \quad z = 1; \quad \theta^s = 0, \quad \text{on} \quad z = 0; \\
&\theta = \theta^s, \quad \theta_{,z} = \tilde{K}\theta^s_{,z} \quad \text{on} \quad z = h_s/d.
\end{aligned} \tag{8}
$$

In (8) the coefficient $\tilde{K}$ is defined by $\tilde{K} = KC_1$. The solution to (6)–(8) is also supposed periodic in the $(x, y)-$ plane and satisfies a plane tiling shape, cf. Straughan [37], p. 51. Let V be the three-dimensional domain given by the plane tiling shape together with the region $z \in (0, 1)$. Further let V be divided into two sub-regions V_1 and V_2 which occupy the domain below the interface $z = h_s/d$ and that above, respectively. Let Γ denote the interface in V at $z = h_s/d$.

Patil and Rees [25] observe that the problem similar to (6)–(8), but with $2N - 1$ combinations of layers, is such that the linear instability is stationary. This is easily seen here since we write $\theta = e^{\sigma t}\theta(\mathbf{x})$, $\theta^s = e^{\sigma t}\theta^s(\mathbf{x})$ with a similar representation for u_i and π, and then linearize (6). The linear instability problem consists of solving

$$
\begin{aligned}
&M_{ij}u_j = -\pi_{,i} + Rk_i\theta, \\
&u_{i,i} = 0, \\
&\sigma\theta = Rw + \Delta\theta,
\end{aligned} \tag{9}
$$

in $\mathbb{R}^2 \times \{z \in (h_s/d, 1)\}$, and

$$
\sigma\theta^s = K\Delta\theta^s, \tag{10}
$$

in $\mathbb{R}^2 \times \{z \in (0, h_s/d)\}$, together with the boundary conditions (8). By supposing $u_i, \theta, \pi, \theta^s$ are complex we multiply $(9)_1$ by u_j^* and integrate over V_2, likewise multiply $(9)_3$ by θ^* and integrate over V_2, and add. We further multiply (10) by $C_1\theta_s^*$, where $*$ here denotes complex conjugate, and integrate the result over V_1. By adding the results together and using conditions (8) we take the imaginary part of the result to find $\sigma \in \mathbb{R}$. Thus, the strong form of the principle of exchange of stabilities holds, cf. Galdi and Straughan [13].

If we adopt the notation of Galdi and Straughan [13] and write (6)–(8) in the form

$$
A\mathbf{u}_t = L\mathbf{u} + N(\mathbf{u})
$$

where the operator variable $\mathbf{u}$ is here given by $\mathbf{u} = (u, v, w, \theta, \theta^s)^T$, then the linear operator L is given by

$$
L\mathbf{u} = \begin{pmatrix} -M_{11} & -M_{12} & -M_{13} & 0 & 0 \\ -M_{12} & -M_{22} & -M_{23} & 0 & 0 \\ -M_{13} & -M_{23} & -M_{33} & R & 0 \\ 0 & 0 & R & \Delta & 0 \\ 0 & 0 & 0 & 0 & \tilde{K}\Delta \end{pmatrix} \begin{pmatrix} u \\ v \\ w \\ \theta \\ \theta^s \end{pmatrix}
$$

Actually the operator A is not strictly in the form employed by Galdi and Straughan [13]. Here $A = \mathrm{diag}(0, 0, 0, 1, 1)$ and only has non-zero entries in the last two rows. Nevertheless, the key results of Galdi and Straughan [13] continue to hold here also. Using boundary conditions (8) it is not difficult to show L is a symmetric, sectorial operator. Further, $(\mathbf{u}, N(\mathbf{u})) = 0$ in the inner product on $L^2(V_2)$, so that the conditions of theorem 2 of Galdi and Straughan [13] hold. Hence, we conclude that the linear instability boundary coincides with the global nonlinear stability one. If the Rayleigh number is in the stability region below the linear instability threshold then one may show that the energy

