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Irrotational Blowup of the Solution to Compressible Euler Equation

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Abstract. Compressible Euler equation is studied. First, we examine the validity of physical laws such as the conservations of total mass and energy and also the decay of total pressure. Then we show the non-existence of global-in-time irrotational solution with positive mass.

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1. Introduction

The purpose of the present paper is to study the compressible Euler equation

$$\rho_t + \nabla \cdot (\rho v) = 0$$
$$\rho(v_t + v \cdot \nabla v) + \nabla p = 0 \qquad \text{in } \mathbf{R}^3 \times (0, T). \tag{1.1}$$

Here $\rho = \rho(x,t), v = v(x,t)$, and $p = p(x,t)$ stand for the mass density, velocity, and pressure, respectively. Then we take the isentropic case

$$p = \rho^\gamma, \qquad \gamma > 1. \tag{1.2}$$

The second equation is reduced to

$$v_t + v \cdot \nabla v + \nabla \frac{\gamma \rho^{\gamma-1}}{\gamma - 1} = 0, \tag{1.3}$$

assuming $\rho > 0$. Then by

$$v \cdot \nabla v = \nabla \frac{1}{2}|v|^2 - v \times b$$
$$b = \nabla \times v, \tag{1.4}$$

it holds that

$$v_t + \nabla \left(\frac{1}{2}|v|^2 + \frac{\gamma \rho^{\gamma-1}}{\gamma - 1} \right) = v \times b. \tag{1.5}$$

The flow is called irrotational if $b = \nabla \times v = 0$, or equivalently, the velocity v takes the scalar potential denoted by ψ:

$$v = -\nabla \psi. \tag{1.6}$$

By the vortex theorem of Helmholtz if the initial velocity is irrotational then this velocity is always irrotational (see [1,3]).

One of the result obtained in this paper is the following theorem concerning non-existence of classical irrotational flow (ρ, v) global-in-time, where (ρ_0, v_0) denotes the initial value

$$(\rho, v)|_{t=0} = (\rho_0, v_0). \tag{1.7}$$

Theorem 1.1. *There is no global-in-time C^1-solution (ρ, v) to (1.1)–(1.2) satisfying*

$$\begin{aligned} & e = \frac{\rho}{2}|v|^2 + \frac{\rho^\gamma}{\gamma - 1} \in L^1(\mathbf{R}^3 \times (0, \ell)) \\ & \limsup_{R\uparrow+\infty} \sup_{t\in[0,\ell)} \|(\rho, v)(\cdot, t)\|_{L^\infty(R<|x|<2R)} < +\infty \end{aligned} \tag{1.8}$$

for any $\ell > 0$, and

$$\begin{aligned} 0 < \rho_0 = \rho_0(x) \in L^1(\mathbf{R}^3, (1 + |x|^2)dx) \\ E = \int_{\mathbf{R}^3} e(x, 0)dx < +\infty \\ v_0 = -\nabla\psi_0, \quad \psi_0 \in L^\infty(\mathbf{R}^3). \end{aligned} \tag{1.9}$$

This result holds also to

$$\begin{aligned} \rho_t + \nabla \cdot (\rho v) = 0 \quad & \text{in } \Omega \times (0, T) \\ \rho(v_t + v \cdot \nabla v) + \nabla p = 0 \quad & \text{in } \Omega \times (0, T) \\ \rho v \cdot \nu = 0 \quad & \text{on } \partial\Omega \times (0, T) \end{aligned} \tag{1.10}$$

with (1.2) and (1.7), where $\Omega = \mathbf{R}^3 \setminus \mathcal{O}$ is the outer domain with bounded star-shaped obstacle $\mathcal{O}$ (see Theorem 4.1).

Generally, if ρ_0 is uniformly close to a positive constant, the decay property of the linear part is dominant, which leads to the existence of the solution global-in-time [5,6,10]. Sideris [9], on the other hand, showed non-existence of C^1-solution global-in-time, assuming that $\rho_0 > 0$ is constant near $x = \infty$ and v_0 is out-going. Then Xin [12] studied arbitrary space dimension n and showed that if ρ_0 has compact support there is no non-trivial solution to (1.1)–(1.2) satisfing $(\rho, v) \in C^1([0, \infty), H^m(\mathbf{R}^n)), m > [\frac{n}{2}] + \max(1, \frac{4}{\gamma-1})$. In these cases, C^1-gap of ρ on its support boundary may be suspected.

Theorem 1.1 is an intermediate result of these cases, where ρ is positive everywhere but decays at $x = \infty$. From the proof, on the other hand, C^1-regularity of (ρ, v) is replaced by a weaker condition, (2.2) below. Regarding these profiles, we may assume either discontinuity (i.e., shock) or blowup of L^∞-norm of ρ in Theorem 1.1. (In one-space dimension, we have shock, see [11].)

Theorem 1.1 holds even for the Beltrami flow (see Theorem 4.2). Theorem 1.1, however, does not hold to (1.10) if Ω is a bounded domain. In fact, on bounded domain we have the stationary solution $\rho = \text{constant} > 0, v = 0$. In accordance with this property we can prove directly that there is no stationary solution satisfying the assumption of Theorem 1.1. In fact, such a stationary solution must satisfy

$$-\Delta\psi = \nabla \cdot v = 0, \qquad \psi \in L^\infty(\mathbf{R}^3),$$

and, hence $\psi = \text{constant}$ by Liouville's theorem. Then we obtain $v = -\nabla\psi = 0$ and hence $\rho = \text{constant} > 0$ by $\nabla p = 0$. Therefore, it follows that

$$E \equiv \int_{\mathbf{R}^3} e(x)dx = +\infty,$$

a contradiction. Actually, non-existence of a non-trivial stationary solution arises in more general setting (see Theorem 4.3 below).

The argument employed for the proof of Theorem 1.1 is valid to the above described case of [9], that is, ρ_0 is a constant and $v_0 = 0$ near $x = \infty$. In this case conditions (1.8) concerning the behavior at $x = \infty$ of the solution (ρ, v) are not necessary and the negativity of the deffect energy (1.12) takes place of the out-going condition imposed by [9]. More precisely we have the following theorem.

Theorem 1.2. *There is no global-in-time C^1-solution (ρ, v) to (1.1)–(1.2) satisfying*

$$\begin{aligned} v_0(x) = 0, \quad \rho_0(x) = \overline{\rho} \quad \textit{for } |x| \gg 1 \\ \nabla \times v_0 = 0, \qquad \rho_0 = \rho_0(x) > 0 \quad \textit{in } \mathbf{R}^3, \end{aligned} \tag{1.11}$$

and

$$E = \int_{\mathbf{R}^3} \frac{\rho_0}{2}|v_0|^2 + \frac{\rho_0^\gamma}{\gamma-1} - \frac{\overline{\rho}^\gamma}{\gamma-1}\ dx < 0, \tag{1.12}$$

where $\overline{\rho} > 0$ *is a constant.*

For the proof of above theorems we shall provide a new formulation to the compressible irrotational flow. It is a Hamilton system with the energy density acting as a Hamiltonian. Then the fundamental properties of the solution, that is, the conservations of total mass and energy together with the decay of total pressure imply the blowup of the solution in finite time.

