J. Math. Fluid Mech. 16 (2014), 105–143

1422-6928/14/010105-39
DOI 10.1007/s00021-013-0146-1

Journal of Mathematical Fluid Mechanics

The Zero Surface Tension Limit of Two-Dimensional Interfacial Darcy Flow

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Communicated by A. Constantin

Abstract. We perform energy estimates for a sharp-interface model of two-dimensional, two-phase Darcy flow with surface tension. A proof of well-posedness of the initial value problem follows from these estimates. In general, the time of existence of these solutions will go to zero as the surface tension parameter vanishes. We then make two additional estimates, in the case that a stability condition is satisfied by the initial data: we make an additional energy estimate which is uniform in the surface tension parameter, and we make an estimate for the difference of two solutions with different values of the surface tension parameter. These additional estimates allow the zero surface tension limit to be taken, showing that solutions of the initial value problem in the absence of surface tension are the limit of solutions of the initial value problem with surface tension as surface tension vanishes.

1. Introduction

We consider a sharp-interface model of two-phase incompressible fluid flow, in which the fluid velocities are given by Darcy's Law. There are two primary settings in which fluid velocities are modeled by Darcy's Law: flow in a porous medium, and Hele-Shaw flow (i.e., flow of fluid between two closely-spaced, parallel sheets of glass) [14,16]. In the present contribution, we consider the effect of surface tension at the interface, and we show that if a condition is satisfied by the initial data, then the flow without surface tension can be recovered by taking the limit as surface tension vanishes.

The fluids are taken to be two-dimensional and of infinite vertical extent. For simplicity, we consider periodic boundary conditions in the horizontal direction. In each fluid region, we have the following expression for the velocity from Darcy's Law:

$$\mathbf{v}_i(x_1, x_2, t) = -\frac{b^2}{12\nu_i}\nabla\left(p_1(x_1, x_2, t) + \rho_i g x_2\right).$$

Here, the subscript i indicates which fluid region is being described; we let $i = 1$ indicate the lower fluid and $i = 2$ indicate the upper fluid. The point (x_1, x_2) is taken to be in fluid region i at time t. The fluid viscosities are denoted by ν_i, and the fluid densities are denoted by ρ_i. For each fluid, p_i is the pressure. The constant g is the acceleration due to gravity. Finally, we mention that the constant b is a physical parameter. When Darcy flow is taken as a model of flow in a porous medium, b is related to the porosity and permeability of the medium. When Darcy flow instead describes flow in a Hele-Shaw cell, b is related to the thickness of the gap between the plates of glass.

The case without surface tension is well-posed only if a stability condition is satisfied; if the fluids had equal densities, or if gravity were not present, then the stability condition would state that the more viscous fluid must displace the less viscous fluid. For the linearization of the flow, this condition was introduced by Saffman and Taylor [44]. It was verified in [3] that the initial value problem is well-posed if the nonlinear version of the stability condition is satisfied. If the stability condition is violated, then the problem is known to be ill-posed [40,46].

The author gratefully acknowledges support from the National Science Foundation through grants DMS-1008387 and DMS-1016267.

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Many authors have considered the well-posedness of interfacial Darcy flow problems previously, both at short times and for all time. Furthermore, studies include the two-dimensional and three-dimensional cases, the cases with and without surface tension, and the case of a single fluid and the case of two fluids. The case with two fluids and without surface tension is sometimes known as the Muskat or Muskat-Leibenzon problem. The Muskat problem is relevant to the current study, because when we show that we are able to take the limit as surface tension vanishes, we will in the end have a new proof of existence of solutions for the Muskat problem for short times. Some papers which treat the well-posedness question are the work of Bailly, in which the short-time well-posedness is established in both two and three dimensions, and the papers of Yi, in which the two-dimensional Muskat problem is shown to be well-posed for short times and for all time (with a smallness condition for the global result) [12,54,55]. Another recent proof of existence of global, small-data solutions for the Muskat problem is [20]. Aside from the Muskat problem, for the one-phase problem or the problem with fixed, positive surface tension, solutions have been shown to exist in papers by Duchon and Robert, Constantin and Pugh, Escher and Simonett, Kim, and Xie [21,27,29–31,41,51].

Another paper proving existence of solutions for the Muskat problem is [25]. In this paper, the authors prove existence of solutions in 2D, and a novel feature is how they treat the self-intersection condition for the interface. In order to prove that classical solutions for the free-surface problem exist, it is necessary to preclude self-intersections of the interface. This is typically done by ensuring that a chord-arc condition is satisfied by the solution at all times; the chord-arc condition was used, for instance, in the landmark papers of Wu proving well-posedness of the irrotational water wave problem in two and three spatial dimensions [49,50]. This condition can be enforced by either "hard" or "soft" means, i.e., by either establishing estimates, or by using theorems of functional analysis. Cordoba et al., in the paper [25] as well as in other papers such as [22] and [23], choose a "hard" solution to this issue, proving an estimate for the time evolution of the L^{∞}-norm of the chord-arc quantity. We choose instead what is primarily a "soft" solution to the issue, relying on a careful use of the Picard theorem to ensure that interfaces we consider stay well away from self-intersections. Similarly, when studying flows for which the stability condition is satisfied, since the related quantity is time-dependent, we again have a choice between "hard" and "soft" methods. Cordoba et al. choose a "hard" approach, making estimates of the growth of the relevant quantity, while we choose a "soft" approach, again relying primarily upon the Picard theorem.

All of the above papers either considered the case with surface tension or the case without surface tension; we instead consider now the relationship between the two cases. This has been done previously for other fluids problems. The author and Masmoudi have previously shown that the zero surface tension limit of water waves can be taken [6,8]. For compressible free-boundary Euler flow in three spatial dimensions, Coutand, Hole, and Shkoller have recently shown that the zero surface tension limit can be taken [26], and Hadzic and Shkoller have shown that the zero surface tension limit can be taken for the Stefan problem [34]. As in the present work, the essential element of these proofs is an estimate for the problem which is uniform in the surface tension parameter.

Also, for the case of Darcy flow, the zero surface tension limit has been studied in some cases. Siegel, Tanveer, and Dai studied the zero surface tension limit of Hele-Shaw flow in the unstable case [47,48]. In this setting, the initial value problem is ill-posed, but some smooth solutions are known to exist. Given a smooth solution of the problem without surface tension, Siegel, Tanveer, and Dai use the solution at time zero as the initial condition for the problem with surface tension, and then study the limit of these solutions as surface tension vanishes. They find that the effect of surface tension is singular, in that the limit as surface tension vanishes is not the solution without surface tension with which they began the process. The present result is complementary, in that we study a different case (in which the initial data satisfies the stability condition), and reach the opposite conclusion. Ceniceros and Hou have made additional studies of the unstable case, confirming that the zero surface tension limit is indeed singular [17,18].

We formulate the evolution equations by following the approach of Hou, Lowengrub, and Shelley (HLS). In [37] and [38], they developed and implemented a non-stiff numerical method for the solution of the initial value problem for two-dimensional interfacial Darcy flow and vortex sheets with surface tension. The formulation involves describing the location of the free surface by using the tangent angle that the interface forms with the horizontal and the arclength element of the curve, rather than the Cartesian coordinates. Furthermore, an artificial tangential velocity is used in order to enforce a normalized arclength parameterization, rather than using, for instance, a Lagrangian parameterization. While these ideas were introduced for the purpose of removing the stiffness from numerical methods, they have since found much use in analysis as well, for instance, in the papers [2,3,6,33,19,23,25,28,52], and [53]. Although the current contribution addresses only the case of fluids in two spatial dimensions, we mention that the HLS ideas have also been extended now to three-dimensional flows, both numerically [43,39,9,10], and analytically [4,7,8,24].

In the HLS formulation, we write the evolution equations by isolating the leading-order terms, i.e., the terms with the most derivatives. We also are careful to indicate which terms in the evolution equation are present only because of the surface tension force, as opposed to terms which are present in any case. These considerations lead to an extensive effort to rewrite the evolution equations. We then introduce mollifiers into the evolution equations, so that we may use the Picard theorem for ordinary differential equations on a Banach space to prove the existence of solutions. Energy estimates are then performed, without assuming the stability condition is satisfied by the initial data. It is found that the growth of the norm of the solutions to the mollified equations can be bounded, uniformly in the mollification parameter. With this uniform control in hand, it is then possible to prove the existence of solutions for the original, non-mollified initial value problem by sending the mollification parameter to zero. Additional, similar energy estimates then imply that the solutions are unique and depend continuously on the initial data.

We then turn to the case in which the stability condition is satisfied by the initial data. In this case, we are able to repeat the energy estimates, this time finding additionally that the estimates can also be made uniformly with respect to the surface tension parameter. This additional uniformity allows the limit to be taken as surface tension vanishes. The vanishing surface tension limit of interfacial Darcy flow with surface tension is thus found to be the interfacial Darcy flow without surface tension.

While we have attempted to discuss the most relevant references above, it surely is not possible to survey all of the prior literature on Hele-Shaw flows. We refer the reader to the bibliography developed by Gillow and Howison, with over 600 references [32].

The remainder of this paper is organized as follows. In Sect. 2, we give a helpful model problem, which demonstrates the spirit of the different energy estimates we will make for our physical problem. In Sect. 3, we present the equations of motion for two-dimensional interfacial Darcy flow. In Sect. 4, we develop a variety of estimates which will be useful many times. In Sect. 5, we prove well-posedness of two-dimensional interfacial Darcy flow for a fixed, positive value of the surface tension coefficient, with no assumption of the stability condition. Then, in Sect. 6, we show that when the stability condition is satisfied by the initial data, the limit can be taken as surface tension vanishes, and the limiting flow is the interfacial Darcy flow without surface tension. Finally, we make some concluding remarks in Sect. 7.

The author is grateful to Michael Siegel for helpful conversations.

2. An Instructive Example

Let $\tau \geq 0$ be a constant, let $c_1 \geq 0$ be a constant, and let c_2 be a constant. Consider the linear equation

$$u_t = -\tau\Lambda^3(u) + c_1\tau\Lambda^2(u) + c_2\Lambda(u), \tag{1}$$

where Λ is the operator with symbol $\hat{\Lambda} = |k|$. (Note that now and in the remainder of the paper, subscripts of the spatial or temporal variable imply differentiation, as u_t here indicates $\partial u/\partial t$.) We take the equation with spatially periodic boundary conditions, and with initial data $u(x, 0) = u_0(x)$. Let $u(x, t)$ be the solution; note that the solution can easily be written down by using the Fourier series. Let $s > 0$ be an integer; below, we will estimate the H^s norm of the solution. Note that the leading-order term in the evolution equation (i.e., the term with Λ^3) offers parabolic smoothing when $\tau > 0$. In that case, the next-order term (i.e., the term with Λ^2) is, however, a backwards parabolic term, since $\tau c_1 \geq 0$. The lowest order term is either a forward parabolic term or a backward parabolic term, depending on the sign of c_2. For $\tau > 0$, the problem is well-posed, since the leading-order term makes the entire problem a forward parabolic problem, regardless of the sign of c_2. If $c_2 > 0$, however, we will need to use the leading-order term to control the lowest-order term, and then the estimates will depend badly on τ; solutions will blow up as τ goes to zero. On the other hand, if $c_2 \leq 0$, then the estimates can be made uniformly in τ, and the limit of the solutions could be taken as τ goes to zero. Furthermore, we mention that with $c_2 \leq 0$, the initial value problem is well-posed in the case $\tau = 0$, and an estimate for the solution could be made in this case in its own right, without considering the limit $\tau \to 0^+$.

We demonstrate these estimates, defining the energy to be

$$E(t) = \frac{1}{2} \int_X u^2(x,t) + (\partial_x^s u(x,t))^2 \, dx, \tag{2}$$

where X is the spatial domain (a periodic interval). Clearly, this energy is equivalent to the square of the usual H^s norm of the solution.

Case 1 we start with the simplest case, $\tau = 0$ and $c_2 \leq 0$. Then, it is elementary that

$$\frac{dE}{dt} = c_2 \int_X \left(\Lambda^{1/2}(u)\right)^2 + \left(\partial_x^s \Lambda^{1/2}(u)\right)^2 \, dx \leq 0.$$

Notice that in addition to showing that solutions are bounded in H^s, if $c_2 < 0$, this estimate can also be used to show that solutions gain derivatives at positive times.

Case 2 the next case that we consider is $\tau > 0$ and $c_2 \leq 0$. We still use (2), so we have

$$\begin{aligned} \frac{dE}{dt} &= \tau \left[\int_X - \left(\Lambda^{3/2}(u)\right)^2 - \left(\partial_x^s \Lambda^{3/2}(u)\right)^2 + c_1 \left(\Lambda(u)\right)^2 + c_1 \left(\partial_x^s \Lambda(u)\right)^2 \, dx \right] \\ &\quad + c_2 \int_X \left(\Lambda^{1/2}(u)\right)^2 + \left(\partial_x^s \Lambda^{1/2}(u)\right)^2 \, dx. \end{aligned} \tag{3}$$

We estimate this by again using $c_2 \leq 0$, and we also notice that $(\Lambda(u))^2$ can be controlled by the energy, since s is at least one:

$$\frac{dE}{dt} \leq C\tau E + \tau \left[\int_X - \left(\partial_x^s \Lambda^{3/2}(u)\right)^2 + c_1 (\partial_x^s \Lambda(u))^2 \, dx \right].$$

We let $v = \partial_x^s u$, and we use the Plancherel theorem to rewrite the remaining integral as a sum:

$$\frac{dE}{dt} \leq C\tau E + \tau \left[\sum_{k=-\infty}^{\infty} \left(-|k|^3 + c_1 k^2\right) |\hat{v}|^2(k) \right].$$

Since c_1 is constant, there exists $\bar{C}$ such that for all k, we have $-|k|^3 + c_1 k^2 \leq \bar{C}$. Also, notice that $\sum |\hat{v}|^2(k) \leq 2E$. We therefore conclude that there exists a constant $\tilde{C}$ such that

$$\frac{dE}{dt} \leq \tilde{C}\tau E,$$

so we may conclude that for any t,

$$E(t) \leq E(0) e^{\tilde{C}\tau t}.$$

Clearly, as $\tau \to 0^+$, this bound on the H^s norm of u is uniform with respect to τ.

Case 3 we now consider our final case, $\tau > 0$ and $c_2 > 0$. We begin from (3), and we begin by estimating the terms $c_1(\Lambda(u))^2$ and $c_2(\Lambda^{1/2}(u))^2$ by the energy, and by using the inequality $-(\Lambda^{3/2}(u))^2 \leq 0$:

$$\frac{dE}{dt} \leq C\tau E + \tau\left[\int_X -\left(\partial_x^s\Lambda^{3/2}(u)\right)^2 + c_1\left(\partial_x^s\Lambda(u)\right)^2\ dx\right] + c_2\int_X \left(\partial_x^s\Lambda^{1/2}(u)\right)^2\ dx.$$

We again use the Plancherel theorem, and we again denote $v = \partial_x^s u$:

$$\frac{dE}{dt} \leq C\tau E + \sum_{k=-\infty}^{\infty}\Big(-\tau|k|^3 + c_1\tau k^2 + c_2|k|\Big)|\hat{v}(k)|^2.$$

For fixed $\tau > 0$, we can treat this as we did previously. That is, there exists a constant $\bar{C} = \bar{C}(\tau)$ such that for all k,

$$-\tau|k|^3 + c_1\tau k^2 + c_2|k| \leq \bar{C}(\tau),$$

so that

$$\frac{dE}{dt} \leq \Big(C\tau + \bar{C}(\tau)\Big)E,$$

and thus

$$E(t) \leq E(0)\exp\left\{\Big(C\tau + \bar{C}(\tau)\Big)t\right\}.$$

This estimate is not uniform in τ, however, since it is evident that $\lim_{\tau\to 0^+}\bar{C}(\tau) = +\infty$.

Remark 1. In Case 3, it is not only true that we are unable to prove that there is a limit as $\tau \to 0^+$, but also that for many pieces of initial data, we can see that the limit definitely fails to exist. For example, say the initial data is given by $\hat{u}_0(k) = 1/|k|^{100}$, for $k \neq 0$. Then, u_0 is in H^s for many reasonable choices of s, and the solution $u(\cdot, t)$ will also be in H^s at positive times, for any fixed, positive value of τ. An explicit calculation of the solution, however, shows that the solutions blow up in H^s as $\tau \to 0^+$, for any $t > 0$.

Remark 2. After specifying the evolution equations for the interfacial Darcy flow in Sect. 3 below, a substantial effort will be made to rewrite the equations in order to make them as similar as possible to (1). The ultimate goal of this effort is to arrive at formula (58) below, in which $\theta^{\varepsilon}_{\alpha\alpha}$ plays the role of u (the variables θ and α will be defined in Sect. 3, and ε will be introduced in Sect. 5).

3. The Equations of Motion

In this section, we present the exact equations of motion for the physical problem being studied, two-dimensional interfacial Darcy flow with surface tension. In Fig. 1, we show a simple schematic of the situation; the two fluids are separated by a sharp interface, are horizontally periodic, and are infinitely deep.

The location of the interface is given by the parameterized curve $(x(\alpha, t), y(\alpha, t))$, where t is time and α is the spatial parameter. This curve is 2π-periodic in the horizontal direction, meaning that for all α and t, we have

$$x(\alpha + 2\pi, t) = x(\alpha, t) + 2\pi, \qquad y(\alpha + 2\pi, t) = y(\alpha, t).$$

We define the arclength element $s_\alpha(\alpha, t) = \left(x_\alpha^2(\alpha, t) + y_\alpha^2(\alpha, t)\right)^{1/2}$. We use the following unit tangent and normal vectors:

$$\hat{\mathbf{t}} = \frac{(x_\alpha, y_\alpha)}{s_\alpha}, \qquad \hat{\mathbf{n}} = \frac{(-y_\alpha, x_\alpha)}{s_\alpha}.$$

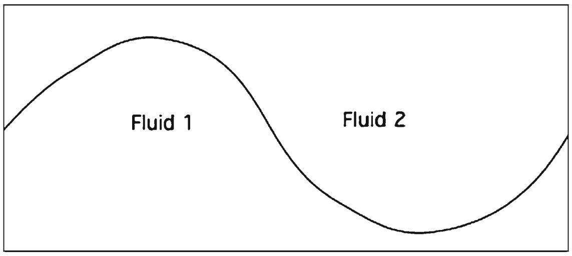

FIG. 1. A simple schematic of the geometry under consideration. The two fluids are separated by a sharp interface, which is periodic in the horizontal direction. The fluids are of infinite vertical extent

It is also useful to introduce the tangent angle formed between the curve and the horizontal,

$$\theta(\alpha,t) = \arctan\left(\frac{y_\alpha(\alpha,t)}{x_\alpha(\alpha,t)}\right).$$

We can express $\hat{\mathbf{t}}$ and $\hat{\mathbf{n}}$ easily in terms of θ :

$$\hat{\mathbf{t}} = (\cos(\theta), \sin(\theta)), \qquad \hat{\mathbf{n}} = (-\sin(\theta), \cos(\theta)).$$

From this formula follows a version of the classical Frenet-Serret formulas,

$$\hat{\mathbf{t}}_\alpha = \theta_\alpha \hat{\mathbf{n}}, \qquad \hat{\mathbf{n}} = -\theta_\alpha \hat{\mathbf{t}}. \tag{4}$$

The motion of the curve is described by its normal velocity, U, and tangential velocity, V:

$$(x,y)_t = U\hat{\mathbf{n}} + V\hat{\mathbf{t}}. \tag{5}$$

Using (5) and the definition of θ, we can infer the following evolution equation for θ:

$$\theta_t = \frac{U_\alpha + V\theta_\alpha}{s_\alpha}. \tag{6}$$

The normal velocity is determined by the fluid dynamics, but we can choose the tangential velocity to maintain a preferred parameterization.

We introduce some notation for the mean value of a function; for a given periodic function f, we let

$$\langle\langle f\rangle\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(\alpha)\ d\alpha.$$

Then, we define the projection $\mathbb{P}$ to be projection off the mean; for any periodic f, we have

$$\mathbb{P}f = f - \langle\langle f\rangle\rangle.$$

Of course, letting I be the identity operator, we could write this as

$$\langle\langle f\rangle\rangle = (I - \mathbb{P})f.$$

We also introduce the operator ∂_α^{-1}, the zero-mean integration operator which acts on mean-zero periodic functions. Given any $n \geq 1$, we obviously have the estimate

$$\|\partial_\alpha^{-1} f\|_{n+1} \leq \|f\|_n.$$

For any f, we can write

$$f = \langle\langle f\rangle\rangle + \partial_\alpha^{-1}\partial_\alpha f. \tag{7}$$

Remark 3. It is to be understood that if we ever write ∂_α^{-1} applied to a function which does not necessarily have zero mean, then there is an implicit application of $\mathbb{P}$. That is, if f does not necessarily have zero mean, and if we write $\partial_\alpha^{-1} f$, then this is to be understood as meaning $\partial_\alpha^{-1}\mathbb{P}f$.

The parameterization which we prefer is a normalized arclength parameterization. Since we are considering flows which are periodic in the horizontal direction, we consider L, the length of one period of the interface. This is defined by

$$L(t) = \int_0^{2\pi} s_\alpha(\alpha,t)\ d\alpha.$$

We can infer the evolution equation for s_α from the definition of s_α and (5); we find the following:

$$s_{\alpha t} = V_\alpha - \theta_\alpha U.$$

Using this, we calculate

$$L_t = -\int_0^{2\pi} \theta_\alpha U\ d\alpha = -2\pi\langle\langle\theta_\alpha U\rangle\rangle.$$

Since our preferred parameterization is a normalized arclength parameterization, we desire to have at all times

$$s_\alpha(\alpha,t) = \frac{L(t)}{2\pi}.$$

This is then achieved if $s_\alpha(\alpha,0) = L(0)/2\pi$ for all α, and if

$$s_{\alpha t} = \frac{L_t}{2\pi} = -\frac{1}{2\pi}\int_0^{2\pi} \theta_\alpha U\ d\alpha = -\langle\langle\theta_\alpha U\rangle\rangle.$$

Since $s_{\alpha t} = V_\alpha - \theta_\alpha U$, this implies that the tangential velocity must be chosen so that

$$V_\alpha = \theta_\alpha U - \langle\langle\theta_\alpha U\rangle\rangle = \mathbb{P}(\theta_\alpha U).$$

To be definite, we define V to be the integral of V_α which has zero mean:

$$V = \partial_\alpha^{-1}\mathbb{P}(\theta_\alpha U).$$

We will frequently find complex notation to be helpful. Towards this end, we introduce the mapping $\Phi : \mathbb{R}^2 \to \mathbb{C}$ defined by

$$\Phi(a,b) = a + ib.$$

We let

$$z(\alpha,t) = \Phi(x(\alpha,t), y(\alpha,t)) = x(\alpha,t) + iy(\alpha,t).$$

We will denote the complex conjugate with *, as in $\Phi(a,b)^* = a - ib$ or $z^* = x - iy$.

The jump conditions for the velocity at the interface are that there is no jump in the normal component of the velocity, but there can be a jump in the tangential component. Since the fluid velocity in the bulk of each fluid is given by a gradient, at first glance one might think that this implies the flow is exactly irrotational; however, since there is a jump in the tangential component of the velocity at the interface, the vorticity is measure-valued (i.e., the vorticity is equal to the Dirac mass of the interface, multiplied by some amplitude). All of this is to say that the interface is a vortex sheet. We call the vortex sheet strength $\gamma(\alpha,t)$; this is the amplitude that multiplies the Dirac mass.

Related to the curve z, we introduce the shifted (or centered) curve z_d, given by

$$z_d(\alpha,t) = z(\alpha,t) - z(0,t).$$

We consider only interfaces which satisfy a non-self-intersection condition. In particular, as in many papers in the field, such as [2,23,49], we insist that a chord-arc condition be satisfied. To this end, we introduce the first divided difference of the interface, $q_1[z_d](\alpha,\alpha')$,

$$q_1[z_d](\alpha,\alpha') = \frac{z_d(\alpha) - z_d(\alpha')}{\alpha - \alpha'}. \tag{8}$$

The chord-arc condition that we will use is the condition that q_1 be bounded away from zero, for all α and α'. Obviously, this precludes self-intersections of the interface; it also precludes cusps and corners. We will remark more on the definition of z_d below, but note for now that q_1 could have been defined in terms of either z or z_d with no difference; that is, notice that the value $z(0,t)$ has no effect upon q_1.

