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Strong Solutions for a Phase Field Navier–Stokes Vesicle–Fluid Interaction Model

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Abstract. In this paper we study a mathematical model for the dynamics of vesicle membranes in a 3D incompressible viscous fluid. The system is in the Eulerian formulation, involving the coupling of the incompressible Navier–Stokes system with a phase field equation. This equation models the vesicle deformations under external flow fields. We prove the local in time existence and uniqueness of strong solutions. Moreover, we show that, given $T > 0$, for initial data which are small (in terms of T), these solutions are defined on $[0, T]$ (almost global existence).

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1. Some Background and Statement of the Main Results

The study of hydrodynamical properties of vesicle membranes is important for many applications, in particular in biological and physiological subjects. A vesicle is an elastic membrane containing a liquid and surrounded by another liquid. Such a vesicle can be found in nature or it can be created in laboratory. They can store and/or transport substances. Modeling vesicles is also a first step in order to study and understand the behavior of more complex cells such as red cells. Recent papers have been devoted to both experimental studies, see for instance [7] or [8], to the modeling and finally to the mathematical analysis of the obtained models, see for instance [3–6, 10].

To model the elastic deformation of the vesicle in the fluid is not an easy task. One possibility could be to consider equations of elasticity for the membrane and the Navier–Stokes equations outside the membrane. However this model could be very difficult to study and to simulate numerically due to the free-boundary corresponding the interfaces between the fluid and the vesicle. Moreover, in this coupling, the equations of elasticity are written in Lagrangian variables whereas the equations for the fluid are written in Eulerian variables. Another approach to model this vesicle–fluid interaction is to use the phase-field theory: the membrane of the vesicle is described by a phase field function φ. In this model, φ takes value $+1$ inside the vesicle membrane and -1 outside, with a thin transition layer of width characterized by a small positive parameter ϵ. The fluid is modeled by the incompressible Navier–Stokes equations written in the whole domain containing the fluid and the vesicle.

We consider the model introduced in [3], based on a phase-field approach and which is described by the following system of equations of unknowns $\mathbf{u}$ (the velocity of the fluid) and φ (the phase field function):

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$$
\begin{cases}
\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} = \nabla p + \nu\Delta\mathbf{u} + W(\varphi)\nabla\varphi & \text{in } [0,T]\times\Omega,\\
\nabla\cdot\mathbf{u} = 0 & \text{in } [0,T]\times\Omega,\\
\varphi_t + \mathbf{u}\cdot\nabla\varphi = -\gamma W(\varphi) & \text{in } [0,T]\times\Omega,\\
\mathbf{u}(0,x) = \mathbf{u}_0(x) & \text{in } \Omega,\\
\mathbf{u}(t,x) = 0 & \text{on } [0,T]\times\partial\Omega,\\
\varphi(0,x) = \varphi_0(x) & \text{in } \Omega,\\
\varphi(t,x) = -1 & \text{on } [0,T]\times\partial\Omega,\\
\Delta\varphi(t,x) = 0 & \text{on } [0,T]\times\partial\Omega.
\end{cases}
\tag{1.1}
$$

In the above equations we denote:

$$
\begin{aligned}
W(\varphi) &= kg(\varphi) + M_1[V(\varphi)-\alpha] + M_2[B(\varphi)-\beta]f(\varphi),\\
&= k\epsilon\Delta^2\varphi - \frac{k}{\epsilon}\Delta(\varphi^3) + \frac{k}{\epsilon}\Delta\varphi + \frac{k}{\epsilon^2}(3\varphi^2-1)f(\varphi)\\
&\quad + M_1[V(\varphi)-\alpha] + M_2[B(\varphi)-\beta]f(\varphi),
\end{aligned}
\tag{1.2}
$$

with

$$V(\varphi) = \int_\Omega \varphi\, dx, \tag{1.3a}$$

$$B(\varphi) = \int_\Omega \left(\frac{\epsilon}{2}|\Delta\varphi|^2 + \frac{1}{4\epsilon}(\varphi^2-1)^2\right) dx, \tag{1.3b}$$

$$f(\varphi) = -\epsilon\Delta\varphi + \frac{1}{\epsilon}(\varphi^2-1)\varphi, \tag{1.3c}$$

$$g(\varphi) = -\Delta f(\varphi) + \frac{1}{\epsilon^2}(3\varphi^2-1)f(\varphi). \tag{1.3d}$$

where ϵ is a small positive parameter depending on the thickness of the membrane of the vesicle. The constant $k > 0$ is the bending modulus of the vesicle. The constant α is defined by $\alpha = \int_\Omega \varphi_0 dx$. Note that $\alpha < 0$ if the vesicle is "small" compared to the remaining part of Ω. The constant β represents the total surface area of the vesicle. Moreover, M_1 and M_2 are two penalty constants used to enforce the volume and the surface of the vesicle to remain constant. More precisely, in this model, the energy of the vesicle is

$$\mathcal{E}(\varphi) := \frac{k}{2\varepsilon}\int_\Omega |f(\varphi)|^2\ dx + \frac{1}{2}M_1[V(\varphi)-\alpha]^2 + \frac{1}{2}M_2[B(\varphi)-\beta]^2,$$

where the first term in the right-hand side of the above formula
is an approximation of the Helfrich bending elasticity energy of the membrane, whereas $B(\varphi)$ is an approximation of the surface area of the vesicle. Formally, the term $W(\varphi)$ is obtained by differentiating $\mathcal{E}$ with respect to φ. For more details on the derivation of the above equations, see [3, 4].

The well-posedness of the system (1.1) has been first studied in [3]. In this work the authors have obtained the existence of weak solutions by using Galerkin's method, in the flavor of the classical techniques introduced in [2]. In the same work, the uniqueness of solutions has been proved in a class smaller than the one used for existence. The aim of the present work is to prove the local in time existence and uniqueness of a stronger concept of solution. Note that our existence and uniqueness results are valid in the same spaces. We also show the existence of almost global in time strong solutions provided that the initial data and the constant $(\mu(\Omega)+\alpha)^2$ are small enough.

In order to give the precise statement of our main result, we need some notations. In the remaining part of this work, $\Omega \subset \mathbb{R}^3$ is an open bounded set with smooth boundary $\partial\Omega$. We set

$$E(t) = \|\mathbf{u}(t)\|^2_{H^1(\Omega)} + \|\varphi(t)\|^2_{H^{2+\frac{3}{8}}(\Omega)}. \tag{1.4}$$

Our first main result is:

Theorem 1.1. (Local existence and uniqueness of strong solutions) *Suppose that*

$$\begin{cases} \mathbf{u}_0 \in H^1_0(\Omega), \quad \nabla\cdot\mathbf{u}_0 = 0 \quad \text{in } \Omega, \\ \varphi_0 \in \tilde{H}^{2+\frac{3}{8}}(\Omega). \end{cases} \tag{1.5}$$

Then there exists $T = T\left(\|\mathbf{u}_0\|_{H^1(\Omega)}, \|\varphi_0\|_{H^{2+\frac{3}{8}}(\Omega)}\right) > 0$ *such that* (1.1) *has a unique maximal strong solution:*

$$\begin{cases} \mathbf{u} \in L^2(0,T;H^2(\Omega)) \cap C([0,T];H^1(\Omega)) \cap H^1(0,T;L^2(\Omega)), \\ p \in L^2(0,T;H^1(\Omega)), \\ \varphi \in L^2(0,T;\tilde{H}^{4+\frac{3}{8}}(\Omega)) \cap C([0,T];\tilde{H}^{2+\frac{3}{8}}(\Omega)) \cap H^1(0,T;H^{\frac{3}{8}}(\Omega)). \end{cases} \tag{1.6}$$

Moreover, one of the following results holds:

1. $T = +\infty$,
2. $\lim\limits_{t\to T^-} E(t) = +\infty$.

The spaces $\tilde{H}^s(\Omega)$ used above are closed subspaces of the classical Sobolev spaces $H^s(\Omega)$. We refer to the next section for the precise definition of these spaces (see (2.4)).

Remark 1.2. The power 3/8 appearing in the spaces for φ comes from the estimates on the nonlinear terms. More precisely, one of the most difficult terms to handle is $\Delta^2\varphi\nabla\varphi$, which appears in the right-hand side of the Navier–Stokes equations. To obtain strong solutions we need L^2 space-time estimates for this term. This fact suggests to work with φ in

$$L^2(0,T;\tilde{H}^{4+s}(\Omega)) \cap C([0,T];\tilde{H}^{2+s}(\Omega)) \cap H^1(0,T;H^s(\Omega)),$$

with $s \geq \frac{1}{4}$ (see the proof of Lemma 4.3 and more precisely inequality (4.8)). In fact, to carry out the fixed point procedure, we need $s > \frac{1}{4}$.

On the other hand, we can not consider s too large. In fact, to prove the existence of strong solution, we use a fixed point strategy where all the nonlinear terms are put in the right-hand sides. Thus a first step is to solve a linear problem with given right-hand sides: these are Corollaries 2.7 and 2.8. However these right-hand sides coming from the nonlinear terms do not satisfy the compatibility conditions (at the boundary) needed to obtain very regular solutions. In particular, the right-hand side for φ could not be in $L^2(\tilde{H}^s)$ for $s \geq \frac{1}{2}$ which implies the restriction $s < \frac{1}{2}$.

For all these considerations, it appears that we need to impose $\frac{1}{4} < s < \frac{1}{2}$. We take here $s = \frac{3}{8}$ but all the theory should work for s in the interval $(\frac{1}{4}, \frac{1}{2})$.

Remark 1.3. In [3], the authors prove the existence of weak solution of (1.1) with the following regularity:

$$\mathbf{u} \in L^2(0,T;V(\Omega)) \cap W^{1,\frac{4}{3}}(0,T;V(\Omega)')$$

and

$$\varphi \in L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega)).$$

Our second main result states that if the initial data and the volume of the vesicle are small in a certain sense, then we have almost global existence of strong solutions:

Theorem 1.4. (Almost global existence of strong solutions) *Given* $T > 0$, *there exists* $q = q(T) > 0$ *such that if*

$$(\mu(\Omega) + \alpha)^2 + \|\mathbf{u}_0\|^2_{H^1(\Omega)} + \|\varphi_0\|^2_{H^{2+\frac{3}{8}}(\Omega)} \leq q, \tag{1.7}$$

then the maximal strong solution (see Theorem 1.1) associated with $(\mathbf{u}_0, \varphi_0)$ *and* α *exists in the interval* $[0, T]$.

Remark 1.5. We do not obtain a result of global in time existence. The main reason for this comes from the definition of $W(\varphi)$ and more precisely from the term $M_1\,[V(\varphi)-\alpha]$. Indeed, when dealing with the a priori estimates (Sect. 3), this term gives some constants in the right-hand side which can not be absorbed by the viscous term as for the classical Navier–Stokes equations (see (3.10) for details).

2. Notation and Preliminaries

We first recall for later use some interpolation inequalities involving Sobolev spaces, see [9, pp.22–23].

Proposition 2.1. *Assume $s_1>0$, $s_2>0$ and set $s=(1-\vartheta)s_1+\vartheta s_2$ for $\vartheta\in(0,1)$. Then there exists $C=C(\vartheta,s_1,s_2,\Omega)$ such that, for evey u in $H^{s_1}(\Omega)\cap H^{s_2}(\Omega)$, we have*

$$\|u\|_{H^s(\Omega)}\le C\|u\|^{1-\vartheta}_{H^{s_1}(\Omega)}\|u\|^{\vartheta}_{H^{s_2}(\Omega)}.$$

Remark 2.2. In particular for $s_1=4+\frac{3}{8}$, $s_2=2+\frac{1}{4}$ and $\vartheta=\frac{16}{17}$, we obtain

$$\|\phi\|_{H^4(\Omega)}\le\|\phi\|_{H^{4+\frac{1}{4}}(\Omega)}\le C\|\phi\|^{\frac{16}{17}}_{H^{4+\frac{3}{8}}(\Omega)}\|\phi\|^{\frac{1}{17}}_{H^{2+\frac{1}{4}}(\Omega)}. \tag{2.1}$$

For $s_1=2$, $s_2=1$ and $\vartheta=\frac{3}{4}$, we obtain

$$\|u\|_{H^{1+\frac{3}{4}}(\Omega)}\le C\|u\|^{\frac{3}{4}}_{H^2(\Omega)}\|u\|^{\frac{1}{4}}_{H^1(\Omega)}\le C\|u\|^{\frac{16}{17}}_{H^2(\Omega)}\|u\|^{\frac{1}{17}}_{H^1(\Omega)}. \tag{2.2}$$

The following proposition will be used to estimate the nonlinear terms.

