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Well-Posedness of the Hydrostatic MHD Equations

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Abstract. The well-posedness of the equations of fluid mechanics in the hydrostatic limit is well known to be a difficult problem. Partial results, both positive and negative, will be reviewed below. In this paper, it is shown that, for ideal magnetohydrodynamics, a magnetic field parallel to the flow direction can ensure well-posedness. The only condition required is that the flow is subalfvenic. The result has some relevance to viscoelastic flows of the upper convected Maxwell fluid, which, in the infinite Weissenberg number limit, is related to ideal MHD.

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1. Introduction

The hydrostatic approximation arises naturally when studying flows where the depth of the region of interest is small compared to horizontal dimensions. Examples include atmospheric and oceanic flows, boundary layers, and blood flow. Notwithstanding this in principle wide applicability, the hydrostatic equations, without additional assumptions such as depth averaging or regularizing eddy viscosity, are not widely used for analysis.

The reason for this is that these equations are, in general, not well-posed. Oliger and Sundström [12] consider the imposition of inflow and outflow conditions for the hydrostatic Euler equations and find that no local boundary condition is suitable for formulating a well-posed problem. However, even without inflow boundaries, well-posedness need not hold. The boundary conditions are of crucial importance. While the problem of flow between a wall and a free surface is well-posed [6,15], flow between two free surfaces is always ill-posed [15]. The case of flow between two walls is more complicated. Brenier [1] established local well-posedness provided Rayleigh's stability criterion holds, i.e. if the velocity profile is convex; see also the related work of Grenier [3,4]. On the other hand, it is shown in [14] that the equations are ill-posed if the velocity profile satisfies the long wave instability criterion of Heisenberg [5], i.e. if the equation

$$\int_0^1 (U(x,y)-c)^{-2}\,dy = 0 \tag{1}$$

has non-real roots.

Viscous effects do not restore the well-posedness of the hydrostatic equations; indeed, the Prandtl equations have been found to be ill-posed even when the inviscid hydrostatic equations are well-posed [2]! On the other hand, it has been noted that a longitudinal magnetic field can suppress inviscid shear flow instabilities [7]. Since the high Weissenberg number limit of certain non-Newtonian flows is formally

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equivalent to ideal MHD [10,11], this effect is also of interest in viscoelastic flows [9,13]. I also refer to [16] for a proof of linear stability of somewhat more general MHD flows with a magnetic field in the flow direction.

The stabilizing effect of a magnetic field on shear flow raises the question whether this effect can also restore well-posedness of the hydrostatic equations. In this paper, we shall prove an affirmative result for two dimensional hydrostatic ideal MHD flows bounded by parallel walls. The only assumption required on the data is that the flow is subalfvenic. The proof of well-posedness will be based on well-established abstract results for hyperbolic PDEs [8], which become applicable after a suitable transformation of the equations, which is based on Brenier's [1] semi-Lagrangian description.

2. Formulation of the Problem

We consider ideal magnetohydrodynamic flow in the strip $-\infty < x < \infty, 0 < y < \epsilon$. The governing equations are

$$\begin{aligned} \mathbf{H}_t + (\mathbf{u}\cdot\nabla)\mathbf{H} - (\mathbf{H}\cdot\nabla)\mathbf{u} &= 0, \\ \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - (\mathbf{H}\cdot\nabla)\mathbf{H} + \nabla p &= \mathbf{0}, \\ \operatorname{div}\mathbf{u} &= 0. \end{aligned} \tag{2}$$

Here, $\mathbf{u} = (u, v)$ denotes the velocity, $\mathbf{H}$ denotes the magnetic field, p denotes the pressure, and we have nondimensionalized such that the constant density and permeability become equal to 1. We impose the nonpenetration condition of zero normal velocity, $v = 0$, on the walls. We shall also assume that $\operatorname{div}\mathbf{H} = 0$ and that the normal component of $\mathbf{H}$ vanishes on the walls. These conditions need not be "imposed," they are automatically preserved if they hold for the initial data. We can then write $\mathbf{H}$ in terms of a potential,

$$\mathbf{H} = (a_y, -a_x), \tag{3}$$

the potential a is then constant on the walls and satisfies the equation

$$a_t + (\mathbf{u}\cdot\nabla)a = 0. \tag{4}$$

The hydrostatic approximation is obtained by scaling y, v and a with ϵ and formally setting $\epsilon = 0$. In the rescaled variables, the flow domain is $0 < y < 1$, and the equations become

$$\begin{aligned} a_t + ua_x + va_y &= 0, \\ u_t + uu_x + vu_y - a_y a_{xy} + a_x a_{yy} - p_x &= 0, \\ p_y &= 0, \\ u_x + v_y &= 0. \end{aligned} \tag{5}$$

On the boundaries, we have $v = 0$. Moreover, we are interested in situations where the horizontal component of the magnetic field is always positive, $a_y > 0$. We can set $a = 0$ on the lower boundary, and then $a = A > 0$ on the upper boundary.