This paper is organized as follows. In the next section we show that the above fundamental properties are valid under the assumption of (1.8). Then Theorem 1.1 is proven in Sect. 3. The final section is devoted to auxiliary results described above, Theorems 1.2–4.3. Two fundamental equalities, one of which is (1.4), are shown in Appendix.

2. Preliminaries

The classical solution (ρ, v) to (1.1) satisfies a weak form for

$$\begin{aligned} \rho_t + \nabla\cdot(\rho v) &= 0 \\ (\rho v)_t + \nabla\cdot\rho v\otimes v + \nabla p &= 0 \quad \text{in } \mathbf{R}^n\times(0,T) \end{aligned} \tag{2.1}$$

for $n = 3$. In fact we have

$$\rho v, \rho, p, e \in W^{1,1}_{loc}(\mathbf{R}^n\times[0,+\infty)) \tag{2.2}$$

with

$$t\in[0,+\infty)\mapsto(\rho(\cdot,t),(\rho v))\in L^1_{loc}(\mathbf{R}^n)\times L^1_{loc}(\mathbf{R}^n)^n$$

locally weakly absolutely continuous and

$$\rho v\otimes v, p\in L^\infty_{loc}([0,+\infty), L^1_{loc}(\mathbf{R}^n))$$
$$\rho|_{t=0} = \rho_0, \qquad \rho v|_{t=0} = \rho_0 v_0,$$

and it holds that

$$\frac{d}{dt}\int_{\mathbf{R}^n}\rho\varphi\ dx = \int_{\mathbf{R}^n}\rho v\cdot\nabla\varphi\ dx$$
$$\frac{d}{dt}\int_{\mathbf{R}^n}\rho v\cdot\psi\ dx = \int_{\mathbf{R}^n}\rho v\otimes v\cdot\nabla\psi + p\nabla\cdot\psi\ dx, \quad \text{a.e. } t\in[0,+\infty)$$

for any $\phi\in C_0^\infty(\mathbf{R}^n)$ and $\psi\in C_0^\infty(\mathbf{R}^n)^n$.

In this section we use the above weak form of the solution (ρ, v) to (1.1)–(1.2) satisfying $T = +\infty$ and $\rho = \rho(x,t) \geq 0$. From our results, condition (1.8) implies the total mass and energy conservations and decay of the total pressure as $t \uparrow +\infty$. Here we do not need to assume $\nabla\times v = 0$ nor $\rho > 0$. Even the space dimension n can be arbitrary, although later we shall use the results mostly for $n = 3$. The second assumption of (1.8), furthermore, can be replaced by the weaker condition

$$\limsup_{R\uparrow+\infty}\sup_{t\in[0,\ell)}\left\|\left(\frac{x}{|x|}\cdot v\right)_-(\cdot,t)\right\|_{L^\infty(R<|x|<2R)} < +\infty, \tag{2.3}$$

where $[a]_- = \max\{0, -a\}$.

First we show the total energy conservation.

Lemma 2.1. *Assuming the first relation of (1.8) and (2.3), we have*

$$\int_{\mathbf{R}^n} e(x,t)dx = E, \qquad t \in [0,\ell). \tag{2.4}$$

Proof. A direct calculation guarantees

$$\left(\frac{\rho}{2}|v|^2 + \frac{p}{\gamma-1}\right)_t + \nabla\cdot\left(\frac{\rho}{2}|v|^2 + \frac{\gamma p}{\gamma-1}\right)v = 0, \qquad \text{in } \mathbf{R}^n \times (0,+\infty)$$

by (2.1) (see [4] for example). We take the cut-off function $\varphi_R = \varphi_R(x), \varphi_R(x) = \varphi(x/R)$. Here, $\varphi = \varphi(x)$ is smooth, radially symmetric, and monotone decreasing in $r = |x|$, satisfying

$$\begin{aligned} &0 \le \varphi \le 1 \text{ in } \mathbf{R}^n, \qquad |\nabla\varphi| \le C_0\varphi^{1/2} \text{ in } \mathbf{R}^n, \\ &\varphi \equiv 1 \text{ in } B(0,1), \qquad \text{supp } \varphi \subset \overline{B(0,2)}, \end{aligned}$$

where $C_0 > 0$ is a constant. Such $\varphi = \varphi(x)$ is obtained as $\varphi = \tilde{\varphi}^2$, where $0 \le \tilde{\varphi} = \tilde{\varphi}(x) \le 1$ is a radially symmetric C^∞-function decreasing in $r = |x|$ with the support radius 2 and equal to 1 in $B(0,1)$. Then it holds that

$$\begin{gathered} |\nabla\varphi_R| \le C_0R^{-1}\varphi_R^{1/2} \text{ in } \mathbf{R}^n \\ \text{supp } |\nabla\varphi_R| \subset \overline{B(0,2R)}\backslash B(0,R) \end{gathered}$$

and

$$\frac{dE_R}{dt} = \int_{\mathbf{R}^n} \left(\frac{\rho}{2}|v|^2 + \frac{\gamma p}{\gamma-1}\right) v\cdot\nabla\varphi_R\, dx, \tag{2.5}$$

where

$$E_R(t) = \int_{\mathbf{R}^n} e(x,t)\varphi_R(x)dx.$$

We obtain

$$\begin{aligned} \left|\frac{dE_R}{dt}\right| &= \left|\int_{\mathbf{R}^n} \left(\frac{\rho}{2}|v|^2 + \frac{\gamma p}{\gamma-1}\right) v\cdot\nabla\varphi_R dx\right| \\ &\le C_0R^{-1}\gamma \int_{R<|x|<2R} e\left[v\cdot\frac{x}{|x|}\right]_- dx \\ &\le C_0R^{-1}\gamma \left\|\left(\frac{x}{|x|}\cdot v\right)_-\right\|_{L^\infty(R<|x|<2R)} \int_{R<|x|<2R} e\, dx \end{aligned}$$

and hence

$$\sup_{t\in[0,\ell)} |E_R(t) - E_R(0)| \le C_0R^{-1}\gamma \cdot \sup_{t\in[0,\ell)} \left\|\left(\frac{x}{|x|}\cdot v\right)_-(\cdot,t)\right\|_{L^\infty(R<|x|<2R)} \cdot \int_0^\ell \|e(\cdot,t)\|_{L^1(R<|x|<2R)}dt.$$

The right-hand side tends to 0 as $R \uparrow +\infty$ by the assumption, and hence (2.4) follows from the monotone convergence theorem. $\square$

We turn to the total mass conservation described below.