Proposition 2.3. (Banach Algebra) *If $s>\frac{3}{2}$, then $H^s(\Omega)$ is a Banach Algebra. More precisely, there exists a positive constant $C=C(\Omega)$ such that for all $u\in H^s(\Omega)$ and $v\in H^s(\Omega)$, we have $uv\in H^s(\Omega)$ and*

$$\|uv\|_{H^s(\Omega)}\le C\|u\|_{H^s(\Omega)}\|v\|_{H^s(\Omega)}.$$

For the proof of the above proposition, we refer to [9, pp.45–46].

Let $\mathcal{A}$ be the Dirichlet Laplacian in Ω, that is:

$$\mathcal{A}:D(\mathcal{A})=H^2(\Omega)\cap H^1_0(\Omega)\to L^2(\Omega),\quad u\mapsto-\Delta u. \tag{2.3}$$

It is well-known that $\mathcal{A}$ is a positive self-adjoint operator and we set

$$\tilde{H}^s(\Omega)=D(\mathcal{A}^{\frac{s}{2}}).$$

Using classical results on $\mathcal{A}$ and on interpolation theory (see, for instance, [1, pp.111–115]), we have the following characterization of $\tilde{H}^s(\Omega)$, for $s-\frac{1}{2}\notin\mathbb{N}$:

$$\tilde{H}^s(\Omega)=\left\{u\in H^s(\Omega);\ (-\Delta)^j u=0\text{ on }\partial\Omega,\quad\text{for }0\le 2j<s-\frac{1}{2},j\in\mathbb{N}\right\}. \tag{2.4}$$

In particular, we use in this paper the spaces

$$\begin{aligned}\tilde{H}^{\frac{3}{8}}(\Omega)&=H^{\frac{3}{8}}(\Omega),\\ \tilde{H}^{2+\frac{3}{8}}(\Omega)&=H^{2+\frac{3}{8}}(\Omega)\cap H^1_0(\Omega),\\ \tilde{H}^{4+\frac{3}{8}}(\Omega)&=\left\{u\in H^{4+\frac{3}{8}}(\Omega)\ ;\ u=\Delta u=0\text{ on }\partial\Omega\right\}.\end{aligned}$$

Proposition 2.4. *The space $\tilde{H}^s(\Omega)$ is a closed subspace of $H^s(\Omega)$. Moreover, if we set*

$$\|\phi\|_{\tilde{H}^s(\Omega)}:=\left\|A^{\frac{s}{2}}\phi\right\|_{L^2(\Omega)},$$

then we have the following inequality

$$\|\phi\|_{H^s(\Omega)}\le C\|\phi\|_{\tilde{H}^s(\Omega)}\quad(\phi\in\tilde{H}^s(\Omega)). \tag{2.5}$$

For the proof of the above proposition, we refer, for instance, to [9, pp. 31–33]. The inequality (2.5) comes from elliptic regularity of $\mathcal{A}$ and from classical interpolation theorems.

Remark 2.5. If there is no confusion, we simply denote $H^s(\Omega)$ by H^s and $\tilde{H}^s(\Omega)$ by $\tilde{H}^s$. We also denote $L^2(0,T;H^s(\Omega))$ and $C([0,T];H^s(\Omega))$ by $L^2(H^s)$ and $C(H^s)$, respectively.

We also recall some classical notations from [2]. Define

$$\mathscr{V}(\Omega) = \left\{\mathbf{u} \in (C_0^\infty(\Omega))^3 \ \mid \nabla\cdot\mathbf{u} = 0\right\}.$$

Let $H(\Omega)$ be the closure of $\mathscr{V}(\Omega)$ in $[L^2(\Omega)]^3$. We denote by $V(\Omega)$ the closure of $\mathscr{V}(\Omega)$ in $[H_0^1(\Omega)]^3$. We also denote by P the orthogonal projector from $[L^2(\Omega)]^3$ onto $H(\Omega)$ (the Leray projector). Consider the Stokes operator $A : D(A) \to H(\Omega)$, where $D(A) = V(\Omega)\cap H^2(\Omega)$, $A\mathbf{u} = -P\Delta\mathbf{u}$, $\forall\mathbf{u}\in D(A)$. If we apply the projector P to the first equation in (1.1) and if we set $\phi = \varphi + 1$, then we can rewrite (1.1) in the following form:

$$\begin{cases} \mathbf{u}_t + \nu A\mathbf{u} = -P(\mathbf{u}\cdot\nabla\mathbf{u}) + P(W(\phi-1)\nabla\phi) & \text{in } [0,T]\times\Omega,\\ \phi_t + k\gamma\epsilon\mathcal{A}^2\phi = -\mathbf{u}\cdot\nabla\phi - \gamma L(\phi-1) & \text{in } [0,T]\times\Omega,\\ \mathbf{u}(0,x) = \mathbf{u}_0(x) & \text{in } \Omega,\\ \phi(0,x) = \phi_0(x) & \text{in } \Omega,\\ \mathbf{u}(t,x) = 0 & \text{on } [0,T]\times\partial\Omega,\\ \phi(t,x) = 0 & \text{on } [0,T]\times\partial\Omega,\\ \Delta\phi(t,x) = 0 & \text{on } [0,T]\times\partial\Omega. \end{cases} \tag{2.6}$$

We write for later use $W(\phi) = k\epsilon\Delta^2\phi + L(\phi)$, with $L(\phi)$ given by:

$$\begin{aligned} L(\phi) =\ & -\frac{k}{\epsilon}\Delta(\phi^3) + \frac{k}{\epsilon}\Delta\phi + \frac{k}{\epsilon^2}(3\phi^2-1)f(\phi)\\ & + M_1[V(\phi)-\alpha] + M_2[B(\phi)-\beta]f(\phi)\\ =\ & J_1(\phi) + J_2(\phi) + J_3(\phi) + J_4(\phi) + J_5(\phi). \end{aligned} \tag{2.7}$$

where $J_1(\phi) = -\frac{k}{\epsilon}\Delta(\phi^3)$, $J_2(\phi) = \frac{k}{\epsilon}\Delta\phi$, $J_3(\phi) = \frac{k}{\epsilon^2}(3\phi^2-1)f(\phi)$, $J_4(\phi) = M_1[V(\phi)-\alpha]$ and $J_5(\phi) = M_2[B(\phi)-\beta]f(\phi)$.

We next recall some tools of functional analysis and semigroup theory. For the proof, see Lemma 3.3 and Theorem 3.1 of [1]. Denote by E a Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Let $A_0 : D(A_0) \to E$ be a strictly positive operator (this means that A_0 is self-adjoint and $(A_0\phi,\phi) \geq c\|\phi\|^2$ for some $c > 0$, $\forall\phi\in D(A_0)$).

Proposition 2.6. *With the above notation, for every $f \in L^2(0,T;E)$, $u_0 \in D(A_0^{\frac{1}{2}})$, the abstract Cauchy problem:*

$$\begin{cases} u_t + A_0 u = f,\\ u(0) = u_0. \end{cases} \tag{2.8}$$

has a unique solution $u \in L^2(0,T;D(A_0)) \cap C([0,T];D(A_0^{\frac{1}{2}}) \cap H^1(0,T;E)$. Moreover, there exists a constant $K > 0$, such that:

$$\begin{aligned} &\|u\|^2_{L^2(0,T;D(A_0))} + \|u\|^2_{C([0,T];D(A_0^{\frac{1}{2}}))} + \|u\|^2_{H^1(0,T;E)}\\ &\leq K\left(\|u_0\|^2_{D(A_0^{\frac{1}{2}})} + \|f\|^2_{L^2(0,T;E)}\right). \end{aligned}$$

As a consequence, we have:

Corollary 2.7. *Suppose $\mathbf{u}_0 \in V(\Omega)$, $\boldsymbol{f} \in L^2(0,T;H(\Omega))$ and let A be the Stokes operator defined in Sect. 1. Then the Cauchy problem*

$$\begin{cases} \mathbf{u}_t + \nu A\mathbf{u} = \mathbf{f},\\ \mathbf{u}(0) = \mathbf{u}_0. \end{cases} \tag{2.9}$$

has a unique solution $\mathbf{u} \in L^2(0,T;D(A)) \cap C([0,T];V(\Omega)) \cap H^1(0,T;H(\Omega))$*. Moreover, there exists* $K>0$ *such that:*

$$\|\mathbf{u}\|^2_{L^2(0,T;H^2)} + \|\mathbf{u}\|^2_{C([0,T];H^1)} + \|\mathbf{u}\|^2_{H^1(0,T;H(\Omega))} \leq K\left(\|\mathbf{u}_0\|^2_{H^1} + \|\mathbf{f}\|^2_{L^2(0,T;L^2)}\right). \tag{2.10}$$

Proof. It suffices to notice that the operator $A: D(A) \to H$ is self-adjoint and strictly positive (see [2, pp.31–34] for the proof) and to apply the above proposition. □

Corollary 2.8. *Suppose* $\phi_0 \in \tilde{H}^{2+\frac{3}{8}}(\Omega)$ *and* $g \in L^2(0,T;H^{\frac{3}{8}}(\Omega))$*, then the Cauchy problem*

$$\begin{cases} \phi_t + k\gamma\epsilon\mathcal{A}^2\phi = g, \\ \phi(0,x) = \phi_0(x). \end{cases} \tag{2.11}$$

has a unique solution

$$\phi \in L^2(0,T;\tilde{H}^{4+\frac{3}{8}}(\Omega)) \cap C([0,T];\tilde{H}^{2+\frac{3}{8}}(\Omega)) \cap H^1(0,T;\tilde{H}^{\frac{3}{8}}(\Omega))$$

with the following estimate:

$$\|\phi\|^2_{L^2(0,T;H^{4+\frac{3}{8}})} + \|\phi\|^2_{C([0,T];H^{2+\frac{3}{8}})} + \|\phi\|^2_{H^1(0,T;H^{\frac{3}{8}})} \leq K\left(\|\phi_0\|^2_{\tilde{H}^{2+\frac{3}{8}}} + \|g\|^2_{L^2(0,T;H^{\frac{3}{8}})}\right). \tag{2.12}$$

Proof. First let us recall that $H^{\frac{3}{8}}(\Omega) = \tilde{H}^{\frac{3}{8}}(\Omega)$ (Proposition 2.4). We also recall that $\mathcal{A}$ is the Dirichlet Laplacian defined by (2.3). By the definition of $\tilde{H}^s(\Omega)$, we have:

$$\mathcal{A}^2: \quad D(\mathcal{A}^{2+\frac{3}{16}}) = \tilde{H}^{4+\frac{3}{8}} \to D(\mathcal{A}^{\frac{3}{16}}) = \tilde{H}^{\frac{3}{8}}.$$

Applying Proposition 2.6 with $A_0 = \mathcal{A}^2$ and $E = H^{\frac{3}{8}}(\Omega)$, we have:

$$\|\phi\|^2_{L^2(0,T;D(\mathcal{A}^{2+\frac{3}{16}}))} + \|\phi\|^2_{C([0,T];D(\mathcal{A}^{1+\frac{3}{16}}))} + \|\phi\|^2_{H^1(0,T;D(\mathcal{A}^{\frac{3}{16}}))} \leq K\left(\|\phi_0\|^2_{D(\mathcal{A}^{1+\frac{3}{16}})} + \|g\|^2_{L^2(0,T;D(\mathcal{A}^{\frac{3}{16}}))}\right). \tag{2.13}$$

By Proposition 2.4, the above inequality implies

$$\|\phi\|^2_{L^2(0,T;H^{4+\frac{3}{8}})} + \|\phi\|^2_{C([0,T];H^{2+\frac{3}{8}})} + \|\phi\|^2_{H^1(0,T;H^{\frac{3}{8}})} \leq K\left(\|\phi_0\|^2_{\tilde{H}^{2+\frac{3}{8}}} + \|g\|^2_{L^2(0,T;H^{\frac{3}{8}})}\right).$$

□

3. Proof of the Main Theorems

We first give some estimates of the nonlinear terms in (1.1). These estimates are essential in our fixed point procedure and will be proved in the next sections. Suppose:

$$\mathbf{u} \in L^2(0,T;[H^2(\Omega)]^3) \cap C([0,T];[H^1(\Omega)]^3) \cap H^1(0,T;[L^2(\Omega)]^3), \tag{3.1}$$

$$\phi \in L^2(0,T;H^{4+\frac{3}{8}}(\Omega)) \cap C([0,T];H^{2+\frac{3}{8}}(\Omega)) \cap H^1(0,T;H^{\frac{3}{8}}(\Omega)). \tag{3.2}$$

Recall the definition of $W(\phi)$ and $L(\phi)$ from Sect. 1. We have the following two theorems.

