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Large Time Behavior of Density-Dependent Incompressible Navier–Stokes Equations on Bounded Domains

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Abstract. The large-time asymptotic behavior of classical solutions to the density-dependent incompressible Navier–Stokes equations driven by an external force on bounded domains in 2-D is studied. It is shown that the velocity field and its first-order derivatives converge to zero as time goes to infinity for large initial data and external forces.

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1. Introduction

Density-dependent incompressible Navier–Stokes equations occur in the mathematical modeling of the motion for a viscous incompressible non-homogeneous fluid flow. In reality, flows are often affected by external forces such as gravity. In this paper, we consider the two-dimensional density-dependent incompressible Navier–Stokes equations driven by external forces, which express the momentum balance and the conservation of mass as follows:

$$\begin{cases} \rho(U_t + U \cdot \nabla U) + \nabla P = \mu \Delta U + \rho \vec{f}, \\ \rho_t + U \cdot \nabla \rho = 0, \\ \nabla \cdot U = 0. \end{cases} \tag{1.1}$$

Here, $U = (u_1, u_2)$ is the vector velocity field, ρ is the density, the constant $\mu > 0$ models viscosity, and $\vec{f}$ stands for external forces.

In real world, flows often move in bounded domains with constraints from boundaries, where initial-boundary value problems appear. Solutions to initial-boundary value problems usually exhibit different behaviors and much richer phenomena comparing with the Cauchy problem. In this paper, we consider system (1.1) on a bounded domain in 2-D. The system is supplemented by the following initial and boundary conditions:

$$\begin{cases} (U, \rho)(\mathbf{x}, 0) = (U_0, \rho_0)(\mathbf{x}), \quad m \le \rho_0(\mathbf{x}) \le M, \quad \mathbf{x} \in \overline{\Omega}; \\ U|_{\partial\Omega} = 0, \quad t \ge 0, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$, and m, M are positive constants.

System (1.1) has drawn the attention of applied mathematicians over the past decades because of its physical background and mathematical feature. It generalizes the standard incompressible Navier–Stokes to inhomogeneous fluid flows. Both Cauchy problem and initial-boundary value problems have been studied in the literature, regarding the existence, uniqueness and regularity of solutions to the model. We refer the readers to [2–9,11–14] and references therein for details. However, the large-time asymptotic

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behavior of classical solutions to (1.1)–(1.2) has not been well-understood, especially in the presence of large initial data and external forces. The major challenge encountered in the asymptotic analysis is the coupling of the density function to the velocity equation by external forces, without which it is standard to show the convergence of the velocity field. This characteristic feature hampers the investigation of the large-time behavior of (1.1).

In this paper, we shall give a definite answer to the question of large-time behavior of (1.1). We shall show that, due to viscosity and boundary effects, the kinetic energy will converge to zero as time tends to infinity, in spite of the magnitudes of the initial data and external forces. The result improves the one obtained by Danchin in [8], where the asymptotic behavior of (1.1) is proved under the assumption that the initial velocity and external forces are small while the density fluctuation can be large.

Throughout this paper, $\|\cdot\|_{L^p}$, $\|\cdot\|_{L^\infty}$ and $\|\cdot\|_{W^{s,p}}$ denote the norms of the usual Lebesgue measurable spaces $L^p(\Omega)$, $L^\infty(\Omega)$ and the usual Sobolev space $W^{s,p}(\Omega)$, respectively. For $p=2$, we denote the norms $\|\cdot\|_{L^2}$ and $\|\cdot\|_{W^{s,2}}$ by $\|\cdot\|$ and $\|\cdot\|_{H^s}$, respectively. The function spaces under consideration are $C([0,T];H^s(\Omega))$ and $L^2([0,T];H^s(\Omega))$, equipped with norms $\sup_{0\le t\le T}\|\Psi(\cdot,t)\|_{H^s}$ and $(\int_0^T\|\Psi(\cdot,\tau)\|_{H^s}^2 d\tau)^{1/2}$, respectively. Unless specified, c_i will denote generic constants which are independent of ρ, U and t, but may depend on $\Omega, \mu, m, M, \vec{f}$ and initial data.

The main results of this paper are summarized in the following theorem.

Theorem 1.1. *Suppose that $\vec{f}$ is a potential flow and is independent of time, i.e., $\vec{f}=\nabla\phi$ for some function $\phi:\Omega\to\mathbb{R}$. In addition, suppose that $\|\phi\|_{H^2}\le F_1$ for some positive constant $F_1<\infty$. If the initial data $(\rho_0,U_0)\in H^3(\Omega)$ is compatible with the boundary condition, then there exists a unique solution (ρ,U) to (1.1)–(1.2) globally in time such that $\rho\in C([0,T);H^3(\Omega))$, $m\le\rho\le M$, and $U\in C([0,T);H^3(\Omega))\cap L^2([0,T;H^4(\Omega))$ for any $T\ge 0$. Moreover, it holds that*

$$\lim_{t\to\infty}\|U(\cdot,t)\|^2=0,\quad \lim_{t\to\infty}\|\nabla U(\cdot,t)\|^2=0,\quad \lim_{t\to\infty}\|U_t(\cdot,t)\|^2=0. \tag{1.3}$$

Remark 1.2. In the results obtained above, no smallness assumption is made upon the initial data and the external force. The external forcing term $\vec{f}$ includes important applications such as $\vec{f}=\mathbf{e}_2=(0,1)^{\mathrm{T}}$, which stands for the effect of gravitational force. The proof of the existence result can be found in the literature, see for example [14], we will focus on the asymptotic behavior.