Since the interface is a vortex sheet, the normal velocity, U, must be the normal component of the Birkhoff–Rott integral, $\mathbf{W}$ (see [45] for details). That is, $U = \mathbf{W}\cdot\hat{\mathbf{n}}$, with

$$\Phi(\mathbf{W})^*(\alpha,t) = \frac{1}{4\pi i}\mathrm{PV}\int_0^{2\pi} \gamma(\alpha',t)\cot\left(\frac{1}{2}(z(\alpha,t)-z(\alpha',t))\right)\,d\alpha'. \tag{9}$$

We understand the Birkhoff–Rott integral, $\mathbf{W}$, by considering it to be something like the Hilbert transform. Given a periodic function f, the periodic Hilbert transform, H, is given by the following formula (see [36] for more information):

$$Hf(\alpha) = \frac{1}{2\pi}\mathrm{PV}\int f(\alpha')\cot\left(\frac{1}{2}(\alpha-\alpha')\right)\,d\alpha'.$$

The Hilbert transform is a multiplier in Fourier space, with symbol $\widehat{H}(k) = -i\,\mathrm{sgn}(k)$. As in [2], we have the following useful formula for $\mathbf{W}_\alpha$:

$$\mathbf{W}_\alpha = \frac{\pi}{L}H(\gamma_\alpha)\hat{\mathbf{n}} - \frac{\pi}{L}H(\gamma\theta_\alpha)\hat{\mathbf{t}} + \mathbf{m}. \tag{10}$$

In order to give the definition of $\mathbf{m}$, we first must define some relevant integral operators.

Given the curve z, we define the operator $\mathcal{K}[z]$:

$$\mathcal{K}[z]f(\alpha) = \frac{1}{4\pi i}\int_0^{2\pi} f(\alpha')\left[\cot\left(\frac{1}{2}(z(\alpha)-z(\alpha'))\right) - \frac{1}{z_\alpha(\alpha')}\cot\left(\frac{1}{2}(\alpha-\alpha')\right)\right]\,d\alpha'. \tag{11}$$

Notice that (assuming some regularity on z) the integral in (11) is not a singular integral, since each of the two cotangents in brackets have the same singularity, which cancels upon subtracting. Also, notice again that z_d could have been used as easily as z; that is,

$$\mathcal{K}[z] = \mathcal{K}[z_d].$$

We also need to define the commutator of the Hilbert transform and multiplication by a function,

$$[H,\phi]f(\alpha) = H(\phi f)(\alpha) - \phi(\alpha)H(f)(\alpha).$$

Using the definition of the Hilbert transform, we can see that this can be written as the following integral operator:

$$[H,\phi]f(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} f(\alpha')\left[(\phi(\alpha')-\phi(\alpha))\cot\left(\frac{1}{2}(\alpha-\alpha')\right)\right]\,d\alpha'.$$

Notice that (again, assuming some regularity on ϕ) this is not a singular integral, since the singularity in the cotangent is canceled by the difference $\phi(\alpha')-\phi(\alpha)$. In Sect. 4.1 below, we will give estimates for both $\mathcal{K}[z]$ and $[H,\phi]$, showing that these are both smoothing operators.

The formula for $\mathbf{m}$ can then be written as

$$\Phi(\mathbf{m})^* = z_\alpha\mathcal{K}[z]\left(\left(\frac{\gamma}{z_\alpha}\right)_\alpha\right) + \frac{z_\alpha}{2i}\left[H,\frac{1}{z_\alpha^2}\right]\left(z_\alpha\left(\frac{\gamma}{z_\alpha}\right)_\alpha\right). \tag{12}$$

We have the following equation for γ, the vortex sheet strength [38]:

$$\gamma = \tau\kappa_\alpha - Ry_\alpha - 2A_\mu s_\alpha\mathbf{W}\cdot\hat{\mathbf{t}}. \tag{13}$$

We note that the derivation of (13) uses the Laplace-Young jump condition, which states that the pressure jump across the interface is equal to the curvature of the interface multiplied by the surface tension coefficient. Here, κ is the curvature of the interface, τ is the constant, non-negative coefficient of surface tension, and R and A_μ are given by the formulas

$$R = (\rho_1 - \rho_2)g, \qquad A_\mu = \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2}.$$

The quantities ρ_i and μ_i are the density of fluid i and the viscosity of fluid i, respectively, for $i \in \{1, 2\}$. Of course, g is the constant acceleration due to gravity. We rewrite (13) by using the formulas $\kappa = \theta_\alpha/s_\alpha, y_\alpha = s_\alpha \sin(\theta)$, and $s_\alpha = L/2\pi$:

$$\gamma = \frac{2\pi\tau}{L}\theta_{\alpha\alpha} - \frac{RL}{2\pi}\sin(\theta) - \frac{A_\mu L}{\pi}\mathbf{W}\cdot\hat{\mathbf{t}}. \tag{14}$$

Notice that this equation for γ is actually an integral equation for γ, since γ appears in the right-hand side through the definition of $\mathbf{W}$. This integral equation is known to be solvable [13]. We will discuss this further in Sect. 4.1 below.

We now rewrite (6). To begin, we use the definition $U = \mathbf{W}\cdot\hat{\mathbf{n}}$, the Frenet equation $\hat{\mathbf{n}}_\alpha = -\theta_\alpha\hat{\mathbf{t}}$, and the equation $s_\alpha = L/2\pi$, finding the following:

$$\theta_t = \frac{2\pi}{L}\mathbf{W}_\alpha\cdot\hat{\mathbf{n}} + \frac{2\pi}{L}\left(V - \mathbf{W}\cdot\hat{\mathbf{t}}\right)\theta_\alpha. \tag{15}$$

We now use the formula (10) for $\mathbf{W}_\alpha$ to substitute in the first term on the right-hand side of (15):

$$\theta_t = \frac{2\pi^2}{L^2}H(\gamma_\alpha) + \frac{2\pi}{L}(V - \mathbf{W}\cdot\hat{\mathbf{t}})\theta_\alpha + \frac{2\pi}{L}\mathbf{m}\cdot\hat{\mathbf{n}}. \tag{16}$$

We differentiate (14) with respect to α, finding

$$\gamma_\alpha = \frac{2\pi\tau}{L}\theta_{\alpha\alpha\alpha} - \frac{RL}{2\pi}\theta_\alpha\cos(\theta) - \frac{A_\mu L}{\pi}\left(\mathbf{W}_\alpha\cdot\hat{\mathbf{t}} + \theta_\alpha U\right). \tag{17}$$

We substitute (17) into (16), arriving at the following:

$$\begin{aligned}\theta_t = &\left(\frac{4\pi^3\tau}{L^3}\right)H(\theta_{\alpha\alpha\alpha}) + \left(\frac{2\pi}{L}\right)H\left(\left\{\frac{-R\cos(\theta)}{2} - A_\mu U\right\}\theta_\alpha\right)\\ &- \left(\frac{2\pi^2 A_\mu}{L^2}\right)\mathbb{P}(\gamma\theta_\alpha) + \left(\frac{2\pi}{L}\right)(V - \mathbf{W}\cdot\hat{\mathbf{t}})\theta_\alpha + \left(\frac{2\pi}{L}\right)\mathbf{m}\cdot\hat{\mathbf{n}} - \left(\frac{2\pi^3 A_\mu}{L^3}\right)H(\mathbf{m}\cdot\hat{\mathbf{t}}).\end{aligned} \tag{18}$$

This is almost our final form for the θ_t equation, however, since we want to carefully track dependence on the surface tension coefficient, we want to rewrite several terms on the right-hand side of (18) to isolate contributions from the surface tension term.

In particular, we write $\mathbf{W} = \tau\mathbf{W}^{\text{s.t.}} + \widetilde{\mathbf{W}}$. We define $\mathbf{W}^{\text{s.t.}}$ to be the part of $\mathbf{W}$ which corresponds to the contribution of $\frac{2\pi\tau}{L}\theta_{\alpha\alpha}$ from the equation for γ (having factored out the τ):

$$\Phi(\mathbf{W}^{\text{s.t.}})^*(\alpha, t) = \frac{1}{2iL}\text{PV}\int_0^{2\pi}\theta_{\alpha\alpha}(\alpha', t)\cot\left(\frac{1}{2}(z(\alpha, t) - z(\alpha', t))\right)\,d\alpha'.$$

The remainder, $\widetilde{\mathbf{W}}$, is simply defined as being the difference,

$$\widetilde{\mathbf{W}} = \mathbf{W} - \tau\mathbf{W}^{\text{s.t.}}.$$

We then make the corresponding decomposition $U = \tau U^{\text{s.t.}} + \widetilde{U}$, where

$$U^{\text{s.t.}} = \mathbf{W}^{\text{s.t.}}\cdot\hat{\mathbf{n}}, \qquad \widetilde{U} = \widetilde{\mathbf{W}}\cdot\hat{\mathbf{n}}.$$

In the same way, we decompose V as $V = \tau V^{\text{s.t.}} + \widetilde{V}$, where

$$V^{\text{s.t.}} = \partial_\alpha^{-1}\mathbb{P}(\theta_\alpha U^{\text{s.t.}}), \qquad \widetilde{V} = \partial_\alpha^{-1}\mathbb{P}(\theta_\alpha\widetilde{U}).$$

Finally, we write $L_t = \tau L_t^{\text{s.t.}} + \widetilde{L}_t$, where

$$L_t^{\text{s.t.}} = -2\pi\langle\!\langle\theta_\alpha U^{\text{s.t.}}\rangle\!\rangle, \qquad \widetilde{L}_t = -2\pi\langle\!\langle\theta_\alpha\widetilde{U}\rangle\!\rangle.$$

We make these substitutions into (18). In particular, we rewrite the following terms from the right-hand side of (18): (a) the term $A_\mu U$ that appears inside the Hilbert transform, (b) the γ that appears inside the operator $\mathbb{P}$, and (c) $V - \mathbf{W}\cdot\hat{\mathbf{t}}$. We also make the definition

$$k(\alpha,t) = k[\theta](\alpha,t) = \frac{2\pi}{L}\left\{\frac{-R\cos(\theta)}{2} - A_\mu \widetilde{U}\right\}. \tag{19}$$

We note that $\mathbb{P}(\theta_{\alpha\alpha}\theta_\alpha) = \theta_{\alpha\alpha}\theta_\alpha$, and that $\mathbb{P}(\sin(\theta)\theta_\alpha) = \sin(\theta)\theta_\alpha$, since these are perfect derivatives. All of these considerations yield the following:

$$\begin{aligned}\theta_t =& \left(\frac{4\pi^3\tau}{L^3}\right)H(\theta_{\alpha\alpha\alpha}) - \tau\left[\frac{4\pi^3 A_\mu}{L^3}\theta_{\alpha\alpha}\theta_\alpha + \frac{2\pi A_\mu}{L}H(U^{\text{s.t.}}\theta_\alpha)\right]\\ &+\left[H(k\theta_\alpha) + \frac{2\pi\tau}{L}(V^{\text{s.t.}} - \mathbf{W}^{\text{s.t.}}\cdot\hat{\mathbf{t}})\theta_\alpha + \frac{2\pi A_\mu^2\tau}{L}\mathbb{P}(\theta_\alpha \mathbf{W}^{\text{s.t.}}\cdot\hat{\mathbf{t}})\right]\\ &+\frac{\pi A_\mu R}{L}\sin(\theta)\theta_\alpha + \frac{2\pi A_\mu^2}{L}\mathbb{P}(\theta_\alpha\widetilde{\mathbf{W}}\cdot\hat{\mathbf{t}}) + \frac{2\pi}{L}(\widetilde{V} - \widetilde{\mathbf{W}}\cdot\hat{\mathbf{t}})\theta_\alpha + \frac{2\pi}{L}\mathbf{m}\cdot\hat{\mathbf{n}} - \frac{2\pi^3 A_\mu}{L^3}H(\mathbf{m}\cdot\hat{\mathbf{t}}).\end{aligned} \tag{20}$$

For future reference, we note that the evolution equation without surface tension (i.e., in the case $\tau = 0$) is the following:

$$\theta_t = H(k\theta_\alpha) + \frac{\pi A_\mu R}{L}\sin(\theta)\theta_\alpha + \frac{2\pi A_\mu^2}{L}\mathbb{P}(\theta_\alpha\widetilde{\mathbf{W}}\cdot\hat{\mathbf{t}}) + \frac{2\pi}{L}(\widetilde{V} - \widetilde{\mathbf{W}}\cdot\hat{\mathbf{t}})\theta_\alpha + \frac{2\pi}{L}\widetilde{\mathbf{m}}\cdot\hat{\mathbf{n}} - \frac{2\pi^3 A_\mu}{L^3}H(\widetilde{\mathbf{m}}\cdot\hat{\mathbf{t}}). \tag{21}$$

4. Preliminary Estimates

In this section, we will present estimates which will be repeatedly useful throughout the sequel. We begin by noting that we mainly use the L^2-based Sobolev spaces, and we denote these by H^j, for $j \geq 0$. The associated norm is $\|\cdot\|_j$. We denote the L^∞ norm as $|\cdot|_\infty$. We will need the following interpolation inequality for Sobolev spaces: if $f \in H^\ell$, and if $m \in \mathbb{R}$ such that $\ell > m > 0$, then there exists a positive constant such that

$$\|f\|_m \leq c\|f\|_\ell^{m/\ell}\|f\|_0^{1-m/\ell}. \tag{22}$$

This is a standard inequality and the proof may be found many places, one of which is [2].

Remark 4. We make a remark about our regularity assumptions. Throughout the sequel, beginning in Sect. 4.2, we will be making estimates for the solution of the initial value problem in the space H^s. Here, $s \in \mathbb{N}$ is fixed, and it is assumed to be "sufficiently large." What this means is that there exists an absolute constant, $\bar{S}$, such that as long as $s \geq \bar{S}$, the arguments that we present will go through. The reason that s must be sufficiently large is so that various results, such as the lemmas to be presented in Sect. 4.1 below, or the Sobolev embedding theorem, may be invoked. We do not count the minimal possible value of $\bar{S}$, but surely $s \geq 6$ is sufficient.

4.1. Estimates for Integral Operators

We begin by noting that when reconstructing z from θ, we are only able to find z up to a constant. This constant, however, is irrelevant to the calculation of $\mathbf{W}$, since only the difference $z(\alpha,t) - z(\alpha',t)$ appears in the definition of $\mathbf{W}$. Therefore, it is sufficient to know z_d in order to calculate $\mathbf{W}$, since $z(\alpha,t) - z(\alpha',t) = z_d(\alpha,t) - z_d(\alpha',t)$.

The lemmas which will appear in this section have been proved, at least in closely related versions, in previous works by the author and others, such as [1,2,15]. To begin, we have the following lemma, which was proved as Lemma 3.5 of [2]:

Lemma 1. *Let $n \geq 2$ be an integer. Assume $z_d \in H^n$. Then $\mathcal{K}[z_d] : H^1 \to H^{n-1}$ and $\mathcal{K}[z_d] : H^0 \to H^{n-2}$, with the estimates*

$$\|\mathcal{K}[z_d]f\|_{n-1} \leq C_1\|f\|_1 \exp\{C_2\|z_d\|_n\},$$
$$\|\mathcal{K}[z_d]f\|_{n-2} \leq C_1\|f\|_0 \exp\{C_2\|z_d\|_n\}.$$

Remark 5. The proof of Lemma 1 is based on the facts that the kernel of the integral operator $\mathcal{K}[z_d]$ is nonsingular, and is bounded in H^{n-2} when z_d is in H^n. If we were to have a more singular f, say $f \in H^{-2}$, then we could simply begin by integrating by parts, placing more derivatives on the kernel before making estimates. Then, the kernel would still be nonsingular, and would just be less regular. As a result, we find that $\mathcal{K}[z_d]$ maps from H^{-2} to H^{n-4}, with the estimate

$$\|\mathcal{K}[z_d]f\|_{n-4} \leq C_1\|f\|_{-2} \exp\{C_2\|z_d\|_n\}. \tag{23}$$

This will be relevant during the proof of Lemma 16 below.

We also need a Lipschitz estimate for $\mathcal{K}$; this estimate was proved in [1].

Lemma 2. *Let θ and θ' be in H^1. Let L and L' be the corresponding lengths of the associated curves z_d and z_d', and let q_1 and q_1' be the associated chord-arc quantities. Assume there exists positive constants $\bar{c}_1$ and $\bar{c}_2$ such that $L < \bar{c}_1$ and $L' < \bar{c}_1$, and for all α and α',*

$$|q_1(\alpha, \alpha')| > \bar{c}_2, \qquad |q_1'(\alpha, \alpha')| > \bar{c}_2.$$

Then the following Lipschitz estimate holds, for any $f \in H^1$:

$$\|\mathcal{K}[z_d]f - \mathcal{K}[z_d']f\|_1 \leq c\|\theta - \theta'\|_1\|f\|_1.$$

We also have the following lemma, which was proved as Lemma 3.7 of [2]:

Lemma 3. *Let $n \geq 1$ be an integer. Let $\phi \in H^n$ be given. Then $[H, \phi] : H^0 \to H^{n-1}$ and $[H, \phi] : H^{-1} \to H^{n-2}$, with the estimates*

$$\|[H, \phi]f\|_{n-1} \leq c\|\phi\|_n\|f\|_0,$$

$$\|[H, \phi]f\|_{n-2} \leq c\|\phi\|_n\|f\|_{-1}. \tag{24}$$

Remark 6. As in Remark 5, we note that if f is less regular, then a version of the commutator estimate still holds. This is true for the same reason as in Remark 5, namely that the operator $[H, \phi]$ is really an integral operator with nonsingular kernel. In the case of an f with low regularity (such as $f \in H^{-2}$), we may first integrate by parts before making estimates. Indeed, this is exactly how (24) is proved, and for $f \in H^{-2}$, we need only integrate by parts once more than we did to find (24). This results in the following estimate:

$$\|[H, \phi]f\|_{n-3} \leq c\|\phi\|_n\|f\|_{-2}. \tag{25}$$

Again, this will be useful during the proof of Lemma 16 below.

Notice that in Lemma 3, only low regularity of the function f is assumed. If f does have higher regularity, then it can be used to conclude that $[H, \phi]$ actually maps into H^n when $\phi \in H^n$. This is the subject of the next lemma, which generalizes Corollary 3.8 of [2]:

Lemma 4. *Let $j \geq 1$ be an integer. Let $n \geq 2j$ be an integer. Let $\phi \in H^n$ be given. Then, $[H, \phi] : H^{n-j} \to H^n$, with the estimate*

$$\|[H, \phi]f\|_n \leq c\|\phi\|_n\|f\|_{n-j}.$$

Proof. We apply ∂_x^n, which we break up as $\partial_x^j\partial_x^{n-j}$. We also use the product rule for ∂_x^{n-j}. These considerations yield the following:

$$\partial_x^n[H,\phi]f = \partial_x^j\left(\partial_x^{n-j}\left(H(\phi f)-\phi H(f)\right)\right) = \partial_x^j \sum_{\ell=0}^{n-j}\binom{n-j}{\ell}\left(H\left((\partial_x^\ell\phi)(\partial_x^{n-j-\ell}f)\right)\right.$$
$$\left. -(\partial_x^\ell\phi)H(\partial_x^{n-j-\ell}f)\right).$$

We further break this up, considering $\ell<j$ and $\ell\geq j$ separately (notice that $n-j\geq j$):

$$\partial_x^n[H,\phi]f = \left(\partial_x^j\sum_{\ell=0}^{j-1}\binom{n-j}{\ell}\left[H,\partial_x^\ell\phi\right]\partial_x^{n-j-\ell}f\right)$$
$$+\left(\partial_x^j\sum_{\ell=j}^{n-j}\binom{n-j}{\ell}H\left((\partial_x^\ell\phi)(\partial_x^{n-j-\ell}f)\right)\right)-\left(\partial_x^j\sum_{\ell=j}^{n-j}\binom{n-j}{\ell}(\partial_x^\ell\phi)H(\partial_x^{n-j-\ell}f)\right). \tag{26}$$

For the second and third summations on the right-hand side of (26), we have $\partial_x^\ell\phi\in H^j$ for all ℓ, and $\partial_x^{n-j-\ell}f\in H^j$ for all ℓ. Since $j\geq 1$, we have that H^j is an algebra, so the summands in the second and third summations are all in H^j. For the first summation, we have that $\partial_x^{n-j-\ell}f\in H^0$ for all ℓ, and since $\phi\in H^n$ with $n\geq 2j$, we also have $\partial_x^\ell\phi\in H^{j+1}$ for all ℓ. Therefore, Lemma 3 applies, and we find that the summands in the first summation are all in H^j. Putting this all together, we conclude that $\partial_x^n[H,\phi]f$ is in H^0, with the corresponding bound. This completes the proof. □

We must now introduce another integral operator, which we will call $\mathcal{J}$. Because of the presence of $\mathbf{W}$ on the right-hand side of (14), the equation is an integral equation for γ. If we define the operator $\mathcal{J}$ by

$$\mathcal{J}[z_d]f(\alpha) = -\mathrm{Re}\left\{iz_{d,\alpha}(\alpha)\mathrm{PV}\int f(\alpha')\cot\left(\frac{1}{2}(z_d(\alpha)-z_d(\alpha'))\right)\,d\alpha'\right\},$$

then (14) is of the form

$$\left(I+\frac{A_\mu}{2\pi}\mathcal{J}[z_d]\right)\gamma = F,$$

for some F. Here, I is the identity operator.

Lemma 5. *Assume $z_d\in H^n$ for $n\geq 3$. The operator $\left(I+\frac{A_\mu}{2\pi}\mathcal{J}[z_d]\right)^{-1}$ is bounded from H^0 to H^0, with the estimate*

$$\left\|\left(I+\frac{A_\mu}{2\pi}\mathcal{J}[z_d]\right)^{-1}F\right\|_0 \leq c_1\exp\{c_2\|z_d\|_3\}\|F\|_0.$$

We do not prove this lemma here, but we refer the reader to the discussion in the papers [13] and [23].

4.2. Estimates and Formulas for Quantities Related to θ

Throughout the sequel, we will need a variety of estimates and formulas for quantities related to θ, such as for $\mathbf{W}\cdot\hat{\mathbf{t}}$, as one example. We establish such estimates and formulas in this section

To begin, we define $\widetilde{\gamma}=\gamma-\frac{2\pi\tau}{L}\theta_{\alpha\alpha}$. To be a bit more precise, using (14), we define $\widetilde{\gamma}$ to be the following quantity:

$$\widetilde{\gamma}=\widetilde{\gamma}[\theta] = -\frac{RL}{2\pi}\sin(\theta)-\frac{A_\mu L}{\pi}\mathbf{W}\cdot\hat{\mathbf{t}}. \tag{27}$$

As the notation in (27) suggests, we view $\widetilde{\gamma}$ to be an operator which acts on θ. This viewpoint will be useful later, when we define a regularized version of the evolutionary problem.

Lemma 6. *If $\theta \in H^s$, then $\widetilde{\gamma}[\theta] \in H^s$.*

Proof. We know from Lemma 5 that there exists $\gamma \in H^0$ which satisfies (14). Inspecting the right-hand side of (14), we see that in order to prove higher regularity of $\widetilde{\gamma}$, it is sufficient to study the regularity of $\mathbf{W} \cdot \hat{\mathbf{t}}$.