Theorem 3.1. Suppose $\psi \in H^{4+\frac{3}{8}}(\Omega)$ and $\mathbf{v} \in H^2(\Omega)$, then there exists a constant $C = C(\Omega, \epsilon, k, M_1, M_2, \beta, \gamma)$ such that:

$$\|W(\psi-1)\nabla\psi\|_{L^2} \le C\Big[\|\psi\|_{H^{2+\frac{3}{8}}}(\|\psi\|_{H^{2+\frac{3}{8}}}+1)^4\|\psi\|_{H^{4+\frac{3}{8}}} + (\|\psi\|_{H^{2+\frac{3}{8}}}+|\alpha+\mu(\Omega)|)^7\|\psi\|_{H^{2+\frac{3}{8}}}\Big], \tag{3.3a}$$

$$\|\mathbf{v}\cdot\nabla\mathbf{v}\|_{L^2} \le C\|\mathbf{v}\|_{H^1}\|\mathbf{v}\|_{H^2}, \tag{3.3b}$$

$$\|\mathbf{v}\cdot\nabla\psi\|_{H^{\frac{3}{8}}} \le C\|\mathbf{v}\|_{H^{1+\frac{3}{4}}}\|\psi\|_{H^{2+\frac{3}{8}}}, \tag{3.3c}$$

$$\|L(\psi-1)\|_{H^{\frac{3}{8}}} \le C\Big[(\|\psi\|_{H^{2+\frac{3}{8}}}+1)^6\|\psi\|_{H^{2+\frac{3}{8}}} + \|\psi\|_{H^2}^2\|\psi\|_{H^4} + |\alpha+\mu(\Omega)|\Big]. \tag{3.3d}$$

Theorem 3.2. *Under the assumptions* (3.1), (3.2) *and* $T \le 1$, *there exists a positive constant* $C = C(\Omega, \epsilon, k, M_1, M_2, \beta, \gamma)$ *such that:*

$$\|W(\phi-1)\nabla\phi\|_{L^{\frac{17}{8}}(0,T;L^2)} \le C\left(\|\phi\|_{C\left([0,T];H^{2+\frac{3}{8}}\right)} + \|\phi\|_{L^2\left(0,T;H^{4+\frac{3}{8}}\right)} + 1\right)^8, \tag{3.4a}$$

$$\|\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^{\frac{17}{8}}(0,T;L^2)} \le C\left(\|\mathbf{u}\|_{C([0,T];H^1)} + \|\mathbf{u}\|_{L^2(0,T;H^2)}\right)^2, \tag{3.4b}$$

$$\|\mathbf{u}\cdot\nabla\phi\|_{L^{\frac{17}{8}}\left(0,T;H^{\frac{3}{8}}\right)} \le C\|\phi\|_{C\left([0,T];H^{2+\frac{3}{8}}\right)}\left(\|\mathbf{u}\|_{C([0,T];H^1)} + \|\mathbf{u}\|_{L^2(0,T;H^2)}\right), \tag{3.4c}$$

$$\|L(\phi-1)\|_{L^{\frac{17}{8}}\left(0,T;H^{\frac{3}{8}}\right)} \le C\left(\|\phi\|_{C\left([0,T];H^{2+\frac{3}{8}}\right)} + \|\phi\|_{L^2(0,T;H^{4+\frac{3}{8}})} + 1\right)^7. \tag{3.4d}$$

We also need the estimates of differences of the nonlinear terms. Let $T \le 1$ and let $R > 1$ be a constant, which will be made precise later. We assume that

$$\begin{cases} \mathbf{u}_i \in L^2(0,T;H^2) \cap C([0,T];H^1) \cap H^1(0,T;L^2), \\ \phi_i \in L^2(0,T;H^{4+\frac{3}{8}}) \cap C([0,T];H^{2+\frac{3}{8}}) \cap H^1(0,T;H^{\frac{3}{8}}). \end{cases} \tag{3.5}$$

$$\begin{cases} \max\left\{\|\mathbf{u}_i\|_{L^2(0,T;H^2)}, \|\mathbf{u}_i\|_{C([0,T];H^1)}, \|\mathbf{u}_i\|_{H^1(0,T;L^2)}\right\} \le CR, \\ \max\left\{||\phi_i||_{L^2(0,T;H^{4+\frac{3}{8}})}, ||\phi_i||_{C([0,T];H^{2+\frac{3}{8}})}, ||\phi_i||_{H^1(0,T;H^{\frac{3}{8}})}\right\} \le CR, \end{cases} \tag{3.6}$$

where C is a positive constant.

Set $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$ and $\phi = \phi_1 - \phi_2$. We have the following estimates:

Theorem 3.3. *There exists a constant* $C = C(\epsilon, k, \Omega, M_1, M_2, \beta, \gamma)$ *such that:*

$$\|L(\phi_1) - L(\phi_2)\|_{L^{\frac{17}{8}}(0,T;H^{\frac{3}{8}})} \le CR^6\left(\|\phi\|_{C([0,T];H^{2+\frac{3}{8}})} + \|\phi\|_{L^2(0,T;H^{4+\frac{3}{8}})}\right), \tag{3.7a}$$

$$\|L(\phi_1)\nabla\phi_1 - L(\phi_2)\nabla\phi_2\|_{L^{\frac{17}{8}}(0,T;L^2)} \le CR^7\left(\|\phi\|_{C([0,T];H^{2+\frac{3}{8}})} + \|\phi\|_{L^2(0,T;H^{4+\frac{3}{8}})}\right), \tag{3.7b}$$

$$\|\Delta^2\phi_1\nabla\phi_1 - \Delta^2\phi_2\nabla\phi_2\|_{L^{\frac{17}{8}}(0,T;L^2)} \le CR\left(\|\phi\|_{C([0,T];H^{2+\frac{1}{4}})} + \|\phi\|_{L^2(0,T;H^{4+\frac{3}{8}})}\right), \tag{3.7c}$$

$$\|\mathbf{u}_1\cdot\nabla\mathbf{u}_1 - \mathbf{u}_2\cdot\nabla\mathbf{u}_2\|_{L^{\frac{17}{8}}(0,T;L^2)} \le CR\left(\|\mathbf{u}\|_{L^2(0,T;H^2)} + \|\mathbf{u}\|_{C([0,T];H^1)}\right), \tag{3.7d}$$

$$\|\mathbf{u}_1\cdot\nabla\phi_1 - \mathbf{u}_2\cdot\nabla\phi_2\|_{L^{\frac{17}{8}}(0,T;H^{\frac{3}{8}})} \le CR\left(\|\mathbf{u}\|_{C([0,T];H^1)} + \|\mathbf{u}\|_{L^2(0,T;H^2)} + \|\phi\|_{C([0,T];H^{2+\frac{3}{8}})}\right). \tag{3.7e}$$

Note that we have the same estimate for $L(\phi_1 - 1) - L(\phi_2 - 1)$ as for $L(\phi_1) - L(\phi_2)$ by applying (3.7a) to $\phi_1 - 1$ and $\phi_2 - 1$. Since the proof of the above three theorems are technical, they are postponed at Sects. 4 and 5. We use them to prove here our main results.

Proof of Theorem 1.1. Without loss of generality, we can assume that $T \leq 1$. By the classical results on the Leray projector (see for instance Chapter 5 of [2]), we only need to prove that (2.6) has a unique strong solution:

$$\begin{cases} \mathbf{u} \in L^2(0,T;H^2(\Omega)) \cap C([0,T];V(\Omega)) \cap H^1(0,T;H(\Omega)), \\ \phi \in L^2(0,T;\tilde{H}^{4+\frac{3}{8}}(\Omega)) \cap C([0,T];\tilde{H}^{2+\frac{3}{8}}(\Omega)) \cap H^1(0,T;H^{\frac{3}{8}}(\Omega)). \end{cases}$$

Consider the Hilbert space

$$E = H(\Omega) \times H^{\frac{3}{8}}(\Omega),$$

and

$$B(0,R) = \left\{ v = (\mathbf{u},\phi) \in L^2(0,T;E) \middle| \; \|v\|_{L^2(0,T;E)} \leq R \right\}.$$

We define the following nonlinear map from B(0,R) to $L^2(0,T;E)$:

$$F: \begin{pmatrix} \mathbf{f} \\ g \end{pmatrix} \longmapsto \begin{pmatrix} -P(\mathbf{u}\cdot\nabla\mathbf{u}) + P(W(\phi-1)\nabla\phi) \\ -\mathbf{u}\cdot\nabla\phi - \gamma L(\phi-1) \end{pmatrix},$$

where $\mathbf{u}$ and ϕ satisfy (2.9) and (2.11) respectively. To obtain the conclusion of this theorem we use Propositions 3.4 and 3.5 below, which allow the application of the Banach fixed point theorem to obtain the local in time existence of strong solutions. □

Proposition 3.4. *There exists a positive constant* $C = C(\Omega,k,\epsilon,\beta,\gamma,M_1,M_2)$ *such that, if we choose* $R \geq 1 + \|\phi_0\|_{H^3} + \|\mathbf{u}_0\|_{H^1}$ *and* $T \leq CR^{-238}$*, then F maps B(0,R) into itself.*

Proof. By using the estimates (3.4), (2.10) and (2.12) (since $\mathbf{u}$ and ϕ satisfy (2.9) and (2.11)) we have:

$$\begin{aligned} &\|W(\phi-1)\nabla\phi\|_{L^{\frac{17}{8}}(L^2)} + \|\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^{\frac{17}{8}}(L^2)} \\ &\quad + \|\mathbf{u}\cdot\nabla\phi\|_{L^{\frac{17}{8}}(H^{\frac{3}{8}})} + \|L(\phi-1)\|_{L^{\frac{17}{8}}(H^{\frac{3}{8}})} \\ &\leq C(\|\mathbf{u}\|_{L^2(H^2)} + \|\mathbf{u}\|_{C(H^1)} + \|\phi\|_{L^2(H^{4+\frac{3}{8}})} + \|\phi\|_{C(H^{2+\frac{1}{4}})} + 1)^8 \\ &\leq C(\|\mathbf{f}\|_{L^2(L^2)} + \|g\|_{L^2(H^{\frac{3}{8}})} + \|\phi_0\|_{H^{2+\frac{3}{8}}} + \|\mathbf{u}_0\|_{H^1} + 1)^8. \end{aligned}$$

If we choose $R \geq 1 + \|\phi_0\|_{H^3} + \|\mathbf{u}_0\|_{H^1}$, then we have:

$$\|F(\mathbf{f},g)\|_{L^{\frac{17}{8}}(0,T;E)} \leq CR^8.$$

Applying Hölder inequality, we obtain:

$$\|F(\mathbf{f},g)\|_{L^2(0,T;E)} \leq T^{\frac{1}{34}} \|F(\mathbf{f},g)\|_{L^{\frac{17}{8}}(0,T;E)} \leq T^{\frac{1}{34}} CR^8.$$

Thus, if we choose $T \leq CR^{-238}$, then F actually maps B(0,R) into itself. □

Proposition 3.5. *Assume that* $R \geq 1 + \|\phi_0\|_{H^3} + \|\mathbf{u}_0\|_{H^1}$*. There exists a positive constant* $C = C(\Omega,k,\epsilon,\beta,\gamma,M_1,M_2)$ *such that, if we choose* $T \leq CR^{-238}$*, then* $F_{|B(0,R)}$ *is a contraction.*

Proof. Assume $(\mathbf{f}_1,g_1)$, $(\mathbf{f}_2,g_2) \in L^2(0,T;E)$, and consider the solution $(\mathbf{u}_1,\phi_1)$ and $(\mathbf{u}_2,\phi_2)$ of (2.9) and (2.11) associated with $(\mathbf{f}_1,g_1)$ and $(\mathbf{f}_2,g_2)$. By using (2.10), (2.12) and that $R \geq 1 + \|\phi_0\|_{H^3} + \|\mathbf{u}_0\|_{H^1}$, we deduce that (3.6) holds true.