The trick in rewriting the equations is to use a as an independent variable instead of y. This is analogous to the approach used by Brenier [1], where the vorticity played the same role. Our problem is now posed on the strip $-\infty < x < \infty, 0 < a < A$, and the vertical position y of a fluid particle is an unknown function: $y = Z(x, a, t)$. We shall use the notation $c = \partial Z/\partial a$. Let $\phi(x, y, t)$ be a smooth function and let $\psi(x, a, t)$ be the corresponding function in the new variables, i.e. $\phi(x, Z(x, a, t), t) = \psi(x, a, t)$. Using the chain rule, we establish the following relations:

$$\begin{aligned} \phi_y &= \frac{1}{c}\psi_a, \\ \phi_t &= \psi_t - \frac{Z_t}{c}\psi_a, \\ \phi_x &= \psi_x - \frac{Z_x}{c}\psi_a. \end{aligned} \tag{6}$$

By using this in the first equation of (5), we obtain

$$v = Z_t + uZ_x. \tag{7}$$

Using these relationships in the second equation of (5), we find

$$u_t + uu_x + \frac{1}{c^3}c_x + \chi(x,t) = 0, \tag{8}$$

where $\chi = p_x$ is unknown but independent of a. Here we have retained the notation u for the horizontal velocity, but we regard it as a function of x, t and a, i.e. u_t denotes the time derivative for fixed x and a, not fixed x and y. Finally, we can show that, as in [1], we have the relation

$$c_t + (uc)_x = 0. \tag{9}$$

We have now derived a system in which the independent variables are u and c, and derivatives are only with respect to t and x. The dependence on a comes in only through the unknown function $\chi(x,t)$, which is implicitly determined by the constraint

$$\int_0^A c(x,a,t)\,da = 1. \tag{10}$$

To maintain this condition, it is clearly necessary that

$$\int_0^A u(x,a,t)c(x,a,t)\,da \tag{11}$$

is independent of x. As in all channel flow problems, we are free to impose either a flow rate or an average pressure gradient, and we choose the former. We thus fix (11) to equal a fixed constant Q. We now find it convenient to make the substitution $w = uc$. After this substitution, we finally end up with the following set of equations:

$$\begin{aligned} w_t + \left(\frac{w^2-1}{c}\right)_x + \chi(x,t)c &= 0, \\ c_t + w_x &= 0. \end{aligned} \tag{12}$$

Since $\int w_t\,da = 0$, we can explicitly determine χ:

$$\chi(x,t) = \frac{\int_0^A \left(\frac{w^2-1}{c}\right)_x da}{\int_0^A c\,da}. \tag{13}$$

3. Statement of the Main Result

To state an existence theorem for solutions of (12), (13), we first define suitable function spaces. Since we want to apply theorems which pertain to solutions taking values in linear spaces, we subtract the averages from w and c. Let $r = w - Q/A, s = c - 1/A$. We obtain the new equations

$$
\begin{aligned}
r_t + \left(\frac{(r+Q/A)^2-1}{s+1/A}\right)_x + \chi(x,t)(s+1/A) &= 0, \\
s_t + r_x &= 0,
\end{aligned}
\tag{14}
$$

where

$$
\chi(x,t) = \int_0^A \left(\frac{(r+Q/A)^2-1}{s+1/A}\right)_x da. \tag{15}
$$

We impose periodic boundary conditions with a given period L in the x direction. All Sobolev spaces shall refer to functions on $\mathbb{R} \times (0, A)$ with periodicity in the x direction. Let X_n be the space all functions ϕ such that ϕ and its first n derivatives with respect to x lie in H^1 (with respect to both x and a), so, for instance $\phi \in X_2$ means that $\phi, \phi_x, \phi_{xx}, \phi_{xxx}, \phi_a, \phi_{xa}$ and ϕ_{xxa} are square integrable. Our basic spaces of functions are

$$
Y_n = \left\{ (r,s) \in (X_n)^2 \mid \int_0^A r(x,a)\, da = \int_0^A s(x,a)\, da = 0 \right\}. \tag{16}
$$

We shall prove the following existence result:

Theorem 1. *Consider given initial data $r(x,a,0) = r_0(x,a), s(x,a,0) = s_0(x,a)$ such that $(r_0, s_0) \in Y_2$. Assume, moreover, that $s_0 + 1/A > 0, (r_0 + Q/A)^2 < 1$. Then, for some time $T > 0$, there exists a solution of (14), (15) such that $(r,s) \in C([0,T], Y_2)$.*

4. An Abstract Existence Result

Our proof will be based on the application of the following abstract existence theorem from [8]. This result concerns evolution problems of the form

$$
\dot{u} = A(t,u)u + f(t,u), \tag{17}
$$

where u takes values in a Banach space, $A(t,u)$ is the infinitesimal generator of a C_0-semigroup, and f is a "perturbation" term. We say that $A \in G(X, M, \omega)$ if

$$
\|e^{At}\|_{L(X)} \le M e^{\omega t}. \tag{18}
$$

The construction of the solution is by an iteration of the form

$$
\dot{u}^{n+1} = A(t,u^n)u^{n+1} + f(t,u^n), \tag{19}
$$

with fixed initial condition $u^n(0) = u_0$.