We add and subtract in (9), and we use that definition of $\mathcal{K}[z]$, to find

$$\Phi(\mathbf{W})^* = \frac{1}{2i} H\left(\frac{\gamma}{z_\alpha}\right) + \mathcal{K}[z](\gamma). \tag{28}$$

We take the dot product of $\mathbf{W}$ with $\hat{\mathbf{t}}$:

$$\mathbf{W} \cdot \hat{\mathbf{t}} = \mathrm{Re}\left\{\frac{\pi z_\alpha}{Li} H\left(\frac{\gamma}{z_\alpha}\right)\right\} + \mathrm{Re}\left\{\frac{2\pi z_\alpha}{L}\mathcal{K}[z](\gamma)\right\}.$$

We pull the factor $1/z_\alpha$ outside the Hilbert transform, incurring a commutator; also, notice that $\mathrm{Re}\left\{\frac{\pi}{Li} H(\gamma)\right\} = 0$. This yields the following:

$$\mathbf{W} \cdot \hat{\mathbf{t}} = \mathrm{Re}\left\{\frac{\pi z_\alpha}{Li}\left[H, \frac{1}{z_\alpha}\right]\gamma\right\} + \mathrm{Re}\left\{\frac{2\pi z_\alpha}{L}\mathcal{K}[z](\gamma)\right\}. \tag{29}$$

Recall that $z_\alpha = \frac{L}{2\pi}(\cos(\theta), \sin(\theta))$, so we have $z_\alpha \in H^s$, and thus $z_d \in H^{s+1}$. Furthermore, since $|z_\alpha| = L/2\pi \geq 1$, we have $1/z_\alpha \in H^s$ as well. From Lemmas 1 and 3, we can then find constants c_1 and c_2 such that

$$\|\mathbf{W} \cdot \hat{\mathbf{t}}\|_{s-1} \leq c_1 \exp\{c_2\|\theta\|_s\}.$$

Since $\mathbf{W} \cdot \hat{\mathbf{t}} \in H^{s-1}$, we conclude from (14) that $\gamma \in H^{s-2}$. Then, we look again at (29), and we can use Lemma 1 again (in the same fashion as before), but we can now use Lemma 4, and we find $\mathbf{W} \cdot \hat{\mathbf{t}} \in H^s$, with

$$\|\mathbf{W} \cdot \hat{\mathbf{t}}\|_s \leq c_1 \exp\{c_2\|\theta\|_s\}.$$

This completes the proof. □

Remark 7. Just as we bounded $\mathbf{W} \cdot \hat{\mathbf{t}}$ in the previous lemma, we have the following estimates:

$$\|\mathbf{W}^{\text{s.t.}} \cdot \hat{\mathbf{t}}\|_s \leq c_1 \exp\{c_2\|\theta\|_s\},$$
$$\|\widetilde{\mathbf{W}} \cdot \hat{\mathbf{t}}\|_s \leq c_1 \exp\{c_2\|\theta\|_s\}.$$

Notice that we have the following formulas:

$$(V - \mathbf{W} \cdot \hat{\mathbf{t}})_\alpha = \frac{L_t}{2\pi} - \mathbf{W}_\alpha \cdot \hat{\mathbf{t}},$$
$$(V^{\text{s.t.}} - \mathbf{W}^{\text{s.t.}} \cdot \hat{\mathbf{t}})_\alpha = \frac{L_t^{\text{s.t.}}}{2\pi} - \mathbf{W}_\alpha^{\text{s.t.}} \cdot \hat{\mathbf{t}},$$
$$(\widetilde{V} - \widetilde{\mathbf{W}} \cdot \hat{\mathbf{t}})_\alpha = \frac{\widetilde{L}_t}{2\pi} - \widetilde{\mathbf{W}}_\alpha \cdot \hat{\mathbf{t}}.$$

We also need to define $\mathbf{m}^{\text{s.t.}}$ and $\widetilde{\mathbf{m}}$. Similarly to the previous definitions, we have $\mathbf{m} = \tau\mathbf{m}^{\text{s.t.}} + \widetilde{\mathbf{m}}$. This decomposition comes about by using $\widetilde{\gamma}$ to define $\widetilde{\mathbf{m}}$:

$$\Phi(\widetilde{\mathbf{m}})^* = z_\alpha \mathcal{K}[z]\left(\left(\frac{\widetilde{\gamma}}{z_\alpha}\right)_\alpha\right) + \frac{z_\alpha}{2i}\left[H, \frac{1}{z_\alpha^2}\right]\left(z_\alpha\left(\frac{\widetilde{\gamma}}{z_\alpha}\right)_\alpha\right). \tag{30}$$

The definition of $\mathbf{m}^{\text{s.t.}}$ is then given by

$$\Phi(\mathbf{m}^{\text{s.t.}})^* = \frac{2\pi z_\alpha}{L}\mathcal{K}[z]\left(\left(\frac{\theta_{\alpha\alpha}}{z_\alpha}\right)_\alpha\right) + \frac{\pi z_\alpha}{Li}\left[H, \frac{1}{z_\alpha^2}\right]\left(z_\alpha\left(\frac{\theta_{\alpha\alpha}}{z_\alpha}\right)_\alpha\right). \tag{31}$$

By Lemmas 1 and 4, we see that each of $\mathbf{m}^{\text{s.t.}}$ and $\widetilde{\mathbf{m}}$ are in H^s, with the estimates

$$\|\mathbf{m}^{\text{s.t.}}\|_s \le c_1 \exp\{c_2\|\theta\|_s\}, \qquad \|\widetilde{\mathbf{m}}\|_s \le c_1 \exp\{c_2\|\theta\|_s\}. \tag{32}$$

From (10), we see that

$$\mathbf{W}_\alpha \cdot \hat{\mathbf{t}} = -\frac{\pi}{L} H(\gamma\theta_\alpha) + \mathbf{m}\cdot\hat{\mathbf{t}}.$$

We have the corresponding formulas

$$\mathbf{W}_\alpha^{\text{s.t.}} \cdot \hat{\mathbf{t}} = -\frac{2\pi^2}{L^2} H(\theta_{\alpha\alpha}\theta_\alpha) + \mathbf{m}^{\text{s.t.}}\cdot\hat{\mathbf{t}}, \tag{33}$$

$$\widetilde{\mathbf{W}}_\alpha \cdot \hat{\mathbf{t}} = -\frac{\pi}{L} H(\widetilde{\gamma}\theta_\alpha) + \widetilde{\mathbf{m}}\cdot\hat{\mathbf{t}}.$$

This allows us to rewrite the above formulas, so that we find

$$(V^{\text{s.t.}} - \mathbf{W}^{\text{s.t.}}\cdot\hat{\mathbf{t}})_\alpha = \frac{L_t^{\text{s.t.}}}{2\pi} + \frac{2\pi^2}{L^2} H(\theta_{\alpha\alpha}\theta_\alpha) - \mathbf{m}^{\text{s.t.}}\cdot\hat{\mathbf{t}},$$

$$(\widetilde{V} - \widetilde{\mathbf{W}}\cdot\hat{\mathbf{t}})_\alpha = \frac{\widetilde{L}_t}{2\pi} + \frac{\pi}{L} H(\widetilde{\gamma}\theta_\alpha) - \widetilde{\mathbf{m}}\cdot\hat{\mathbf{t}}.$$

We note that it will be helpful in the sequel if we rewrite (33) by pulling θ_α through the Hilbert transform in the first term on the right-hand side of (33):

$$\mathbf{W}_\alpha^{\text{s.t.}} \cdot \hat{\mathbf{t}} = -\frac{2\pi^2}{L^2}\theta_\alpha H(\theta_{\alpha\alpha}) + \mathbf{m}^{\text{s.t.}}\cdot\hat{\mathbf{t}} - \frac{2\pi^2}{L^2}[H,\theta_\alpha]\theta_{\alpha\alpha}. \tag{34}$$

We also want a helpful formula for $U_\alpha^{\text{s.t.}}$. Since $U^{\text{s.t.}} = \mathbf{W}^{\text{s.t.}}\cdot\hat{\mathbf{n}}$, we clearly have $U_\alpha^{\text{s.t.}} = \mathbf{W}_\alpha^{\text{s.t.}}\cdot\hat{\mathbf{n}} - \theta_\alpha(\mathbf{W}^{\text{s.t.}}\cdot\hat{\mathbf{t}})$. We also have

$$\mathbf{W}_\alpha^{\text{s.t.}}\cdot\hat{\mathbf{n}} = \frac{2\pi^2}{L^2} H(\theta_{\alpha\alpha\alpha}) + \mathbf{m}^{\text{s.t.}}\cdot\hat{\mathbf{n}}.$$

Combining these formulas, we find

$$U_\alpha^{\text{s.t.}} = \frac{2\pi^2}{L^2} H(\theta_{\alpha\alpha\alpha}) - \theta_\alpha(\mathbf{W}^{\text{s.t.}}\cdot\hat{\mathbf{t}}) + \mathbf{m}^{\text{s.t.}}\cdot\hat{\mathbf{n}}. \tag{35}$$

Using (7) with (35), we see that

$$U^{\text{s.t.}} = \langle\langle U^{\text{s.t.}}\rangle\rangle + \frac{2\pi^2}{L^2} H(\theta_{\alpha\alpha}) + \partial_\alpha^{-1}\left(-\theta_\alpha(\mathbf{W}^{\text{s.t.}}\cdot\hat{\mathbf{t}}) + \mathbf{m}^{\text{s.t.}}\cdot\hat{\mathbf{n}}\right). \tag{36}$$

We give the name Q to the lower-order terms on the right-hand side, so that

$$U^{\text{s.t.}} = \frac{2\pi^2}{L^2} H(\theta_{\alpha\alpha}) + Q. \tag{37}$$

Remark 8. Strictly speaking, we will not need these formulas and estimates until Sect. 6.2 below. More immediately, we introduce a mollified version of the evolution equations, and we establish formulas corresponding to those in the present section for the mollified problem. This will be done next, in Sect. 5.

5. Well-Posedness with Surface Tension

In this section, we establish the existence of solutions for the initial value problem in the presence of surface tension. We also establish uniqueness of these solutions, and continuous dependence on the initial data. We do this without assuming that the stability condition is satisfied by the initial data. We begin by introducing mollifiers, forming an evolution equation for θ^ε. We then rewrite this evolution equation, differentiating twice to find the evolution equation for $\theta^\varepsilon_{\alpha\alpha}$. We continue by extracting the most singular terms. The ultimate goal of this endeavor is to arrive at formula (58) below. We will then establish some related auxiliary estimates; this will leave us ready to perform the energy estimate for the full problem which corresponds to the estimate of Case 3 of Sect. 2. We make this estimate in Sect. 5.4. Then, in Sect. 5.5, we establish existence of solutions. We subsequently discuss uniqueness and continuous dependence.

5.1. The Mollified Evolution Equation

In this section, we introduce the mollified problem. Earlier in this manuscript, we have introduced a variety of nonlocal quantities, such as $\mathbf{W}, V$, and so on. We will now define new versions of all of these quantities, and these new versions will all depend, either explicitly or implicitly, on ε, our positive mollification parameter. First, however, we must define the curve z^ε, which is determined from θ^ε.

We need to be careful in defining z^ε because it is not the case that any 2π-periodic tangent angle gives rise to a 2π-periodic curve. (However, for the exact evolution equation, if the tangent angle initially corresponds to a 2π-periodic curve, then the tangent angle will correspond to a 2π-periodic curve at positive times as well.) We begin with the observation that the value $z(0)$ is irrelevant to the evolution of θ; that is, the only way that z arises is either through z_α or $z(\alpha)-z(\alpha')$. Therefore, we recall the definition $z_d(\alpha,t)=z(\alpha,t)-z(0,t)$, and we notice that $z_\alpha=z_{d,\alpha}$ and $z(\alpha)-z(\alpha')=z_d(\alpha)-z_d(\alpha')$. Also, z is 2π-periodic if and only if z_d is 2π-periodic. So, we want to define z_d^ε based on θ^ε. To begin, we find a formula for the length of the curve in the non-mollified problem. From the periodicity, we have (for any t)

$$2\pi = x(2\pi,t)-x(0,t)=\int_0^{2\pi} x_\alpha(\alpha,t)\ d\alpha.$$

Since we have $\cos(\theta)=x_\alpha/s_\alpha=2\pi x_\alpha/L$, we can write this as

$$2\pi=\frac{L}{2\pi}\int_0^{2\pi}\cos(\theta(\alpha,t))\ d\alpha.$$

Solving this for L yields

$$L=\frac{4\pi^2}{\int_0^{2\pi}\cos(\theta(\alpha,t))\ d\alpha}, \tag{38}$$

and we therefore make the definition

$$L^\varepsilon(t)=\frac{4\pi^2}{\int_0^{2\pi}\cos(\theta^\varepsilon(\alpha,t))\ d\alpha}.$$

Next, we define $z_d^\varepsilon(\alpha,t)$ by

$$z_d^\varepsilon(\alpha,t)=\frac{L^\varepsilon}{2\pi}\int_0^{\alpha}\cos(\theta^\varepsilon(\alpha',t))+i\mathbb{P}(\sin(\theta^\varepsilon(\alpha',t)))\ d\alpha'. \tag{39}$$

With this definition of the curve, we clearly have $z_d^\varepsilon(\alpha+2\pi,t)=z_d^\varepsilon(\alpha,t)+2\pi$. We have the corresponding unit tangent and normal vectors,

$$\Phi(\hat{\mathbf{t}}^\varepsilon)=\frac{z^\varepsilon_{d,\alpha}}{|z^\varepsilon_{d,\alpha}|},\qquad \Phi(\hat{\mathbf{n}}^\varepsilon)=\frac{iz^\varepsilon_{d,\alpha}}{|z^\varepsilon_{d,\alpha}|}.$$

We will revisit this definition of z_d^ε below, as we will be able to eliminate the presence of the operator $\mathbb{P}$ for solutions of the mollified evolution equation.

For $\varepsilon>0$, we introduce the mollifier χ_ε. This is a standard mollifier, and could be defined in a few different ways. For instance, $\chi_\varepsilon f$ could be the operator which truncates the Fourier series of f for modes beyond $1/\varepsilon$. To be definite, however, we specify that χ_ε is the periodic convolution with an approximate

Dirac mass, scaled so that the support of the approximate Dirac mass has width ε and the maximum value is on the order of $1/\varepsilon$.

We write the mollified evolution equation as

$$\theta_t^\varepsilon = \mathfrak{B}^\varepsilon + \mu^\varepsilon, \tag{40}$$

where

$$\begin{aligned}\mathfrak{B}^\varepsilon =& \left(\frac{4\pi^3\tau}{(L^\varepsilon)^3}\right)\chi_\varepsilon^2 H(\theta_{\alpha\alpha\alpha}^\varepsilon) - \tau\chi_\varepsilon\left[\frac{4\pi^3 A_\mu}{(L^\varepsilon)^3}(\chi_\varepsilon\theta_{\alpha\alpha}^\varepsilon)(\chi_\varepsilon\theta_\alpha^\varepsilon) + \frac{2\pi A_\mu}{L^\varepsilon}H(U^{\mathrm{s.t.},\varepsilon}\chi_\varepsilon\theta_\alpha^\varepsilon)\right]\\ &+\chi_\varepsilon\left[H\left(k^\varepsilon\chi_\varepsilon\theta_\alpha^\varepsilon\right) + \frac{2\pi\tau}{L^\varepsilon}(V^{\mathrm{s.t.},\varepsilon} - \mathbf{W}^{\mathrm{s.t.},\varepsilon}\cdot\hat{\mathbf{t}}^\varepsilon)\chi_\varepsilon\theta_\alpha^\varepsilon + \frac{2\pi A_\mu^2\tau}{L^\varepsilon}\mathbb{P}((\chi_\varepsilon\theta_\alpha^\varepsilon)\mathbf{W}^{\mathrm{s.t.},\varepsilon}\cdot\hat{\mathbf{t}}^\varepsilon)\right]\\ &+\frac{\pi A_\mu R}{L^\varepsilon}\chi_\varepsilon\left(\sin(\chi_\varepsilon\theta^\varepsilon)\chi_\varepsilon\theta_\alpha^\varepsilon\right) + \frac{2\pi A_\mu^2}{L^\varepsilon}\chi_\varepsilon\mathbb{P}((\chi_\varepsilon\theta_\alpha^\varepsilon)\widetilde{\mathbf{W}}^\varepsilon\cdot\hat{\mathbf{t}}^\varepsilon)\\ &+\frac{2\pi}{L^\varepsilon}\chi_\varepsilon\left[(\widetilde{V}^\varepsilon - \widetilde{\mathbf{W}}^\varepsilon\cdot\hat{\mathbf{t}}^\varepsilon)\chi_\varepsilon\theta_\alpha^\varepsilon\right] + \frac{2\pi}{L^\varepsilon}\chi_\varepsilon\left[\mathbf{m}^\varepsilon\cdot\hat{\mathbf{n}}^\varepsilon\right] - \frac{2\pi^3 A_\mu}{(L^\varepsilon)^3}\chi_\varepsilon H(\mathbf{m}^\varepsilon\cdot\hat{\mathbf{t}}^\varepsilon).\end{aligned} \tag{41}$$

There are several terms above which have not yet been defined; they will be defined shortly, either in the present subsection, or in Sect. 5.2 below. Notice that $\mathfrak{B}^\varepsilon$ is a mollified version of the right-hand side of (20); for our later convenience, we also introduce the notation $\mathfrak{B}$ to refer to the right-hand side of (20). In (41), the placement of the operators χ_ε may perhaps seem arbitrary or unusual, but they are carefully placed so that the estimates we are about to undertake will work out. It would be reasonable to think that there is no need for the additional term μ^ε, as $\mathfrak{B}^\varepsilon$ provides a mollified version of the evolution equation for θ. However, it will be helpful if the evolution of θ^ε maintains the property that $\mathbb{P}(\sin(\theta^\varepsilon)) = \sin(\theta^\varepsilon)$; the term μ^ε enforces this condition. We note that μ^ε will depend only on t, and not on α.

To begin, we recall that under the exact, non-mollified evolution, we have $\theta_t = (U_\alpha + V\theta_\alpha)/s_\alpha$, and $V_\alpha = s_{\alpha t}/s_\alpha + \theta_\alpha U$. If we consider the evolution of $\int_0^{2\pi}\sin(\theta)\,d\alpha$, we have

$$\frac{d}{dt}\int_0^{2\pi}\sin(\theta)\ d\alpha = \int_0^{2\pi}\cos(\theta)\theta_t\ d\alpha = \frac{1}{s_\alpha}\int_0^{2\pi}\cos(\theta)U_\alpha + \cos(\theta)V\theta_\alpha\ d\alpha.$$

We integrate both terms on the right-hand side by parts, noting that $\cos(\theta)\theta_\alpha$ is a perfect derivative:

$$\frac{d}{dt}\int_0^{2\pi}\sin(\theta)\ d\alpha = \frac{1}{s_\alpha}\int_0^{2\pi}\sin(\theta)\theta_\alpha U - \sin(\theta)V_\alpha\ d\alpha.$$

We substitute for V_α and see an important cancellation; this leaves only

$$\frac{d}{dt}\int_0^{2\pi}\sin(\theta)\ d\alpha = -\frac{s_{\alpha t}}{s_\alpha}\int_0^{2\pi}\sin(\theta)\ d\alpha.$$

Clearly, then, for smooth solutions of the non-mollified equation, if initially $\langle\langle\sin(\theta)\rangle\rangle = 0$, then this property is maintained by the evolution. For the mollified equation, we do not have the simple structure that we used in the present calculation, and we must actively enforce this condition. We begin the above calculation again, this time for a solution of the mollified evolution, and we use the fact that μ^ε is to be independent of α :

$$\frac{d}{dt}\int_0^{2\pi}\sin(\theta^\varepsilon)\ d\alpha = \int_0^{2\pi}\cos(\theta^\varepsilon)\theta_t^\varepsilon\ d\alpha = \int_0^{2\pi}\cos(\theta^\varepsilon)\mathfrak{B}^\varepsilon\ d\alpha + \mu^\varepsilon\int_0^{2\pi}\cos(\theta^\varepsilon)\ d\alpha.$$

Setting this equal to zero, and recalling the definition of L^ε, we have the definition of μ^ε:

$$\mu^\varepsilon = -\frac{\int_0^{2\pi} \cos(\theta^\varepsilon)\mathfrak{B}^\varepsilon \ d\alpha}{4\pi^2 L^\varepsilon}. \tag{42}$$

We said previously that we would revisit the expression (39) for the curve to be reconstructed from θ^ε. We now give another, simpler such expression. By construction, because of the presence of μ^ε, if θ^ε is the solution of (40), then we have $\mathbb{P}(\sin(\theta^\varepsilon)) = \sin(\theta^\varepsilon)$. In light of this, we are able to state the definition of z_d^ε without needing $\mathbb{P}$:

$$z_d^\varepsilon(\alpha, t) = \frac{L^\varepsilon}{2\pi}\int_0^\alpha \cos(\theta^\varepsilon(\alpha', t)) + i\sin(\theta^\varepsilon(\alpha', t)) \ d\alpha'.$$

Note that the benefit of this is that we are able to write

$$z_{d,\alpha}^\varepsilon = \frac{L^\varepsilon}{2\pi}(\cos(\theta^\varepsilon) + i\sin(\theta^\varepsilon)), \tag{43}$$

implying that $|z_{d,\alpha}^\varepsilon|$ is independent of α, as desired. The presence of the operator $\mathbb{P}$ would have complicated this. An immediate consequence of (43) is that we can write

$$\hat{\mathbf{t}}^\varepsilon = (\cos(\theta^\varepsilon), \sin(\theta^\varepsilon)), \qquad \hat{\mathbf{n}}^\varepsilon = (-\sin(\theta^\varepsilon), \cos(\theta^\varepsilon)).$$

Another consequence is, since $|z_{d,\alpha}^\varepsilon| = L^\varepsilon/2\pi$, and since the curve is 2π-periodic, we must have

$$L^\varepsilon \geq 2\pi. \tag{44}$$

Note that z_d^ε is bounded in terms of θ^ε and L^ε. This can be proved directly by using either formula (39) or (43). The estimate one finds is, for $\theta \in H^s$,

$$\|z_d^\varepsilon\|_{s+1} \leq cL(1 + \|\theta^\varepsilon\|_s).$$

Of course, this is true without the superscripts of ε as well.

5.2. The Mollified Birkhoff–Rott Integral and Its Consequences

We need similar formulas to those previously established. To begin, we define $\mathbf{W}^\varepsilon$ to be $\mathbf{W}^\varepsilon = \tau\mathbf{W}^{\text{s.t.},\varepsilon} + \widetilde{\mathbf{W}}^\varepsilon$, with

$$\Phi(\mathbf{W}^{\text{s.t.},\varepsilon})^*(\alpha, t) = \frac{1}{2iL^\varepsilon}\text{PV}\int_0^{2\pi} \chi_\varepsilon\theta_{\alpha\alpha}^\varepsilon(\alpha', t)\cot\left(\frac{1}{2}(z_d^\varepsilon(\alpha, t) - z_d^\varepsilon(\alpha', t))\right) \ d\alpha',$$

$$\Phi(\widetilde{\mathbf{W}}^\varepsilon)^*(\alpha, t) = \frac{1}{4\pi i}\text{PV}\int \widetilde{\gamma}[\theta^\varepsilon](\alpha')\cot\left(\frac{1}{2}(z_d^\varepsilon(\alpha) - z_d^\varepsilon(\alpha'))\right) \ d\alpha'.$$

We then define $U^{\text{s.t.},\varepsilon} = \mathbf{W}^{\text{s.t.},\varepsilon}\cdot\hat{\mathbf{n}}^\varepsilon$ and $V^{\text{s.t.},\varepsilon} = \partial_\alpha^{-1}\mathbb{P}(\theta_\alpha^\varepsilon U^{\text{s.t.},\varepsilon})$. Correspondingly, we define $\widetilde{U}^\varepsilon = \widetilde{\mathbf{W}}^\varepsilon\cdot\hat{\mathbf{n}}^\varepsilon$ and $\widetilde{V}^\varepsilon = \partial_\alpha^{-1}\mathbb{P}(\theta_\alpha^\varepsilon\widetilde{U}^\varepsilon)$. Now that we have defined $\widetilde{U}^\varepsilon$, we may define k^ε :

$$k^\varepsilon = \frac{2\pi}{L}\left\{-\frac{R\cos(\theta^\varepsilon)}{2} - A_\mu\widetilde{U}^\varepsilon\right\}. \tag{45}$$

Notice, however, that this is the same as saying the following:

$$k^\varepsilon = k[\theta^\varepsilon]. \tag{46}$$

We also will need definitions of $\mathbf{m}^{\mathrm{s.t.},\varepsilon}$ and $\widetilde{\mathbf{m}}^{\varepsilon}$. We define them as follows:

$$\Phi(\widetilde{\mathbf{m}}^{\varepsilon})^{*}=z^{\varepsilon}_{d,\alpha}\mathcal{K}[z^{\varepsilon}_{d}]\left(\left(\frac{\widetilde{\gamma}[\theta^{\varepsilon}]}{z^{\varepsilon}_{d,\alpha}}\right)_{\alpha}\right)+\frac{z^{\varepsilon}_{d,\alpha}}{2i}\left[H,\frac{1}{(z^{\varepsilon}_{d,\alpha})^{2}}\right]\left(z^{\varepsilon}_{d,\alpha}\left(\frac{\widetilde{\gamma}[\theta^{\varepsilon}]}{z^{\varepsilon}_{d,\alpha}}\right)_{\alpha}\right),$$

$$\Phi(\mathbf{m}^{\mathrm{s.t.},\varepsilon})^{*}=\frac{2\pi z^{\varepsilon}_{d,\alpha}}{L^{\varepsilon}}\mathcal{K}[z^{\varepsilon}_{d}]\left(\left(\frac{\chi_{\varepsilon}\theta^{\varepsilon}_{\alpha\alpha}}{z_{d,\alpha}}\right)_{\alpha}\right)+\frac{\pi z^{\varepsilon}_{d,\alpha}}{Li}\left[H,\frac{1}{(z^{\varepsilon}_{d,\alpha})^{2}}\right]\left(z^{\varepsilon}_{d,\alpha}\left(\frac{\chi_{\varepsilon}\theta^{\varepsilon}_{\alpha\alpha}}{z^{\varepsilon}_{d,\alpha}}\right)_{\alpha}\right).$$

Of course, as before, we have the definition of $\mathbf{m}^{\varepsilon}$ given $\mathbf{m}^{\mathrm{s.t.},\varepsilon}$ and $\widetilde{\mathbf{m}}^{\varepsilon}$:

$$\mathbf{m}^{\varepsilon}=\tau\mathbf{m}^{\mathrm{s.t.},\varepsilon}+\widetilde{\mathbf{m}}^{\varepsilon}.$$

We note that this was the final quantity from (41) which had been undefined; therefore, we have now fully specified the mollified evolution.