We prove Theorem 1.1 via the method of energy estimate. The main difficulties of the proof come from the coupling between the velocity and density equations by convection, external force and boundary effects. We overcome the barrier by recovering the free energy associated with the system. Current proof involves applications of Sobolev and Ladyzhenskaya type inequalities, see Lemma 2.3, and results on Stokes equations by Temam [16], see lemma 2.2. The temporal independence of the potential function ϕ is crucial in our analysis, with whose help we will be able to recover the free energy formulation (entropy–entropy flux pair) associated with the system. By combining this key ingredient with the L^2 estimate of U we will show that $\|U(\cdot,t)\|^2\in W^{1,1}(0,\infty)$, which implies $\|U(\cdot,t)\|^2\to 0$ as $t\to\infty$. For higher order estimates, because of the lack of the spatial derivatives of the solution at the boundary, the energy estimates proceed as follows: we first apply the standard energy estimate on the temporal derivatives of the solution. We then apply the Temam's results on Stokes equation to recover the spatial derivatives. Such a process will be repeated twice, and then the coupled estimates will be composed into a desired one leading to the decay of the first order derivatives of the velocity field. The result suggests that viscosity is strong enough to compensate the effects of external forces and density fluctuation to cause the slowing down of the flow.

The rest of the paper is organized as follows. In Sect. 2, we give some basic facts that will be used throughout the paper. Then we establish the asymptotic behavior of the solution in Sect. 3.

2. Preliminaries

In this section, we will list several facts which will be used in the proof of Theorem 1.1. First we recall Poincaré inequality, which is standard (c.f. [10]).

Lemma 2.1. *Let Ω be any bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. For any function $g \in H_0^1(\Omega)$, there exists a constant $c_0 = c_0(\Omega, p)$ such that*

$$\|g\|_{L^p}^2 \leq c_0\|\nabla g\|_{L^p}^2, \quad \forall\, 1 \leq p < \infty.$$

Second, we recall some useful results from [16].

Lemma 2.2. *Let Ω be any bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. Consider the Stokes problem*

$$\begin{cases} -\mu\Delta U + \nabla P = F \quad in\ \Omega \\ \nabla \cdot U = 0 \quad in\ \Omega \\ U = 0 \quad on\ \partial\Omega. \end{cases}$$

If $F \in W^{m,p}$, then $U \in W^{m+2,p}, P \in W^{m+1,p}$ and there exists a constant $c_1 = c_1(\mu, p, m, \Omega)$ such that

$$\|U\|_{W^{m+2,p}} + \|P\|_{W^{m+1,p}} \leq c_1\|F\|_{W^{m,p}}$$

for any $p \in (1, \infty)$ and the integer $m \geq -1$.

We also need the following Sobolev embeddings and Ladyzhenskaya inequalities which are well-known and standard (c.f. [1,15]).

Lemma 2.3. *Let Ω be any bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. Then the following embeddings and inequalities hold*:

(i) $\|h\|_{L^p}^2 \leq c_2\|h\|_{H^1}^2, \quad \forall\, 1 \leq p < \infty, \quad \forall\, h \in H^1(\Omega)$;
(ii) $\|h\|_{L^\infty}^2 \leq c_3\|h\|_{W^{1,p}}^2, \quad \forall\, 2 < p < \infty, \quad \forall\, h \in W^{1,p}(\Omega)$;
(iii) $\|h\|_{L^4}^2 \leq c_4\|h\|\|\nabla h\|, \quad \forall\, h \in H_0^1(\Omega)$;
(iv) $\|h\|_{L^4}^2 \leq c_5\left(\|h\|\|\nabla h\| + \|h\|^2\right), \quad \forall\, h \in H^1(\Omega)$,

for some constants $c_i, i = 2, \ldots, 5$, depending only on Ω and p.

3. Asymptotic Analysis

In this section, we prove Theorem 1.1. The proof is based on several steps of energy estimates which are stated as a sequence of lemmas. We start with the decay estimate of $\|U(\cdot, t)\|$.

