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Point Singularities of 3D Stationary Navier–Stokes Flows

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Abstract. This article characterizes the singularities of very weak solutions of 3D stationary Navier–Stokes equations in a punctured ball which are sufficiently small in weak L^3.

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1. Introduction

We consider point singularities of very weak solutions of the 3D stationary Navier–Stokes equations in a finite region Ω in $\mathbb{R}^3$. The Navier–Stokes equations for the velocity $u : \Omega \to \mathbb{R}^3$ and pressure $p : \Omega \to \mathbb{R}$ with external force $f : \Omega \to \mathbb{R}^3$ are

$$-\Delta u + (u \cdot \nabla)u + \nabla p = f, \quad \operatorname{div} u = 0, \quad (x \in \Omega). \tag{1.1}$$

A *very weak solution* is a vector function u in $L^2_{loc}(\Omega)$ which satisfies (1.1) in distribution sense:

$$\int -u \cdot \Delta\varphi + u_j u_i \partial_j \varphi_i = \langle f, \varphi \rangle, \quad \forall \varphi \in C^\infty_{c,\sigma}(\Omega), \tag{1.2}$$

and $\int u \cdot \nabla h = 0$ for any $h \in C^\infty_c(\Omega)$. Here the force f is allowed to be a distribution and

$$C^\infty_{c,\sigma}(\Omega) = \{\varphi \in C^\infty_c(\Omega, \mathbb{R}^3) : \operatorname{div} \varphi = 0\}. \tag{1.3}$$

In this definition the pressure is not needed. Denote $B_R = \{x \in \mathbb{R}^3 : |x| < R\}$ and $B^c_R = \mathbb{R}^3 \setminus B_R$ for $R > 0$.

We are concerned with the behavior of very weak solutions which solve (1.1) in the punctured ball $B_2 \setminus \{0\}$ with zero force, i.e., $f = 0$. There are a lot of studies on this problem [4,5,8,15,16]. A typical result is to show that, under some conditions, the solution is a very weak solution across the origin without singular forcing supported at the origin (removable singularity), and is regular, i.e., locally bounded, under possibly more assumptions (regularity). Dyer–Edmunds [5] proved removable singularity and regularity assuming both $u, p \in L^{3+\varepsilon}(B_2)$ for some $\varepsilon > 0$. Shapiro [15,16] proved removable singularity and regularity assuming $u \in L^{3+\varepsilon}(B_2)$ for some $\varepsilon > 0$ and $u(x) = o(|x|^{-1})$ as $x \to 0$, without assumption on p. Choe and Kim [4] proved removable singularity assuming $u \in L^3(B_2)$ *or* $u(x) = o(|x|^{-1})$ as $x \to 0$, and regularity assuming $u \in L^{3+\varepsilon}(B_2)$ for some $\varepsilon > 0$. Kim and Kozono [8] recently proved removable singularity under the same assumptions as [4], and regularity assuming $u \in L^3(B_2)$ or u is small in weak L^3. As mentioned in [8], their result is optimal in the sense that if their assumption is replaced by

$$|u(x)| \le C_* |x|^{-1} \tag{1.4}$$

for $0 < |x| < 2$, then the singularity is not removable in general, due to the existence of *Landau solutions*, which is the family of explicit singular solutions calculated by Landau [11], and can be found in standard textbooks, see e.g., [12, p. 82] or [1, p. 206].

The purpose of this article is to characterize the singularity and to identify the leading order behavior of very weak solutions satisfying the threshold assumption (1.4) when the constant C_* is sufficiently small. We show that it is given by Landau solutions.

We now recall Landau solutions in order to state our main theorems. Landau solutions can be parametrized by vectors $b \in \mathbb{R}^3$ in the following way: For each $b \in \mathbb{R}^3$ there exists a unique (-1)-homogeneous solution U^b of (1.1) together with an associated pressure P^b which is (-2)-homogeneous, such that U^b, P^b are smooth in $\mathbb{R}^3\backslash\{0\}$ and they solve

$$-\Delta u + (u\cdot\nabla)u + \nabla p = b\delta, \quad \operatorname{div} u = 0, \tag{1.5}$$

