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Compressible Navier–Stokes Equations on Thin Domains

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Abstract. We consider the barotropic Navier–Stokes system describing the motion of a compressible viscous fluid confined to a straight layer $\Omega_\varepsilon = \omega \times (0, \varepsilon)$, where ω is a particular 2-D domain (a periodic cell, bounded domain or the whole 2-D space). We show that the weak solutions in the 3D domain converge to a (strong) solutions of the 2-D Navier–Stokes system on ω as $\varepsilon \to 0$ on the maximal life time of the strong solution.

Keywords. Compressible Navier–Stokes system, dimension reduction, thin domains.

1. Introduction

This paper is devoted to the problem of the limit passage from $3D$ two $2D$ in the compressible fluid flows. We shall consider compressible Navier–Stokes equations in thin domains

$$\Omega_\varepsilon = \omega \times (0, \varepsilon),\ \varepsilon > 0, \tag{1.1}$$

where ω is a fixed domain in R^2, and investigate the situation when $\varepsilon \to 0$.

These domains are supposed to be filled with a compressible viscous gas, whose evolution through the time interval $[0, T]$, $T > 0$ is described by the isentropic compressible Navier–Stokes system for the unknown functions, density $\varrho = \varrho(t, x)$ and velocity $\mathbf{u} = \mathbf{u}(t, x)$, $t \in [0, T]$, $x \in \Omega_\varepsilon$:

$$\partial t\varrho + \mathrm{div}(\varrho\mathbf{u}) = 0 \quad \text{in } (0, T) \times \Omega_\varepsilon, \tag{1.2}$$

$$\partial_t(\varrho\mathbf{u}) + \mathrm{div}(\varrho\mathbf{u} \otimes \mathbf{u}) + \nabla p(\varrho) = \mathrm{div}\mathbb{S}(\nabla\mathbf{u}) \quad \text{in } (0, T) \times \Omega_\varepsilon. \tag{1.3}$$

Equations are completed with the initial conditions

$$\varrho(0, x) = \tilde\varrho_{0,\varepsilon}(x), \quad \mathbf{u}(0, x) = \tilde{\mathbf{u}}_{0,\varepsilon}(x), \quad x \in \Omega_\varepsilon \tag{1.4}$$

and boundary conditions that will be specified later.

In (1.3), $\mathbb{S}$ is the viscous stress tensor given by the *Newton law*

$$\mathbb{S}(\nabla\mathbf{u}) = \mu\left(\mathbb{D}(\nabla\mathbf{u}) - \frac{2}{3}\mathrm{div}\mathbf{u}\mathbb{I}\right) + \eta\mathrm{div}\mathbf{u}\mathbb{I},\ \mathbb{D}(\nabla\mathbf{u}) = \nabla\mathbf{u} + (\nabla\mathbf{u})^T \tag{1.5}$$

with the shear viscosity coefficient $\mu > 0$, the bulk viscosity coefficient $\eta \geq 0$ and $\mathbb{I}$ the identity matrix.

Finally, p denotes the pressure, a given function of density ϱ characterizing the gaz. Anticipating the existence theory of weak solutions in 3-D domains, we shall suppose that

$$p \in C[0, \infty) \cap C^2(0, \infty),\ p(0) = 0,\ p'(\varrho) > 0 \text{ for all } \varrho > 0,$$
$$\lim_{\varrho\to\infty} \frac{p'(\varrho)}{\varrho^{\gamma-1}} = p_\infty > 0,\ \gamma > \frac{3}{2}. \tag{1.6}$$

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Rigorous justification of the limit passage from the $3D$-fluid motion to a planar one seems to be of obvious practical importance. However, for the compressible fluid flows, to the best of our knowledge, there are no results concerning the 3D-2D limit passage and only a few results concerning the 3D-1D reduction, see Vodák [22] and recently [2]. There are numerous studies of the *incompressible* fluid flows on thin domains, where the limit motion becomes planar, see Iftimie et al. [10], Raugel and Sell [19] and the references therein. This work develops and adapts the ideas introduced in [2] to the problematics of 3D-2D reduction in the compressible fluid flows.

Analysis of similar dimension reduction problems in the elasticity theory leans on variants of the Korn inequality that provides estimates on the gradient of a vector function $\mathbf{v}$ in terms of its symmetric part, specifically,

$$\|\nabla\mathbf{v}\|_{L^2(\Omega_\varepsilon;R^{3\times 3})} \le c(\varepsilon)\left\|\nabla\mathbf{v}+(\nabla\mathbf{v})^T\right\|_{L^2(\Omega_\varepsilon;R^{3\times 3})}. \tag{1.7}$$

Clearly, the validity of (1.7) requires the kernel of the linear operator $\mathbf{v}\mapsto\nabla\mathbf{v}+(\nabla\mathbf{v})^T$ to be empty on the space of vector fields satisfying the given boundary data. Even if (1.7) holds for any fixed $\varepsilon>0$, the constant $c(\varepsilon)$ blows up for $\varepsilon\to 0$ unless some necessary restrictions are imposed on the field $\mathbf{v}$, and this is true even if the set ω is not rotationally symmetric, cf. Lewicka and Müller [13].

It is not difficult to see that the problems arising in the context of *compressible* fluids would need a stronger analogue of (1.7), namely

$$\|\nabla\mathbf{v}\|_{L^2(\Omega_\varepsilon;R^{3\times 3})} \le c(\varepsilon)\left\|\nabla\mathbf{v}+(\nabla\mathbf{v})^T-\frac{2}{3}\mathrm{div}\mathbf{v}\mathbb{I}\right\|_{L^2(\Omega_\varepsilon;R^{3\times 3})}, \tag{1.8}$$

obviously related to the shear viscosity component of the viscous stress tensor, see Dain [4], Reshetnyak [20]. In view of the above mentioned difficulties related to the validity of (1.7) or (1.8), our approach relies on the structural stability of the family of solutions of the barotropic Navier–Stokes system encoded in the *relative entropy inequality* introduced in [6,8]. This method is basically independent of the specific form of the viscous stress and of possible "dissipative" bounds for the Navier–Stokes system.

In this investigation we make a choice to rescale the equations to a fixed domain. Introducing the change of variables

$$\Omega_\varepsilon \ni (x_h,\varepsilon x_3)\mapsto (x_h,x_3)\in\Omega:=\Omega_1,\quad \text{where } x_h=(x_1,x_2), \tag{1.9}$$

and denoting the new density and velocity again by ϱ, $\mathbf{u}$, we may rewrite system (1.2–1.4) as follows:

$$\partial_t\varrho+\mathrm{div}_\varepsilon(\varrho\mathbf{u})=0 \quad \text{in } (0,T)\times\Omega, \tag{1.10}$$

$$\partial_t(\varrho\mathbf{u})+\mathrm{div}_\varepsilon(\varrho\mathbf{u}\otimes\mathbf{u})+\nabla_\varepsilon p(\varrho)=\mathrm{div}_\varepsilon\mathbb{S}(\nabla_\varepsilon\mathbf{u}) \quad \text{in}(0,T)\times\Omega, \tag{1.11}$$

$$\varrho(0,x)=\varrho_{0,\varepsilon}(x)\quad \mathbf{u}(0,x)=\mathbf{u}_{0,\varepsilon}(x),\quad x\in\Omega \tag{1.12}$$

[where $\varrho_{0,\varepsilon}(x)=\tilde{\varrho}_{0,\varepsilon}(x_h,\varepsilon x_3)$, $\mathbf{u}_{0,\varepsilon}(x)=\tilde{\mathbf{u}}_{0,\varepsilon}(x_h,\varepsilon x_3)$, cf. (1.4)]. The boundary conditions will be specified later.

Here and hereafter, we denote

$$\nabla_\varepsilon=\left(\nabla_h,\frac{1}{\varepsilon}\partial_{x_3}),\ \nabla_h=(\partial_{x_1},\partial_{x_2}\right),$$

$$\mathrm{div}_\varepsilon\mathbf{u}=\mathrm{div}_h\mathbf{v}_h+\frac{1}{\varepsilon}\partial_{x_3}v_3,\ \mathbf{v}_h=(v_1,v_2),\ \mathrm{div}_h\mathbf{v}_h=\partial_{x_1}v_1+\partial_{x_2}v_2.$$

The goal of this work is to investigate the limit process $\varepsilon\to 0$ in the system of equations (1.10–1.12), provided the initial data $[\varrho_{0,\varepsilon},\mathbf{u}_{0,\varepsilon}](x)$ converge in a certain sense to $[r_0,\mathbf{v}_0](x)=[\mathbf{v}_{0,h},0](x_h)$. Since the target initial data do not depend on the vertical variable x_3, it is natural to expect that the sequence $[\varrho_\varepsilon,\mathbf{u}_\varepsilon](t,x)$ of (weak) solutions to (1.10–1.12) will converge to $[r,\mathbf{V}](t,x_h)$, $\mathbf{V}=[\mathbf{w},0]$, where the couple

$[r(t,x_h),\mathbf{w}(t,x_h)]$ solves the 2-D compressible Navier–Stokes equations on the domain ω:

$$\partial_t r + \mathrm{div}_h(r\mathbf{w}) = 0 \text{ in } (0,T)\times\omega, \tag{1.13}$$

$$r\partial_t\mathbf{w} + r\mathbf{w}\cdot\nabla_h\mathbf{w} + \nabla_h p(r) = \mathrm{div_h}\mathbb{S}_h(\nabla_h\mathbf{w}) \text{ in } (0,T)\times\omega, \tag{1.14}$$

$$r(0,x_h) = r_0(x_h),\ \mathbf{w}(0,x_h) = \mathbf{w}_0 := \mathbf{v}_{0,h}(x_h),\ x_h\in\omega, \tag{1.15}$$

where

$$\mathbb{S}_h(\nabla_h\mathbf{w}) = \mu\Big(\nabla_h\mathbf{w} + (\nabla_h\mathbf{w})^T - \mathrm{div}_h\mathbf{w}\Big) + (\eta + \frac{\mu}{3})\mathrm{div}_h\mathbf{w}\mathbb{I}_h,$$

and $\mathbb{I}_h$ is the identity matrix.

Our goal is to justify the above (formal) limit in the framework of weak solutions of the *primitive system* (1.10), (1.11), (1.12).

We shall consider several geometrical situations: periodic layers in Sect. 2, layers over bounded domains ω with no-slip conditions on $\partial\omega$ in Sect. 3, layers over bounded domains with slip conditions in Sect. 4, and some particular cases of unbounded layers in Sect. 5. Finally, in the last section we discuss the possible generalizations of the pressure law (1.6) in order accommodate adiabatic coefficients $\gamma\geq 1$.

We finish the introduction with some remarks on the notation. As far as the functional spaces are concerned, we deal with the classical Lebesgue and Sobolev spaces and we use the standard notation that can be find e.g. in the book of Adams [1]. The notation of Bocher types spaces is again standard, the same as in the book [17]. Finally, in various estimates, the symbols c, c' denote generic positive constants always independent of the small parameter ε; they may take different values in different formulas.

2. Periodic Layers

We consider system (1.2–1.3), (1.4) of thin domains Ω_ε [see (1.1)], where $\omega = [0,1]^2\Big|_{0,1}$ is a 2-D periodic cell of period 1 in both directions. It is completed with the no slip boundary conditions on the bottom and top of the layer

$$\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega_\varepsilon} = 0,\ [\mathbb{S}(\nabla\mathbf{u})\cdot\mathbf{n}]\times\mathbf{n}|_{\partial\Omega_\varepsilon} = 0, \tag{2.1}$$

After rescaling, this situation corresponds to system (1.10–1.12) on Ω [see (1.9)] completed with boundary conditions

$$\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = 0,\ [\mathbb{S}(\nabla_\varepsilon\mathbf{u})\cdot\mathbf{n}]\times\mathbf{n}|_{\partial\Omega} = 0, \tag{2.2}$$

Since ω is a periodic cell, conditions (2.2) means that

$$u_3|_{\omega\times\{0,1\}} = 0,\ \ \Big(\partial_{x_k}u_3 + \frac{1}{\varepsilon}\partial_{x_3}u_k\Big)|_{\omega\times\{0,1\}} = 0,\ k=1,2, \tag{2.3}$$

where $\mathbf{u}(\cdot,x_3)$, $x_3\in(0,1)$ are 1-periodic functions in x_h-variable.

