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On the Two-Dimensional Euler Equations with Spatially Almost Periodic Initial Data

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Abstract. We consider the nonstationary Euler equations in $\mathbb{R}^2$ with almost periodic unbounded vorticity. We show that a unique solution is always spatially almost periodic at any time when the almost periodic initial data belongs to some function space. In order to prove this, we demonstrate the continuity with respect to initial data which do not decay at spatial infinity. The proof of the continuity with respect to initial data is based on that of Vishik's uniqueness theorem.

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1. Introduction

In the present paper, we consider the persistency of almost periodicity instead of periodicity to a two-dimensional ideal incompressible fluid. The motion of an ideal fluid in $\mathbb{R}^n$ is described by the Euler equations:

$$\text{(E)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p = 0, \quad \operatorname{div} u = 0 \quad \text{in } x \in \mathbb{R}^n,\ t \in (0,T), \\ \qquad\qquad u\,|_{t=0} = u_0, \end{cases}$$

where $u = (u^1(x,t), u^2(x,t), \cdots, u^n(x,t))$ and $p = p(x,t)$ denote the velocity vector field and the pressure of fluid at the point $(x,t) \in \mathbb{R}^n \times (0,T)$, respectively, while $u_0 = (u_0^1(x), u_0^2(x), \cdots, u_0^n(x))$ is the given initial velocity vector field.

Flows expressed by almost periodic functions are not only physically but also mathematically interesting. For example, several studies have considered the persistency of spatially almost periodicity on the Navier–Stokes equations [12, 13, 14, 15, 16] and general abstract evolution equations [10]. In particular Giga–Inui–Mahalov–Matsui [13] established the unique local existence for the Cauchy problem of the incompressible Navier–Stokes equations with the Coriolis force when the initial data may not decrease at spatial infinity so that almost periodic data is allowed. Moreover they showed a uniform estimate for existence time in any size of the Coriolis force. This result is very important to consider fast singular oscillating limits and prove global existence for large fixed Coriolis term when initial data is almost periodic function.

On the other hand, the persistency of almost periodicity on the Euler equations has not yet been considered. The purpose of the present paper is to prove the persistency of spatially almost periodicity of solutions to 2-D Euler equations. This is less obvious compared with the periodic case. In order to prove this, we demonstrate the continuity with respect to initial data. The proof of the continuity with respect to initial data is based on Vishik [27]. Now we give sketch of the proof. First, we decompose the difference between two solutions by using Littlewood–Paley decomposition into high and low frequency parts. For the low frequency part, we use Bony's paraproduct formula developed by Bahouri–Chemin [1] and a priori estimate of transport equations. It is easy to estimate the high frequency part. Combining these two estimates, we can apply a generalized version of Osgood's lemma which has never been used before. The lemma admits that vorticity can be taken from Yudovich class which is strictly bigger than BUC, bounded uniformly continuous functions. By our method, we obtain the continuity with respect to initial data in $L^1(0,T;B^0_{\infty,1})$. By an easy calculation, we can improve this estimate up to $L^\infty(0,T;B^0_{\infty,1})$.

Many researchers have investigated the two-dimensional Euler equations when the initial data has the decay property: $|u(x)| \to 0$ as $|x| \to \infty$ and $|\mathrm{rot}\ u(x)| \to 0$ as $|x| \to \infty$ in some sense. For example, DiPerna–Majda [8] showed that if $\omega_0 = \mathrm{rot}\ u_0 \in L^1 \cap L^p$ for $1 < p < \infty$, then there exist a weak solution u on $[0,\infty)$ with

$$u \in L^\infty(0,\infty; W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2)), \quad \omega = \mathrm{rot}\ u \in L^\infty(0,\infty; L^p(\mathbb{R}^2)).$$

Note that Giga–Miyakawa–Osada [17] independently proved a result similar to [8] without the assumption $\omega_0 \in L^1$ by using a different method. Chae [3] proved that if $\omega_0 \in L\log L$, then there exists a weak solution u on $[0,\infty)$ with

$$u \in L^\infty(0,\infty; L^2(\mathbb{R}^2)).$$

Concerning the uniqueness theorem, for the Euler equations in a bounded domain Ω, Yudovich [29] showed that a solution u satisfying

$$u \in L^\infty(0,T;L^2), \quad \omega = \mathrm{rot}\ u \in L^\infty(0,T;L^\infty)$$

is uniquely determined by the initial data u_0. Moreover, in [30], he proved the uniqueness theorem for unbounded vorticity rot u. He showed that, for the Euler equations in a bounded domain $\Omega \subset \mathbb{R}^n$, a solution u satisfying

$$u \in L^\infty(0,T;L^2(\Omega)), \quad \mathrm{rot}\ u \in L^\infty(0,T;V^\Theta)$$

is uniquely determined by the initial data u_0. Here, V^Θ introduced by Yudovich, is wider than $L^\infty(\Omega)$ and includes $\log^+\log^+(1/|x|)$. For details, see [30]. Recently, Vishik [27] showed the new uniqueness theorem for the solutions to (E) in the entire n-dimensional space $\mathbb{R}^n$. He proved that the uniqueness holds in the class

$$\omega \in L^\infty(0,T; L^p(\mathbb{R}^n) \cap B_\Gamma(\mathbb{R}^n)) \quad \text{for some } 1 < p < n, \tag{1.1}$$

where B_Γ is a space of Besov type and is wider than $B^0_{\infty,\infty}$ and **bmo**. Moreover, in the case $n = 2$, he also proved the global existence of solutions to (E) in the class (1.1). Besov spaces are also utilized, for example, by [26], [28], and [4].

On the other hand, with respect to flows having *non-decaying* velocity at spatial infinity, Chemin [6] proved the unique local existence of solutions in $C([0,T); C^{1+\alpha}(\mathbb{R}^n))$ to (E) with non-decaying initial data in $C^{1+\alpha}(\mathbb{R}^n)$ ($\alpha > 0$). Recently, Pak–Park [21] showed the unique local existence in $B^1_{\infty,1}(\mathbb{R}^n)$. In the two-dimensional case, Serfati [23] proved the unique global existence of a solution to (E) in $\mathbb{R}^2$ with initial data $(u_0, \omega_0) \in L^\infty \times L^\infty$. The first author [25] showed the global existence of a solution to (E) in $\mathbb{R}^2$ with $(u_0, \omega_0) \in L^\infty \times Y^\Theta_{ul}$, where Y^Θ_{ul} is wider than L^∞ and contains unbounded functions. (See also [31].) For the definition of Y^Θ_{ul}, see the next section. In the present paper, we will consider the persistency of almost periodicity of the solutions given in [25]. We also consider the uniqueness of the solution given in [25] in the case of $\Theta(q) = \log(q + e)$.

