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Analyticity and Gevrey-Class Regularity for the Second-Grade Fluid Equations

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Abstract. We address the global persistence of analyticity and Gevrey-class regularity of solutions to the two and three-dimensional visco-elastic second-grade fluid equations. We obtain an explicit novel lower bound on the radius of analyticity of the solutions that does not vanish as $t \to \infty$, and which is independent of the Rivlin–Ericksen material parameter α. Applications to the damped incompressible Euler equations are also given.

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1. Introduction

In this paper we address the regularity of an asymptotically smooth system arising in non-Newtonian fluid mechanics, which is not smoothing in finite time, but admits a compact global attractor (in the two-dimensional case). More precisely, we consider the system of visco-elastic second-grade fluids

$$\partial_t(u - \alpha^2 \Delta u) - \nu \Delta u + \operatorname{curl}(u - \alpha^2 \Delta u) \times u + \nabla p = 0, \tag{1.1}$$

$$\operatorname{div} u = 0, \tag{1.2}$$

$$u(0, x) = u_0(x), \tag{1.3}$$

where $\alpha > 0$ is a material parameter, $\nu \geq 0$ is the kinematic viscosity, the vector field u represents the velocity of the fluid, and the scalar field p represents the pressure. Here $(x, t) \in \mathbb{T}^d \times [0, \infty)$, where $\mathbb{T}^d = [0, 2\pi]^d$ is the d-dimensional torus, and $d \in \{2, 3\}$. Without loss of generality we consider velocities that have zero-mean on $\mathbb{T}^d$.

Fluids of second-grade are a particular class of non-Newtonian Rivlin–Ericksen fluids (cf. [48]) of differential type, and the above precise form has been justified by Dunn and Fosdick [18]. The local existence in time, and the uniqueness of strong solutions of the Eqs. (1.1)–(1.3) in a two or three-dimensional bounded domain with no slip boundary conditions has been addressed by Cioranescu and Ouazar [14]. Moreover, in the two-dimensional case, they obtained the global in time existence of solutions (see also [13, 24, 25, 29]). Moise et al. [40] have shown later that in two dimensions these equations admit a compact global attractor $\mathcal{A}_\alpha$ (see also [2, 15, 22, 26, 27, 30, 39, 45, 47]). The question of regularity and finite-dimensional behavior of $\mathcal{A}_\alpha$ was studied by Paicu et al. in [45], where it was shown that the compact global attractor in $H^3(\mathbb{T}^2)$ is contained in any Sobolev space $H^m(\mathbb{T}^2)$ provided that the material coefficient α is small enough, and the forcing term is regular. Moreover, on the global attractor, the second-grade fluid system can be reduced to a finite-dimensional system of ordinary differential equations with an infinite delay. As a consequence, the existence of a finite number of determining modes for the equation of fluids of grade two was established in [45].

Note that the Eqs. (1.1)–(1.3) essentially differ from the α-Navier–Stokes system (cf. Foias et al. [20, 21], and references therein). Indeed, the equations governing the second-grade fluids do not contain the regularizing term $-\nu\Delta(u - \alpha^2 \Delta u)$ (cf. [21]), but instead $-\nu \Delta u$, and thus the problem is not semi-linear. Moreover, the dissipative term $-\nu\Delta$ is very weak—it behaves like a damping term—and the system is not smoothing in finite time, that is, for generic initial data in H^3 the solution does not become analytic in finite time (as opposed to parabolic equations [19,23,44]). The α-models are used, in particular, as an alternative to the usual Navier–Stokes equations for numerical modeling of turbulence phenomena in pipes and channels. Note that the physics underlying the second-grade fluid equations and the α-models are quite different. There are numerous papers devoted to the asymptotic behavior of the α-models, including Camassa–Holm equations, α-Navier–Stokes equations, α-Bardina equations (cf. [9,20,21,34,37]).

In this paper we characterize the domain of analyticity and Gevrey-class regularity of solutions to the second-grade fluids equation, and of the Euler equation with a damping term. We prove that if the initial data u_0 is of Gevrey-class s, with $s \geq 1$, then the unique smooth solution $u(t)$ remains of Gevrey-class s for all $t < T_*$, where $T_* \in (0, \infty]$ is the maximal time of existence in the Sobolev norm of the solution.

The main novelty of our result is that if $\nu > 0$, and $d = 2$, or if $d = 3$ and u_0 is small in a certain norm (these are the cases when $T_* = \infty$), then the lower bound on the radius of analyticity does not vanish as $t \to \infty$. Instead, it is bounded from below for all time by a constant that depends solely on ν, α, the analytic norm, and the radius of analyticity of the initial data. In contrast, we note that the shear flow example of DiPerna and Majda [17] (see also [5]) may be used to construct explicit solutions to the incompressible two and three-dimensional Euler equations (in the absence of damping) whose radius of analyticity is decaying for all time, and hence vanishes as $t \to \infty$. We emphasize that when $0 \leq \alpha \leq 1$ our lower bound on the radius of analyticity is independent of α, which gives the framework in which we prove the convergence of analytic solutions to the second-grade fluid equations to those of the corresponding Navier–Stokes equations, in the limit $\alpha \to 0$, when $d = 2$.

When $d = 3$ and the initial data is not small, the solution might a-priori blow up in finite time. Here we obtain an explicit lower bound for the real-analyticity radius of the solution which for all $\nu, \alpha > 0$ decays algebraically in $\exp(\int_0^t \|\nabla u(s)\|_{L^\infty} ds)$. A similar lower bound on the analyticity radius for solutions to the incompressible Euler equations was obtained by Kukavica and Vicol [32,33], but with an additional algebraic decay in time (see also [1,3,4,6,36]).

The main results of our paper are given bellow (for the definitions see the following sections).

Theorem 1.1 (The two-dimensional case). *Fix $\nu > 0$, $0 \leq \alpha \leq 1$, and assume that u_0 is of Gevrey-class s for some $s \geq 1$, with radius $\tau_0 > 0$. Then there exists a unique global in time Gevrey-class s solution $u(t)$ to (1.1)–(1.3), such that for all $t \geq 0$ the radius of Gevrey-class regularity is bounded from below by*

$$\tau(t) \geq \frac{\tau_0}{1 + C_0 \tau_0},$$

where $C_0 > 0$ is a constant depending on ν and the initial data via (3.24) below.

Note that in this case we obtain the global in time control of the radius of analyticity, which is moreover uniform in α. This allows us to prove the convergence as $\alpha \to 0$ of the solutions of the second-grade fluid to solutions of the corresponding Navier–Stokes equations in analytic norms (cf. Sect. 3.3). The convergence of solutions to the Euler-α equations to the corresponding Euler equations, in the limit $\alpha \to 0$, has been addressed in [37]. The corresponding theorem for the damped Euler equations is given in Sect. 5.

Theorem 1.2 (The three-dimensional case). *Fix $\nu, \alpha > 0$, and assume that ω_0 is of Gevrey-class s, for some $s \geq 1$. Then the unique solution $\omega(t) \in C([0, T^*); L^2(\mathbb{T}^3))$ to (2.6)–(2.8) is of Gevrey-class s for all $t < T^*$, where $T^* \in (0, \infty]$ is the maximal time of existence of the Sobolev solution. Moreover, the radius $\tau(t)$ of Gevrey-class s regularity of the solution is bounded from below as*

$$\tau(t) \geq \frac{\tau_0}{C_0} e^{-C \int_0^t \|\nabla u(s)\|_{L^\infty} ds},$$

where $C > 0$ is a dimensional constant, and $C_0 > 0$ has additional explicit dependence on the initial data, α, and ν via (4.24) below.

The proofs of the above theorems are based on the Fourier-based method introduced by Foias and Temam [23] to study the analyticity of the Navier–Stokes equations, and which was further refined by Levermore and Oliver [36] for the Euler equations (see also [11,19,32,34,35,42–44]). We emphasize that the technique of analytic estimates may be used to obtain the existence of global solutions for the Navier–Stokes equation with some type of large initial data [12,46].

2. Preliminaries

In this section we introduce the notations that are used throughout the paper. We denote the usual Lebesgue spaces by $L^p(\mathbb{T}^d) = L^p$, for $1 \leq p \leq \infty$. The L^2-inner product is denoted by $\langle \cdot, \cdot \rangle$. The Sobolev spaces $H^r(\mathbb{T}^d) = H^r$ of *mean-free functions* are classically characterized in terms of the Fourier series

$$H^r(\mathbb{T}^d) = \left\{ v(x) = \sum_{k \in \mathbb{Z}^d} \widehat{v}_k e^{ik\cdot x} \ : \ \overline{\widehat{v}_k} = \widehat{v}_{-k},\ \widehat{v}_0 = 0, \right.$$
$$\left. \|v\|_{H^r}^2 = (2\pi)^3 \sum_{k \in \mathbb{Z}^d} |k|^{2r} |\widehat{v}_k|^2 < \infty \right\}.$$

We let $\lambda_1 > 0$ be the first positive eigenvalue of the Stokes operator, which in the periodic setting coincides with $-\Delta$ [16,49]. For simplicity we consider $\mathbb{T}^d = [0, 2\pi]^d$, and hence $\lambda_1 = 1$. The Poincaré inequality then reads $\|v\|_{L^2} \leq \|\nabla v\|_{L^2}$ for all $v \in H^1$. Throughout the paper we shall denote by Λ the operator $(-\Delta)^{1/2}$, i.e., the Fourier multiplier operator with symbol $|k|$. We will denote by C a generic sufficiently large positive dimensional constant, which does not depend on α, ν. Moreover, the curl of a vector field v will be denoted by $\operatorname{curl} v = \nabla \times v$.

2.1. Dyadic Decompositions and Para-Differential Calculus

Fix a smooth nonnegative radial function χ with support in the ball $\left\{|\xi| \leq \frac{4}{3}\right\}$, which is identically 1 in $\left\{|\xi| \leq \frac{3}{4}\right\}$, and such that the map $r \mapsto \chi(|r|)$ is non-increasing over $\mathbb{R}_+$. Let $\varphi(\xi) = \chi(\xi/2) - \chi(\xi)$. We classically have

$$\sum_{q \in \mathbb{Z}} \varphi(2^{-q}\xi) = 1 \quad \text{for all} \quad \xi \in \mathbb{R}^d \backslash \{0\}. \tag{2.1}$$

We define the spectral localization operators Δ_q and S_q $(q \in \mathbb{Z})$ by

$$\Delta_q\, u := \varphi(2^{-q}D)u = \sum_{k \in \mathbb{Z}^d} \widehat{u}(k) e^{ikx} \varphi(2^{-q}|k|)$$

and

$$S_q\, u := \chi(2^{-q}D)u = \sum_{k \in \mathbb{Z}^d} \widehat{u}(k) e^{ikx} \chi(2^{-q}|k|).$$

We have the following quasi-orthogonality property:

$$\Delta_k \Delta_q u \equiv 0 \quad \text{if } |k - q| \geq 2; \quad \text{and} \quad \Delta_k(S_{q-1}u \Delta_q v) \equiv 0 \quad \text{if } |k - q| \geq 5. \tag{2.2}$$

We recall the very useful *Bernstein inequality*.

Lemma 2.1. *Let $n \in \mathbb{N}$, $q \in \mathbb{Z}$, $1 \leq p_1 \leq p_2 \leq \infty$, and $\psi \in C_c^\infty(\mathbb{R}^d)$. There exists a constant C depending only on n, d and $\sup \psi$ such that*

$$\|D^n \psi(2^{-q}D)u\|_{L^{p_2}} \leq C 2^{qs} \|\psi(2^{-q}D)u\|_{L^{p_1}},$$

and

$$C^{-1}2^{qs}\|\varphi(2^{-q}D)u\|_{L^{p_1}} \le \sup_{|\alpha|=n} \|\partial^\alpha\varphi(2^{-q}D)u\|_{L^{p_2}} \le C2^{qs}\|\varphi(2^{-q}D)u\|_{L^{p_1}}.$$

where $s = n + d\,(1/p_1 - 1/p_2)$.