3.1. Decay of $\|U(\cdot, t)\|$

First, since ϕ is independent of t, using the continuity equation we have

$$\frac{d}{dt}\left(\int_\Omega \rho\phi d\mathbf{x}\right) = \int_\Omega \rho_t\phi d\mathbf{x} = -\int_\Omega \nabla \cdot (\rho U)\phi d\mathbf{x}.$$

Since $U|_{\partial\Omega} = 0$, after integration by parts we have

$$\frac{d}{dt}\left(\int_\Omega \rho\phi d\mathbf{x}\right) = \int_\Omega \rho U \cdot \nabla\phi d\mathbf{x}. \tag{3.1}$$

Taking L^2 inner product of $(1.1)_1$ with U, after integration by parts we have

$$\int_\Omega \rho\frac{d}{dt}\left(|U|^2\right) d\mathbf{x} - \int_\Omega \nabla \cdot (\rho U)\left(|U|^2\right) d\mathbf{x} + 2\mu\|\nabla U\|^2 = 2\int_\Omega \rho U \cdot \nabla\phi d\mathbf{x}. \tag{3.2}$$

Using the continuity equation and (3.1) we update (3.2) as

$$\frac{d}{dt}\left(\int_{\Omega}\rho|U|^2 d\mathbf{x} - 2\int_{\Omega}\rho\phi d\mathbf{x}\right) + 2\mu\|\nabla U\|^2 = 0. \tag{3.3}$$

We remark that (3.3) gives the free energy formulation of the original system.

Upon integrating (3.3) in time we have

$$\int_{\Omega}\rho|U|^2 d\mathbf{x} - 2\int_{\Omega}\rho\phi d\mathbf{x} + 2\mu\int_0^t \|\nabla U\|^2 d\tau = \int_{\Omega}\rho_0|U_0|^2 d\mathbf{x} - 2\int_{\Omega}\rho_0\phi d\mathbf{x},$$

which implies that

$$\int_{\Omega}\rho|U|^2 d\mathbf{x} + 2\mu\int_0^t \|\nabla U\|^2 d\tau \le \int_{\Omega}\rho_0|U_0|^2 d\mathbf{x} + 2\left(\|\rho_0\|_{L^\infty} + \|\rho\|_{L^\infty}\right)\|\phi\|_{L^1}.$$

Since $m \le \rho \le M$, using the condition on ϕ we get from above that

$$\int_{\Omega}\rho|U|^2 d\mathbf{x} + 2\mu\int_0^t \|\nabla U\|^2 d\tau \le c_6, \quad \forall\, t \ge 0. \tag{3.4}$$

Since $U|_{\partial\Omega} = 0$, by Lemma 2.1 we have

$$\int_0^t\int_{\Omega}\rho|U|^2 d\mathbf{x}d\tau \le M\int_0^t \|U\|^2 d\tau \le c_0 M\int_0^t \|\nabla U\|^2 d\tau,$$

which, together with (3.4), implies that

$$\int_0^t\int_{\Omega}\rho|U|^2 d\mathbf{x}d\tau \le c_7, \quad \forall\, t \ge 0. \tag{3.5}$$

Let

$$E_1(t) \equiv \int_{\Omega}\rho|U|^2 d\mathbf{x}.$$

Then, from (3.4) and (3.5) we see that

$$0 \le E_1(t) \le c_6, \quad \int_0^t E_1(\tau)d\tau \le c_7, \quad \forall\, t \ge 0. \tag{3.6}$$

Therefore

$$\int_0^t \left[E_1(\tau)\right]^2 d\tau \le c_6 c_7, \quad \forall\, t \ge 0. \tag{3.7}$$

Moreover, from (3.3) and (3.1) we have

$$\frac{d}{dt}E_1(t) = 2\int_{\Omega}\rho U\cdot\nabla\phi d\mathbf{x} - 2\mu\|\nabla U\|^2. \tag{3.8}$$

Using (3.4), (3.7)–(3.8) we have

$$\begin{aligned}\int_0^t \left| \frac{d}{d\tau} [E_1(\tau)]^2 \right| d\tau &= 2\int_0^t E_1(\tau) \left| \frac{d}{dt} E_1(\tau) \right| d\tau \\ &\leq 2\int_0^t E_1(\tau) \left(2 \left| \int_\Omega \rho U \cdot \nabla \phi d\mathbf{x} \right| + 2\mu \|\nabla U\|^2 \right) d\tau \\ &\leq 4\int_0^t E_1(\tau) \left| \int_\Omega \rho U \cdot \nabla \phi d\mathbf{x} \right| d\tau + 2c_6^2. \end{aligned} \tag{3.9}$$

For the first term on the RHS of (3.9), using (3.4) and the condition on ϕ we have

$$\left| \int_\Omega \rho U \cdot \nabla \phi d\mathbf{x} \right| \leq \|\sqrt{\rho} U\| \|\sqrt{\rho}\|_\infty \|\nabla \phi\| \leq \sqrt{c_6 M} \|\nabla \phi\| \leq c_8,$$

which implies, by (3.6) and (3.9), that

$$\int_0^t \left| \frac{d}{d\tau} [E_1(\tau)]^2 \right| d\tau \leq 4c_7 c_8 + 2c_6^2 \equiv c_9, \quad \forall\, t \geq 0,$$

which, together with (3.7), yields

$$[E_1(t)]^2 \in W^{1,1}(0, \infty).$$

Therefore, we have

$$\lim_{t\to\infty} \left(\int_\Omega \rho |U|^2 d\mathbf{x} \right)(t) = \lim_{t\to\infty} E_1(t) = 0.$$

Since $\rho \geq m$, we conclude that

$$\lim_{t\to\infty} \|U(\cdot, t)\|^2 = 0.$$

We summarize the above results in the following lemma.