2. Preliminaries and main results

Before presenting our results, we present some definitions. Let $B(x,r)$ denote a ball of radius r centered at x, and let

$$L^p_{ul} := L^p_{\text{unif,loc}} = \{ f \in L^1_{\text{loc}}(\mathbb{R}^2); \|f\|_{L^p_{ul}} < \infty \},$$
$$\|f\|_{L^p_{ul}} := \sup_{x \in \mathbb{R}^2} \left(\int_{|x-y|<1} |f(y)|^p dy \right)^{1/p}.$$

For $m = 0, 1, 2, \cdots$, and $0 < \alpha < 1$, let $C^{m+\alpha}$ denote the uniform Hölder space:

$$\left\{ f\,;\ \sum_{|\beta| \le m} \|\partial^\beta f\|_\infty + \sum_{|\beta|=m} \sup_{x \ne y} \frac{|\partial^\beta f(x) - \partial^\beta f(y)|}{|x-y|^\alpha} < \infty \right\}.$$

We first introduce the definition of a weak solution to (E) as follows:

Definition 2.1 (weak solution). u is called a weak solution to (E) in $\mathbb{R}^n$ on $[0,T]$ with initial data u_0, if u satisfies the following conditions:

(i) $u \in L^2_{\text{loc}}(\mathbb{R}^n \times [0,T])$

(ii) $\displaystyle\int_0^T \int_{\mathbb{R}^n} \left\{ -u \cdot \frac{\partial}{\partial t}\varphi - \sum_{k,l=1}^n u^k u^l \frac{\partial}{\partial x_l} \varphi^k \right\} dxdt = \int_{\mathbb{R}^n} u_0 \cdot \varphi(0) dx$

for all $\varphi \in C_0^\infty([0,T) \times \mathbb{R}^n)$ with $\nabla \cdot \varphi = 0$,

(iii) $\nabla \cdot u = 0$ in $\mathcal{D}'((0,T) \times \mathbb{R}^n)$.

Now, we recall the Littlewood–Paley decomposition $\psi, \phi_j \in \mathcal{S}$, $j = 0, 1, \cdots$, such that

$$\operatorname{supp} \hat{\psi} \subset \{|\xi| \le 5/6\}, \quad \operatorname{supp} \hat{\phi} \subset \{3/5 \le |\xi| \le 5/3\}, \quad \phi_j(x) = 2^{nj}\phi(2^j x),$$

$$\begin{aligned} 1 &= \hat{\psi}(\xi) + \sum_{j=0}^{\infty} \hat{\phi}_j(\xi) \quad (\xi \in \mathbb{R}^n), \\ 1 &= \sum_{j=-\infty}^{\infty} \hat{\phi}_j(\xi) \quad (\xi \in \mathbb{R}^n \setminus \{0\}), \end{aligned} \tag{2.1}$$

where $\hat{f}$ denotes the Fourier transform of f.

Let

$$\begin{aligned} \Delta_{-1} f &:= \psi * f, \\ \Delta_j f &:= \phi_j * f, \quad j \ge 0, \\ \Delta_j f &:= 0, \qquad\quad j \le -2, \\ S_k f &:= \sum_{j=-1}^{k} \Delta_j f \end{aligned}$$

for $f \in \mathcal{S}'$.

Definition 2.2 (Besov Space cf. [2]). The inhomogeneous and homogeneous Besov spaces $B^s_{p,\rho}$ and $\dot{B}^s_{p,\rho}$ are defined as follows:

$$B^s_{p,\rho} \equiv \{f \in \mathcal{S}'; \|f\|_{B^s_{p,\rho}} < \infty\}, \quad \dot{B}^s_{p,\rho} \equiv \{f \in \mathcal{S}'; \|f\|_{\dot{B}^s_{p,\rho}} < \infty\},$$

where

$$\|f\|_{B^s_{p,\rho}} = \|\psi * f\|_p + \left(\sum_{j=0}^{\infty} \|2^{js}\phi_j * f\|_p^\rho\right)^{1/\rho}, \quad \|f\|_{\dot{B}^s_{p,\rho}} = \left(\sum_{j=-\infty}^{\infty} \|2^{js}\phi_j * f\|_p^\rho\right)^{1/\rho}$$

for $s \in \mathbb{R}$, $1 \le p, \rho \le \infty$.

Note that

$$\left\{ f \in \mathcal{S}';\ \|f\|_{\dot{B}^s_{p,\rho}} < \infty,\ f = \sum_{j=-\infty}^{\infty} \phi_j * f \text{ in } \mathcal{S}' \right\} \cong \dot{B}^s_{p,\rho}/\mathcal{P}, \tag{2.2}$$

holds if

$$s < n/p, \quad \text{or} \quad s = n/p \ \text{ and } \ \rho = 1. \tag{2.3}$$

For details, see [18]. Here, $\mathcal{P}$ denotes the set of all polynomials. Hence, when s, p, and ρ satisfy (2.3), we may modify the definition of homogeneous Besov space as

$$\dot{B}^s_{p,\rho} \equiv \left\{ f \in \mathcal{S}';\ \|f\|_{\dot{B}^s_{p,\rho}} < \infty,\ f = \sum_{j=-\infty}^{\infty} \phi_j * f \text{ in } \mathcal{S}' \right\}. \tag{2.4}$$

Hereinafter we use (2.4) as the definition of $\dot{B}^s_{p,\rho}$ when s, p, and ρ satisfy (2.3). Then, if s, p, and ρ satisfy (2.3), $\dot{B}^s_{p,\rho}$ is a Banach space and

$$\|f\|_{\dot{B}^s_{p,q}} = 0 \text{ if and only if } f = 0 \text{ in } \mathcal{S}'.$$

Next, we define the Riesz operator $R_k := \partial_k(-\Delta)^{-1/2}$ $(k = 1, 2 \cdots, n)$ on Besov spaces. Let s, p, and ρ satisfy (2.3), and let $f \in \dot{B}^s_{p,\rho}$. Then, $R_k f$ can be defined by

$$R_k f \equiv \sum_{j=-\infty}^{\infty} (R_k \tilde{\phi}_j) * \phi_j * f \text{ in } \mathcal{S}' \tag{2.5}$$

where $\tilde{\phi}_j = \phi_{j-1} + \phi_j + \phi_{j+1}$. We note that $\hat{\tilde{\phi}}_j \hat{\phi}_j = \hat{\phi}_j$. Using this definition, we see that R_k is a bounded operator in $\dot{B}^s_{p,\rho}$ as a subspace of $\mathcal{S}'$, if s, p, and ρ satisfy (2.3). In particular, R_k is bounded in $\dot{B}^0_{\infty,1}$.

Yudovich [30] introduced a function space to prove the uniqueness theorem for the Euler equations. Here, we introduce a slightly modified version of his space. See also [19, 20].