Taking a derivative of $U^{\mathrm{s.t.},\varepsilon}$, we clearly have

$$U^{\mathrm{s.t.},\varepsilon}_{\alpha}=\mathbf{W}^{\mathrm{s.t.},\varepsilon}_{\alpha}\cdot\hat{\mathbf{n}}^{\varepsilon}-\theta^{\varepsilon}_{\alpha}\mathbf{W}^{\mathrm{s.t.},\varepsilon}\cdot\hat{\mathbf{t}}^{\varepsilon}.$$

We can write, similarly to the unmollified case, the following expression for $\mathbf{W}^{\mathrm{s.t.},\varepsilon}_{\alpha}\cdot\hat{\mathbf{n}}^{\varepsilon}$:

$$\mathbf{W}^{\mathrm{s.t.},\varepsilon}_{\alpha}\cdot\hat{\mathbf{n}}^{\varepsilon}=\frac{2\pi^{2}}{(L^{\varepsilon})^{2}}\chi_{\varepsilon}H(\theta^{\varepsilon}_{\alpha\alpha\alpha})+\mathbf{m}^{\mathrm{s.t.},\varepsilon}\cdot\hat{\mathbf{n}}^{\varepsilon}.$$

Putting these equations together, we arrive at our expression for $U^{\mathrm{s.t.},\varepsilon}_{\alpha}$:

$$U^{\mathrm{s.t.},\varepsilon}_{\alpha}=\frac{2\pi^{2}}{(L^{\varepsilon})^{2}}\chi_{\varepsilon}H(\theta^{\varepsilon}_{\alpha\alpha\alpha})-\theta^{\varepsilon}_{\alpha}\mathbf{W}^{\mathrm{s.t.},\varepsilon}\cdot\hat{\mathbf{t}}^{\varepsilon}+\mathbf{m}^{\mathrm{s.t.},\varepsilon}\cdot\hat{\mathbf{n}}^{\varepsilon}. \tag{47}$$

Integrating with respect to α, we find the corresponding expression for $U^{\mathrm{s.t.},\varepsilon}$:

$$U^{\mathrm{s.t.},\varepsilon}=\langle\!\langle U^{\mathrm{s.t.},\varepsilon}\rangle\!\rangle+\frac{2\pi^{2}}{(L^{\varepsilon})^{2}}\chi_{\varepsilon}H(\theta^{\varepsilon}_{\alpha\alpha})+\partial_{\alpha}^{-1}\left(-\theta^{\varepsilon}_{\alpha}(\mathbf{W}^{\mathrm{s.t.},\varepsilon}\cdot\hat{\mathbf{t}}^{\varepsilon})+\mathbf{m}^{\mathrm{s.t.},\varepsilon}\cdot\hat{\mathbf{n}}^{\varepsilon}\right). \tag{48}$$

We also have the following:

$$\mathbf{W}^{\mathrm{s.t.},\varepsilon}_{\alpha}\cdot\hat{\mathbf{t}}^{\varepsilon}=-\frac{2\pi^{2}}{(L^{\varepsilon})^{2}}\theta^{\varepsilon}_{\alpha}\chi_{\varepsilon}H(\theta^{\varepsilon}_{\alpha\alpha})+\mathbf{m}^{\mathrm{s.t.},\varepsilon}\cdot\hat{\mathbf{t}}^{\varepsilon}-\frac{2\pi^{2}}{(L^{\varepsilon})^{2}}[H,\theta^{\varepsilon}_{\alpha}]\chi_{\varepsilon}\theta^{\varepsilon}_{\alpha\alpha}. \tag{49}$$

5.3. Higher Derivatives

It is helpful to apply one spatial derivative to (40). Notice that since μ^{ε} does not depend on α, it will make no contribution, and we will simply have $\theta^{\varepsilon}_{\alpha,t}=\mathfrak{B}^{\varepsilon}_{\alpha}$. Furthermore, we note that since $\partial_{\alpha}\mathbb{P}=\partial_{\alpha}$, the operator $\mathbb{P}$ will not appear in the differentiated equation. With these considerations in mind, applying the derivative, we find

$$\begin{aligned}\theta^{\varepsilon}_{\alpha,t}=&\frac{4\pi^{3}\tau}{(L^{\varepsilon})^{3}}\chi_{\varepsilon}^{2}H\partial_{\alpha}^{4}(\theta^{\varepsilon})-\tau\chi_{\varepsilon}\left[\frac{4\pi^{3}A_{\mu}}{(L^{\varepsilon})^{3}}(\chi_{\varepsilon}\theta^{\varepsilon}_{\alpha})(\chi_{\varepsilon}\theta^{\varepsilon}_{\alpha\alpha\alpha})+\frac{2\pi A_{\mu}}{L^{\varepsilon}}H(U^{\mathrm{s.t.},\varepsilon}_{\alpha}\chi_{\varepsilon}\theta^{\varepsilon}_{\alpha})\right]\\&-\tau\chi_{\varepsilon}\left[\frac{4\pi^{3}A_{\mu}}{(L^{\varepsilon})^{3}}(\chi_{\varepsilon}\theta^{\varepsilon}_{\alpha\alpha})^{2}\right]-\tau\chi_{\varepsilon}\left[\frac{2\pi A_{\mu}}{L^{\varepsilon}}H(U^{\mathrm{s.t.},\varepsilon}\chi_{\varepsilon}\theta^{\varepsilon}_{\alpha\alpha})\right]\\&+\chi_{\varepsilon}H(k^{\varepsilon}\chi_{\varepsilon}\theta^{\varepsilon}_{\alpha\alpha})+\chi_{\varepsilon}H(k^{\varepsilon}_{\alpha}\chi_{\varepsilon}\theta^{\varepsilon}_{\alpha})-\frac{2\pi\tau}{L^{\varepsilon}}\chi_{\varepsilon}\left[(\mathbf{W}^{\mathrm{s.t.},\varepsilon}_{\alpha}\cdot\hat{\mathbf{t}}^{\varepsilon})\chi_{\varepsilon}\theta^{\varepsilon}_{\alpha}\right]+\frac{\tau L^{\mathrm{s.t.},\varepsilon}_{t}}{L^{\varepsilon}}\chi_{\varepsilon}^{2}\theta^{\varepsilon}_{\alpha}\\&+\frac{2\pi\tau}{L^{\varepsilon}}\chi_{\varepsilon}\left[(V^{\mathrm{s.t.},\varepsilon}-\mathbf{W}^{\mathrm{s.t.},\varepsilon}\cdot\hat{\mathbf{t}}^{\varepsilon})\chi_{\varepsilon}\theta^{\varepsilon}_{\alpha\alpha}\right]\end{aligned}$$

$$+\frac{2\pi A_\mu^2\tau}{L^\varepsilon}\chi_\varepsilon\left[(\chi_\varepsilon\theta^\varepsilon_{\alpha\alpha})\mathbf{W}^{\mathrm{s.t.},\varepsilon}\cdot\hat{\mathbf{t}}^\varepsilon+(\chi_\varepsilon\theta^\varepsilon_\alpha)\partial_\alpha(\mathbf{W}^{\mathrm{s.t.},\varepsilon}\cdot\hat{\mathbf{t}}^\varepsilon)\right]$$
$$+\partial_\alpha\chi_\varepsilon\left[\frac{\pi A_\mu R}{L^\varepsilon}\sin(\chi_\varepsilon\theta^\varepsilon)\chi_\varepsilon\theta^\varepsilon_\alpha+\frac{2\pi A_\mu^2}{L^\varepsilon}(\chi_\varepsilon\theta^\varepsilon_\alpha)\widetilde{\mathbf{W}}^\varepsilon\cdot\hat{\mathbf{t}}^\varepsilon\right.$$
$$\left.+\frac{2\pi}{L^\varepsilon}(\widetilde{V}^\varepsilon-\widetilde{\mathbf{W}}^\varepsilon\cdot\hat{\mathbf{t}}^\varepsilon)\chi_\varepsilon\theta^\varepsilon_\alpha+\frac{2\pi}{L^\varepsilon}\mathbf{m}^\varepsilon\cdot\hat{\mathbf{n}}^\varepsilon-\frac{2\pi^3A_\mu}{(L^\varepsilon)^3}H(\mathbf{m}^\varepsilon\cdot\hat{\mathbf{t}}^\varepsilon)\right]. \tag{50}$$

We continue to rearrange this, to isolate the terms which must be treated carefully in the energy estimate. In particular, we use (49), (47), and (48) to substitute for $\mathbf{W}^{\mathrm{s.t.},\varepsilon}_\alpha\cdot\hat{\mathbf{t}}, U^{\mathrm{s.t.},\varepsilon}_\alpha$, and $U^{\mathrm{s.t.},\varepsilon}$ in (50). In addition, we make another substitution. Since the mean value of the Hilbert transform of any function is zero, we can write

$$H(U^{\mathrm{s.t.},\varepsilon}\chi_\varepsilon\theta^\varepsilon_{\alpha\alpha})=\partial_\alpha^{-1}\partial_\alpha H(U^{\mathrm{s.t.},\varepsilon}\chi_\varepsilon\theta^\varepsilon_{\alpha\alpha})=\partial_\alpha^{-1}H(U^{\mathrm{s.t.},\varepsilon}_\alpha\chi_\varepsilon\theta^\varepsilon_{\alpha\alpha})+\partial_\alpha^{-1}H(U^{\mathrm{s.t.},\varepsilon}\chi_\varepsilon\theta^\varepsilon_{\alpha\alpha\alpha}). \tag{51}$$

We substitute (47) into the first term on the right-hand side of (51), and we pull $U^{\mathrm{s.t.},\varepsilon}$ through the Hilbert transform in the second term on the right-hand side of (51), incurring a commutator. This yields the following:

$$H(U^{\mathrm{s.t.},\varepsilon}\chi_\varepsilon\theta^\varepsilon_{\alpha\alpha})=\frac{2\pi^2}{(L^\varepsilon)^2}\partial_\alpha^{-1}H\left((\chi_\varepsilon\theta^\varepsilon_{\alpha\alpha})H(\chi_\varepsilon\theta^\varepsilon_{\alpha\alpha\alpha})\right)+\partial_\alpha^{-1}\left(U^{\mathrm{s.t.},\varepsilon}H(\chi_\varepsilon\theta^\varepsilon_{\alpha\alpha\alpha})\right)$$
$$+\partial_\alpha^{-1}H\left((\chi_\varepsilon\theta^\varepsilon_{\alpha\alpha})\Big(-\theta^\varepsilon_\alpha(\mathbf{W}^{\mathrm{s.t.},\varepsilon}\cdot\hat{\mathbf{t}}^\varepsilon)+\mathbf{m}^{\mathrm{s.t.},\varepsilon}\cdot\hat{\mathbf{n}}^\varepsilon\Big)\right)+\partial_\alpha^{-1}[H,U^{\mathrm{s.t.},\varepsilon}]\chi_\varepsilon\theta^\varepsilon_{\alpha\alpha\alpha}. \tag{52}$$

We pull $\chi_\varepsilon\theta^\varepsilon_{\alpha\alpha}$ through the Hilbert transform in the first term on the right-hand side of (52), incurring a commutator. We also use the fact that $H^2=-I$ when applied to functions with zero mean. These considerations yield the following:

$$H(U^{\mathrm{s.t.},\varepsilon}\chi_\varepsilon\theta^\varepsilon_{\alpha\alpha})=-\frac{2\pi^2}{(L^\varepsilon)^2}\partial_\alpha^{-1}\left((\chi_\varepsilon\theta^\varepsilon_{\alpha\alpha})\chi_\varepsilon\theta^\varepsilon_{\alpha\alpha\alpha}\right)+\partial_\alpha^{-1}\left(U^{\mathrm{s.t.},\varepsilon}H(\chi_\varepsilon\theta^\varepsilon_{\alpha\alpha\alpha})\right)$$
$$+\frac{2\pi^2}{(L^\varepsilon)^2}\partial_\alpha^{-1}[H,\chi_\varepsilon\theta^\varepsilon_{\alpha\alpha}]H(\chi_\varepsilon\theta^\varepsilon_{\alpha\alpha\alpha})+\partial_\alpha^{-1}[H,U^{\mathrm{s.t.},\varepsilon}]\chi_\varepsilon\theta^\varepsilon_{\alpha\alpha\alpha}$$
$$+\partial_\alpha^{-1}H\left((\chi_\varepsilon\theta^\varepsilon_{\alpha\alpha})\Big(-\theta^\varepsilon_\alpha(\mathbf{W}^{\mathrm{s.t.},\varepsilon}\cdot\hat{\mathbf{t}}^\varepsilon)+\mathbf{m}^{\mathrm{s.t.},\varepsilon}\cdot\hat{\mathbf{n}}^\varepsilon\Big)\right). \tag{53}$$

Similarly to this calculation, but more simply, we want to use (47) to expand $H(U^{\mathrm{s.t.},\varepsilon}_\alpha\chi_\varepsilon\theta^\varepsilon_\alpha)$. We have

$$H(U^{\mathrm{s.t.},\varepsilon}_\alpha\chi_\varepsilon\theta^\varepsilon_\alpha)=\frac{2\pi^2}{(L^\varepsilon)^2}H\left((\chi_\varepsilon\theta^\varepsilon_\alpha)H(\chi_\varepsilon\theta^\varepsilon_{\alpha\alpha\alpha})\right)-H((\chi_\varepsilon\theta^\varepsilon_\alpha)\theta^\varepsilon_\alpha\mathbf{W}^{\mathrm{s.t.},\varepsilon}\cdot\hat{\mathbf{t}}^\varepsilon)+H((\chi_\varepsilon\theta^\varepsilon_\alpha)\mathbf{m}^{\mathrm{s.t.},\varepsilon}\cdot\hat{\mathbf{n}}^\varepsilon).$$

In the first term on the right-hand side, we pull $\chi_\varepsilon\theta^\varepsilon_\alpha$ through the Hilbert transform, incurring a commutator. We also use the fact that $H^2=-I$ for mean-zero functions. This yields the following:

$$H(U^{\mathrm{s.t.},\varepsilon}_\alpha\chi_\varepsilon\theta^\varepsilon_\alpha)=-\frac{2\pi^2}{(L^\varepsilon)^2}(\chi_\varepsilon\theta^\varepsilon_\alpha)\chi_\varepsilon\theta^\varepsilon_{\alpha\alpha\alpha}-H((\chi_\varepsilon\theta^\varepsilon_\alpha)\theta^\varepsilon_\alpha\mathbf{W}^{\mathrm{s.t.},\varepsilon}\cdot\hat{\mathbf{t}}^\varepsilon)$$
$$+H((\chi_\varepsilon\theta^\varepsilon_\alpha)\mathbf{m}^{\mathrm{s.t.},\varepsilon}\cdot\hat{\mathbf{n}}^\varepsilon)+\frac{2\pi^2}{(L^\varepsilon)^2}[H,\chi_\varepsilon\theta^\varepsilon_\alpha]H(\chi_\varepsilon\theta^\varepsilon_{\alpha\alpha\alpha}). \tag{54}$$

We rewrite (50) according to the above considerations. We find

$$\theta^\varepsilon_{\alpha,t}=\chi_\varepsilon\left[\frac{4\pi^3\tau}{(L^\varepsilon)^3}H\partial_\alpha^4(\chi_\varepsilon\theta^\varepsilon)+\tau\Upsilon^\varepsilon_1\chi_\varepsilon\theta^\varepsilon_{\alpha\alpha\alpha}+\Upsilon^\varepsilon_2+\Upsilon^\varepsilon_3+\Upsilon^\varepsilon_4\right], \tag{55}$$

where we will give formulas for the Υ^ε_i shortly. We first mention that Υ^ε_1 is clearly just the coefficient of $\tau\chi_\varepsilon\theta^\varepsilon_{\alpha\alpha\alpha}$. Next, Υ^ε_2 is essentially a collection of terms which include $H(\chi_\varepsilon\theta^\varepsilon_{\alpha\alpha})$; it is not this simple, however, since we also include the term $\partial_\alpha^{-1}(U^{\mathrm{s.t.},\varepsilon}H(\chi_\varepsilon\theta^\varepsilon_{\alpha\alpha\alpha}))$ from (53). The distinction between Υ^ε_3 and Υ^ε_4 is that Υ^ε_3 consists of transport terms, and Υ^ε_4 consists of smooth terms; these are all routine to estimate, but they need to be treated differently in the energy estimates, since the transport terms require an integration by parts that the smooth terms do not.

Using (54), we find that the formula for Υ^ε_1 is

$$\Upsilon^\varepsilon_1 = -\frac{4\pi^3 A_\mu}{(L^\varepsilon)^3}\chi_\varepsilon\theta^\varepsilon_\alpha + \frac{4\pi^3 A_\mu}{(L^\varepsilon)^3}\chi_\varepsilon\theta^\varepsilon_\alpha = 0. \tag{56}$$

Remark 9. Even if these two terms did not cancel, this would not be an obstacle to the rest of the proof. The fact that Υ^ε_1 is zero corresponds, in the example of Sect. 2, to having the value $c_1 = 0$. In the example, we were able to bound the relevant term even when $c_1 > 0$, which is the more difficult case.

We write $H(k^\varepsilon\chi_\varepsilon\theta^\varepsilon_{\alpha\alpha}) = k^\varepsilon H(\chi_\varepsilon\theta^\varepsilon_{\alpha\alpha}) + [H, k^\varepsilon]\chi_\varepsilon\theta^\varepsilon_{\alpha\alpha}$, and we use (53) and (49), collecting terms that are essentially proportional to $H(\chi_\varepsilon\theta^\varepsilon_{\alpha\alpha})$ into Υ^ε_2 :

$$\Upsilon^\varepsilon_2 = k^\varepsilon H(\chi_\varepsilon\theta^\varepsilon_{\alpha\alpha}) - \frac{2\pi\tau A_\mu}{L^\varepsilon}\partial_\alpha^{-1}(U^{\mathrm{s.t.},\varepsilon}H(\chi_\varepsilon\theta^\varepsilon_{\alpha\alpha\alpha})) + \frac{4\pi^3\tau}{(L^\varepsilon)^3}(\theta^\varepsilon_\alpha\chi_\varepsilon\theta^\varepsilon_\alpha)H(\chi_\varepsilon\theta^\varepsilon_{\alpha\alpha}).$$

The term Υ^ε_3 is the collection terms that are essentially transport terms:

$$\begin{aligned}\Upsilon^\varepsilon_3 = &-\frac{4\pi^3\tau A_\mu}{(L^\varepsilon)^3}(\chi_\varepsilon\theta^\varepsilon_{\alpha\alpha})^2 + \frac{4\pi^3\tau A_\mu}{(L^\varepsilon)^3}\partial_\alpha^{-1}((\chi_\varepsilon\theta^\varepsilon_{\alpha\alpha})\chi_\varepsilon\theta^\varepsilon_{\alpha\alpha\alpha}) + \frac{2\pi\tau}{L^\varepsilon}(V^{\mathrm{s.t.},\varepsilon} - \mathbf{W}^{\mathrm{s.t.},\varepsilon}\cdot\hat{\mathbf{t}}^\varepsilon)\chi_\varepsilon\theta^\varepsilon_{\alpha\alpha}\\ &+\frac{2\pi\tau A_\mu^2}{L^\varepsilon}(\mathbf{W}^{\mathrm{s.t.},\varepsilon}\cdot\hat{\mathbf{t}}^\varepsilon)\chi_\varepsilon\theta^\varepsilon_{\alpha\alpha} + \frac{\pi A_\mu R}{L^\varepsilon}\sin(\chi_\varepsilon\theta^\varepsilon)\chi_\varepsilon\theta^\varepsilon_{\alpha\alpha}\\ &+\frac{2\pi A_\mu^2}{L^\varepsilon}(\widetilde{\mathbf{W}}^\varepsilon\cdot\hat{\mathbf{t}}^\varepsilon)\chi_\varepsilon\theta^\varepsilon_{\alpha\alpha} + \frac{2\pi}{L^\varepsilon}(\widetilde{V}^\varepsilon - \widetilde{\mathbf{W}}^\varepsilon\cdot\hat{\mathbf{t}}^\varepsilon)\chi_\varepsilon\theta^\varepsilon_{\alpha\alpha}.\end{aligned}$$

Of course, Υ^ε_4 consists of all remaining terms:

$$\begin{aligned}\Upsilon^\varepsilon_4 = &\frac{2\pi\tau A_\mu}{L^\varepsilon}H((\chi_\varepsilon\theta^\varepsilon_\alpha)\theta^\varepsilon_\alpha\mathbf{W}^{\mathrm{s.t.},\varepsilon}\cdot\hat{\mathbf{t}}^\varepsilon) - \frac{2\pi\tau A_\mu}{L^\varepsilon}H((\chi_\varepsilon\theta^\varepsilon_\alpha)\mathbf{m}^{\mathrm{s.t.},\varepsilon}\cdot\hat{\mathbf{n}}^\varepsilon)\\ &-\frac{4\pi^3\tau A_\mu}{(L^\varepsilon)^3}[H, \chi_\varepsilon\theta^\varepsilon_\alpha]H(\chi_\varepsilon\theta^\varepsilon_{\alpha\alpha\alpha}) - \frac{4\pi^3\tau A_\mu}{(L^\varepsilon)^3}\partial_\alpha^{-1}[H, \chi_\varepsilon\theta^\varepsilon_{\alpha\alpha}]H(\chi_\varepsilon\theta^\varepsilon_{\alpha\alpha\alpha})\\ &-\frac{2\pi\tau A_\mu}{L^\varepsilon}\partial_\alpha^{-1}H\left((\chi_\varepsilon\theta^\varepsilon_{\alpha\alpha})(-\theta^\varepsilon_\alpha(\mathbf{W}^{\mathrm{s.t.},\varepsilon}\cdot\hat{\mathbf{t}}^\varepsilon) + \mathbf{m}^{\mathrm{s.t.},\varepsilon}\cdot\hat{\mathbf{n}}^\varepsilon)\right) - \frac{2\pi\tau A_\mu}{L^\varepsilon}\partial_\alpha^{-1}[H, U^{\mathrm{s.t.},\varepsilon}]\chi_\varepsilon\theta^\varepsilon_{\alpha\alpha\alpha}\\ &+[H, k^\varepsilon]\chi_\varepsilon\theta^\varepsilon_{\alpha\alpha} + H(k^\varepsilon_\alpha\chi_\varepsilon\theta^\varepsilon_\alpha) - \frac{2\pi\tau}{L^\varepsilon}(\chi_\varepsilon\theta^\varepsilon_\alpha)\mathbf{m}^{\mathrm{s.t.},\varepsilon}\cdot\hat{\mathbf{t}}^\varepsilon + \frac{4\pi^3\tau}{(L^\varepsilon)^3}(\chi_\varepsilon\theta^\varepsilon_\alpha)[H, \theta^\varepsilon_\alpha]\chi_\varepsilon\theta^\varepsilon_{\alpha\alpha}\\ &+\frac{\tau L_t^{\mathrm{s.t.},\varepsilon}}{L^\varepsilon}\chi_\varepsilon^2\theta^\varepsilon_\alpha + \frac{2\pi A_\mu^2\tau}{L^\varepsilon}(\chi_\varepsilon\theta^\varepsilon_\alpha)\partial_\alpha(\mathbf{W}^{\mathrm{s.t.},\varepsilon}\cdot\hat{\mathbf{t}}^\varepsilon) + \frac{\pi A_\mu R}{L^\varepsilon}(\chi_\varepsilon\theta^\varepsilon_\alpha)^2\cos(\chi_\varepsilon\theta^\varepsilon)\\ &+\frac{2\pi A_\mu^2}{L^\varepsilon}(\chi_\varepsilon\theta^\varepsilon_\alpha)(\widetilde{\mathbf{W}}^\varepsilon\cdot\hat{\mathbf{t}}^\varepsilon)_\alpha + \frac{2\pi}{L^\varepsilon}(\chi_\varepsilon\theta^\varepsilon_\alpha)(\widetilde{V}^\varepsilon - \widetilde{\mathbf{W}}^\varepsilon\cdot\hat{\mathbf{t}}^\varepsilon)_\alpha + \frac{2\pi}{L^\varepsilon}(\mathbf{m}^\varepsilon\cdot\hat{\mathbf{n}}^\varepsilon)_\alpha - \frac{2\pi^3 A_\mu}{(L^\varepsilon)^3}H(\mathbf{m}^\varepsilon\cdot\hat{\mathbf{t}}^\varepsilon)_\alpha.\end{aligned} \tag{57}$$

Now, we differentiate (55) with respect to α and we also use again the notation $\Lambda = H\partial_\alpha$, arriving at our desired formula,

$$\theta^\varepsilon_{\alpha,t} = \chi_\varepsilon\left[-\frac{4\pi^3\tau}{(L^\varepsilon)^3}\Lambda^3(\chi_\varepsilon\theta^\varepsilon_{\alpha\alpha}) + \Upsilon^\varepsilon_5\Lambda(\chi_\varepsilon\theta^\varepsilon_{\alpha\alpha}) + \Upsilon^\varepsilon_6\chi_\varepsilon\theta^\varepsilon_{\alpha\alpha\alpha} + \Upsilon^\varepsilon_7\right]; \tag{58}$$

we will give the definitions of Υ^ε_5, Υ^ε_6, and Υ^ε_7 next.