Lemma 3.1. *Under the assumptions of Theorem 1.1, there exists a constant $\alpha_1 > 0$ independent of t such that*

$$\|U(\cdot, t)\|^2 + \int_0^t \|\nabla U(\cdot, \tau)\|^2 d\tau \leq \alpha_1, \quad \forall\, t \geq 0, \text{ and } \lim_{t\to\infty} \|U(\cdot, t)\|^2 = 0.$$

Remark 3.2. It is well known that the 3-D version of (1.1) has a weak solution defined for all $t \geq 0$ (c.f. [14]). We remark that, since the dimensionality does not affect the proof of the above results, by repeating the arguments in Subsect. 3.1, one can show that Lemma 3.1 still holds for weak solutions for the 3-D model.

3.2. Uniform Estimates

The uniform estimates established in this subsection will be used to prove the decay of the first order derivatives of U.

First, by taking L^2 inner product of $(1.1)_1$ with U_t we get

$$\frac{\mu}{2}\frac{d}{dt}\|\nabla U\|^2 + \int_\Omega \rho|U_t|^2 d\mathbf{x} = \int_\Omega \rho\nabla\phi\cdot U_t d\mathbf{x} - \int_\Omega \rho(U\cdot\nabla U)\cdot U_t d\mathbf{x}. \tag{3.10}$$

For the first term on the RHS of (3.10), since ϕ is independent of t, using $(1.1)_2$ and $(1.1)_3$ we have

$$\begin{aligned}\int_\Omega \rho\nabla\phi\cdot U_t d\mathbf{x} &= \frac{d}{dt}\left(\int_\Omega \rho\nabla\phi\cdot U d\mathbf{x}\right) - \int_\Omega \rho_t\nabla\phi\cdot U d\mathbf{x}\\ &= \frac{d}{dt}\left(\int_\Omega \rho\nabla\phi\cdot U d\mathbf{x}\right) + \int_\Omega \nabla\cdot(\rho U)(\nabla\phi\cdot U) d\mathbf{x}\\ &= \frac{d}{dt}\left(\int_\Omega \rho\nabla\phi\cdot U d\mathbf{x}\right) - \int_\Omega \rho U\cdot\nabla(\nabla\phi\cdot U) d\mathbf{x}.\end{aligned}$$

So we update (3.10) as

$$\begin{aligned}&\frac{d}{dt}\left(\frac{\mu}{2}\|\nabla U\|^2 - \int_\Omega \rho\nabla\phi\cdot U d\mathbf{x}\right) + \int_\Omega \rho|U_t|^2 d\mathbf{x}\\ &\quad = -\int_\Omega \rho U\cdot\nabla(\nabla\phi\cdot U) d\mathbf{x} - \int_\Omega \rho(U\cdot\nabla U)\cdot U_t d\mathbf{x}.\end{aligned} \tag{3.11}$$

Using the condition on ϕ, Lemmas 2.1 and 2.3 we have

$$\begin{aligned}&\left|-\int_\Omega \rho U\cdot\nabla(\nabla\phi\cdot U) d\mathbf{x}\right|\\ &\quad\le M\int_\Omega \left(|U|^2|D^2\phi| + |U||\nabla\phi||\nabla U|\right) d\mathbf{x}\\ &\quad\le M\left(\|U\|_{L^4}^2\|D^2\phi\| + \|U\|_{L^4}^2\|\nabla\phi\|_{L^4}^2 + \|\nabla U\|^2\right)\\ &\quad\le M\left(c_{F_1}\|U\|_{L^4}^2 + \|\nabla U\|^2\right)\\ &\quad\le M\left(c_{F_1}c_4\|U\|\|\nabla U\| + \|\nabla U\|^2\right)\\ &\quad\le M(c_{F_1}c_4c_0+1)\|\nabla U\|^2 \equiv c_{10}\|\nabla U\|^2,\end{aligned} \tag{3.12}$$

where c_{F_1} is a constant depending only on F_1.