Definition 2.3. Let $\Theta(p) \geq 1$ be a nondecaying function on $[1, \infty)$.
$Y^\Theta_{ul}(\mathbb{R}^n) \equiv \left\{f \in \bigcap_{1\leq p<\infty} L^p_{ul}(\mathbb{R}^n); \|f\|_{Y^\Theta_{ul}(\mathbb{R}^n)} < \infty\right\}$, where

$$\|f\|_{Y^\Theta_{ul}(\mathbb{R}^n)} := \sup_{p\geq 1} \frac{\|f\|_{L^p_{ul}(\mathbb{R}^n)}}{\Theta(p)},$$

$$\|f\|_{L^p_{ul}(\mathbb{R}^n)} := \sup_{x\in\mathbb{R}^n} \left(\int_{|y-x|\leq 1} |f(y)|^p dy\right)^{1/p}.$$

Then, clearly it holds that

$$L^\infty(\mathbb{R}^n) \subset Y^\Theta_{ul}(\mathbb{R}^n).$$

Moreover, we observe that $Y^\Theta_{ul}(\mathbb{R}^2)$ includes $\log(e + \log(e + 1/|x|))$ if $\Theta(p) = \log(p + e)$.

We now recall the space of Besov type introduced by Vishik [27].

Definition 2.4 (Vishik [27]). Let $\Gamma(\alpha)(\geq 1)$ be a nondecreasing function on $[1, \infty)$. $B_\Gamma = \{f \in \mathcal{S}'; \|f\|_{B_\Gamma} < \infty\}$ is introduced by the norm

$$\|f\|_{B_\Gamma} = \sup_{N=1,2,\cdots} \frac{\|\Delta_{-1} f\|_\infty + \sum_{j=0}^N \|\Delta_j f\|_\infty}{\Gamma(N)}.$$

Note that $\|f\|_{B_\Gamma} \leq C\|f\|_{Y^\Theta_{ul}} \leq C\|f\|_\infty$, if $\Gamma(N) = N\Theta(N)$. See Proposition 2.15.

Let us next define "*Almost periodic functions*".

Definition 2.5. Let X be some Banach space included in BUC. We say that $u \in X(\mathbb{R}^n)$ is *almost periodic* in X if a function u satisfies the following property.

For any $\epsilon > 0$, there exists a number $l(\epsilon) > 0$ with the property that any cube of length $l(\epsilon)$ of $\mathbb{R}^n$ contains at least one point with abscissa ξ such that

$$\|u(\cdot + \xi) - u(\cdot)\|_X < \epsilon.$$

The totality of almost periodic functions in X is denoted by $AP_X(\mathbb{R}^n)$.

We need the following well known lemma to treat the *frequency set*.

Lemma 2.6 ([7, Theorem 1. 24]). *Assume that* $X = BUC$ *(with the norm* $\|\cdot\|_{L^\infty}$*). For all* $u \in AP_{BUC}(\mathbb{R}^n)$*, there exists a sequence of trigonometric polynomials*

$$\sigma_m(x) = \sum_{k=1}^{\ell(m)} r_{k,m} c_k e^{i\lambda_k x}, \quad \{\lambda_k\}_k \subset \mathbb{R}^n, \quad \{c_k\}_k \subset \mathbb{C}^n, \quad \{r_{k,m}\}_{k,m} \subset \mathbb{Q}, \tag{2.6}$$

such that the numbers $r_{k,m}$ *converge to* 1 $(m \to \infty)$ *for all* $k \in \mathbb{N}$ *and* $\sigma_m(x)$ *converges uniformly to* $u(x)$ *on* $\mathbb{R}^n$ *as* $m \to \infty$ $(\ell(m) \to \infty)$*. The numbers* c_k *are uniquely determined as follows:*

$$c_k = \lim_{\gamma \to \infty} \frac{1}{|Q_\gamma|} \int_{Q_\gamma} u(x) e^{i\lambda_k x} dx, \tag{2.7}$$

where $Q_\gamma = \{x \in \mathbb{R}^n : |x_j| < \gamma, j = 1, \cdots n\}$*.* $\{r_{k,m}\}$ *depend on* $\{\lambda_k\}_k$ *and* m*, but not on* $\{c_k\}_k$*.*

Remark 2.7. Since $B^0_{\infty,1}$ is continuously embedded in BUC, the function $u \in AP_{B^0_{\infty,1}}(\mathbb{R}^n)$ can be expressed as $\lim_{m\to\infty} \sigma_m(x)$, where $\sigma_m(x)$ is determined as (2.6).

The following lemma is key in the present paper.

Lemma 2.8. *Let* F *be a nonnegative and continuous function on* $(0, T)$ *with* $\int_0^T F(t)dt < \infty$*, and let* $\Theta(s)$ *be continuous for* $s \geq 0$ *and nondecreasing on* $[0, \gamma]$ *for some* $\gamma > 0$ *and* $\Theta(0) = 0$*,* $\Theta(s) > 0$ *if* $s > 0$*. Assume that there exists a function* $g(\alpha)$ *such that*

$$\lim_{\alpha \to +0} g(\alpha) = 0, \quad g(\alpha) > \alpha \ \text{ for sufficiently small } \ \alpha > 0, \tag{2.8}$$

$$\liminf_{\alpha \to +0} \int_\alpha^{g(\alpha)} ds/\Theta(s) > \int_0^T F(t)dt. \tag{2.9}$$

Then, there exists a constant $\alpha_0 = \alpha_0(F, \Theta, g, \gamma, T) > 0$ *such that the inequality*

$$\sup_{0 \leq t \leq T} v(t) \leq g(\alpha) \tag{2.10}$$

holds for all $(v, \alpha) \in C([0,T)) \times (0, \alpha_0]$ *satisfying*

$$0 \le v(t) \le \alpha + \int_0^t F(\tau)\Theta(v(\tau))d\tau \quad \textit{for all} \quad 0 \le t \le T. \tag{2.11}$$

Remark 2.9. If $\Theta(s) = s$, $g(\alpha) = e^{2\int_0^T F dt} \cdot \alpha$ satisfies the hypothesis of the above lemma. If $\Theta(s) = s \log(\frac{1}{s} + 1)$, $g(\alpha) = \alpha^p$ $(0 < p < 1)$ satisfies the hypothesis of the above lemma. If $\Theta(s) = s \log(\frac{1}{s} + e) \cdot \log(\log(\frac{1}{s} + e) + e)$, $g(\alpha) = 1/\log(1/\alpha + e)$ satisfies the hypothesis of the above lemma.