Lemma 2.2. *Under the assumption of the first relation of (1.8) and (2.3), it holds that*

$$\|\rho(\cdot,t)\|_1 = M < +\infty, \qquad t \in [0,\ell), \tag{2.6}$$

where

$$M = \int_{\mathbf{R}^n} \rho_0(x)dx \tag{2.7}$$

Proof. Since the first relation of (1.8) implies $\sqrt{\rho}v \in L^2(\mathbf{R}^n \times (0,\ell))$, we have

$$\lim_{R\uparrow+\infty} R^{-1} \int_0^\ell \|\sqrt{\rho}v(\cdot,t)\|_{L^2(R<|x|<2R)}dt = 0. \tag{2.8}$$

For $M_R = M_R(t)$ defined by

$$M_R(t) = \int_{\mathbf{R}^n} \rho(x,t)\varphi_R(x)dx$$

it holds that

$$\left|\frac{dM_R}{dt}\right| = \left|\int_{\mathbf{R}^n} \rho v \cdot \nabla\varphi_R dx\right| \le C_0 R^{-1} \int_{\mathbf{R}^n} (\rho\varphi_R)^{1/2}\sqrt{\rho}|v|\ dx.$$

Then we have

$$\left|\frac{dM_R}{dt}\right| \le C_0 R^{-1} M_R^{1/2} \|\sqrt{\rho}v\|_{L^2(R<|x|<2R)}$$

or

$$\left|\frac{dM_R^{1/2}}{dt}\right| \le \frac{C_0 R^{-1}}{2} \|\sqrt{\rho}v\|_{L^2(R<|x|<2R)},$$

and hence

$$\sup_{t\in[0,\ell)} |M_R(t)^{1/2} - M_R(0)^{1/2}| \le \frac{C_0 R^{-1}}{2} \int_0^\ell \|(\sqrt{\rho}v(\cdot,t)\|_{L^2(R<|x|<2R)}dt.$$

Sending $R \uparrow +\infty$, we obtain (2.6) by (2.8) and the monotone convergence theorem. □

Finally, we show the decay of total pressure by the method of [12,7], using

$$(|x-tv|^2\rho)_t + \nabla\cdot|x-tv|^2\rho v = \nabla\cdot\left(2txp - \frac{2\gamma t^2}{\gamma-1}vp\right) - \left(\frac{2t^2p}{\gamma-1}\right)_t - 2\left(n - \frac{2}{\gamma-1}\right)tp \tag{2.9}$$

derived from (1.1)–(1.2). The proof of (2.9) is given in appendix for completeness.

Lemma 2.3. *Let the first relation of (1.8) and (2.3) hold for any ℓ, and let*

$$0 \le \rho_0 = \rho_0(x) \in L^1(\mathbf{R}^n, (1+|x|^2)dx). \tag{2.10}$$

Then it holds that

$$\int_{\mathbf{R}^n} p(x,t)\ dx = O(t^{-\min\{2,2-\alpha\}}), \qquad t \uparrow +\infty, \tag{2.11}$$

where $\alpha = 2 - n(\gamma-1)$.

Proof. First, we have

$$\frac{d}{dt}\int_{\mathbf{R}^n}\left(|x-tv|^2\rho+\frac{2t^2p}{\gamma-1}\right)\varphi_R\,dx+\frac{n(\gamma-1)-2}{t}\int_{\mathbf{R}^n}\frac{2t^2p}{\gamma-1}\varphi_R\,dx$$
$$=\int_{R<|x|<2R}\left(-2tx+\frac{2\gamma t^2}{\gamma-1}v\right)p\cdot\nabla\varphi_R+|x-tv|^2\rho v\cdot\nabla\varphi_R\,dx \tag{2.12}$$

by (2.9). Let

$$A_R(t)=\int_{\mathbf{R}^n}\left(|x-tv|^2\rho+\frac{2t^2p}{\gamma-1}\right)\varphi_R\,dx.$$

In the case of $\gamma\geq 1+2/n$, it holds that

$$\frac{dA_R}{dt}\leq C_0\left\{2t+\frac{2\gamma t^2}{\gamma-1}\left\|\left(\frac{x}{|x|}\cdot v\right)_-\right\|_{L^\infty(R<|x|<2R)}\right\}\int_{R<|x|<2R}p\,dx$$
$$+2\int_{R<|x|<2R}(|x|^2+t^2|v|^2)\rho[v\cdot\nabla\varphi_R]_+dx, \tag{2.13}$$

where $[\cdot]_+=\max\{0,\cdot\}$. Here the first term on the right-hand side is estimated above by

$$C_0\left\{2\ell+\frac{2\gamma\ell^2}{\gamma-1}\left\|\left(\frac{x}{|x|}\cdot v\right)_-\right\|_{L^\infty(R<|x|<2R)}\right\}\cdot(\gamma-1)\int_{R<|x|<2R}e\,dx$$

for $t\in[0,\ell)$. Then we use

$$\int_{R<|x|<2R}|x|^2\rho[v\cdot\nabla\varphi_R]_+dx$$
$$\leq 2C_0\left\|\left(\frac{x}{|x|}\cdot v\right)_-\right\|_{L^\infty(R<|x|<2R)}\int_{R<|x|<2R}|x|\rho\varphi_R^{1/2}dx$$
$$\leq 2C_0\left\|\left(\frac{x}{|x|}\cdot v\right)_-\right\|_{L^\infty(R<|x|<2R)}\cdot M^{1/2}\cdot\left\{\int_{R<|x|<2R}|x|^2\rho\varphi_R dx\right\}^{1/2},$$

and

$$\int_{R<|x|<2R}|x|^2\rho\varphi_R dx\leq 2\int_{\mathbf{R}^n}|x-tv|^2\rho\varphi_R dx+2t^2\int_{R<|x|<2R}|v|^2\rho\,dx.$$

It holds that

$$\int_{R<|x|<2R}|x|^2\rho[v\cdot\nabla\varphi_R]_+dx\leq 2\sqrt{2}C_0\left\|\left(\frac{x}{|x|}\cdot v\right)_-\right\|_{L^\infty(R<|x|<2R)}M^{1/2}$$
$$\cdot\left\{\int_{\mathbf{R}^n}|x-tv|^2\rho\varphi_R\,dx+\ell^2\int_{R<|x|<2R}|v|^2\rho\,dx\right\}^{1/2}$$

with

$$\left\{ \int_{\mathbf{R}^n} |x - tv|^2 \rho\varphi_R \ dx + \ell^2 \int_{R<|x|<2R} |v|^2 \rho \ dx \right\}^{1/2}$$
$$\leq \int_{\mathbf{R}^n} |x - tv|^2 \rho\varphi_R \ dx + 2\ell^2 \int_{R<|x|<2R} e \ dx + \frac{1}{4}.$$

It holds also that

$$\int_{R<|x|<2R} |v|^2 \rho [v \cdot \nabla\varphi_R]_+ dx$$
$$\leq C_0 R^{-1} \left\| \left(\frac{x}{|x|} \cdot v\right)_- \right\|_{L^\infty(R<|x|<2R)} \int_{R<|x|<2R} |v|^2 \rho \ dx$$
$$\leq 2C_0 R^{-1} \left\| \left(\frac{x}{|x|} \cdot v\right)_- \right\|_{L^\infty(R<|x|<2R)} \int_{R<|x|<2R} e \ dx.$$