Similarly, by Cauchy–Schwarz inequality we have

$$\begin{aligned}\left|-\int_\Omega \rho(U\cdot\nabla U)\cdot U_t d\mathbf{x}\right| &\le \frac{1}{4}\int_\Omega \rho|U_t|^2 d\mathbf{x} + \int_\Omega \rho|U\cdot\nabla U|^2 d\mathbf{x}\\ &\le \frac{1}{4}\int_\Omega \rho|U_t|^2 d\mathbf{x} + M\|U\|_{L^4}^2\|\nabla U\|_{L^4}^2.\end{aligned} \tag{3.13}$$

Plugging (3.12) and (3.13) into (3.11) we have

$$\begin{aligned}&\frac{d}{dt}\left(\frac{\mu}{2}\|\nabla U\|^2 - \int_\Omega \rho\nabla\phi\cdot U d\mathbf{x}\right) + \frac{3}{4}\int_\Omega \rho|U_t|^2 d\mathbf{x}\\ &\quad\le c_{10}\|\nabla U\|^2 + M\|U\|_{L^4}^2\|\nabla U\|_{L^4}^2.\end{aligned} \tag{3.14}$$

For the second term on the RHS of (3.14), using Lemmas 2.1, 2.3 and 3.1 we have

$$\begin{aligned} M\|U\|_{L^4}^2\|\nabla U\|_{L^4}^2 &\le Mc_4c_5\|U\|\|\nabla U\|\left(\|\nabla U\|\|D^2U\|+\|\nabla U\|^2\right)\\ &\le c_{11}\|\nabla U\|^2\|D^2U\|+c_{12}\|\nabla U\|^4. \end{aligned} \tag{3.15}$$

Since $U|_{\partial\Omega}=0$, by Lemma 2.2 and (3.15) we have

$$\begin{aligned} \|U\|_{H^2} &\le \sqrt{c_1}\left(\|\rho U_t\|+\|\rho U\cdot\nabla U\|+\|\rho\nabla\phi\|\right)\\ &\le \sqrt{c_1}\left(\sqrt{M}\|\sqrt{\rho}U_t\|+M\|U\|_{L^4}\|\nabla U\|_{L^4}+\|\rho\|_{L^4}\|\nabla\phi\|_{L^4}\right)\\ &\le c_{13}\|\sqrt{\rho}U_t\|+c_{14}\|\nabla U\|\|D^2U\|^{1/2}+c_{15}\|\nabla U\|^2+c_{16}\\ &\le c_{13}\|\sqrt{\rho}U_t\|+c_{16}+c_{17}\|\nabla U\|^2+\frac{1}{2}\|U\|_{H^2}. \end{aligned} \tag{3.16}$$

After rearranging terms we get from (3.16) that

$$\|U\|_{H^2}\le c_{18}\|\sqrt{\rho}U_t\|+c_{19}\|\nabla U\|^2+c_{20}. \tag{3.17}$$

Plugging (3.17) into (3.15) we have

$$\begin{aligned} &M\|U\|_{L^4}^2\|\nabla U\|_{L^4}^2\\ &\quad\le c_{11}\|\nabla U\|^2\left(c_{18}\|\sqrt{\rho}U_t\|+c_{19}\|\nabla U\|^2+c_{20}\right)+c_{12}\|\nabla U\|^4\\ &\quad\le c_{21}\|\nabla U\|^2\|\sqrt{\rho}U_t\|+c_{22}\|\nabla U\|^4+c_{23}\|\nabla U\|^2\\ &\quad\le \frac{1}{4}\|\sqrt{\rho}U_t\|^2+c_{24}\|\nabla U\|^4+c_{23}\|\nabla U\|^2. \end{aligned} \tag{3.18}$$

Combining (3.14) and (3.18) we have

$$\begin{aligned} &\frac{d}{dt}\left(\frac{\mu}{2}\|\nabla U\|^2-\int_\Omega\rho\nabla\phi\cdot U\,d\mathbf{x}\right)+\frac{1}{2}\int_\Omega\rho|U_t|^2d\mathbf{x}\\ &\quad\le c_{24}\|\nabla U\|^4+c_{25}\|\nabla U\|^2. \end{aligned} \tag{3.19}$$

Multiplying (3.3) by c_{25}/μ we have

$$\frac{d}{dt}\left(\frac{c_{25}}{\mu}\int_\Omega\rho|U|^2d\mathbf{x}-\frac{2c_{25}}{\mu}\int_\Omega\rho\phi d\mathbf{x}\right)+2c_{25}\|\nabla U\|^2=0. \tag{3.20}$$

Combining (3.19) and (3.20) we have

$$\frac{d}{dt}\left(E_2(t)\right)+\frac{1}{2}\left\|\sqrt{\rho}U_t\right\|^2+c_{25}\|\nabla U\|^2\le c_{24}\|\nabla U\|^4. \tag{3.21}$$

where

$$E_2(t)=\frac{\mu}{2}\|\nabla U\|^2+\frac{c_{25}}{\mu}\int_\Omega\rho|U|^2d\mathbf{x}-\int_\Omega\rho\nabla\phi\cdot U\,d\mathbf{x}-\frac{2c_{25}}{\mu}\int_\Omega\rho\phi d\mathbf{x}.$$

Next, we shall apply Gronwall inequality to (3.21). For this purpose, we observe that

$$\begin{aligned} &\left|-\int_\Omega\rho\nabla\phi\cdot U\,d\mathbf{x}-\frac{2c_{25}}{\mu}\int_\Omega\rho\phi d\mathbf{x}\right|\\ &\quad\le\frac{c_{25}}{2\mu}\int_\Omega\rho|U|^2d\mathbf{x}+c_{26}M\|\nabla\phi\|^2+\frac{2c_{25}M}{\mu}\|\phi\|_{L^1}\\ &\quad\le\frac{c_{25}}{2\mu}\int_\Omega\rho|U|^2d\mathbf{x}+c_{27}. \end{aligned}$$