$$
\frac{C_1}{2}\int_{V_1} \theta_s^2 dx + \frac{1}{2}\int_{V_2} \theta^2 dx
$$

decays exponentially. From (6), one finds

$$
\int_{V_2} M_{ij}u_iu_j dx = R\int_{V_2} \theta w \, dx.
$$

Thus, using the bound on M_{ij}, (2) and the arithmetic-geometric mean inequality one shows

$$\int_{V_2} u_i u_i dx \leq \frac{R^2}{2\mu_0^2} \int_{V_2} \theta^2 dx$$

and so for a Rayleigh number in the stable region one also has exponential decay of $\int_{V_2} u_i u_i dx$.

Remark. We have only dealt with one Darcy porous region and one solid region in this example. However, the main result that the linear instability boundary is the same as the nonlinear one is also true for the thermal convection problem involving an arbitrary number of identical porous layers alternating with identical rigid heat conducting layers, in which the porous layers are of the same depths, are composed of the same fluid and same solid skeleton material, and the rigid partitions are all of the same material and same depth which is, in general, different to that of the porous layers. The proof given above carries over to this situation, *mutatis mutandis*. Since the porous layers are identical and the solid layers are likewise identical we can take the growth rates for each porous layer to be the same and this gives rise to the same cell shape in each layer, and same critical wavenumber.

3. The Brinkman Equations

We now consider the analogous problem to that of Patil and Rees [25] but where the identical porous layers are of Brinkman type. Again, it is sufficient to sketch the proof of equivalence of the nonlinear stability boundary and the linear instability boundary for a one porous layer—one solid layer case, and the more general result for a multilayer system follows in a very similar manner, *mutatis mutandis*. The results of Sect. 2 carry over to the case where the porous layer(s) is one of Brinkman type. The modification is to change Eqs. (6)–(8) to

$$\begin{aligned} &M_{ij}u_j = -\pi_{,i} + Rk_i\theta + \lambda\Delta u_i, \\ &u_{i,i} = 0, \\ &\theta_{,t} + u_i\theta_{,i} = Rw + \Delta\theta, \end{aligned} \tag{11}$$

in $\mathbb{R}^2 \times \{z \in (h_s/d, 1)\} \times \{t > 0\}$, and

$$\theta^s_{,t} = K\Delta\theta^s, \tag{12}$$

in $\mathbb{R}^2 \times \{z \in (0, h_s/d)\} \times \{t > 0\}$, together with the boundary conditions

$$\begin{aligned} &u_i = 0, \quad \text{on} \quad z = h_s/d, 1; \quad \theta = 0, \quad \text{on} \quad z = 1; \quad \theta^s = 0, \quad \text{on} \quad z = 0; \\ &\theta = \theta^s, \quad \theta_{,z} = \tilde{K}\theta^s_{,z} \quad \text{on} \quad z = h_s/d. \end{aligned} \tag{13}$$

Note that we now have conditions of no-slip on the boundaries of the porous layer rather than zero normal component of velocity.

Again one may show exchange of stabilities in a similar manner, and then one shows that theorem 2 of Galdi and Straughan [13] holds. The only modification is to account for the Brinkman term $\lambda\Delta u_i$ in (11). This term is catered for by the amended boundary condition $u_i = 0$ on the upper and lower surfaces of the porous medium.

4. Local Thermal Non-Equilibrium Equations

In this section we consider the equivalent thermal convection problem to that of Sect. 2, but we take the permeability isotropic (for simplicity only), and we allow the fluid temperature and the temperature of the porous skeleton in the part of the layer occupying $z \in (h_s, d)$, to be different. Studies involving different fluid and solid skeleton temperatures are known as local thermal non-equilibrium (LTNE) and due to many applications are numerous in today's literature, see e.g. the references in Scott and Straughan [34].