in $\mathbb{R}^3$ in the sense of distributions, where δ denotes the Dirac δ function. When $b = (0,0,\beta)$ with $\beta \geq 0$, they have the following explicit formulas in spherical coordinates r, θ, ϕ with $x = (r\sin\theta\cos\phi,\ r\sin\theta\sin\phi,\ r\cos\theta)$:

$$U = \frac{2}{r}\left(\frac{A^2-1}{(A-\cos\theta)^2} - 1\right)e_r - \frac{2\sin\theta}{r(A-\cos\theta)}e_\theta, \quad P = \frac{-4(A\cos\theta - 1)}{r^2(A-\cos\theta)^2} \tag{1.6}$$

where $e_r = \frac{x}{r}$ and $e_\theta = (\cos\theta\cos\phi,\ \cos\theta\sin\phi,\ -\sin\theta)$. The parameters $\beta \geq 0$ and $A \in (1,\infty]$ are related by the formula

$$\beta = 16\pi\left(A + \frac{1}{2}A^2\log\frac{A-1}{A+1} + \frac{4A}{3(A^2-1)}\right). \tag{1.7}$$

The formulas for general b can be obtained from rotation. One checks directly that $\left\|rU^b\right\|_{L^\infty}$ is monotone in $|b|$ and $\left\|rU^b\right\|_{L^\infty} \to 0$ (or ∞) as $|b| \to 0$ (or ∞). Recently Sverak [17] proved that Landau solutions are the only solutions of (1.1) in $\mathbb{R}^3\backslash\{0\}$ which are smooth and (-1)-homogeneous in $\mathbb{R}^3\backslash\{0\}$, without assuming axisymmetry. See also [2,9,19] for related results.

If u, p is a solution of (1.1), we will denote by

$$T_{ij}(u,p) = p\delta_{ij} + u_iu_j - \partial_iu_j - \partial_ju_i \tag{1.8}$$

the momentum flux density tensor in the fluid, which plays an important role to determine the equation for (u,p) at 0. Our main result is the following.

Theorem 1.1. *For any $q \in (1,3)$, there is a small $C_* = C_*(q) > 0$ such that, if u is a very weak solution of (1.1) with zero force in $B_2\backslash\{0\}$ satisfying (1.4) in $B_2\backslash\{0\}$, then there is a scalar function p satisfying $|p(x)| \leq C|x|^{-2}$, unique up to a constant, so that (u,p) satisfies (1.5) in B_2 with $b_i = \int_{|x|=1} T_{ij}(u,p)n_j(x)$, and*

$$\left\|u - U^b\right\|_{W^{1,q}(B_1)} + \sup_{x\in B_1}|x|^{3/q-1}|(u-U^b)(x)| \leq CC_*, \tag{1.9}$$

where the constant C is independent of q and u.

The exponent q can be regarded as the degree of the approximation of u by U^b. The closer q gets to 3, the less singular $u - U^b$ is. But in our theorem, $C_*(q)$ shrinks to zero as $q \to 3_-$. Ideally, one would like to prove that $u - U^b \in L^\infty$. However, it seems quite subtle in view of the following model equation for a scalar function,

$$-\Delta v + cv = 0, \quad c = \Delta v/v. \tag{1.10}$$

If we choose $v = \log|x|$, then $c(x) \in L^{3/2}$ and $\lim_{|x|\to 0}|x|^2|c(x)| = 0$, but $v \notin L^\infty$. In Eq. (3.2) for the difference $w = u - U^b$, there is a term $(w\cdot\nabla)U^b$ which has similar behavior as cv above.

In fact, we have the following stronger result. Denote by L^r_{wk} the weak L^r spaces. We claim the same conclusion as in Theorem 1.1 assuming only a small L^3_{wk} bound of u but not the pointwise bound (1.4).