In order to obtain optimal bounds on the nonlinear terms in a system, we use the paradifferential calculus, a tool which was introduced by Bony in [7]. More precisely, the product of two functions f and g may be decomposed according to

$$fg = T_f g + T_g f + R(f,g) \tag{2.3}$$

where the paraproduct operator T is defined by the formula

$$T_f g := \sum_q S_{q-1} f\, \Delta_q g,$$

and the remainder operator, R, by

$$R(f,g) := \sum_q \Delta_q f \tilde{\Delta}_q g \quad \text{with } \tilde{\Delta}_q := \Delta_{q-1} + \Delta_q + \Delta_{q+1}.$$

2.2. Analytic and Gevrey-Class Norms

Classically, a $C^\infty(\mathbb{T}^d)$ function v is in the *Gevrey-class* s, for some $s > 0$ if there exist $M, \tau > 0$ such that

$$|\partial^\beta v(x)| \le M\frac{\beta!^s}{\tau^{|\beta|}},$$

for all $x \in \mathbb{T}^d$, and all multi-indices $\beta \in \mathbb{N}_0^3$. We will refer to τ as the *radius of Gevrey-class regularity* of the function v. When $s = 1$ we recover the class of real-analytic functions, and the *radius of analyticity* τ is (up to a dimensional constant) the radius of convergence of the Taylor series at each point. When $s > 1$ the Gevrey-classes consist of C^∞ functions which are not analytic. It is however more convenient in PDEs to use an equivalent characterization, introduced by Foias and Temam [23] to address the analyticity of solutions of the Navier–Stokes equations. Namely, for all $s \ge 1$ the Gevrey-class s is given by

$$\bigcup_{\tau>0} \mathcal{D}(\Lambda^r e^{\tau\Lambda^{1/s}})$$

for any $r \ge 0$, where

$$\|\Lambda^r e^{\tau\Lambda^{1/s}} v\|_{L^2}^2 = (2\pi)^3 \sum_{k\in\mathbb{Z}^d} |k|^{2r} e^{2\tau|k|^{1/s}} |\widehat{v}_k|^2. \tag{2.4}$$

See [16,19,23,31–33,36,44,49] and references therein for more details on Gevrey-classes. We emphasize that the radius of analyticity gives an estimate on the minimal scale in the flow [28,31], and it also gives the explicit rate of exponential decay of its Fourier coefficients [23].

2.3. Vorticity Formulation

It is convenient to consider the evolution of the vorticity ω, which is defined as

$$\omega = \operatorname{curl}(u - \alpha^2\Delta u) = (I - \alpha^2\Delta)\operatorname{curl} u. \tag{2.5}$$

It follows from (1.1)–(1.2), that ω satisfies the initial value problem

$$\partial_t\omega - \nu\Delta(I-\alpha^2\Delta)^{-1}\omega + (u\cdot\nabla)\omega = (\omega\cdot\nabla)u, \tag{2.6}$$

$$\operatorname{div}\omega = 0, \tag{2.7}$$

$$\omega(0,x) = \omega_0(x) = \operatorname{curl}(u_0 - \alpha^2\Delta u_0) \tag{2.8}$$

on $\mathbb{T}^d \times (0,\infty)$. Additionally, if $d=2$, ω is a scalar, and the right side of (2.6) is zero. Denote by $\mathcal{R}_\alpha$ the operator

$$\mathcal{R}_\alpha = (-\Delta)(I-\alpha^2\Delta)^{-1}. \tag{2.9}$$

It follows from Plancherel's theorem, that for all $v \in L^2$ we have

$$\frac{1}{1+\alpha}\|v\|_{L^2} \le \|\mathcal{R}_\alpha v\|_{L^2} \le \frac{1}{\alpha}\|v\|_{L^2}. \tag{2.10}$$

The velocity is obtained from the vorticity by solving the elliptic problem

$$\operatorname{div} u = 0, \quad \operatorname{curl} u = (I-\alpha^2\Delta)^{-1}\omega, \quad \int_{\mathbb{T}^3} u = 0, \tag{2.11}$$

which in turn classically gives that

$$u = K * (I-\alpha^2\Delta)^{-1}\omega = \mathcal{K}_\alpha \omega, \tag{2.12}$$

where K is the periodic Biot–Savart kernel. Combined with (2.10), the above implies that

$$\|u\|_{H^3} \le \frac{C}{\alpha}\|\omega\|_{L^2}, \tag{2.13}$$

for some universal constant $C>0$. Note that when $\alpha \to 0$ the above estimate becomes obsolete.

3. The Two-Dimensional Case

3.1. The Case α Large

In the two-dimensional case, the evolution equation (2.6) for ω does not include the term $\omega \cdot \nabla u$, which makes the problem tangible, in analogy to the two-dimensional Euler equations. The main result below gives the global well-posedness of solutions evolving from Gevrey-class data, whose radius $\tau(t)$ does not vanish as $t \to \infty$.

Theorem 3.1. *Fix $\nu, \alpha > 0$, and assume that $\omega_0 \in \mathcal{D}(e^{\tau_0 \Lambda^{1/s}})$, for some $s \ge 1$, and $\tau_0 > 0$. Then there exists a unique global in time Gevrey-class s solution $\omega(t)$ to (2.6)–(2.8), such that for all $t \ge 0$ we have $\omega(t) \in \mathcal{D}(e^{\tau(t)\Lambda^{1/s}})$, and moreover we have the lower bound*

$$\tau(t) \ge \tau_0 e^{-CM_0\int_0^t e^{-\nu s/(2+2\alpha^2)}ds/\alpha} \ge \tau_0 e^{-C(2+2\alpha^2)M_0/(\alpha\nu)}, \tag{3.1}$$

where $M_0 = \|e^{\tau_0\Lambda^{1/s}}\omega_0\|_{L^2}$, and C is a universal constant.

Proof of Theorem 3.1. We take the L^2-inner product of $\partial_t\omega + \nu\mathcal{R}_\alpha\omega + (u\cdot\nabla)\omega = 0$ with $e^{2\tau\Lambda^{1/s}}$ and obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|e^{\tau\Lambda^{1/s}}\omega\|_{L^2}^2 - \dot{\tau}\|\Lambda^{1/2s}e^{\tau\Lambda^{1/s}}\omega\|_{L^2}^2 + \langle e^{\tau\Lambda^{1/s}}\mathcal{R}_\alpha\omega, e^{\tau\Lambda^{1/s}}\omega\rangle \\
&\quad = -\langle e^{\tau\Lambda^{1/s}}(u\cdot\nabla\omega), e^{\tau\Lambda^{1/s}}\omega\rangle.
\end{aligned} \tag{3.2}$$

Note that the Fourier multiplier symbol of the operator $\mathcal{R}_\alpha$ is an increasing function of $|k| \ge 1$, and therefore by Plancherel's theorem and Parseval's identity we have

$$\begin{aligned}
\langle e^{\tau\Lambda^{1/s}}\mathcal{R}_\alpha\omega, \Lambda e^{\tau\Lambda^{1/s}}\omega\rangle &= (2\pi)^2 \sum_{k\in\mathbb{Z}^2\setminus\{0\}} \frac{|k|^2}{1+\alpha^2|k|^2}|\widehat{\omega}_k|^2 e^{2\tau|k|^{1/s}} \\
&\ge \frac{(2\pi)^2}{1+\alpha^2}\sum_{k\in\mathbb{Z}^2\setminus\{0\}} |\widehat{\omega}_k|^2 e^{2\tau|k|^{1/s}} = \frac{1}{1+\alpha^2}\|e^{\tau\Lambda^{1/s}}\omega\|_{L^2}^2.
\end{aligned}$$

We therefore have the a priori estimate

$$\frac{1}{2}\frac{d}{dt}\|e^{\tau\Lambda^{1/s}}\omega\|_{L^2}^2 - \dot{\tau}\|\Lambda^{1/2s}e^{\tau\Lambda^{1/s}}\omega\|_{L^2}^2 + \frac{\nu}{1+\alpha^2}\|e^{\tau\Lambda^{1/s}}\omega\|_{L^2}^2 \le |\langle u\cdot\nabla\omega, e^{2\tau\Lambda^{1/s}}\omega\rangle|. \tag{3.3}$$

The following lemma gives a bound on the convection term on the right of (3.3) above.

Lemma 3.2. *For $\omega \in \mathcal{D}(\Lambda^{1/2s}e^{\tau\Lambda^{1/s}})$, and divergence free $u = \mathcal{K}_\alpha\omega$, we have*

$$\left|\langle u\cdot\nabla\omega, e^{2\tau\Lambda^{1/s}}\omega\rangle\right| \le \frac{C\tau}{\alpha}\|e^{\tau\Lambda^{1/s}}\omega\|_{L^2}\|\Lambda^{1/2s}e^{\tau\Lambda^{1/s}}\omega\|_{L^2}^2, \tag{3.4}$$

for some dimensional constant $C > 0$.

The proof of bound (3.4) is the same as the proof of estimate (4.4) below, which is in turn given in the Appendix. Therefore, by (3.3) and (3.4), if we chose τ that satisfies

$$\dot{\tau} + \frac{C\tau}{\alpha}\|e^{\tau\Lambda^{1/s}}\omega\|_{L^2} = 0, \tag{3.5}$$

then we have

$$\frac{1}{2}\frac{d}{dt}\|\omega\|_{X_{s,\tau}}^2 + \frac{\nu}{1+\alpha^2}\|\omega\|_{X_{s,\tau}}^2 \le 0,$$

and hence

$$\|e^{\tau(t)\Lambda^{1/s}}\omega(t)\|_{L^2} \le \|e^{\tau_0\Lambda^{1/s}}\omega_0\|_{L^2}e^{-\gamma t}, \tag{3.6}$$

where we have denoted $\gamma = \nu/(2+2\alpha^2)$. The above estimate and condition (3.5) show that

$$\tau(t) \ge \tau_0 e^{-\frac{C}{\alpha}\|e^{\tau_0\Lambda^{1/s}}\omega_0\|_{L^2}\int_0^t e^{-\gamma s}ds} \ge \tau_0 e^{-C(2+2\alpha^2)\|e^{\tau_0\Lambda^{1/s}}\omega_0\|_{L^2}/(\nu\alpha)}, \tag{3.7}$$

which concludes the proof of the theorem. The above a priori estimates are made rigorous using a classical Fourier–Galerkin approximating sequence. We omit further details. □

3.2. The Case α Small

The lower bound (3.1) on the radius of Gevrey-class regularity converges to 0 as $\alpha \to 0$. In this section we give a new estimate on $\tau(t)$, in the case when α is small.

Theorem 3.3. *Fix $\nu > 0$, $0 \le \alpha < 1$, and assume that $\operatorname{curl} u_0 \in \mathcal{D}(\Delta e^{\tau_0\Lambda^{1/s}})$, for some $s \ge 1$, and $\tau_0 > 0$. Then there exists a unique global in time Gevrey-class s solution $u(t)$ to (1.1)–(1.3), such that for all $t \ge 0$ we have $u(t) \in \mathcal{D}(e^{\tau(t)\Lambda^{1/s}})$, and moreover we have the lower bound*

$$\tau(t) \ge \frac{\tau_0}{1 + C_0\tau_0}, \tag{3.8}$$

where $C_0 = C_0(\nu, \|u_0\|_{H^3}, \|\Lambda e^{\tau_0\Lambda^{1/s}}\operatorname{curl} u_0\|_{L^2}, \|e^{\tau\Lambda^{1/s}}\operatorname{curl}\Delta u_0\|_{L^2})$ is given explicitly in (3.24).

Proof of Theorem 3.3. For simplicity of the presentation, we give the proof in the case $s = 1$. Taking the L^2-inner product of (1.1) with $-e^{2\tau\Lambda}\operatorname{curl}\Delta u$, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|\Lambda e^{\tau\Lambda}\operatorname{curl} u\|_{L^2}^2 + \alpha^2\|e^{\tau\Lambda}\operatorname{curl}\Delta u\|_{L^2}^2\right) + \nu\|e^{\tau\Lambda}\operatorname{curl}\Delta u\|_{L^2}^2 - \dot{\tau}\left(\|\Lambda^{3/2}e^{\tau\Lambda}\operatorname{curl} u\|_{L^2}^2 + \alpha^2\|\Lambda^{1/2}e^{\tau\Lambda}\operatorname{curl}\Delta u\|_{L^2}^2\right) \le T_1 + T_2, \tag{3.9}$$

where

$$T_1 = \alpha^2\left|\langle e^{\tau\Lambda}\left((u\cdot\nabla)\Delta\operatorname{curl} u\right), e^{\tau\Lambda}\Delta\operatorname{curl} u\rangle\right|, \tag{3.10}$$

and

$$T_2 = \left|\langle \Lambda e^{\tau\Lambda}\left((u\cdot\nabla)\operatorname{curl} u\right), \Lambda e^{\tau\Lambda}\operatorname{curl} u\rangle\right|. \tag{3.11}$$

The upper bounds for T_1 and T_2 are given in the following lemma.