The basic equations for thermal convection with LTNE effects are to be found in many places, e.g. Banu and Rees [2], Straughan [38], and for a Darcy porous medium they are

$$\begin{aligned} & v_i = -\frac{K}{\mu}\, p_{,i} + \frac{\rho_f g\alpha K}{\mu}\, T_f k_i, \\ & v_{i,i} = 0, \\ & \epsilon(\rho c)_f T^f_{,t} + (\rho c)_f v_i T^f_{,i} = \epsilon k_f \Delta T_f + \tilde{h}(T_s - T_f), \\ & (1-\epsilon)(\rho c)_s T^s_{,t} = (1-\epsilon) k_s \Delta T_s - \tilde{h}(T_s - T_f), \end{aligned} \tag{14}$$

these holding in $\mathbb{R}^2 \times \{z \in (h_s, d)\} \times \{t > 0\}$. In these equations v_i, p, T^f, T^s are velocity, pressure, fluid temperature, solid skeleton temperature, a sub or superscript f denotes the fluid component whereas sub or superscript s denotes the component of the solid skeleton. The variables $K, \mu, \rho, g, \alpha, \epsilon, c, k$ and $\tilde{h}$ are permeability, dynamic viscosity of the fluid, density, gravity, thermal expansion coefficient of the fluid, porosity, specific heat at constant pressure, thermal conductivity, and an interaction coefficient, respectively. Denoting by T, ρ, c and k the temperature, density, specific heat, and the thermal conductivity of the rigid solid occupying the horizontal region between the planes $z = 0$ and $z = h_s$, the relevant equations for this region are

$$\rho c T_{,t} = k\Delta T,$$

in $\mathbb{R}^2 \times \{z \in (0, h_s)\} \times \{t > 0\}$.

Letting $\mathbf{v} = (u, v, w)$ the appropriate boundary conditions are

$$\begin{aligned} & w = 0, \quad z = h_s, d; \quad T = T_L, \quad z = 0; \quad T_f = T_U, T_s = T_U, \quad z = d; \\ & T_s = T, \quad T_f = T, \quad \text{on } z = h_s; \\ & \epsilon k_f T^f_{,z} + (1-\epsilon) k_s T^s_{,z} = k T_{,z} \quad \text{on} \quad z = h_s. \end{aligned} \tag{15}$$

Here T_L and T_U are constants with $T_L > T_U$. The interface has to be treated with care, cf. Nield and Kuznetsov [23], Nield [22], and we have supposed both T_f and T_s to be continuous with T, the temperature of the solid, at the interface. On the microscopic level this is certainly true, and we assume it in the continuum model. The remaining interface condition represents continuity of heat flux across $z = h_s$. It might be tempting to require only an average condition on the temperature fields like $\epsilon T_f + (1-\epsilon)T_s = T$ at the interface. However, one condition like this is not sufficient to make the problem determinate.

The basic steady solution of form $\bar{v}_i \equiv 0$, $\bar{T}^f = \bar{T}^f(z)$, $\bar{T}^s = \bar{T}^s(z)$ and $\bar{T} = \bar{T}(z)$, is found to be

$$\begin{aligned} & \bar{T}^f = \bar{T}^s = \beta + \alpha z, \quad z \in (h_s, d), \\ & \bar{T} = \alpha_1 z + T_L, \quad z \in (0, h_s), \end{aligned} \tag{16}$$

where

$$\begin{aligned} & \alpha = -\Delta T\, \frac{k}{kh + \hat{k} h_s}, \quad \alpha_1 = -\Delta T\, \frac{\hat{k}}{kh + \hat{k} h_s}, \\ & \beta = T_U + \frac{kd\Delta T}{kh + \hat{k} h_s}, \\ & \hat{k} = \epsilon k_f + (1-\epsilon) k_s, \quad \Delta T = T_L - T_U > 0. \end{aligned}$$