Theorem 1.2. *There is a small $\varepsilon_* > 0$ such that, if u is a very weak solution of* (1.1) *with zero force in $\Omega = B_{2.1}\backslash\{0\}$ satisfying $\|u\|_{L^3_{wk}(\Omega)} =: \varepsilon \le \varepsilon_*$, then u satisfies $|u(x)| \le C_1\varepsilon|x|^{-1}$ in $B_2\backslash\{0\}$ for some C_1. Thus the conclusion of Theorem* 1.1 *holds if $C_1\varepsilon \le C_*(q)$.*

Our results are closely related to the *regularity problem* of very weak solutions, which could be considered when u is only assumed to be in L^2_{loc}. In fact, the problem with the assumption u being large in L^3_{wk} already exhibits a great difficulty. Recall the scaling property of (1.1): If (u, p) is a solution of (1.1), then so is

$$(u_\lambda, p_\lambda)(x) = (\lambda u(\lambda x), \lambda^2 p(\lambda x)), \quad (\lambda > 0). \tag{1.11}$$

The known methods are primarily perturbation arguments. Since L^3_{wk}-quasi-norm is invariant under the above scaling and does not become smaller when restricted to smaller regions, one would need to exploit the structure of the Navier–Stokes equations in order to get a positive answer. Compare the recent result [3] on axisymmetric solutions of nonstationary Navier–Stokes equations, which also considers a borderline case under the natural scaling.

This work is inspired by Korolev–Sverak [9] in which they study the asymptotic as $|x| \to \infty$ of solutions of (1.1) satisfying (1.4) in $\mathbb{R}^3\backslash B_1$, extending [13]. They show that the leading behavior is also given by Landau solutions if C_* is sufficiently small. Our theorem can be considered as a dual version of their result. However, their proof is based on the uniqueness of solutions of the equation on $\mathbb{R}^3$ satisfied by $v = \varphi(u - U^b) + \zeta$ where φ is a cut-off function supported near infinity and ζ is a suitable function chosen to make $\operatorname{div} v = 0$. If one tries the same approach for our problem, since one needs to remove the origin as well as the region $|x| \ge 2$ while extending $u - U^b$, one needs to choose a sequence φ_k with the supports of $1 - \varphi_k$ shrinking to the origin, which produce very singular force terms near the origin.

Specifically, let $\varphi(x)$ be a smooth function which is 0 for $|x| < 1.8$ and 1 for $|x| > 1.9$, and let $\varphi_k(x) = \varphi(2^k x)$. Let $v_k = (\varphi - \varphi_k)(u - U^b) + \zeta_k$ where ζ_k is chosen to make $\operatorname{div} v_k = 0$, and is supported in $E_k = \{x : 1.7 \le |x| \le 2 \text{ or } 1.7 \cdot 2^{-k} \le |x| \le 2^{1-k}\}$. The vector field v_k satisfies an equation in $\mathbb{R}^3$ with a force f_k which is supported in E_k with zero average, and whose height is of order 2^{3k}. One tries to construct a solution v of this equation in the class

$$|v(x)| \le CC_*(|x|^\delta + |x|)^{-1}, \tag{1.12}$$

with uniform in k constants $\delta \in (0, 1)$ and $C > 0$, and prove that it is equal to v_k. This seems possible but delicate.

Instead, we first define the equation for (u, p) at the origin (Lemma 2.3). Since the equation for u is same as U^b near the origin for $b = b(u)$, the δ-functions at the origin cancel in the equation for their difference. We then apply the approach of Kim–Kozono [8] to the difference equation, and prove its unique existence in $W^{1,r}_0(B_2)$ for $3/2 \le r < 3$ and uniqueness in $W^{1,r}_0 \cap L^3_{wk}(B_2)$ for $1 < r < 3/2$, which improves the regularity of the original difference. Above $W^{1,r}_0(B_2)$ is the closure of $C^\infty_c(B_2)$ in the $W^{1,r}(B_2)$-norm.

As an application, we give the following corollary. Recall u_λ for $\lambda > 0$ is defined in (1.11). A solution u on $B_2\backslash\{0\}$ is called *discretely self-similar* if there is a $\lambda_1 \in (0, 1)$ so that $u_{\lambda_1} = u$. Such a solution is completely determined by its values in the annulus $B_1\backslash B_{\lambda_1}$ since $u(\lambda_1^k x) = \lambda_1^{-k} u(x)$. They contain minus-one homogeneous solutions as a special subclass.

Corollary 1.3. *If u satisfies the assumptions of Theorem* 1.1 *and furthermore u is discretely self-similar in $B_2\backslash\{0\}$, then $u \equiv U^b$.*

This corollary also follows from [9] (with domain $\mathbb{R}^3\backslash B_1$ and $\lambda_1 > 1$). In the case of small C_*, this corollary extends the result of Sverak [17] on minus-one homogeneous solutions. The classification of discretely self-similar solutions with large C_* is unknown.