Lemma 3.4. *Let $\nu,\tau>0$, $0\le\alpha<1$, and u be such that $\operatorname{curl} u\in\mathcal{D}(\Lambda^{5/2}e^{\tau\Lambda})$. Then*

$$T_1 \le \frac{\nu}{4}\|e^{\tau\Lambda}\operatorname{curl}\Delta u\|_{L^2}^2 + \frac{C\alpha^4\tau^2}{\nu}\|\Lambda^{1/2}e^{\tau\Lambda}\operatorname{curl}\Delta u\|_{L^2}^2\|e^{\tau\Lambda}\operatorname{curl}\Delta u\|_{L^2}^2, \tag{3.12}$$

and

$$\begin{aligned}T_2 \le{}& \frac{\nu}{4}\|e^{\tau\Lambda}\Delta\operatorname{curl} u\|_{L^2}^2 + \frac{C}{\nu^3}\|\operatorname{curl} u\|_{L^2}^4\|\Lambda e^{\tau\Lambda}\operatorname{curl} u\|_{L^2}^2\\ &+\frac{C\tau^2}{\nu}\|\Lambda^{3/2}e^{\tau\Lambda}\operatorname{curl} u\|_{L^2}^2\|\Lambda e^{\tau\Lambda}\operatorname{curl} u\|_{L^2}^2,\end{aligned} \tag{3.13}$$

where $C>0$ is a universal constant.

We give the proof of the above lemma in the Appendix (cf. Sect. 6.1). Assuming that estimates (3.12) and (3.13) are proven, we obtain from (3.9) that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\Lambda e^{\tau\Lambda}\operatorname{curl} u\|_{L^2}^2+\alpha^2\|e^{\tau\Lambda}\operatorname{curl}\Delta u\|_{L^2}^2\right)+\frac{\nu}{2}\|e^{\tau\Lambda}\operatorname{curl}\Delta u\|_{L^2}^2\\ &\le \frac{C}{\nu^3}\|\operatorname{curl} u\|_{L^2}^4\|\Lambda e^{\tau\Lambda}\operatorname{curl} u\|_{L^2}^2\\ &\quad+\left(\dot\tau+\frac{C\tau^2}{\nu}\|\Lambda e^{\tau\Lambda}\operatorname{curl} u\|_{L^2}^2\right)\|\Lambda^{3/2}e^{\tau\Lambda}\operatorname{curl} u\|_{L^2}^2\\ &\quad+\alpha^2\left(\dot\tau+\alpha^2\frac{C\tau^2}{\nu}\|e^{\tau\Lambda}\operatorname{curl}\Delta u\|_{L^2}^2\right)\|\Lambda^{1/2}e^{\tau\Lambda}\operatorname{curl}\Delta u\|_{L^2}^2.\end{aligned} \tag{3.14}$$

Define

$$Z(t)=\|\Lambda e^{\tau\Lambda}\operatorname{curl} u\|_{L^2}^2$$

and

$$W(t)=\|e^{\tau\Lambda}\operatorname{curl}\Delta u\|_{L^2}^2.$$

We let τ be decreasing fast enough so that

$$\dot\tau(t)+\frac{C\tau(t)^2}{\nu}W(t)=0, \tag{3.15}$$

which by the Poincaré inequality implies

$$\dot\tau+\frac{C\tau^2}{\nu}\|\Lambda e^{\tau\Lambda}\operatorname{curl} u\|_{L^2}^2\le 0,$$

and also

$$\dot\tau+\alpha^2\frac{C\tau^2}{\nu}\|e^{\tau\Lambda}\operatorname{curl}\Delta u\|_{L^2}^2\le 0,$$

since by assumption $\alpha\le 1$. It follows from (3.14) that for all $0\le\alpha\le 1$ we have

$$\frac{1}{2}\frac{d}{dt}(Z+\alpha^2 W)+\frac{\nu}{2}W \le \frac{C}{\nu^3}\|\operatorname{curl} u\|_{L^2}^4 Z \tag{3.16}$$

$$\le \frac{C}{\nu^3}\|\operatorname{curl} u\|_{L^2}^4(Z+\alpha^2 W). \tag{3.17}$$

We recall that $\omega=\operatorname{curl}(I-\alpha^2\Delta)u$ solves the equation

$$\partial_t\omega+\nu\mathcal{R}_\alpha\omega+(u\cdot\nabla)\omega=0 \tag{3.18}$$

which by the classical energy estimates implies

$$\frac{1}{2}\frac{d}{dt}\|\omega(t)\|_{L^2}^2 + \frac{\nu}{1+\alpha^2}\|\omega(t)\|_{L^2}^2 \leq 0 \tag{3.19}$$

and therefore

$$\|\omega(t)\|_{L^2}^2 \leq \|\omega_0\|_{L^2}^2 e^{-2\gamma t} \tag{3.20}$$

where $\gamma = \nu/(2+2\alpha^2)$. Using that $0 \leq \alpha < 1$ and

$$\|\omega\|_{L^2}^2 = \|\operatorname{curl} u\|_{L^2}^2 + 2\alpha^2\|\Delta u\|_{L^2}^2 + \alpha^4\|\operatorname{curl}\Delta u\|_{L^2}^2 \tag{3.21}$$

we obtain the exponential decay rate

$$\|\operatorname{curl} u(t)\|_{L^2} \leq C\|u_0\|_{H^3} e^{-\gamma t}. \tag{3.22}$$

Combining (3.17) and (3.22), and using $\alpha \leq 1$, we get

$$\begin{aligned} Z(t) &\leq (Z(0)+\alpha^2 W(0)) e^{\frac{C}{\nu^3}\int_0^t \|\operatorname{curl} u(s)\|_{L^2}^4\, ds} \\ &\leq (Z(0)+\alpha^2 W(0)) e^{\frac{C}{4\gamma\nu}\|u_0\|_{H^3}^4} \leq (Z(0)+W(0)) e^{\frac{C}{\nu^4} M_0^4}, \end{aligned} \tag{3.23}$$

where we have denoted $M_0 = \|u_0\|_{H^3}$. Plugging the above bound in (3.16) and integrating in time, we obtain

$$\begin{aligned} &Z(t) + \alpha^2 W(t) + \frac{\nu}{2}\int_0^t W(s)\, ds \\ &\quad \leq (Z(0)+W(0)) \left(1 + \frac{C}{\nu^3} e^{CM_0^4/\nu^4} \int_0^t \|\operatorname{curl} u(s)\|_{L^2}^4\, ds\right) \\ &\quad \leq (Z(0)+W(0)) \left(\frac{1}{\nu^2} + \frac{CM_0^4}{\nu^6} e^{CM_0^4/\nu^4}\right)\nu^2 = C_0\nu^2, \end{aligned} \tag{3.24}$$

where $C_0 = C_0(\nu, \|u_0\|_{H^3}, Z(0), W(0)) > 0$ is a constant depending on the data. Thus, by the construction of τ in (3.15) and the above estimate, by possibly enlarging C_0, we have the lower bound

$$\tau(t) = \left(\frac{1}{\tau_0} + \frac{C}{\nu}\int_0^t W(s)\, ds\right)^{-1} \geq \frac{\tau_0}{1+\tau_0 C_0}, \tag{3.25}$$

thereby proving (3.8). We note that this lower bound is independent of $t \geq 0$, and $0 \leq \alpha \leq 1$. This concludes the a priori estimates needed to prove Theorem 3.3. The formal construction of the real-analytic solution is standard and we omit details. The proof of the theorem in the case $s > 1$ follows *mutatis mutandis.* □

3.3. Convergence to the Navier–Stokes Equations as $\alpha \to 0$

In this section we compare in an analytic norm the solutions of the second-grade fluids equations with those of the corresponding Navier–Stokes equations, in the limit as α goes to zero. The fact that the analyticity radius for the solutions of the second-grade fluids is bounded from bellow by a positive constant, for all positive time, will play a fundamental role. We consider $a > 0$ and u_0 such that $e^{a\Lambda}u_0 \in H^3(\mathbb{T}^2)$. We recall that the Navier–Stokes equations

$$\begin{aligned} \partial_t u - \nu\Delta u + \operatorname{curl} u \times u + \nabla p = 0 \\ \operatorname{div} u = 0 \\ u|_{t=0} = u_0, \end{aligned} \tag{3.26}$$

have a unique global regular solution when $u_0 \in L^2(\mathbb{T}^2)$. Moreover, this solution is analytic for every $t > 0$, and if $e^{\delta\Lambda} u_0 \in H^3$ one can prove that $e^{\delta\Lambda} u(t) \in H^3$ for all $t > 0$ (for example, one can use the same proof as in the one in Sect. 5). Let u_α denote the solution of the second-grade fluids equations. Then $z = u_\alpha - u$ is divergence free, satisfies

$$\begin{aligned}
&\partial_t(z - \alpha^2 \Delta z) - \nu \Delta z + \operatorname{curl} z \times u_\alpha + \operatorname{curl} u \times z + \nabla(p_\alpha - p) \\
&\quad = \alpha^2 \partial_t \Delta u + \alpha^2 \operatorname{curl} \Delta u_\alpha \times u_\alpha,
\end{aligned}$$

and the initial condition is $z(\cdot, 0) = 0$. The following product Sobolev estimate (see [11]) will prove to be very useful

$$\|e^{\delta\Lambda}(ab)\|_{H^{s_1+s_2-1}(\mathbb{T}^2)} \le \|e^{\delta\Lambda} a\|_{H^{s_1}(\mathbb{T}^2)} \|e^{\delta\Lambda} b\|_{H^{s_2}(\mathbb{T}^2)}, \tag{3.27}$$

where $s_1 + s_2 > 0$, $s_1 < 1$, $s_2 < 1$. Applying $e^{\delta\Lambda}$ with $0 < \delta < a$ fixed but small enough (given for example by (3.25)) to the equation, denoting by $z^\delta(t) = e^{\delta\Lambda} z(t)$, and considering the $L^2(\mathbb{T}^2)$ energy estimates, using (3.27), the Young inequality, and the classical Sobolev inequalities, we obtain the following estimate

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|z^\delta\|_{L^2}^2 + \alpha^2 \|\nabla z^\delta\|_{L^2}^2) + \nu \|\nabla z^\delta\|_{L^2}^2 \\
&\quad \le \frac{C\alpha^4}{\nu} \|\partial_t \nabla u^\delta\|_{L^2}^2 + \frac{C}{\nu} \|u_\alpha^\delta\|_{H^{\frac12}}^2 \|z^\delta\|_{H^{\frac12}}^2 + \frac{\nu}{50} \|\nabla z^\delta\|_{L^2}^2 \\
&\qquad + \alpha^2 \|\operatorname{curl} \Delta u_\alpha^\delta\|_{L^2} \|u_\alpha^\delta\|_{H^{\frac12}} \|z^\delta\|_{H^{\frac12}} + \|\operatorname{curl} u^\delta\|_{L^2} \|z^\delta\|_{H^{\frac12}}^2 \\
&\quad \le \frac{C\alpha^4}{\nu} \|\partial_t \nabla u^\delta\|_{L^2}^2 + \frac{C}{\nu} \|u_\alpha^\delta\|_{L^2} \|\nabla u_\alpha^\delta\|_{L^2} \|z^\delta\|_{L^2} \|\nabla z^\delta\|_{L^2} + \frac{\nu}{50} \|\nabla z^\delta\|_{L^2}^2 \\
&\qquad + \alpha^2 \|\operatorname{curl} \Delta u_\alpha^\delta\|_{L^2} \|u_\alpha^\delta\|_{L^2}^{\frac12} \|\nabla u_\alpha^\delta\|_{L^2}^{\frac12} \|z^\delta\|_{L^2}^{\frac12} \|\nabla z^\delta\|_{L^2}^{\frac12} \\
&\qquad + \|u^\delta\|_{H^1} \|z^\delta\|_{L^2} \|\nabla z^\delta\|_{L^2} \\
&\quad \le \frac{C\alpha^4}{\nu} \|\partial_t \nabla u^\delta\|_{L^2}^2 + \frac{\nu}{4} \|\nabla z^\delta\|_{L^2}^2 + \frac{C}{\nu^2} \|u_\alpha^\delta\|_{L^2}^2 \|\nabla u_\alpha^\delta\|_{L^2}^2 \|z^\delta\|_{L^2}^2 \\
&\qquad + \frac{C\alpha^4}{\nu} \|\operatorname{curl} \Delta u_\alpha^\delta\|_{L^2}^2 \|u_\alpha^\delta\|_{L^2} \|\nabla u_\alpha^\delta\|_{L^2} + \frac{\nu}{4} \|z^\delta\|_{L^2}^2 + \frac{C}{\nu} \|u^\delta\|_{H^1}^2 \|z^\delta\|_{L^2}^2.
\end{aligned}$$