We also have that the difference $\phi = \phi_1 - \phi_2$ and $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$ satisfy:

$$\begin{cases} \mathbf{u}_t + \nu A\mathbf{u} = \mathbf{f} \\ \phi_t + k\gamma\epsilon\mathcal{A}^2\phi = g \\ \mathbf{u}(0,x) = 0 \\ \phi(0,x) = 0 \end{cases}$$

where $\mathbf{f} = \mathbf{f}_1 - \mathbf{f}_2$ and $g = g_1 - g_2$. Using the estimates (2.10) and (2.12), we obtain,

$$\|\mathbf{u}\|_{L^2(H^2)} + \|\mathbf{u}\|_{C(H^1)} \le C\|\mathbf{f}\|_{L^2(L^2)},$$
$$\|\phi\|_{L^2(H^{4+\frac{3}{8}})} + \|\phi\|_{C(H^{2+\frac{3}{8}})} \le C\|g\|_{L^2(H^{\frac{3}{8}})}.$$

Applying (3.7) to $(\phi_1 - 1, \phi_2 - 1)$, we obtain:

$$\begin{aligned}
&\|F(\mathbf{f}_1, g_1) - F(\mathbf{f}_2, g_2)\|_{L^{\frac{17}{8}}(E)} \\
&\quad \le \|\mathbf{u}_1 \cdot \nabla \mathbf{u}_1 - \mathbf{u}_2 \cdot \nabla \mathbf{u}_2\|_{L^{\frac{17}{8}}(L^2)} + \|W(\phi_1 - 1)\nabla\phi_1 - W(\phi_2 - 1)\nabla\phi_2\|_{L^{\frac{17}{8}}(L^2)} \\
&\qquad + \|\mathbf{u}_1 \cdot \nabla\phi_1 - \mathbf{u}_2 \cdot \nabla\phi_2\|_{L^{\frac{17}{8}}(H^{\frac{3}{8}})} + \|L(\phi_1 - 1) - L(\phi_2 - 1)\|_{L^{\frac{17}{8}}(H^{\frac{3}{8}})} \\
&\quad \le CR^7 \left(\|\mathbf{u}\|_{L^2(H^2)} + \|\mathbf{u}\|_{C(H^1)} + \|\phi\|_{L^2(H^{4+\frac{3}{8}})} + \|\phi\|_{C(H^{2+\frac{1}{4}})} \right) \\
&\quad \le CR^7 \left(\|\mathbf{f}\|_{L^2(L^2)} + \|g\|_{L^2(H^{\frac{3}{8}})} \right).
\end{aligned}$$

We deduce from the above estimates that:

$$\begin{aligned}
&\|F(\mathbf{f}_1, g_1) - F(\mathbf{f}_2, g_2)\|_{L^2(0,T;E)} \\
&\quad \le T^{\frac{1}{34}} \|F(\mathbf{f}_1, g_1) - F(\mathbf{f}_2, g_2)\|_{L^{\frac{17}{8}}(0,T;E)} \\
&\quad \le T^{\frac{1}{34}} CR^7 (\|\mathbf{f}\|_{L^2(L^2)} + \|g\|_{L^2(H^{\frac{3}{8}})}) \\
&\quad = T^{\frac{1}{34}} CR^7 \|(\mathbf{f}_1, g_1) - (\mathbf{f}_2, g_2)\|_{L^2(0,T;E)}.
\end{aligned}$$

So if we choose $T \le CR^{-238}$, then F is a contraction. □

Next we prove Theorem 1.4. We first prove the following proposition:

Proposition 3.6. *Let $\mathbf{u}$ and ϕ be the strong solution constructed in Theorem 1.1, then there exists a positive constant $C > 0$ which does not depend on T such that:*

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}(t)\|_{H^1}^2 + \frac{3}{4}\nu\|A\mathbf{u}(t)\|_{L^2}^2 \le C\|\mathbf{u}(t)\cdot\nabla\mathbf{u}(t)\|_{L^2}^2 + C\|W(\phi(t) - 1)\nabla\phi(t)\|_{L^2}^2, \tag{3.8}$$
$$\frac{1}{2}\frac{d}{dt}\|\phi(t)\|_{H^{2+\frac{3}{8}}}^2 + \frac{3}{4}k\gamma\epsilon\|\mathcal{A}^{2+\frac{3}{16}}\phi(t)\|_{L^2}^2 \le C\|\mathbf{u}(t)\cdot\nabla\phi(t)\|_{H^{\frac{3}{8}}}^2 + C\|L(\phi(t) - 1)\|_{H^{\frac{3}{8}}}^2. \tag{3.9}$$

Proof. Note that $\mathbf{u}_t \in L^2(0, T; H)$, $A\mathbf{u} \in L^2(0, T; H)$. Taking the inner product in $L^2(\Omega)$ of the first equation in (2.6) by $A\mathbf{u}$, we get:

$$< \mathbf{u}_t, A\mathbf{u} > +\nu < A\mathbf{u}, A\mathbf{u} >= - < \mathbf{u}\cdot\nabla\mathbf{u}, A\mathbf{u} > + < W(\phi - 1)\nabla\phi, A\mathbf{u} >.$$

Here we have denoted by $< \cdot, \cdot >$ the standard $L^2(\Omega)$ inner product.

It follows from the above equation that:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|A^{\frac{1}{2}}\mathbf{u}\|_{L^2}^2 + \nu\|A\mathbf{u}\|_{L^2}^2 \\
&\quad \le \|\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^2}\|A\mathbf{u}\|_{L^2} + \|W(\phi - 1)\nabla\phi\|_{L^2}\|A\mathbf{u}\|_{L^2} \\
&\quad \le \frac{2}{\nu}\|\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^2}^2 + \frac{\nu}{8}\|A\mathbf{u}\|_{L^2}^2 + \frac{2}{\nu}\|W(\phi - 1)\nabla\phi\|_{L^2}^2 + \frac{\nu}{8}\|A\mathbf{u}\|_{L^2}^2,
\end{aligned}$$

which implies:

$$\frac{1}{2}\frac{d}{dt}\|A^{\frac{1}{2}}\mathbf{u}\|_{L^2}^2 + \frac{3}{4}\nu\|A\mathbf{u}\|_{L^2}^2 \le \frac{2}{\nu}\|\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^2}^2 + \frac{2}{\nu}\|W(\phi - 1)\nabla\phi\|_{L^2}^2.$$

We have proved (3.8).

To prove the other inequlity, we first recall that $-\mathcal{A}$ is the Dirichlet Laplacian and that there exists a constant C such that

$$\|\phi\|_{H^{4+\frac{3}{8}}}^2 \le C\|\mathcal{A}^{2+\frac{3}{16}}\phi\|_{L^2}^2.$$

According to (3.2), $\phi_t \in L^2(0,T;H^{\frac{3}{8}})$ and $\mathcal{A}^{2+\frac{3}{8}}\phi \in L^2(0,T;H^{-\frac{3}{8}})$. Taking the duality product between the second equation in (2.6) and $\mathcal{A}^{2+\frac{3}{8}}\phi$ yields

$$\begin{aligned}
&(\phi_t, \mathcal{A}^{2+\frac{3}{8}}\phi) + k\gamma\epsilon(\mathcal{A}^2\phi, \mathcal{A}^{2+\frac{3}{8}}\phi)\\
&\quad = -(u\cdot\nabla\phi, \mathcal{A}^{2+\frac{3}{8}}\phi) - \gamma(L(\phi-1), \mathcal{A}^{2+\frac{3}{8}}\phi)\\
&\quad = -< \mathcal{A}^{\frac{3}{16}}(u\cdot\nabla\phi), \mathcal{A}^{2+\frac{3}{16}}\phi > -\gamma < \mathcal{A}^{\frac{3}{16}}(L(\phi-1)), \mathcal{A}^{2+\frac{3}{16}}\phi >\\
&\quad \le \|\mathbf{u}\cdot\nabla\phi\|_{H^{\frac{3}{8}}}\|\phi\|_{H^{4+\frac{3}{8}}} + \gamma\|L(\phi-1)\|_{H^{\frac{3}{8}}}\|\phi\|_{H^{4+\frac{3}{8}}}\\
&\quad \le \frac{k\gamma\epsilon}{8C}\|\phi\|^2_{H^{4+\frac{3}{8}}} + \frac{2C}{k\gamma\epsilon}\|\mathbf{u}\cdot\nabla\phi\|^2_{H^{\frac{3}{8}}} + \frac{k\gamma\epsilon}{8C}\|\phi\|^2_{H^{4+\frac{3}{8}}} + \frac{2\gamma C}{k\epsilon}\|L(\phi-1)\|^2_{H^{\frac{3}{8}}}.
\end{aligned} \tag{3.10}$$

Here we have denoted by $(\cdot,\cdot)$ the pairing of $H^{3/8}(\Omega)$ and $H^{-3/8}(\Omega)$.

Since $\frac{1}{2}\frac{d}{dt}\|\mathcal{A}^{1+\frac{3}{16}}\phi\|^2_{L^2} = (\phi_t, \mathcal{A}^{2+\frac{3}{8}}\phi)$, we deduce from the above inequality that

$$\frac{1}{2}\frac{d}{dt}\|\mathcal{A}^{1+\frac{3}{16}}\phi\|^2_{L^2} + \frac{3}{4}k\gamma\epsilon\|\mathcal{A}^{2+\frac{3}{16}}\phi\|^2_{L^2} \le \frac{2C}{k\gamma\epsilon}\|\mathbf{u}\cdot\nabla\phi\|^2_{H^{\frac{3}{8}}} + \frac{2\gamma C}{k\epsilon}\|L(\phi-1)\|^2_{H^{\frac{3}{8}}}. \tag{3.11}$$

As a consequence, (3.9) is proved. □

Proof of Theorem 1.4. Using Theorem 1.1, we can consider a maximal strong solution $(\mathbf{u},\phi)$ of (1.1) on the time interval $[0,T_{\max})$. Assume $T>0$. We prove here that if we choose the initial data small enough, then we have $T_{\max} > T$ i.e. the solution exists on [0,T].