As another application, we consider a conjecture by Sverak [17, Sect. 5]:

Conjecture 1.4. *If u is a solution of the stationary Navier–Stokes equations* (1.1) *with zero force in $\mathbb{R}^3\backslash\{0\}$ satisfying* (1.4) *with some $C_* > 0$. Then u is a Landau solution.*

We give a partial answer for this problem.

Corollary 1.5. *Conjecture 1.4 is true, provided the constant C_* is sufficiently small.*

The above corollary can be also shown to be true by either our main theorem or the result of Korolev-Sverak [9], see Sect. 3.4. The corresponding conjecture for large C_* is related to the regularity problem of evolutionary Navier–Stokes equations via the usual blow-up procedures.

2. Preliminaries

In this section we collect some lemmas for the proof of Theorem 1.1. The first lemma recalls Hölder and Sobolev type inequalities in Lorentz spaces. We denote the Lorentz spaces by $L^{p,q}$ ($1 < p < \infty$, $1 \le q \le \infty$). Note $L^3_{wk} = L^{3,\infty}$.

Lemma 2.1. *Let $B = B_2 \subset \mathbb{R}^n$, $n \ge 2$.*

(i) *Let $1 < p_1, p_2 < \infty$ with $1/p := 1/p_1 + 1/p_2 < 1$ and let $1 \le r_1, r_2 \le \infty$. For $f \in L^{p_1,r_1}$ and $g \in L^{p_2,r_2}$, we have*

$$\|fg\|_{L^{p,r}(B)} \le C\|f\|_{L^{p_1,r_1}(B)}\|g\|_{L^{p_2,r_2}(B)} \qquad \text{for } r := \min\{r_1, r_2\}, \tag{2.1}$$

where $C = C(p_1, r_1, p_2, r_2)$.

(ii) *Let $1 < r < n$. For $f \in W^{1,r}(B)$, we have*

$$\|f\|_{L^{\frac{nr}{n-r},r}(B)} \le C\|f\|_{W^{1,r}(B)}, \tag{2.2}$$

where $C = C(n, r)$.

Part (i) of Lemma 2.1 was proved in [14]. Part (ii) was proved in [14] for $\mathbb{R}^n$ and in [10,8] for bounded domains.

By this lemma, when $n = 3$ and $1 < r < 3$, we have

$$\|fg\|_{L^r(B)} \le C\,\|f\|_{L^3_{wk}}\,\|g\|_{L^{\frac{3r}{3-r},r}} \le C_r\,\|f\|_{L^3_{wk}(B)}\,\|g\|_{W^{1,r}(B)}\,. \tag{2.3}$$

This estimate first appeared in [8] and plays an important role for our application.

The next lemma is on interior estimates for Stokes system with no assumption on the pressure.

Lemma 2.2. *Assume $v \in L^1$ is a distribution solution of the Stokes system*

$$-\Delta v_i + \partial_i p = \partial_j f_{ij}, \quad \operatorname{div} v = 0 \quad \text{in } B_{2R} \tag{2.4}$$

and $f \in L^r$ for some $r \in (1, \infty)$. Then $v \in W^{1,r}_{loc}$ and, for some constant C_r independent of v and R,

$$\|\nabla v\|_{L^r(B_R)} \le C_r\,\|f\|_{L^r(B_{2R})} + C_r R^{-4+3/r}\,\|v\|_{L^1(B_{2R})}\,. \tag{2.5}$$

This lemma is [18], Theorem 2.2. Although the statement in [18] assumes $v \in W^{1,r}_{loc}$, its proof only requires $v \in L^1$. This lemma can be also considered as [3, Lemma A.2] restricted to time-independent functions.

The following lemma shows the first part of Theorem 1.1, except (1.9). In particular, it shows that (u, p) solves (1.5).