Proof. First assume that for all $\alpha_0 > 0$ there exist a constant $\alpha \in (0, \alpha_0]$ and a function $v \in C([0,T))$ such that (2.11) holds for all $0 \le t \le T$ and $\sup_{0 \le t \le T} v(t) > g(\alpha)$. Then, for all $n > 1/\gamma$ there exist $\alpha_n \in (0, \frac{1}{n}] (\subset (0, \gamma))$ and $v_n \in C([0,T])$ such that

$$0 \le v_n(t) \le \alpha_n + \int_0^t F(\tau)\Theta(v_n(\tau))d\tau \quad \text{for all} \quad 0 \le t \le T, \tag{2.12}$$

$$\sup_{0 \le t \le T} v_n(t) > g(\alpha_n). \tag{2.13}$$

Set $V_n(t) := \alpha_n + \int_0^t F(\tau)\Theta(v_n(\tau))d\tau$. By (2.12) and (2.13) we have

$$V_n(T) = \sup_{0 \le t \le T} V_n(t) \ge \sup_{0 \le t \le T} v_n(t) > g(\alpha_n).$$

Since $V_n(0) = \alpha_n$ and V_n is nondecreasing, we see that there exist $\tau_{n,0}$ and $\tau_{n,1}$ such that

$$0 \le \tau_{n,0} < \tau_{n,1} \le T, \tag{2.14}$$

$$V_n(t) = \alpha_n \text{ for } 0 \le t \le \tau_{n,0}, \quad V_n(\tau_{n,1}) = \min(g(\alpha_n), \gamma), \tag{2.15}$$

$$\alpha_n < V_n(t) < \min(g(\alpha_n), \gamma) \text{ for } \tau_{n,0} < t < \tau_{n,1}. \tag{2.16}$$

Since $\Theta(s)$ is nondecreasing on $[0, \gamma]$, we have $V_n'(t) = F(t)\Theta(v_n(t)) \le F(t)\Theta(V_n(t))$ for $t \in (\tau_{n,0}, \tau_{n,1})$. Then,

$$\int_{\alpha_n}^{\min(g(\alpha_n),\gamma)} \frac{1}{\Theta(s)} ds = \int_{\tau_{n,0}}^{\tau_{n,1}} \frac{V_n'(t)}{\Theta(V_n(t))} dt \le \int_{\tau_{n,0}}^{\tau_{n,1}} F(t)dt \le \int_0^T F(t)dt.$$

Letting $n \to \infty$, we find a contradiction to the assumption (2.9). This proves Lemma 2.8. □

The existence of global solutions of two-dimensional Euler equations in some function space has already been proven.

Theorem 2.10 ([25, Theorem 1.1]). *Let* $n = 2$ *and* $\Theta(q) = \log(q + e)$. *Assume* $u_0 \in L^\infty(\mathbb{R}^2)$ *and* $\omega_0 = \mathrm{rot}\ u_0 \in Y_{ul}^\Theta(\mathbb{R}^2)$. *Then, there exists a weak solution to*

(E) *on* $[0,\infty)$ *in the class*

$$\begin{aligned} u &\in C([0,\infty);L^\infty) \\ \text{rot } u &\in L^\infty_{\text{loc}}([0,\infty);Y^\Theta_{ul}). \end{aligned} \tag{2.17}$$

Remark 2.11. (a) Let $p=\sum_{k,l=1}^2 R_kR_lu^ku^l$. Then, the pair (u,p) satisfies (E) in $\mathcal{D}'((0,T)\times\mathbb{R}^2)$. This is shown in the Appendix. (b) The above global existence theorem also holds for the case $\Theta(q)=q$. See [25].

In order to prove our main theorem, we show the continuity with respect to initial data.

Theorem 2.12. *Let* $\Theta(q)=\log(q+e)$. *Assume that two weak solutions* u *and* v *to* (E) *in* $\mathbb{R}^n$ *on* $[0,T]$ *satisfy*

$$u,\ v\ \in C([0,T];L^\infty), \tag{2.18}$$

$$\text{rot } u,\ \text{rot } v\ \in L^\infty(0,T;Y^\Theta_{ul}) \tag{2.19}$$

and assume that (u,p_1) *and* (v,p_2) *satisfy* (E) *in* $\mathcal{D}'((0,T)\times\mathbb{R}^n)$. *Furthermore, assume*

$$p_1=\sum_{k,l=1}^n R_kR_lu^ku^l,\quad p_2=\sum_{k,l=1}^n R_kR_lv^kv^l \ \text{ in } BMO. \tag{2.20}$$

Then, there exists $\alpha_0=\alpha_0(T)$ *such that if* $T\|u(0)-v(0)\|_{B^0_{\infty,1}}<\alpha_0$, *then it holds that*

$$\|u(t)-v(t)\|_{B^0_{\infty,1}}\le\Phi(T\|u(0)-v(0)\|_{B^0_{\infty,1}})+\|u(0)-v(0)\|_{B^0_{\infty,1}} \tag{2.21}$$

for any $t\in[0,T]$, *where* $\Phi(\alpha)=C\Gamma_1(\log(e+1/h(\alpha)))h(\alpha)$, $h(\alpha)=1/\log(1/\alpha+1)$, $\Gamma_1(\beta):=\beta\log(\beta+e)$.

Remark 2.13. (i) The function $\Phi(\alpha)$ is a monotonically increasing function, and $\Phi(0)=0$.

(ii) Theorem 2.12 with $L^\infty(0,T;Y^\Theta_{ul})$ replaced by $L^\infty(0,T;B_{\Gamma_1})$ holds for $\Gamma_1(N)=N\cdot\log(N+e)$.

(iii) Clearly, Theorem 2.12 yields the *uniqueness* of solutions to (E) within the class (2.18)–(2.19) with (2.20). The global-in-time existence of weak solutions u to the 2-D Euler equations in the class (2.18)–(2.19) with (2.20) was proven in [25]. See also the Appendix of the present paper. Note that Shen–Zhu [24] proved the global existence of axisymmetric solutions to 3-D Euler equations in

$$u\in C\big([0,\infty);L^2(\mathbb{R}^3)\cap B^1_{\infty,\infty}\big),\quad \text{rot } u\in L^\infty_{\text{loc}}\big([0,\infty);L^\infty(\mathbb{R}^3)\big). \tag{2.22}$$

Moreover, they showed the uniqueness of such solutions within the class of axisymmetric solutions in (2.22). Since they assumed axisymmetry on the uniqueness class and the boundedness of vorticity, the present uniqueness result is an improvement of the uniqueness result of Shen–Zhu. (It is straightforward to see that any weak solution in (2.22) has an associate pressure represented by (2.20).)

(iv) Condition (2.20) is necessary to prove the uniqueness. If this condition is not imposed, then the uniqueness fails. In fact, for any vector $\vec{a}$, $(u,p) = (\vec{a}t, -\vec{a}\cdot x)$ is a solution with initial data $u(0) = 0$. See, for example, [11].

(v) By the above uniqueness result, it is easy to prove that the solution u to (E) is periodic in x if u_0 is periodic.

(vi) We can generalize Theorem 2.12 as follows. For

$$\Theta(p) = \underbrace{\log(e+p)\cdot\log(e+\log(e+p))\cdot\log(e+\log(e+\log(e+p)))\cdots}_{\text{finite times iterated}},$$

(2.21) is replaced by

$$\|u(t)-v(t)\|_{B^0_{\infty,1}} \le \tilde{\Phi}\big(T\|u(0)-v(0)\|_{B^0_{\infty,1}}\big) + \|u(0)-v(0)\|_{B^0_{\infty,1}}, \qquad (2.23)$$

where $\tilde{\Phi}$ is some appropriate function with $\lim_{\delta\to+0}\tilde{\Phi}(\delta) = +0$.

We now state our main theorem.