We remark that the constant c_{27} does not depend on t. Therefore, we have

$$E_2(t) + c_{27} \geq \frac{\mu}{2}\|\nabla U\|^2 + \frac{c_{25}}{2\mu}\int_{\Omega} \rho|U|^2 d\mathbf{x}. \tag{3.22}$$

Using (3.22) we update (3.21) as

$$\frac{d}{dt}\left(E_2(t) + c_{27}\right) + \frac{1}{2}\|\sqrt{\rho}U_t\|^2 + c_{25}\|\nabla U\|^2 \leq c_{28}\|\nabla U\|^2\left(E_2(t) + c_{27}\right). \tag{3.23}$$

Applying Gronwall inequality to (3.23) and using Lemma 3.1 we have

$$(E_2(t) + c_{27}) \leq c_{29}, \quad \text{and} \quad \int_0^t \|\sqrt{\rho}U_t\|^2 + \|\nabla U\|^2 d\tau \leq c_{30}.$$

In particular, by (3.22) we have

Lemma 3.3. *Under the assumptions of Theorem 1.1, there exists a constant $\alpha_2 > 0$ independent of t such that*

$$\|\nabla U(\cdot,t)\|^2 + \int_0^t \|U_t(\cdot,\tau)\|^2 d\tau \leq \alpha_2, \quad \forall\ t \geq 0. \tag{3.24}$$

3.3. Decay of $\|\nabla U(\cdot,t)\|$ and $\|U_t(\cdot,t)\|$

The proof of the decay of $\|\nabla U(\cdot,t)\|$ and $\|U_t(\cdot,t)\|$ is based on applications of Lemmas 2.3 and 3.3. First, by taking temporal derivative of $(1.1)_1$ we have

$$\rho U_{tt} + \rho U\cdot\nabla U_t + \rho U_t\cdot\nabla U + \rho_t\left(U_t + U\cdot\nabla U\right) + \nabla P_t = \mu\Delta U_t + \rho_t\nabla\phi. \tag{3.25}$$

Taking L^2 inner product of (3.25) with U_t and using $(1.1)_2$ we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}U_t\|^2 + \mu\|\nabla U_t\|^2 \\
&\quad = \int_{\Omega} \rho_t\nabla\phi\cdot U_t d\mathbf{x} - \int_{\Omega} \rho(U_t\cdot\nabla U)\cdot U_t d\mathbf{x} - \\
&\qquad \int_{\Omega} \rho_t|U_t|^2 d\mathbf{x} - \int_{\Omega} \rho_t(U\cdot\nabla U)\cdot U_t d\mathbf{x} \\
&\quad \equiv \sum_{i=1}^{4} R_i.
\end{aligned}$$

Since $U_t|_{\partial\Omega} = 0$, using $(1.1)_2$, Lemmas 2.1 and 2.3 we estimate R_1 as

$$\begin{aligned}
|R_1| &= \left| \int_\Omega \nabla \cdot (\rho U) \nabla \phi \cdot U_t d\mathbf{x} \right| \\
&= \left| \int_\Omega \rho U \cdot \nabla (\nabla \phi \cdot U_t) d\mathbf{x} \right| \\
&\leq c_{31} \left(\int_\Omega |U||D^2\phi||U_t| d\mathbf{x} + \int_\Omega |U||\nabla\phi||\nabla U_t| d\mathbf{x} \right) \\
&\leq c_{32} \|D^2\phi\| \|U\|_{L^4} \|U_t\|_{L^4} + c_{33}(\varepsilon) \|U\|_{L^4}^2 \|\nabla\phi\|_{L^4}^2 + \varepsilon \|\nabla U_t\|^2 \\
&\leq c_{34} \|U\|_{H^1} \|U_t\|_{H^1} + c_{35}(\varepsilon) \|U\|_{H^1}^2 + \varepsilon \|\nabla U_t\|^2 \\
&\leq c_{36} \|\nabla U\| \|\nabla U_t\| + c_{37}(\varepsilon) \|\nabla U\|^2 + \varepsilon \|\nabla U_t\|^2 \\
&\leq c_{38}(\varepsilon) \|\nabla U\|^2 + 2\varepsilon \|\nabla U_t\|^2.
\end{aligned}$$

In a similar fashion, for R_2, using Lemma 3.3 we have

$$\begin{aligned}
|R_2| &= \left| \int_\Omega \rho (U_t \cdot \nabla U) \cdot U_t d\mathbf{x} \right| \\
&\leq M \|\nabla U\| \|U_t\|_{L^4}^2 \\
&\leq c_{39} \|U_t\| \|\nabla U_t\| \\
&\leq c_{40}(\varepsilon) \|U_t\|^2 + \varepsilon \|\nabla U_t\|^2.
\end{aligned}$$