2.1. Preliminaries, Main Results

2.1.1. Weak Solutions to the Primitive System.

Denote

$$W^{1,2}_{\mathbf{n}}(\Omega;R^3) = \{\mathbf{v}\in W^{1,2}(\Omega;R^3)\big|\mathbf{v}\cdot\mathbf{n}|_{\partial\Omega} = 0\}.$$

Definition 2.1. We say that $[\varrho,\mathbf{u}]$ is a *finite energy weak solution* to the compressible Navier–Stokes (1.10–1.11) with initial conditions (1.12) and boundary conditions (2.2) in the space time cylinder $(0,T)\times\Omega$ if the following holds:

- the functions $[\varrho, \mathbf{u}]$ belong to the regularity class

$$\left\{ \begin{array}{c} \varrho \in L^\infty([0,T];L^\gamma(\Omega)),\ \varrho \geq 0 \text{ a.a. in } (0,T)\times\Omega,\ \gamma > \frac{3}{2}, \\ \\ \mathbf{u} \in L^2(0,T;W^{1,2}_{\mathbf{n}}(\Omega;R^3)),\ \varrho\mathbf{u}^2 \in L^\infty([0,T];L^1(\Omega)); \end{array} \right\} \tag{2.4}$$

- $\varrho \in C_{\text{weak}}([0,T];L^\gamma(\Omega))$ and the *continuity equation* (1.10) is satisfied in the weak sense,

$$\int_\Omega \varrho\varphi \ \mathrm{d}x(\tau) - \int_\Omega \varrho_{\varepsilon,0}\varphi(0,\cdot)\ \mathrm{d}x = \int_0^\tau \int_\Omega \varrho\Big(\partial_t\varphi + \mathbf{u}\cdot\nabla_\varepsilon\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t. \tag{2.5}$$

for all $\tau \in [0,T]$ and any test function $\varphi \in C^\infty_c([0,T]\times\overline{\Omega})$;
- $\varrho\mathbf{u} \in C_{\text{weak}}([0,T];L^{2\gamma/(\gamma+1)}(\Omega))$ and the *momentum equation* (1.11) holds in the sense that

$$\int_\Omega \varrho\mathbf{u}\cdot\varphi\ \mathrm{d}x(\tau) - \int_\Omega \varrho_{\varepsilon,0}\mathbf{u}_{\varepsilon,0}\varphi(0,\cdot)\ \mathrm{d}x$$

$$= \int_0^T\int_\Omega \Big(\varrho\partial_t\varphi + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla_\varepsilon\varphi + p(\varrho_\varepsilon)\mathrm{div}_\varepsilon\varphi\ \mathrm{d}x\ \mathrm{d}t - \int_0^\tau\int_\Omega \mathbb{S}(\nabla_\varepsilon\mathbf{u}) : \nabla_\varepsilon\varphi\ \mathrm{d}x\ \mathrm{d}t \tag{2.6}$$

for all $\tau \in [0,T]$ and for any $\varphi \in C^\infty_c([0,T]\times\overline{\Omega};R^3)$, $\varphi_3|_{\omega\times\{0,1\}} = 0$;
- the *energy inequality*

$$\int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + H(\varrho)\right](\tau)\ \mathrm{d}x + \int_0^\tau\int_\Omega \mathbb{S}(\nabla_\varepsilon\mathbf{u}) : \nabla_\varepsilon\mathbf{u}\ \mathrm{d}x\ \mathrm{d}t \leq \int_\Omega \left[\frac{1}{2}\varrho_{0,\varepsilon}|\mathbf{u}_{0,\varepsilon}|^2 + H(\varrho_{0,\varepsilon})\right]\ \mathrm{d}x \tag{2.7}$$

holds for a.a. $\tau \in (0,T)$.

Notice that the definition of weak solutions for the system (1.2–1.3), (1.4), (2.1) before rescaling can be obtained replacing Ω by Ω_ε, ∇_ε by ∇, div_ε by div and $\varrho_{\varepsilon,0}$ by $\tilde{\varrho}_{\varepsilon,0}$, $\mathbf{u}_{\varepsilon,0}$ by $\tilde{\mathbf{u}}_{\varepsilon,0}$.

The reader may consult the monograph [15] by Lions, Feireisl [5] and [17], for the mathematical theory of compressible viscous fluids in the framework of weak solutions. In particular, the weak solutions for the system (1.2–1.4), (2.1) before rescaling are known to exist globally in time for any finite energy initial data.

Consequently, the system (1.10–1.12), (2.2) after rescaling possesses finite energy weak solutions as well. The corresponding theorem reads:

Proposition 2.1. *Let* $\mathbb{S}$ *satisfy* (1.5) *and* p *verify* (1.6). *Suppose that the initial data satisfy*

$$\varrho_{0,\varepsilon} \geq 0,\ \int_\Omega \varrho_{0,\varepsilon} = M_\varepsilon > 0,\ \int_\Omega \left(\frac{1}{2}\varrho_{0,\varepsilon}\mathbf{u}^2_{0,\varepsilon} + H(\varrho_{0,\varepsilon})\right)\ \mathrm{d}x < \infty. \tag{2.8}$$

Then the problem (1.10–1.12), (2.2) *admits at least one finite energy weak solution on the arbitrary time interval* $(0,T)$.

2.1.2. Relative Entropy Inequality. Motivated by [8] we introduce the relative entropy functional

$$\mathcal{E}\left(\varrho,\mathbf{u}\middle|r,\mathbf{U}\right) = \int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}-\mathbf{U}|^2 + E(\varrho,r)\right]\ \mathrm{d}x, \tag{2.9}$$

where

$$E(\varrho,r) = H(\varrho) - H'(r)(\varrho - r) - H(r),\ H(\varrho) = \varrho\int_1^\varrho \frac{p(s)}{s^2}\mathrm{d}s.$$

(The reader can consult Dafermos [3], Germain [9], Mellet and Vasseur [16] for the utilisation of the notion of relative entropies in other contexts in the mathematical fluid mechanics).

Now, the crucial observation is that *any* finite energy weak solution defined through (2.4–2.7) satisfies the so called relative entropy inequality. More precisely, we have the following theorem (that follows by rescaling from see [6, Section 3.2.1]):

Proposition 2.2. *Let all assumptions of Proposition 2.1 be satisfied and let $[\varrho, \mathbf{u}]$ be a finite energy weak solution of the system (1.10)–(1.12), (2.2). Then $[\varrho, \mathbf{u}]$ satisfies the* relative entropy inequality

$$\mathcal{E}\left(\varrho, \mathbf{u}\Big| r, \mathbf{U}\right)(\tau) + \int_0^\tau \int_\Omega \left(\mathbb{S}\Big(\nabla_\varepsilon(\mathbf{u}-\mathbf{U})\Big) : \nabla_\varepsilon(\mathbf{u}-\mathbf{U})\ \mathrm{d}x\ \mathrm{d}t\right.$$
$$\leq \mathcal{E}\left([\varrho, \mathbf{u}]\Big|[r, \mathbf{U}]\right)(0) + \int_0^\tau \mathcal{R}(\varrho, \mathbf{u}, r, \mathbf{U})\ \mathrm{d}t, \tag{2.10}$$

with the remainder term

$$\mathcal{R}(\varrho, \mathbf{u}, r, \mathbf{U}) := \int_\Omega \varrho\Big(\partial_t \mathbf{U} + \mathbf{u}\cdot\nabla_\varepsilon \mathbf{U}\Big)\cdot(\mathbf{U}-\mathbf{u})\ \mathrm{d}x + \int_\Omega \mathbb{S}(\nabla_\varepsilon \mathbf{U}) : \nabla_\varepsilon(\mathbf{U}-\mathbf{u})\ \mathrm{d}x$$
$$+ \int_\Omega \left((r-\varrho)\partial_t H'(r) + \nabla_\varepsilon H'(r)\cdot(r\mathbf{U}-\varrho\mathbf{u})\right)\ \mathrm{d}x - \int_\Omega \mathrm{div}_\varepsilon \mathbf{U}\Big(p(\varrho)-p(r)\Big)\ \mathrm{d}x, \tag{2.11}$$

with any pair of test functions

$$r \in C^1([0,T]\times\overline{\Omega}),\ r > 0,\ \mathbf{U} \in C^1([0,T]\times\overline{\Omega}; R^3),\ \mathbf{U}\cdot\mathbf{n}|_{\partial\Omega} = 0.$$

Remark 2.1. Notice that the set of test functions in the relative entropy inequality can be enlarged by density argument to:

$$\begin{aligned}
&0 < \underline{r} < r < \overline{r} < \infty,\ \partial_t r \in L^1(0,T; L^{\frac{\gamma}{\gamma-1}}(\Omega)),\ \nabla r \in L^2(0,T; L^{\frac{2\gamma}{\gamma-1}}(\Omega; R^3))\\
&\mathbf{U} \in L^2(0,T; W^{1,2}(\Omega),\ \partial_t \mathbf{U} \in L^2(0,T; L^6(\Omega; R^3)),\\
&\nabla \mathbf{U} \in L^2(0,T; L^{\frac{3\gamma}{2\gamma-3}}(\Omega; R^{3\times 3})),\ \mathrm{div}\mathbf{U} \in L^1(0,T; L^\infty(\Omega)).
\end{aligned} \tag{2.12}$$

We remark that this class is not optimal. Nevertheless, we shall see that it is sufficient for our purposes.

2.1.3. Solutions of the Target System. It is well known since eighties that the target system (1.13–1.15) admits a unique strong solution on a maximal time interval $[0, T_{\max})$ that depends on the size of the initial data. The following theorem can be deduced as Theorem 2.5 in Valli and Zajaczkowski [21]:

Proposition 2.3. *Let D be a positive constant. Suppose that $p \in C^2(0,\infty)$ and that*

$$r_0 \in W^{2,2}(\omega),\ \inf_\omega r_0 > 0,\ \mathbf{w}_0 \in W^{3,2}(\omega; R^2). \tag{2.13}$$

Then there exists $T = T_{\max}(D)$ such that if

$$\|r_0\|_{W^{2,2}(\omega)} + \|\mathbf{w}_0\|_{W^{3,2}(\omega;R^2)} + 1/\inf_\omega r_0 \leq D, \tag{2.14}$$

then the problem (1.13–1.15) admits a unique strong solution (in the sense a.e. in $(0,T)\times\omega$) in the class

$$\begin{aligned}
r \in C([0,T); W^{2,2}(\omega)),\ \mathbf{w} \in C([0,T); W^{2,2}(\omega; R^2)) \cap L^2(0,T; W^{3,2}(\omega; R^2))\\
\partial_t r \in C([0,T); W^{1,2}(\omega)),\ \partial_t \mathbf{w} \in L^2(0,T; W^{2,2}(\omega; R^2)).
\end{aligned} \tag{2.15}$$

In particular,

$$0 < \underline{r} \equiv \inf_{(t,x_h)\in(0,T)\times\omega} r(t,x_h) \leq \sup_{(t,x_h)\in(0,T)\times\omega} r(t,x_h) \equiv \overline{r}. \tag{2.16}$$

2.1.4. Main Result. We are now in a position to formulate the main result of this section.

Theorem 2.1. *Suppose that the pressure p satisfies hypotheses (1.6) and that the stress tensor is given by (1.5). Let r_0, $\mathbf{w}_0$ satisfy assumptions (2.13) and let $T_{\max} > 0$ be the life time of the strong solution to problem (1.13–1.15) corresponding to $[r_0, \mathbf{w}_0]$ determined in Proposition 2.3. Let $(\varrho_\varepsilon, \mathbf{u}_\varepsilon)$ be a sequence of weak solutions to the 3-D compressible Navier Stokes equations (1.10–1.12), (2.2) emanating from the initial data $[\varrho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon}]$. Suppose that*

$$\mathcal{E}\Big(\varrho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon}\Big| r_0, \mathbf{V}_0\Big) \to 0, \tag{2.17}$$

where $\mathbf{V}_0 = [\mathbf{w}_0, 0]$.

Then

$$\mathrm{esssup}_{t\in(0,T_{\max})}\mathcal{E}(\varrho_\varepsilon, \mathbf{u}_\varepsilon \Big| r, \mathbf{V}) \to 0, \tag{2.18}$$

where $\mathbf{V}(t,x) = [\mathbf{w}(t, x_h), 0]$ and where the couple $(r, \mathbf{w})$ satisfies the 2-D compressible Navier–Stokes system (1.13–1.15) on the periodic cell ω on the time interval $[0, T_{\max})$.

Before coming to the proof in the next section, we comment the above theorem and formulate two additional Corollaries and two Remarks which shed more light on the result.

Remark 2.2. In order to see more clearly the sense of the limit above, we notice that (2.18) implies, for example

$$\begin{aligned} \varrho_\varepsilon &\to r \text{ strongly in } L^\infty(0,T; L^\gamma(\Omega)),\\ \sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon &\to \sqrt{r}\mathbf{V} \text{ strongly in } L^\infty(0,T; L^2(\Omega; R^3)),\\ \varrho_\varepsilon\mathbf{u}_\varepsilon &\to r\mathbf{V} \text{ strongly in } L^\infty(0,T; L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^3)). \end{aligned}$$

Remark 2.3. We notice that the 3D-2D reduction increases the two dimensional bulk viscosity of the fluid from η to $\eta + \mu/3$, cf. (1.13–1.15).

Corollary 2.1. *Suppose that the pressure p, the stress tensor $\mathbb{S}$ satisfy assumptions of Theorem 2.1. Assume that $[\varrho_{\varepsilon,0}, \mathbf{u}_{\varepsilon,0}]$, $\varrho_\varepsilon \geq 0$ verify*

$$\int_0^1 \varrho_{\varepsilon,0}(x)\mathrm{d}x_3 \rightharpoonup r_0 \textit{ weakly in } L^1(\omega; R^3), \tag{2.19}$$

$$\int_0^1 \varrho_{\varepsilon,0}\mathbf{u}_{\varepsilon,0}\mathrm{d}x_3 = r_0\mathbf{V}_0 \textit{ weakly in } L^1(\omega; R^3),$$

where $\mathbf{V}_0 = [\mathbf{w}_0, 0]$ and $[r_0, \mathbf{w}_0]$ belongs to the regularity class (2.13), and

$$\int_\Omega \Big[\frac{1}{2}\varrho_{\varepsilon,0}\mathbf{u}_{\varepsilon,0}^2 + H(\varrho_{\varepsilon,0})\Big]\mathrm{d}x \to \int_\omega \Big[\frac{1}{2}r_0\mathbf{w}_0^2 + H(r_0)\Big]\mathrm{d}x_h. \tag{2.20}$$

Let finally $[\varrho_\varepsilon, \mathbf{u}_\varepsilon]$ be a sequence of weak solutions to the 3-D compressible Navier Stokes equations (1.10–1.12), (2.2) emanating from the initial data $[\varrho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon}]$. Then (2.18) holds.