From the above estimate and the Poincaré inequality, we deduce that

$$\begin{aligned}
&\frac{d}{dt}(\|z^\delta\|_{L^2}^2 + \alpha^2 \|\nabla z^\delta\|_{L^2}^2) + \gamma \left(\|z^\delta\|_{L^2}^2 + \alpha^2 \|\nabla z^\delta\|_{L^2}^2\right) \\
&\quad \le \left(\frac{C}{\nu} \|u^\delta\|_{H^1}^2 + \frac{C}{\nu^2} \|u_\alpha^\delta\|_{L^2}^2 \|\nabla u_\alpha^\delta\|_{L^2}^2\right) \|z^\delta\|_{L^2}^2 \\
&\qquad + \frac{C\alpha^4}{\nu} \left(\|\partial_t \nabla u^\delta\|_{L^2}^2 + \|\operatorname{curl} \Delta u_\alpha^\delta\|_{L^2}^2 \|u_\alpha^\delta\|_{L^2} \|\nabla u_\alpha^\delta\|_{L^2}\right),
\end{aligned}$$

holds for $t > 0$, where we let $\gamma = \nu/(2 + 2\alpha^2) > 0$. Integrating this inequality from 0 to t and using the Grönwall inequality, we obtain

$$\begin{aligned}
&\|z^\delta(t)\|_{L^2}^2 + \alpha^2 \|\nabla z^\delta(t)\|_{L^2}^2 \\
&\quad \le \int_0^t \left(\frac{C}{\nu} \|u^\delta\|_{H^1}^2 + \frac{C}{\nu^2} \|u_\alpha^\delta\|_{L^2}^2 \|\nabla u_\alpha^\delta\|_{L^2}^2\right) \|z^\delta\|_{L^2}^2 \, ds \\
&\qquad + \frac{C\alpha^4}{\nu} \int_0^t e^{\gamma(s-t)} \left(\|\partial_t \nabla u^\delta(s)\|_{L^2}^2 + \|\operatorname{curl} \Delta u_\alpha^\delta\|_{L^2}^2 \|u_\alpha^\delta\|_{L^2} \|\nabla u_\alpha^\delta\|_{L^2}\right) ds.
\end{aligned}$$

Using one more time the Grönwall lemma, we deduce from the above estimate that, for $t \geq 0$

$$\|z^\delta(t)\|_{L^2}^2 + \alpha^2 \|\nabla z^\delta(t)\|_{L^2}^2 \leq \exp\left(\int_0^t \left(\frac{C}{\nu} \|u^\delta\|_{H^1}^2 + \frac{C}{\nu^2} \|u_\alpha^\delta\|_{L^2}^2 \|\nabla u_\alpha^\delta\|_{L^2}^2 \right) ds \right) \times \frac{C\alpha^4}{\nu} \int_0^t e^{\gamma(s-t)} \left(\|\partial_t \nabla u^\delta(s)\|_{L^2}^2 + \|\operatorname{curl} \Delta u_\alpha^\delta\|_{L^2}^2 \|u_\alpha^\delta\|_{L^2} \|\nabla u_\alpha^\delta\|_{L^2} \right) ds. \tag{3.28}$$

We recall the estimate (3.24) on u_α^δ, which gives

$$\|\Delta u_\alpha^\delta\|_{L^2}^2 + \alpha^2 \|\operatorname{curl} \Delta u_\alpha^\delta\|_{L^2}^2 + \nu \int_0^t \|\operatorname{curl} \Delta u_\alpha^\delta\|_{L^2}^2 \leq M_0. \tag{3.29}$$

The equation on u_α gives that

$$\partial_t u_\alpha = (I - \alpha^2 \Delta)^{-1} [\nu \Delta u_\alpha - \mathbb{P}(\operatorname{curl}(u_\alpha - \alpha^2 \Delta u_\alpha) \times u_\alpha)].$$

Using estimates (3.27), (3.29), and the fact that the operator $\alpha \nabla (I - \alpha^2 \Delta)^{-1}$ is uniformly bounded on $L^2(\mathbb{T}^2)$, we obtain that $\alpha \|\partial_t \nabla u_\alpha^\delta\|_{L^2} \leq C M_0$. When $\alpha \leq 1$, inequality (3.28) together with the above uniform bounds and the corresponding property for the Navier–Stokes equation, namely $\int_0^t \|u^\delta\|_{H^1}^2 \leq M$, implies that

$$\|z^\delta(t)\|_{L^2}^2 + \alpha \|\nabla z^\delta(t)\|_{L^2}^2 \leq \alpha^2 K_0 e_1^K, \tag{3.30}$$

where K_0 and K_1 are positive constants depending only on $\|e^{a\Lambda} u_0\|_{H^3}$. Thus, we obtain the convergence in the analytic norm as $\alpha \to 0$ of the solution of the second-grade fluid to the solutions of Navier–Stokes equations, with same analytic initial data u_0, such that $e^{a\Lambda} u_0 \in H^3$.

4. The Three-Dimensional Case

4.1. Global in Time Results for Small Initial Data

In this section we state our main result in the case $\nu > 0$, with small initial data: There exists a global in time solution whose Gevrey-class radius is bounded from below by a positive constant for all time. A similar result for small data is obtained in [41].

Theorem 4.1. *Fix $\nu, \alpha > 0$, and assume that $\omega_0 \in \mathcal{D}(\Lambda^{1/2s} e^{\tau_0 \Lambda^{1/s}})$, for some $s \geq 1$, and $\tau_0 > 0$. There exists a positive sufficiently large dimensional constant κ, such that if*

$$\kappa \|\omega_0\|_{L^2} \leq \frac{\nu\alpha}{2(1+\alpha^2)}, \tag{4.1}$$

then there exists a unique global in time Gevrey-class s solution $\omega(t)$ to (2.6)–(2.8), such that for all $t \geq 0$ we have $\omega(t) \in \mathcal{D}(e^{\tau(t)\Lambda^{1/s}})$, and moreover we have the lower bound

$$\tau(t) \geq \tau_0 e^{-\kappa(4+4\alpha^2) M_0/(\nu\alpha)} \tag{4.2}$$

for all $t \geq 0$, where $M_0 = \|e^{\tau_0 \Lambda^{1/s}} \omega_0\|_{L^2}$.

The smallness condition (4.1) ensures that $\|\omega(t)\|_{L^2}$ decays exponentially in time, and hence by the Sobolev and Poincaré inequalities the same decay holds for $\|\nabla u(t)\|_{L^\infty}$. Therefore, as opposed to the case of large initial data treated in Sect. 4.2, in this case there is no loss in expressing the radius of Gevrey-class regularity in terms of the vorticity $\omega(t)$. It is thus more transparent to prove Theorem 4.1 by just using the operator Λ (cf. [36]), instead of using the operators Λ_m (cf. [32]) which are used to prove Theorem 4.3 below.

Proof of Theorem 4.1. Similarly to (3.3), we have the a priori estimate

$$\frac{1}{2}\frac{d}{dt}\|e^{\tau\Lambda^{1/s}}\omega\|_{L^2}^2+\frac{\nu}{1+\alpha^2}\|e^{\tau\Lambda^{1/s}}\omega\|_{L^2}^2 \leq \dot{\tau}\|\Lambda^{1/2s}e^{\tau\Lambda^{1/s}}\omega\|_{L^2}^2+|(u\cdot\nabla\omega,e^{2\tau\Lambda^{1/s}}\omega)|+|(\omega\cdot\nabla u,e^{2\tau\Lambda^{1/s}}\omega)|. \tag{4.3}$$

The convection term and the vorticity stretching term are estimated in the following lemma.

Lemma 4.2. *There exists a positive dimensional constant C such that for $\omega\in Y_{s,\tau}$, and $u=\mathcal{K}_\alpha$ is divergence-free, we have*

$$|(u\cdot\nabla\omega,e^{2\tau\Lambda^{1/s}}\omega)|\leq\frac{C\tau}{\alpha}\|e^{\tau\Lambda^{1/s}}\omega\|_{L^2}\|\Lambda^{1/2s}e^{\tau\Lambda^{1/s}}\omega\|_{L^2}^2, \tag{4.4}$$

and

$$|(\omega\cdot\nabla u,e^{2\tau\Lambda^{1/s}}\omega)|\leq\frac{C}{\alpha}\|\omega\|_{L^2}\|e^{\tau\Lambda^{1/s}}\omega\|_{L^2}^2 +\frac{C\tau}{\alpha}\|e^{\tau\Lambda^{1/s}}\omega\|_{L^2}\|\Lambda^{1/2s}e^{\tau\Lambda^{1/s}}\omega\|_{L^2}^2. \tag{4.5}$$

The proof of the above lemma is similar to [36, Lemma 8], but for the sake of completeness a sketch is given in the Appendix (cf. Sect. 6.2).

The smallness condition (4.1) implies via the Sobolev and Poincaré inequalities that $\|\nabla u_0\|_{L^\infty}\leq\nu/(2+2\alpha^2)$, if κ is chosen sufficiently large. Let $\gamma=\nu/(2+2\alpha^2)$. It follows from standard energy inequalities that $\|\omega(t)\|_{L^2}\leq\|\omega_0\|_{L^2}e^{-\gamma t/2}\leq\|\omega_0\|_{L^2}$. Combining this estimate with (4.3), (4.4), and (4.5), we obtain

$$\frac{1}{2}\frac{d}{dt}\|e^{\tau\Lambda^{1/s}}\omega\|_{L^2}^2+\gamma\|e^{\tau\Lambda^{1/s}}\omega\|_{L^2}^2\leq\left(\dot{\tau}+\frac{C\tau}{\alpha}\|e^{\tau\Lambda^{1/s}}\omega\|_{L^2}\right)\|\Lambda^{1/2s}e^{\tau\Lambda^{1/s}}\omega\|_{L^2}^2, \tag{4.6}$$

where we have used that κ was chosen sufficiently large, i.e., $\kappa\geq C$. The above a-priori estimate gives the global in time Gevrey-class s solution $\omega(t)\in\mathcal{D}(e^{\tau(t)\Lambda^{1/s}})$, if the radius of Gevrey-class regularity $\tau(t)$ is chosen such that

$$\dot{\tau}+\frac{C\tau}{\alpha}\|e^{\tau\Lambda^{1/s}}\omega\|_{L^2}\leq 0. \tag{4.7}$$

Since under this condition we have

$$\|e^{\tau(t)\Lambda^{1/s}}\omega(t)\|_{L^2}\leq\|e^{\tau_0\Lambda^{1/s}}\omega_0\|_{L^2}e^{-\gamma t/2}$$

for all $t\geq 0$, it is sufficient to let $\tau(t)$ be such that $\dot{\tau}+CM_0e^{-\gamma t/2}\tau/\alpha=0$, where we let $M_0=\|e^{\tau_0\Lambda^{1/s}}\omega_0\|_{L^2}$. We obtain

$$\tau(t)=\tau_0e^{-CM_0\int_0^t e^{-\gamma s/2}ds/\alpha}, \tag{4.8}$$

and in particular the radius of analyticity does not vanish as $t\to\infty$, since it is bounded as

$$\tau(t)\geq\tau_0e^{-2CM_0/(\gamma\alpha)}=\tau_0e^{-CM_0(4+4\alpha^2)/(\nu\alpha)}, \tag{4.9}$$

for all $t\geq 0$, thereby concluding the proof of Theorem 4.1. □

4.2. Large Initial Data

The main theorem of this section deals with the case of large initial data, where only the local in time existence of solutions is known (cf. [13,14]). We prove the persistence of Gevrey-class regularity: as long as the solution exists and does not blow-up in the Sobolev norm, it does not blow-up in the Gevrey-class norm. Similarly to the Euler equations, the finite time blow-up remains an open problem.

Theorem 4.3. *Fix $\nu, \alpha > 0$, and assume that ω_0 is of Gevrey-class s, for some $s \geq 1$. Then the unique solution $\omega(t) \in C([0, T^*); L^2(\mathbb{T}^3))$ to (2.6)–(2.8) is of Gevrey-class s for all $t < T^*$, where $T^* \in (0, \infty]$ is the maximal time of existence of the Sobolev solution. Moreover, the radius $\tau(t)$ of Gevrey-class s regularity of the solution is bounded from below as*

$$\tau(t) \geq \frac{\tau_0}{C_0} e^{-C \int_0^t \|\nabla u(s)\|_{L^\infty} ds}, \tag{4.10}$$

where $C > 0$ is a dimensional constant, and $C_0 > 0$ has additional explicit dependence on the initial data, α, and ν via (4.24) below.

We note that the radius of Gevrey-class regularity is expressed in terms of $\|\nabla u\|_{L^\infty}$, as opposed to an exponential in terms of higher Sobolev norms of the velocity. Hence Theorem 4.3 may be viewed as a blow-up criterion: if the initial data is of Gevrey-class s (its Fourier coefficients decay at the exponential rate $e^{-\tau_0 |k|^{1/s}}$), and at time T_* the Fourier coefficients of the solution $u(T_*)$ do not decay sufficiently fast, then the Sobolev norm of the solution must blow up at T_*.