First, Υ^ε_5 is deduced from the definition of Υ^ε_2, and is given by

$$\Upsilon^\varepsilon_5 = k^\varepsilon - \frac{2\pi\tau A_\mu}{L^\varepsilon}U^{\mathrm{s.t.},\varepsilon} + \frac{4\pi^3\tau}{(L^\varepsilon)^3}\theta^\varepsilon_\alpha\chi_\varepsilon\theta^\varepsilon_\alpha. \tag{59}$$

Next, we have the formula for Υ_6^ε, which is deduced from Υ_3^ε :

$$\Upsilon_6^\varepsilon = -\frac{4\pi^3\tau A_\mu}{(L^\varepsilon)^3}\chi_\varepsilon\theta_{\alpha\alpha}^\varepsilon + \frac{2\pi\tau}{L^\varepsilon}(V^{\text{s.t.},\varepsilon} - \mathbf{W}^{\text{s.t.},\varepsilon}\cdot\hat{\mathbf{t}}^\varepsilon) + \frac{2\pi\tau A_\mu^2}{L^\varepsilon}\mathbf{W}^{\text{s.t.},\varepsilon}\cdot\hat{\mathbf{t}}^\varepsilon$$
$$+\frac{\pi A_\mu R}{L^\varepsilon}\sin(\chi_\varepsilon\theta^\varepsilon) + \frac{2\pi A_\mu^2}{L^\varepsilon}\widetilde{\mathbf{W}}^\varepsilon\cdot\hat{\mathbf{t}}^\varepsilon + \frac{2\pi}{L^\varepsilon}(\widetilde{V}^\varepsilon - \widetilde{\mathbf{W}}^\varepsilon\cdot\hat{\mathbf{t}}^\varepsilon).$$

Once again, all of the remaining terms are incorporated into the final term, which is now Υ_7^ε. The remaining terms include the derivative of Υ_4^ε, in addition to leftover terms from the derivatives of Υ_2^ε and Υ_3^ε:

$$\Upsilon_7^\varepsilon = \Upsilon_{4,\alpha}^\varepsilon + k_\alpha^\varepsilon H(\chi_\varepsilon\theta_{\alpha\alpha}^\varepsilon) + \frac{4\pi^3\tau}{(L^\varepsilon)^3}(H\chi_\varepsilon\theta_{\alpha\alpha}^\varepsilon)\partial_\alpha(\theta_\alpha^\varepsilon\chi_\varepsilon\theta_\alpha^\varepsilon) + \frac{2\pi\tau}{L^\varepsilon}(\chi_\varepsilon\theta_{\alpha\alpha}^\varepsilon)\partial_\alpha(V^{\text{s.t.},\varepsilon} - \mathbf{W}^{\text{s.t.},\varepsilon}\cdot\hat{\mathbf{t}}^\varepsilon)$$
$$+\frac{2\pi\tau A_\mu^2}{L^\varepsilon}(\chi_\varepsilon\theta_{\alpha\alpha}^\varepsilon)\partial_\alpha(\mathbf{W}^{\text{s.t.},\varepsilon}\cdot\hat{\mathbf{t}}^\varepsilon) + \frac{\pi A_\mu R}{L^\varepsilon}(\chi_\varepsilon\theta_{\alpha\alpha}^\varepsilon)(\chi_\varepsilon\theta_\alpha^\varepsilon)\cos(\chi_\varepsilon\theta^\varepsilon)$$
$$+\frac{2\pi A_\mu^2}{L^\varepsilon}(\chi_\varepsilon\theta_{\alpha\alpha}^\varepsilon)\partial_\alpha(\widetilde{\mathbf{W}}^\varepsilon\cdot\hat{\mathbf{t}}^\varepsilon) + \frac{2\pi}{L^\varepsilon}(\chi_\varepsilon\theta_{\alpha\alpha}^\varepsilon)\partial_\alpha(\widetilde{V}^\varepsilon - \widetilde{\mathbf{W}}^\varepsilon\cdot\hat{\mathbf{t}}^\varepsilon).$$

5.4. The Energy Estimate with $\tau > 0$

Let $\bar{d}_1, \bar{d}_2$, and $\bar{d}_3$ be positive numbers. Let the open set $\mathcal{O} \subseteq H^s(X)$ be defined as the subset of $H^s(X)$ such that for all $\theta \in \mathcal{O}$, the following conditions hold:

$$\|\theta\|_s < \bar{d}_1, \qquad L < \bar{d}_2, \qquad |q_1[z_d](\alpha,\alpha')| = \left|\frac{z_d[\theta](\alpha) - z_d[\theta](\alpha')}{\alpha-\alpha'}\right| > \bar{d}_3,\ \forall\alpha,\alpha'. \tag{60}$$

Notice that for a typical θ, it will not be the case that $\langle\langle\sin(\theta)\rangle\rangle = 0$, but we would like the curve which we construct from θ to be 2π-periodic. Therefore, we use the method of constructing a curve outlined in (39), which applies a projection in order to remove the mean of $\sin(\theta)$. Also, the length L used in (60) is the length defined by (38).

As in Sect. 2, for convenience, we will denote the domain for the spatial variable, α, by X. Of course, X is the periodic interval $[0, 2\pi]$.

Given this open set, we are able to use the Picard theorem to prove existence of solutions of the initial value problem, with the solutions in the set $\mathcal{O}$ at each time. (Note that the particular version of the Picard theorem being used is Theorem 3.1 from [42]; closely related versions of the Picard theorem can be found, for instance, in [35] or [56].) To verify the hypotheses of the Picard theorem, we need to know that the right-hand side of the evolution equation maps into the space H^s, and is Lipschitz. These properties are not difficult to establish for the mollified equation, and we do not provide the details here. The conclusion of our application of the Picard theorem is the following:

Lemma 7. *Let $\tau > 0$ and $\varepsilon > 0$ be given. Let $\theta_0 \in \mathcal{O}$ satisfying $\langle\langle\sin(\theta_0)\rangle\rangle = 0$ be given. Then, there exists $T^\varepsilon > 0$ and $\theta^\varepsilon \in C^1((-T^\varepsilon, T^\varepsilon); \mathcal{O})$ such that for all $t \in (-T^\varepsilon, T^\varepsilon)$, the equation (40) is satisfied by θ^ε, and such that $\theta^\varepsilon(\cdot, 0) = \theta_0$.*

We remark again that since these solutions θ^ε satisfy (40), it is the case that $\langle\langle\sin(\theta^\varepsilon)\rangle\rangle = 0$ at positive times. We now seek to establish that these solutions θ^ε all exist on a common time interval, and to this end, we prove an energy estimate. Before doing this, it is helpful to have a lemma about the regularity of the various Υ_i.

Lemma 8. *Let θ^ε in $\mathcal{O}$ be given, such that $\langle\langle\sin(\theta^\varepsilon)\rangle\rangle = 0$. We have $\Upsilon_4^\varepsilon \in H^{s-1}$, with the norm bounded uniformly with respect to $\tau \in [0, 1]$ and $\varepsilon \in [0, 1]$. For $i \in \{5, 6, 7\}$, we have $\Upsilon_i^\varepsilon \in H^{s-2}$, with the norms bounded uniformly with respect to $\tau \in [0, 1]$ and $\varepsilon \in [0, 1]$.*

Proof. We will not carry out every detail of the proof of this lemma, but we will give the idea. We begin by noting that $\frac{1}{L^\varepsilon}$ is bounded above, which can be seen from (44).

For Υ_4^ε, the proof is lengthy, but straightforward. As can be seen in (57), there are many terms on the right-hand side of the definition of Υ_4^ε. Using (57), we write $\Upsilon_4^\varepsilon = \sum_{j=1}^{17} \Xi_j$; each of the 17 terms corresponds to one of the terms on the right-hand side of (57) in the straightforward way.

To begin, we have from the Sobolev algebra property,

$$\|\Xi_1\|_{s-1} = \left\| \frac{2\pi\tau A_\mu}{L^\varepsilon} H((\chi_\varepsilon\theta_\alpha^\varepsilon)\theta_\alpha^\varepsilon \mathbf{W}^{\mathrm{s.t.},\varepsilon} \cdot \hat{\mathbf{t}}^\varepsilon) \right\|_{s-1} \le c\|\theta^\varepsilon\|_s^2 \|\mathbf{W}^{\mathrm{s.t.},\varepsilon} \cdot \hat{\mathbf{t}}^\varepsilon\|_{s-1}.$$

An estimate for $\mathbf{W}^{\mathrm{s.t.},\varepsilon} \cdot \hat{\mathbf{t}}^\varepsilon$ can be proved along the same lines as in Lemma 6 and Remark 7. This implies that $\|\Xi_1\|_{s-1}$ is bounded in terms of $\bar{d}_1$. For Ξ_2, we have

$$\|\Xi_2\|_{s-1} = \left\| -\frac{2\pi\tau A_\mu}{L^\varepsilon} H((\chi_\varepsilon\theta_\alpha^\varepsilon)\mathbf{m}^{\mathrm{s.t.},\varepsilon} \cdot \hat{\mathbf{n}})^\varepsilon \right\|_{s-1} \le c\|\theta^\varepsilon\|_s (1 + \|\theta^\varepsilon\|_{s-1}\|)\mathbf{m}^{\mathrm{s.t.},\varepsilon}\|_{s-1}.$$

An estimate for $\mathbf{m}^{\mathrm{s.t},\varepsilon}$ (and also an estimate for $\widetilde{\mathbf{m}}^\varepsilon$, although it is not needed for Ξ_2) can be proven completely analogously to (32), and these imply that Ξ_2 is bounded in H^{s-1} in terms of $\bar{d}_1$.

For Ξ_3 and Ξ_4, we need to use the commutator estimates of Lemma 4. We begin with the estimate for Ξ_3 :

$$\|\Xi_3\|_{s-1} = \left\| -\frac{4\pi^3\tau A_\mu}{(L^\varepsilon)^3}[H, \chi_\varepsilon\theta_\alpha^\varepsilon] H(\chi_\varepsilon\theta_{\alpha\alpha\alpha}^\varepsilon) \right\|_{s-1} \le c\|\chi_\varepsilon\theta_\alpha^\varepsilon\|_{s-1}\|\chi_\varepsilon\theta_{\alpha\alpha\alpha}^\varepsilon\|_{s-3} \le c\|\theta^\varepsilon\|_s^2 \le c\bar{d}_1^2.$$

(Notice that we used Lemma 4 with $n = s - 1$ and $k = 2$; the lemma requires $n \ge 2k$, so we must have $s \ge 5$.) We turn to Ξ_4:

$$\begin{aligned} \|\Xi_4\|_{s-1} &= \left\| -\frac{4\pi^3\tau A_\mu}{(L^\varepsilon)^3} \partial_\alpha^{-1}[H, \chi_\varepsilon\theta_{\alpha\alpha}^\varepsilon] H(\chi_\varepsilon\theta_{\alpha\alpha\alpha}^\varepsilon) \right\|_{s-1} \le c\, \|[H, \chi_\varepsilon\theta_{\alpha\alpha}^\varepsilon] H(\chi_\varepsilon\theta_{\alpha\alpha\alpha}^\varepsilon)\|_{s-2} \\ &\le c\|\chi_\varepsilon\theta_{\alpha\alpha}^\varepsilon\|_{s-2}\|\chi_\varepsilon\theta_{\alpha\alpha\alpha}^\varepsilon\|_{s-3} \le c\bar{d}_1^2. \end{aligned}$$

Here, we used Lemma 4 with $n = s - 2$ and $k = 1$.

We have the following definition of Ξ_5 :

$$\Xi_5 = -\frac{2\pi\tau A_\mu}{L^\varepsilon} \partial_\alpha^{-1} H\left((\chi_\varepsilon\theta_{\alpha\alpha}^\varepsilon)(-\theta_\alpha^\varepsilon(\mathbf{W}^{\mathrm{s.t.},\varepsilon} \cdot \hat{\mathbf{t}}^\varepsilon) + \mathbf{m}^{\mathrm{s.t.},\varepsilon} \cdot \hat{\mathbf{n}}^\varepsilon) \right).$$

This can be bounded in terms of $\bar{d}_1$ because of the presence of the ∂_α^{-1} operator, and also because of the previously discussed bounds for $\mathbf{W}^{\mathrm{s.t.},\varepsilon} \cdot \hat{\mathbf{t}}^\varepsilon$ and $\mathbf{m}^{\mathrm{s.t.},\varepsilon}$.

We next have the following, which again uses Lemma 4:

$$\|\Xi_6\|_{s-1} = \left\| -\frac{2\pi\tau A_\mu}{L^\varepsilon} \partial_\alpha^{-1}[H, U^{\mathrm{s.t.},\varepsilon}]\chi_\varepsilon\theta_{\alpha\alpha}^\varepsilon \right\|_{s-1} \le c\left\| [H, U^{\mathrm{s.t.},\varepsilon}]\chi_\varepsilon\theta_{\alpha\alpha}^\varepsilon \right\|_{s-2} \le c\|U^{\mathrm{s.t.},\varepsilon}\|_{s-2}\|\theta^\varepsilon\|_s.$$

Using (48), we can easily bound $\|U^{\mathrm{s.t.},\varepsilon}\|_{s-2}$ in terms of $\bar{d}_1$, so we can thus bound Ξ_6 in terms of $\bar{d}_1$ as well.

Both of Ξ_7 and Ξ_8 involve k^ε. To begin to estimate these, we note that we can write

$$\Phi(\widetilde{\mathbf{W}}^\varepsilon)^* = \frac{1}{2i} H\left(\frac{\widetilde{\gamma}[\theta^\varepsilon]}{z_{d,\alpha}^\varepsilon} \right) + \mathcal{K}[z_d^\varepsilon](\widetilde{\gamma}[\theta^\varepsilon]),$$

similarly to the formula (28). Using Lemmas 1 and 6, and the definition $\widetilde{U}^\varepsilon = \widetilde{\mathbf{W}}^\varepsilon \cdot \hat{\mathbf{n}}^\varepsilon$, we see that $\widetilde{U}^\varepsilon$ is bounded in H^s in terms of $\bar{d}_1$. From (45), which is the definition of k^ε, we see then that k^ε is bounded in H^s in terms of $\bar{d}_1$. For Ξ_7, we can use Lemma 4 again, yielding the following:

$$\|\Xi_7\|_{s-1} = \|[H, k^\varepsilon]\chi_\varepsilon\theta^\varepsilon_{\alpha\alpha}\|_{s-1} \le c\|k^\varepsilon\|_{s-1}\|\theta^\varepsilon\|_s.$$

Given the above discussion on the boundedness of k^ε, we see that Ξ_7 is bounded in terms of $\bar{d}_1$. Next, we see from the above discussion and the following definition of Ξ_8 that it is bounded in H^{s-1} in terms of $\bar{d}_1$:

$$\Xi_8 = H(k^\varepsilon_\alpha\chi_\varepsilon\theta^\varepsilon_\alpha).$$

We will stop providing details at this point, but the rest of the proof for Υ^ε_4, and indeed, the rest of the proof of the lemma, follows in the same fashion. □

As in Sect. 2, we let the energy E be defined as

$$E(t) = \frac{1}{2}\int\limits_X (\theta^\varepsilon(\alpha,t))^2 + (\partial^s_\alpha\theta^\varepsilon(\alpha,t))^2\ d\alpha.$$

Of course, this energy is the square of a norm for θ^ε in the space H^s, and this norm is equivalent to the usual one. We are now ready for the energy estimate:

Lemma 9. *Let $\tau > 0$ and $\varepsilon > 0$ be given. Let $\theta^\varepsilon \in C([0,T];\mathcal{O})$ solve (40), where T may depend on both ε and τ. Then, there exists constants $c_1 > 0, c_2 > 0$, and $c_3 > 0$ depending only on $s, \bar{d}_1, \bar{d}_2$, and $\bar{d}_3$, and there exists $\bar{C} > 0$ which can also depend on τ, such that for all $t \in [0,T]$, we have*

$$\frac{dE}{dt} \le c_1\exp\{c_2E\} + \bar{C}E - c_3\tau\int\limits_X (\Lambda^{3/2}\partial^s_\alpha\chi_\varepsilon\theta^\varepsilon)^2\ d\alpha. \tag{61}$$

Proof. Clearly, we have

$$\frac{dE}{dt} = \int\limits_X \theta^\varepsilon\theta^\varepsilon_t + (\partial^s_\alpha\theta^\varepsilon)\partial^s_\alpha\theta^\varepsilon_t\ d\alpha = \int\limits_X \theta^\varepsilon\theta^\varepsilon_t + (\partial^s_\alpha\theta^\varepsilon)\partial^{s-2}_\alpha\theta^\varepsilon_{\alpha\alpha,t}\ d\alpha.$$

Since s has been taken to be sufficiently large, we can see that $\|\theta^\varepsilon_t\|_0$ is bounded in terms of the energy; so,

$$\int\limits_X \theta^\varepsilon\theta^\varepsilon_t\ d\alpha \le c_1\exp\{c_2E\}. \tag{62}$$

Using (58), and using that fact that χ_ε is self-adjoint, we calculate the following:

$$\int\limits_X (\partial^s_\alpha\theta^\varepsilon)\partial^{s-2}_\alpha\theta^\varepsilon_{\alpha\alpha,t}\ d\alpha = -\frac{4\pi^3\tau}{(L^\varepsilon)^3}\int\limits_X (\partial^s_\alpha\chi_\varepsilon\theta^\varepsilon)(\partial^s_\alpha\Lambda^3\chi_\varepsilon\theta)\ d\alpha + \int\limits_X (\partial^s_\alpha\chi_\varepsilon\theta^\varepsilon)\partial^{s-2}_\alpha\left(\Upsilon^\varepsilon_5\Lambda(\chi_\varepsilon\theta^\varepsilon_{\alpha\alpha})\right)\ d\alpha$$
$$+\int\limits_X (\partial^s_\alpha\chi_\varepsilon\theta^\varepsilon)\partial^{s-2}_\alpha\left(\Upsilon^\varepsilon_6\chi_\varepsilon\theta^\varepsilon_{\alpha\alpha\alpha}\right)\ d\alpha + \int\limits_X (\partial^s_\alpha\chi_\varepsilon\theta^\varepsilon)\partial^{s-2}_\alpha\Upsilon^\varepsilon_7\ d\alpha. \tag{63}$$

We estimate each of the integrals on the right-hand side of (63). To begin, we notice from Lemma 8 that we can easily estimate the last integral on the right-hand side of (63):

$$\int\limits_X (\partial^s_\alpha\chi_\varepsilon\theta^\varepsilon)\partial^{s-2}_\alpha\Upsilon^\varepsilon_7\ d\alpha \le \|\theta^\varepsilon\|_s\|\Upsilon^\varepsilon_7\|_{s-2} \le c_1\exp\{c_2E\}. \tag{64}$$

For the third integral on the right-hand side of (63), we use the product rule in order to expand $\partial^{s-2}_\alpha(\Upsilon^\varepsilon_6\chi_\varepsilon\theta^\varepsilon_{\alpha\alpha\alpha})$:

$$\int\limits_X (\partial^s_\alpha\chi_\varepsilon\theta^\varepsilon)\partial^{s-2}_\alpha\left(\Upsilon^\varepsilon_6\chi_\varepsilon\theta^\varepsilon_{\alpha\alpha\alpha}\right)\ d\alpha = \int\limits_X \Upsilon^\varepsilon_6(\partial^s_\alpha\chi_\varepsilon\theta^\varepsilon)(\partial^{s+1}_\alpha\chi_\varepsilon\theta^\varepsilon)\ d\alpha$$
$$+\sum_{j=1}^{s-2}\binom{s-2}{j}\int\limits_X (\partial^s_\alpha\chi_\varepsilon\theta^\varepsilon)(\partial^j_\alpha\Upsilon^\varepsilon_6)(\partial^{s+1-j}_\alpha\chi_\varepsilon\theta^\varepsilon)\ d\alpha. \tag{65}$$

The first integral on the right-hand side of (65) can be integrated by parts, since $(\partial_\alpha^s\chi_\varepsilon\theta^\varepsilon)\partial_\alpha^{s+1}\chi_\varepsilon\theta^\varepsilon = \frac{1}{2}\partial_\alpha\left((\partial_\alpha^s\chi_\varepsilon\theta^\varepsilon)^2\right)$. After performing this integration by parts, the resulting integral is bounded in terms of the energy. Furthermore, all of the integrals in the sum on the right-hand side of (65) are bounded in terms of the energy. For both of these bounds, we have used Lemma 8. So, we have established

$$\int_X (\partial_\alpha^s\chi_\varepsilon\theta^\varepsilon)\partial_\alpha^{s-2}\left(\Upsilon_6^\varepsilon\chi_\varepsilon\theta_{\alpha\alpha\alpha}^\varepsilon\right)\,d\alpha \le c_1\exp\{c_2E\}. \tag{66}$$

The other two integrals on the right-hand side of (63) require more careful attention. We begin with the second of these, which we again expand with the product rule:

$$\int_X (\partial_\alpha^s\chi_\varepsilon\theta^\varepsilon)\partial_\alpha^{s-2}(\Upsilon_5^\varepsilon\Lambda(\chi_\varepsilon\theta_{\alpha\alpha}^\varepsilon))\,d\alpha = \int_X \Upsilon_5^\varepsilon(\partial_\alpha^s\chi_\varepsilon\theta^\varepsilon)\Lambda(\partial_\alpha^s\chi_\varepsilon\theta^\varepsilon)\,d\alpha + \sum_{j=1}^{s-2}\binom{s-2}{j}\int_X (\partial_\alpha^s\chi_\varepsilon\theta^\varepsilon)(\partial_\alpha^j\Upsilon_5^\varepsilon)\Lambda(\partial_\alpha^{s-j}\chi_\varepsilon\theta^\varepsilon)\,d\alpha. \tag{67}$$

For the first integral on the right-hand side of (67), we estimate it with Young's Inequality:

$$\int_X \Upsilon_5^\varepsilon(\partial_\alpha^s\chi_\varepsilon\theta^\varepsilon)\Lambda(\partial_\alpha^s\chi_\varepsilon\theta^\varepsilon)\,d\alpha \le \frac{1}{2}\int_X (\Upsilon_5^\varepsilon)^2(\partial_\alpha^s\chi_\varepsilon\theta^\varepsilon)^2\,d\alpha + \frac{1}{2}\int_X (\partial_\alpha^{s+1}\chi_\varepsilon\theta^\varepsilon)^2\,d\alpha.$$

By Lemma 8, the first of these is bounded in terms of the energy; furthermore, all of the integrals in the sum on the right-hand side of (67) are bounded in terms of the energy. These considerations yield the following bound:

$$\int_X (\partial_\alpha^s\chi_\varepsilon\theta^\varepsilon)\partial_\alpha^{s-2}(\Upsilon_5^\varepsilon\Lambda(\chi_\varepsilon\theta_{\alpha\alpha}^\varepsilon))\,d\alpha \le c_1\exp\{c_2E\} + \frac{1}{2}\int_X (\partial_\alpha^{s+1}\chi_\varepsilon\theta^\varepsilon)^2\,d\alpha. \tag{68}$$

We now consider the first integral on the right-hand side of (63). We simply rewrite this integral, using the fact that the operator $\Lambda^{3/2}$ is self-adjoint:

$$-\frac{4\pi^3\tau}{(L^\varepsilon)^3}\int_X (\partial_\alpha^s\chi_\varepsilon\theta^\varepsilon)\partial_\alpha^{s-2}\Lambda^3(\chi_\varepsilon\theta^\varepsilon)\,d\alpha = -\frac{4\pi^3\tau}{(L^\varepsilon)^3}\int_X (\Lambda^{3/2}\partial_\alpha^s\chi_\varepsilon\theta^\varepsilon)^2\,d\alpha. \tag{69}$$

We now add the inequalities (64), (66), (68), and (69), using these with (63). This yields the inequality

$$\int_X (\partial_\alpha^s\theta^\varepsilon)\partial_\alpha^{s-2}\theta_{\alpha\alpha,t}^\varepsilon\,d\alpha \le c_1\exp\{c_2E\} + \int_X \left(-\frac{4\pi^3\tau}{(L^\varepsilon)^3}(\Lambda^{3/2}(\partial_\alpha^s\chi_\varepsilon\theta^\varepsilon))^2\right) + \frac{1}{2}(\partial_\alpha\partial_\alpha^s\chi_\varepsilon\theta^\varepsilon)^2\,d\alpha. \tag{70}$$

Similarly to the estimate of Case 3 of the example in Sect. 2, we let $v = \partial_\alpha^s\chi_\varepsilon\theta^\varepsilon$, and we rewrite the integral on the right-hand side of (70) by using the Plancherel theorem. We also add (70) with (62):

$$\frac{dE}{dt} \le c_1\exp\{c_2E\} + \sum_{\xi=-\infty}^{\infty}\left(-\frac{4\pi^3\tau}{(L^\varepsilon)^3}|\xi|^3 + \frac{1}{2}|\xi|^2\right)|\hat{v}(\xi)|^2.$$

From the definition of the open set $\mathcal{O}$, we know $L^\varepsilon < \bar{d}_2$, and therefore

$$-\frac{4\pi^3\tau}{(L^\varepsilon)^3} < -\frac{4\pi^3\tau}{\bar{d}_2^3}.$$

Now, there exists $\bar{C}(\tau) > 0$ such that for all $\xi \in \mathbb{Z}$, we have

$$-\frac{2\pi^3\tau}{\bar{d}_2^3}|\xi|^3 + \frac{1}{2}|\xi|^2 \le \bar{C}(\tau).$$

Notice that by the Plancherel theorem and the definition of E, and by the inequality $\|\chi_\varepsilon\theta^\varepsilon\|_s \leq \|\theta^\varepsilon\|_s$, the sum $\sum_{\xi=-\infty}^{\infty} |\hat{v}(\xi)|^2$ is bounded by $2E$. We use the Plancherel theorem once more, and we conclude that we have proven that

$$\frac{dE}{dt} \leq c_1 \exp\{c_2 E\} + 2\bar{C}(\tau)E - \frac{2\pi^3\tau}{\bar{d}_2^3} \int\limits_X (\Lambda^{3/2}\partial_\alpha^s \chi_\varepsilon \theta^\varepsilon)^2 \, d\alpha.$$

This completes the proof of the lemma. □

Remark 10. While the constants c_1 and c_2 are independent of τ, it is clear that $\lim_{\tau\to 0^+} \bar{C}(\tau) = +\infty$. Therefore, the present estimate is certainly not uniform in τ.