Summing up the above estimates, we obtain $C = C(\ell) > 0$ independent of $R \gg 1$ such that

$$\frac{dA_R}{dt} \leq C\|e\|_{L^1(R<|x|<2R)} + A_R + \frac{1}{4}, \qquad 0 \leq t < \ell$$

by (2.3). Then it follows that

$$e^{-\ell} \sup_{t\in[0,\ell)} A_R(t) \leq A_R(0) + C \int_0^\ell \|e(\cdot,t)\|_{L^1(R<|x|<2R)} dt + \frac{\ell}{4}.$$

Sending $R \uparrow +\infty$, we obtain

$$\sup_{t\in[0,\ell)} \int_{\mathbf{R}^n} |x - tv|^2 \rho + \frac{2t^2\rho}{\gamma - 1} \ dx < +\infty \tag{2.14}$$

by the monotone convergence theorem and the assumption (2.10), where $\ell > 0$ is arbitrary. Using (2.3), Lemma 2.1, and (2.14), we have

$$\lim_{R\uparrow+\infty} \int_0^\ell dt \int_{R<|x|<2R} (|x|^2 + t^2|v|^2)\rho[v \cdot \nabla\varphi_R]_+ dx = 0$$

by the dominated convergence theorem.

Now it holds that

$$\frac{dA_R}{dt} \leq C\|e\|_{L^1(R<|x|<2R)} + B_R(t)$$

by (2.13), with $B_R = B_R(t)$ satisfying

$$\lim_{R\uparrow+\infty} \int_0^\ell B_R(t) dt = 0.$$

Thus we obtain

$$A_R(t) \leq A_R(0) + \int_0^\ell C\|e(\cdot,t)\|_{L^1(R<|x|<2R)} + B_R(t) \ dt, \quad 0 \leq t < \ell$$

which implies

$$A(t) \equiv \int_{\mathbf{R}^n} |x - tv|^2 + \frac{2t^2 p}{\gamma - 1}\ dx$$
$$\leq A(0) = \int_{\mathbf{R}^n} |x|^2 \rho_0 dx < +\infty, \quad 0 \leq t < \ell.$$

Hence it follows that

$$\int_{\mathbf{R}^n} p(x,t) dx \leq \frac{\gamma - 1}{2} t^{-2} \cdot \int_{\mathbf{R}^n} |x|^2 \rho_0\ dx, \qquad t > 0.$$

In the other case of $1 < \gamma < 1 + \frac{2}{n}$, we use $\alpha = 2 - n(\gamma - 1) \in (0,2)$. Then (2.12) reads

$$\frac{d}{dt} t^{-\alpha} \int_{\mathbf{R}^n} \left(|x - tv|^2 \rho + \frac{2t^2 p}{\gamma - 1} \right) \varphi_R\ dx$$
$$= t^{-\alpha} \int_{R<|x|<2R} \left(-2tx + \frac{2\gamma t^2}{\gamma - 1} v \right) p \cdot \nabla \varphi_R + |x - tv|^2 \rho v \cdot \nabla \varphi_R\ dx.$$

Translating t to $t+1$, now we obtain

$$\frac{d}{dt}(t+1)^{-\alpha} \int_{\mathbf{R}^n} \left(|x - (t+1)v|^2 \rho + \frac{2(t+1)^2 p}{\gamma - 1} \right) \varphi_R\ dx$$
$$= (t+1)^{-\alpha} \int_{R<|x|<2R} \left(-2(t+1)x + \frac{2\gamma (t+1)^2}{\gamma - 1} v \right) p \cdot \nabla \varphi_R$$
$$+ |x - (t+1)v|^2 \rho v \cdot \nabla \varphi_R\ dx$$

which implies

$$B(t+1) \equiv (t+1)^{-\alpha} \int_{\mathbf{R}^n} \left(|x - (t+1)v|^2 \rho + \frac{2(t+1)^2 p}{\gamma - 1} \right)\ dx$$
$$\leq B(0) = \int_{\mathbf{R}^n} |x - v_0|^2 \rho_0 + \frac{2\rho_0^\gamma}{\gamma - 1}\ dx < +\infty, \quad t > 0$$

similarly. Then it follows that

$$\int_{\mathbf{R}^n} p(x,t) dx = O(t^{\alpha - 2})$$

and the proof is complete. □

The condition (2.3) may be weakend as

$$\limsup_{R \uparrow +\infty} \sup_{t \in [0,T)} \left\| \left(\frac{x}{|x|} \cdot v \right)_{-} (\cdot, t) \right\|_{L^\infty (R<|x|<R+1)} < +\infty$$

just for (2.4) or (2.6) to guarantee. In fact, we have only to use the cut-off function $\varphi^R = \varphi^R(x)$ for $\varphi_R = \varphi_R(x)$, smooth, radially symmetric, and monotone decreasing in $r = |x|$, satisfying

$$0 \leq \varphi^R \leq 1 \text{ in } \mathbf{R}^n, \qquad |\nabla \varphi^R| \leq C_1 \ \text{ in } \mathbf{R}^n,$$
$$\varphi^R \equiv 1 \text{ in } B(0,R), \qquad \text{supp } \varphi^R \subset \overline{B(0,R+1)},$$

where $C_1 > 0$ is a constant. Also the condition $e \in L^1(\mathbf{R}^n \times (0, \ell))$ may be replaced by

$$\lim_{R \uparrow +\infty} \int_0^\ell \|e(\cdot, t)\|_{L^1(R<|x|<2R)} dt = 0$$

for Lemma 2.3 to hold.

3. Proof of Theorem 1.1

We come back to the irrotational, global-in-time C^1-solution to (1.1)–(1.2). First, the positivity of $\rho = \rho(x, t)$ follows from the first equation of (1.1) (see [2], for example).

Lemma 3.1. *If $\rho_0 = \rho_0(x) > 0$ in $\mathbf{R}^3$ then it holds that $\rho = \rho(x, t) > 0$ in $\mathbf{R}^3 \times (0, T)$.*

The second equation of (1.1) is now reduced to (1.5), and hence we obtain

$$\begin{aligned} &\rho_t + \nabla \cdot (\rho v) = 0 \\ &v_t + \nabla \left(\frac{1}{2}|v|^2 + \frac{\gamma \rho^{\gamma-1}}{\gamma - 1} \right) = 0 \quad \text{in } \mathbf{R}^3 \times (0, T) \end{aligned} \tag{3.1}$$

by $b = \nabla \times v = 0$. Theorem 1.1 is thus obtained by the following lemma.