Lemma 2.3. *If u is a very weak solution of (1.1) with zero force in $B_2\backslash\{0\}$ satisfying (1.4) in $B_2\backslash\{0\}$ (with C_* allowed to be large), there is a scalar function p satisfying $|p(x)| \le CC_*|x|^{-2}$, unique up to a constant, such that (u, p) satisfies (1.5) in B_2 with $b_i = \int_{|x|=1} T_{ij}(u, p)n_j(x)$. Moreover, T_{ij} satisfies $|T_{ij}(x)| \le C'C_*|x|^{-2}$ and u, p are smooth in $B_2\backslash\{0\}$. Here the positive constants C and C' depend on C_* but not on (u, p). Their dependence on C_* can be dropped if $C_* \in (0, 1)$.*

Proof. For each $R \in (0, 1/2]$, u is a very weak solution in $B_2 - \bar{B}_R$ in L^∞. Lemma 2.2 shows u is a weak solution in $W^{1,2}_{loc}$. The usual theory shows that u is smooth and there is a scalar function p_R, unique up to a constant, so that (u, p_R) solves (1.1) in $B_2 - \bar{B}_R$, see e.g. [7]. By the scaling argument in Sverak-Tsai [18] using Lemma 2.2, we have for $x \in B_{3R} - B_{2R}$,

$$|\nabla^k u(x)| \le \frac{C_k C_*}{|x|^{k+1}} \qquad \text{for } k = 1, 2, \ldots, \tag{2.6}$$

where $C_k = C_k(C_*)$ are independent of $R \in (0, 1/2]$ and its dependence on C_* can be dropped if $C_* \in (0, 1)$. Varying R, (2.6) is valid for $x \in B_{3/2}\backslash\{0\}$. For $0 < R < R'$, by uniqueness of p'_R, the difference $p_R|_{B_2-\bar{B}_{R'}} - p_{R'}$ is a constant. Thus we can fix the constant by requiring $p_R = p_{1/2}$ in $B_2\backslash \bar{B}_{1/2}$, and define $p(x) = p_R(x)$ for any $x \in B_2\backslash\{0\}$ with $R = |x|/2$. By the equation, $|\nabla p(x)| \le CC_*|x|^{-3}$. Integrating from $|x| = 1$ we get $|p(x)| \le CC_*|x|^{-2}$. In particular

$$|T_{ij}(u, p)(x)| \le CC_*|x|^{-2} \qquad \text{for } x \in B_{3/2}\backslash\{0\}. \tag{2.7}$$

Denote $NS(u) = -\Delta u + (u \cdot \nabla)u + \nabla p$. We have $NS(u)_i = \partial_j T_{ij}(u)$ in the sense of distributions. Thus, by divergence theorem and $NS(u) = 0$ in $B_2\backslash\{0\}$,

$$b_i = \int\limits_{|x|=1} T_{ij}(u, p)n_j(x) = \int\limits_{|x|=R} T_{ij}(u, p)n_j(x) \tag{2.8}$$

for any $R \in (0, 2)$. Let ϕ be any test function in $C^\infty_c(B_1)$. For small $\varepsilon > 0$,

$$\begin{aligned}
\langle NS(u)_i, \phi\rangle &= -\int T_{ij}(u)\partial_j\phi \\
&= -\int\limits_{B_1\backslash B_\varepsilon} T_{ij}(u)\partial_j\phi - \int\limits_{B_\varepsilon} T_{ij}(u)\partial_j\phi \\
&= \int\limits_{B_1\backslash B_\varepsilon} \partial_j T_{ij}(u)\phi + \int\limits_{\partial B_\varepsilon} T_{ij}(u)\phi n_j - \int\limits_{\partial B_1} T_{ij}(u)\phi n_j - \int\limits_{B_\varepsilon} T_{ij}(u)\partial_j\phi.
\end{aligned}$$

In the last line, the first integral is zero since $NS(u) = 0$ and the third integral is zero since $\phi = 0$. By the pointwise estimate (2.7), the last integral is bounded by $C\varepsilon^{3-2}$. On the other hand, by (2.8),

$$\int\limits_{\partial B_\varepsilon} T_{ij}(u)\phi n_j \to b_i\phi(0) \quad \text{as} \quad \varepsilon \to 0. \tag{2.9}$$

Thus (u, p) solves (1.5) and we have proved the lemma. □

It follows from the proof that $|b| \le CC_*$ for $C_* < 1$. With this lemma, we have completely proved Theorem 1.1 in the case $q < 3/2$. In the case $3/2 \le q < 3$, it remains to prove (1.9).