Theorem 2.14. *Let* $\Theta(q) = \log(q+e)$. *Assume that* $u_0 \in AP_{B^0_{\infty,1}}(\mathbb{R}^2)$ *and* rot $u_0 \in Y^{\Theta}_{ul}$, *which has a Fourier expansion in the form*

$$u_0(x) = \lim_{m\to\infty}\sum_{k=1}^{\ell(m)} r_{k,m}c_k e^{i\lambda_k x}, \quad \Lambda := \{\lambda_k\}_k \subset \mathbb{R}^2,$$

where $c_k \in \mathbb{C}^2$ *for* $k = 1, 2, \cdots$. *(By simple analogy, we set* $\{c_\lambda\}_{\lambda\in\Lambda} := \{c_k\}_k$*.)*

Assume, moreover, that (u,p) *is a unique solution to* (E) *with*

$$u \in C([0,\infty); L^\infty), \quad \text{rot } u \in L^\infty_{\text{loc}}([0,\infty); Y^{\Theta}_{ul}), \quad p = \sum_{k,l=1}^{2} R_k R_l u^k u^l.$$

Then, the solution $u(t)$ *also belongs to* $AP_{B^0_{\infty,1}} \subset AP_{BUC}$ $(t > 0)$, *which has a Fourier expansion of the form*

$$u(x,t) = \lim_{m\to\infty}\sum_{j=1}^{\ell(m)} r_{j,m}c_j(t)e^{i\mu_j x}, \quad G := \{\mu_j\}_j \subset \mathbb{R}^2,$$

where $c_j(t) \in \mathbb{C}^2\times[0,\infty)$ *for* $j = 1, 2, \cdots$. *(By simple analogy, we set* $\{c_\mu(t)\}_{\mu\in G} := \{c_j(t)\}_j$*.) Moreover,* $c_j(t) \in C([0,\infty))$ *for* $j = 1, 2, \cdots$, *a frequency set of initial data* Λ *is included in* G, *and* $c_\mu(0) = c_\lambda$ *if* $\mu = \lambda \in \Lambda$, $c_\mu(0) = 0$ *if* $\mu \notin \Lambda$.

Before closing this section, we state a basic proposition.

Proposition 2.15. (i) *Let* $1 \le q \le \infty$, $j = 0, \pm 1, \pm 2, \cdots$, $\phi \in \mathcal{S}$, *and let* $f \in L^q_{ul}(\mathbb{R}^n)$. *Then, the following holds:*

$$\|\phi_j * f\|_\infty \le \begin{cases} C 2^{nj/q} \|f\|_{L^q_{ul}} & \text{for all } \ j \ge 0, \\ C\|f\|_{L^q_{ul}} & \text{for all } \ j \le -1, \end{cases} \tag{2.24}$$

where C *is independent of* q, j, *and* f. *Here,* $\phi_j(\cdot) = 2^{nj}\phi(2^j \cdot)$.

(ii) *Let* Γ *and* Θ *be nondecreasing functions on* $[1, \infty)$ *with* $\Gamma(N) \ge N\Theta(N)$ *for all* $N = 1, 2, \cdots$. *Then,*

$$\|f\|_{B_\Gamma} \le C\|f\|_{Y^\Theta_{ul}}. \tag{2.25}$$

Proof of Proposition 2.15. The proof of (i) is given in [25]. Here, we only prove (ii). By (2.24), we have

$$\begin{aligned} \|\Delta_{-1} f\|_\infty + \sum_{0 \le j \le N} \|\Delta_j f\|_\infty &\le C\|f\|_{L^q_{ul}} + C \sum_{0 \le j \le N} 2^{nj/q} \|f\|_{L^q_{ul}} \\ &\le CN 2^{nN/q} \|f\|_{L^q_{ul}} \le CN 2^{nN/q} \Theta(q) \|f\|_{Y^\Theta_{ul}} \end{aligned}$$

for all $q \ge 1$, where C is independent of q and N. Then, letting $q = N$, we obtain the desired inequality (2.25). $\square$

3. Proof of Theorems

Proof of Theorem 2.12. The proof is based on Vishik [27]. Let $w := u - v$ and $p = p_1 - p_2$. Then, by (E) we have

$$\partial_t \Delta_l w + \sum_{k=1}^n \partial_k \Delta_l (u^k w) + \sum_{k=1}^n \partial_k \Delta_l (w^k v) + \nabla \Delta_l p = 0 \tag{3.1}$$

for all $l \ge -1$. Let M be a constant such that

$$M \ge \sup_{0 \le t \le T} \big(\|u\|_\infty + \|v\|_\infty + \|\text{rot } u\|_{Y^\theta_{ul}} + \|\text{rot } v\|_{Y^\theta_{ul}} \big). \tag{3.2}$$

Then, for $j \ge 0$,

$$\begin{aligned} \|S_j \nabla u\|_\infty &\le \sum_{m=-1}^{j} \|\Delta_m \nabla u\|_\infty \le C \sum_{m=-1}^{j} 2^m \|\Delta_m u\|_\infty \\ &\le C\bigg(\|\Delta_{-1} u\|_\infty + \sum_{m=0}^{j} \|\Delta_m \text{rot } u\|_\infty \bigg) \\ &\le CM\Gamma_1(j), \end{aligned} \tag{3.3}$$

where

$$\Gamma_1(j) = j \log(j + e)$$

Similarly, we have, for all $j \geq 0$,

$$\|S_j \nabla v\|_\infty + \|S_j \nabla w\|_\infty \leq CM\Gamma_1(j). \tag{3.4}$$

By the above calculation, we also have

$$\sum_{m=-1}^{j} 2^m \|\Delta_m u\|_\infty + \sum_{m=-1}^{j} 2^m \|\Delta_m v\|_\infty + \sum_{m=-1}^{j} 2^m \|\Delta_m w\|_\infty \leq CM\Gamma_1(j). \tag{3.5}$$

By Bahouri–Chemin [1] and Vishik [27], we can transform (3.1) into

$$\partial_t \Delta_l w + (S_{l-2}u) \cdot \nabla \Delta_l w = \sum_{k=1}^{n} (I^{l,k} + J^{l,k}) + K^l, \tag{3.6}$$

where

$$\begin{aligned} I^{l,k} &= (S_{l-2}u) \cdot \nabla \Delta_l w - \partial_k \Delta_l (u_k w), \\ J^{l,k} &= -\partial_k \Delta_l (w^k v), \\ K^l &= -\nabla \Delta_l p. \end{aligned} \tag{3.7}$$

Step 1. (Estimate of $J^{l,k}$.) From Bony's paraproduct formula, we obtain

$$\begin{aligned} J^{l,k} =& -\partial_k \Delta_l \sum_{|j-l|\leq 3} S_{j-2} w^k \cdot \Delta_j v - \partial_k \Delta_l \sum_{|j-l|\leq 3} \Delta_j w^k \cdot S_{j-2} v \\ & - \partial_k \Delta_l \sum_{|j-l|\leq 1,\ \max(j,j')\geq l-3} \Delta_j w^k \cdot \Delta_{j'} v \\ =:& \, J_1^{l,k} + J_2^{l,k} + J_3^{l,k}. \end{aligned} \tag{3.8}$$