Lemma 3.2. *There is no global-in-time C^1-solution to (3.1) with $\rho \geq 0$, satisfying (1.8), (2.4), (2.6), and (2.11).*

First,

$$\psi = \int_0^t \frac{1}{2}|v|^2 + \frac{\gamma \rho^{\gamma-1}}{\gamma - 1} \, dt' + \psi_0 \in W^{1,1}_{loc}(\mathbf{R}^3 \times [0, +\infty))) \tag{3.2}$$

satisfies

$$\nabla \psi_t = -v_t$$

and hence

$$v = -\nabla \psi \tag{3.3}$$

holds by the last equality of (1.9). Adding a constant to ψ_0, recalling (1.9), we may assume

$$\begin{aligned} \psi_t &= \frac{1}{2}|v|^2 + \frac{\gamma \rho^{\gamma-1}}{\gamma - 1} \\ \psi|_{t=0} &= \psi_0 \geq 0 \end{aligned} \tag{3.4}$$

and then $\psi = \psi(x, t) \geq 0$ follows everywhere. We have, finally,

$$\limsup_{R \uparrow +\infty} \sup_{t \in [0, \ell)} \|\psi(\cdot, t)\|_{L^\infty(R<|x|<2R)} < +\infty \tag{3.5}$$

for any $\ell > 0$ by the second requirement of (1.8) and the definition (3.2) of ψ. Now we take the following lemma.

Lemma 3.3. *Under the assumption of Theorem 3.2 it holds that*

$$\|(\rho\psi)(\cdot, t)\|_1 \leq \|\rho_0\|_1 \|\psi_0\|_\infty + \int_0^t \left[-E + \frac{\gamma + 1}{\gamma - 1} \|p(\cdot, t')\|_1 \right] dt' \tag{3.6}$$

as $t \uparrow +\infty$.

Proof. From the first equation of (3.1), (3.3), and (3.4), it follows that

$$\begin{aligned}\frac{d}{dt}\int_{\mathbf{R}^3}(\rho\psi)\varphi_R\,dx &= \int_{\mathbf{R}^n}(\rho_t\psi+\rho\psi_t)\varphi_R\,dx\\ &= \int_{\mathbf{R}^3}\rho v\cdot\nabla(\psi\varphi_R)+\rho\left[\frac{1}{2}|v|^2+\frac{\gamma}{\gamma-1}\rho^{\gamma-1}\right]\varphi_R\,dx\\ &= \int_{\mathbf{R}^n}\left[-\frac{\rho}{2}|v|^2+\frac{\gamma}{\gamma-1}\rho^{\gamma}\right]\varphi_R+\rho v\cdot\psi\nabla\varphi_R\,dx. \end{aligned}\tag{3.7}$$

Then we argue similarly to the proof of Lemma 2.3, putting

$$D_R(t)=\int_{\mathbf{R}^3}(\rho\psi\varphi_R)(x,t)dx.$$

Thus, first, we have

$$\frac{dD_R}{dt}=\int_{\mathbf{R}^3}\left[-\frac{\rho}{2}|v|^2+\frac{\gamma}{\gamma-1}\rho^{\gamma}\right]\varphi_R+\rho v\cdot\psi\nabla\varphi_R dx$$

with

$$\int_{\mathbf{R}^3}\left[-\frac{\rho}{2}|v|^2+\frac{\gamma}{\gamma-1}\rho^{\gamma}\right]\varphi_R dx=\int_{\mathbf{R}^n}\left[-e+\frac{\gamma+1}{\gamma-1}p\right]\varphi_R dx$$

$$\begin{aligned}\int_{\mathbf{R}^3}\rho v\cdot\psi\nabla\varphi_R dx\leq{}&\left\|\left(\frac{x}{|x|}\cdot v\right)_-\right\|_{L^\infty(R<|x|<2R)}\\ &\cdot\|\psi\|_{L^\infty(R<|x|<2R)}\cdot\frac{1}{R}\int_{R<|x|<2R}\rho\,dx.\end{aligned}$$

Then, operating $\int_0^t dt'\cdot$ and sending $R\uparrow+\infty$ in (3.7), we end up with

$$\begin{aligned}\int_{\mathbf{R}^n}\rho\psi\,dx &= \int_{\mathbf{R}^n}\rho_0\psi_0\,dx+\int_0^t dt'\int_{\mathbf{R}^n}-\frac{\rho}{2}|v|^2+\frac{\gamma}{\gamma-1}\rho^{\gamma}\,dx\\ &= \int_{\mathbf{R}^n}\rho_0\psi_0\,dx+\int_0^t dt'\int_{\mathbf{R}^n}-e+\frac{\gamma+1}{\gamma-1}p\,dx.\end{aligned}$$

Then (3.6) follows from $\rho\psi\geq 0$. □

Now we are able to give the following proof.

Proof of Theorem 3.2. Since $E>0$ the right-hand side of (3.6) is negative for $t\gg 1$ by (2.11), a contradiction. □

4. Proof of Theorem 1.2 and Other Results

In this section we show the auxiliary results described in Sect. 1 and prove also Theorem 1.2.

First, the proof of Lemmas 2.1 and 2.2 is valid even to (1.10), $\Omega=\mathbf{R}^3\setminus\mathcal{O}$ with bounded domain $\mathcal{O}$. The argument in the proof of Lemma 2.3 works also if $\mathcal{O}$ is star-shaped. To confirm this fact we just note that (2.9) implies formally

$$\frac{d}{dt}\int\limits_{\Omega} |x-tv|^2\rho + \frac{2t^2p}{\gamma-1}\, dx + \int\limits_{\Omega} \frac{n(\gamma-1)-2}{t}\cdot\frac{2t^2p}{\gamma-1}\, dx$$
$$= -\int\limits_{\partial\mathcal{O}} 2t(x\cdot\nu)p\, dS \le 0$$

which is justified by the cut-off argument in the previous section. Thus we obtain the following theorem.

Theorem 4.1. *Let $\Omega = \mathbf{R}^3\backslash\mathcal{O}$ with the star-shaped bounded domain $\mathcal{O}$. Then there is no global-in-time C^1-solution (ρ, v) to (1.10) with (1.2) satisfying*

$$e = \frac{\rho}{2}|v|^2 + \frac{\rho^\gamma}{\gamma-1} \in L^1(\Omega\times(0,\ell))$$
$$\limsup_{R\uparrow+\infty}\sup_{t\in[0,\ell)} \|(\rho,v)(\cdot,t)\|_{L^\infty(R<|x|<2R)} < +\infty$$

for any $\ell > 0$ and

$$0 < \rho_0 = \rho_0(x) \in L^1(\Omega, (1+|x|^2)dx)$$
$$E = \int\limits_{\Omega} e(x,0)dx < +\infty$$
$$v_0 = -\nabla\psi_0,\ \ \psi_0 \in L^\infty(\Omega).$$

The irrotational condition $v = -\nabla\psi$ can be replaced by the assumption

$$v\times b = 0 \tag{4.1}$$

for $b = \nabla\times v$ in the Proof of Theorem 1.1. If (4.1) is the case, this fluid is called the Beltrami flow. Any flow is the Beltrami if the space dimension n is equal to 1 or 2.