Assume by contradiction that $T_{\max}\le T$, Adding (3.8) to (3.9) and using (3.3), we obtain:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\mathbf{u}\|^2_{H^1} + \|\phi\|^2_{H^{2+\frac{3}{8}}}) + \frac{3}{4}\nu\|\mathbf{u}\|^2_{H^2} + \frac{3}{4}k\gamma\epsilon\|\phi\|^2_{H^{4+\frac{3}{8}}}\\
&\quad \le C\|\mathbf{u}(t)\cdot\nabla\mathbf{u}(t)\|^2_{L^2} + C\|W(\phi-1)\nabla\phi(t)\|^2_{L^2} + C\|\mathbf{u}(t)\cdot\nabla\phi(t)\|^2_{H^{\frac{3}{8}}} + C\|L(\phi-1)\|_{H^{\frac{3}{8}}}\\
&\quad \le C\|\mathbf{u}\|^2_{H^1}\|\mathbf{u}\|^2_{H^2} + C\|\phi\|^2_{H^{2+\frac{3}{8}}}(\|\phi\|_{H^{2+\frac{3}{8}}}+1)^8\|\phi\|^2_{H^{4+\frac{3}{8}}}\\
&\qquad + C(\|\phi\|_{H^{2+\frac{3}{8}}} + |\alpha+\mu(\Omega)|)^{14}\|\phi\|^2_{H^{2+\frac{3}{8}}} + C\|\mathbf{u}\|^2_{H^2}\|\phi\|^2_{H^{2+\frac{3}{8}}}\\
&\qquad + C(\|\phi\|_{H^{2+\frac{3}{8}}}+1)^{12}\|\phi\|^2_{H^{2+\frac{3}{8}}} + C\|\phi\|^4_{H^2}\|\phi\|^2_{H^4} + C(\alpha+\mu(\Omega))^2\\
&\quad \le C\left(\|\mathbf{u}\|^2_{H^1} + \|\phi\|^2_{H^{2+\frac{3}{8}}}\right)\|\mathbf{u}\|^2_{H^2} + C\|\phi\|^2_{H^{2+\frac{3}{8}}}(\|\phi\|_{H^{2+\frac{3}{8}}}+1)^8\|\phi\|^2_{H^{4+\frac{3}{8}}}\\
&\qquad + C(\|\phi\|_{H^{2+\frac{3}{8}}} + |\alpha+\mu(\Omega)|)^{14}\|\phi\|^2_{H^{2+\frac{3}{8}}} + C(\alpha+\mu(\Omega))^2.
\end{aligned} \tag{3.12}$$

Here the constant C does not depend on α and which will be fixed for the end of the proof. We also consider $q>0$ small enough so that

$$q \le \frac{\nu}{4C}, \quad q(\sqrt{q}+1)^8 \le \frac{k\gamma\epsilon}{4C} \tag{3.13}$$

We recall that E is defined by (1.4) and we set $\mathbf{G} = \left\{t\in[0,T_{\max}) | \sup_{[0,t]} E \le q\right\}$. We prove below that if we assume

$$E(0) + 2TC\left((\sqrt{q} + |\alpha+\mu(\Omega)|)^{14} q + (\alpha+\mu(\Omega))^2\right) < q, \tag{3.14}$$

then $\mathbf{G} = [0,T_{\max})$. To prove this, we use a connectedness argument: we verify the following three properties:

1. $0\in\mathbf{G}$,
2. $\mathbf{G}$ is closed in $[0,T_{\max})$,
3. $\mathbf{G}$ is open in $[0,T_{\max})$.

Conditions 1. and 2. are easy to check so we skip their proof and we only prove Condition 3. Assume $t_1 \in \mathbf{G}$ then from (3.13), we deduce that for $t \in [0, t_1]$

$$\begin{cases} C(\|\mathbf{u}(t)\|_{H^1}^2 + \|\phi(t)\|_{H^{2+\frac{3}{8}}}^2) \leq \dfrac{\nu}{4}, \\ C\|\phi(t)\|_{H^{2+\frac{3}{8}}}^2 (\|\phi(t)\|_{H^{2+\frac{3}{8}}} + 1)^8 \leq \frac{k\gamma\epsilon}{4}. \end{cases}$$

With these inequalities, (3.12) can be written as:

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\mathbf{u}(t)\|_{H^1}^2 + \|\phi(t)\|_{H^{2+\frac{3}{8}}}^2) + \frac{1}{2}\nu\|\mathbf{u}(t)\|_{H^2}^2 + \frac{1}{2}k\gamma\epsilon\|\phi(t)\|_{H^{4+\frac{3}{8}}}^2 \\ &\quad \leq C(\sqrt{q} + |\alpha + \mu(\Omega)|)^{14} q + C(\alpha + \mu(\Omega))^2. \end{aligned} \tag{3.15}$$

Relation (3.15) implies that for $t \in [0, t_1]$:

$$\frac{d}{dt}E(t) \leq 2C\left((\sqrt{q} + |\alpha + \mu(\Omega)|)^{14} q + (\alpha + \mu(\Omega))^2\right)$$

Using (3.14), the above relation implies

$$E(t_1) \leq E(0) + 2C\left((\sqrt{q} + |\alpha + \mu(\Omega)|)^{14} q + (\alpha + \mu(\Omega))^2\right) T < q.$$

It follows that there exists $\delta > 0$ such that $E(t') \leq q$, $\forall t' \in (-\delta + t_1, \delta + t_1)$. Consequently $\mathbf{G}$ is open in $[0, T_{\max})$. Since $[0, T_{\max})$ is a connected set, we have $\mathbf{G} = [0, T_{\max})$. On the other hand, by the definition of $\mathbf{G}$, we have $E(t)$ is bounded on $[0, T_{\max})$, which contradicts Theorem 1.1. We thus have $T_{\max} > T$, which proves Theorem 1.4. $\square$

Remark 3.7. In the inequality (3.10), one can see the obstacle to get the global existence for small data as for the Navier–Stokes equations. In fact, contrary to the Navier–Stokes equations, the estimates of the nonlinear terms

$$\begin{aligned} &C\left(\|\mathbf{u}\|_{H^1}^2 + \|\phi\|_{H^{2+\frac{3}{8}}}^2\right)\|\mathbf{u}\|_{H^2}^2 + C\|\phi\|_{H^{2+\frac{3}{8}}}^2(\|\phi\|_{H^{2+\frac{3}{8}}} + 1)^8\|\phi\|_{H^{4+\frac{3}{8}}}^2 \\ &\quad + C(\|\phi\|_{H^{2+\frac{3}{8}}} + |\alpha + \mu(\Omega)|)^{14}\|\phi\|_{H^{2+\frac{3}{8}}}^2 + C(\alpha + \mu(\Omega))^2. \end{aligned}$$

can not be absorbed by the viscous term $\frac{3}{4}\nu\|\mathbf{u}\|_{H^2}^2 + \frac{3}{4}k\gamma\epsilon\|\phi\|_{H^{4+\frac{3}{8}}}^2$, even for $\|\mathbf{u}\|_{H^1}$ and $\|\phi\|_{H^{2+\frac{3}{8}}}$ small. We can control the two first terms of the right-hand side of (3.10) by using this strategy but the two other terms and in particular $C(\alpha + \mu(\Omega))^2$ have no reason to be smaller than the viscous term.

4. Proof of Theorems 3.1 and 3.2

In this part, we will detail the proof of the estimates of the nonlinear terms defined by (1.2). We use the notation C for any positive constant depending on Ω and physical parameters $\epsilon, \gamma, \beta, k, \Omega, M_1, M_2, \nu$ (but not depending on α).

We first estimate $f(\phi)$, $B(\phi)$ defined by (1.3):

Lemma 4.1. *There exists a constant $C = C(\Omega, \epsilon)$ such that:*

$$\max\{B(\psi), B(\psi - 1)\} \leq C(\|\psi\|_{H^2}^4 + 1), \tag{4.1}$$

$$\max\left\{\|f(\psi)\|_{H^{\frac{3}{8}}}, \|f(\psi - 1)\|_{H^{\frac{3}{8}}}\right\} \leq C\left(\|\psi\|_{H^{2+\frac{3}{8}}} + \|\psi\|_{H^{2+\frac{3}{8}}}^3\right), \tag{4.2}$$

for $\psi \in H^4(\Omega)$. Furthermore, we have

$$\max\{\|f(\psi)\|_{H^2}, \|f(\psi - 1)\|_{H^2}\} \leq C(\|\psi\|_{H^4} + \|\psi\|_{H^2}^3), \quad \psi \in H^4(\Omega). \tag{4.3}$$

Proof. The proof is essentially based on the fact that $H^2(\Omega)$ is a Banach Algebra. Since the proof of the four estimates are similar, we only prove (4.2). We have:

$$\|f(\psi)\|_{H^{\frac{3}{8}}} \le \epsilon\|\Delta\psi\|_{H^{\frac{3}{8}}} + \frac{1}{\epsilon}\|\psi^3\|_{H^{\frac{3}{8}}} + \frac{1}{\epsilon}\|\psi\|_{H^{\frac{3}{8}}} \le C\left(\|\psi\|_{H^{2+\frac{3}{8}}} + \|\psi\|^3_{H^{2+\frac{3}{8}}}\right).$$

Since

$$f(\psi-1) = -\epsilon\Delta\psi + \frac{1}{\epsilon}(\psi^3 - 3\psi^2 + 2\psi),$$

we have the same estimate for $f(\psi-1)$. □

We next estimate the term $W(\phi)\nabla\phi = k\epsilon\Delta^2\phi\nabla\phi + L(\phi)\nabla\phi$, where $L(\phi)$ is defined by (2.7).

Proposition 4.2. *We have:*

$$\begin{aligned}\|W(\psi-1)\nabla\psi\|_{L^2} \le{}& C\|\psi\|_{H^{2+\frac{3}{8}}}(\|\psi\|_{H^{2+\frac{3}{8}}}+1)^4\|\psi\|_{H^{4+\frac{3}{8}}} \\ &+ C(\|\psi\|_{H^{2+\frac{3}{8}}} + |\alpha+\mu(\Omega)|)^7\|\psi\|_{H^{2+\frac{3}{8}}},\end{aligned} \tag{4.4}$$

for every $\psi \in H^{4+\frac{3}{8}}(\Omega)$. *Moreover, for every* ϕ *satisfying* (3.2) *and* $T \le 1$, *we have:*

$$\|W(\phi-1)\nabla\phi\|_{L^{\frac{17}{8}}(L^2)} \le C\left(\|\phi\|_{C(H^{2+\frac{3}{8}})} + \|\phi\|_{L^2(H^{4+\frac{3}{8}})} + 1\right)^8. \tag{4.5}$$

The proof of the above proposition will be split into six lemmas.

Lemma 4.3. *We have:*

$$\|\Delta^2\psi\ \nabla\psi\|_{L^2(\Omega)} \le C\|\psi\|_{H^{4+\frac{3}{8}}}\|\psi\|_{H^{2+\frac{3}{8}}}, \quad \psi\in H^{4+\frac{3}{8}}(\Omega). \tag{4.6}$$

Moreover, for $T\le 1$ *and* ϕ *satisfying* (3.2), *we have*

$$\|\Delta^2\phi\ \nabla\phi\|_{L^{\frac{17}{8}}(L^2)} \le C\|\phi\|^{\frac{16}{17}}_{L^2(H^{4+\frac{3}{8}})}\|\phi\|^{\frac{18}{17}}_{C(H^{2+\frac{1}{4}})}. \tag{4.7}$$

Proof. By Hölder's inequality and Sobolev inequality, we have:

$$\begin{aligned}\|\Delta^2\psi\ \nabla\psi\|_{L^2(\Omega)} &\le \|\Delta^2\psi\|_{L^{\frac{12}{5}}}\|\nabla\psi\|_{L^{12}} \le C\|\Delta^2\psi\|_{H^{\frac{1}{4}}}\|\nabla\psi\|_{H^{\frac{5}{4}}} \\ &\le C\|\psi\|_{H^{4+\frac{1}{4}}}\|\psi\|_{H^{2+\frac{1}{4}}},\end{aligned} \tag{4.8}$$

which implies (4.6). Using next (2.1), we have:

$$\begin{aligned}\|\Delta^2\psi\ \nabla\psi\|_{L^2(\Omega)} &\le C\|\psi\|^{\frac{16}{17}}_{H^{4+\frac{3}{8}}}\|\psi\|^{\frac{1}{17}}_{H^{2+\frac{1}{4}}}\|\psi\|_{H^{2+\frac{1}{4}}} \\ &= C\|\psi\|^{\frac{16}{17}}_{H^{4+\frac{3}{8}}}\|\psi\|^{\frac{18}{17}}_{H^{2+\frac{1}{4}}}.\end{aligned} \tag{4.9}$$

Applying next (4.9) with $\psi = \phi(t,\cdot)$, we have:

$$\|\Delta^2\phi(t)\ \nabla\phi(t)\|_{L^2(\Omega)} \le C\|\phi(t)\|^{\frac{16}{17}}_{H^{4+\frac{3}{8}}}\|\phi(t)\|^{\frac{18}{17}}_{H^{2+\frac{1}{4}}}, \quad t\in[0,T]\ a.e.$$

This implies (4.7). □

Recall from (2.7) that $J_1(\phi) = -\frac{k}{\epsilon}\Delta(\phi^3)$.