3. Proof of Main Theorem

In this section, we present the proof of Theorem 1.1. We first prove that solutions belong to $W^{1,q}$. We next apply this result to obtain the pointwise estimate. For what follows, denote

$$w = u - U, \quad U = U^b, \tag{3.1}$$

where U^b is the Landau solution with b given by (2.8).

By Lemma 2.3, there is a function $\tilde{p}$ such that $(w, \tilde{p})$ satisfies in B_2 that

$$\begin{gathered}
-\Delta w + U \cdot \nabla w + w \cdot \nabla(U + w) + \nabla\tilde{p} = 0, \quad \operatorname{div} w = 0, \\
|w(x)| \le \frac{CC_*}{|x|}, \quad |\tilde{p}(x)| \le \frac{CC_*}{|x|^2}.
\end{gathered} \tag{3.2}$$

Note that the δ-functions at the origin cancel. In order to check the estimate w, we use (1.7) to expand $A = A(\beta)$ in (1.6) as $A(\beta) = C\beta^{-1} + O(1)$ for small $\beta > 0$. By (1.6) and Lemma 2.3, we have $|U^b(x)| \lesssim |b||x|^{-1} \lesssim C_*|x|^{-1}$ for small $|b| > 0$.

3.1. $W^{1,q}$ Regularity

In this subsection we will show $w \in W^{1,q}(B_1)$. Fix a cut off function φ with $\varphi = 1$ in $B_{9/8}$ and $\varphi = 0$ in $B^c_{11/8}$. We localize w by introducing

$$v = \varphi w + \zeta \tag{3.3}$$

where ζ is a solution of the problem $\operatorname{div}\zeta = -\nabla\varphi \cdot w$. By Galdi [6, Ch. 3] Theorem 3.1, there exists such a ζ satisfying

$$\operatorname{supp}\zeta \subset B_{3/2}\backslash B_1, \quad \|\nabla\zeta\|_{L^{100}} \le C\|\nabla\varphi\cdot w\|_{L^{100}} \le CC_*. \tag{3.4}$$

The vector v is supported in $\bar{B}_{3/2}$, satisfies $v \in W^{1,r} \cap L^3_{wk}$ for $r < 3/2$ by (1.4), (2.6) and (3.4), and

$$-\Delta v + U\cdot\nabla v + v\cdot\nabla(U+v) + \nabla\pi = f, \quad \operatorname{div} v = 0, \tag{3.5}$$

where $\pi = \varphi\tilde{p}$, and

$$\begin{aligned} f = &-2(\nabla\varphi\cdot\nabla)w - (\Delta\varphi)w + (U\cdot\nabla\varphi)w + (\varphi^2-\varphi)w\cdot\nabla w + (w\cdot\nabla\varphi)w \\ &+ \tilde{p}\nabla\varphi - \Delta\zeta + (U\cdot\nabla)\zeta + \zeta\cdot\nabla(U+\varphi w+\zeta) + \varphi w\cdot\nabla\zeta \end{aligned} \tag{3.6}$$

is supported in the annulus $\bar{B}_{3/2}\backslash B_1$. One verifies directly that, for some C_1,

$$\sup_{1\le r\le 100} \|f\|_{W_0^{-1,r}(B_2)} \le C_1 C_*. \tag{3.7}$$

Our proof is based on the following lemmas.

Lemma 3.1 (Unique existence). *For any $3/2 \le r < 3$, for sufficiently small $C_* = C_*(r) > 0$, there is a unique solution v of (3.5) and (3.7) in the set*

$$V = \{v \in W_0^{1,r}(B_2), \quad \|v\|_V := \|v\|_{W_0^{1,r}(B_2)} \le C_2 C_*\} \tag{3.8}$$

for some $C_2 > 0$ independent of C_* and $r \in [3/2, 3)$.