To prove Theorem 4.3, let us first introduce the functional setting. For fixed $s \geq 1$, $\tau \geq 0$, and $m \in \{1, 2, 3\}$, we define via the Fourier transform the space

$$\mathcal{D}(\Lambda_m e^{\tau \Lambda_m^{1/s}}) = \Bigg\{ \omega \in C^\infty(\mathbb{T}^d) : \operatorname{div} \omega = 0, \int_{\mathbb{T}^d} \omega = 0,$$
$$\left\| \Lambda_m e^{\tau \Lambda_m^{1/s}} \omega \right\|_{L^2}^2 = (2\pi)^d \sum_{k \in \mathbb{Z}^d} |k_m|^2 e^{2\tau |k_m|^{1/s}} |\widehat{\omega}_k|^2 < \infty \Bigg\},$$

where $\widehat{\omega}_k$ is the k^{th} Fourier coefficient of ω, and Λ_m is the Fourier-multiplier operator with symbol $|k_m|$. For s, τ as before, also define the normed spaces $Y_{s,\tau} \subset X_{s,\tau}$ by

$$X_{s,\tau} = \bigcap_{m=1}^{3} \mathcal{D}(\Lambda_m e^{\tau \Lambda_m^{1/s}}), \qquad \|\omega\|_{X_{s,\tau}}^2 = \sum_{m=1}^{3} \left\| \Lambda_m e^{\tau \Lambda_m^{1/s}} \omega \right\|_{L^2}^2, \tag{4.11}$$

and

$$Y_{s,\tau} = \bigcap_{m=1}^{3} \mathcal{D}(\Lambda_m^{1+s/2} e^{\tau \Lambda_m^{1/s}}), \qquad \|\omega\|_{Y_{s,\tau}}^2 = \sum_{m=1}^{3} \left\| \Lambda_m^{1+s/2} e^{\tau \Lambda_m^{1/s}} \omega \right\|_{L^2}^2, \tag{4.12}$$

It follows from the triangle inequality that if $\omega \in X_{s,\tau}$ then ω is a function of Gevrey-class s, with radius proportional to τ (up to a dimensional constant). If instead of the $X_{s,\tau}$ norm we use $\|\Lambda e^{\tau \Lambda^{1/s}} \omega\|_{L^2}$ (cf. [16,36]), then the lower bound for the radius of Gevrey-class regularity will decay exponentially in $\|\omega\|_{H^1}$ (i.e., a higher Sobolev norm of the velocity). It was shown in [32] that using the spaces $X_{s,\tau}$ it is possible give lower bounds on τ that depend algebraically on the higher Sobolev norms of u, and exponentially on $\|\nabla u(t)\|_{L^\infty}$, which in turn gives a better estimate on the analyticity radius.

Proof of Theorem 4.3. Assume that the initial datum ω_0 is of Gevrey-class s, for some $s \geq 1$, with $\omega_0 \in Y_{s,\tau_0}$, for some $\tau_0 = \tau(0) > 0$. We take the L^2-inner product of (2.6) with $\Lambda_m^2 e^{2\tau(t) \Lambda_m^{1/s}} \omega(t)$ and obtain

$$\begin{aligned}(\partial_t \omega, \Lambda_m^2 e^{2\tau \Lambda_m^{1/s}} \omega) &+ \nu (\mathcal{R}_\alpha \omega, \Lambda_m^2 e^{2\tau \Lambda_m^{1/s}} \omega) \\ &= -(u \cdot \nabla \omega, \Lambda_m^2 e^{2\tau \Lambda_m^{1/s}} \omega) + (\omega \cdot \nabla u, \Lambda_m^2 e^{2\tau \Lambda_m^{1/s}} \omega).\end{aligned}$$

For simplicity, we omit the t-dependence of τ and ω. The above implies

$$\begin{aligned}&(\partial_t \Lambda_m e^{\tau \Lambda_m^{1/s}} \omega, \Lambda_m e^{\tau \Lambda_m^{1/s}} \omega) \\ &\quad - \dot{\tau} (\Lambda_m^{1+s/2} e^{\tau \Lambda_m^{1/s}} \omega, \Lambda_m^{1+s/2} e^{\tau \Lambda_m^{1/s}} \omega) + \nu (\mathcal{R}_\alpha \Lambda_m e^{\tau \Lambda_m^{1/s}} \omega, \Lambda_m e^{\tau \Lambda_m^{1/s}} \omega) \\ &= -(u \cdot \nabla \omega, \Lambda_m^2 e^{2\tau \Lambda_m^{1/s}} \omega) + (\omega \cdot \nabla u, \Lambda_m^2 e^{2\tau \Lambda_m^{1/s}} \omega).\end{aligned} \tag{4.13}$$

Note that the Fourier multiplier symbol of the operator $\mathcal{R}_\alpha$ is an increasing function of $|k| \geq 1$, and therefore by Plancherel's theorem and Parseval's identity we have

$$(\mathcal{R}_\alpha \Lambda_m e^{\tau\Lambda_m^{1/s}}\omega, \Lambda_m e^{\tau\Lambda_m^{1/s}}\omega) = (2\pi)^3 \sum_{k\in\mathbb{Z}^3\setminus\{0\}} \frac{|k|^2}{1+\alpha^2|k|^2}|k_m|^2|\widehat{\omega}_k|^2 e^{2\tau|k|^s}$$
$$\geq \frac{(2\pi)^3}{1+\alpha^2}\sum_{k\in\mathbb{Z}^3\setminus\{0\}}|k_m|^2|\widehat{\omega}_k|^2 e^{2\tau|k|^s} = \frac{1}{1+\alpha^2}\|\Lambda_m e^{\tau\Lambda_m^{1/s}}\omega\|_{L^2}^2.$$

The above estimate combined with (4.13) gives for all $m \in \{1,2,3\}$, the a-priori estimate

$$\frac{1}{2}\frac{d}{dt}\|\Lambda_m e^{\tau\Lambda_m^{1/s}}\omega\|_{L^2}^2 + \frac{\nu}{1+\alpha^2}\|\Lambda_m e^{\tau\Lambda_m^{1/s}}\omega\|_{L^2}^2 - \dot{\tau}\|\Lambda_m^{1+s/2}e^{\tau\Lambda_m^{1/s}}\omega\|_{L^2}^2$$
$$\leq T_1 + T_2, \tag{4.14}$$

where we have denoted

$$T_1 = \left|(u\cdot\nabla\omega, \Lambda_m^2 e^{2\tau\Lambda_m^{1/s}}\omega)\right|, \quad \text{and} \quad T_2 = \left|(\omega\cdot\nabla u, \Lambda_m^2 e^{2\tau\Lambda_m^{1/s}}\omega)\right|. \tag{4.15}$$

The convection term T_1, and the vorticity stretching term T_2 are estimated using the fact that $\operatorname{div} u = 0$, and that $u = \mathcal{K}_\alpha\omega$.

Lemma 4.4. *For all $m \in \{1,2,3\}$ and $\omega \in Y_{s,\tau}$, we have*

$$T_1 + T_2 \leq C\,\|\nabla u\|_{L^\infty}\|\omega\|_{X_{s,\tau}}^2 + \frac{C}{\alpha}(1+\tau)\,\|\omega\|_{H^1}^2\,\|\omega\|_{X_{s,\tau}}$$
$$+\left(C\tau\,\|\nabla u\|_{L^\infty} + \frac{C\tau^2}{\alpha}\,\|\omega\|_{H^1} + \frac{C\tau^2}{\alpha}\,\|\omega\|_{X_{s,\tau}}\right)\|\omega\|_{Y_{s,\tau}}^2, \tag{4.16}$$

where $C > 0$ is a dimensional constant.

This lemma in the context of the Euler equations was proven by Kukavica and Vicol [32, Lemma 2.5], but for the sake of completeness we sketch the proof in the Appendix (cf. Sect. 6.3). The novelty of this lemma is that the term $\|\nabla u\|_{L^\infty}$ is paired with τ, while the term $\|\omega\|_{H^1}$ is paired with τ^2. This gives the exponential dependence on the gradient norm and the algebraic dependence of the Sobolev norm. By summing over $m = 1,2,3$ in (4.14), and using (4.16), we have proven the a-priori estimate

$$\frac{1}{2}\frac{d}{dt}\|\omega\|_{X_{s,\tau}}^2 + \frac{\nu}{1+\alpha^2}\|\omega\|_{X_{s,\tau}}^2$$
$$\leq C\|\nabla u\|_{L^\infty}\|\omega\|_{X_{s,\tau}}^2 + \frac{C}{\alpha}(1+\tau)\,\|\omega\|_{H^1}^2\,\|\omega\|_{X_{s,\tau}}$$
$$+\left(\dot{\tau} + C\tau\,\|\nabla u\|_{L^\infty} + \frac{C\tau^2}{\alpha}\,\|\omega\|_{H^1} + \frac{C\tau^2}{\alpha}\,\|\omega\|_{X_{s,\tau}}\right)\|\omega\|_{Y_{s,\tau}}^2. \tag{4.17}$$

Therefore, if the radius of Gevrey-class regularity is chosen to decay fast enough so that

$$\dot{\tau} + C\tau\,\|\nabla u\|_{L^\infty} + \frac{C\tau^2}{\alpha}\,\|\omega\|_{H^1} + \frac{C\tau^2}{\alpha}\,\|\omega\|_{X_{s,\tau}} \leq 0, \tag{4.18}$$

then for all $\nu > 0$ we have

$$\frac{d}{dt}\|\omega\|_{X_{s,\tau}} + 2\gamma\|\omega\|_{X_{s,\tau}} \leq C\|\nabla u\|_{L^\infty}\|\omega\|_{X_{s,\tau}} + \frac{C}{\alpha}(1+\tau_0)\,\|\omega\|_{H^1}^2, \tag{4.19}$$

where as before $\gamma = \nu/(2+2\alpha^2)$. Hence by Grönwall's inequality

$$\|\omega(t)\|_{X_{s,\tau(t)}} \leq M(t)e^{-2\gamma t}$$
$$\times\left(\|\omega_0\|_{X_{s,\tau_0}} + \frac{C}{\alpha}(1+\tau_0)\int_0^t \|\omega(s)\|_{H^1}^2 e^{2\gamma s}M(s)^{-1}ds\right).$$

where for the sake of compactness we have denoted

$$M(t) = e^{C\int_0^t \|\nabla u(s)\|_{L^\infty}\, ds}.$$

Thus it is sufficient to consider the Gevrey-class radius $\tau(t)$ that solves

$$\begin{aligned} 0 &= \dot{\tau}(t) + C\tau(t)\,\|\nabla u(t)\|_{L^\infty} + \frac{C\tau^2(t)}{\alpha}\,\|\omega(t)\|_{H^1} \\ &\quad + \frac{C\tau^2(t)}{\alpha} M(t) e^{-2\gamma t}\left(\|\omega_0\|_{X_{s,\tau_0}} + \frac{C}{\alpha}(1+\tau_0)\int_0^t \|\omega(s)\|_{H^1}^2 e^{2\gamma s} M(s)^{-1} ds\right) \end{aligned} \tag{4.20}$$

The explicit dependence of τ is hence algebraically on $\|\omega\|_{H^1}$ and exponentially on $\|\nabla u\|_{L^\infty}$ via

$$\begin{aligned} \tau(t) = M(t)^{-1} \Bigg(&\frac{1}{\tau_0} + \frac{C(1+\tau_0)}{\alpha^2} \int_0^t e^{-2\gamma s} \int_0^s \|\omega(s')\|_{H^1}^2 M(s')^{-1} e^{2\gamma s'}\, ds'\, ds \\ &+ \frac{C}{\alpha}\int_0^t \|\omega(s)\|_{H^1} M(s)^{-1} + e^{-2\gamma s}\|\omega_0\|_{X_{s,\tau_0}}\, ds \Bigg)^{-1}. \end{aligned} \tag{4.21}$$

A more compact lower bound for $\tau(t)$ is obtained by noting that if $\nu \geq 0$ we have

$$\|\omega(t)\|_{H^1}^2 \leq M(t) e^{-2\gamma t} \|\omega_0\|_{H^1}^2 \tag{4.22}$$

for all $t \geq 0$. Assuming (4.22) holds, if $\nu > 0$ (and hence $\gamma > 0$), then

$$\begin{aligned} \tau(t) &\geq M(t)^{-1}\left(\frac{1}{\tau_0} + C\frac{\|\omega_0\|_{H^1} + \|\omega_0\|_{X_{s,\tau_0}}}{\alpha\gamma} + C\frac{(1+\tau_0)\|\omega_0\|_{H^1}^2}{4\alpha^2\gamma^2}\right)^{-1} \\ &\geq \frac{\tau_0}{C_0} M(t)^{-1} \end{aligned} \tag{4.23}$$

where the constant $C_0 = C_0(\nu, \alpha, \tau_0, \omega_0)$ is given explicitly by

$$C_0 = 1 + C\tau_0(\|\omega_0\|_{H^1} + \|\omega_0\|_{X_{s,\tau_0}})\frac{1+\alpha^2}{\nu\alpha} + C\tau_0(1+\tau_0)\|\omega_0\|_{H^1}^2 \frac{(1+\alpha^2)^2}{\nu^2\alpha^2}. \tag{4.24}$$

The proof of the theorem is hence complete, modulo the proof of estimate (4.22), which is given in the Appendix (cf. Sect. 6.4). □

5. Applications to the Damped Euler Equations

The initial value problem for the *damped* Euler equations in terms of the vorticity $\omega = \operatorname{curl} u$ is

$$\partial_t \omega + \nu\omega + (u\cdot\nabla)\omega = (\omega\cdot\nabla)u \tag{5.1}$$

$$u = K_d * \omega \tag{5.2}$$

$$\omega(0) = \omega_0 = \operatorname{curl} u_0, \tag{5.3}$$

where K_d is the $\mathbb{T}^d$-periodic Biot–Savart kernel, and $\nu \geq 0$ is a fixed positive parameter. Here u and ω are $\mathbb{T}^d$-periodic functions with $\int_{\mathbb{T}^d} u = 0$, and $d = 2, 3$. When $d = 2$ the vorticity is a scalar and the term on the right of (5.1) is absent. It is a classical result that if $d = 2$, and for any $\nu \geq 0$, the initial value problem (5.1)–(5.3) has a global in time smooth solution in the Sobolev space H^r, with $r > 2$. We refer the reader to [10,38] for details. Moreover, in the case $d = 3$, and $\nu > 0$, if the initial data satisfies $\|\nabla u_0\|_{L^\infty} < \nu/\kappa$ for some sufficiently large positive dimensional constant κ, it follows from standard energy estimates that (5.1)–(5.3) has a global in time smooth solution in H^r, with $r > 5/2$.