If (4.1) is the case and

$$v_0 = -\nabla\psi_0 + \nabla\times\xi_0 \tag{4.2}$$

denotes the Helmholtz decomposition, system (3.1) implies, instead of (3.3),

$$v = -\nabla\psi + \nabla\times\xi_0.$$

Then (3.7) is replaced by

$$\frac{d}{dt}\int\limits_{\mathbf{R}^3} (\rho\psi)\varphi_R\, dx = \int\limits_{\mathbf{R}^3} (\rho_t\psi + \rho\psi_t)\varphi_R\, dx$$
$$= \int\limits_{\mathbf{R}^3} \left[-\frac{\rho}{2}|v|^2 + \frac{\gamma}{\gamma-1}\rho^\gamma + \rho v\cdot\nabla\times\xi_0\right]\varphi_R + \rho v\cdot\psi\nabla\varphi_R\, dx.$$

Here, it holds that

$$-\frac{\rho}{2}|v|^2 + \frac{\gamma}{\gamma-1}\rho^\gamma + \rho v\cdot\nabla\times\xi_0$$
$$\le \left(-\frac{1}{2}+\varepsilon\right)\rho|v|^2 + \frac{\rho}{4\varepsilon}|\nabla\times\xi_0|^2 + \frac{\gamma\rho^\gamma}{\gamma-1}$$
$$= -(1-2\varepsilon)e + \frac{\rho}{4\varepsilon}|\nabla\times\xi_0|^2 + \frac{1-2\varepsilon+\gamma}{\gamma-1}\rho^\gamma,$$

and, hence

$$\|(\rho\psi)(\cdot,t)\|_1 \le \|\rho_0\|_1\|\psi_0\|_\infty + \int\limits_0^t \left[-(1-2\varepsilon)E + \frac{M}{4\varepsilon}\|\nabla\times\xi_0\|_\infty^2 + \frac{\gamma+1}{\gamma-1}\|p(\cdot,t')\|_1\right] dt',$$

where $\varepsilon > 0$ is arbitrary. Adjusting $\varepsilon > 0$ so that

$$-(1-2\varepsilon)E + \frac{M}{4\varepsilon}\|\nabla \times \xi_0\|_\infty^2 < 0,$$

we obtain the following theorem.

Theorem 4.2. *There is no global-in-time C^1-Beltrami flow (ρ, v) to (1.1)–(1.2) satisfying (1.8), (4.2) and*

$$\begin{aligned} 0 < \rho_0 &\in L^1(\mathbf{R}^3, (1+|x|^2)dx) \\ \psi_0 &\in L^\infty(\mathbf{R}^3) \\ E/M &> 2\|\nabla \times \xi_0\|_\infty^2, \end{aligned}$$

where

$$\begin{aligned} E &= \int_{\mathbf{R}^3} \frac{\rho_0}{2}|v_0|^2 + \frac{\rho_0^\gamma}{\gamma - 1}\ dx \\ M &= \int_{\mathbf{R}^3} \rho_0\ dx. \end{aligned}$$

The argument in Sect. 2, on the other hand, is applicable to the classical stationary flow. Thus if (ρ, v) is such a flow satisfying

$$\begin{aligned} 0 \le \rho &\in L^1(\mathbf{R}^n) \\ E &= \int_{\mathbf{R}^n} \frac{\rho}{2}|v|^2 + \frac{\rho^\gamma}{\gamma-1}\ dx < +\infty \\ \limsup_{R\uparrow+\infty} &\left\| \left(\frac{x}{|x|}\cdot v\right)_- \right\|_{L^\infty(R<|x|<2R)} < +\infty, \end{aligned} \tag{4.3}$$

then it holds that $\|p\|_1 = 0$ by Lemma 2.3. Hence we obtain the following theorem valid to any n, where the flow may not be irrotational.

Theorem 4.3. *There is no non-trivial stationary C^1-solution to (1.1)–(1.2) satisfying (4.3).*

We conclude this section with the following proof.

Proof of Theorem 1.2. First, we have $\rho > 0$ in $\mathbf{R}^3 \times (0, T)$ by $\rho_0(x) > 0$ in $\mathbf{R}^3$. Hence (2.1) is written as

$$\begin{aligned} \rho_t + \nabla\cdot(\rho v) &= 0 \\ (\rho v)_t + \nabla\cdot\rho^{-1}\rho v\otimes\rho v + \nabla(p - \overline{p}) &= 0 \quad \text{in } \mathbf{R}^3\times(0,T), \end{aligned}$$

where $\overline{p} = \overline{\rho}^\gamma$. It forms a system of conservation law concerning $u = (\hat{\rho}, (\hat{\rho}+\overline{\rho})v)$ denoted by

$$u_t + \nabla\cdot f(u) = 0, \qquad f(0) = 0$$

where $\hat{\rho} = \rho - \overline{\rho}$.

Since Proposition of [8] is applicable to this system, the propagtion speed of the support of $(\hat{\rho}, (\hat{\rho}+\rho_0)v)$ is finite and hence $v(\cdot, t), \rho(\cdot, t) - \overline{\rho}$ keeps 0 near $x = \infty$. More precisely, if

$$v_0(x) = 0, \quad \rho_0(x) = \overline{\rho}, \qquad |x| > R$$

is the case then it holds that

$$v(x,t) = 0, \quad \rho(x,t) = \overline{\rho}, \qquad |x| > R + \sigma t, \tag{4.4}$$

where $\sigma = (\gamma\overline{\rho}^{\gamma-1})^{1/2}$ (see [9]).

The well-definedness and the time invariance of

$$E = \int_{\mathbf{R}^3} \frac{\rho}{2}|v|^2 + \frac{\rho^\gamma}{\gamma-1} - \frac{\overline{\rho}^\gamma}{\gamma-1} \, dx$$

$$M = \int_{\mathbf{R}^3} \rho - \overline{\rho} \, dx$$

arise without assuming (1.8) by (4.4). Inequality (2.11), furthermore, is replaced by

$$\int_{\mathbf{R}^3} p - \overline{p} \, dx = O(t^{-\min\{2,2-\alpha\}}), \qquad t \uparrow +\infty,$$

which is proven by

$$(|x-tv|^2\rho)_t + \nabla \cdot |x-tv|^2 \rho v = \nabla \cdot \left(2tx(p-\overline{p}) - \frac{2\gamma t^2}{\gamma-1} vp\right) - \left(\frac{2t^2(p-\overline{p})}{\gamma-1}\right)_t - 2\left(n - \frac{2}{\gamma-1}\right) t(p-\overline{p}).$$

Hence it holds that

$$\begin{aligned} \int_{|x|<R+\sigma t} \overline{\rho}^\gamma dx &= \int_{\mathbf{R}^3} \overline{\rho}^\gamma - \rho^\gamma dx + \int_{|x|<R+\sigma t} \rho^\gamma dx \\ &= \int_{|x|<R+\sigma t} \rho^\gamma dx + o(1) \end{aligned} \tag{4.5}$$

and

$$E = \int_{\mathbf{R}^3} \frac{\rho}{2}|v|^2 + \frac{\rho^\gamma}{\gamma-1} - \frac{\overline{\rho}^\gamma}{\gamma-1} dx = \int_{\mathbf{R}^3} \frac{\rho}{2}|v|^2 dx + o(1). \tag{4.6}$$