The estimate of R_3 also follows in a similar fashion

$$\begin{aligned}
|R_3| &= \left| \int_\Omega \rho_t |U_t|^2 d\mathbf{x} \right| \\
&= \left| \int_\Omega \rho U \cdot \nabla (|U_t|^2) d\mathbf{x} \right| \\
&\leq c_{41} \|\nabla U_t\| \|U\|_{L^4} \|U_t\|_{L^4} \\
&\leq c_{42}(\varepsilon) \|U\|_{L^4}^2 \|U_t\|_{L^4}^2 + \varepsilon \|\nabla U_t\|^2 \\
&\leq c_{43}(\varepsilon) \|U_t\| \|\nabla U_t\| + \varepsilon \|\nabla U_t\|^2 \\
&\leq c_{44}(\varepsilon) \|U_t\|^2 + 2\varepsilon \|\nabla U_t\|^2.
\end{aligned}$$

Lastly, we estimate R_4 as follows:

$$\begin{aligned}
|R_4| &= \left| \int_\Omega \rho_t (U \cdot \nabla U) \cdot U_t d\mathbf{x} \right| \\
&= \left| \int_\Omega \rho U \cdot \nabla \left((U \cdot \nabla U) \cdot U_t \right) d\mathbf{x} \right|
\end{aligned}$$

$$\leq M\left(\int_\Omega |U||\nabla U|^2|U_t|d\mathbf{x} + \int_\Omega |U|^2|D^2U||U_t|d\mathbf{x} + \int_\Omega |U|^2|\nabla U||\nabla U_t|d\mathbf{x}\right). \tag{3.26}$$

Using (3.17) and (3.24) we estimate the first term on the RHS of (3.26) as

$$\begin{aligned}
&M\int_\Omega |U||\nabla U|^2|U_t|d\mathbf{x}\\
&\leq M\|\nabla U\|_{L^4}^2\|U\|_{L^4}\|U_t\|_{L^4}\\
&\leq c_{45}\left(\|\nabla U\|\|D^2U\| + \|\nabla U\|^2\right)\|\nabla U\|\|\nabla U_t\|\\
&\leq c_{46}\left(\|\sqrt{\rho}U_t\| + \|\nabla U\|^2 + 1\right)\|\nabla U\|\|\nabla U_t\|\\
&\leq c_{47}\|\sqrt{\rho}U_t\|\|\nabla U\|\|\nabla U_t\| + c_{48}\|\nabla U\|\|\nabla U_t\|\\
&\leq c_{49}(\varepsilon)\|\sqrt{\rho}U_t\|^2\|\nabla U\|^2 + c_{50}(\varepsilon)\|\nabla U\|^2 + \varepsilon\|\nabla U_t\|^2.
\end{aligned} \tag{3.27}$$

For the second term we have

$$\begin{aligned}
&M\int_\Omega |U|^2|D^2U||U_t|d\mathbf{x}\\
&\leq M\|D^2U\|\|U\|_{L^8}^2\|U_t\|_{L^4}\\
&\leq c_{51}\left(\|\sqrt{\rho}U_t\| + \|\nabla U\|^2 + 1\right)\|\nabla U\|^2\|\nabla U_t\|\\
&\leq c_{52}\|\sqrt{\rho}U_t\|\|\nabla U\|\|\nabla U_t\| + c_{53}\|\nabla U\|\|\nabla U_t\|\\
&\leq c_{54}(\varepsilon)\|\sqrt{\rho}U_t\|^2\|\nabla U\|^2 + c_{55}(\varepsilon)\|\nabla U\|^2 + \varepsilon\|\nabla U_t\|^2,
\end{aligned} \tag{3.28}$$

where we have used the Sobolev embedding $H^1(\Omega) \hookrightarrow L^p(\Omega)$ $(1 \leq p < \infty)$ in 2-D. Finally, we have

$$\begin{aligned}
&M\int_\Omega |U|^2|\nabla U||\nabla U_t|d\mathbf{x}\\
&\leq c_{56}(\varepsilon)\|\nabla U\|_{L^4}^2\|U\|_{L^8}^4 + \varepsilon\|\nabla U_t\|^2\\
&\leq c_{57}(\varepsilon)\left(\|\nabla U\|\|D^2U\| + \|\nabla U\|^2\right)\|\nabla U\|^4 + \varepsilon\|\nabla U_t\|^2\\
&\leq c_{58}(\varepsilon)\|\sqrt{\rho}U_t\|^2\|\nabla U\|^2 + c_{59}(\varepsilon)\|\nabla U\|^2 + \varepsilon\|\nabla U_t\|^2.
\end{aligned} \tag{3.29}$$

Plugging (3.27)–(3.29) into (3.26) we have

$$|R_4| \leq c_{60}(\varepsilon)\|\nabla U\|^2\|\sqrt{\rho}U_t\|^2 + c_{61}(\varepsilon)\|\nabla U\|^2 + 3\varepsilon\|\nabla U_t\|^2.$$