For results concerning the analyticity and Gevrey-class regularity of (5.1)–(5.3), with $\nu = 0$, i.e. the classical incompressible Euler equations, we refer the reader to [1,3,4,6,32,36]. Note that in this case one can construct explicit solutions (cf. [5,17]) to (5.1)–(5.3) whose radius of analyticity is decaying for all time and hence vanishes as $t \to \infty$, both for $d = 2$ and $d = 3$. In this section we show that if $\nu > 0$, and either $d = 2$, or if $d = 3$ and the initial data is small compared to ν, then this is not possible: there exists a positive constant such that the radius of analyticity of the solution never drops below it. The following is our main result.

Theorem 5.1. *Assume that $\nu > 0$, and that the divergence-free ω_0 is of Gevrey-class s, for some $s \geq 1$. If additionally, one of the following conditions is satisfied,*

1. $d = 2$
2. *$d = 3$ and $\|\nabla u_0\|_{L^\infty} \leq \nu/\kappa$, for some sufficiently large positive constant κ,*

then there exists a unique global in time Gevrey-class s solution to (5.1)–(5.3), with $\omega(t) \in \mathcal{D}(\Lambda^r e^{\tau(t)\Lambda^{1/s}})$ for all $t \geq 0$, and moreover we have the lower bound

$$\tau(t) \geq \tau(0)e^{-\bar{C}\int_0^t e^{-\nu s/2}ds} \geq \tau(0)e^{-2\bar{C}/\nu}, \tag{5.4}$$

where $\bar{C} > 0$ is a constant depending only on ω_0.

Proof of Theorem 5.1. Let us first treat the case when $d = 2$, with $\nu > 0$ fixed. Since $\operatorname{div} u = 0$, it classically follows from (5.1) that for all $1 \leq p \leq \infty$ we have

$$\|\omega(t)\|_{L^p} \leq \|\omega_0\|_{L^p} e^{-\nu t}, \tag{5.5}$$

$t \geq 0$, and for any $r > 0$ the Sobolev energy inequality holds

$$\frac{1}{2}\frac{d}{dt}\|\omega(t)\|_{H^r}^2 + \nu\|\omega(t)\|_{H^r}^2 \leq C\|\nabla u(t)\|_{L^\infty}\|\omega(t)\|_{H^r}^2, \tag{5.6}$$

where C is a positive dimensional constant depending on r. Moreover, if $r > 1$ the classical potential estimate(cf. [8,38])

$$\|\nabla u\|_{L^\infty} \leq C\|\omega\|_{L^2} + C\|\omega\|_{L^\infty} + C\|\omega\|_{L^\infty}\log\left(1 + \frac{\|\omega\|_{H^r}}{\|\omega\|_{L^\infty}}\right)$$

combined with (5.5) shows that

$$\begin{aligned}\|\nabla u(t)\|_{L^\infty} &\leq Ce^{-\nu t}\left(\|\omega_0\|_{L^2} + \|\omega_0\|_{L^\infty} + \|\omega_0\|_{L^\infty}\log\left(1 + \frac{e^{\nu t}\|\omega(t)\|_{H^r}}{\|\omega_0\|_{L^\infty}}\right)\right)\\ &\leq CC_0e^{-\nu t}\left(2 + \log\left(1 + e^{\nu t}\|\omega(t)\|_{H^r}/C_0\right)\right),\end{aligned} \tag{5.7}$$

where $C_0 = \max\{\|\omega_0\|_{L^2}, \|\omega_0\|_{L^\infty}\} > 0$. Multiplying (5.6) by $e^{\nu t}$ and combining with the above estimate (5.7), upon letting $y(t) = e^{\nu t}\|\omega(t)\|_{H^r}/C_0$, we obtain

$$\dot{y}(t) \leq Ce^{-\nu t}y(t)\left(2 + \log(1 + y(t))\right).$$

By Grönwall's inequality, the above implies that there exists a positive constant $C_1 = C(C_0, \nu, \|\omega_0\|_{H^r})$ such that $y(t) \leq C_1/C_0$ for all $t \geq 0$, and therefore by the definition of $y(t)$ we have

$$\|\omega(t)\|_{H^r} \leq C_1 e^{-\nu t}, \tag{5.8}$$

for all $t \geq 0$. Similarly, by (5.7), there exists $C_2 = C(C_0, C_1) > 0$ such that for all $t \geq 0$ we have

$$\|\nabla u(t)\|_{L^\infty} \leq C_2 e^{-\nu t}. \tag{5.9}$$

We now turn to the corresponding Gevrey-class estimates. For $r > 5/2$, and initial vorticity satisfying $\|\Lambda^{r+1/2s}e^{\tau_0\Lambda^{1/s}}\omega_0\|_{L^2} < \infty$, the following estimate can be deduced from [36]

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\Lambda^r e^{\tau\Lambda^{1/s}}\omega\|_{L^2}^2 + \nu\|\Lambda^r e^{\tau\Lambda^{1/s}}\omega\|_{L^2}^2\\ &\quad\leq C\|\omega\|_{H^r}^3 + \left(\dot{\tau} + C\tau\|\Lambda^r e^{\tau\Lambda^{1/s}}\omega\|_{L^2}\right)\|\Lambda^{r+1/2s}e^{\tau\Lambda^{1/s}}\omega\|_{L^2}^2.\end{aligned} \tag{5.10}$$

Therefore, if $\tau(t)$ decays fast enough so that $\dot{\tau}(t) + C\tau(t)\|\Lambda^r e^{\tau(t)\Lambda^{1/s}}\omega(t)\|_{L^2} \le 0$ for all $t \ge 0$, then using (5.8) we have

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^r e^{\tau(t)\Lambda^{1/s}}\omega(t)\|_{L^2}^2 + \nu\|\Lambda^r e^{\tau(t)\Lambda^{1/s}}\omega(t)\|_{L^2}^2 \le CC_1^3 e^{-3\nu t}, \tag{5.11}$$

and hence there exists a positive constant $C_3 = C(C_1, \nu, \|\Lambda^r e^{\tau_0\Lambda^{1/s}}\omega_0\|_{L^2})$ such that for all $t \ge 0$

$$\|\Lambda^r e^{\tau(t)\Lambda^{1/s}}\omega(t)\|_{L^2} \le C_3 e^{-\nu t/2}. \tag{5.12}$$

Then it is sufficient to impose

$$\dot{\tau}(t) + CC_3\tau(t)e^{-\nu t/2} = 0, \tag{5.13}$$

and hence we obtain the lower bound for the radius of Gevrey-class regularity

$$\tau(t) \ge \tau_0 e^{-CC_3\int_0^t e^{-\nu s/2}ds}. \tag{5.14}$$

In particular it follows that for all $t \ge 0$,

$$\tau(t) \ge \tau_0 e^{-2CC_3/\nu}, \tag{5.15}$$

which proves the first part of the theorem. The case $d = 3$ is treated similarly: the estimate (5.10) holds also if $d = 3$, so the missing ingredient is the exponential decay of the Sobolev norms. But as noted earlier, the smallness condition on $\|\nabla u\|_{L^\infty}$, not only gives the global in time existence of H^r solutions, but also their exponential decay. We omit further details. □

6. Appendix

6.1. Proof of Lemma 3.4

Proof of (3.12). Recall that we need to bound the quantity

$$\begin{aligned} T_1 &= \alpha^2 \left|\langle e^{\tau\Lambda}\left((u\cdot\nabla)\Delta \operatorname{curl} u\right), e^{\tau\Lambda}\Delta \operatorname{curl} u\rangle\right| \\ &= \alpha^2 \left|\langle e^{\tau\Lambda}\left((u\cdot\nabla)\Delta \operatorname{curl} u\right)(u\cdot\nabla)e^{\tau\Lambda}\Delta \operatorname{curl} u, e^{\tau\Lambda}\Delta \operatorname{curl} u\rangle\right|, \end{aligned} \tag{6.1}$$

since $\operatorname{div} u = 0$. By Plancherel's theorem we have

$$T_1 \le C\alpha^2 \sum_{j+k=l;\, j,k,l\neq 0} \left(e^{\tau|l|} - e^{\tau|k|}\right)|\widehat{u}_j \cdot j||k|^2|k\times\widehat{u}_k||l|^2|l\times\widehat{u}_l|e^{\tau|l|}. \tag{6.2}$$

Since $|e^{\tau|l|} - e^{\tau|k|}| \le C\tau|j|e^{\max\{|k|,|l|\}}$, we obtain

$$\begin{aligned} T_1 &\le C\alpha^2\tau \sum_{j+k=l;\, j,k,l\neq 0} |j|^2|\widehat{u}_j|e^{\tau|j|}|k|^2|k\times\widehat{u}_k|e^{\tau|k|}|l|^2|l\times\widehat{u}_l|e^{\tau|l|} \\ &\le C\alpha^2\tau \sum_{j+k=l;\, j,k,l\neq 0;\, |l|\ge|k|} |j|^{3/2}|\widehat{u}_j|e^{\tau|j|}|k|^2|k\times\widehat{u}_k|e^{\tau|k|}|l|^{5/2}|l\times\widehat{u}_l|e^{\tau|l|} \\ &\le C\alpha^2\tau\|\Lambda^{1/2}e^{\tau\Lambda}\operatorname{curl}\Delta u\|_{L^2}\|e^{\tau\Lambda}\operatorname{curl}\Delta u\|_{L^2}\sum_{j\neq 0}|j|^{3/2}|\widehat{u}_j|e^{\tau|j|} \\ &\le C\alpha^2\tau\|\Lambda^{1/2}e^{\tau\Lambda}\operatorname{curl}\Delta u\|_{L^2}\|e^{\tau\Lambda}\operatorname{curl}\Delta u\|_{L^2}^2. \end{aligned} \tag{6.3}$$

In the above we have used the triangle inequality $|j|^{1/2} \le |k|^{1/2} + |l|^{1/2}$, the Cauchy–Schwartz inequality, and the fact that in the two-dimensional case we have $\sum_{j\in\mathbb{Z}^2\setminus\{0\}}|j|^{-3} < \infty$. By estimating the right side of (6.3) as

$$\frac{\nu}{4}\|e^{\tau\Lambda}\operatorname{curl}\Delta u\|_{L^2}^2 + \frac{C\alpha^4\tau^2}{\nu}\|\Lambda^{1/2}e^{\tau\Lambda}\operatorname{curl}\Delta u\|_{L^2}^2\|e^{\tau\Lambda}\operatorname{curl}\Delta u\|_{L^2}^2,$$

the proof of (3.12) is concluded. □

Proof of (3.13). Recall that we need to bound the quantity T_2, which can be written as

$$T_2 = \left|\langle \Lambda e^{\tau\Lambda}\left((u\cdot\nabla)\operatorname{curl} u\right), \Lambda e^{\tau\Lambda}\operatorname{curl} u\rangle - \langle (u\cdot\nabla)\Lambda e^{\tau\Lambda}\operatorname{curl} u, \Lambda e^{\tau\Lambda}\operatorname{curl} u\rangle\right|, \tag{6.4}$$

using the fact that $\operatorname{div} u = 0$. By Plancherel's theorem we have

$$T_2 \le C \sum_{j+k=l;\, j,k,l\neq 0} \left(|l|e^{\tau|l|} - |k|e^{\tau|k|}\right) |\widehat{u}_j\cdot j||k\times\widehat{u}_k||l||l\times\widehat{u}_l|e^{\tau|l|}. \tag{6.5}$$