Corollary 2.1 follows directly from Theorem 2.1. It is enough to observe that (2.19–2.20) imply (2.17). Indeed, recalling that $[r_0, \mathbf{w}_0]$ is independent of x_3 and realizing that

$$\begin{aligned} \mathcal{E}(\varrho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon}\Big| r_0, \mathbf{V}_0) &= \int_\Omega \Big[\frac{1}{2}\varrho_{0,\varepsilon}\mathbf{u}_{0,\varepsilon}^2 + H(\varrho_{0,\varepsilon})\Big]\mathrm{d}x - \int_\Omega \Big[\frac{1}{2}r_0\mathbf{w}_0^2 + H(r_0)\Big]\mathrm{d}x\\ &\quad - \int_\Omega H'(r_0)(\varrho_\varepsilon - r_0)\,\mathrm{d}x + \int_\Omega \Big[\frac{1}{2}r_0\mathbf{w}_0^2 + \frac{1}{2}\varrho_{0,\varepsilon}\mathbf{w}_0^2 - \varrho_{0,\varepsilon}\mathbf{u}_{0,\varepsilon}\cdot\mathbf{V}_0\Big]\,\mathrm{d}x, \end{aligned}$$

we may use (2.20) at the first line and (2.19) at the second line, to show the both lines converge to 0.

Corollary 2.1 can be reformulated in terms of the sequence of solutions $[\tilde{\varrho}_\varepsilon, \tilde{\mathbf{u}}_\varepsilon]$ of the non rescaled original problem (1.2–1.4), (2.1).

Corollary 2.2. *Suppose that the pressure p, the stress tensor $\mathbb{S}$ satisfy assumptions of Theorem 2.1. Assume that $[\tilde{\varrho}_{\varepsilon,0}, \tilde{\mathbf{u}}_{\varepsilon,0}]$, $\tilde{\varrho}_\varepsilon \geq 0$ verify*

$$\frac{1}{\varepsilon}\int_0^\varepsilon \tilde{\varrho}_{\varepsilon,0}(x)\mathrm{d}x_3 \rightharpoonup r_0 \text{ weakly in} L^1(\omega), \tag{2.21}$$

$$\frac{1}{\varepsilon}\int_0^\varepsilon \tilde{\varrho}_{\varepsilon,0}\tilde{\mathbf{u}}_{\varepsilon,0}\mathrm{d}x_3 = r_0\mathbf{V}_0 \text{ weakly in} L^1(\omega;R^3),$$

where $\mathbf{V}_0 = [\mathbf{w}_0, 0]$ and $[r_0, \mathbf{w}_0]$ belongs to the regularity class (2.13), and

$$\frac{1}{\varepsilon}\int_{\Omega_\varepsilon}\Big[\frac{1}{2}\tilde{\varrho}_{\varepsilon,0}\tilde{\mathbf{u}}_{\varepsilon,0}^2 + H(\tilde{\varrho}_{\varepsilon,0})\Big]\mathrm{d}x \to \int_\omega \Big[\frac{1}{2}r_0\mathbf{w}_0^2 + H(r_0)\Big]\mathrm{d}x_h. \tag{2.22}$$

Let finally $(\tilde{\varrho}_\varepsilon, \tilde{\mathbf{u}}_\varepsilon)$ be a sequence of weak solutions to the 3-D compressible Navier Stokes equations (1.2–1.4), (2.1) emanating from the initial data $[\tilde{\varrho}_{0,\varepsilon}, \tilde{\mathbf{u}}_{0,\varepsilon}]$.

Then

$$\mathrm{esssup}_{t\in(0,T_{\max})}\mathcal{E}_\varepsilon(\tilde{\varrho}_\varepsilon, \tilde{\mathbf{u}}_\varepsilon \Big| r, \mathbf{V}) \to 0 \tag{2.23}$$

with

$$\mathcal{E}_\varepsilon(\varrho, \mathbf{u}\Big| r, \mathbf{U}) = \frac{1}{\varepsilon}\int_{\Omega_\varepsilon}\Big[\frac{1}{2}\varrho|\mathbf{u}-\mathbf{U}|^2 + H(\varrho) - H'(r)(\varrho - r) - H(r)\Big]\mathrm{d}x,$$

where $\mathbf{V}(t,x) = [\mathbf{w}(t,x_h), 0]$ and where the couple $[r, w]$ satisfies the 2-D compressible Navier–Stokes system (1.13–1.15) on the periodic cell ω on the time interval $[0, T_{\max})$.

2.2. Proof of Theorem 2.1

2.2.1. Korn and Poincaré Type Inequalities.

We start by the following Korn type inequality.

Lemma 2.1. *Let ω be a periodic cell and $\Omega = \omega \times (0,1)$. Then there exists $c > 0$ such that for all $\mathbf{v} \in W^{1,2}_{\mathbf{n}}(\Omega;R^3)$)*

$$\int_\Omega \mathbb{S}(\nabla_\varepsilon \mathbf{v}) : \nabla_\varepsilon \mathbf{v}\ \mathrm{d}x\mathrm{d}x = \mu\|\nabla_\varepsilon\mathbf{v}\|^2_{L^2(\Omega;R^{3\times3})} + (\frac{1}{3}\mu + \eta)\|\mathrm{div}_\varepsilon\mathbf{v}\|^2_{L^2(\Omega)},$$

where $\|\mathbb{A}\|^2_{L^2(\Omega;R^{3\times3})} = \sum_{i,j=1}^3\int_\Omega |A_{ij}|^2\mathrm{d}x$.

Proof of Lemma 2.1. We write

$$\mathbb{S}(\nabla_\varepsilon\mathbf{v}) = \mu\mathbb{T}(\nabla_\varepsilon\mathbf{v}) + \eta\mathrm{div}_\varepsilon\mathbf{v}\mathbb{I}, \text{ where } \mathbb{T}(\mathbb{A}) = \mathbb{D}(\mathbb{A}) - \frac{2}{3}\mathrm{tr}(\mathbb{A}).$$

Employing the integration by parts, taking advantage of the periodicity of domain ω and of the Navier boundary conditions on the boundary of the layer, we get by an easy calculation

$$\int_\Omega \mathbb{T}(\nabla_\epsilon\mathbf{v}) : \nabla_\epsilon\mathbf{v}\mathrm{d}x = \mu\left(\int_\Omega |\nabla_\epsilon\mathbf{v}|^2 dx + \frac{1}{3}\int_\Omega(\mathrm{div}_\epsilon\mathbf{v})^2 dx\right);$$

whence

$$\int_\Omega \mathbb{S}(\nabla_\varepsilon \mathbf{v}) : \nabla_\varepsilon \mathbf{v} \, \mathrm{d}x = \mu ||\nabla_\varepsilon \mathbf{v}||^2_{L^2(\Omega)} + (\frac{1}{3}\mu + \eta)||\mathrm{div}_\varepsilon \mathbf{v}||^2_{L^2(\Omega)}.$$

Lemma 2.1 is proved.

The next Poincaré type inequality follows from [7, Appendix, Theorem 10.14] by an easy straightforward argument.

Lemma 2.2. *Let Ω be the same as in Lemma 2.1. Let $K > 0$, $M > 0$, $\gamma > 6/5$. There exists $c = c(K, M, \gamma) > 0$ such that for any*

$$\mathbf{v} \in W^{1,2}(\Omega; R^3), \ \varrho \geq 0, \ \int_\Omega \varrho \mathrm{d}x \geq M, \ \int_\Omega \varrho^\gamma \mathrm{d}x \leq K \tag{2.24}$$

and $\varepsilon \in (0, 1)$, there holds

$$\|\mathbf{v}\|^2_{L^2(\Omega;R^3)} \leq c \left(\|\nabla_\varepsilon \mathbf{v}\|^2_{L^2(\Omega;R^{3\times 3})} + \int_\Omega \varrho \mathbf{v}^2 \mathrm{d}x \right). \tag{2.25}$$

Putting together both Lemmas 2.1 and 2.2 we obtain:

Lemma 2.3. *Let $\gamma > 6/5$. For the same domain as in Lemma 2.2, and for any $K > 0$, $M > 0$, there exists $c = c(K, M, \gamma) > 0$ such that for any couple in the class (2.25) and for any $\varepsilon \in (0, 1)$ we have*

$$\|\mathbf{v}\|^2_{W^{1,2}(\Omega;R^3)} \leq c \left(\int_\Omega \mathbb{S}(\nabla_\varepsilon \mathbf{v}) : \nabla_\varepsilon \mathbf{v} \, \mathrm{d}x + \int_\Omega \varrho \mathbf{v}^2 \mathrm{d}x \right). \tag{2.26}$$

This Lemma will be useful later in Sect. 2.2.5 for estimating the remainder.

2.2.2. Relative Entropy Inequality With Special Test Functions. In view of Remark 2.1, we may use the couple $[r, \mathbf{V}]$, where $\mathbf{V} = [\mathbf{w}, 0]$ and $[r, \mathbf{w}]$ is the strong solution of the target problem (1.13–1.15) on the periodic cell ω, as the test in the relative entropy inequality (2.10).

We start by observing that the couple $(r, \mathbf{V})$ satisfies, by virtue of of (1.13–1.14),

$$\partial_t r + \mathrm{div}(r\mathbf{V}) = 0 \text{ in } (0, T) \times \Omega, \tag{2.27}$$

$$r\partial_t \mathbf{V} + r\mathbf{V} \cdot \nabla \mathbf{V} + \nabla p(r) = \mathrm{div}\mathbb{S}(\nabla \mathbf{V}) \text{ in } (0, T) \times \Omega. \tag{2.28}$$

Multiplying (2.28) scalarly by $\mathbf{u}_\varepsilon - \mathbf{V}$ and integrating over Ω, we get

$$\int_\Omega \Big(r\partial_t \mathbf{V} + r\mathbf{V} \cdot \nabla_\varepsilon \mathbf{V} + \nabla_\varepsilon p(r) \Big) \cdot (\mathbf{u}_\varepsilon - \mathbf{V}) \, \mathrm{d}x + \int_\Omega \mathbb{S}(\nabla_\varepsilon \mathbf{V}) : \nabla_\varepsilon (\mathbf{u}_\varepsilon - \mathbf{V}) \, \mathrm{d}x = 0, \tag{2.29}$$

where we have used the integration by parts in the last integral.

Next we rewrite the relative entropy inequality (2.10) with test functions r, $\mathbf{U} = \mathbf{V}$. In view of (2.29), we may rewrite the remainder (2.11) as follows

$$\begin{aligned}\mathcal{R}(\varrho_\varepsilon, \mathbf{u}_\varepsilon, r, \mathbf{V}) := & \int_\Omega \varrho_\varepsilon (\mathbf{u}_\varepsilon - \mathbf{V}) \cdot \nabla_\varepsilon \mathbf{V} \cdot (\mathbf{V} - \mathbf{u}_\varepsilon) \, \mathrm{d}x \\ & + \int_\Omega \rho_\varepsilon (\partial_t \mathbf{V} + \mathbf{V} \cdot \nabla_\varepsilon \mathbf{V}) \cdot (\mathbf{V} - \mathbf{u}_\varepsilon) \, \mathrm{d}x + \int_\Omega (r\partial_t \mathbf{V} + r\mathbf{V} \cdot \nabla_\varepsilon \mathbf{V} + \nabla_\varepsilon p(r)) \cdot (\mathbf{u}_\varepsilon - \mathbf{V}) \, \mathrm{d}x \\ & + \int_\Omega \frac{r - \rho_\varepsilon}{r} \partial_t p(r) + \frac{\nabla_\varepsilon p(r)}{r} (r\mathbf{V} - \rho_\varepsilon \mathbf{u}_\varepsilon) \, \mathrm{d}x + \int_\Omega (p(r) - p(\rho_\varepsilon)) \mathrm{div}_\varepsilon \mathbf{V} \, \mathrm{d}x\end{aligned}$$

$$= \int_\Omega (\rho_\varepsilon - r)(\partial_t \mathbf{V} + \mathbf{V}\cdot\nabla_\varepsilon \mathbf{V})\cdot(\mathbf{V}-\mathbf{u}_\varepsilon)\ \mathrm{d}x + \int_\Omega \rho_\varepsilon(\mathbf{u}_\varepsilon - \mathbf{V})\cdot\nabla_\varepsilon\mathbf{V}\cdot(\mathbf{V}-\mathbf{u}_\varepsilon)\ \mathrm{d}x$$
$$+ \int_\Omega \frac{r-\rho_\varepsilon}{r}\partial_t p(r) + \frac{\nabla_\varepsilon p(r)}{r}(r-\rho_\varepsilon)\mathbf{u}_\varepsilon\ \mathrm{d}x + \int_\Omega (p(r)-p(\rho_\varepsilon))\mathrm{div}_\varepsilon\mathbf{V}\ \mathrm{d}x.$$