Collecting the estimates for R_i and plugging them into (3.25) we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}U_t\|^2 + \mu\|\nabla U_t\|^2\\
&\quad\leq c_{62}(\varepsilon)\|\nabla U\|^2\|\sqrt{\rho}U_t\|^2 + c_{63}(\varepsilon)\left(\|\nabla U\|^2 + \|U_t\|^2\right) + 8\varepsilon\|\nabla U_t\|^2.
\end{aligned}$$

Choosing $\varepsilon = \mu/16$ we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}U_t\|^2 + \frac{\mu}{2}\|\nabla U_t\|^2\\
&\quad\leq c_{64}\|\nabla U\|^2\|\sqrt{\rho}U_t\|^2 + c_{65}\left(\|\nabla U\|^2 + \|U_t\|^2\right).
\end{aligned} \tag{3.30}$$

Applying Gronwall inequality to (3.30) and using Lemmas 3.1 and 3.3 we have

$$\|U_t(\cdot,t)\|^2 \le c_{66}, \text{ and } \int_0^t \|\nabla U_t(\cdot,\tau)\|^2 d\tau \le c_{67}, \quad \forall\, t \ge 0. \tag{3.31}$$

Since

$$\int_0^t \left|\frac{d}{d\tau}\|\nabla U(\cdot,\tau)\|^2\right| d\tau \le 2\int_0^t \|\nabla U(\cdot,\tau)\|\|\nabla U_t(\cdot,\tau)\| d\tau$$
$$\le \int_0^t \|\nabla U(\cdot,\tau)\|^2 + \|\nabla U_t(\cdot,\tau)\|^2 d\tau,$$

by (3.4) and (3.31) we have

$$\int_0^t \left|\frac{d}{d\tau}\|\nabla U(\cdot,\tau)\|^2\right| d\tau \le c_{68}, \quad \forall\, t \ge 0,$$

which, together with (3.4), implies that

$$\|\nabla U(\cdot,t)\|^2 \in W^{1,1}(0,\infty).$$

Hence, we have

$$\lim_{t\to\infty} \|\nabla U(\cdot,t)\|^2 = 0.$$

From Poincaré inequality and (3.31) we have

$$\int_0^t \|U_t(\cdot,\tau)\|^2 d\tau \le c_{69}, \quad \forall\, t \ge 0, \tag{3.32}$$

which implies that

$$\int_0^t \|\sqrt{\rho} U_t(\cdot,\tau)\|^2 d\tau \le c_{70}, \quad \forall\, t \ge 0. \tag{3.33}$$

It is easy to see from (3.30)–(3.31), (3.32)–(3.33) and Lemmas 3.1–3.3 that

$$\int_0^t \left|\frac{d}{d\tau}\|\sqrt{\rho} U_t(\cdot,\tau)\|^2\right| d\tau \le c_{71}.$$

Therefore, we have

$$\|\sqrt{\rho} U_t(\cdot,t)\|^2 \in W^{1,1}(0,\infty),$$

which implies that

$$\lim_{t\to\infty} \|\sqrt{\rho} U_t(\cdot,t)\|^2 = 0.$$

Since $\rho \ge m$, it thus holds that

$$\lim_{t\to\infty} \|U_t(\cdot,t)\|^2 = 0.$$

Collecting the above results we conclude

Lemma 3.4. *Under the assumptions of Theorem 1.1, there exists a constant $\alpha_3 > 0$ independent of t such that*

$$\|U_t(\cdot,t)\|^2 + \int_0^t \|\nabla U_t(\cdot,\tau)\|^2 d\tau \le \alpha_3, \quad \forall\ t \ge 0, \quad \text{and}$$

$$\lim_{t\to\infty} \|\nabla U(\cdot,t)\|^2 = 0,\ \lim_{t\to\infty} \|U_t(\cdot,t)\|^2 = 0.$$

Lemmas 3.1–3.4 conclude our main result, Theorem 1.1.

4. Discussion

In this paper, we showed that the velocity field and its first order derivatives, associated with the 2-D density-dependent incompressible Navier–Stokes equations driven by external forces on a bounded domain, tend to zero as time goes to infinity due to viscosity and boundary effects, under the assumption that the external forcing term is a time-independent gradient flow. The result holds for large amplitudes of initial data and external forces, which improves previous results regarding large-time asymptotic behavior of solutions to the modeling equations. We proved the result by the method of energy estimate involving two major ingredients: (1) recovery of the free energy formulation associated with the system, and (2) the fact that $f(t) \in W^{1,1}(0,\infty) \Rightarrow \lim_{t\to\infty} f(t) = 0$. However, there are still many unanswered questions regarding (1.1). For example: (1) it is not clear whether Theorem 1.1 holds if the density is not strictly bounded from below, or the problem is set on the whole space; (2) the explicit decay rate of the velocity is not identified; (3) the long-time dynamics of the density is unknown based on our analysis. We leave the investigations for the future.

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