By the mean value theorem

$$\left||l|e^{\tau|l|} - |k|e^{\tau|k|}\right| \le |j|(1+\tau\max\{|l|,|k|\})e^{\tau\max\{|l|,|k|\}},$$

and therefore by the triangle inequality we obtain

$$\begin{aligned} T_2 \le C &\sum_{j+k=l;\, j,k,l\neq 0} |\widehat{u}_j||j|^2 e^{\tau|j|}|k\times\widehat{u}_k|e^{\tau|k|}|l||l\times\widehat{u}_l|e^{\tau|l|} \\ &+ C\tau \sum_{j+k=l;\, j,k,l\neq 0} |\widehat{u}_j||j|^2 e^{\tau|j|}(|j|+|k|)|k\times\widehat{u}_k|e^{\tau|k|}|l||l\times\widehat{u}_l|e^{\tau|l|}. \end{aligned} \tag{6.6}$$

By symmetry, and using $e^x \le 1 + xe^x$ for all $x \ge 0$, we get

$$\begin{aligned} T_2 \le C &\sum_{j+k=l;\, j,k,l\neq 0;\, |j|\le|l|} |\widehat{u}_j||j|e^{\tau|j|}|k\times\widehat{u}_k||l|^2|l\times\widehat{u}_l|e^{\tau|l|} \\ &+ C\tau \sum_{j+k=l;\, j,k,l\neq 0;\, |j|\le|k|,|l|} |\widehat{u}_j||j|^{1/2}e^{\tau|j|}|k|^{3/2}|k\times\widehat{u}_k|e^{\tau|k|}|l|^2|l\times\widehat{u}_l|e^{\tau|l|}, \end{aligned}$$

and by the Cauchy–Schwartz inequality, it follows that

$$\begin{aligned} T_2 \le C&\|\operatorname{curl} u\|_{L^2}\|e^{\tau\Lambda}\operatorname{curl}\Delta u\|_{L^2}\sum_{j\neq 0}|\widehat{u}_j||j|e^{\tau|j|} \\ &+ C\tau\|\Lambda^{3/2}e^{\tau\Lambda}\operatorname{curl} u\|_{L^2}\|e^{\tau\Lambda}\operatorname{curl}\Delta u\|_{L^2}\sum_{j\neq 0}|\widehat{u}_j||j|^{1/2}e^{\tau|j|}. \end{aligned} \tag{6.7}$$

Note that in the two-dimensional case, by the Cauchy–Schwartz inequality we have

$$\begin{aligned} \sum_{j\neq 0}|\widehat{u}_j||j|e^{\tau|j|} &= \sum_{j\neq 0}\left(|j||\widehat{u}_j|^{1/2}e^{\tau|j|/2}\right)\left(|j|^{3/2}|\widehat{u}_j|^{1/2}e^{\tau|j|/2}\right)|j|^{-3/2} \\ &\le C\|\Lambda e^{\tau\Lambda}\operatorname{curl} u\|_{L^2}^{1/2}\|e^{\tau\Lambda}\operatorname{curl}\Delta u\|_{L^2}^{1/2}. \end{aligned} \tag{6.8}$$

Similarly,

$$\sum_{j\neq 0}|j|^{1/2}|\widehat{u}_j|e^{\tau|j|} \le \sum_{j\neq 0}|j|^2|\widehat{u}_j|e^{\tau|j|}|j|^{-3/2} \le C\|\Lambda e^{\tau\Lambda}\operatorname{curl} u\|_{L^2}, \tag{6.9}$$

and therefore

$$\begin{aligned} T_2 \le C&\|\operatorname{curl} u\|_{L^2}\|\Lambda e^{\tau\Lambda}\operatorname{curl} u\|_{L^2}^{1/2}\|e^{\tau\Lambda}\operatorname{curl}\Delta u\|_{L^2}^{3/2} \\ &+ C\tau\|\Lambda^{3/2}e^{\tau\Lambda}\operatorname{curl} u\|_{L^2}\|\Lambda e^{\tau\Lambda}\operatorname{curl} u\|_{L^2}\|e^{\tau\Lambda}\operatorname{curl}\Delta u\|_{L^2}. \end{aligned} \tag{6.10}$$

The above estimate and Young's inequality concludes the proof of (3.13). □

6.2. Proof of Lemma 4.2

For convenience of notation we let $\zeta = 1/s$, so that $\zeta \in (0,1]$. Since $\operatorname{div} u = 0$, cf. [32,36] we have $(u \cdot \nabla e^{\tau\Lambda^\zeta}\omega, e^{\tau\Lambda^\zeta}\omega) = 0$, and therefore

$$T_1 = \left|(u \cdot \nabla\omega, e^{2\tau\Lambda^\zeta}\omega)\right| = \left|(u \cdot \nabla\omega, e^{2\tau\Lambda^\zeta}\omega) - (u \cdot \nabla e^{\tau\Lambda^\zeta}\omega, e^{\tau\Lambda^\zeta}\omega)\right|.$$

As in [23,32,36], using Plancherel's theorem we write the above term as

$$T_1 = \left|(2\pi)^3 i \sum_{j+k=l} (\widehat{u}_j \cdot k)(\widehat{\omega}_k \cdot \bar{\widehat{\omega}}_l) e^{\tau|l|^\zeta}\left(e^{\tau|l|^\zeta} - e^{\tau|k|^\zeta}\right)\right|, \tag{6.11}$$

where the sum is taken over all $j,k,l \in \mathbb{Z}^3\setminus\{0\}$. Using the inequality $e^x - 1 \le xe^x$ for $x \ge 0$, the mean-value theorem, and the triangle inequality $|k+j|^\zeta \le |k|^\zeta + |j|^\zeta$, we estimate

$$\left|e^{\tau|l|^\zeta} - e^{\tau|k|^\zeta}\right| \le \tau\left||l|^\zeta - |k|^\zeta\right| e^{\tau\max\{|l|^\zeta,|k|^\zeta\}} \le C\tau \frac{|j|}{|k|^{1-\zeta} + |l|^{1-\zeta}} e^{\tau|j|^\zeta} e^{\tau|k|^\zeta},$$

for all $\zeta \in (0,1]$, where $C > 0$ is a dimensional constant. By (6.11), the triangle inequality, and the Cauchy–Schwartz inequality we obtain

$$\begin{aligned}
T_1 &\le C\tau \sum_{j+k=l} |j||\widehat{u}_j|e^{\tau|j|^\zeta}|\widehat{\omega}_k|e^{\tau|k|^\zeta}|\widehat{\omega}_l|e^{\tau|l|^\zeta}\frac{|k|}{|k|^{1-\zeta}+|l|^{1-\zeta}}\\
&\le C\tau \sum_{j+k=l} |j||\widehat{u}_j|e^{\tau|j|^\zeta}|\widehat{\omega}_k|e^{\tau|k|^\zeta}|\widehat{\omega}_l|e^{\tau|l|^\zeta}|k|^{\zeta/2}\left(|j|^{\zeta/2}+|l|^{\zeta/2}\right)\\
&\le C\tau\|e^{\tau\Lambda^\zeta}\omega\|_{L^2}\|\Lambda^{\zeta/2}e^{\tau\Lambda^\zeta}\omega\|_{L^2}\sum_{j\neq 0}|j|^{1+\zeta/2}|\widehat{u}_j|e^{\tau|j|^\zeta}\\
&\quad + C\tau\|\Lambda^{\zeta/2}e^{\tau\Lambda^\zeta}\omega\|_{L^2}^2\sum_{j\neq 0}|j||\widehat{u}_j|e^{\tau|j|^\zeta}\\
&\le C\tau\|e^{\tau\Lambda^\zeta}\omega\|_{L^2}\|\Lambda^{\zeta/2}e^{\tau\Lambda^\zeta}\omega\|_{L^2}\|\Lambda^{3+\zeta/2}e^{\tau\Lambda^\zeta}u\|_{L^2}\\
&\quad + C\tau\|\Lambda^{\zeta/2}e^{\tau\Lambda^\zeta}\omega\|_{L^2}^2\|\Lambda^3 e^{\tau\Lambda^\zeta}u\|_{L^2}
\end{aligned} \tag{6.12}$$

In the above we used the fact that $\sum_{j\neq 0,\, j\in\mathbb{Z}^3}|j|^{-4} < \infty$. We recall that by (2.12) we have $u = \mathcal{K}_\alpha\omega$, and therefore for $\alpha > 0$ we have

$$\|\Lambda^3 u\|_{L^2} \le \frac{C}{\alpha}\|\omega\|_{L^2},$$

and similarly

$$\|\Lambda^3 e^{\tau\Lambda^\zeta}u\|_{L^2} \le \frac{C}{\alpha}\|e^{\tau\Lambda^\zeta}\omega\|_{L^2}, \quad \text{and} \quad \|\Lambda^{3+\zeta/2}e^{\tau\Lambda^\zeta}u\|_{L^2} \le \frac{C}{\alpha}\|\Lambda^{\zeta/2}e^{\tau\Lambda^\zeta}\omega\|_{L^2}. \tag{6.13}$$

By combining (6.12) and (6.13) above, we obtain for all $\tau \ge 0$, and $\zeta \in (0,1]$ that

$$T_1 \le \frac{C\tau}{\alpha}\|e^{\tau\Lambda^\zeta}\omega\|_{L^2}\|\Lambda^{\zeta/2}e^{\tau\Lambda^\zeta}\omega\|_{L^2}^2, \tag{6.14}$$

for some sufficiently large dimensional constant C, thereby proving (4.4), since $\zeta = 1/s$.

The estimate for the vorticity stretching term is similar. By the triangle inequality and the the estimate $e^x \leq 1 + xe^x$ for all $x \geq 0$, we have

$$\begin{aligned} T_2 = \left|(\omega \cdot u, e^{2\tau\Lambda^\zeta}\omega)\right| &= \left|(2\pi)^3 i \sum_{j+k=l} (\widehat{\omega}_j \cdot k)(\widehat{u}_k \cdot \bar{\widehat{\omega}}_l) e^{2\tau|l|^\zeta}\right| \\ &\leq C \sum_{j+k=l} |\widehat{\omega}_j| e^{\tau|j|^\zeta} |k| |\widehat{u}_k| e^{\tau|k|^\zeta} |\widehat{\omega}_l| e^{\tau|l|^\zeta} \\ &\leq C \sum_{j+k=l} |\widehat{\omega}_j| e^{\tau|j|^\zeta} |k| |\widehat{u}_k| |\widehat{\omega}_l| e^{\tau|l|^\zeta} \\ &\quad + C\tau \sum_{j+k=l} |\widehat{\omega}_j| e^{\tau|j|^\zeta} |k|^{1+\zeta} |\widehat{u}_k| e^{\tau|k|^\zeta} |\widehat{\omega}_l| e^{\tau|l|^\zeta} \\ &\leq \frac{C}{\alpha} \|\omega\|_{L^2} \|e^{\tau\Lambda^\zeta}\omega\|_{L^2}^2 + \frac{C\tau}{\alpha} \|e^{\tau\Lambda^\zeta}\omega\|_{L^2} \|\Lambda^{\zeta/2} e^{\tau\Lambda^\zeta}\omega\|_{L^2}^2. \end{aligned} \tag{6.15}$$

In the last inequality above we also used $\|\Lambda^3 u\|_{L^2} \leq C\|\omega\|_{L^2}/\alpha$. This proves (4.5) and hence concludes the proof of the lemma.