From $\rho > 0$ and $\nabla \times v = 0$, we have

$$\begin{aligned} \rho_t + \nabla \cdot \rho v &= 0 \\ \psi_t &= \frac{1}{2}|v|^2 + \frac{\gamma \rho^{\gamma-1}}{\gamma-1} \\ v = -\nabla\psi \quad &\text{in } \mathbf{R}^3 \times (0,T) \end{aligned} \tag{4.7}$$

similarly to the previous section. We may assume $\psi_0 = 0$ for $|x| > R$, and then it holds that

$$\psi = \frac{\gamma \overline{\rho}^{\gamma-1}}{\gamma-1} t, \qquad |x| > R + \sigma t. \tag{4.8}$$

We have, furthermore,

$$\begin{aligned} &\frac{d}{dt} \int_{\mathbf{R}^3} \rho(\psi - \frac{\gamma\overline{\rho}^{\gamma-1}}{\gamma-1} t) \, dx \\ &= \int_{\mathbf{R}^3} -\psi \nabla \cdot \rho v + \rho(\frac{1}{2}|v|^2 + \frac{\gamma\rho^{\gamma-1}}{\gamma-1} - \frac{\gamma\overline{\rho}^{\gamma-1}}{\gamma-1}) \, dx \\ &= \int_{\mathbf{R}^3} -\frac{\rho}{2}|v|^2 + \frac{\gamma\rho}{\gamma-1}(\rho^{\gamma-1} - \overline{\rho}^{\gamma-1}) \, dx \\ &= -E + \int_{\mathbf{R}^3} \frac{\gamma\overline{\rho}^{\gamma-1}}{\gamma-1}(\overline{\rho} - \rho) \, dx + o(1) \\ &= -E - \frac{\gamma\overline{\rho}^{\gamma-1}}{\gamma-1} M + o(1) \end{aligned}$$

by (4.7) and (4.6) which implies

$$
\begin{aligned}
-Et + o(t) &= \int_{\mathbf{R}^3} \rho(\psi - \frac{\gamma\overline{\rho}^{\gamma-1}}{\gamma-1}t)\ dx + \frac{\gamma\overline{\rho}^{\gamma-1}}{\gamma-1}Mt \\
&= \int_{|x|<R+\sigma t} \rho(\psi - \frac{\gamma\overline{\rho}^{\gamma-1}}{\gamma-1}t) + \frac{\gamma\overline{\rho}^{\gamma-1}}{\gamma-1}(\rho-\overline{\rho})t\ dx \\
&= \int_{|x|<R+\sigma t} \rho\psi - \frac{\gamma\overline{\rho}^{\gamma-1}}{\gamma-1}t\ dx.
\end{aligned} \tag{4.9}
$$

recalling (4.4) and (4.8).

Here we use

$$
\psi_t \geq \frac{\gamma\rho^{\gamma-1}}{\gamma-1}
$$

derived from (4.7). It follows that

$$
\psi_t\rho t \geq \frac{\gamma\rho^{\gamma}}{\gamma-1}t
$$

and hence

$$
\begin{aligned}
\rho\psi - \frac{\gamma\rho^{\gamma}}{\gamma-1}t &\geq \rho\psi - \psi_t\rho t = 2\rho\psi - \frac{\partial}{\partial t}(\psi\rho t) + \psi\rho_t t \\
&= 2\rho\psi - \frac{\partial}{\partial t}\{\psi(\rho-\overline{\rho})t\} - \overline{\rho}\frac{\partial}{\partial t}(\psi t) + \psi\rho_t t \\
&= 2\rho\psi - \frac{\partial}{\partial t}\{\psi(\rho-\overline{\rho})t\} - \overline{\rho}\frac{\partial}{\partial t}\{(\psi - \frac{\gamma\overline{\rho}^{\gamma-1}}{\gamma-1}t)t\} - 2t\frac{\gamma\overline{\rho}^{\gamma}}{\gamma-1} + \psi\rho_t t.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
\int_{|x|<R+\sigma t} \rho\psi - \frac{\gamma\rho^{\gamma}}{\gamma-1}t\ dx &\geq 2\int_{|x|<R+\sigma t} \rho\psi - \frac{\gamma\overline{\rho}^{\gamma}}{\gamma-1}t\ dx \\
&\quad + \int_{\mathbf{R}^3} -\frac{\partial}{\partial t}\{\psi(\rho-\overline{\rho})t + \overline{\rho}(\psi - \frac{\gamma\overline{\rho}^{\gamma-1}}{\gamma-1}t)t\} + \psi\rho_t t\ dx \\
&= 2\int_{|x|<R+\sigma t} \rho\psi - \frac{\gamma\rho^{\gamma}}{\gamma-1}t\ dx + \int_{\mathbf{R}^3} -\frac{\partial}{\partial t}\{\psi(\rho-\overline{\rho})t \\
&\quad + \overline{\rho}(\psi - \frac{\gamma\overline{\rho}^{\gamma-1}}{\gamma-1}t)t\} + \psi\rho_t t\ dx + o(t)
\end{aligned}
$$

by (4.5) which means

$$
\begin{aligned}
&\int_{|x|<R+\sigma t} \rho\psi - \frac{\gamma\rho^{\gamma}}{\gamma-1}t\ dx \\
&\quad \leq \int_{\mathbf{R}^3} \frac{\partial}{\partial t}\left\{\psi\left(\rho-\overline{\rho}\right)t + \overline{\rho}\left(\psi - \frac{\gamma\overline{\rho}^{\gamma-1}}{\gamma-1}t\right)t\right\} - \psi\rho_t t\ dx + o(t)
\end{aligned}
$$

$$= \frac{d}{dt}\int_{\mathbf{R}^3} \psi\,(\rho-\overline{\rho})\,t + \overline{\rho}\left(\psi - \frac{\gamma\overline{\rho}^{\gamma-1}}{\gamma-1}t\right) t\ dx + t\int_{\mathbf{R}^3} \rho|v|^2\ dx + o(t)$$

$$= \frac{d}{dt}\int_{\mathbf{R}^3} \left(\psi\rho - \frac{\gamma\overline{\rho}^{\gamma}}{\gamma-1}t\right) t\ dx + t\int_{\mathbf{R}^3} \rho|v|^2\ dx + o(t),$$

recalling $\rho_t = -\nabla\cdot\rho v$ and $v = -\nabla\psi$. By (4.6) and (4.5) we thus obtain

$$\begin{aligned}
&\int_{|x|<R+\sigma t} \rho\psi - \frac{\gamma\rho^{\gamma}}{\gamma-1}t\ dx \\
&\quad \le \frac{d}{dt}\int_{\mathbf{R}^3} \left(\psi\rho - \frac{\gamma\overline{\rho}^{\gamma}}{\gamma-1}t\right) t\ dx + 2tE + o(t) \\
&\quad = \int_{\mathbf{R}^3} \psi\rho - \frac{\gamma\overline{\rho}^{\gamma}}{\gamma-1}t\ dx + t\frac{d}{dt}\int_{\mathbf{R}^3} \psi\rho - \frac{\gamma\overline{\rho}^{\gamma}}{\gamma-1}t\ dx + 2tE + o(t).
\end{aligned}$$