We put $v_i = \bar{v}_i + u_i$, $p = \bar{p} + \pi$, $T^f = \bar{T}^f + \theta$, $T^s = \bar{T}^s + \phi$ and $T = \bar{T} + \psi$ and then from Eqs. (14)–(16) we derive the equations and boundary conditions governing the perturbation variables. We employ the non-dimensionalizations

$$\begin{aligned}
&u_i = Uu_i^*, \quad x_i = x_i^* d, \quad \pi = P\pi^*, \quad \theta = \theta^* T^\sharp,\\
&\phi = \phi^* T^\sharp, \quad \psi = \psi^* T^\sharp, \quad t = t^* \mathcal{T}, \quad P = \frac{\mu d U}{K},\\
&U = \frac{\epsilon k_f}{d(\rho c)_f}, \quad \mathcal{T} = \frac{(\rho c)_f}{k_f}\, d^2, \quad \gamma = \frac{\epsilon k_f}{(1-\epsilon)k_s}, \quad H = \frac{hd^2}{\epsilon k_f},\\
&A = \frac{\rho_s c_s k_f}{k_s \rho_f c_f}, \quad B = \frac{\rho c}{\epsilon \rho_f c_f}, \quad b = \frac{k}{\epsilon k_f},\\
&T^\sharp = Ud\sqrt{\frac{c_f k \Delta T \mu}{\epsilon k_f (kh + \hat{k} h_s) g \alpha K}},\\
&Ra = R^2 = d^2 \rho_f^2\, \frac{c_f k g \alpha K \Delta T}{\epsilon \mu k_f (kh + \hat{k} h_s)}.
\end{aligned}$$

One may then show that the non-dimensional perturbation equations may be written in the form

$$\begin{aligned}
&u_i = -\pi_{,i} + Rk_i\theta,\\
&u_{i,i} = 0,\\
&\theta_{,t} + u_i\theta_{,i} = Rw + \Delta\theta + H(\phi - \theta),\\
&\frac{A}{\gamma}\phi_{,t} = \frac{1}{\gamma}\Delta\phi - H(\phi - \theta),
\end{aligned} \tag{17}$$

in $\mathbb{R}^2 \times \{z \in (h_s/d, 1)\} \times \{t > 0\}$, and

$$B\psi_{,t} = b\Delta\psi, \tag{18}$$

in $\mathbb{R}^2 \times \{z \in (0, h_s/d)\} \times \{t > 0\}$, together with the boundary conditions,

$$\begin{aligned}
&w = 0, \quad z = h_s/d, 1; \quad \theta = 0, \quad z = 1; \quad \phi = 0, \quad z = 1;\\
&\psi = 0, \quad z = 0;\\
&\theta = \psi, \phi = \psi, \quad z = h_s/d;\\
&\theta_{,z} + \frac{1}{\gamma}\phi_{,z} = b\psi_{,z}, \quad z = h_s/d.
\end{aligned} \tag{19}$$

With the equations in the form (17)–(19) it is straightforward to show the strong form of the principle of exchange of stabilities holds for the linearized problem, cf. Galdi and Straughan [13]. Furthermore, one may show the linear operator attached to (17)–(19) is symmetric and then theorem 2 of Galdi and Straughan [13] applies to allow us to deduce that the linear instability boundary also yields the global nonlinear stability boundary. The natural energy to use is

$$\frac{1}{2}\|\theta\|_2^2 + \frac{A}{2\gamma}\|\phi\|_2^2 + \frac{B}{2}\|\psi\|_1^2$$

where $\|\cdot\|_1$ and $\|\cdot\|_2$ denote the norms on $L^2(V_1)$ and $L^2(V_2)$, respectively. If the Rayleigh number Ra is less than the linear instability threshold then from use of theorem 2 of Galdi and Straughan [13] one may deduce this energy decays to zero at least exponentially. It is then an easy matter to show from Eqs. $(17)_{1,2}$ that $\|\mathbf{u}\|_2$ likewise decays in a similar manner. Thus, even for the more complicated class of layered convection problems involving LTNE porous materials and rigid solids one has the powerful result that the linear instability threshold completely captures the physics of the onset of convection. Of course, in practice, one still has to solve the linear instability problem. For the problem of the current section this involves a second order differential equation for w, and similar equations for θ, ϕ and ψ, yielding an 8th order eigenvalue problem for R. Clearly, we require 8 boundary conditions and this we have. We point out that a very good numerical method to solve this eigenvalue problem is that of Chebyshev tau, cf. Dongarra et al. [8].