6.3. Proof of Lemma 4.4

For ease of notation we let $\zeta = 1/s$, so that $\zeta \in (0, 1]$. Following notations in Sect. 4, for any $m \in \{1, 2, 3\}$, we need to estimate

$$T_1 = (u \cdot \nabla\omega, \Lambda_m^2 e^{2\tau\Lambda_m^\zeta}\omega), \tag{6.16}$$

and

$$T_2 = (\omega \cdot \nabla u, \Lambda_m^2 e^{2\tau\Lambda_m^\zeta}\omega). \tag{6.17}$$

First we bound the term T_1. Note that since $\operatorname{div} u = 0$, we have

$$(u \cdot \nabla\Lambda_m e^{\tau\Lambda_m^\zeta}\omega, \Lambda_m e^{\tau\Lambda_m^\zeta}\omega) = 0,$$

and therefore by Plancherel's theorem (see also [32]) we obtain

$$\begin{aligned} T_1 &= (u \cdot \nabla\omega, \Lambda_m^2 e^{2\tau\Lambda_m^\zeta}\omega) - (u \cdot \nabla\Lambda_m e^{\tau\Lambda_m^\zeta}\omega, \Lambda_m e^{\tau\Lambda_m^\zeta}\omega) \\ &= i(2\pi)^3 \sum_{j+k=l} \left(|l_m| e^{\tau|l_m|^\zeta} - |k_m| e^{\tau|l_m|^\zeta}\right) (\widehat{u}_j \cdot k)(\widehat{\omega}_k \cdot \bar{\widehat{\omega}}_l) |l_m| e^{\tau|l_m|^\zeta} \end{aligned} \tag{6.18}$$

where the summation is taken over all $j, k, l \in \mathbb{Z}^3 \setminus \{0\}$. We split the Fourier symbol arising from the commutator, namely $|l_m| e^{\tau|l_m|^\zeta} - |k_m| e^{\tau|l_m|^\zeta}$, in four parts (cf. [32]) by letting

$$\begin{aligned} T_{11} &= i(2\pi)^3 \sum_{j+k=l} (|l_m| - |k_m|)\, e^{\tau|k_m|^\zeta} (\widehat{u}_j \cdot k)(\widehat{\omega}_k \cdot \bar{\widehat{\omega}}_l) |l_m| e^{\tau|l_m|^\zeta}, \\ T_{12} &= i(2\pi)^3 \sum_{j+k=l} |l_m| e^{\tau|k_m|^\zeta} \left(e^{\tau(|l_m|^\zeta - |k_m|^\zeta)} - 1 - \tau(|l_m|^\zeta - |k_m|^\zeta)\right) \\ &\quad \times (\widehat{u}_j \cdot k)(\widehat{\omega}_k \cdot \bar{\widehat{\omega}}_l) |l_m| e^{\tau|l_m|^\zeta}, \\ T_{13} &= i(2\pi)^3 \sum_{j+k=l} \tau |k_m|^{1-\zeta/2} e^{\tau|k_m|^\zeta} \left(|l_m|^\zeta - |k_m|^\zeta\right) \\ &\quad \times (\widehat{u}_j \cdot k)(\widehat{\omega}_k \cdot \bar{\widehat{\omega}}_l) |l_m|^{1+\zeta/2} e^{\tau|l_m|^\zeta}, \end{aligned}$$

$$T_{14} = i(2\pi)^3 \sum_{j+k=l} \tau(|l_m| - |k_m|)e^{\tau|k_m|^\zeta} \left(|l_m|^{1-\zeta/2} - |k_m|^{1-\zeta/2}\right) \\ \times(\widehat{u}_j \cdot k)(\widehat{\omega}_k \cdot \bar{\widehat{\omega}}_l)|l_m|^{1+\zeta/2}e^{\tau|l_m|^\zeta}.$$

To isolate the term $\|\nabla u\|_{L^\infty}$ arising from T_{11} and T_{13}, we need to use the inverse Fourier transform and hence may not directly bound these two terms in absolute value. The key idea is to use the one-dimensional identity (cf. [32])

$$|j_m + k_m| - |k_m| = j_m \operatorname{sgn}(k_m) + 2(j_m + k_m) \operatorname{sgn}(j_m)\chi_{\{\operatorname{sgn}(k_m+j_m)\operatorname{sgn}(k_m)=-1\}}, \tag{6.19}$$

an notice that on the region $\{\operatorname{sgn}(k_m + j_m)\operatorname{sgn}(k_m) = -1\}$, we have $0 \le |k_m| \le |j_m|$. Define the operator H_m as the fourier multiplier with symbol $\operatorname{sgn}(k_m)$, which is hence bounded on L^2. From (6.18), the definition of T_{11}, and (6.19), it follows that

$$T_{11} = (\partial_m u \cdot \nabla H_m e^{\tau\Lambda_m^\zeta}\omega, \Lambda_m e^{\tau\Lambda_m^\zeta}\omega) \\ + i(2\pi)^3 \sum_{j+k=l;\{\operatorname{sgn}(k_m+j_m)\operatorname{sgn}(k_m)=-1\}} 2(j_m + k_m)\operatorname{sgn}(j_m)e^{\tau|k_m|^\zeta} \\ \times(\widehat{u}_j \cdot k)(\widehat{\omega}_k \cdot \bar{\widehat{\omega}}_l)|l_m|e^{\tau|l_m|^\zeta}. \tag{6.20}$$

The first term in the above equality is bounded by the Hölder inequality from above by $\|\nabla u\|_{L^\infty}\|\omega\|^2_{X_{s,\tau}}$. The second term is bounded in absolute value, by making use of $e^{\tau|k_m|^\zeta} \le e + \tau^2|k_m|^{2\zeta}e^{\tau|k_m|^\zeta}$, and of $|k_m| \le |j_m|$, by the quantity

$$C\|\omega\|_{H^1}\|\omega\|_{X_{s,\tau}} \left(\sum_{j\ne0} |j_m||\widehat{u}_j|\right) + C\tau^2\|\omega\|^2_{Y_{s,\tau}} \left(\sum_{j\ne0} |j_m|^{1+\zeta}|\widehat{u}_j|\right). \tag{6.21}$$

By the Cauchy–Schwartz inequality, and the fact that $2(\zeta - 3) < -3$ for all $\zeta \in (0, 1]$, we have

$$\sum_{j\ne0} |j_m|^{1+\zeta}|\widehat{u}_j| = \sum_{j\ne0} |j_m|^{1+\zeta}|j|^{3-\zeta}|\widehat{u}_j||j|^{-3+\zeta} \\ \le C\|\Lambda_m^{1+\zeta}\Lambda^{3-\zeta}u\|_{L^2} \le C\|\omega\|_{H^1}/\alpha, \tag{6.22}$$

and similarly $\sum_{j\ne0} |j_m||\widehat{u}_j| \le C\|\omega\|_{H^1}/\alpha$. Therefore

$$|T_{11}| \le C\|\nabla u\|_{L^\infty}\|\omega\|^2_{X_{s,\tau}} + \frac{C}{\alpha}\|\omega\|^2_{H^1}\|\omega\|_{X_{s,\tau}} + \frac{C}{\alpha}\tau^2\|\omega\|_{H^1}\|\omega\|^2_{Y_{s,\tau}}. \tag{6.23}$$

To bound T_{13} one proceeds exactly the same if $s = \zeta = 1$. If $\zeta \in (0, 1)$, (6.19) may not be applied directly to $|l_m|^\zeta - |k_m|^\zeta$. In this case, by the mean value theorem, for any $|l_m|, |k_m| \ge 0$, there exists $\theta_{m,k,l} \in (0, 1)$ such that

$$|l_m|^\zeta - |k_m|^\zeta = \zeta(|l_m| - |k_m|)|k_m|^{\zeta-1} \\ + \zeta(|l_m| - |k_m|)\left((\theta_{m,k,l}|k_m| + (1 - \theta_{m,k,l})|l_m|)^{\zeta-1} - |k_m|^{\zeta-1}\right). \tag{6.24}$$

We apply (6.19) to the first term in the above identity, while the second term is bounded in absolute value by $\zeta(1 - \zeta)|j_m|^2|k_m|^{\zeta-1}/\min\{|k_m|, |l_m|\}$. The rest of the T_{13} estimate is the same as the one for T_{11} and one similarly obtains

$$|T_{13}| \le C\|\nabla u\|_{L^\infty}\|\omega\|^2_{X_{s,\tau}} + \frac{C}{\alpha}\|\omega\|^2_{H^1}\|\omega\|_{X_{s,\tau}} + \frac{C}{\alpha}\tau^2\|\omega\|_{H^1}\|\omega\|^2_{Y_{s,\tau}}. \tag{6.25}$$

The term T_{12} is estimated in absolute value, by making use of the inequality $|e^x - 1 - x| \le x^2e^{|x|}$, and of $||l_m|^\zeta - |k_m|^\zeta| \le C|j_m|/(|k_m|^{1-\zeta} + |l_m|^{1-\zeta})$. It follows from the Cauchy–Schwartz inequality applied in the Fourier variables that

$$|T_{12}| \leq \frac{C}{\alpha}\tau^2\|\omega\|_{X_{s,\tau}}\|\omega\|^2_{Y_{s,\tau}}. \tag{6.26}$$

Similarly, by using that $e^x - 1 \leq xe^x$ for all $x \geq 0$, it follows that

$$|T_{14}| \leq \frac{C}{\alpha}\tau\|\omega\|^2_{H^1}\|\omega\|_{X_{s,\tau}} + \frac{C}{\alpha}\tau^2\|\omega\|_{H^1}\|\omega\|^2_{Y_{s,\tau}}. \tag{6.27}$$

Combining the estimates (6.23), (6.26), (6.25), and (6.27), and using that $\tau(t) \leq \tau(0) \leq C$, we obtain the desired estimate on T_1. To estimate T_2, we proceed similarly. Here we do not have a commutator, and all terms are estimated in absolute value in Fourier space. We omit details and refer the interested reader to [32, Proof of Lemma 2.5].

6.4. Proof of Estimate (4.22)

If we take the inner product of (2.6) with ω, and then with $\Delta\omega$, using the fact that $\int u\nabla\omega\Delta\omega = -\int \partial_k u_i\, \partial_i\omega_j\, \partial_k\omega_j$ by integrating by parts, we obtain

$$\frac{d}{2dt}\|\omega\|^2_{H^1} + \frac{\nu}{1+\alpha^2}\|\omega\|^2_{H^1} \leq C\|\nabla u\|_{L^\infty}\|\omega\|^2_{H^1} + |\langle\partial_k(\omega\cdot\nabla u), \partial_k\omega\rangle|. \tag{6.28}$$

The proof of (4.22) follows from the above estimate by using Hölder's inequality and Grönwall's inequality and assuming that we have

$$\|\omega\cdot\nabla u\|_{H^1} \leq C\|\nabla u\|_{L^\infty}\|\omega\|_{H^1}. \tag{6.29}$$

The latter can be proved by using the Bony's para-differential calculus [10]. This inequality is equivalent to proving that

$$\|\Delta_q(\omega\cdot\nabla u)\|_{L^2} \leq C2^{-q}a_q\|\nabla u\|_{L^\infty}\|\omega\|_{H^1},$$

for some $0 \leq a_q \in \ell^2(\mathbb{N})$ with $\sum a_q^2 \leq 1$. Let $\Delta_q(ab) = \Delta_q T_a b + \Delta_q T_b a + \Delta_q R(a,b)$, where

$$\Delta_q R(a,b) = \sum_{q'>q-3}\Delta_q(\Delta_{q'}a\tilde{\Delta}_{q'}b),$$

and

$$\Delta_q T_a b = \sum_{|q-q'|\leq 4}\Delta_q(S_{q'-1}b\Delta_{q'}a).$$

We have $\Delta_q(\omega\nabla u) = \Delta_q T_\omega\nabla u + \Delta_q T_{\nabla u}\omega + \Delta_q R(\nabla u, \omega)$. Using a Bernstein type inequality we have

$$\|S_{q'-1}\omega\|_{L^\infty} \leq C2^{2q'}\|\nabla u\|_{L^\infty}$$

and also

$$\|\Delta_{q'}\nabla u\|_{L^2} \leq C2^{-2q'}\sup_{|\alpha|=2}\|\Delta_{q'}\partial^\alpha\nabla u\|_{L^2} \leq C\alpha^{-1}2^{-2q}\|\Delta_{q'}\omega\|_{L^2}.$$

So, we obtain

$$\|\Delta_q T_\omega\nabla u\|_{L^2} \leq C\|\nabla u\|_{L^\infty}\|\Delta_{q'}\omega\|_{L^2} \leq C2^{-q}a_q\|\nabla u\|_{L^\infty}\|\omega\|_{H^1},$$

where $a_q \in \ell^2(\mathbb{N})$. Similarly, we have

$$\|\Delta_q T_{\nabla u}\omega\|_{L^2} \leq C\|\nabla u\|_{L^\infty}\|\Delta_{q'}\omega\|_{L^2} \leq C2^{-q}a_q\|\nabla u\|_{L^\infty}\|\omega\|_{H^1}.$$

Concerning the rest term, we have

$$\begin{aligned}\|\Delta_q R(\omega,\nabla u)\|_{L^2} &\leq \sum_{q'>q-3} \|\Delta_{q'}\omega\|_{L^\infty}\|\tilde{\Delta}_{q'}\nabla u\|_{L^2}\\ &\leq \sum_{q'>q-3} \|\nabla u\|_{L^\infty}\|\Delta_{q'}\omega\|_{L^2}\\ &\leq C\sum_{q'>q-3} 2^{-q'}a_{q'}\|\nabla u\|_{L^\infty}\|\omega\|_{H^1} \leq C2^{-q}\tilde{a}_q\|\nabla u\|_{L^\infty}\|\omega\|_{H^1} \qquad (6.30)\end{aligned}$$

where $\tilde{a}_q = \sum_{q'>q-3} 2^{-(q'-q)}a_{q'} \in \ell^2(\mathbb{N})$. This completes the proof.

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