Using additionally (2.27), one gets

$$\mathcal{R}(\rho_\varepsilon,\mathbf{u}_\varepsilon,r,\mathbf{V}) = \int_\Omega (\rho_\varepsilon - r)(\partial_t \mathbf{V} + \mathbf{V}\cdot\nabla_\varepsilon \mathbf{V})\cdot(\mathbf{V}-\mathbf{u}_\varepsilon)\ \mathrm{d}x + \int_\Omega \rho_\varepsilon(\mathbf{u}_\varepsilon - \mathbf{V})\cdot\nabla_\varepsilon\mathbf{V}\cdot(\mathbf{V}-\mathbf{u}_\varepsilon)\ \mathrm{d}x$$
$$+ \int_\Omega \frac{\nabla_\varepsilon p(r)}{r}(r-\rho_\varepsilon)\mathbf{u}_\varepsilon\ \mathrm{d}x + \int_\Omega (p(r)-p(\rho_\varepsilon))\mathrm{div}_\varepsilon\mathbf{V}\ \mathrm{d}x$$
$$+ \int_\Omega \frac{\nabla_\varepsilon p(r)}{r}\cdot\mathbf{V}(\rho_\varepsilon - r)\ \mathrm{d}x + \int_\Omega p'(r)(\rho_\varepsilon - r)\mathrm{div}_\varepsilon\mathbf{V}\ \mathrm{d}x.$$

Resuming the above calculation, we rewrite the relative entropy inequality (2.10) in the form

$$\mathcal{E}\Big(\varrho_\varepsilon,\mathbf{u}_\varepsilon\Big|r,\mathbf{V}\Big)(\tau) + \int_0^\tau\int_\Omega \mathbb{S}\Big(\nabla_\varepsilon(\mathbf{u}_\varepsilon-\mathbf{V})\Big):\nabla_\varepsilon(\mathbf{u}_\varepsilon-\mathbf{V})\ \mathrm{d}x\mathrm{d}t$$
$$\le \mathcal{E}\Big(\varrho_{0,\varepsilon},\mathbf{u}_{0,\varepsilon}\Big|r_0,\mathbf{V}_0\Big) + \int_0^\tau \mathcal{R}(\varrho_\varepsilon,\mathbf{u}_\varepsilon,r,\mathbf{V})\mathrm{d}t \tag{2.30}$$

where the remainder reads

$$\mathcal{R}(\rho_\varepsilon,\mathbf{u}_\varepsilon,r,\mathbf{V}) = \int_\Omega (\rho_\varepsilon - r)(\partial_t \mathbf{V} + \mathbf{V}\cdot\nabla_\varepsilon \mathbf{V})\cdot(\mathbf{V}-\mathbf{u}_\varepsilon)\ \mathrm{d}x + \int_\Omega \rho_\varepsilon(\mathbf{u}_\varepsilon - \mathbf{V})\cdot\nabla_\varepsilon\mathbf{V}\cdot(\mathbf{V}-\mathbf{u}_\varepsilon)\ \mathrm{d}x$$
$$+ \int_\Omega \frac{\nabla_\varepsilon p(r)}{r}(r-\rho_\varepsilon)\cdot(\mathbf{u}_\varepsilon-\mathbf{V})\ \mathrm{d}x - \int_\Omega (p(\rho_\varepsilon) - p'(r)(\rho_\varepsilon - r) - p(r))\mathrm{div}_\varepsilon\mathbf{V}\ \mathrm{d}x. \tag{2.31}$$

2.2.3. Main Ideas: Towards the Gronwall Inequality. The goal now is to use Lemma 2.3 in order to find an estimate of the left hand side of (2.10) from below in the form

$$\mathcal{E}(\varrho_\varepsilon,\mathbf{u}_\varepsilon\Big|r,\mathbf{V})(\tau) + c\int_0^\tau \|\mathbf{u}_\varepsilon-\mathbf{V}\|^2_{W^{1,2}(\Omega)}\mathrm{d}t - c'\int_0^\tau \mathcal{E}(\varrho_\varepsilon,\mathbf{u}_\varepsilon\Big|r,\mathbf{V})\mathrm{d}t \tag{2.32}$$

and of the right hand side from above in the form

$$h_\varepsilon(\tau) + \delta\int_0^\tau \|\mathbf{u}_\varepsilon-\mathbf{V}\|^2_{W^{1,2}(\Omega)}\mathrm{d}t + c'(\delta)\int_0^\tau a(t)\mathcal{E}(\varrho_\varepsilon,\mathbf{u}_\varepsilon\Big|r,\mathbf{V})\mathrm{d}t \tag{2.33}$$

with any $\delta > 0$, where $c > 0$ is independent of δ, $c' = c'(\delta) > 0$, $a \in L^1(0,T)$ and

$$h_\varepsilon \to 0 \text{ in } L^\infty(0,T).$$

If we succeed to establish these bounds, we deduce from the relative entropy inequality (2.10) estimate

$$\mathcal{E}(\varrho_\varepsilon,\mathbf{u}_\varepsilon\Big|r,\mathbf{V})(\tau) \le h_\varepsilon(\tau) + c\int_0^\tau a(t)\mathcal{E}(\varrho_\varepsilon,\mathbf{u}_\varepsilon\Big|r,\mathbf{V})\mathrm{d}t \tag{2.34}$$

that implies (2.18) by using the Gronwall inequality. In the rest of this section, we shall perform this programme.

2.2.4. Essential and Residual Sets. We begin with an algebraic inequality whose straightforward proof is left to the reader.

Lemma 2.4. *Let $0 < a < b < \infty$. Then there exists $c = c(a,b) > 0$ such that for all $\varrho \in [0,\infty)$ and $r \in [a,b]$ there holds*

$$E(\varrho|r) \geq c(a,b)\Big(1_{\mathcal{O}_{\rm res}} + \varrho^{\gamma}1_{\mathcal{O}_{\rm res}} + (\varrho - r)^2 1_{\mathcal{O}_{\rm ess}}\Big),$$

where $E(\varrho|r)$ is defined in (2.9) and

$$\mathcal{O}_{\rm ess} = [a/2, 2b],\ \mathcal{O}_{\rm res} = R_+\backslash[a/2, 2b].$$

Next we introduce essential and residual sets in Ω. Let $0 < \underline{\varrho} < \overline{\varrho} < \infty$. We define for a.e. $t \in (0,T)$ the residual and essential subsets of Ω as follows:

$$N^{\varepsilon}_{\rm ess}(t) = \{x \in \Omega \,\Big|\, \frac{1}{2}\underline{\varrho} \leq \varrho_\varepsilon(t,x) \leq 2\overline{\varrho}\},\ N^{\varepsilon}_{\rm res}(t) = \Omega \setminus N^{\varepsilon}_{\rm ess}(t) \tag{2.35}$$

and denote for a function h defined a.e. in $(0,T)\times\Omega$,

$$[h]_{\rm ess} = h1_{N^{\varepsilon}_{\rm ess}},\ [h]_{\rm res} = h1_{N^{\varepsilon}_{\rm res}}.$$

Here and hereafter we shall use essential and residual sets with

$$\underline{\varrho} = \underline{r}, \quad \overline{\varrho} = \overline{r}, \tag{2.36}$$

where $\underline{r}$ and $\overline{r}$ are defined in Proposition 2.3. In particular, Lemma 2.4 implies,

$$c\int\limits_{\Omega}\big([1]_{\rm res} + [\varrho_\varepsilon^{\gamma}]_{\rm res} + [\varrho_\varepsilon - r]^2_{\rm ess}\big)\ {\rm d}x \leq \int\limits_{\Omega} E(\varrho_\varepsilon, \mathbf{u}_\varepsilon \Big| r, \mathbf{U})\ {\rm d}x, \tag{2.37}$$

with some $c = c(\underline{\varrho}, \overline{\varrho}) > 0$.

2.2.5. Estimates of the Remainder. We are now in position to estimate the remainder (2.31). We shall do it in four steps.

Step 1: We shall first estimate the "essential part" of the first term: Since $V_3 = 0$ and $\partial_{x_3}\mathbf{V} = 0$, we have

$$\begin{aligned}
&\int\limits_0^\tau\int\limits_\Omega 1_{\rm ess}(\rho_\varepsilon - r)(\partial_t\mathbf{V} + \mathbf{V}\cdot\nabla_\varepsilon\mathbf{V})\cdot(\mathbf{V} - \mathbf{u}_\varepsilon)\ {\rm d}x{\rm d}t\\
&\quad\leq \int\limits_0^\tau \Big\|\partial_t\mathbf{V}_h + \mathbf{V}_h\cdot\nabla_h\mathbf{V}_h\|_{L^\infty(\Omega;R^2)}\Big\|\Big[\rho_\varepsilon - r\Big]_{\rm ess}\Big\|_{L^2(\Omega)}\Big\|\mathbf{u}_{\varepsilon h} - \mathbf{V}_h\Big\|_{L^2(\Omega;R^2)}{\rm d}t\\
&\quad\leq \delta\int\limits_0^\tau\Big\|\mathbf{u}_{\varepsilon h} - \mathbf{V}_h\Big\|^2_{L^2(\Omega;R^2)}{\rm d}t + c(\delta)\int\limits_0^\tau a(t)\mathcal{E}\Big(\varrho_\varepsilon, \mathbf{u}_\varepsilon\Big|r, \mathbf{V}\Big){\rm d}t,
\end{aligned} \tag{2.38}$$

where

$$a = \|\partial_t\mathbf{V}_h + \mathbf{V}_h\cdot\nabla_h\mathbf{V}_h\|^2_{L^\infty(\Omega;R^2)} \in L^1(0,T).$$

For the "residual" part of the first term, we get

$$\begin{aligned}
&\int_0^\tau \int_\Omega 1_{\rm res}(\rho_\varepsilon - r)(\partial_t \mathbf{V} + \mathbf{V}\cdot\nabla_\varepsilon \mathbf{V})\cdot(\mathbf{V}-\mathbf{u}_\varepsilon)\,\mathrm{d}x\mathrm{d}t\\
&\quad\le \int_0^\tau \Big\|\partial_t \mathbf{V}_h + \mathbf{V}_h\cdot\nabla_h \mathbf{V}_h\|_{L^\infty(\Omega;R^3)}\Big\|\Big[\rho_\varepsilon - r\Big]_{\rm res}\Big\|_{L^{6/5}(\Omega)}\Big\|\mathbf{u}_{\varepsilon h} - \mathbf{V}_h\Big\|_{L^6(\Omega;R^2)}\mathrm{d}t\\
&\quad\le \delta\int_0^\tau \Big\|\mathbf{u}_{\varepsilon h} - \mathbf{V}_h\Big\|^2_{W^{1,2}(\Omega;R^2)}\mathrm{d}t + c(\delta)\int_0^\tau a(t)\Big[\mathcal{E}\Big(\varrho_\varepsilon,\mathbf{u}_\varepsilon\Big| r,\mathbf{V}\Big)\Big]^{10/3}\mathrm{d}t\\
&\quad\le \delta\int_0^\tau \Big\|\mathbf{u}_{\varepsilon h} - \mathbf{V}_h\Big\|^2_{W^{1,2}(\Omega;R^2)}\mathrm{d}t + c(\delta)\int_0^\tau a(t)\mathcal{E}\Big(\varrho_\varepsilon,\mathbf{u}_\varepsilon\Big| r,\mathbf{V}\Big)\mathrm{d}t \qquad (2.39)
\end{aligned}$$

provided $\gamma \ge 6/5$. Here we have used Hölder and Young inequalities, continuous imbedding $W^{1,2}(\Omega) \subset L^6(\Omega)$ and the fact that

$$\mathcal{E}\Big(\varrho_\varepsilon,\mathbf{u}_\varepsilon\Big| r,\mathbf{V}\Big) \in L^\infty(0,T),$$

by virtue of (2.7), (2.15–2.16).

Step 2:

$$\int_0^\tau \int_\Omega \varrho_\varepsilon(\mathbf{u}_\varepsilon - \mathbf{V})\cdot\nabla_\varepsilon \mathbf{V}\cdot(\mathbf{V}-\mathbf{u}_\varepsilon)\,\mathrm{d}x\mathrm{d}t \le c\int_0^\tau a(t)\mathcal{E}\Big(\varrho_\varepsilon,\mathbf{u}_\varepsilon\Big| r,\mathbf{V}\Big)\mathrm{d}t \qquad (2.40)$$

with

$$a = \|\nabla_h \mathbf{V}_h\|_{L^\infty(\Omega;R^{2\times 2})} \in L^\infty(0,T).$$

Step 3: Similarly as in Step 1,

$$\begin{aligned}
&\int_0^\tau \int_\Omega \frac{\nabla_\varepsilon p(r)}{r}(r-\rho_\varepsilon)\cdot(\mathbf{u}_\varepsilon - \mathbf{V})\,\mathrm{d}x\mathrm{d}t\\
&\quad\le \delta\int_0^\tau \Big\|\mathbf{u}_{\varepsilon h} - \mathbf{V}_h\Big\|^2_{W^{1,2}(\Omega;R^2)}\mathrm{d}t + c(\delta)\int_0^\tau a(t)\mathcal{E}\Big(\varrho_\varepsilon,\mathbf{u}_\varepsilon\Big| r,\mathbf{V}\Big)\mathrm{d}t, \qquad (2.41)
\end{aligned}$$

where

$$a = \Big\|\frac{\nabla_h p(r)}{r}\Big\|^2_{L^\infty(\Omega;R^2)} \in L^\infty(0,T).$$

Step 4:

As far as the last term is concerned, we use: 1) The Taylor formula together with the regularity C^2 of the pressure p [see (1.6)], in order to estimate the essential part

$$\begin{aligned}
-\int_0^\tau \int_\Omega \Big[p(\rho_\varepsilon) - p'(r)(\rho_\varepsilon - r) - p(r)\Big]_{\rm ess}\mathrm{div}_\varepsilon \mathbf{V}\,\mathrm{d}x\mathrm{d}t &\le c\int_0^\tau \|\mathrm{div}_h \mathbf{V}_h\|_{L^\infty(\Omega)}\Big\|\Big[\varrho_\varepsilon - r\Big]_{\rm ess}\Big\|^2_{L^2(\Omega)}\\
&\le c\int_0^\tau a(t)\mathcal{E}\Big(\varrho_\varepsilon,\mathbf{u}_\varepsilon\Big| r,\mathbf{V}\Big)\mathrm{d}t, \quad a = \|\mathrm{div}_h \mathbf{V}_h\|_{L^\infty(\Omega)}.
\end{aligned}$$

(2.42)

2) The pointwise bound

$$\Big|[p(\rho_\varepsilon) - p'(r)(\rho_\varepsilon - r) - p(r)]_{\rm res}\Big| \le c\Big(1_{\rm res} + [\varrho]^\gamma_{\rm res}\Big)$$

[cf. (1.6), (2.15–2.16)], in order to estimate the residual part

$$-\int_0^\tau\int_\Omega \Big[p(\rho_\varepsilon) - p'(r)(\rho_\varepsilon - r) - p(r)\Big]_{\rm res} {\rm div}_h \mathbf{V}\ {\rm d}x{\rm d}t$$
$$\le c\int_0^\tau \|{\rm div}_h \mathbf{V}_h\|_{L^\infty(\Omega)} \int_\Omega \Big(1_{\rm res}\varrho_\varepsilon^\gamma + 1_{\rm res}\Big)\ {\rm d}x{\rm d}t \le c\int_0^\tau a(t)\mathcal{E}\Big(\varrho_\varepsilon, \mathbf{u}_\varepsilon\Big| r, \mathbf{V}\Big){\rm d}t. \tag{2.43}$$

Coming back with these estimates to the relative entropy inequality (2.30), we easily verify the validity of (2.34) with

$$a = \|\partial_t \mathbf{V}_h + \mathbf{V}_h \cdot \nabla_h \mathbf{V}_h\|^2_{L^\infty(\Omega;R^2)} + \|\nabla_h \mathbf{V}\|_{L^\infty(\Omega;R^{2\times2})} + \left\|\frac{\nabla_h p(r)}{r}\right\|^2_{L^\infty(\Omega;R^2)} + \|{\rm div}_h \mathbf{V}_h\|_{L^\infty(\Omega)}$$

and

$$h_\varepsilon = \mathcal{E}(\varrho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon}\Big| r, \mathbf{V}).$$

This finishes the proof of Theorem 2.1.

3. Bounded Layers With No-Slip Boundary Conditions

In this section we consider the compressible Navier–Stokes system (1.2–1.4) on $\Omega_\varepsilon = \omega \times (0, \varepsilon)$, where $\omega \subset R^2$ is a bounded domain. It is completed with the slip boundary conditions on the boundary $\omega \times \{0, \varepsilon\}$ and no slip boundary conditions on the boundary $\partial\omega \times (0, \varepsilon)$:

$$\mathbf{u}|_{\partial\omega\times(0,\varepsilon)} = 0,\ \mathbf{u}\cdot\mathbf{n}|_{\omega\times\{0,\varepsilon\}} = 0,\ [\mathbb{S}(\nabla\mathbf{u})\cdot\mathbf{n}]\times\mathbf{n}|_{\omega\times\{0,\varepsilon\}} = 0. \tag{3.1}$$

After rescaling, this problem is equivalent to the system (1.10–1.12) on the domain $\Omega = \omega \times (0, 1)$ the with boundary conditions

$$\mathbf{u}|_{\partial\omega\times(0,1)} = 0,\ \mathbf{u}\cdot\mathbf{n}|_{\omega\times\{0,1\}} = 0,\ [\mathbb{S}(\nabla\mathbf{u})\cdot\mathbf{n}]\times\mathbf{n}|_{\omega\times\{0,1\}} = 0. \tag{3.2}$$

3.1. Weak Solutions and Relative Entropy

Definition 3.1. The couple $[\varrho, \mathbf{u}]$ is a finite energy weak solution of the problem (1.10–1.12) with boundary conditions (3.2) if:

- $[\varrho, \mathbf{u}]$ belongs to the functional spaces (2.4), where we replace $W^{1,2}_{\mathbf{n}}(\Omega; R^3))$ with

$$W^{1,2}_{0,\mathbf{n}}(\Omega; R^3) \equiv \{\mathbf{v} \in W^{1,2}(\Omega; R^3)\Big|\mathbf{v}|_{\partial\omega\times(0,1)} = 0,\ \mathbf{v}\cdot\mathbf{n}|_{\omega\times\{0,1\}} = 0\}.$$

- Weak formulation (2.5) of the continuity Eq. (1.10) remains without changes;
- Weak formulation (2.6) of the momentum Eq. (1.11) holds with test functions

$$\varphi \in C^\infty_c([0, T] \times \overline{\Omega}; R^3),\ \varphi|_{[0,T]\times\partial\omega\times(0,1)} = 0,\ \varphi_3|_{[0,T]\times\omega\times\{0,1\}} = 0;$$

- Energy inequality (2.7) holds.

Again, from Lions [15], Feireisl, [5,17] and [11], one can deduce *existence of weak solutions* for the above problem with $\mathbb{S}$, p and $[\varrho_{\varepsilon,0}, \mathbf{u}_{\varepsilon,0}]$ verifying assumptions (1.5), (1.6), (2.8), provided the domain ω is Lipschitz. Moreover, due to [6], any weak solution satisfies relative entropy inequality (2.10) with the remainder (2.11), where the test functions satisfy

$$\begin{aligned} & r \in C^1([0,T]\times\overline{\Omega}),\ r > 0 \\ & \mathbf{U} \in C^1([0,T]\times\overline{\Omega}; R^3),\ \mathbf{U}|_{\partial\omega\times(0,1)} = 0,\ U_3|_{\omega\times\{0,1\}} = 0. \end{aligned} \tag{3.3}$$

Finally, as in Remark 2.1, it can be shown by the density argument, that the test function $(r, \mathbf{U})$ can be taken in the larger regularity class (2.12).

3.2. Target System and the Main Result

The expected target system is system (1.13–1.15) endowed with the no slip boundary conditions:

$$\mathbf{w}|_{\partial\omega} = 0. \tag{3.4}$$

Theorem 2.5 in Valli and Zajaczkowski [21] states:

Proposition 3.1. *Let D be a positive constant. Suppose that $p \in C^2(0,\infty)$, $\partial\omega \in C^3$. Assume that the initial conditions $[r_0, \mathbf{w}_0]$ belong to the regularity class (2.13) and satisfy the compatibility condition*

$$\frac{1}{r_0}\Big(\nabla_h p(r_0) - \mathrm{div}_h \mathbb{S}_h(\nabla_h \mathbf{w}_0) + r_0 \mathbf{w}_0 \cdot \nabla_h \mathbf{w}_0\Big)\Big|_{\partial\omega} = 0. \tag{3.5}$$

Then there exists $T = T_{\max}(D)$ such that if

$$\|r_0\|_{W^{2,2}(\omega)} + \|\mathbf{w}_0\|_{W^{3,2}(\omega;R^2)} + 1/\inf_{\omega} r_0 \le D,$$

then the problem (1.13–1.15), (3.4) admits a unique strong solution (in the sense a.e. in $(0,T)\times\omega$) in the class (2.15), (2.16).

We are now in a position to formulate the main result of Sect. 3.

Theorem 3.1. *Let $\partial\omega \in C^3$. Suppose that the pressure p satisfies hypotheses (1.6) and that the stress tensor is given by (1.5). Let r_0, $\mathbf{w}_0$ satisfy assumptions (2.13), (3.5) and let $T_{\max} > 0$ be the life time of the strong solution to problem (1.13–1.15), (3.4) corresponding to $[r_0, \mathbf{w}_0]$ determined in Proposition 3.1. Let $[\varrho_\varepsilon, \mathbf{u}_\varepsilon]$ be a sequence of weak solutions to the 3-D compressible Navier Stokes equations (1.10–1.12), (3.2) emanating from the initial data $[\varrho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon}]$. Suppose that initial data satisfy (2.17).*
Then

$$\mathrm{esssup}_{t\in(0,T_{\max})}\mathcal{E}(\varrho_\varepsilon, \mathbf{u}_\varepsilon \Big| r, \mathbf{V}) \to 0, \tag{3.6}$$

where $\mathbf{V}(t,x) = [\mathbf{w}(t,x_h), 0]$ and where the couple $(r, \mathbf{w})$ satisfies the 2-D compressible Navier–Stokes system (1.13–1.15) with the boundary conditions (3.4) on the time interval $[0, T_{\max})$.

Remark 3.1. We notice that all conclusions of Remarks 2.2–2.3 and Corollaries 2.1–2.2 remain valid also in the case of bounded layers with the no slip boundary conditions.

3.3. Proof of Theorem 3.1

The great lines of the proof follow the proof of Theorem 2.1. However, we are able to obtain only a weaker uniform lower bound for the integrals involving the quantity $\mathbb{S}(\nabla_\varepsilon \mathbf{v}) : \nabla_\varepsilon \mathbf{v}$. Consequently, finding the convenient upper bound will be more involved. In this Section, we focus essentially on these two points

3.3.1. Some Auxiliary Results. We observe that the following identity holds pointwise on Ω:

$$\mathbb{S}(\nabla_\varepsilon \mathbf{v}) : \nabla_\varepsilon \mathbf{v} = \eta |\mathrm{div}_\varepsilon \mathbf{v}|^2 + \frac{\mu}{2}\left|\mathbb{D}(\nabla_\varepsilon \mathbf{v}) - \frac{2}{3}\mathrm{div}_\varepsilon \mathbf{v}\mathbb{I}\right|^2 \tag{3.7}$$

This implies the following lemma:

Lemma 3.1. *Let $\mathbb{S}$, $\mathbb{D}$ be defined in (1.5), where $\mu > 0$, $\eta > 0$. Then we have: For all $\mathbf{v} \in W^{1,2}(\Omega; R^3)$ and $\varepsilon \in (0,1)$, there holds*

$$\eta \|\mathrm{div}_\varepsilon \mathbf{v}\|^2_{L^2(\Omega)} \leq \int_\Omega \mathbb{S}(\nabla_\varepsilon \mathbf{v}) : \nabla_\varepsilon \mathbf{v}\ \mathrm{d}x$$

and

$$\frac{\mu}{2}\|\mathbb{D}(\nabla_\varepsilon \mathbf{v})\|^2_{L^2(\Omega)} \leq 2\left(1 + \frac{\mu}{3\eta}\right)\int_\Omega \mathbb{S}(\nabla_\varepsilon \mathbf{v}) : \nabla_\varepsilon \mathbf{v}\ \mathrm{d}x.$$

Next, simply by integration by parts, we show the identity

$$\|\mathbb{D}_h(\nabla_h \mathbf{v})\|^2_{L^2(\omega,R^2)} = 2\Big(\|\nabla_h \mathbf{v}\|^2_{L^2(\omega,R^{2\times 2})} + \|\mathrm{div_h}\mathbf{v}\|^2_{\mathrm{L}^2(\omega)}\Big) \tag{3.8}$$

Finally, we recall the standard Poincaré inequality,

$$\|\mathbf{v}\|_{L^2(\omega;R^2} \leq c\|\nabla_h \mathbf{v}\|^2_{L^2(\omega,R^{2\times 2})} \tag{3.9}$$

Both formulas (3.8), (3.9) hold for all $\mathbf{v} \in W^{1,2}_0(\omega; R^2)$ provided ω is a Lipschitz domain.

Putting together these results, we may write the following lemma.

Lemma 3.2. *Let $\Omega = \omega \times (0,1)$, where ω is a bounded Lipschitz domain and let $\eta > 0$. Then there exists $c = c(\omega) > 0$ such that for all $\mathbf{v} \in W^{1,2}(\Omega; R^3)$, $\mathbf{v}|_{\partial\omega\times(0,1)} = 0$ and $\varepsilon \in (0,1)$,*

$$\|\mathbf{v}_h\|^2_{L^2(\Omega,R^2)} + \|\nabla_h \mathbf{v}_h\|^2_{L^2(\Omega,R^{2\times 2})} \leq c\int_\Omega \mathbb{S}(\nabla_\varepsilon \mathbf{v}) : \nabla_\varepsilon \mathbf{v}\ \mathrm{d}x, \tag{3.10}$$

where the tensor $\mathbb{S}$ is defined in (1.5).

Remark 3.2. Besides (3.10) we have also trivially

$$\|\partial_3 v_3\|^2_{L^2(\Omega)} \leq c\varepsilon^2 \int_\Omega \mathbb{S}(\nabla_\varepsilon \mathbf{v}) : \nabla_\varepsilon \mathbf{v}\ \mathrm{d}x,$$

and moreover, if $v_3|_{\omega\times\{0,1\}} = 0$,

$$\|v_3\|^2_{L^2(\Omega)} \leq c\varepsilon^2 \int_\Omega \mathbb{S}(\nabla_\varepsilon \mathbf{v}) : \nabla_\varepsilon \mathbf{v}\ \mathrm{d}x.$$

These additional estimates will not be exploited throughout the proof.

Remark 3.3. In contrast to Lemma 2.3 dealing with the periodic case, Lemma 3.2 does not provide the estimate for the whole $W^{1,2}$- norm of neither $\mathbf{v}$ nor $\mathbf{v}_h$. The challenge now is to estimate the remainder in the relative entropy inequality in a different way than in Sect. 2 by using only the above “incomplete” estimate.