By (3.5), we have

$$\begin{aligned} \sum_{l=-1}^{N} \|J_1^{l,k}\|_\infty &\leq \sum_{l=-1}^{N} 2^l \sum_{|j-l|\leq 3} \|S_{j-2} w^k\|_\infty \|\Delta_j v\|_\infty \\ &\leq C\|w\|_\infty \sum_{l=-1}^{N} 2^l \sum_{|j-l|\leq 3} \|\Delta_j v\|_\infty \\ &\leq C\|w\|_\infty \sum_{l=-1}^{N} \sum_{|j-l|\leq 3} 2^{j+3} \|\Delta_j v\|_\infty \leq CM\Gamma_1(N+3)\|w\|_{B^0_{\infty,1}}. \end{aligned} \tag{3.9}$$

By (3.4) and $\sum_k \partial_k w_k = 0$, we have

$$\begin{aligned}\sum_{l=-1}^{N}\|J_2^{l,k}\|_\infty &\le \sum_{l=-1}^{N}\sum_{|j-l|\le 3}\|\Delta_l(\Delta_j w^k\cdot\partial_k S_{j-2}v)\|_\infty\\ &\le C\sum_{l=-1}^{N}\sum_{|j-l|\le 3}\|\Delta_j w\|_\infty\|S_{j-2}\nabla v\|_\infty\\ &\le CM\Gamma_1(N+1)\sum_{l=-1}^{N}\sum_{|j-l|\le 3}\|\Delta_j w\|_\infty \le CM\Gamma_1(N+1)\|w\|_{B^0_{\infty,1}}. \end{aligned} \tag{3.10}$$

By (3.5), we have

$$\begin{aligned}\sum_{l=-1}^{N}\|J_3^{l,k}\|_\infty &\le \sum_{l=-1}^{N}\sum_{|j-j'|\le 3,\ \max(j,j')\ge l-3} 2^l\|\Delta_j w^k\|_\infty\|\Delta_{j'}v\|_\infty\\ &\approx C\sum_{l=-1}^{N}\sum_{j\ge l-3}2^l\|\Delta_j w\|_\infty\|\Delta_j v\|_\infty\\ &= \sum_{l=-1}^{N}\sum_{\alpha\ge -3}2^l\|\Delta_{l+\alpha}w\|_\infty\|\Delta_{l+\alpha}v\|_\infty\\ &= \sum_{\alpha\ge -3}2^{-\alpha}\sum_{l=-1}^{N}\|\Delta_{l+\alpha}w\|_\infty 2^{l+\alpha}\|\Delta_{l+\alpha}v\|_\infty\\ &\le CM\|w\|_{B^0_{\infty,1}}\sum_{\alpha\ge -3}2^{-\alpha}\Gamma_1(N+\alpha)\\ &\le CM\|w\|_{B^0_{\infty,1}}\Gamma_1(N). \end{aligned} \tag{3.11}$$

Here, we used the following inequality:

$$\sum_{\alpha\ge -3}2^{-\alpha}\Gamma_1(N+\alpha)\le \int_0^\infty 2^{-x}(N+x)\log(N+x)dx \le N\log N\le \Gamma_1(N).$$

Hence, by (3.9), (3.10), and (3.11), we obtain

$$\sum_{l=-1}^{N}\|J^{l,k}\|_\infty \le CM\|w\|_{B^0_{\infty,1}}\Gamma_1(N). \tag{3.12}$$

Step 2. (Estimate of K^l.) Recall that $\tilde{\Delta}_l = \Delta_{l-1}+\Delta_l+\Delta_{l+1}$ ($\tilde{\phi}_l = \phi_{l-1} + \phi_l + \phi_{l+1}$). Since

$$K^l = -\nabla\Delta_l\sum_{i,i'=1}^{n}R_iR_{i'}(w^iu^{i'}+v^iw^{i'}) = -\nabla\Delta_l(-\Delta)^{-1}\sum_{i,i'=1}^{n}\partial_i\partial_{i'}(w^iu^{i'}+v^iw^{i'}),$$

we have, for $l \geq 0$,

$$K^l = -\nabla(-\Delta)^{-1} \sum_{i,i'=1}^{n} \partial_{i'} \tilde{\Delta}_l \partial_i \Delta_l (w^i u^{i'}) - \nabla(-\Delta)^{-1} \sum_{i,i'=1}^{n} \partial_{i'} \tilde{\Delta}_l \partial_i \Delta_l (v^i w^{i'}). \tag{3.13}$$

Recall that $\|\nabla(-\Delta)^{-1}\partial_i \tilde{\Delta}_l\|_{B(L^\infty, L^\infty)} \leq C$ for all $l \geq 0$, where C is independent of l. Then, we see

$$\sum_{l=0}^{N} \|K^l\|_\infty \leq C \sum_{l=0}^{N} \sum_{i,i'=1}^{n} \|\partial_i \Delta_l (w^i u^{i'})\|_\infty + C \sum_{l=0}^{N} \sum_{i,i'=1}^{n} \|\partial_{i'} \Delta_l (w^{i'} v^i)\|_\infty. \tag{3.14}$$

In the same manner used to estimate $J^{l,k} = -\partial_k \Delta_l (w^k v)$ in Step 1, we obtain

$$\sum_{l=0}^{N} \|K^l\|_\infty \leq CM\Gamma_1(N)\|w\|_{B^0_{\infty,1}}. \tag{3.15}$$

For $l = -1$, we observe that

$$\begin{aligned} \|K^l\|_\infty &= \left\| \sum_{i,i'=1}^{n} R_i R_{i'} \nabla \Delta_{-1} (w^i u^{i'} + v^i w^{i'}) \right\|_\infty = \|R_i R_{i'} \nabla \psi * (w^i u^{i'} + v^i w^{i'})\|_\infty \\ &\leq \|R_i R_{i'} \nabla \psi\|_{L^1} \|(w^i u^{i'} + v^i w^{i'})\|_\infty \leq C\|w\|_\infty (\|u\|_\infty + \|v\|_\infty), \end{aligned} \tag{3.16}$$

since $\|R_i R_{i'} \nabla \psi\|_{L^1} \leq C \|\nabla \psi\|_{\mathcal{H}^1} < \infty$. Here, $\mathcal{H}^1$ denotes the Hardy space. From (3.15) and (3.16), we obtain

$$\sum_{l=-1}^{N} \|K^l\|_\infty \leq CM\Gamma_1(N)\|w\|_{B^0_{\infty,1}}. \tag{3.17}$$

Step 3. (Estimate of I^l.) From [27, p. 793 Theorm 6.1], we see that

$$\begin{aligned} \sum_{k=1}^{n} \|I^{l,k}\|_\infty \leq C & \sum_{|j-l| \leq 3} \left\{ \|S_{j-2} \nabla w\|_\infty \|\Delta_j u\|_\infty + \|S_{j-2} \nabla u\|_\infty \|\Delta_j w\|_\infty \right\} \\ & + C 2^l \sideset{}{'}\sum_{j \geq l-3,\ |j-j'| \leq 1} 2^{-j} \|\Delta_j \nabla u\|_\infty \|\Delta_{j'} w\|_\infty. \end{aligned}$$

When $j = -1$, the factor $\|\Delta_j \nabla u\|_\infty$ in the last sum should be replaced by $\|\Delta_{-1} u\|_\infty$. Hereinafter, the notation $\sideset{}{'}\sum$ is used to indicate this convention: $\|\Delta_{-1} u\|_\infty$ instead of $\|\Delta_{-1} \nabla u\|_\infty$.