Remark 11. The integral on the right-hand side of (61) demonstrates a version of the expected parabolic smoothing for this problem. By integrating (61) in time, because of the presence of this integral, we see that the mollified solution of the approximate evolution equation, $\chi_\varepsilon\theta^\varepsilon$, is in fact in the space $L^2([0,T];H^{s+3/2})$, with the norm in this space bounded independently of ε. Of course, $\chi_\varepsilon\theta^\varepsilon$ is in the space $H^{s+3/2}$ pointwise in time because of the presence of the mollifier, but without this estimate, the norm in $H^{s+3/2}$ would depend badly on ε. (We mention that we have not yet shown that T can be taken to be independent of ε, but we will do this next.)

5.5. Existence of Solutions with $\tau > 0$

Thus far, we have proved the existence of a solution θ^ε to the initial value problem associated to the mollified evolution equation, for any $\varepsilon > 0$, and this solution exists on a time interval $[0, T^\varepsilon]$. Furthermore, we have proved that the time derivative of the H^s-norm of θ^ε is bounded, independently of ε. Our next task is to combine these facts to find that the solutions θ^ε all exist on a common time interval; this is the content of our next lemma.

Lemma 10. *Let θ^ε be as in Lemma 7. There exists $T_* > 0$ such that for all $\varepsilon > 0, \theta^\varepsilon$ is a solution of (40) on the time interval $[0, T_*]$, and $\theta^\varepsilon \in C([0,T_*];\mathcal{O}) \cap C^1([0,T_*];H^{s-3})$.*

Proof. By the continuation theorem for autonomous differential equations on Banach spaces (see Theorem 3.3 of [42]), the solution θ^ε can be continued so long as it does not leave the set $\mathcal{O}$. There are three conditions in (60) defining the set $\mathcal{O}$, and we must check that these cannot be violated arbitrarily quickly.

For any given $\varepsilon > 0$, define T_*^ε to be the maximal time of existence for θ^ε in the set $\mathcal{O}$. Assume that the solutions can leave the set $\mathcal{O}$ arbitrarily quickly. This means there exists a sequence $\varepsilon_n > 0$ such that $\varepsilon_n \to 0$ and such that $T_*^{\varepsilon_n} \to 0$ as $n \to \infty$. The uniform bound of the previous section, however, immediately implies that $\|\theta^\varepsilon\|_s$ cannot reach the value $\bar{d}_1$ arbitrarily fast (that is, if it could reach the value d_1 arbitrarily fast, then the time derivative of E would have to be able to become arbitrarily large, which is ruled out by the estimate). Similarly, if L^ε were to become equal to $\bar{d}_2$ arbitrarily fast, then the time derivative, L_t^ε, would need to be arbitrarily large, but again, this is not the case. Similarly, the energy also controls the time derivative of the chord-arc quantity, q_1, so the chord-arc condition in (60) cannot be violated arbitrarily fast. Such a sequence ε_n is therefore seen to be impossible. We conclude that there exists $T_* > 0$ such that for all $\varepsilon > 0$, we have $\theta^\varepsilon \in C([0,T_*],\mathcal{O})$. □

Remark 12. Following up on Remark 11, we see that the mollified solutions of the approximate evolution equation, $\chi_\varepsilon\theta^\varepsilon$, are bounded in $L^2([0,T_*];H^{s+3/2})$, and the bound is uniform with respect to ε.

We are now able to prove the existence of a limiting solution, θ.

Theorem 11. *Let $\theta_0 \in \mathcal{O}$ satisfy $\langle\langle \sin(\theta_0)\rangle\rangle = 0$. Let $T_* > 0$ be as in Lemma 10. Then there exists $\theta \in C([0,T_*];\bar{\mathcal{O}}) \cap C^1([0,T_*];H^{s-3})$ such that $\theta(\cdot,0) = \theta_0$ and such that θ satisfies (20).*

Proof. From the uniform bound on the H^s-norm of the solutions in the definition of $\mathcal{O}$, we conclude that $|\theta_t^\varepsilon|_\infty$ and $|\theta_\alpha^\varepsilon|_\infty$ are uniformly bounded, with respect to both ε and t, over the interval $[0, T_*]$. Therefore, θ^ε forms an equicontinuous family. By the Arzela-Ascoli theorem, there exists a subsequence (which we do not relabel) and a limit, θ, such that $\theta^\varepsilon \to \theta$ in $C(X \times [0, T_*])$. This implies $\theta^\varepsilon \to \theta$ in $C([0, T]; H^0)$, and using the uniform H^s-estimate together with the elementary interpolation inequality (22), this implies $\theta^\varepsilon \to \theta$ in $C([0, T_*]; H^{s'})$, for any s' satisfying $0 \le s' < s$.

Next, we establish that the limiting solution satisfies the unmollified evolution equation. Recall that $\mathfrak{B}^\varepsilon$ is defined in (41), and that $\mathfrak{B}$ is defined to be the right-hand side of (20). We write

$$\theta^\varepsilon(\alpha, t) \;=\; \theta_0(\alpha) + \int_0^t \theta_t^\varepsilon(\alpha, s)\ ds \;=\; \theta_0(\alpha) + \int_0^t \mathfrak{B}^\varepsilon(\alpha, s) + \mu^\varepsilon(s)\ ds.$$

Having established convergence in $H^{s'}$ for sufficiently large s', we are able to pass to the limit in this equation, finding

$$\theta(\alpha, t) = \theta_0(\alpha) + \int_0^t \lim_{\varepsilon \to 0} \theta_t^\varepsilon(\alpha, s)\ ds = \theta_0(\alpha) + \int_0^t \mathfrak{B} + \lim_{\varepsilon \to 0} \mu^\varepsilon\ ds.$$

We give the name μ to the limit of μ^ε; considering the discussion at the end of Sect. 5.1, we see that

$$\mu = -\frac{\int_0^{2\pi} \cos(\theta)\mathfrak{B}\ d\alpha}{4\pi^2 L} = \frac{L_t}{4\pi^2 L^2} \int_0^{2\pi} \sin(\theta)\ d\alpha.$$

As discussed at the end of Sect. 5.1, we recall that for all $\varepsilon > 0$, we have $\int_0^{2\pi} \sin(\theta^\varepsilon)\ d\alpha = 0$. We can thus pass to the limit to find $\int_0^{2\pi} \sin(\theta)\ d\alpha = 0$. We conclude that $\mu = 0$. This implies that the limit, θ, satisfies the appropriate evolution equation, $\theta_t = \mathfrak{B}$.

Finally, we remark on the highest regularity. The solutions of the mollified equation are in the space H^s at each time in $[0, T_*]$, uniformly bounded with respect to ε. This means that at each time, there is a subsequence which converges weakly in H^s, and this limit must be θ. Therefore, θ is in H^s pointwise in time. What remains is to show that $\theta \in C([0, T]; H^s)$; we do not include the details, but this can be done by adapting the corresponding argument for regularity of solutions for the Navier-Stokes equations in Chapter 3 of [42]. The steps are to first show that θ is weakly continuous in time with values in H^s; this follows easily from the uniform bound and the strong continuity in $H^{s'}$. Then, it is shown that the solution is strongly right-continuous in time at $t = 0$; this follows from the energy estimate and Fatou's Lemma. The final step is to use parabolic smoothing; in Remark 12, we see that $\chi_\varepsilon \theta^\varepsilon$ is uniformly bounded in the space $L^2([0, T_*]; H^{s+3/2})$. Since this is a Hilbert space, we see that our subsequence of $\chi_\varepsilon \theta^\varepsilon$ has a subsequence with a weak limit in this space, and this weak limit must be θ. The existence theory can then be repeated in higher regularity spaces starting from almost any positive time, t, with initial data $\theta(\cdot, t)$. Using the uniqueness theorem (which is Theorem 12 below), the solution starting from time t and the solution starting from time zero must be the same. It can then be concluded that the solution starting from time t is continuous in H^s (since H^s would no longer be the highest regularity), and we are able to do this for any arbitrarily small value of t. Together with the right-continuity at time zero, this argument implies $\theta \in C([0, T_*]; H^s)$. In addition to [42], we remark that the author has recently used this same argument for highest regularity in the papers [5] and [11]. □

5.6. Uniqueness and Continuous Dependence with $\tau > 0$

Theorem 12. *Let $\theta_0 \in \mathcal{O}$ and $\theta_1 \in \mathcal{O}$ be given, and assume that $\langle\langle \sin(\theta_i) \rangle\rangle = 0$ for $i \in \{0, 1\}$. The solution of the initial value problem (20) with $\theta(\cdot, 0) = \theta_0$ is unique. Moreover, if $T > 0$ such that $\theta \in C([0, T]; \mathcal{O})$*

is the solution corresponding to data θ_0 *and if* $\theta' \in C([0,T];\mathcal{O})$ *is the solution corresponding to data* θ_1*, then there exists* $c > 0$ *such that*

$$\sup_{t\in[0,T]} \|\theta - \theta'\|_1 \leq c\|\theta_0 - \theta_1\|_1.$$

Remark 13. The proof of this theorem is the same as the proof of Theorem 14 below, and so we do not include it here. In Theorem 14, we prove an estimate for the difference of two solutions, where the solutions correspond to two different values of the surface tension parameter. In the present case, the solutions would correspond to two different pieces of initial data. The estimate of the proof of Theorem 14 is more general, and thus implies the present result. Theorem 14 is for the H^1 norm of the difference of two solutions; the continuous dependence result of Theorem 12 can easily be extended to higher regularity by interpolation.

6. The Zero Surface Tension Limit

We revisit the energy estimate of Sect. 5.4 in the case that the stability condition is satisfied. This will allow us to show that the solutions found above exist on a uniform time interval. We will then be able to take the limit as τ vanishes.

6.1. Uniform Time of Existence

We define another open set, $\mathcal{O}_k$; this will be a subset of our previous open set, $\mathcal{O}$. We let $\bar{d}_4 > 0$ be given. Then, we make the definition

$$\mathcal{O}_k = \{f \in \mathcal{O} : \forall\alpha, k[f](\alpha) < -\bar{d}_4\}.$$

This inequality, that k be negative and uniformly bounded away from zero, is the stability condition that we discussed in the introduction; it is the nonlinear generalization of the condition of Saffman and Taylor [44]. We repeat the previous energy estimate, considering now $\theta \in \mathcal{O}_k$, and considering sufficiently small τ. Let us be precise about the τ to be considered. For any $f \in \mathcal{O}_k$, we have the bounds (60), since $\mathcal{O}_k \subseteq \mathcal{O}$. Considering the formula (46) and the definition of Υ_5^ε in (59), we see that if $\theta^\varepsilon \in \mathcal{O}_k$, then since $k[\theta^\varepsilon] < -\bar{d}_4$, and since $U^{\mathrm{s.t},\varepsilon}$ is bounded in terms of θ^ε (uniformly in ε) there exists $\tau^* \in (0,1)$ such that for all $\tau \in (0,\tau^*)$, we have the pointwise estimate $\Upsilon_5^\varepsilon < -\frac{\bar{d}_4}{2} < 0$.

Theorem 13. *Let* τ^* *be as above. Let* $\theta_0 \in \mathcal{O}_k$ *be given, such that* $\langle\langle\sin(\theta_0)\rangle\rangle = 0$*. Then there exists* $T > 0$ *such that for all* $\tau \in (0,\tau^*)$*, the solution* θ^τ *of the initial value problem given by (20) and the condition* $\theta^\tau(\cdot,0) = \theta_0$ *exists on* $[0,T]$*, with* $\theta^\tau \in C([0,T];\bar{\mathcal{O}}_k) \cap C^1([0,T];H^{s-3})$.

Proof. We perform the energy estimate in the same way as before, except for the term which includes Υ_5^ε. The exact nature of the difference is that previously, when estimating (67), we had used Young's Inequality, and now we will instead use the estimate $\Upsilon_5^\varepsilon < -\frac{\bar{d}_4}{2}$. We begin by naming the following integral I, and by using the formula $\Lambda = H\partial_\alpha$:

$$I = \int_X \Upsilon_5^\varepsilon(\partial_\alpha^s\chi_\varepsilon\theta^\varepsilon)\Lambda(\partial_\alpha^s\chi_\varepsilon\theta^\varepsilon)\ d\alpha = \int_X \Upsilon_5^\varepsilon(\partial_\alpha^s\chi_\varepsilon\theta^\varepsilon)\partial_\alpha H(\partial_\alpha^s\chi_\varepsilon\theta^\varepsilon)\ d\alpha.$$

Now, we write $\Upsilon_5^\varepsilon = -\left(\sqrt{-\Upsilon_5^\varepsilon}\right)^2$:

$$I = -\int_X \left(\sqrt{-\Upsilon_5^\varepsilon}\partial_\alpha^s\chi_\varepsilon\theta^\varepsilon\right)\left(\sqrt{-\Upsilon_5^\varepsilon}\partial_\alpha H\partial_\alpha^s\chi_\varepsilon\theta^\varepsilon\right)\ d\alpha.$$

We bring a factor of $\sqrt{-\Upsilon_5^{\varepsilon}}$ through ∂_α :

$$I=-\int\limits_X\left(\sqrt{-\Upsilon_5^{\varepsilon}}\partial_\alpha^s\chi_\varepsilon\theta^\varepsilon\right)\partial_\alpha\left(\sqrt{-\Upsilon_5^{\varepsilon}}H\partial_\alpha^s\chi_\varepsilon\theta^\varepsilon\right)\,d\alpha-\frac{1}{2}\int\limits_X\Upsilon_{5,\alpha}^{\varepsilon}(\partial_\alpha^s\chi_\varepsilon\theta^\varepsilon)(H\partial_\alpha^s\chi_\varepsilon\theta^\varepsilon)\,d\alpha.$$

Notice that the second integral on the right-hand side is bounded in terms of the energy. For the first integral on the right-hand side, we now pull a factor of $\sqrt{-\Upsilon_5^{\varepsilon}}$ through the Hilbert transform as well:

$$I=-\int\limits_X\left(\sqrt{-\Upsilon_5^{\varepsilon}}\partial_\alpha^s\chi_\varepsilon\theta^\varepsilon\right)\Lambda\left(\sqrt{-\Upsilon_5^{\varepsilon}}\partial_\alpha^s\chi_\varepsilon\theta^\varepsilon\right)\,d\alpha+\int\limits_X\left(\sqrt{-\Upsilon_5^{\varepsilon}}\partial_\alpha^s\chi_\varepsilon\theta^\varepsilon\right)\partial_\alpha[H,\sqrt{-\Upsilon_5^{\varepsilon}}](\partial_\alpha^s\chi_\varepsilon\theta^\varepsilon)\,d\alpha$$
$$-\frac{1}{2}\int\limits_X\Upsilon_{5,\alpha}^{\varepsilon}(\partial_\alpha^s\chi_\varepsilon\theta^\varepsilon)(H\partial_\alpha^s\chi_\varepsilon\theta^\varepsilon)\,d\alpha.$$

Now, the second integral on the right-hand side is also bounded in terms of the energy, because of the smoothing properties of these commutators. The first integral on the right-hand side, meanwhile, is non-positive; this is an important change from the previous estimate. Therefore, we are able to estimate I as

$$I\leq c_1\exp\{c_2E\}-\int\limits_X\left(\Lambda^{1/2}\left(\sqrt{-\Upsilon_5^{\varepsilon}}\partial_\alpha^s\chi_\varepsilon\theta^\varepsilon\right)\right)^2\,d\alpha.$$

This estimate completely obviates the need for Young's inequality, and (70) is replaced by

$$\int\limits_X(\partial_\alpha^s\theta^\varepsilon)\partial_\alpha^{s-2}\theta_{\alpha\alpha,t}^\varepsilon\,d\alpha\leq c_1\exp\{c_2E\}-\int\limits_X\frac{4\pi^3\tau}{(L^\varepsilon)^3}(\Lambda^{3/2}(\partial_\alpha^s\chi_\varepsilon\theta^\varepsilon))^2\,d\alpha$$
$$-\int\limits_X\left(\Lambda^{1/2}\left(\sqrt{-\Upsilon_5^{\varepsilon}}\partial_\alpha^s\chi_\varepsilon\theta^\varepsilon\right)\right)^2\,d\alpha.$$

As before, since $L^\varepsilon<\bar{d}_2$, we have the simple inequality

$$-\frac{4\pi^3}{(L^\varepsilon)^3}<-\frac{4\pi^3}{\bar{d}_2^3}.$$

We therefore conclude that

$$\frac{dE}{dt}\leq c_1\exp\{c_2E\}-\frac{4\pi^3\tau}{\bar{d}_2^3}\int\limits_X(\Lambda^{3/2}\partial_\alpha^s\chi_\varepsilon\theta^\varepsilon)^2\,d\alpha-\int\limits_X\left(\Lambda^{1/2}\left(\sqrt{-\Upsilon_5^{\varepsilon}}\partial_\alpha^s\chi_\varepsilon\theta^\varepsilon\right)\right)^2\,d\alpha,\tag{71}$$

where c_1 and c_2 are independent of $\tau\in(0,\tau^*)$. □

Remark 14. As in Remarks 11 and 12, we can see parabolic smoothing from this estimate, although we now have two different ways to see this. With $\tau>0$, we consider the first integral on the right-hand side of (71), and we conclude that $\chi_\varepsilon\theta^\varepsilon$ is in $L^2([0,T_*];H^{s+3/2})$, with the norm in this space bounded independently of ε. Our estimate of this norm depends badly on τ. We are able to see smoothing independent of τ, however, if we use the second integral on the right-hand side of (71). We can conclude that $\chi_\varepsilon\theta^\varepsilon$ is in $L^2([0,T_*];H^{s+1/2})$, with the norm in this space bounded independently of ε and independently of τ.

6.2. Cauchy Sequence as $\tau\to 0$

We now consider the behavior of solutions as τ vanishes. We therefore need to consider solutions with different values of the parameter τ, and we will use superscripts for this purpose. That is, given some $\tau>0$, the solution of the initial value problem for the non-mollified equation (20) will be denoted by θ^τ, and related quantites based on θ^τ will be denoted, for example, as $\mathbf{W}^\tau$ or L^τ. In our next theorem, we demonstrate that as τ vanishes, any sequence of θ^τ forms a Cauchy sequence in the space $C([0,T];H^1)$.

Theorem 14. *Let $\theta_0 \in \mathcal{O}_k$ be given, such that $\langle\!\langle \sin(\theta_0) \rangle\!\rangle = 0$. Let $T > 0$ be as in Theorem 13. Let τ and τ' be in $(0, \tau^*)$. Let θ^τ and $\theta^{\tau'}$ be the corresponding solutions of the initial value problem (20) with initial data θ_0, with the solutions valid over the time interval $[0, T]$, as in Theorem 13. For any $\eta > 0$, there exists $\delta > 0$ such that if $|\tau - \tau'| < \delta$, then*

$$\sup_{t\in[0,T]} \|\theta^\tau - \theta^{\tau'}\|_1 < \eta.$$

Proof. We need to show that we can make $\|\theta^\tau(\cdot, t) - \theta^{\tau'}(\cdot, t)\|_1$ arbitrarily small, uniformly in $t \in [0, T]$, by taking τ and τ' sufficiently close together. In order to estimate $\|\theta^\tau - \theta^{\tau'}\|_1$, we first need a good expression for $(\theta^\tau - \theta^{\tau'})_t$. To this end, we begin by substituting from (18), and we write the result as

$$(\theta^\tau - \theta^{\tau'})_t = B_1 + B_2 + B_3 + B_4 + B_5 + B_6;$$

each of these B_i correspond to one of the six terms on the right-hand side of (18). We will now write each of these out and manipulate each to get it to a useful form.