3.3.2. Application of the Relative Entropy. In view of Remark 3.3 we shall modify the procedure of Sect. 2.2.3 as follows: We shall estimate the left hand side of the relative entropy inequality (2.10) with test functions $[r, \mathbf{V}]$, $\mathbf{V} = [\mathbf{w}, 0]$ from below by

$$\mathcal{E}\left(\varrho, \mathbf{u}\middle| r, \mathbf{V}\right)(\tau) + c\int_0^\tau \|\mathbf{u}_{\varepsilon h} - \mathbf{V}_h\|^2_{L^2(\Omega;R^2)}\mathrm{d}t - c'\int_0^\tau \mathcal{E}\left(\varrho_\varepsilon, \mathbf{u}_\varepsilon\middle| r, \mathbf{V}\right)\mathrm{d}t \tag{3.11}$$

[this follows trivially from (2.32)], and the right hand side from above by

$$h_\varepsilon(\tau) + \delta\int_0^\tau \|\mathbf{u}_{\varepsilon h} - \mathbf{V}_h\|^2_{L^2(\Omega)}\mathrm{d}t + c'(\delta)\int_0^\tau a(t)\mathcal{E}(\varrho_\varepsilon, \mathbf{u}_\varepsilon\big| r, \mathbf{V})\mathrm{d}t \tag{3.12}$$

with any $\delta > 0$, where $c > 0$ is independent of δ, $c' = c'(\delta) > 0$, $a \in L^1(0, T)$ and

$$h_\varepsilon \to 0 \text{ in } L^\infty(0, T).$$

This process leads again to the inequality (2.34), that finishes the proof. The "only" point in the proof is therefore to find the bound (3.12) of the remainder (2.31). This will be done in the next section.

3.3.3. Estimate of the Remainder. In order to get the estimate (3.12) of the remainder (2.31), we proceed in three steps. As in Sect. 2.2.5, we shall systematically use that $V_3 = 0$ and $\partial_{x_3}\mathbf{V} = 0$.

Step 1: The essential part of the first term $\int_0^\tau\int_\Omega 1_{\mathrm{ess}}(\rho_\varepsilon - r)(\partial_t\mathbf{V} + \mathbf{V}\cdot\nabla_\varepsilon\mathbf{V})\cdot(\mathbf{V} - \mathbf{u}_\varepsilon)\,\mathrm{d}x\mathrm{d}t$ is estimated in the same way as in formula (2.38).

Concerning the residual part, we shall estimate the integrals over the sets $\{\varrho \le \underline{\varrho}/2\}$ and $\{\varrho \ge 2\overline{\varrho}\}$ separately. [Numbers $\underline{\varrho}$, $\overline{\varrho}$ are defined in (2.36) and essential/residual sets in (2.35)].

$$\begin{aligned}
&\int_0^\tau\int_\Omega 1_{\{\varrho\le\underline{\varrho}/2\}}(\rho_\varepsilon - r)(\partial_t\mathbf{V} + \mathbf{V}\cdot\nabla_\varepsilon\mathbf{V})\cdot(\mathbf{V} - \mathbf{u}_\varepsilon)\,\mathrm{d}x\mathrm{d}t\\
&\quad\le 2\overline{\varrho}\int_0^\tau\int_\Omega 1_{\mathrm{res}}\left|\partial_t\mathbf{V} + \mathbf{V}\cdot\nabla_\varepsilon\mathbf{V}\right|\left|\mathbf{V} - \mathbf{u}_\varepsilon\right|\,\mathrm{d}x\mathrm{d}t\\
&\quad\le \int_0^\tau \left\|\partial_t\mathbf{V}_h + \mathbf{V}_h\cdot\nabla_h\mathbf{V}_h\right\|_{L^\infty(\Omega;R^2)}\left\|1_{\mathrm{res}}\right\|_{L^2(\Omega)}\left\|\mathbf{u}_{\varepsilon h} - \mathbf{V}_h\right\|_{L^2(\Omega;R^2)}\mathrm{d}t\\
&\quad\le \delta\int_0^\tau \left\|\mathbf{u}_{\varepsilon h} - \mathbf{V}_h\right\|^2_{L^2(\Omega;R^2)}\mathrm{d}t + c(\delta)\int_0^\tau a(t)\mathcal{E}\left(\varrho_\varepsilon, \mathbf{u}_\varepsilon\middle| r, \mathbf{V}\right)\mathrm{d}t,
\end{aligned}$$

where a is given in (2.38).

Finally,

$$\begin{aligned}
&\int_0^\tau\int_\Omega 1_{\{\varrho\ge 2\overline{\varrho}\}}(\varrho_\varepsilon - r)(\partial_t\mathbf{V} + \mathbf{V}\cdot\nabla_\varepsilon\mathbf{V})\cdot(\mathbf{V} - \mathbf{u}_\varepsilon)\,\mathrm{d}x\mathrm{d}t\\
&\quad\le 2\int_0^\tau\int_\Omega 1_{\mathrm{res}}\sqrt{\varrho_\varepsilon}\left|\partial_t\mathbf{V} + \mathbf{V}\cdot\nabla_\varepsilon\mathbf{V}\right|\sqrt{\varrho_\varepsilon}\left|\mathbf{V} - \mathbf{u}_\varepsilon\right|\,\mathrm{d}x\mathrm{d}t
\end{aligned}$$

$$\leq \int_0^\tau \left\|\partial_t \mathbf{V}_h + \mathbf{V}_h \cdot \nabla_h \mathbf{V}_h\|_{L^\infty(\Omega;R^2)}\right\| \left[\varrho\right]_{\rm res}\|_{L^1(\Omega)}^{1/2} \left\|\varrho_\varepsilon\Big(\mathbf{u}_{\varepsilon h} - \mathbf{V}_h\Big)^2\right\|_{L^1(\Omega}^{1/2} {\rm d}t$$
$$\leq c\int_0^\tau a(t)\mathcal{E}\Big(\varrho_\varepsilon, \mathbf{u}_\varepsilon\Big| r, \mathbf{V}\Big){\rm d}t$$

with the same a as before. In all above three formulas, we have employed (2.37) in the passage to their last lines.

Step 2: Estimates of the second

$$\int_0^\tau \int_\Omega \rho_\varepsilon(\mathbf{u}_\varepsilon - \mathbf{V}) \cdot \nabla_\varepsilon \mathbf{V} \cdot (\mathbf{V} - \mathbf{u}_\varepsilon)\ {\rm d}x{\rm d}t$$

and the fourth

$$\int_0^\tau \int_\Omega (p(\rho_\varepsilon) - p'(r)(\rho_\varepsilon - r) - p(r)){\rm div}_\varepsilon \mathbf{V}\ {\rm d}x{\rm d}t$$

terms of the remainder are the same as (2.40) and (2.42–2.43) in Sect. 2.2.5.

Step 3: Estimate of the third term folows the same lines as the estimates effectuated in Step 1, namely

$$\int_0^\tau \int_\Omega \frac{\nabla_\varepsilon p(r)}{r}(r-\rho_\varepsilon) \cdot (\mathbf{u}_\varepsilon - \mathbf{V})\ {\rm d}x{\rm d}t \leq \delta \int_0^\tau \left\|\mathbf{u}_{\varepsilon h} - \mathbf{V}_h\right\|_{L^2(\Omega;R^2)}^2 {\rm d}t + c(\delta)\int_0^\tau a(t)\mathcal{E}\Big(\varrho_\varepsilon, \mathbf{u}_\varepsilon\Big| r, \mathbf{V}\Big){\rm d}t,$$

where

$$a(t) = \left\|\frac{1}{r}\nabla_h p(r)\right\|_{L^\infty(\Omega;R^2)}^2.$$

Collecting these estimates in the relative entropy inequality (2.30), we obtain formula (2.34) with

$$h_\varepsilon = \mathcal{E}\Big(\varrho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon}\Big| r_0, \mathbf{V}_0\Big),$$

and conclude by the Gronwall lemma.

4. Bounded Layers With Slip Boundary Conditions

We start again with system (1.2–1.4) on the domain $\Omega_\varepsilon = \omega \times (0,\varepsilon)$ where $\omega \in R^2$ is bounded. The boundary conditions read

$$\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega_\varepsilon} = 0,\ [\mathbb{S}(\nabla\mathbf{u})\cdot\mathbf{n}]\times\mathbf{n}|_{\partial\Omega_\varepsilon} = 0. \tag{4.1}$$

Again, after rescaling, this problem is equivalent to the system (1.10–1.12) on the domain $\Omega = \omega \times (0,1)$ with boundary conditions

$$\mathbf{u}\cdot\mathbf{n}|_{\omega\times\{0,1\}\cup\partial\omega\times(0,1)\}} = 0,\ [\mathbb{S}(\nabla\mathbf{u})\cdot\mathbf{n}]\times\mathbf{n}|_{\omega\times\{0,1\}\cup\partial\omega\times(0,1)} = 0. \tag{4.2}$$

4.1. Weak Solutions, Relative Entropy, Target System and the Main Result

4.2. Weak Solutions and Relative Entropy

Definition 4.1. The couple $[\varrho, \mathbf{u}]$ is a finite energy weak solution of the problem (1.10–1.12) with boundary conditions (3.2) if:

- $[\varrho, \mathbf{u}]$ belongs to the functional spaces (2.4), where we replace $W^{1,2}_{\mathbf{n}}(\Omega; R^3))$ with

$$W^{1,2}_{\mathbf{n}}(\Omega; R^3) \equiv \{\mathbf{v} \in W^{1,2}(\Omega; R^3) \Big| \mathbf{v} \cdot \mathbf{n}|_{\partial\Omega} = 0\}.$$

- Weak formulation (2.5) of the continuity Eq. (1.10) remains without changes;
- Weak formulation (2.6) of the momentum Eq. (1.11) holds with test functions

$$\varphi \in C^\infty_c([0, T] \times \overline{\Omega}; R^3),\ \varphi \cdot \mathbf{n}|_{[0,T]\times\partial\Omega} = 0.$$

- Energy inequality (2.7) holds.

Existence of weak solutions for the above problem with $\mathbb{S}$, p and $[\varrho_{\varepsilon,0}, \mathbf{u}_{\varepsilon,0}]$ verifying assumptions (1.5), (1.6), (2.8) can be deduced from [15], Feireisl [5,17] and [11], provided the domain ω is of class $C^{2,\nu}$, $\nu \in (0, 1)$. Moreover, due to [6], any weak solution satisfies relative entropy inequality (2.10) with the remainder (2.11), where the test functions satisfy

$$\begin{aligned} & r \in C^1([0, T] \times \overline{\Omega}),\ r > 0 \\ & \mathbf{U} \in C^1([0, T] \times \overline{\Omega}; R^3),\ \mathbf{U} \cdot \mathbf{n}|_{\partial\omega\times(0,1)} = 0, \end{aligned} \tag{4.3}$$

Finally, as in Remark 2.1, it can be shown by the density argument, that the test function $[r, \mathbf{U}]$ can be taken in the larger regularity class (2.12).

The expected target system is system (1.13–1.15) endowed with the no slip boundary conditions:

$$\mathbf{w} \cdot \mathbf{n}|_{\partial\omega} = 0, \quad \Big(\mathbb{S}_h(\nabla_h \mathbf{w})\mathbf{n}\Big) \times \mathbf{n}|_{\partial\omega} = 0. \tag{4.4}$$

One can again deduce from Valli and Zajaczkowski [21, Theorem 2.5] that under assumptions $p \in C^2(0, \infty)$, $\partial\omega \in C^3$ with initial conditions $[r_0, \mathbf{w}_0]$ belonging to the regularity class (2.13) and satisfying the compatibility condition

$$\frac{1}{r_0}\Big(\nabla_h p(r_0) - \mathrm{div}_h \mathbb{S}_h(\nabla_h \mathbf{w}_0) + r_0 \mathbf{w}_0 \cdot \nabla_h \mathbf{w}_0\Big) \cdot \mathbf{n}\Big|_{\partial\omega} = 0, \tag{4.5}$$

the problem (1.13–1.15), (4.4) admits a unique strong solution in the class (2.15), (2.16) on the maximal time interval $[0, T = T_{\max})$ [that depends on the size of initial data as described in formula (2.14)].

We are now in a position to formulate the main result of Sect. 4.

Theorem 4.1. *Let $\partial\omega \in C^3$. Suppose that the pressure p satisfies hypotheses (1.6) and that the stress tensor is given by (1.5) with $\eta > 0$. Let r_0, $\mathbf{w}_0$ satisfy assumptions (2.13), (4.5). Let $[\varrho_\varepsilon, \mathbf{u}_\varepsilon]$ be a sequence of weak solutions to the 3-D compressible Navier Stokes equations (1.10–1.12), (4.2) emanating from the initial data $[\varrho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon}]$ that converge to $[r_0, \mathbf{w}_0]$ in the sense (2.17).*

Then $[\varrho_\varepsilon, \mathbf{u}_\varepsilon]$ converges to $[r, \mathbf{w}]$ in the sense (2.18), where $[\varrho, \mathbf{w}]$ is the unique strong solution of the 2-D compressible Navier–Stokes system (1.13–1.15) with boundary conditions (4.4) on the maximal existence time interval of the strong solution $[0, T = T_{\max})$.

Remark 4.1. Remarks 2.2–2.3 and Corollaries 2.1–2.2 remain valid also in the case of bounded layers with the slip boundary conditions.