Then, we have

$$\sum_{k=1}^{n} \|I^{l,k}\|_\infty \le C \sum_{|j-l|\le 3} \left\{ \|w\|_\infty 2^j \|\Delta_j u\|_\infty + \|S_{j-2}\nabla u\|_\infty \|\Delta_j w\|_\infty \right\} + C2^l \sum_{j\ge l-3,\ |j-j'|\le 1} \|\Delta_j u\|_\infty \|\Delta_{j'} w\|_\infty =: X_1^l + X_2^l + X_3^l. \tag{3.18}$$

It is straightforward to see that

$$\sum_{l=-1}^{N} X_1^l \le CM\Gamma_1(N+3)\|w\|_{B^0_{\infty,1}} \tag{3.19}$$

and

$$\sum_{l=-1}^{N} X_2^l \le C \sum_{l=-1}^{N} \sum_{|j-l|\le 3} M\Gamma_1(j)\|\Delta_j w\|_\infty \le CM\Gamma(N) \sum_{l=-1}^{N} \sum_{|j-l|\le 3} \|\Delta_j w\|_\infty \le CM\Gamma_1(N)\|w\|_{B^0_{\infty,1}}.$$

By the same argument as in (3.11), we obtain

$$\sum_{l=-1}^{N} X_3^l = \sum_{l=-1}^{N} \sum_{j\ge l-3,\ |j-j'|\le 1} 2^l \|\Delta_j u\|_\infty \|\Delta_{j'} w\|_\infty \approx \sum_{l=-1}^{N} \sum_{j\ge l-3} 2^l \|\Delta_j u\|_\infty \|\Delta_j w\|_\infty \le CM\Gamma_1(N)\|w\|_{B^0_{\infty,1}}. \tag{3.20}$$

Then, we have

$$\sum_{l=-1}^{N} \|I^l\|_\infty \le CM\Gamma_1(N)\|w\|_{B^0_{\infty,1}}. \tag{3.21}$$

We consider the trajectory flow $\{\Phi_l(x,t)\}$ along $S_{l-2}u$ defined by the solution of the ordinary differential equations

$$\begin{cases} \dfrac{\partial}{\partial t}\Phi_l(x,t) = (S_{l-2}u)(\Phi_l(x,t),t) \\ \Phi_l(x,0) = x. \end{cases}$$

By (3.6) we have

$$\Delta_l w(\Phi_l(x,t),t) = \Delta_l w(x,0) + \int_0^t \left(\sum_{k=1}^{n} (I^{l,k} + J^{l,k}) + K^l \right) (\Phi_l(x,\tau),\tau)\,d\tau.$$

which yields

$$\|\Delta_l w(t)\|_\infty \le \|\Delta_l w(0)\|_\infty + \int_0^t \left(\sum_{k=1}^n \left(\|I^{l,k}\|_\infty + \|J^{l,k}\|_\infty\right) + \|K^l\|_\infty\right) d\tau.$$

From (3.12), (3.17), and (3.21), we obtain

$$\sum_{l=-1}^N \|\Delta_l w(t)\|_\infty \le \|w(0)\|_{B^0_{\infty,1}} + CM\Gamma_1(N)\int_0^t \|w(\tau)\|_{B^0_{\infty,1}} d\tau. \tag{3.22}$$

Now, we calculate the high-frequency part. By (3.4), we have

$$\begin{aligned}\sum_{l>N}\|\Delta_l w(t)\|_\infty &\le C\sum_{l>N}2^{-l}\|\Delta_l \nabla w(t)\|_\infty\\ &\le C\sum_{l>N}2^{-l}\|S_{l+1}\nabla w(t)\|_\infty \le C\sum_{l>N}2^{-l}\Gamma_1(l)M.\end{aligned}$$

Since

$$\begin{aligned}\sum_{l\ge N}2^{-l}\Gamma_1(l) \le \int_N^\infty 2^{-x}x\log x\ dx &= 2^{-N}\int_0^\infty 2^{-x}(N+x)\log(N+x)dx\\ &\sim 2^{-N}N\log N \le 2^{-N}\Gamma_1(N),\end{aligned}$$

we have

$$\sum_{l>N}\|\Delta_l w(t)\|_\infty \le CM2^{-N}\Gamma_1(N). \tag{3.23}$$

Hence, from (3.22) and (3.23), we obtain

$$\|w(t)\|_{B^0_{\infty,1}} \le \|w(0)\|_{B^0_{\infty,1}} + CM\Gamma_1(N)\left(\int_0^t \|w(\tau)\|_{B^0_{\infty,1}}d\tau + 2^{-N}\right). \tag{3.24}$$

for all $N = 1, 2, \cdots$, where C is independent of N. Let $h(t) := \int_0^t \|w(\tau)\|_{B^0_{\infty,1}}d\tau$, and let N be chosen such that

$$\begin{aligned} &N = 1, && \text{if } h(t) \ge 1,\\ &N \sim \log_2(1/h(t)), && \text{if } 0 < h(t) < 1,\\ &N = \infty, && \text{if } h(t) = 0.\end{aligned} \tag{3.25}$$

Then,

$$\|w(t)\|_{B^0_{\infty,1}} \le \|w(0)\|_{B^0_{\infty,1}} + C\Gamma_1(\log(e + 1/h(t)))h(t), \tag{3.26}$$

which yields $h(t) \le T\|w(0)\|_{B^0_{\infty,1}} + C\displaystyle\int_0^t \Gamma_1(\log(e + 1/h(\tau)))h(\tau)d\tau$. Therefore, from Lemma 2.8 we obtain

$$h(T) \le \frac{1}{\log\left(\frac{1}{T\|u(0)-v(0)\|_{B^0_{\infty,1}}} + e\right)}. \tag{3.27}$$

Combining the estimate (3.26), we have (2.21). □

Proof of Theorem 2.14. Let $T > 0$ be fixed, and let $\epsilon > 0$ be given arbitrarily. First, we consider the case $\epsilon \leq \alpha_0/T$, where α_0 is given in Lemma 1.2. Since u_0 is almost periodic in $B^0_{\infty,1}$ and by Theorem 1.4, there exists a number $l(\epsilon) > 0$ with the property that any cube of length $l(\epsilon)$ of $\mathbb{R}^2$ contains at least one point with abscissa ξ such that

$$\|u_0(\cdot + \xi) - u_0(\cdot)\|_{B^0_{\infty,1}} < \epsilon.$$

Thus, we have

$$\|u(\cdot + \xi, t) - u(\cdot, t)\|_{B^0_{\infty,1}} \leq \Phi(T\|u_0(\cdot + \xi) - u_0(\cdot)\|_{B^0_{\infty,1}}) + \epsilon < \Phi(T\epsilon) + \epsilon.$$