Here, the first term on the right-hand side is equal to

$$\int_{|x|<R+\sigma t} \psi\rho - \frac{\gamma\rho^{\gamma}}{\gamma-1}t\ dx + o(t)$$

by (4.4), (4.8), and (4.5). Hence it follows that

$$\begin{aligned}
&\int_{|x|<R+\sigma t} \rho\psi - \frac{\gamma\rho^{\gamma}}{\gamma-1}t\ dx \\
&\quad \le \int_{|x|<R+\sigma t} \psi\rho - \frac{\gamma\overline{\rho}^{\gamma}}{\gamma-1}t\ dx + t\frac{d}{dt}\int_{\mathbf{R}^3} \psi\rho - \frac{\gamma\overline{\rho}^{\gamma}}{\gamma-1}t\ dx + 2tE + o(t),
\end{aligned}$$

or equivalently,

$$-2E + o(1) \le \frac{d}{dt}\int_{\mathbf{R}^3} \psi\rho - \frac{\gamma\overline{\rho}^{\gamma}}{\gamma-1}t\ dx.$$

Then, we have

$$\begin{aligned}
-2Et + o(t) &\le \int_{\mathbf{R}^3} \psi\rho - \frac{\gamma\overline{\rho}^{\gamma}}{\gamma-1}t\ dx + o(t) \\
&= \int_{|x|<R+\sigma t} \psi\rho - \frac{\gamma\overline{\rho}^{\gamma}}{\gamma-1}t\ dx + o(t) = -Et + o(t)
\end{aligned}$$

by (4.9). It thus follows that $E \ge o(1)$, and hence $E \ge 0$ from $T = +\infty$, a contradiction. $\square$

Appendix A. Proof of (1.4)

Writing $v = (v^j)_j$ and $x = (x_i)_i$, we have

$$v \cdot \nabla v - \nabla \left(\frac{1}{2}|v|^2\right) = \left(\sum_j v^j \left(\frac{\partial v^i}{\partial x_j} - \frac{\partial v^j}{\partial x_i}\right)\right)_i$$

$$= \begin{pmatrix} v^2 \left(\frac{\partial v^1}{\partial x_2} - \frac{\partial v^2}{\partial x_1}\right) + v^3 \left(\frac{\partial v^1}{\partial x_3} - \frac{\partial v^3}{\partial x_1}\right) \\ v^1 \left(\frac{\partial v^2}{\partial x_1} - \frac{\partial v^1}{\partial x_2}\right) + v^3 \left(\frac{\partial v^2}{\partial x_3} - \frac{\partial v^3}{\partial x_2}\right) \\ v^1 \left(\frac{\partial v^3}{\partial x_1} - \frac{\partial v^1}{\partial x_3}\right) + v^2 \left(\frac{\partial v^3}{\partial x_2} - \frac{\partial v^2}{\partial x_3}\right) \end{pmatrix}$$

$$= - \begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix} \times \begin{pmatrix} \frac{\partial v^3}{\partial x_2} - \frac{\partial v^2}{\partial x_3} \\ \frac{\partial v^1}{\partial x_3} - \frac{\partial v^3}{\partial x_1} \\ \frac{\partial v^2}{\partial x_1} - \frac{\partial v^1}{\partial x_2} \end{pmatrix}$$

which means (1.4).

Appendix B. Proof of (2.9)

Since

$$\begin{aligned} (|x - tv|^2 \rho)_t &= |x - tv|^2 \rho_t + 2(x - tv) \cdot (x - tv)_t \rho \\ &= |x - tv|^2 \rho_t - 2(x - tv) \cdot (v + tv_t)\rho \end{aligned}$$

and

$$\begin{aligned} \nabla \cdot (|x - tv|^2 \rho v) &= |x - tv|^2 \nabla \cdot (\rho v) + \sum_{i,j} \rho v^j \partial_j (x_i - tv^i)^2 \\ &= |x - tv|^2 \nabla \cdot (\rho v) + 2 \sum_{i,j} \rho v^j (x_i - tv^i)(\delta_{ij} - t\partial_j v^i) \\ &= |x - tv|^2 \nabla \cdot (\rho v) + 2(x - tv) \cdot (v - t(v \cdot \nabla)v)\rho \end{aligned}$$

it holds that

$$\begin{aligned} &(|x - tv|^2 \rho)_t + \nabla \cdot (|x - tv|^2 \rho v) \\ &\quad = |x - tv|^2 (\rho_t + \nabla \cdot (\rho v)) - 2(x - tv) \cdot \{(v + tv_t)\rho - (v - t(v \cdot \nabla)v)\rho\} \\ &\quad = -2t(x - tv) \cdot \{\rho(v_t + (v \cdot \nabla)v)\} \\ &\quad = 2t(x - tv) \cdot \nabla p. \end{aligned} \tag{B.1}$$

Here we have

$$(x \cdot \nabla)p = \nabla \cdot (xp) - (\nabla \cdot x)p = \nabla \cdot (xp) - np, \tag{B.2}$$

while

$$\begin{aligned} (v \cdot \nabla)p &= \gamma \rho^{\gamma-1}(v \cdot \nabla)\rho = \gamma \rho^{\gamma-1}(\nabla \cdot (\rho v) - \rho \nabla \cdot v) \\ &= -\gamma \rho^{\gamma-1}\rho_t - \gamma p \nabla \cdot v = -p_t - \gamma \nabla \cdot (pv) + \gamma (v \cdot \nabla)p, \end{aligned}$$

implies

$$(v \cdot \nabla)p = \frac{p_t}{\gamma - 1} + \frac{\gamma}{\gamma - 1} \nabla \cdot (pv). \tag{B.3}$$

Equalities (B.2)–(B.3) imply

$$\begin{aligned}
&2t(x-tv)\cdot\nabla p \\
&\quad = 2t\nabla\cdot(xp) - 2npt - \frac{2t^2 p_t}{\gamma-1} - \frac{2\gamma t^2}{\gamma-1}\nabla\cdot(vp) \\
&\quad = \nabla\cdot\left(2txp - \frac{2\gamma t^2}{\gamma-1}vp\right) - \frac{2t^2}{\gamma-1}p_t - 2npt \\
&\quad = \nabla\cdot\left(2txp - \frac{2\gamma t^2}{\gamma-1}vp\right) - 2\left(n - \frac{2}{\gamma-1}\right)tp - \left(\frac{2t^2 p}{\gamma-1}\right)_t.
\end{aligned} \tag{B.4}$$

Then we obtain (2.9) by (B.1) and (B.4).

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