4.3. Proof of Theorem 4.1

First, recall the classical Korn inequality on a bounded Lipschitz domain ω. In particular, if $\omega \subset R^2$ it reads: There exists $c = c(\omega)$ such that for all $\mathbf{v} \in W^{1,2}(\omega; R^2)$, there holds

$$\|\mathbf{v}\|_{W^{1,2}(\omega;R^2)} \leq c(\Big(\|\mathbb{D}(\nabla_h \mathbf{v})\|_{L^2(\omega,R^{2\times 2})} + \|\mathbf{v}\|^2_{L^2(\omega;R^2)}\Big). \tag{4.6}$$

Next, we deduce from this inequality, the following result:

Lemma 4.1. *Let $\omega \in R^2$ be a bounded Lipschitz domain that is not radially symmetric. Then there exists $c = c(\omega) > 0$ such that for all $\mathbf{v} \in W^{1,2}(\omega; R^2)$, $\mathbf{w} \cdot \mathbf{n}|_{\partial\omega} = 0$ there holds:*

$$\|\mathbf{v}\|_{W^{1,2}(\omega;R^2)} \leq c\|\mathbb{D}_h(\nabla \mathbf{v})\|_{L^2(\omega;R^{2\times 2})}.$$

Proof of Lemma 4.1. If the conclusion is not true then there exits a sequence $\mathbf{v}_n \in W^{1,2}_{\mathbf{n}}(\omega; R^2)$ such that

$$||\mathbf{v}_n||_{W^{1,2}(\omega;R^2)} = 1$$
$$\|\mathbb{D}_h(\nabla \mathbf{v}_n)\|_{L^2(\omega;R^{2\times 2})} < \frac{1}{n}$$

Consequently, there is $\mathbf{v} \in W^{1,2}(\omega, R^2)$ such that $\mathbf{v}_n \to \mathbf{v}$ in $L^2(\omega, R^2)$ and $\mathbf{v}_n \to \mathbf{v}$ weakly in $W^{1,2}(\omega, R^2)$. Using (4.6) we deduce that $\mathbf{v}_n$ is a Cauchy sequence in $W^{1,2}(\omega, R^2)$. Therefore, $\mathbf{v} \in W^{1,2}(\omega, R^2)$ and

$$\mathbf{v}_n \to \mathbf{v} \in W^{1,2}(\omega, R^2).$$

On the other hand

$$\nabla \mathbf{v} + (\nabla \mathbf{v})^T = 0;$$

whence $\mathbf{v}$ is a rigid rotation $\mathbf{v} = \mathbf{b} \times x$, $\mathbf{b} \in R^3$. This contradicts condition $\mathbf{v} \cdot \mathbf{n}|_{\partial\omega} = 0$ for the non circular domains. Lemma 4.1 is proved.

Collecting results of Lemmas 3.1 and 4.1 we obtain an estimate that will be used later for the estimating of the remainder of in the relative entropy inequality, namely:

Lemma 4.2. *Let $\mathbb{S}$ satisfy (1.5), where $\mu > 0$, $\eta > 0$. Let $\Omega = \omega \times (0, 1)$, where ω is a bounded Lipschitz domain that is not a circle. Then there exists $c = c(\omega) > 0$ such that for all $\mathbf{v} \in W^{1,2}(\Omega; R^3)$, $\mathbf{v}_h \cdot \mathbf{n}_h|_{\partial\omega\times(0,1)} = 0$ and $\varepsilon \in (0, 1)$,*

$$\|\mathbf{v}_h\|^2_{L^2(\Omega;R^2)} + \|\nabla_h \mathbf{v}_h\|^2_{L^2(\Omega,R^{2\times 2})} \leq \int_\Omega \mathbb{S}(\nabla_\varepsilon \mathbf{v}) : \nabla_\varepsilon \mathbf{v} \, \mathrm{d}x.$$

Since Lemma 4.2 provides exactly the same estimate as Lemma 3.2, the proof of Theorem 4.1 is exactly the same as the proof of Theorem 3.1.

5. Unbounded Layers

In this Section we consider the compressible Navier–Stokes system (1.2–1.4) on unbounded layers with slip boundary conditions on the boundary of the layer, more precisely

$$\Omega_\varepsilon = R^2 \times (0, \varepsilon),\ \mathbf{u} \cdot \mathbf{n}|_{\partial\Omega_\varepsilon} = 0,\ [\mathbb{S}(\nabla \mathbf{u}) \cdot \mathbf{n}] \times \mathbf{n}|_{\partial\Omega_\varepsilon} = 0 \tag{5.1}$$

Since the domain is unbounded, the system has to be completed with conditions at infinity,

$$\mathbf{u}(t, x) \to 0,\ \varrho(t, x) \to \overline{\varrho} > 0 \text{ as } |x| \to \infty. \tag{5.2}$$

After rescaling, we get system (1.10–1.12) with the same boundary conditions on the boundary of Ω:

$$\Omega = R^2 \times (0, 1),\ \mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0,\ [\mathbb{S}(\nabla \mathbf{u}) \cdot \mathbf{n}] \times \mathbf{n}|_{\partial\Omega} = 0, \tag{5.3}$$

and conditions (5.2) as $|x| \to \infty$.

5.1. Weak Solutions to the Primitive System

5.1.1. Weak Solutions.

We introduce again the Hilbert space

$$W^{1,2}_{\mathbf{n}}(\Omega;R^3)=\{\mathbf{v}\in W^{1,2}(\Omega;R^3),\ \mathbf{v}\cdot\mathbf{n}|_{\partial\Omega}=0\}=W^{1,2}(\Omega,R^2)\times H^1_0(\Omega).$$

As we are on an unbounded domain, the definition of weak solutions has to be modified as follows, cf. e.g. [17].

Definition 5.1. We say that $[\varrho,\mathbf{u}]$ is a *finite energy weak solution* to the compressible Navier–Stokes (1.10–1.11) with initial conditions (1.12) and boundary conditions (5.2–5.3) in the space time cylinder $(0,T)\times\Omega$ if the following holds:

- the functions $[\varrho,\mathbf{u}]$ belong to the regularity class

$$\left\{\begin{array}{c}\varrho-\overline{\varrho}\in L^\infty([0,T];L^\gamma(\Omega)+L^2(\Omega)),\ \ \varrho\geq 0 \text{ a.a. in } (0,T)\times\Omega,\ \ \gamma>\frac{3}{2},\\[2mm] \mathbf{u}\in L^2(0,T;W^{1,2}_{\mathbf{n}}(\Omega;R^3)),\ \ \varrho\mathbf{u}\in L^\infty(0,T;L^2(\Omega)+L^{\frac{2\gamma}{\gamma+1}}(\Omega));\end{array}\right\} \tag{5.4}$$

- $\varrho-\overline{\varrho}\in C_{\mathrm{weak}}([0,T];L^\gamma(\Omega)+L^2(\Omega))$ and the *continuity Eq.* (1.10) is satisfied in the weak sense,

$$\int_\Omega \varrho\varphi\ \mathrm{d}x(\tau)-\int_\Omega \varrho_{\varepsilon,0}\varphi(0,\cdot)\ \mathrm{d}x=\int_0^\tau\int_\Omega \varrho\Big(\partial_t\varphi+\mathbf{u}\cdot\nabla_\varepsilon\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t. \tag{5.5}$$

  for all $\tau\in[0,T]$ and any test function $\varphi\in C^\infty_c([0,T]\times\overline{\Omega})$;
- $\varrho\mathbf{u}\in C_{\mathrm{weak}}([0,T];L^{2\gamma/(\gamma+1)}(\Omega)+L^2(\Omega))$ and the *momentum equation* (1.11) holds in the sense that

$$\int_\Omega \varrho\mathbf{u}\cdot\varphi\ \mathrm{d}x(\tau)-\int_\Omega \varrho_{\varepsilon,0}\mathbf{u}_{\varepsilon,0}\varphi(0,\cdot)\ \mathrm{d}x$$
$$=\int_0^T\int_\Omega\Big(\varrho\partial_t\varphi+\varrho\mathbf{u}\otimes\mathbf{u}:\nabla_\varepsilon\varphi+p(\varrho_\varepsilon)\mathrm{div}_\varepsilon\varphi\ \mathrm{d}x\ \mathrm{d}t-\int_0^\tau\int_\Omega \mathbb{S}(\nabla_\varepsilon\mathbf{u}):\nabla_\varepsilon\varphi\ \mathrm{d}x\ \mathrm{d}t \tag{5.6}$$

  for all $\tau\in[0,T]$ and for any $\varphi\in C^\infty_c([0,T]\times\overline{\Omega};R^3)$, $\varphi_3|_{R^2\times\{0,1\}}=0$;
- the *energy inequality*

$$\int_\Omega\left[\frac{1}{2}\varrho|\mathbf{u}|^2+E(\varrho,\overline{\varrho})\right](\tau)\ \mathrm{d}x+\int_0^\tau\int_\Omega \mathbb{S}(\nabla_\varepsilon\mathbf{u}):\nabla_\varepsilon\mathbf{u}\ \mathrm{d}x\ \mathrm{d}t$$
$$\leq\int_\Omega\left[\frac{1}{2}\varrho_{0,\varepsilon}|\mathbf{u}_{0,\varepsilon}|^2+E(\varrho_{0,\varepsilon},\overline{\varrho})\right]\ \mathrm{d}x \tag{5.7}$$

  holds for a.a. $\tau\in(0,T)$.

5.1.2. Existence of Weak Solutions and the Relative Entropy Inequality.

Proposition 5.1. *Let $\mathbb{S}$ satisfy (1.5) and p verify (1.6). Suppose that the initial data satisfy*

$$\varrho_{\varepsilon,0}\geq 0,\quad \int_\Omega\Big(\frac{1}{2}\varrho_{\varepsilon,0}\mathbf{u}^2_{\varepsilon,0}+E(\varrho_{\varepsilon,0},\overline{\varrho})\Big)\ \mathrm{d}x<\infty. \tag{5.8}$$

Then the problem (1.10–1.12), (5.2–5.3) admits at least one finite energy weak solution on the arbitrary time interval $(0,T)$.

Moreover $[\varrho,\mathbf{u}]$ satisfy the relative entropy inequality (2.10) with the remainder term (2.11) for any couple $(r,\mathbf{U})$ such that

$$r>0,\ \ r-\overline{\varrho}\in C^\infty_c([0,T]\times\overline{\Omega}),\ \ \mathbf{U}\in C^\infty_c([0,T]\times\overline{\Omega},R^3),\ \ \mathbf{U}\cdot\mathbf{n}|_{\partial\Omega}=0.$$

Existence of weak solutions is a particular case of [11, Theorem 6.3] that treats more general heat conducting case. The relative entropy inequality is proved in [6].

5.2. Target System and the Main Result

The expected target system is system (1.13–1.15) on $\omega = R^2$ with conditions at infinity

$$r(x_h) \to \overline{\varrho},\ \mathbf{w}(x_h) \to 0 \text{ as } |x| \to \infty.$$

If the initial data satisfy conditions

$$\int_{R^2} E(r_0, \overline{\varrho}) \mathrm{d}x < \infty \tag{5.9}$$

$$\nabla_h r_0 \in W^{1,2}(R^2),\ \inf_{R^2} r_0 > 0,\ \mathbf{w}_0 \in W^{3,2}(R^2; R^2), \tag{5.10}$$

then there exists an unique (classical) solutions to this system on a short time interval $[0, T = T_{\max})$ (depending on the initial data) in the class

$$\nabla r \in C([0,T); W^{1,2}(\omega)),\ \mathbf{w} \in C([0,T); W^{2,2}(\omega; R^2)) \cap L^2(0,T; W^{3,2}(\omega; R^2))$$
$$\partial_t r \in C([0,T); W^{1,2}(\omega)),\ \partial_t \mathbf{w} \in L^2(0,T; W^{2,2}(\omega; R^2)), \tag{5.11}$$
$$0 < \underline{r} \equiv \inf_{(t,x_h)\in(0,T)\times\omega} r(t,x_h) \le \sup_{(t,x_h)\in(0,T)\times\omega} r(t,x_h) \equiv \overline{r}, \tag{5.12}$$
$$\int_{R^2} E(r, \overline{\varrho}) \mathrm{d}x \in L^\infty(0,T). \tag{5.13}$$

This fact can be obtained following the proof of [21, Theorem 2.5].

We have the following Theorem

Theorem 5.1. *Suppose that p and S satisfy hypotheses (1.6) and (1.5), respectively. Let $(r, \mathbf{w})$ a solution of the target system (1.13–1.15) on $\omega = R^2$, on the time interval $(0, T)$ belonging to the class (5.11–5.13) emanating from the initial data $[r_0, \mathbf{w}_0]$ satisfying (5.9–5.10). Let $(\varrho_\varepsilon, \mathbf{u}_\varepsilon)$ be a weak solution of the compressible Navier–Stokes equations (1.10–1.12), (5.2–5.3) emanating from the initial data $[\varrho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon}]$ satisfying (5.8). Finally assume that*

$$\mathcal{E}(\varrho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon} \,\Big|\, r_0, \mathbf{V}_0) \to 0,$$

where $\mathbf{V}_0 = [\mathbf{w}_0, 0]$.

Then

$$\mathrm{esssup}_{t\in(0,T)} \mathcal{E}(\varrho_\varepsilon, \mathbf{u}_\varepsilon \,\Big|\, r, \mathbf{V}) \to 0,$$

where $\mathbf{V} = [\mathbf{w}, 0]$.