5. Conclusions

In this paper the problem of nonlinear stability in thermal convection in a horizontal layer composed of identical horizontal layers of porous material and identical layers of a rigid solid has been addressed. For three important classes of porous material it has been shown that the physics of the onset of convection is accurately captured by utilizing a linearized instability analysis. This is deduced by appealing to symmetry results of Galdi and Straughan [13] and concluding the linear instability thresholds are the same as the global nonlinear stability ones. The three classes of porous material involve a Darcy one which typically describes a dense porous material like sandstone, and Brinkman and LTNE theories which are potentially likely to hold for porous materials with a porosity close to 1, such as man made porous metallic foams, cf. Straughan [39], pp. 1–5. Since such materials have much use in the heat transfer and in the insulation industries, the results given herein are of value. We close with some remarks.

Remark 1. We reiterate that although Sects. 2–4 have treated only a two layer problem we can deduce the same results for a combination of $N-1$ identical solid layers, between N identical Darcy layers, Brinkman layers, or LTNE layers, the porous and solid layers having, in general, different depths, providing these layers are governed by the equations presented here. We could also derive the same sort of result as that of this paper if the porous layers were of LTNE type with a Brinkman term present. Such equations for the single porous layer case are dealt with in e.g. Banu and Rees [2], Straughan [38].

Remark 2. In connection with the first remark, we deduce complete information on the nonlinear stability problem by solving the linear instability one. One still has, of course, to solve the linear instability problem and we point out that an ideal method to do this is via a Chebyshev tau method coupled to a matrix solver like the $QZ-$algorithm, cf. Dongarra et al. [8]. To solve a many layer problem one simply maps each sub-layer alternately to $(-1, 1)$ then $(1, -1)$, expands the appropriate functions in Chebyshev series in each layer and truncates employing the boundary and interface equations to complete the matrix eigenvalue problem. For example, we might consider a 5-layer problem of porous depths h, solid depths h_s, with layers 2 and 4 composed of a rigid solid, and layers 1,3,5 composed of a LTNE porous material with Brinkman terms. This is analogous to the situation sketched in Patil and Rees [25], p. 236, only they consider a Darcy porous material. For the linear instability problem we then have a second order ordinary differential equation in layers 2 and 4, and fourth order ordinary differential equations for w together with second order ordinary differential equations for θ and ϕ in layers 1, 3 and 5. This is an involved differential eigenvalue problem of order 28 and we obtain 28 boundary and interface conditions. There are conditions on $w, w' = \partial w/\partial z$, and θ and ϕ on $z = 0$ and 1, making 8 boundary conditions, and interface conditions on w, w', θ, ϕ and ψ at the four internal interfaces, making 20 interface conditions. The Chebyshev tau D^2 numerical method is particularly suited to solving such an eigenvalue problem.

Remark 3. In each of Sects. 2–4 we have neglected the fluid inertia. Inclusion of such inertia may be important in some problems, see e.g. Straughan [42] and the references therein. It presents no difficulty whatsoever to establish the results of this article in the presence of fluid inertia.

Remark 4. We have followed Patil and Rees [25] and taken the solid layers to be rigid. It would be interesting to allow these layers to deform due to temperature gradients. Of course, this would complicate the problem since one would need to employ the theory for a nonlinear thermoelastic body to describe the solid layer(s).

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B. Straughan
Department of Mathematics
University of Durham
Durham DH1 3LE
UK
e-mail: brian.straughan@durham.ac.uk

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