Lemma 4.4. *We have:*

$$\|J_1(\psi-1)\ \nabla\psi\|_{L^2(\Omega)} \le C\|\psi\|^4_{H^{2+\frac{1}{4}}} + C\|\psi\|^3_{H^{2+\frac{1}{4}}}, \quad \psi\in H^{2+\frac{1}{4}}(\Omega). \tag{4.10}$$

Moreover, for $\phi\in C([0,T];H^{2+\frac{1}{4}}(\Omega))$ *and* $T\le 1$ *we have:*

$$\|J_1(\phi-1)\ \nabla\phi\|_{L^{\frac{17}{8}}(L^2)} \le C\|\phi\|^4_{C(H^{2+\frac{1}{4}})} + C\|\phi\|^3_{C(H^{2+\frac{1}{4}})}. \tag{4.11}$$

Proof. Using Hölder's inequality, Sobolev inequality and the fact that $H^{2+\frac{1}{4}}(\Omega)$ is a Banach Algebra, we have:

$$\begin{aligned}\|\Delta(\psi-1)^3\ \nabla\psi\|_{L^2} &\le \|\Delta(\psi^3+3\psi^2+3\psi)\|_{L^{\frac{12}{5}}}\|\nabla\psi\|_{L^{12}}\\ &\le C\left(\|\psi\|^3_{H^{2+\frac{1}{4}}}+1\right)\|\psi\|_{H^{2+\frac{1}{4}}},\end{aligned}$$

which proves (4.10). Applying it with $\psi=\phi(t,\cdot)$, we have:

$$\|J_1(\phi-1)\ \nabla\phi\|_{L^2}\le C\left(\|\phi\|^3_{H^{2+\frac{1}{4}}}+1\right)\|\phi\|_{H^{2+\frac{1}{4}}},\quad t\in[0,T]\ a.e.$$

which clearly implies (4.11). □

Recall from (2.7) that $J_2(\phi)=\dfrac{k}{\epsilon}\Delta\phi$, where $f(\phi)$ is defined by (1.3c).

Lemma 4.5. *We have*

$$\|J_2(\psi-1)\ \nabla\psi\|_{L^2}\le C\|\psi\|^2_{H^{2+\frac{1}{4}}},\quad \psi\in H^{2+\frac{1}{4}}(\Omega). \tag{4.12}$$

Moreover, for every $\phi\in C([0,T];H^{2+\frac{1}{4}}(\Omega))$*, we have*

$$\|J_2(\phi-1)\nabla\phi\|_{C(L^2)}\le C\|\phi\|^2_{C(H^{2+\frac{1}{4}})}. \tag{4.13}$$

Proof. With the same arguments as in Lemma 4.4, we have:

$$\|\Delta\psi\ \nabla\psi\|_{L^2}\le\|\Delta\psi\|_{L^{\frac{12}{5}}}\|\nabla\psi\|_{L^{12}}\le C\|\Delta\psi\|_{H^{\frac{1}{4}}}\|\nabla\psi\|_{H^{\frac{5}{4}}}\le C\|\psi\|^2_{H^{2+\frac{1}{4}}},$$

which clearly implies (4.12) and (4.13). □

Recall from (2.7) that $J_3(\phi)=\dfrac{k}{\epsilon^2}(3\phi^2-1)f(\phi)$.

Lemma 4.6. *We have:*

$$\|J_3(\psi-1)\nabla\psi\|_{L^2}\le C\left(\|\psi\|^3_{H^{2+\frac{3}{8}}}+\|\psi\|_{H^{2+\frac{3}{8}}}\right)^2,\quad \psi\in H^{4+\frac{3}{8}}(\Omega). \tag{4.14}$$

Moreover, for ϕ *satisfying* (3.2) *and* $T\le 1$*, we have*

$$\|J_3(\phi-1)\nabla\phi\|_{L^{\frac{17}{8}}(L^2)}\le C\left(\|\phi\|^3_{C([0,T];H^{2+\frac{3}{8}})}+\|\phi\|_{C([0,T];H^{2+\frac{3}{8}})}\right)^2. \tag{4.15}$$

Proof. We only prove (4.14) since it easily leads to (4.15). By Sobolev inequality, we have:

$$\begin{aligned}\|f(\psi-1)(3(\psi-1)^2-1)\nabla\psi\|_{L^2} &\le \|f(\psi-1)\|_{L^{\frac{8}{3}}}\|\nabla((\psi-1)^3-\psi+1)\|_{L^8}\\ &\le C\|f(\psi-1)\|_{H^{\frac{3}{8}}}\|\psi^3-3\psi^2+2\psi\|_{H^{2+\frac{3}{8}}}.\end{aligned}$$

then (4.14) is proved by applying (4.2) and the fact that $H^{2+\frac{3}{8}}(\Omega)$ is a Banach algebra. □

Recall from (2.7) that $J_4(\phi)=M_1[V(\phi)-\alpha]$, the following lemma holds:

Lemma 4.7. *We have*

$$\|J_4(\psi-1)\nabla\psi\|_{L^2(\Omega)}\le C(\|\psi\|_{L^2}+|\mu(\Omega)+\alpha|)\|\psi\|_{H^1},\quad (\psi\in H^1(\Omega)).$$

Moreover, for every ϕ *satisfying* (3.2) *and* $T\le 1$*, we have*

$$\|J_4(\phi-1)\nabla\phi\|_{L^{\frac{17}{8}}(L^2)}\le C\|\phi\|_{C(H^1)}(\|\phi\|_{C(L^2)}+|\mu(\Omega)+\alpha|).$$

Proof. The proofs are similar to the proofs of the lemmas above, so we skip them. □

Recall from (2.7) that $J_5(\psi)=M_2\,(B(\psi)-\beta)\,f(\psi)$.

Lemma 4.8. *We have*

$$\|J_5(\psi-1)\nabla\psi\|_{L^2} \le C\left(\|\psi\|^8_{H^{2+\frac{3}{8}}} + \|\psi\|^2_{H^{2+\frac{3}{8}}}\right), \quad \psi \in H^{2+\frac{3}{8}}(\Omega). \tag{4.16}$$

Moreover, for ϕ *satisfying* (3.2) *and* $T \le 1$, *we have*

$$\|J_5(\phi-1)\nabla\phi\|_{L^{\frac{17}{8}}(L^2)} \le C\left(\|\psi\|^8_{C([0,T];H^{2+\frac{3}{8}})} + \|\psi\|^2_{C([0,T];H^{2+\frac{3}{8}})}\right). \tag{4.17}$$

Proof. Inequality (4.17) can be easily derived from (4.16). Therefore we have just to check (4.16). By Hölder's inequality and Sobolev inequality:

$$\begin{aligned}\|(B(\psi-1)-\beta)f(\psi-1)\cdot\nabla\psi\|_{L^2} &\le |B(\psi-1)-\beta|\,\|f(\psi-1)\|_{L^{\frac{8}{3}}}\|\nabla\psi\|_{L^8}\\ &\le |B(\psi-1)-\beta|\,\|f(\psi-1)\|_{H^{\frac{3}{8}}}\|\nabla\psi\|_{H^{\frac{9}{8}}}.\end{aligned}$$

By (4.1) and (4.2), we have:

$$\|J_5(\psi-1)\nabla\psi\|_{L^2} \le C\left(\|\psi\|^4_{H^2}+1\right)\left(\|\psi\|_{H^{2+\frac{3}{8}}} + \|\psi\|^3_{H^{2+\frac{3}{8}}}\right)\|\psi\|_{H^{2+\frac{3}{8}}},$$

which implies (4.16) and, consequently, (4.17). □

We next recall some classical estimates.

Proposition 4.9. *We have*

$$\|\mathbf{v}_1\cdot\nabla\mathbf{v}_2\|_{L^2} \le C\|\mathbf{v}_1\|_{H^1}\|\mathbf{v}_2\|_{H^2}, \quad (\mathbf{v}_1,\mathbf{v}_2 \in [H^2(\Omega)]^3). \tag{4.18}$$

Moreover, for every $\mathbf{u}_1, \mathbf{u}_2$ *satisfying* (3.1), *we have*

$$\|\mathbf{u}_1\cdot\nabla\mathbf{u}_2\|_{L^{\frac{17}{8}}(L^2)} \le C\|\mathbf{u}_1\|_{C(H^1)}\left(\|\mathbf{u}_1\|_{C(H^1)} + \|\mathbf{u}_2\|_{L^2(H^2)}\right). \tag{4.19}$$

Proof. By Hölder's inequality, Sobolev inequality and (2.2), we have

$$\begin{aligned}\|\mathbf{v}_1\cdot\nabla\mathbf{v}_2\|_{L^2} \le \|\mathbf{v}_1\|_{L^4}\|\nabla\mathbf{v}_2\|_{L^4} &\le C\|\mathbf{v}_1\|_{H^1}\|\mathbf{v}_2\|_{H^{1+\frac{3}{4}}}\\ &\le C\|\mathbf{v}_1\|_{H^1}\|\mathbf{v}_2\|^{\frac{1}{17}}_{H^1}\|\mathbf{v}_2\|^{\frac{16}{17}}_{H^2},\end{aligned}$$

which implies (4.18). Estimate (4.19) is proved by applying the above estimate with $\mathbf{v}_1 = \mathbf{u}_1(t,\cdot)$ and $\mathbf{v}_2 = \mathbf{u}_2(t,\cdot)$. □

We next estimate the term $\mathbf{u}\cdot\nabla\phi$.

Proposition 4.10. *We have*

$$\|\mathbf{v}\cdot\nabla\psi\|_{H^{\frac{3}{8}}} \le C\|\mathbf{v}\|_{H^{1+\frac{3}{4}}}\|\psi\|_{H^{2+\frac{3}{8}}}, \quad \mathbf{v}\in[H^2(\Omega)]^3,\ \psi\in H^{2+\frac{3}{8}}(\Omega). \tag{4.20}$$

Moreover, for $\mathbf{u}$, ϕ *satisfying* (3.1) *and* (3.2) *respectively, we have*

$$\|\mathbf{u}\cdot\nabla\phi\|_{L^{\frac{17}{8}}(H^{\frac{3}{8}})} \le C\|\phi\|_{C(H^{2+\frac{3}{8}})}\left(\|\mathbf{u}\|_{C(H^1)} + \|\mathbf{u}\|_{L^2(H^2)}\right). \tag{4.21}$$

Proof. Denote $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{u} = (u_1, u_2, u_3)$. By Hölder's inequality, Sobolev inequality and (2.2), we have for $i, j, k \in \{1,2,3\}$,

$$\|v_k\partial_i\partial_j\psi\|_{L^2} \le \|v_k\|_{L^8}\|\partial_i\partial_j\psi\|_{L^{\frac{8}{3}}} \le C\|\mathbf{v}\|_{H^{\frac{9}{8}}}\|\psi\|_{H^{2+\frac{3}{8}}}. \tag{4.22}$$

$$\|\partial_i v_j\partial_k\psi\|_{L^2} \le \|\partial_i v_j\|_{L^{\frac{12}{5}}}\|\partial_k\psi\|_{L^{12}} \le C\|\mathbf{v}\|_{H^{1+\frac{3}{4}}}\|\psi\|_{H^{2+\frac{1}{4}}}. \tag{4.23}$$

We also have

$$\|v_i\partial_j\psi\|_{L^2} \le \|v_i\|_{L^4}\|\partial_j\psi\|_{L^4} \le C\|\mathbf{v}\|_{H^1}\|\psi\|_{H^2}.$$

By the Leibnitz formula, $\partial_k(v_i\partial_j\psi) = \partial_k v_i\partial_j\psi + v_i\partial_k\partial_j\psi$. With the above estimates, we have:

$$\|v_k\partial_j\psi\|_{H^1} \le \|v_k\partial_j\psi\|_{L^2} + \sum_{i=1}^{3}\|\partial_i(v_k\partial_j\psi)\|_{L^2} \le C\|\mathbf{v}\|_{H^{1+\frac{3}{4}}}\|\psi\|_{H^{2+\frac{3}{8}}},$$

which clearly implies (4.20). Applying (2.2) to (4.22) and (4.23), we obtain

$$\|v_k\partial_i\partial_j\psi\|_{L^2} \le C\|\mathbf{v}\|_{H^2}^{\frac{16}{17}}\|\mathbf{v}\|_{H^1}^{\frac{1}{17}}\|\psi\|_{H^{2+\frac{3}{8}}},$$

$$\|\partial_i v_j\partial_k\psi\|_{L^2} \le C\|\mathbf{v}\|_{H^2}^{\frac{16}{17}}\|\mathbf{v}\|_{H^1}^{\frac{1}{17}}\|\psi\|_{H^{2+\frac{1}{4}}}.$$

Applying to $\mathbf{v} = \mathbf{u}(t,\cdot)$ and $\psi = \phi(t,\cdot)$ we have

$$\|u_i\partial_j\partial_k\phi\|_{L^{\frac{17}{8}}(L^2)} \le C\|\mathbf{u}\|_{C(H^1)}^{\frac{1}{17}}\|\mathbf{u}\|_{L^2(H^2)}^{\frac{16}{17}}\|\phi\|_{C(H^{2+\frac{3}{8}})},$$

$$\|\partial_i u_j\partial_k\phi\|_{L^{\frac{17}{8}}(L^2)} \le C\|\mathbf{u}\|_{C(H^1)}^{\frac{1}{17}}\|\mathbf{u}\|_{L^2(H^2)}^{\frac{16}{17}}\|\phi\|_{C(H^{2+\frac{1}{4}})},$$

which implies (4.21). □

We next estimate the term $L(\phi)$ defined by (2.7).