Remark 5.1. In accordance with Remark 2.2 we have

$$\varrho_\varepsilon \to r \ in\ L^\infty(0,T; L^2 + L^\gamma(\Omega))$$

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \to r\mathbf{V} \ in\ L^\infty(0,T; L^2 + L^{\frac{2\gamma}{\gamma+1}}(\Omega))$$

5.3. Proof of Theorem 5.1

5.3.1. Uniform Estimates. The bounds that can be deduced from the inequality (5.7) are collected in the following lemma.

Lemma 5.1. *There exits c independent of $\epsilon \in (0,1)$ such that*

$$\|[1]_{res}\|_{L^\infty(0,T;L^1(\Omega))} \leq c, \tag{5.14}$$

$$\|[\varrho_\varepsilon - \overline{\varrho}]_{ess}\|_{L^\infty(0,T;L^2(\Omega))} \leq c, \tag{5.15}$$

$$\|[\varrho_\varepsilon]_{res}\|_{L^\infty(0,T;L^\gamma(\Omega))} \leq c, \tag{5.16}$$

$$\text{esssup}_{t\in(0,T)} \int\limits_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon^2 \, \mathrm{d}x \leq c,$$

$$\int\limits_0^T \int\limits_\Omega \mathbb{S}(\nabla_\varepsilon \mathbf{u}_\varepsilon) : \nabla_\varepsilon \mathbf{u}_\varepsilon \, \mathrm{d}x\mathrm{d}t \leq c.$$

In spite of the fact that we shall need in the sequel solely estimates (5.14–5.16), we have collected in the above lemma all uniform estimates available for the sequence of the finite energy weak solutions.

The straightforward proof of the following algebraic lemma is left to the reader.

Lemma 5.2. *Let $0 < a < b < \infty$. There exits $c = c(a,b) > 0$ such that*

$$\forall \varrho \geq 0, (r_1, r_2) \in [a,b]^2, \ E(\varrho, r_2) \leq c\Big(E(\varrho, r_1) + E(r_1, r_2)\Big).$$

We deduce from Lemma 5.1 and Lemma 5.2 the following result that will be used during the estimates of the remainder.

Lemma 5.3. *Let the couple r, $\mathbf{V} = [\mathbf{w}, 0]$ belong to the class (5.11–5.13). Then*

$$\|\mathcal{E}(\varrho_\varepsilon, \mathbf{u}_\varepsilon \Big| r, \mathbf{V})\|_{L^\infty(0,T)} \leq c$$

uniformly with respect to $\varepsilon \in (0,1)$.

5.3.2. Korn and Poincaré Type Inequalities. We start with a Korn type inequality:

Lemma 5.4. *There exists $c > 0$ such that for all $\mathbf{v} \in W^{1,2}_{\mathbf{n}}(\Omega; R^3)$*

$$\int\limits_\Omega \mathbb{S}(\nabla_\varepsilon \mathbf{v}) : \nabla_\varepsilon \mathbf{v} \, \mathrm{d}x = \mu \|\nabla_\varepsilon \mathbf{v}\|^2_{L^2(\Omega;R^{3\times 3})} + \left(\frac{1}{3}\mu + \eta\right) \|\mathrm{div}_\varepsilon \mathbf{v}\|^2_{L^2(\Omega)}.$$

Proof of Lemma 5.4. Since the set $\{\mathbf{v} \in C^\infty_c(\overline{\Omega}; R^3) \,|\, v_3|_{R^2\times\{0,1\}}\}$ is dense in $W^{1,2}_{\mathbf{n}}(\Omega; R^3)$, it is enough to prove lemma with $\mathbf{v} \in C^\infty_c(\overline{\Omega}, R^2) \times C^\infty_c(\Omega, R)$. The proof is the same as the proof of Lemma 2.1.

Next we show a Poincaré type inequality including the sequence ϱ_ε.

Lemma 5.5. *There exists $c > 0$ such that for all $\epsilon \in (0,1)$ we have,*

$$\forall \mathbf{v} \in W^{1,2}(\Omega, R^3), \ ||\mathbf{v}||^2_{L^2(\Omega)} \leq c \left(||\nabla_\varepsilon \mathbf{v}||^2_{L^2(\Omega)} + \int\limits_\Omega \varrho_\varepsilon \mathbf{v}^2 dx \right), a.a\ t \in (0,T).$$

Proof of Lemma 5.5. Let $A \in R^*_+$ and $\mathcal{V} = (0,A)^2 \times (0,1)$. We denote $\tilde{N}^\varepsilon_{res}(t) = \{x \in \Omega, \ \varrho_\varepsilon(x,t) \leq \frac{\overline{\varrho}}{2}\} \subset N^\varepsilon_{res}(t)$. Due to (5.14), there exists $\Gamma > 0$ such that

$$\text{esssup}_{t\in(0,T)} |N^\varepsilon_{res}(t)| < \Gamma.$$

We chose A in such a way that

$$|\mathcal{V}| \geq 2\Gamma.$$

We denote

$$\mathcal{V}_b = \mathcal{V} + Ab,\ b \in Z^2 \times \{0\}$$

and we remark that

$$\Omega = \Big(\cup_{b \in Z^2 \times \{0\}} \mathcal{V}_b \Big) \cup \mathcal{N},\ |\mathcal{N}| = 0,\ \mathcal{V}_b \cap \mathcal{V}_{b'} = \emptyset \text{ if } b \neq b'.$$

In view of (5.14)

$$|\mathcal{V}_b \cap N^{\varepsilon}_{ess}(t)| \geq \Gamma,\ \text{for a.a } t \in (0,T);$$

whence

$$\int_{\mathcal{V}_b} \varrho_\varepsilon(t,x)dx \geq \frac{\overline{\varrho}}{2}\Gamma,\ \text{for a.a. } t \in (0,T).$$

Moreover

$$\int_{\mathcal{V}_b} \varrho^\gamma_\varepsilon(t,x)dx = \int_{\mathcal{V}_b \cap N^\varepsilon_{res}(t)} \varrho^\gamma_\varepsilon(t,x)dx + \int_{\mathcal{V}_b \cap N^\varepsilon_{ess}(t)} \varrho^\gamma_\varepsilon(t,x)dx \leq C + (2\overline{\varrho})^\gamma |\mathcal{V}|.$$

Now, according to Lemma using 2.2, there exits $c = c(\Gamma, \gamma, |\mathcal{V}|) > 0$ such that for $t \in (0,T)$ and for all $\mathbf{v} \in W^{1,2}(\mathcal{V}_b, R^3)$,

$$||\mathbf{v}||^2_{L^2(\mathcal{V}_b,R^3)} \leq c \left(||\nabla_\varepsilon \mathbf{v}||^2_{L^2(\mathcal{V}_b)} + \int_{\mathcal{V}_b} \varrho_\varepsilon \mathbf{v}^2 dx \right).$$

We obtain the statement of Lemma 5.5 by summing the above estimates over $b \in Z^2 \times \{0\}$. This completes the proof.

5.3.3. Estimates of the Remainder. One can show by a density argument that the solution $[r, \mathbf{V}]$, $\mathbf{V} = [\mathbf{w}, 0]$ of the target system in the class (5.11), (5.13) may be used as a test function in the relative entropy inequality (2.10). Lemmas 5.4 and Lemma 5.5 provide the same estimates as Lemma 2.3 in the case of the periodic layer. When estimating the remainder (2.31) we can proceed step by step as in Sect. 2.2.5.

6. Relaxing the Hypotheses on the Pressure

Estimates effectuated in Sect. 3.3.2 indicate that one can considerably relax the hypotheses (1.6) imposed on the pressure, provided one takes the existence of weak solutions to the primitive system (1.2–1.5) for granted. In fact, in all cases considered in this paper, it is enough to suppose that the pressure satisfies the hypotheses $(1.6)_1$ and

$$c_1 + c_2\varrho + c_3 H(\varrho) \geq p(\varrho),\ \text{for } \varrho > \overline{R}, \tag{6.1}$$

where $\overline{R}$, c_1, c_2, c_3 are some fixed positive constants. Similar observation in the case of 3−D–1−D dimension reduction has been done in Bella et al. [2].

Indeed, without any growth condition like $(1.6)_2$ [and without (6.1)], we still get the following lemma:

Lemma 6.1. *Let*

$$p \in C[0,\infty) \cap C^1(0,\infty), \quad p'(\varrho) > 0.$$

Then, with the notation of Lemma 2.4, we have for all $\varrho \in [0,\infty)$ and $r \in [a,b]$:

$$E(\varrho|r) \geq c(a,b)\Big(1_{\mathcal{O}_{\rm res}} + \varrho 1_{\mathcal{O}_{\rm res}} + (\varrho - r)^2 1_{\mathcal{O}_{\rm ess}}\Big).$$

Proof of Lemma 6.1. If $\varrho \in [a/2, 2b]$ we use the strict convexity of H to obtain that

$$E(\varrho|r) \geq c|\varrho - r|^2 \text{ where } c = c(a,b) > 0.$$

If $\varrho \in R_+ \setminus [a/2, 2b]$, we observe that

$$\partial_\varrho E(\varrho|r) = H'(\varrho) - H'(r), \quad \partial_r E(\varrho|r) = H''(r)(r - \varrho),$$

where $s \to H'(s)$ is an increasing function on $(0, \infty)$. Now, relying on the monotonicity of functions $s \to E(s|r)$ and $s \to E(\varrho|s)$ induced by the above formulas, we consider two situations. 1) If $\varrho > 2b$, we observe that $E(\varrho|2b) > 0$, whence $H(\varrho) + p(2b) > H'(2b)\varrho$. Consequently,

$$p(2b) - p(b) + E(\varrho|r) \geq p(2b) - p(b) + E(\varrho|b) = H(\varrho) + p(2b) - H'(b)\varrho \geq \Big(H'(2b) - H'(b)\Big)\varrho.$$

This inequality and the fact that $E(\varrho, r) \geq E(2b, b) > 0$, $p(2b) > p(b)$, $H'(2b) > H'(b)$ yield

$$E(\varrho|r) \geq c(1 + \varrho)$$

with some $c = c(b) > 0$. 2) If $\varrho < a/2$ then

$$E(\varrho|r) \geq E(a/2|a) \geq \frac{E(a/2|a)}{a}\varrho + \frac{E(a/2|a)}{2} \geq c(1 + \varrho)$$

with some $c = c(a) > 0$. This completes the proof.

The above relaxed hypothesis can be investigated in combination with the situations studied in Sects. 3, 4 and 5. For the sake of brevity, we choose the system (1.2–1.5) with no slip boundary conditions (3.2) to illustrate the modifications needed in the proofs.

We shall start with a possible definition of a weak solution, modifying slightly Definition 3.1: We shall require that

- $\varrho \geq 0$, $\varrho, p(\varrho) \in L^\infty(0,T; L^1(\Omega))$, $\mathbf{u} \in L^2(0,T; W^{1,2}_{0,\mathbf{n}}(\Omega; R^3))$, $\varrho\mathbf{u} \in L^\infty(0,T; L^1(\Omega; R^3))$;
- $\varrho \in C_{\rm weak}([0,T]; L^1(\Omega))$ and weak formulation (2.5) of the continuity Eq. (1.10) is valid;
- $\varrho\mathbf{u} \in C_{\rm weak}([0,T]; L^1(\Omega; R^3))$ and weak formulation (2.6) of the momentum Eq. (1.11) holds with test functions

$$\varphi \in C^\infty_c([0,T] \times \overline{\Omega}; R^3), \ \varphi|_{[0,T]\times\partial\omega\times(0,1)} = 0, \ \varphi_3|_{[0,T]\times\omega\times\{0,1\}} = 0;$$

- Energy inequality (2.7) holds.

We shall suppose the existence of weak solutions to system (1.2–1.5), (3.2) for granted. It is easy to verify that any weak solution satisfies the relative entropy inequality (2.10–2.11) with test functions (3.3) whose regularity can be relaxed. In particular, we can take for the test functions the couple r, $\mathbf{U} = \mathbf{V}$, $\mathbf{V} = [\mathbf{w}, 0]$, where $[r, \mathbf{w}]$ is the strong solution of the system (1.13–1.15), (3.4) in the regularity class (2.15–2.16) guaranteed by the Proposition 3.1. We set in the definition of the residual and essential sets $\underline{\varrho} = \underline{r}$, $\overline{\varrho} = \max\{\overline{r}, \overline{R}\}$. Hence, in view of Lemma 6.1, we have the following bound

$$c\int_\Omega \left(\Big[1\Big]_{\rm res} + \Big[\varrho_\varepsilon\Big]_{\rm res} + \Big[\varrho_\varepsilon - r\Big]^2_{\rm ess}\right) \,{\rm d}x \leq \int_\Omega E(\varrho_\varepsilon, \mathbf{u}_\varepsilon \Big| r, \mathbf{U}) \,{\rm d}x, \tag{6.2}$$

that will replace the bound (2.37) in the estimate of the remainder. The only point, where we need the additional assumption (6.1) is the estimate of the residual part of the expression in Step 2 of Sect. 3.3.3.

After this preparation, we can repeat the argumentation of Sect. 3.3.2. We can conclude that *Theorem* 3.1 *remains valid*, if we replace $(1.6)_{1-2}$ by *a weaker assumption* $(1.6)_1$, (6.1). Likewise, one can reformulate in the same way *Theorem* 2.1 *dealing with periodic layers*, *Theorem* 4.1 *dealing with slip boundary conditions* and *Theorem* 5.1 *dealing with unbounded layers*.

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