We begin with B_1. We have

$$B_1 = -\frac{4\pi^3\tau}{(L^\tau)^3}\Lambda^3(\theta^\tau) + \frac{4\pi^3\tau'}{(L^{\tau'})^3}\Lambda^3(\theta^{\tau'}).$$

We add and subtract:

$$B_1 = -\frac{4\pi^3(\tau - \tau')}{(L^\tau)^3}\Lambda^3(\theta^\tau) - 4\pi^3\tau'\left(\frac{1}{(L^\tau)^3} - \frac{1}{(L^{\tau'})^3}\right)\Lambda^3(\theta^\tau) - \frac{4\pi^3\tau'}{(L^{\tau'})^3}\Lambda^3(\theta^\tau - \theta^{\tau'}).$$

We turn to B_2. We write the second term on the right-hand side of (18) as

$$\frac{2\pi}{L}H\left(\left\{-\frac{R\cos(\theta)}{2} - A_\mu U\right\}\theta_\alpha\right) = H(k\theta_\alpha) - \frac{2\pi\tau A_\mu}{L}H(U^{\mathrm{s.t.}}\theta_\alpha).$$

We then have our first expression for B_2, namely

$$B_2 = H(k^\tau\theta^\tau_\alpha) - H(k^{\tau'}\theta^{\tau'}_\alpha) - \frac{2\pi\tau A_\mu}{L^\tau}H(U^{\mathrm{s.t.},\tau}\theta^\tau_\alpha) + \frac{2\pi\tau' A_\mu}{L^{\tau'}}H(U^{\mathrm{s.t.},\tau'}\theta^{\tau'}_\alpha).$$

Of course, we add and subtract:

$$\begin{aligned} B_2 = {} & H((k^\tau - k^{\tau'})\theta^\tau_\alpha) + H(k^{\tau'}(\theta^\tau_\alpha - \theta^{\tau'}_\alpha)) \\ & -\frac{2\pi(\tau - \tau')A_\mu}{L^\tau}H(U^{\mathrm{s.t.},\tau}\theta^\tau_\alpha) - 2\pi\tau' A_\mu\left(\frac{1}{L^\tau} - \frac{1}{L^{\tau'}}\right)H(U^{\mathrm{s.t.},\tau}\theta^\tau_\alpha) \\ & -\frac{2\pi\tau' A_\mu}{L^{\tau'}}H((U^{\mathrm{s.t.},\tau} - U^{\mathrm{s.t.},\tau'})\theta^\tau_\alpha) - \frac{2\pi\tau' A_\mu}{L^{\tau'}}H(U^{\mathrm{s.t.},\tau'}(\theta^\tau_\alpha - \theta^{\tau'}_\alpha)). \end{aligned}$$

We rewrite the term with $U^{\mathrm{s.t.},\tau} - U^{\mathrm{s.t.},\tau'}$ by using (37):

$$\begin{aligned} B_2 = {} & H((k^\tau - k^{\tau'})\theta^\tau_\alpha) + H(k^{\tau'}(\theta^\tau_\alpha - \theta^{\tau'}_\alpha)) \\ & -\frac{2\pi(\tau - \tau')A_\mu}{L^\tau}H(U^{\mathrm{s.t.},\tau}\theta^\tau_\alpha) - 2\pi\tau' A_\mu\left(\frac{1}{L^\tau} - \frac{1}{L^{\tau'}}\right)H(U^{\mathrm{s.t.},\tau}\theta^\tau_\alpha) \\ & -\frac{2\pi\tau' A_\mu}{L^{\tau'}}H(U^{\mathrm{s.t.},\tau'}(\theta^\tau_\alpha - \theta^{\tau'}_\alpha)) - \frac{2\pi\tau' A_\mu}{L^{\tau'}}H((Q^\tau - Q^{\tau'})\theta^\tau_\alpha) \\ & -\frac{4\pi^3\tau' A_\mu}{L^{\tau'}}\left(\frac{1}{(L^\tau)^2} - \frac{1}{(L^{\tau'})^2}\right)H(\theta^\tau_\alpha H(\theta^\tau_{\alpha\alpha})) - \frac{4\pi^3\tau' A_\mu}{(L^{\tau'})^3}H(\theta^\tau_\alpha H(\theta^\tau_{\alpha\alpha} - \theta^{\tau'}_{\alpha\alpha})). \end{aligned}$$

To get our final form for B_2, we now pull some factors through Hilbert transforms, incurring commutators:

$$\begin{aligned}B_2 &= H((k^{\tau}-k^{\tau'})\theta_{\alpha}^{\tau}) + k^{\tau'}H(\theta_{\alpha}^{\tau}-\theta_{\alpha}^{\tau'})\\ &\quad -\frac{2\pi(\tau-\tau')A_{\mu}}{L^{\tau}}H(U^{\mathrm{s.t.},\tau}\theta_{\alpha}^{\tau}) - 2\pi\tau' A_{\mu}\left(\frac{1}{L^{\tau}}-\frac{1}{L^{\tau'}}\right)H(U^{\mathrm{s.t.},\tau}\theta_{\alpha}^{\tau})\\ &\quad -\frac{2\pi\tau' A_{\mu}}{L^{\tau'}}U^{\mathrm{s.t.},\tau'}H(\theta_{\alpha}^{\tau}-\theta_{\alpha}^{\tau'}) - \frac{2\pi\tau' A_{\mu}}{L^{\tau'}}H((Q^{\tau}-Q^{\tau'})\theta_{\alpha}^{\tau})\\ &\quad -\frac{4\pi^3\tau' A_{\mu}}{L^{\tau'}}\left(\frac{1}{(L^{\tau})^2}-\frac{1}{(L^{\tau'})^2}\right)H(\theta_{\alpha}^{\tau}H(\theta_{\alpha\alpha}^{\tau})) + \frac{4\pi^3\tau' A_{\mu}}{(L^{\tau'})^3}\theta_{\alpha}^{\tau}(\theta_{\alpha\alpha}^{\tau}-\theta_{\alpha\alpha}^{\tau'})\\ &\quad +[H,k^{\tau'}](\theta_{\alpha}^{\tau}-\theta_{\alpha}^{\tau'}) - \frac{2\pi\tau' A_{\mu}}{L^{\tau'}}[H,U^{\mathrm{s.t.},\tau'}](\theta_{\alpha}^{\tau}-\theta_{\alpha}^{\tau'}) - \frac{4\pi^3\tau' A_{\mu}}{(L^{\tau'})^3}[H,\theta_{\alpha}^{\tau}]H(\theta_{\alpha\alpha}^{\tau}-\theta_{\alpha\alpha}^{\tau'}).\end{aligned}$$

For B_3, we begin by recalling the definition

$$\gamma = \frac{2\pi\tau}{L}\theta_{\alpha\alpha} + \tilde{\gamma}.$$

Using this, we can write

$$-\frac{2\pi^2 A_{\mu}}{L^2}\mathbb{P}(\gamma\theta_{\alpha}) = -\frac{4\pi^3 A_{\mu}\tau}{L^3}\mathbb{P}(\theta_{\alpha\alpha}\theta_{\alpha}) - \frac{2\pi^2 A_{\mu}}{L^2}\mathbb{P}(\tilde{\gamma}\theta_{\alpha}),$$

and we note that the mean of $\theta_{\alpha\alpha}\theta_{\alpha}$ is zero (since it is a perfect derivative). So, we have the following for B_3:

$$B_3 = -\frac{4\pi^3 A_{\mu}\tau}{(L^{\tau})^3}\theta_{\alpha}^{\tau}\theta_{\alpha\alpha}^{\tau} + \frac{4\pi^3 A_{\mu}\tau'}{(L^{\tau'})^3}\theta_{\alpha}^{\tau'}\theta_{\alpha\alpha}^{\tau'} - \frac{2\pi^2 A_{\mu}}{(L^{\tau})^2}\mathbb{P}(\tilde{\gamma}^{\tau}\theta_{\alpha}^{\tau}) + \frac{2\pi^2 A_{\mu}}{(L^{\tau'})^2}\mathbb{P}(\tilde{\gamma}^{\tau'}\theta_{\alpha}^{\tau'}).$$

Once again, we add and subtract:

$$\begin{aligned}B_3 &= -\frac{4\pi^3 A_{\mu}(\tau-\tau')}{(L^{\tau})^3}\theta_{\alpha}^{\tau}\theta_{\alpha\alpha}^{\tau} - 4\pi^3 A_{\mu}\tau'\left(\frac{1}{(L^{\tau})^3}-\frac{1}{(L^{\tau'})^3}\right)\theta_{\alpha}^{\tau}\theta_{\alpha\alpha}^{\tau}\\ &\quad -\frac{4\pi^3 A_{\mu}\tau'}{(L^{\tau'})^3}(\theta_{\alpha}^{\tau}-\theta_{\alpha}^{\tau'})\theta_{\alpha\alpha}^{\tau} - \frac{4\pi^3 A_{\mu}\tau'}{(L^{\tau'})^3}\theta_{\alpha}^{\tau'}(\theta_{\alpha\alpha}^{\tau}-\theta_{\alpha\alpha}^{\tau'}) - 2\pi^2 A_{\mu}\left(\frac{1}{(L^{\tau})^2}-\frac{1}{(L^{\tau'})^2}\right)\mathbb{P}(\tilde{\gamma}^{\tau}\theta_{\alpha}^{\tau})\\ &\quad -\frac{2\pi^2 A_{\mu}}{(L^{\tau'})^2}\mathbb{P}((\tilde{\gamma}^{\tau}-\tilde{\gamma}^{\tau'})\theta_{\alpha}^{\tau}) - \frac{2\pi^2 A_{\mu}}{(L^{\tau'})^2}\mathbb{P}(\tilde{\gamma}^{\tau'}(\theta_{\alpha}^{\tau}-\theta_{\alpha}^{\tau'})).\end{aligned}$$

We rewrite the last term on the right-hand side, by using the formula $\mathbb{P}f = f - \langle\langle f\rangle\rangle$:

$$\begin{aligned}B_3 &= -\frac{4\pi^3 A_{\mu}(\tau-\tau')}{(L^{\tau})^3}\theta_{\alpha}^{\tau}\theta_{\alpha\alpha}^{\tau} - 4\pi^3 A_{\mu}\tau'\left(\frac{1}{(L^{\tau})^3}-\frac{1}{(L^{\tau'})^3}\right)\theta_{\alpha}^{\tau}\theta_{\alpha\alpha}^{\tau}\\ &\quad -\frac{4\pi^3 A_{\mu}\tau'}{(L^{\tau'})^3}(\theta_{\alpha}^{\tau}-\theta_{\alpha}^{\tau'})\theta_{\alpha\alpha}^{\tau} - \frac{4\pi^3 A_{\mu}\tau'}{(L^{\tau'})^3}\theta_{\alpha}^{\tau'}(\theta_{\alpha\alpha}^{\tau}-\theta_{\alpha\alpha}^{\tau'}) - 2\pi^2 A_{\mu}\left(\frac{1}{(L^{\tau})^2}-\frac{1}{(L^{\tau'})^2}\right)\mathbb{P}(\tilde{\gamma}^{\tau}\theta_{\alpha}^{\tau})\\ &\quad -\frac{2\pi^2 A_{\mu}}{(L^{\tau'})^2}\mathbb{P}((\tilde{\gamma}^{\tau}-\tilde{\gamma}^{\tau'})\theta_{\alpha}^{\tau}) - \frac{2\pi^2 A_{\mu}}{(L^{\tau'})^2}\tilde{\gamma}^{\tau'}(\theta_{\alpha}^{\tau}-\theta_{\alpha}^{\tau'}) + \frac{2\pi^2 A_{\mu}}{(L^{\tau'})^2}\langle\langle\tilde{\gamma}^{\tau'}(\theta_{\alpha}^{\tau}-\theta_{\alpha}^{\tau'})\rangle\rangle.\end{aligned}$$

It is helpful to rewrite $V - \mathbf{W}\cdot\hat{\mathbf{t}}$ before beginning to give the formula for B_4. We use (7) for this, finding

$$V - \mathbf{W}\cdot\hat{\mathbf{t}} = \langle\langle V - \mathbf{W}\cdot\hat{\mathbf{t}}\rangle\rangle + \partial_{\alpha}^{-1}\left(\frac{L_t}{2\pi} - \mathbf{W}_{\alpha}\cdot\hat{\mathbf{t}}\right).$$

From previous considerations, we have the formula

$$\mathbf{W}_{\alpha}\cdot\hat{\mathbf{t}} = -\frac{2\pi^2\tau}{L^2}H(\theta_{\alpha\alpha}\theta_{\alpha}) - \frac{\pi}{L}H(\tilde{\gamma}\theta_{\alpha}) + \mathbf{m}\cdot\hat{\mathbf{t}}.$$

We observe that we can calculate $\partial_\alpha^{-1} H(\theta_{\alpha\alpha}\theta_\alpha)$ exactly, and we find

$$V - \mathbf{W}\cdot\hat{\mathbf{t}} = \langle\!\langle V - \mathbf{W}\cdot\hat{\mathbf{t}}\rangle\!\rangle + \frac{\pi^2\tau}{L^2} H(\theta_\alpha^2) + \partial_\alpha^{-1}\left(\frac{L_t}{2\pi} + \frac{\pi}{L} H(\widetilde{\gamma}\theta_\alpha) - \mathbf{m}\cdot\hat{\mathbf{t}}\right).$$

We then write B_4 as

$$\begin{aligned}
B_4 &= \frac{2\pi}{L^\tau}(V - \mathbf{W}\cdot\hat{\mathbf{t}})^\tau\theta_\alpha^\tau - \frac{2\pi}{L^{\tau'}}(V - \mathbf{W}\cdot\hat{\mathbf{t}})^{\tau'}\theta_\alpha^{\tau'} \\
&= 2\pi\left(\frac{1}{L^\tau} - \frac{1}{L^{\tau'}}\right)(V - \mathbf{W}\cdot\hat{\mathbf{t}})^\tau\theta_\alpha^\tau + \frac{2\pi}{L^{\tau'}}\left(\langle\!\langle V - \mathbf{W}\cdot\hat{\mathbf{t}}\rangle\!\rangle^\tau - \langle\!\langle V - \mathbf{W}\cdot\hat{\mathbf{t}}\rangle\!\rangle^{\tau'}\right)\theta_\alpha^\tau \\
&\quad + \frac{2\pi}{L^{\tau'}}\cdot\frac{\pi^2(\tau-\tau')}{(L^\tau)^2}(H((\theta_\alpha^\tau)^2))\theta_\alpha^\tau + \frac{2\pi^3\tau'}{L^{\tau'}}\left(\frac{1}{(L^\tau)^2} - \frac{1}{(L^{\tau'})^2}\right)(H((\theta_\alpha^\tau)^2))\theta_\alpha^\tau \\
&\quad + \frac{2\pi^3\tau'}{(L^{\tau'})^3}\left(H\left((\theta_\alpha^\tau)^2 - (\theta_\alpha^{\tau'})^2\right)\right)\theta_\alpha^\tau \\
&\quad + \frac{2\pi}{L^{\tau'}}\partial_\alpha^{-1}\left[\left(\frac{L_t}{2\pi} + \frac{\pi}{L}H(\widetilde{\gamma}\theta_\alpha) - \mathbf{m}\cdot\hat{\mathbf{t}}\right)^\tau - \left(\frac{L_t}{2\pi} + \frac{\pi}{L}H(\widetilde{\gamma}\theta_\alpha) - \mathbf{m}\cdot\hat{\mathbf{t}}\right)^{\tau'}\right]\theta_\alpha^\tau \\
&\quad + \frac{2\pi}{L^{\tau'}}(V - \mathbf{W}\cdot\hat{\mathbf{t}})^{\tau'}(\theta_\alpha^\tau - \theta_\alpha^{\tau'}).
\end{aligned}$$

We rewrite this by factoring $(\theta_\alpha^\tau)^2 - (\theta_\alpha^{\tau'})^2$, and then pulling $\theta_\alpha^\tau + \theta_\alpha^{\tau'}$ through the Hilbert transform:

$$\begin{aligned}
B_4 &= 2\pi\left(\frac{1}{L^\tau} - \frac{1}{L^{\tau'}}\right)(V - \mathbf{W}\cdot\hat{\mathbf{t}})^\tau\theta_\alpha^\tau + \frac{2\pi}{L^{\tau'}}\left(\langle\!\langle V - \mathbf{W}\cdot\hat{\mathbf{t}}\rangle\!\rangle^\tau - \langle\!\langle V - \mathbf{W}\cdot\hat{\mathbf{t}}\rangle\!\rangle^{\tau'}\right)\theta_\alpha^\tau \\
&\quad + \frac{2\pi}{L^{\tau'}}\cdot\frac{\pi^2(\tau-\tau')}{(L^\tau)^2}(H((\theta_\alpha^\tau)^2))\theta_\alpha^\tau + \frac{2\pi^3\tau'}{L^{\tau'}}\left(\frac{1}{(L^\tau)^2} - \frac{1}{(L^{\tau'})^2}\right)(H((\theta_\alpha^\tau)^2))\theta_\alpha^\tau \\
&\quad + \frac{2\pi^3\tau'}{(L^{\tau'})^3}\theta_\alpha^\tau(\theta_\alpha^\tau + \theta_\alpha^{\tau'})H\left(\theta_\alpha^\tau - \theta_\alpha^{\tau'}\right) \\
&\quad + \frac{2\pi}{L^{\tau'}}\partial_\alpha^{-1}\left[\left(\frac{L_t}{2\pi} + \frac{\pi}{L}H(\widetilde{\gamma}\theta_\alpha) - \mathbf{m}\cdot\hat{\mathbf{t}}\right)^\tau - \left(\frac{L_t}{2\pi} + \frac{\pi}{L}H(\widetilde{\gamma}\theta_\alpha) - \mathbf{m}\cdot\hat{\mathbf{t}}\right)^{\tau'}\right]\theta_\alpha^\tau \\
&\quad + \frac{2\pi}{L^{\tau'}}(V - \mathbf{W}\cdot\hat{\mathbf{t}})^{\tau'}(\theta_\alpha^\tau - \theta_\alpha^{\tau'}) + \frac{2\pi^3\tau'}{(L^{\tau'})^3}\theta_\alpha^\tau[H, \theta_\alpha^\tau + \theta_\alpha^{\tau'}](\theta_\alpha^\tau - \theta_\alpha^{\tau'}).
\end{aligned}$$

We simply write B_5 and B_6 as follows:

$$\begin{aligned}
B_5 &= \frac{2\pi}{L^\tau}\mathbf{m}^\tau\cdot\hat{\mathbf{n}}^\tau - \frac{2\pi}{L^{\tau'}}\mathbf{m}^{\tau'}\cdot\hat{\mathbf{n}}^{\tau'}, \\
B_6 &= -\frac{2\pi^3 A_\mu}{(L^\tau)^3}H(\mathbf{m}^\tau\cdot\hat{\mathbf{t}}^\tau) + \frac{2\pi^3 A_\mu}{(L^{\tau'})^3}H(\mathbf{m}^{\tau'}\cdot\hat{\mathbf{t}}^{\tau'}).
\end{aligned}$$

We now want to collect those terms which have a similar character together. In particular, we want to write $(\theta^\tau - \theta^{\tau'})_t$ as

$$\begin{aligned}
(\theta^\tau - \theta^{\tau'})_t = &-\frac{4\pi^3\tau'}{(L^{\tau'})^3}\Lambda^3(\theta^\tau - \theta^{\tau'}) + \tau'\Upsilon_8(\theta_{\alpha\alpha}^\tau - \theta_{\alpha\alpha}^{\tau'}) + \Upsilon_9\Lambda(\theta^\tau - \theta^{\tau'}) \\
&+ \Upsilon_{10}(\theta_\alpha^\tau - \theta_\alpha^{\tau'}) + (\tau - \tau')\Upsilon_{11} + \Upsilon_{12}.
\end{aligned} \tag{72}$$

We now give the formulas for each of $\Upsilon_8, \Upsilon_9, \Upsilon_{10}, \Upsilon_{11}$, and Υ_{12}. To begin, we see that there is one term from B_2 and one term from B_3 which constitute Υ_8:

$$\Upsilon_8 = \frac{4\pi^3 A_\mu}{(L^{\tau'})^3}\theta_\alpha^\tau - \frac{4\pi^3 A_\mu}{(L^{\tau'})^3}\theta_\alpha^{\tau'} = \frac{4\pi^3 A_\mu}{(L^{\tau'})^3}(\theta_\alpha^\tau - \theta_\alpha^{\tau'}).$$

For Υ_9, we collect two terms from B_2 and one term from B_4:

$$\Upsilon_9 = k^{\tau'} - \frac{2\pi\tau' A_\mu}{L^{\tau'}} U^{\mathrm{s.t.},\tau'} + \frac{2\pi^3\tau'}{(L^{\tau'})^3}\theta^\tau_\alpha(\theta^\tau_\alpha + \theta^{\tau'}_\alpha).$$

For Υ_{10}, we collect two terms from B_3 and one term from B_4:

$$\Upsilon_{10} = -\frac{4\pi^3 A_\mu \tau'}{(L^{\tau'})^3}\theta^\tau_{\alpha\alpha} - \frac{2\pi^2 A_\mu}{(L^{\tau'})^2}\tilde{\gamma}^{\tau'} + \frac{2\pi}{L^{\tau'}}(V - \mathbf{W}\cdot\hat{\mathbf{t}})^{\tau'}.$$

For Υ_{11}, we collect one term each from B_1, B_2, B_3, and B_4:

$$\Upsilon_{11} = -\frac{4\pi^3}{(L^\tau)^3}\Lambda^3(\theta^\tau) - \frac{2\pi A_\mu}{L^\tau}H(U^{\mathrm{s.t.},\tau}\theta^\tau_\alpha) - \frac{4\pi^3 A_\mu}{(L^\tau)^3}\theta^\tau_\alpha\theta^\tau_{\alpha\alpha} + \frac{2\pi^3}{L^{\tau'}(L^\tau)^2}\theta^\tau_\alpha H((\theta^\tau_\alpha)^2).$$

Since θ^τ and $\theta^{\tau'}$ are both in the set $\mathcal{O}_k$, and since s is sufficiently large, we see that there exists $M > 0$ such that for $i \in \{8, 9, 10, 11\}$ and for $j \in \{0, 1, 2\}$, we have the estimate

$$|\partial^j_\alpha \Upsilon_i|_\infty \le M.$$

Of course, this M is related to the constant $\bar{d}_1$ from (60).

Of course, Υ_{12} consists of all remaining terms; we will not list them explicitly now, but we do write out the definition below in (79). We need to establish the following estimate for Υ_{12} :

$$\|\Upsilon_{12}\|_1 \le c\|\theta^\tau - \theta^{\tau'}\|_1. \tag{73}$$

In order to establish this bound, we need to establish corresponding bounds for related quantities, such as $L^\tau - L^{\tau'}, k^\tau - k^{\tau'}$, and so on. We will establish (73) in Lemma 16 following the current proof; for the moment, we will assume that it holds.

We next differentiate (72), finding the following:

$$(\theta^\tau - \theta^{\tau'})_{\alpha,t} = -\frac{4\pi^3\tau'}{(L^{\tau'})^3}\Lambda^3(\theta^\tau_\alpha - \theta^{\tau'}_\alpha) + \tau'\Upsilon_8(\theta^\tau_{\alpha\alpha\alpha} - \theta^{\tau'}_{\alpha\alpha\alpha}) + \Upsilon_9\Lambda(\theta^\tau_\alpha - \theta^{\tau'}_\alpha)$$
$$+ (\Upsilon_{10} + \tau'\Upsilon_{8,\alpha})(\theta^\tau_{\alpha\alpha} - \theta^{\tau'}_{\alpha\alpha}) + \Upsilon_{9,\alpha}\Lambda(\theta^\tau - \theta^{\tau'}) + \Upsilon_{10,\alpha}(\theta^\tau_\alpha - \theta^{\tau'}_\alpha) + (\tau - \tau')\Upsilon_{11,\alpha} + \Upsilon_{12,\alpha}. \tag{74}$$

We are ready to make our estimate. Define $E_d = \frac{1}{2}\|\theta^\tau - \theta^{\tau'}\|_1^2$,

$$E_d = \frac{1}{2}\int_X (\theta^\tau - \theta^{\tau'})^2 + (\theta^\tau_\alpha - \theta^{\tau'}_\alpha)^2 \, d\alpha.$$

We differentiate this with respect to time:

$$\frac{dE_d}{dt} = \int_X (\theta^\tau - \theta^{\tau'})(\theta^\tau - \theta^{\tau'})_t + (\theta^\tau_\alpha - \theta^{\tau'}_\alpha)(\theta^\tau_{\alpha,t} - \theta^{\tau'}_{\alpha,t}) \, d\alpha.$$

We first use (74), and we write

$$\int_X (\theta^\tau_\alpha - \theta^{\tau'}_\alpha)(\theta^\tau_{\alpha,t} - \theta^{\tau'}_{\alpha,t}) \, d\alpha = Z_1 + Z_2 + Z_3 + Z_4 + Z_5 + Z_6 + Z_7 + Z_8,$$

where each of the Z_i terms corresponds to one of the eight terms on the right-hand side of (74).

We can immediately bound Z_5, Z_6, and Z_8 (using (73) for Z_8):

$$Z_5 + Z_6 + Z_8 \le cE_d.$$

We also can immediately bound Z_7 :

$$Z_7 \le c|\tau - \tau'|E_d^{1/2}.$$

To bound Z_4, we simply integrate by parts once, and we then find

$$Z_4 \le cE_d.$$

Of course, Z_1, Z_2, and Z_3 must be treated more carefully. We begin by rearranging Z_1 using the self-adjointedness of powers of Λ:

$$Z_1 = -\frac{4\pi^3\tau'}{(L^{\tau'})^3}\int\limits_X (\Lambda^{3/2}(\theta_\alpha^\tau - \theta_\alpha^{\tau'}))^2 \, d\alpha.$$

By the Plancherel theorem, and using $L^{\tau'} < \bar{d}_2$, we write this as

$$Z_1 = -\frac{4\pi^3\tau'}{(L^{\tau'})^3}\sum_{\xi=-\infty}^{\infty} |\xi|^3|\hat{v}(\xi)|^2 \leq -\frac{4\pi^3\tau'}{\bar{d}_2^3}\sum_{\xi=-\infty}^{\infty} |\xi|^3|\hat{v}(\xi)|^2, \tag{75}$$

where $v = \theta_\alpha^\tau - \theta_\alpha^{\tau'}$.