Theorem 2. *Let $Y \subset Z \subset Z' \subset X$ be four real Banach spaces, all of them reflexive and separable, with continuous and dense inclusions. We assume that*

1. *Z' is an interpolation space between Y and X (i.e. linear operators which are bounded on both Y and X are also bounded on Z').*

   Let $N(X)$ be the set of all norms on X equivalent to the given one. On $N(X)$ we introduce a distance function

$$
d(\|\cdot\|_\alpha, \|\cdot\|_\beta) := \ln \max\{\sup_{z \neq 0} \|z\|_\alpha / \|z\|_\beta, \sup_{z \neq 0} \|z\|_\beta / \|z\|_\alpha\}. \tag{20}
$$

Let W be an open set in Y. We assume that there is a real number β and positive numbers λ_N, $\mu_N, \ldots$ such that the following hold for all $t, t' \in [0,T]$ and $w, w' \in W$.

2. *$N(t,w) \in N(X)$, and*

$$d(N(t,w), \|\cdot\|_X) \le \lambda_N,$$
$$d(N(t',w'), N(t,w)) \le \mu_N[|t'-t| + \|w'-w\|_Z]. \tag{21}$$

3. *There is an isomorphism $S(t,w) \in B(Y,X)$, with*

$$\|S(t,w)\|_{Y,X} \le \lambda_S,\ \|S(t,w)^{-1}\|_{X,Y} \le \lambda'_S,$$
$$\|S(t',w') - S(t,w)\|_{Y,X} \le \mu_S[|t'-t| + \|w'-w\|_Z]. \tag{22}$$

4. *$A(t,w) \in G(X_{N(t,w)}, 1, \beta)$.*
5. *$S(t,w)A(t,w)S(t,w)^{-1} = A(t,w) + B(t,w)$, where $B(t,w)$ is a bounded operator in X and $\|B(t,w)\|_X \le \lambda_B$.*
6. *$A(t,w) \in B(Y,Z)$ with*

$$\|A(t,w)\|_{Y,Z} \le \lambda_A,\ \|A(t,w') - A(t,w)\|_{Y,Z'} \le \mu_A\|w'-w\|_{Z'}. \tag{23}$$

Moreover, the mapping $t \to A(t,w) \in B(Y,X)$ is continuous in norm.

7. *$f(t,w) \in Y$, with*

$$\|f(t,w)\|_Y \le \lambda_f,\ \|f(t,w') - f(t,w)\|_{Z'} \le \mu_f\|w'-w\|_{Z'}, \tag{24}$$

and the mapping $t \to f(t,w) \in X$ is continuous.

If all of the above assumptions are satisfied, and $u_0 \in W \subset Y$, then there is a $T' \in (0,T]$ such that (17) has a unique solution u on $[0,T']$ with $u \in C([0,T'];W) \cap C^1([0,T'];X)$. Here T' may depend on all the constants involved in the assumptions and on the distance between u_0 and the boundary of W. The mapping $u_0 \to u(t)$ is Lipschitz continuous in the Z'-norm, uniformly for $t \in [0,T']$. The solution is obtained by the iteration (19).

5. Proof of the Existence Theorem

To apply the abstract result, we set $X = Y_0, Z = Z' = Y_1$, and $Y = Y_2$. Moreover, we define $f = 0$ and

$$A(\rho,\sigma)(r,s) = \left(\frac{2(\rho+Q/A)}{\sigma+1/A} r_x - \frac{(\rho+Q/A)^2-1}{(\sigma+1/A)^2} s_x - \chi(x,t)(\sigma+1/A), r_x\right), \tag{25}$$

where

$$\chi(x,t) = \int_0^A \frac{2(\rho+Q/A)}{\sigma+1/A} r_x - \frac{(\rho+Q/A)^2-1}{(\sigma+1/A)^2} s_x \, da. \tag{26}$$

We set $S = (\frac{\partial}{\partial x} + \lambda)^2$, where λ is any non-imaginary number. Finally, the norm N is associated with the inner product

$$\langle\langle (r,s),(r',s')\rangle\rangle = \left\langle \frac{1}{\sigma+1/A} r, r'\right\rangle + \left\langle \frac{1-(\rho+Q/A)^2}{(\sigma+1/A)^3} s, s'\right\rangle, \tag{27}$$

where $\langle\cdot,\cdot\rangle$ denotes the ordinary inner product in X_0.