Lemma 3.2 (Uniqueness). *Let $1 < r < 3/2$. If both v_1 and v_2 are solutions of (3.5) and (3.7) in $W_0^{1,r}\cap L^3_{wk}$ and $C_* + \|v_1\|_{L^3_{wk}} + \|v_2\|_{L^3_{wk}}$ is sufficiently small, then $v_1 = v_2$.*

Assuming the above lemmas, we get $W^{1,q}$ regularity as follows. First we have a solution $\tilde{v}$ of (3.5) in $W_0^{1,q}(B_2)$ by Lemma 3.1. On the other hand, both $v = \varphi w + \zeta$ and $\tilde{v}$ are small solutions of (3.5) in $W_0^{1,r} \cap L^3_{wk}(B_2)$ for $r = 5/4$, and thus $v = \tilde{v}$ by Lemma 3.2. Thus $v \in W_0^{1,q}(B_2)$ and $w \in W^{1,q}(B_1)$.

Proof of Lemma 3.1. Consider the following mapping Φ: For each $v \in V$, let $\bar{v} = \Phi v$ be the unique solution in $W_0^{1,r}(B_2)$ of the Stokes system

$$-\Delta\bar{v} + \nabla\bar{\pi} = f - \nabla\cdot(U\otimes v + v\otimes(U+v)), \quad \operatorname{div}\bar{v} = 0. \tag{3.9}$$

By estimates for the Stokes system, see Galdi [6, Ch.4] Theorem 6.1, in particular (6.9), for $1 < r < \infty$, we have

$$\|\bar{v}\|_{W_0^{1,r}(B_2)} \le N_r\|f\|_{W_0^{-1,r}} + N_r\|\nabla\cdot(U\otimes v + v\otimes(U+v))\|_{W_0^{-1,r}} \tag{3.10}$$

for some constant $N_r > 0$ which is uniformly bounded for r in any compact regions of $(1,\infty)$. By (3.7) and Lemma 2.1, in particular (2.3), for $1 < r < 3$,

$$\begin{aligned} \|\bar{v}\|_{W_0^{1,r}(B_2)} &\le N_r C_1 C_* + N_r\|U\otimes v + v\otimes(U+v)\|_{L^r} \\ &\le N_r C_1 C_* + N_r C_r(\|U\|_{L^3_{wk}} + \|v\|_{L^3_{wk}})\|v\|_V. \end{aligned} \tag{3.11}$$

We now choose $C_2 = 2(C_1+1)\sup_{3/2\le r<3} N_r$. Since $V \subset L^3_{wk}$ if $r \ge 3/2$, we get $\bar v = \Phi v \in V$ if C_* is sufficiently small.

We next consider the difference estimate. Let $v_1, v_2 \in V$, $\bar v_1 = \Phi v_1$, and $\bar v_2 = \Phi v_2$. Then

$$\|\Phi v_1 - \Phi v_2\|_{W^{1,r}} \le CC_r(\|U\|_{L^3_{wk}} + \|v_1\|_{L^3_{wk}} + \|v_2\|_{L^3_{wk}})\|v_1 - v_2\|_{W^{1,r}}. \tag{3.12}$$

Taking C_* sufficiently small for $3/2 \le r < 3$, we get $\|\Phi v_1 - \Phi v_2\|_V \le \frac{1}{2}\|v_1 - v_2\|_V$, which shows that Φ is a contraction mapping in V and thus has a unique fixed point. We have proved the unique existence of the solution for (3.5)–(3.7) in V. □

Remark. Since the constant C_r from Lemma 2.1 (ii) blows up as $r \to 3_-$, our C_* shrinks to zero as $r \to 3_-$.

Proof of Lemma 3.2. By the difference estimate (3.12), we have

$$\|v_1 - v_2\|_{W^{1,r}} \le C(\|U\|_{L^3_{wk}} + \|v_1\|_{L^3_{wk}} + \|v_2\|_{L^3_{wk}})\|v_1 - v_2\|_{W^{1,r}}. \tag{3.13}$$

Thus, if $C(\|U\|_{L^3_{wk}} + \|v_1\|_{L^3_{wk}} + \|v_2\|_{L^3_{wk}}) < 1$, we conclude $v_1 = v_2$. □

3.2. Pointwise Bound

In this subsection, we will prove pointwise bound of w using $\|w\|_{W^{1,q}} \lesssim C_*$.

For any fixed $x_0 \in B_{1/2}\backslash\{0\}$, let $R = |x_0|/4$ and $E_k = B(x_0, kR)$, $k = 1, 2$.