Proposition 4.11. *We have*

$$\|L(\psi-1)\|_{H^{\frac{3}{8}}} \le C(\|\psi\|_{H^{2+\frac{3}{8}}}+1)^6\|\psi\|_{H^{2+\frac{3}{8}}} + C\|\psi\|_{H^2}^2\|\psi\|_{H^4} + C\,|\mu(\Omega)+\alpha|\,, \tag{4.24}$$

for $\psi \in H^4(\Omega)$. Moreover, for every ϕ satisfying (3.2) and $T \le 1$, we have

$$\|L(\phi-1)\|_{L^{\frac{17}{8}}(H^{\frac{3}{8}})} \le C\left(\|\phi\|_{C(H^{2+\frac{3}{8}})} + \|\phi\|_{L^2(H^{4+\frac{3}{8}})} + 1\right)^7. \tag{4.25}$$

Proof. By the fact that $H^{2+\frac{3}{8}}(\Omega)$ is a Banach Algebra, we have for $J_1(\psi-1)$:

$$\|J_1(\psi-1)\|_{H^{\frac{3}{8}}} = C\|\Delta(\psi-1)^3\|_{H^{\frac{3}{8}}} \le C\|\psi\|_{H^{2+\frac{3}{8}}}^3 + C\|\psi\|_{H^{2+\frac{3}{8}}}. \tag{4.26}$$

For $J_2(\psi-1)$, we have

$$\|J_2(\psi-1)\|_{H^{\frac{3}{8}}} \le C\|\psi\|_{H^{2+\frac{3}{8}}}. \tag{4.27}$$

By using (4.2) and the Banach algebra property of $H^2(\Omega)$, we obtain

$$\begin{aligned}
&\|(3(\psi-1)^2-1)f(\psi-1)\|_{H^{\frac{3}{8}}}\\
&\quad\le 3\|\psi^2 f(\psi-1)\|_{H^2} + C\|f(\psi-1)\|_{H^{\frac{3}{8}}}\\
&\quad\le 3\|\psi^2\|_{H^2}\|f(\psi-1)\|_{H^2} + C\|f(\psi-1)\|_{H^{\frac{3}{8}}}\\
&\quad\le C\|\psi\|_{H^2}^2\left(\|\psi\|_{H^4}+\|\psi\|_{H^2}^3\right) + C\|f(\psi-1)\|_{H^{\frac{3}{8}}},
\end{aligned}$$

which implies:

$$\|J_3(\psi-1)\|_{H^{\frac{3}{8}}} \le C\|\psi\|_{H^2}^2\left(\|\psi\|_{H^4}+\|\psi\|_{H^2}^3\right) + C\|\psi\|_{H^{2+\frac{3}{8}}} + C\|\psi\|_{H^{2+\frac{3}{8}}}^3. \tag{4.28}$$

For $J_4(\psi-1)$, we have

$$\|J_4(\psi-1)\|_{H^{\frac{3}{8}}} \le C(\|\psi\|_{H^1} + |\alpha+\mu(\Omega)|). \tag{4.29}$$

Similarly we have the following:

$$\begin{aligned}
\|J_5(\psi-1)\|_{H^{\frac{3}{8}}} &\le C|B(\psi-1)-\beta|\cdot\|f(\psi-1)\|_{H^{\frac{3}{8}}}\\
&\le C\left(\|\psi\|_{H^2}^4+1\right)\left(\|\psi\|_{H^{2+\frac{3}{8}}}+\|\psi\|_{H^{2+\frac{3}{8}}}^3\right).
\end{aligned} \tag{4.30}$$

Then (4.24) is proved by adding inequalities (4.26)–(4.30). Applying (4.26), (4.27) and (4.29) with $\psi = \phi(t,\cdot)$, we obtain:

$$\|J_1(\phi-1)\|_{C(H^{\frac{3}{8}})} \le C\|\phi\|_{C(H^{2+\frac{3}{8}})}\left(\|\phi\|_{C(H^{2+\frac{3}{8}})}^2+1\right), \tag{4.31a}$$

$$\|J_2(\phi-1)\|_{C(H^{\frac{3}{8}})} \le C\|\phi\|_{C(H^{2+\frac{3}{8}})}, \tag{4.31b}$$

$$\|J_4(\phi-1)\|_{C(H^{\frac{3}{8}})} \le C(\|\phi\|_{C(H^1)} + |\alpha+\mu(\Omega)|). \tag{4.31c}$$

Applying (4.28) with $\psi = \phi(t,\cdot)$, we have the following

$$
\begin{aligned}
&\|J_3(\phi-1)\|_{L^{\frac{17}{8}}(H^{\frac{3}{8}})}\\
&\quad \le C\|\phi\|_{C(H^2)}^2\left(\|\phi\|_{L^2(H^{4+\frac{3}{8}})}^{\frac{16}{17}}\|\phi\|_{C(H^{2+\frac{1}{4}})}^{\frac{1}{17}}+\|\phi\|_{C(H^2)}^3\right)\\
&\qquad + C\|\phi\|_{C(H^{2+\frac{3}{8}})} + C\|\phi\|_{C(H^{2+\frac{3}{8}})}^3\\
&\quad \le C\left(\|\phi\|_{C(H^{2+\frac{3}{8}})}^2+1\right)\left(\|\phi\|_{L^2(H^{4+\frac{3}{8}})}^{\frac{16}{17}}\|\phi\|_{C(H^{2+\frac{1}{4}})}^{\frac{1}{17}}+\|\phi\|_{C(H^{2+\frac{3}{8}})}^3\right)
\end{aligned}
\tag{4.32}
$$

By (4.30), we have

$$
\|J_5(\phi-1)\|_{C(H^{\frac{3}{8}})} \le C(\|\psi\|_{C(H^2)}^4+1)\left(\|\psi\|_{C(H^{2+\frac{3}{8}})}+\|\psi\|_{C(H^{2+\frac{3}{8}})}^3\right). \tag{4.33}
$$

Then (4.25) is proved by adding (4.31), (4.32) and (4.33) together. □

5. Proof of Theorem 3.3

In this section, we detail the proof of Theorem 3.3. We begin with a lemma. Recall the definitions of $f(\phi)$ and $B(\phi)$ from (1.3), we have the following result.

Lemma 5.1. *For $\psi_1, \psi_2 \in H^4(\Omega)$, we have*

$$
\|\Delta(\psi_1^3)-\Delta(\psi_2^3)\|_{H^{\frac{3}{8}}} \le C\|\psi\|_{H^{2+\frac{3}{8}}}^3 + C\|\psi_1\|_{H^{2+\frac{3}{8}}}\|\psi_2\|_{H^{2+\frac{3}{8}}}\|\psi\|_{H^{2+\frac{3}{8}}}, \tag{5.1}
$$

$$
\|f(\psi_1)-f(\psi_2)\|_{H^2} \le C\left(\|\psi\|_{H^4}+\|\psi\|_{H^2}^2+\|\psi_1\|_{H^2}\|\psi_2\|_{H^2}\|\psi\|_{H^2}\right), \tag{5.2}
$$

$$
|B(\psi_1)-B(\psi_2)| \le C\left(\|\psi_1\|_{H^2}^3+\|\psi_2\|_{H^2}^3+1\right)\|\psi\|_{H^2}. \tag{5.3}
$$

where $\psi = \psi_1 - \psi_2$.

Proof. we have:

$$
\begin{aligned}
\|\Delta(\psi_1^3)-\Delta(\psi_2^3)\|_{H^{\frac{3}{8}}} &\le \|\psi_1^3-\psi_1^3\|_{H^{2+\frac{3}{8}}}\\
&\le \|(\psi_1-\psi_2)^3\|_{H^{2+\frac{3}{8}}} + 3\|\psi_1\psi_2(\psi_1-\psi_2)\|_{H^{2+\frac{3}{8}}}.
\end{aligned}
$$

Then (5.1) is proved by the fact that $H^s(\Omega)$ is a Banach Algebra if $s > \frac{3}{2}$. For $f(\psi)$ we have

$$
\|f(\psi_1)-f(\psi_2)\|_{H^2} \le C\|\Delta\psi\|_{H^2} + C\|\psi_1^3-\psi_2^3\|_{H^2},
$$

which implies (5.2). Estimate (5.3) can be proved similarly, so we skip the details. □

We next estimate the leading order term $\Delta^2\phi\nabla\phi$.

Proposition 5.2. *Under the condition (3.6), there exists a constant $C = C(\Omega)$ such that:*

$$
\|\Delta^2\phi_1\nabla\phi_1 - \Delta^2\phi_2\nabla\phi_2\|_{L^{\frac{17}{8}}(L^2)} \le CR\left(\|\phi\|_{L^2(H^{4+\frac{3}{8}})}+\|\phi\|_{C(H^{2+\frac{1}{4}})}\right).
$$

Proof. Since $\Delta^2\phi_1\nabla\phi_1 - \Delta^2\phi_2\nabla\phi_2 = \Delta^2\phi\nabla\phi_1 + \Delta^2\phi_2\nabla\phi$, the lemma is proved by applying (4.7). □

Recall the definition of $L(\phi)$ from (2.7) We have

Proposition 5.3. *Under the assumption (3.6), there exists a constant C depending on parameters $\epsilon, k, \Omega, M_1, M_2, \beta$ such that:*

$$
\|L(\phi_1)\nabla\phi_1 - L(\phi_2)\nabla\phi_2\|_{L^{\frac{17}{8}}(0,T;L^2)} \le CR^7(\|\phi\|_{C([0,T];H^{2+\frac{3}{8}})}+\|\phi\|_{L^2(0,T;H^{4+\frac{3}{8}})}).
$$

Proof.

$$L(\phi_1)\nabla\phi_1 - L(\phi_2)\nabla\phi_2 = \sum_{i=1}^{5}\left[J_i(\phi_1)\nabla\phi_1 - J_i(\phi_2)\nabla\phi_2\right]$$

We will estimate term by term. Recall from (2.7) the definition of J_1,

$$\|\Delta(\phi_1^3)\nabla\phi_1 - \Delta(\phi_2^3)\nabla\phi_2\|_{L^2} \le \|[\Delta(\phi_1^3) - \Delta(\phi_2^3)]\nabla\phi_2 + \Delta(\phi_1^3)(\nabla(\phi))\|_{L^2}.$$