For Z_2, we integrate by parts once:

$$Z_2 = -\tau'\int\limits_X \Upsilon_8(\theta_{\alpha\alpha}^\tau - \theta_{\alpha\alpha}^{\tau'})^2 \, d\alpha - \tau'\int\limits_X \Upsilon_{8,\alpha}(\theta_\alpha^\tau - \theta_\alpha^{\tau'})(\theta_{\alpha\alpha}^\tau - \theta_{\alpha\alpha}^{\tau'}) \, d\alpha.$$

We integrate the second integral on the right hand side by parts once more:

$$Z_2 = -\tau'\int\limits_X \Upsilon_8(\theta_{\alpha\alpha}^\tau - \theta_{\alpha\alpha}^{\tau'})^2 \, d\alpha + \frac{\tau'}{2}\int\limits_X \Upsilon_{8,\alpha\alpha}(\theta_\alpha^\tau - \theta_\alpha^{\tau'})^2 \, d\alpha.$$

Note that the second integral on the right-hand side can be bounded by cE_d, and that Υ_8 is bounded by a constant uniformly in time. Therefore, we may conclude

$$Z_2 \leq cE_d + C_2\tau'\int\limits_X (\theta_{\alpha\alpha}^\tau - \theta_{\alpha\alpha}^{\tau'})^2 \, d\alpha.$$

By the Plancherel theorem, we can write this as

$$Z_2 \leq cE_d + C_2\tau'\sum_{\xi=-\infty}^{\infty} |\xi|^2|\hat{v}(\xi)|^2, \tag{76}$$

where again $v = \theta_\alpha^\tau - \theta_\alpha^{\tau'}$.

We add (75) and (76), finding the following:

$$Z_1 + Z_2 \leq cE_d + \tau'\sum_{\xi=-\infty}^{\infty}\left(-\frac{4\pi^3}{\bar{d}_2^3}|\xi|^3 + C_2|\xi|^2\right)|\hat{v}(\xi)|^2.$$

There exists a constant, $\bar{C}$, which is independent of τ and τ', such that for all ξ,

$$\left(-\frac{4\pi^3}{\bar{d}_2^3}|\xi|^3 + C_2|\xi|^2\right) \leq \bar{C}.$$

Note also that $\|v\|_0^2 \leq E_d$. These considerations, and another use of the Plancherel theorem, imply

$$Z_1 + Z_2 \leq (c + \tau'\bar{C})E_d;$$

we rename constants to simply write this as $Z_1 + Z_2 \leq cE_d$.

We rewrite Z_3 by adding and subtracting $\Upsilon_5^{\tau'}$:

$$Z_3 = \int\limits_X \Upsilon_5^{\tau'}(\theta_\alpha^\tau - \theta_\alpha^{\tau'})\Lambda(\theta_\alpha^\tau - \theta_\alpha^{\tau'}) \, d\alpha + \int\limits_X (\Upsilon_9 - \Upsilon_5^{\tau'})(\theta_\alpha^\tau - \theta_\alpha^{\tau'})\Lambda(\theta_\alpha^\tau - \theta_\alpha^{\tau'}) \, d\alpha. \tag{77}$$

Of course, $\Upsilon_5^{\tau'}$ refers to the quantity defined by (59), with τ' as the surface tension parameter, but in the case $\varepsilon = 0$; to be perfectly clear, we write this out:

$$\Upsilon_5^{\tau'} = k^{\tau'} - \frac{2\pi\tau' A_\mu}{L^\tau}U^{\mathrm{s.t.},\tau'} + \frac{4\pi^3\tau'}{(L^{\tau'})^3}(\theta_\alpha^{\tau'})^2.$$

To estimate the second integral on the right-hand side of (77), we will use the uniform bound on $\Lambda(\theta^{\tau}_{\alpha}-\theta^{\tau'}_{\alpha})$, and we will also need the following estimate:

$$\|\Upsilon_9 - \Upsilon_5^{\tau'}\|_0 \leq cE_d^{1/2}. \tag{78}$$

To show this, we simply write out

$$\Upsilon_9 - \Upsilon_5^{\tau'} = \frac{2\pi^3\tau'}{(L^{\tau'})^3}\left((\theta^{\tau}_{\alpha})^2 + \theta^{\tau}_{\alpha}\theta^{\tau'}_{\alpha} - 2(\theta^{\tau'}_{\alpha})^2\right).$$

The right-hand side can clearly be bounded in H^0 by the H^1 norm of $\theta^{\tau} - \theta^{\tau'}$. This implies that the second integral on the right-hand side of (77) can be bounded by cE_d.

For the first integral on the right-hand side of (77), we again write $\Upsilon_5^{\tau'} = -(\sqrt{-\Upsilon_5^{\tau'}})^2$. As before, we pass a factor of $\sqrt{-\Upsilon_5^{\tau'}}$ through each of ∂_α and H, in turn:

$$\begin{aligned}
\int\limits_X \Upsilon_5^{\tau'}(\theta^{\tau}_{\alpha} - \theta^{\tau'}_{\alpha})\Lambda(\theta^{\tau}_{\alpha} - \theta^{\tau'}_{\alpha})\ d\alpha &= -\int\limits_X \left(\sqrt{-\Upsilon_5^{\tau'}}(\theta^{\tau}_{\alpha} - \theta^{\tau'}_{\alpha})\right)\left(\sqrt{-\Upsilon_5^{\tau'}}\partial_\alpha H(\theta^{\tau}_{\alpha} - \theta^{\tau'}_{\alpha'})\right)\ d\alpha \\
&= -\int\limits_X \left(\sqrt{-\Upsilon_5^{\tau'}}(\theta^{\tau}_{\alpha} - \theta^{\tau'}_{\alpha})\right)\partial_\alpha\left(\sqrt{-\Upsilon_5^{\tau'}}H(\theta^{\tau}_{\alpha} - \theta^{\tau'}_{\alpha'})\right)\ d\alpha \\
&\quad -\frac{1}{2}\int\limits_X (\theta^{\tau}_{\alpha} - \theta^{\tau'}_{\alpha})(\Upsilon^{\tau'}_{5,\alpha})H(\theta^{\tau}_{\alpha} - \theta^{\tau'}_{\alpha})\ d\alpha \\
&= -\int\limits_X \left(\sqrt{-\Upsilon_5^{\tau'}}(\theta^{\tau}_{\alpha} - \theta^{\tau'}_{\alpha})\right)\Lambda\left(\sqrt{-\Upsilon_5^{\tau'}}(\theta^{\tau}_{\alpha} - \theta^{\tau'}_{\alpha'})\right)\ d\alpha \\
&\quad +\int\limits_X \left(\sqrt{-\Upsilon_5^{\tau'}}(\theta^{\tau}_{\alpha} - \theta^{\tau'}_{\alpha})\right)\partial_\alpha\left[H, \sqrt{-\Upsilon_5^{\tau'}}\right](\theta^{\tau}_{\alpha} - \theta^{\tau'}_{\alpha})\ d\alpha \\
&\quad -\frac{1}{2}\int\limits_X (\theta^{\tau}_{\alpha} - \theta^{\tau'}_{\alpha})(\Upsilon^{\tau'}_{5,\alpha})H(\theta^{\tau}_{\alpha} - \theta^{\tau'}_{\alpha})\ d\alpha.
\end{aligned}$$

Of the three integrals on the right-hand side, the first is non-positive, while the second and third can be bounded immediately in terms of the energy. We conclude

$$Z_3 \leq cE_d.$$

Adding our estimates for all of the Z_i, we conclude that there exist positive constants c_1 and c_2 such that

$$\frac{d}{dt}\frac{1}{2}\int\limits_X (\theta^{\tau}_{\alpha} - \theta^{\tau'}_{\alpha})^2\ d\alpha \leq c_1E_d + c_2|\tau - \tau'|E_d^{1/2}.$$

We could carry out the same estimates, using (72) instead of (74), and we would find that

$$\frac{d}{dt}\frac{1}{2}\int\limits_X (\theta^{\tau} - \theta^{\tau'})^2\ d\alpha \leq c_1E_d + c_2|\tau - \tau'|E_d^{1/2}.$$

Together, these estimates imply

$$\frac{dE_d}{dt} \leq c_1E_d + c_2|\tau - \tau'|E_d^{1/2}.$$

Solving this differential inequality, we see that

$$E_d \leq E_d(0)e^{c_1t} + \frac{c_2|\tau - \tau'|(e^{c_1t} - 1)}{c_1}.$$

Together with the initial condition $E_d(0) = 0$, this immediately implies the conclusion of the theorem. □

We are now able to prove the main result of the present work, which is that the limit as surface tension vanishes for Darcy flow with surface tension is the Darcy flow without surface tension, when the stability condition is satisfied by the initial data.

Theorem 15. *Let $\theta_0 \in \mathcal{O}_k$ be given, such that $\langle\langle \sin(\theta_0)\rangle\rangle = 0$. Let $T > 0$ be as in Theorem 13. For all $\tau \in (0, \tau^*)$, let θ^τ be as in Theorem 13. Let s' be given such that $0 \le s' < s$. There exists $\theta \in C([0,T]; \bar{\mathcal{O}}_k) \cap C^1([0,T]; H^{s-1})$ such that*

$$\lim_{\tau\to 0} \sup_{t\in[0,T]} \|\theta^\tau - \theta\|_{s'} = 0.$$

This θ is the solution of the initial value problem (21), with initial data θ_0.

Proof. From Theorem 14, we see that θ^τ forms a Cauchy sequence in H^1 as $\tau \to 0^+$. Since the solutions θ^τ are bounded in H^s independently of τ, the Sobolev interpolation inequality (22) implies that the sequence θ^τ is in fact a Cauchy sequence in $H^{s'}$. Therefore, there exists a limit, $\theta \in C([0,T]; \bar{\mathcal{O}}_k)$, such that $\theta^\tau \to \theta$ in $H^{s'}$ as $\tau \to 0^+$.

Next, we see that θ solves the initial value problem with $\tau = 0$. We call the right-hand side of (20) by the name $\mathfrak{B}^\tau$, and we call the right-hand side of (21) by the name $\mathfrak{B}^{(\tau=0)}$. Since s is sufficiently large, and since $\theta^\tau \to \theta$ in $C([0,T]; H^{s''})$ for s'' arbitrarily close to s, we see that $\mathfrak{B}^\tau$ converges uniformly to $\mathfrak{B}^{(\tau=0)}$. We integrate (20) in time, using the initial condition, finding

$$\theta^\tau(\cdot,t) = \theta_0 + \int_0^t \mathfrak{B}^\tau(\cdot,s)\, ds.$$

Because of the uniform convergence, we can pass to the limit as $\tau \to 0^+$, finding

$$\theta(\cdot,t) = \theta_0 + \int_0^t \mathfrak{B}^{(\tau=0)}(\cdot,s)\, ds.$$

This implies that θ solves the initial value problem (21) with initial data θ_0.

Finally, we address the highest regularity of θ. As in the proof of Theorem 11, we do not provide the details, but the same argument works here. In particular, we are able to use parabolic smoothing; the observations made in Remark 14 allow us to conclude that the solutions θ^τ are uniformly bounded in $C([0,T]; H^s) \cap L^2([0,T]; H^{s+1/2})$. This is enough regularity with which to apply the argument of [42], as discussed in Theorem 11 (As an alternative, the highest regularity also follows from the well-posedness result for the problem without surface tension [3].). □

All that remains is to prove the bound (73).

Lemma 16. *The estimate (73) holds.*

Proof. We are attempting to show that $\|\Upsilon_{12}\|_1$ can be bounded by $cE_d^{1/2}$. To begin, we write out the whole formula for Υ_{12}:

$$\begin{aligned}\Upsilon_{12} = {} & -4\pi^3\tau'\left(\frac{1}{(L^\tau)^3} - \frac{1}{(L^{\tau'})^3}\right)\Lambda^3(\theta^\tau) + H\left((k^\tau - k^{\tau'})\theta^\tau_\alpha\right) - 2\pi\tau' A_\mu\left(\frac{1}{L^\tau} - \frac{1}{L^{\tau'}}\right)H(U^{\mathrm{s.t.},\tau}\theta^\tau_\alpha)\\
& -\frac{2\pi\tau' A_\mu}{L^{\tau'}}H\left((Q^\tau - Q^{\tau'})\theta^\tau_\alpha\right) - \frac{4\pi^3\tau' A_\mu}{L^{\tau'}}\left(\frac{1}{(L^\tau)^2} - \frac{1}{(L^{\tau'})^2}\right)H\left(\theta^\tau_\alpha H(\theta^\tau_{\alpha\alpha})\right) + [H, k^{\tau'}](\theta^\tau_\alpha - \theta^{\tau'}_\alpha)\\
& -\frac{2\pi\tau' A_\mu}{L^{\tau'}}\left[H, U^{\mathrm{s.t.},\tau}\right](\theta^\tau_\alpha - \theta^{\tau'}_\alpha)\\
& -\frac{4\pi^3\tau' A_\mu}{(L^{\tau'})^3}\left[H, \theta^{\tau'}_\alpha\right]H(\theta^\tau_{\alpha\alpha} - \theta^{\tau'}_{\alpha\alpha}) - 4\pi^3 A_\mu\tau'\left(\frac{1}{(L^\tau)^3} - \frac{1}{(L^{\tau'})^3}\right)\theta^\tau_\alpha\theta^\tau_{\alpha\alpha}\\
& -2\pi^2 A_\mu\left(\frac{1}{(L^\tau)^2} - \frac{1}{(L^{\tau'})^2}\right)\mathbb{P}(\tilde{\gamma}^\tau\theta^\tau_\alpha) - \frac{2\pi^2 A_\mu}{(L^{\tau'})^2}\mathbb{P}\left((\tilde{\gamma}^\tau - \tilde{\gamma}^{\tau'})\theta^\tau_\alpha\right) + \frac{2\pi^2 A_\mu}{(L^{\tau'})^2}\langle\langle\tilde{\gamma}^{\tau'}(\theta^\tau_\alpha - \theta^{\tau'}_\alpha)\rangle\rangle\end{aligned}$$

$$+2\pi\left(\frac{1}{L^{\tau}}-\frac{1}{L^{\tau'}}\right)(V-\mathbf{W}\cdot\hat{\mathbf{t}})^{\tau}\theta_{\alpha}^{\tau}+\frac{2\pi}{L^{\tau'}}\left(\langle\!\langle V-\mathbf{W}\cdot\hat{\mathbf{t}}\rangle\!\rangle^{\tau}-\langle\!\langle V-\mathbf{W}\cdot\hat{\mathbf{t}}\rangle\!\rangle^{\tau'}\right)\theta_{\alpha}^{\tau}$$
$$+\frac{2\pi^3\tau'}{L^{\tau'}}\left(\frac{1}{(L^{\tau})^2}-\frac{1}{(L^{\tau'})^2}\right)(H((\theta_{\alpha}^{\tau})^2))\theta_{\alpha}^{\tau}+\frac{2\pi^3\tau'}{(L^{\tau'})^3}\theta_{\alpha}^{\tau}\left[H,\theta_{\alpha}^{\tau}+\theta_{\alpha}^{\tau'}\right](\theta_{\alpha}^{\tau}-\theta_{\alpha}^{\tau'})$$
$$+\frac{2\pi}{L^{\tau'}}\partial_{\alpha}^{-1}\left[\left(\frac{L_t}{2\pi}+\frac{\pi}{L}H(\widetilde{\gamma}\theta_{\alpha})-\mathbf{m}\cdot\hat{\mathbf{t}}\right)^{\tau}-\left(\frac{L_t}{2\pi}+\frac{\pi}{L}H(\widetilde{\gamma}\theta_{\alpha})-\mathbf{m}\cdot\hat{\mathbf{t}}\right)^{\tau'}\right]\theta_{\alpha}^{\tau}+B_5+B_6. \tag{79}$$

There are nineteen terms on the right-hand side of (79). Each of these nineteen terms includes the difference between some τ-quantity and the corresponding τ'-quantity. In order to prove the lemma, we will need to establish estimates for these differences. To begin, we notice that many of the terms on the right-hand side of (79) include a difference $L^{\tau}-L^{\tau'}$. The estimate

$$|L^{\tau}-L^{\tau'}|\le cE_d^{1/2}$$

follows from (38).

Next, we establish the following:

$$\|\mathbf{m}^{\tau}-\mathbf{m}^{\tau'}\|_1\le cE_d^{1/2}. \tag{80}$$

We use (12) to write out $\mathbf{m}^{\tau}-\mathbf{m}^{\tau'}$, adding and subtracting several times:

$$\begin{aligned}\Phi(\mathbf{m}^{\tau}-\mathbf{m}^{\tau'})^* &= (z_{\alpha}^{\tau}-z_{\alpha}^{\tau'})\mathcal{K}[z_d^{\tau}]\left(\left(\frac{\gamma^{\tau}}{z_{\alpha}^{\tau}}\right)_{\alpha}\right)+z_{\alpha}^{\tau'}\left(\mathcal{K}[z_d^{\tau}]-\mathcal{K}[z_d^{\tau'}]\right)\left(\left(\frac{\gamma^{\tau}}{z_{\alpha}^{\tau}}\right)_{\alpha}\right)\\ &\quad+z_{\alpha}^{\tau'}\mathcal{K}[z_d^{\tau'}]\left(\left(\frac{\gamma^{\tau}}{z_{\alpha}^{\tau}}\right)_{\alpha}-\left(\frac{\gamma^{\tau'}}{z_{\alpha}^{\tau'}}\right)_{\alpha}\right)\\ &\quad+\frac{(z_{\alpha}^{\tau}-z_{\alpha}^{\tau'})}{2i}\left[H,\frac{1}{(z_{\alpha}^{\tau})^2}\right]\left(z_{\alpha}^{\tau}\left(\frac{\gamma^{\tau}}{z_{\alpha}^{\tau}}\right)_{\alpha}\right)\frac{z_{\alpha}^{\tau'}}{2i}\left[H,\frac{1}{(z_{\alpha}^{\tau})^2}-\frac{1}{(z_{\alpha}^{\tau'})^2}\right]\left(z_{\alpha}^{\tau}\left(\frac{\gamma^{\tau}}{z_{\alpha}^{\tau}}\right)_{\alpha}\right)\\ &\quad+\frac{z_{\alpha}^{\tau'}}{2i}\left[H,\frac{1}{(z_{\alpha}^{\tau'})^2}\right]\left(z_{\alpha}^{\tau}\left(\frac{\gamma^{\tau}}{z_{\alpha}^{\tau}}\right)_{\alpha}-z_{\alpha}^{\tau'}\left(\frac{\gamma^{\tau'}}{z_{\alpha}^{\tau'}}\right)_{\alpha}\right)\\ &:=T_1+T_2+T_3+T_4+T_5+T_6.\end{aligned}$$

For each value of j from 1 through 6, we need to show $\|T_j\|_1\le cE_d^{1/2}$. For T_1 and T_4, this estimate is an immediate consequence of the formula $z_{\alpha}=Le^{i\theta}/2\pi$ and the Lipschitz estimate for the exponential. The estimate for T_5 is also immediate, for the same reason, and we just note that for this term, we do not need to use the smoothing properties of the commutator at all. The estimate for T_2 follows by applying Lemma 2. The estimate for T_3 follows from (23). The estimate for T_6 follows from (25). Thus, we have proven (80).

Using the same tools as for $\mathbf{m}^{\tau}-\mathbf{m}^{\tau'}$, we see from (29) that we have the following estimate:

$$\|(\mathbf{W}\cdot\hat{\mathbf{t}})^{\tau}-(\mathbf{W}\cdot\hat{\mathbf{t}})^{\tau'}\|_1\le cE_d^{1/2}.$$

Then, using the definition of $\widetilde{\gamma}$ in (27), we have the corresponding bound for the difference $\widetilde{\gamma}^{\tau}-\widetilde{\gamma}^{\tau'}$:

$$\|\widetilde{\gamma}^{\tau}-\widetilde{\gamma}^{\tau'}\|_1\le cE_d^{1/2}.$$

For the same reasons, using the definition of Q in (36) and (37), we have

$$\|Q^{\tau}-Q^{\tau'}\|_1\le cE_d^{1/2}.$$

To prove an estimate for $\left|L_t^\tau - L_t^{\tau'}\right|$, we begin by finding a helpful expression for L_t^τ and $L_t^{\tau'}$. We have established that $L_t^\tau = -2\pi\langle\langle\theta_\alpha^\tau U^\tau\rangle\rangle$, and we also have written $U^\tau = \tau U^{\mathrm{s.t.},\tau} + \widetilde{U}^\tau$. We also have the equation (37) as an expression for $U^{\mathrm{s.t},\tau}$. Combining these ingredients yields the following:

$$L_t^\tau = -\frac{4\pi^3\tau}{(L^\tau)^2}\langle\langle\theta_\alpha^\tau H(\theta_{\alpha\alpha}^\tau)\rangle\rangle - 2\pi\tau\langle\langle\theta_\alpha^\tau Q^\tau\rangle\rangle - 2\pi\langle\langle\theta_\alpha^\tau \widetilde{U}^\tau\rangle\rangle,$$

and of course we have the exactly corresponding formula with τ'. The most interesting piece of the estimate for $|L_t^\tau - L_t^{\tau'}|$ is then the following:

$$\left|\langle\langle\theta_\alpha^\tau H(\theta_{\alpha\alpha}^\tau)\rangle\rangle - \langle\langle\theta_\alpha^{\tau'} H(\theta_{\alpha\alpha}^{\tau'})\rangle\rangle\right| \leq \left|\langle\langle(\theta_\alpha^\tau - \theta_\alpha^{\tau'})H(\theta_{\alpha\alpha}^\tau)\rangle\rangle\right| + \left|\langle\langle\theta_\alpha^{\tau'} H(\theta_{\alpha\alpha}^\tau - \theta_{\alpha\alpha}^{\tau'})\rangle\rangle\right|.$$

Since we are able to integrate by parts here (recall that the notation $\langle\langle\cdot\rangle\rangle$ indicates an integral), we are able to bound this by $cE_d^{1/2}$.

We omit the remaining details, but the rest of the proof follows similar, and perhaps simpler, lines. □

7. Conclusion

We have demonstrated estimates for the two-dimensional interfacial Darcy flow problem, in two cases. First, we have estimated the norm of the solution in the case that surface tension is present. Second, in the case that the stability condition (k is negative and bounded away from zero) is satisfied, we also demonstrated energy estimates. We actually demonstrated the estimates in both cases not for the solutions of the physical problem, but instead for an approximate solution (i.e., the solution of the initial value problem for an equation with mollifiers), since this is what was needed for the proofs of our theorems.

Using these energy estimates, we first proved that for fixed, positive surface tension ($\tau > 0$), the initial value problem is well-posed in sufficiently regular Sobolev spaces. Then, we studied the behavior of these solutions as τ vanishes in the case that the stability condition is satisfied. There is work in the literature (see the introduction for the references) that strongly indicates that when the stability condition is violated, the solutions with surface tension do not converge to the solution without surface tension as surface tension vanishes. We reach the opposite conclusion by making an estimate of the difference of the solutions corresponding to two different values of the surface tension parameter, and showing that the solutions form a Cauchy sequence as τ vanishes. This gives a new proof of existence of solutions for the problem without surface tension.

The first energy estimate relied strongly on the presence of the surface tension, and if this were the only energy estimate, then the limit would not be able to be taken as surface tension vanished. The second energy estimate, which was still for solutions of the problem in the case that surface tension is present, was uniform in the surface tension parameter. The energy estimates also allow the parabolic nature of the Darcy problem to be shown; the problem with surface tension gains 3/2 of a spatial derivative at any positive time. Additionally, in the case that the stability condition is satisfied, there is a gain of 1/2 of a spatial derivative with a bound independent of the surface tension parameter.

There is a third kind of estimate which is available for 2D interfacial Darcy flow, and that is the estimate for the problem without surface tension. We mention this in order to make the remark that it is not necessary to consider the case with surface tension in order to show existence of solutions for the problem without surface tension. The paper [3] by the author demonstrates energy estimates for the problem without surface tension, and the well-posedness argument is sketched. We note that the sketched argument is essentially the same as the present argument in the $\tau > 0$ case, and this is also essentially the same as the argument used by the author previously in [2] for the vortex sheet with surface tension. If one were to combine the full well-posedness argument of the present work with the estimates of [3] (which is to say, if one were to carry out the details of the well-posedness proof as sketched in [3]), one would indeed arrive at a complete well-posedness proof for the $\tau = 0$ case.

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(accepted: March 29, 2013; published online: August 8, 2013)