Note $q^* \in (3, \infty)$. Let s be the dual exponent of q^*, $1/s + 1/q^* = 1$. We have

$$\|w\|_{L^1(E_2)} \lesssim \|w\|_{L^{q^*}(E_2)} \|1\|_{L^s(E_2)} \lesssim C_* R^{4-3/q}. \tag{3.14}$$

By the interior estimate Lemma 2.2,

$$\|\nabla w\|_{L^{q^*}(E_1)} \lesssim \|f\|_{L^{q^*}(E_2)} + R^{-4+3/q^*}\|w\|_{L^1(E_2)} \tag{3.15}$$

where $f = U \otimes w + w \otimes (U + w)$. Since $|U| + |w| \lesssim C_*|x|^{-1} \lesssim C_* R^{-1}$ in E_2,

$$\|f\|_{L^{q^*}(E_2)} \lesssim C_* R^{-1}\|w\|_{L^{q^*}(E_2)} \lesssim C_*^2 R^{-1}. \tag{3.16}$$

We also have $R^{-4+3/q^*}\|w\|_{L^1(E_2)} \lesssim R^{-4+3/q^*} C_* R^{4-3/q} = C_* R^{-1}$. Thus

$$\|\nabla w\|_{L^{q^*}(E_1)} \lesssim C_* R^{-1}. \tag{3.17}$$

By Gagliardo-Nirenberg inequality in E_1,

$$\|w\|_{L^\infty(E_1)} \lesssim \|w\|^{1-\theta}_{L^{q^*}(E_1)} \|\nabla w\|^{\theta}_{L^{q^*}(E_1)} + R^{-3}\|w\|_{L^1(E_1)}, \tag{3.18}$$

where $1/\infty = (1-\theta)/q^* + \theta(1/q_* - 1/3)$ and thus $\theta = 3/q - 1$. We conclude $\|w\|_{L^\infty(E_1)} \le C_* R^{-\theta}$. Since x_0 is arbitrary, we have proved the pointwise bound, and completed the proof of Theorem 1.1.

Remark. Equivalently, one can define $v(x) = u(x_0 + Rx)$, find the equation of v, estimate v in $L^\infty(B_1)$, and then derive the bound for $w(x_0)$.

3.3. Proof of Theorem 1.2

In this subsection we prove Theorem 1.2. For any $x_0 \in B_2\backslash\{0\}$, let $v(x) = \lambda u(\lambda x + x_0)$ with $\lambda = \min(0.1, |x_0|)/2$. By our choice of λ, v is a very weak solution in B_2 and $\|v\|_{L^3_{wk}(B_2)} \le \varepsilon = \|u\|_{L^3_{wk}(B_{2.1}\backslash\{0\})}$. By [8], we have $\|v\|_{L^\infty(B_1)} \le C_2\varepsilon$ for some constant C_2 if ε is sufficiently small. Thus $|u(x_0)| \le C_2\varepsilon\lambda^{-1} \le 40C_2\varepsilon|x_0|^{-1}$.

3.4. Proof of Corollary 1.5

In this subsection we prove Corollary 1.5. Suppose u satisfies (1.4) with $C_* = C_*(q)$, $q = 2$, given in Theorem 1.1. Let b be given by (2.8), $U = U^b$ and $w = u - U$. Let $u_\lambda = \lambda u(\lambda x)$ be the rescaled solution and $w_\lambda(x) = \lambda w(\lambda x)$. Note U is scaling-invariant. Then $u_\lambda = U + w_\lambda$ also satisfies (1.4) with same C_*. By Theorem 1.1 with $q = 2$, we have the bound

$$|w_\lambda(x)| \le CC_*|x|^{-1/2}, \quad |x| < 1, \tag{3.19}$$

which is uniform in λ. In terms of w and $y = \lambda x$, we get

$$|w(y)| \le CC_*\lambda^{-1}|\lambda^{-1}y|^{-1/2}, \quad |y| \le \lambda. \tag{3.20}$$

Now fix y and let $\lambda \to \infty$. We conclude $w \equiv 0$.

Remark. Note Corollary 1.5 assumes (1.4) in entire space, not just B_2, and thus does not imply $u = U^b$ in Theorem 1.1.

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