By Hölder's inequality and Sobolev inequality,

$$\begin{aligned}&\|\Delta(\phi_1^3)\nabla\phi_1 - \Delta(\phi_2^3)\nabla\phi_2\|_{L^2}\\ &\quad\le \|(\Delta(\phi_1^3) - \Delta(\phi_2^3))\nabla\phi_2 + \Delta(\phi_1^3)(\nabla(\phi))\|_{L^2}\\ &\quad\le \|\Delta(\phi_1^3) - \Delta(\phi_2^3)\|_{L^{\frac{8}{3}}}\|\nabla\phi_2\|_{L^8} + \|\Delta(\phi_1^3)\|_{L^{\frac{8}{3}}}\|\nabla\phi\|_{L^8}\\ &\quad\le C\|\Delta(\phi_1^3) - \Delta(\phi_2^3)\|_{H^{\frac{3}{8}}}\|\phi_2\|_{H^2} + C\|\phi_1^3\|_{H^{\frac{3}{8}}}\|\phi\|_{H^2}.\end{aligned}$$

By the Banach algebra property of $H^s(\Omega)$ and (5.1),

$$\begin{aligned}&\|\Delta(\phi_1^3)\nabla\phi_1 - \Delta(\phi_2^3)\nabla\phi_2\|_{L^2}\\ &\quad\le C\left(\|\phi\|^3_{H^{2+\frac{3}{8}}} + \|\phi_1\|_{H^{2+\frac{3}{8}}}\|\phi_2\|_{H^{2+\frac{3}{8}}}\|\phi\|_{H^{2+\frac{3}{8}}}\right)\|\phi_2\|_{H^{2+\frac{3}{8}}}\\ &\qquad + C\|\phi_1\|^3_{H^{2+\frac{3}{8}}}\|\phi\|_{H^{2+\frac{3}{8}}},\end{aligned} \tag{5.4}$$

which clearly implies

$$\|J_1(\phi_1)\nabla\phi_1 - J_1(\phi_2)\nabla\phi_2\|_{L^{\frac{17}{8}}(L^2)} \le CR^3\|\phi\|_{C(H^{2+\frac{3}{8}})}. \tag{5.5}$$

Recall from (2.7) for the definition of J_3. By Sobolev inequality, (4.3) and (5.2),

$$\begin{aligned}&\|(3\phi_1^2 - 1)f(\phi_1)\cdot\nabla\phi_1 - (3\phi_2^2 - 1)f(\phi_2)\cdot\nabla\phi_2\|_{L^2}\\ &\quad\le \|f(\phi_1) - f(\phi_2)\|_{H^2}\|\nabla(\phi_1^3 - \phi_1)\|_{L^2} + \|\nabla(\phi_1^3 - \phi_1) - \nabla(\phi_2^3 - \phi_2)\|_{L^2}\|f(\phi_2)\|_{H^2}\\ &\quad\le C\left(\|\phi\|_{H^4} + \|\phi\|^2_{H^2} + \|\phi_1\|_{H^2}\|\phi_2\|_{H^2}\|\phi\|_{H^2}\right)\left(\|\phi_1\|^3_{H^2} + \|\phi_1\|_{H^2}\right)\\ &\qquad + C\left(\|\phi\|^3_{H^2} + \|\phi_1\|_{H^2}\|\phi_2\|_{H^2}\|\phi\|_{H^2} + \|\phi\|_{H^2}\right)\left(\|\phi_2\|^3_{H^2} + \|\phi_2\|_{H^4}\right).\end{aligned}$$

Using (2.1) to deal with the term with $H^4(\Omega)$ norm,

$$\begin{aligned}&\|J_3(\phi_1)\nabla\phi_1 - J_3(\phi_2)\nabla\phi_2\|_{L^2(\Omega)}\\ &\quad\le C\left(\|\phi\|^{\frac{16}{17}}_{H^{4+\frac{3}{8}}(\Omega)}\|\phi\|^{\frac{1}{17}}_{H^{2+\frac{1}{4}}(\Omega)} + \|\phi\|^2_{H^2} + \|\phi_1\|_{H^2}\|\phi_2\|_{H^2}\|\phi\|_{H^2}\right)\\ &\qquad\times(\|\phi_1\|^3_{H^2} + \|\phi_1\|_{H^2}) + C(\|\phi\|^3_{H^2} + \|\phi_1\|_{H^2}\|\phi_2\|_{H^2}\|\phi\|_{H^2} + \|\phi\|_{H^2})\\ &\qquad\times\left(\|\phi_2\|^3_{H^2} + \|\phi_2\|^{\frac{16}{17}}_{H^{4+\frac{3}{8}}(\Omega)}\|\phi_2\|^{\frac{1}{17}}_{H^{2+\frac{1}{4}}(\Omega)}\right).\end{aligned} \tag{5.6}$$

By (3.6) we have:

$$\|J_3(\phi_1)\nabla\phi_1 - J_3(\phi_2)\nabla\phi_2\|_{L^{\frac{17}{8}}(L^2)} \le CR^5\left(\|\phi\|_{L^2(H^{4+\frac{3}{8}})} + \|\phi\|_{C(H^{2+\frac{1}{4}})}\right). \tag{5.7}$$

Recall from (2.7) for the definition of J_5. Note that

$$\begin{aligned}&M_2^{-1}[J_5(\phi_1)\nabla\phi_1 - J_5(\phi_2)\nabla\phi_2]\\ &\quad= [B(\phi_1) - \beta]f(\phi_1)\nabla\phi_1 - [B(\phi_2) - \beta]f(\phi_2)\nabla\phi_2\\ &\quad= [B(\phi_1) - \beta]\left\{[f(\phi_1) - f(\phi_2)]\nabla\phi_1 + f(\phi_2)\nabla\phi\right\} + [B(\phi_1) - B(\phi_2)]\,f(\phi_2)\nabla\phi_2.\end{aligned}$$

From the above formula and the Sobolev inequality, it follows that:

$$\begin{aligned} &M_2^{-1}\|J_5(\phi_1)\nabla\phi_1 - J_5(\phi_2)\nabla\phi_2\|_{L^2(\Omega)} \\ &\quad\le |B(\phi_1)-\beta|\,\{\|f(\phi_1)-f(\phi_2)\|_{L^\infty}\|\nabla\phi_1\|_{L^2} + \|f(\phi_2)\|_{L^\infty}\|\nabla\phi\|_{L^2}\} \\ &\qquad + |B(\phi_1)-B(\phi_2)|\|f(\phi_2)\|_{L^\infty}\|\nabla\phi_2\|_{L^2} \\ &\quad\le |B(\phi_1)-\beta|\,[\|f(\phi_1)-f(\phi_2)\|_{H^2}\|\phi_1\|_{H^1} + \|f(\phi_2)\|_{H^2}\|\phi\|_{H^1}] \\ &\qquad + |B(\phi_1)-B(\phi_2)|\|f(\phi_2)\|_{H^2}\|\phi_2\|_{H^1}, \end{aligned}$$

By (4.1), (4.3), (5.2) and (5.3) we have:

$$\begin{aligned} \|J_5(\phi_1)\nabla\phi_1 - J_5(\phi_2)\nabla\phi_2\|_{L^2(\Omega)} &\le C(\|\phi_1\|_{H^2}^4+1)C\left(\|\phi\|_{H^4} + \|\phi\|_{H^2}^2 + \|\phi_1\|_{H^2}\|\phi_2\|_{H^2}\|\phi\|_{H^2}\right) \\ &\quad + C\left(\|\phi_1\|_{H^2}^3 + \|\phi_2\|_{H^2}^3 + 1\right)\|\phi\|_{H^2}(\|\phi_2\|_{H^2} + \|\phi_2\|_{H^2}^3)\|\phi_2\|_{H^1}, \end{aligned}$$

Using the same steps as those used to derive (5.6) we obtain

$$\|J_5(\phi_1)\nabla\phi_1 - J_5(\phi_2)\nabla\phi_2\|_{L^{\frac{17}{8}}(L^2)} \le CR^7\left(\|\phi\|_{L^2(H^{4+\frac{3}{8}})} + \|\phi\|_{C(H^{2+\frac{1}{4}})}\right). \tag{5.8}$$

Since the terms J_2 and J_4 are of lower order and thus easier, we don't detail the proofs of the corresponding estimates,

$$\|J_2(\phi_1)\nabla\phi_1 - J_2(\phi_1)\nabla\phi_1\|_{C(H^{2+\frac{3}{8}})} \le CR\|\phi\|_{C(H^{2+\frac{3}{8}})} \tag{5.9}$$

$$\|J_4(\phi_1)\nabla\phi_1 - J_4(\phi_1)\nabla\phi_1\|_{C(H^{2+\frac{3}{8}})} \le CR\|\phi\|_{C(H^{2+\frac{3}{8}})} \tag{5.10}$$

Then the proposition is proved by combining (5.5), (5.7), (5.8), (5.9) and (5.10). □

The proof of the following proposition is similar to the previous proposition, so we skip the details.

Proposition 5.4. *Under the condition (3.6), there exists a constant $C = C(\epsilon, \beta, k, \Omega, M_1, M_2)$ such that we have the following estimates:*

$$\|L(\phi_1) - L(\phi_2)\|_{L^{\frac{17}{8}}(H^{\frac{3}{8}})} \le CR^6\left(\|\phi\|_{L^2(H^{4+\frac{3}{8}})} + \|\phi\|_{C(H^{2+\frac{3}{8}})}\right).$$

The following proposition can be easily derived from (4.19) and (4.21).

Proposition 5.5. *Under the assumption (3.6), there exists a constant $C = C(\Omega)$ such that*

$$\|\mathbf{u}_1\cdot\nabla\mathbf{u}_1 - \mathbf{u}_2\cdot\nabla\mathbf{u}_2\|_{L^{\frac{17}{8}}(0,T;L^2)} \le CR(\|\mathbf{u}\|_{L^2(0,T;H^2)} + \|\mathbf{u}\|_{C([0,T];H^1)}). \tag{5.11}$$

$$\|\mathbf{u}_1\cdot\nabla\phi_1 - \mathbf{u}_2\cdot\nabla\phi_2\|_{L^{\frac{17}{8}}(0,T;H^{\frac{3}{8}})} \le CR\left(\|\mathbf{u}\|_{C([0,T];H^1)} + \|\mathbf{u}\|_{L^2(0,T;H^2)} + \|\phi\|_{C([0,T];H^{2+\frac{3}{8}})}\right). \tag{5.12}$$

References

[1] Bensoussan, A., Da Prato, G., Delfour, M.C., Mitter, S.K.: Representation and control of infinite-dimensional systems, vol. 1. Systems and Control: Foundations and Applications. Birkhäuser Boston Inc., Boston (1992)

[2] Constantin, P., Foias, C.: Navier–Stokes equations. Chicago Lectures in Mathematics. University of Chicago Press, Chicago (1988)

[3] Du, Q., Li, M., Liu, C.: Analysis of a phase field Navier–Stokes vesicle–fluid interaction model. Discrete Contin. Dyn. Syst. Ser. B, **8**(3), 539–556 (2007) (electronic)

[4] Du, Q., Liu, C., Ryham, R., Wang, X.: Energetic variational approaches in modeling vesicle and fluid interactions. Phys. D **238**(9–10), 923–930 (2009)

[5] Du, Q., Liu, C., Wang, X.: A phase field approach in the numerical study of the elastic bending energy for vesicle membranes. J. Comput. Phys. **198**(2), 450–468 (2004)

[6] Du, Q., Liu, C., Wang, X.,: Retrieving topological information for phase field models. SIAM J. Appl. Math. **65**(6), 1913–1932 (2005) (electronic)

[7] de Haas, K.H., Bloom, C., Ende, D.van den , Duits, M.H.G., Mellema, J.: Deformation of giant lipid bilayer vesicles in shear flow. Phys. Rev. Lett. **56**(6), 7132–7137 (1997)

[8] Lartigue, C., Abkarian, M., Viallat, A.: Tank treading and unbinding of deformable vesicles in shear flow: determination of the lift force. Phys. Rev. Lett. **88**(6) (2002)

[9] Lions, J.-L., Magenes, E.: Non-homogeneous boundary value problems and applications. vol. 1. Springer, New York (1972) (Translated from the French by P. Kenneth, Die Grundlehren der mathematischen Wissenschaften, Band 181)
[10] Wang, X., Du, Q.: Modelling and simulations of multi-component lipid membranes and open membranes via diffuse interface approaches. J. Math. Biol. **56**(3), 347–371 (2008)

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