We now need to verify the assumptions of the abstract existence result from the previous section. Assumption 7 is vacuous and assumptions 1 and 3 are trivial. Next we note that X_1 is a Banach algebra and a multiplier in X_0. From this we easily deduce assumptions 2 and 6. To see assumption 5, we note that $SAS^{-1} - A = (SA - AS)S^{-1}$, and an easy calculation shows that $SA - AS$ is an operator involving only derivatives with respect to x up to second order. Finally, an explicit calculation using integration by parts in x and taking advantage of the fact that $\int r\, da = 0$ (and hence $\int\int r\chi \, da\, dx = 0$) shows that

$$\langle\langle (r,s), A(\rho,\sigma)(r,s)\rangle\rangle \le C\langle\langle (r,s),(r,s)\rangle\rangle. \tag{28}$$

Assumption 4 now follows from the Lumer-Phillips theorem, once we establish that $A(\rho,\sigma)+\lambda$ is onto for λ large enough. This can be proved by a number of standard arguments. We can, for instance approximate

A by operators A_h obtained by replacing the x derivative by a symmetric difference with step size h, and then obtain uniform estimates for the resolvents of the approximate operators.

6. Other Boundary Conditions

So far, we have considered flows bounded by parallel walls. In this section, we shall see what happens when one or both of the walls are replaced by free surfaces. Instead of a boundary condition of zero normal velocity, we then have a boundary condition of zero pressure. We shall keep the requirement that the normal component of $\mathbf{H}$ vanishes.

If one of the boundaries is a free surface, the pressure term vanishes from the horizontal momentum balance. As a consequence, we obtain the system (12) without the term χ and without the integral constraint on c. This is simply a first order system which is well-posed if it is hyperbolic, and it turns out to be always hyperbolic. For a full solution, we must, however, still recover the function $Z(x, a, t)$. If one boundary is a wall and the other a free surface, this is easy. Let us say the lower boundary is a wall, then we simply have

$$Z(x, a, t) = \int_0^a c(x, \alpha, t)\, d\alpha. \tag{29}$$

If we have two free surfaces, then there is the pesky matter of how to determine the integration constant. Indeed, it is this point where the problem may become ill-posed. The proper condition is obtained by integrating the second component of the momentum equation: since the pressure is zero on both surfaces, we should have

$$\int v_t + v v_x + v v_y + a_y a_{xx} - a_x a_{xy}\, dy = 0. \tag{30}$$

In the semi-Lagrangian variables, this condition becomes

$$\int_0^A c \left(\frac{\partial}{\partial t} + u \frac{\partial}{\partial x} \right)^2 Z - \frac{Z_{xx}}{c}\, da = \dots \tag{31}$$

where the dots indicate terms that are given in terms of c and first order derivatives of Z. We can now set

$$Z(x, a, t) = \int_0^a c(x, \alpha, t)\, d\alpha + \beta(x, t), \tag{32}$$

and for the integration constant β we obtain an equation of the form

$$\beta_{tt} \int_0^A c\, da + 2\beta_{xt} \int_0^A uc\, da + \beta_{xx} \int_0^A (u^2 c - \frac{1}{c})\, da = \dots \tag{33}$$

For a well-posed problem, we need this equation to be hyperbolic, i.e.

$$\left(\int_0^A c\, da \right) \left(\int_0^A u^2 c - \frac{1}{c}\, da \right) < \left(\int_0^A uc\, da \right)^2. \tag{34}$$

This is exactly the condition which guarantees stability in the special case of parallel shear flow [13]. In the problem without magnetic field, the term $1/c$ is absent in the second integral on the left side, and the Cauchy-Schwarz inequality guarantees that the equation is elliptic and the problem is ill-posed.

The case of free surface boundaries thus turns out to be much simpler, and we have a sharp characterization of when the equations are well-posed. For the case of wall boundaries, our result is clearly not sharp; for instance, it does not imply Brenier's result [1]. Indeed, in the simpler context of stability of parallel shear flow, Brenier's result applies to the case where stability is guaranteed by Rayleigh's criterion; the result in this paper applies to the case where stability is guaranteed by Howard's semicircle theorem. Neither of them is a necessary condition for stability. A necessary condition is spectral stability, i.e. the absence of unstable eigenvalues. This leads to Heisenberg's criterion mentioned in the introduction, which can easily be generalized to the MHD case [13]. However, even in the limited context of linear stability of parallel shear flows, it is not well understood if, when and how spectral stability actually implies stability.

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