Next, we consider the case $\epsilon > \alpha_0/T$. There exists a number $l(\alpha_0/T) > 0$ with the property that any cube of length $l(\alpha_0/T)$ of $\mathbb{R}^2$ contains at least one point with abscissa ξ such that

$$\|u_0(\cdot + \xi) - u_0(\cdot)\|_{B^0_{\infty,1}} < \alpha_0/T < \epsilon.$$

Thus, we have

$$\|u(\cdot+\xi, t)-u(\cdot, t)\|_{B^0_{\infty,1}} \leq \Phi(T\|u_0(\cdot+\xi)-u_0(\cdot)\|_{B^0_{\infty,1}})+\epsilon < \Phi(\alpha_0)+\epsilon \leq \Phi(T\epsilon)+\epsilon.$$

Therefore, $u(x, t)$ is almost periodic in $B^0_{\infty,1}$ if $u_0(x)$ is almost periodic in $B^0_{\infty,1}$, since $\Phi(T\epsilon) \to 0$ as $\epsilon \to +0$. By [9, Chapter 4, Theorem 4.5, (i) and (ii)], we can say that a frequency set of initial data Λ is included in G. By (2.7), it is easy to see that $c_\mu(t) \in C([0, \infty))$ for $\mu \in G$, $c_\mu(0) = c_\lambda$ if $\mu = \lambda \in \Lambda$, and $c_\mu(0) = 0$ if $\mu \notin \Lambda$. □

4. Appendix

Let u be a weak solution to (E) given in [25] and

$$p = \sum_{k,l=1}^{2} R_k R_l u^k u^l \text{ in } BMO. \tag{4.1}$$

In this section, we verify that the pair (u, p) actually satisfies (E) in $\mathcal{D}'((0, \infty)\times\mathbb{R}^2)$. This fact is non-trivial in the case where u does not decay at spatial infinity, since any weak solution to (E) does not necessarily have an associate pressure represented by (4.1). In fact, $(v, p_1) := (\vec{a}t, -\vec{a} \cdot x)$ satisfies (E), and v is weak solution to (E). But $(v, p_2) := (\vec{a}t, \sum_{k,l=1}^2 R_k R_l v^k v^l)$ does not satisfy (E), if $\vec{a} \neq 0$, since $\nabla \sum_{k,l=1}^2 R_k R_l v^k v^l = 0$.

Theorem 4.1. *Let u be a weak solution to Euler equations given in* [25, Theorem 1.1], *and let p be the function defined by* (4.1). *Then, the pair (u, p) actually satisfies* (E) *in the distributional sense.*

Proof. In [25], for a given initial data $u_0 \in L^\infty$ with rot $u_0 \in Y_{ul}^\Theta$, a solution u is constructed via a limit of an approximation solution sequence $\{u_\epsilon\}$, where u_ϵ is a regular solution to (E) with sufficiently smooth initial data $u_{0,\epsilon} := \rho_\epsilon * u_0 \in C^{1+\alpha}$. Here, $\alpha > 0$ and $\rho_\epsilon * u_0$ is the usual mollification of u_0. The regular solution (u_ϵ, p_ϵ) to (E) with the initial data $u_{0,\epsilon}$ was constructed by Chemin [5], Serfati [22], and Pak–Park [21]. This solution (u_ϵ, p_ϵ) satisfies the relation:

$$\nabla p_\epsilon(t) = R \sum_{i,j=1}^{2} R_j(u_\epsilon^i \partial_i u_\epsilon^j)(t) = \nabla \sum_{i,j=1}^{2} R_i R_j(u_\epsilon^i u_\epsilon^j)(t) \quad \text{in } \dot{B}^0_{\infty,1}. \tag{4.2}$$

See [25, p. 185]. On the other hand, following [25, Proof of Theorem 1.1], we have for all T,

$$\begin{aligned} \sup_{0\le t<T} \|u_\epsilon(t)\|_\infty &\le C(\|u_{0,\epsilon}\|_\infty, \|\text{rot } u_{0,\epsilon}\|_{Y_{ul}^\theta}, T) \\ &\le C(\|u_0\|_\infty, \|\text{rot } u_0\|_{Y_{ul}^\theta}, T), \\ \sup_{0\le t<T} \|\text{rot } u_\epsilon(t)\|_{L^q_{ul}} &\le C(\|u_0\|_\infty, \|\text{rot } u_0\|_{Y_{ul}^\theta}, T), \end{aligned} \tag{4.3}$$

Let $0 < \delta < 1$. By [25, (39),(41)], we see that for all $0 < s, t < T$,

$$\begin{aligned} \|u_\epsilon(t)\|_{C^{1-\delta}} &\le C(\|u_0\|_\infty, \|\text{rot } u_0\|_{Y_{ul}^\theta}), \\ \|u_\epsilon(t) - u_\epsilon(s)\|_{B^{-\delta}_{\infty,\infty}} &\le |t-s| C(\|u_0\|_\infty, \|\text{rot } u_0\|_{Y_{ul}^\theta}, T), \end{aligned} \tag{4.4}$$

where C is independent of ϵ, s, and t. Then, there exists a subsequence $\{u_{\epsilon'}\}$ of $\{u_\epsilon\}$ such that

$$\begin{aligned} &u_{\epsilon'} \to u \text{ weak-}* \text{ in } L^\infty(0,T;L^\infty) \text{ and} \\ &u_{\epsilon'} \to u \text{ strongly in } L^\infty(0,T;L^\infty(B(0,r))) \text{ for all } r>0 \text{ and all } T>0 \end{aligned} \tag{4.5}$$

which implies that

$$u^i_{\epsilon'} u^j_{\epsilon'} \to u^i u^j \text{ weak-}* \text{ in } L^\infty(0,T;L^\infty) \quad \text{for all } T > 0. \tag{4.6}$$

Then, for all $\varphi \in C_0^\infty((0,\infty)\times\mathbb{R}^2)$, we observe that

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}^2} \sum_{i,j=1}^{2} R_i R_j(u^i_{\epsilon'} u^j_{\epsilon'})(x,t)\ \nabla\cdot\varphi(x,t)dxdt \\ &= \sum_{i,j=1}^{2} \int_0^\infty \int_{\mathbb{R}^2} (u^i_{\epsilon'} u^j_{\epsilon'})\ R_i R_j(\nabla\cdot\varphi)dxdt \to \sum_{i,j=1}^{2} \int_0^\infty \int_{\mathbb{R}^2} (u^i u^j)\ R_i R_j(\nabla\cdot\varphi)dxdt \\ &= \int_0^\infty \int_{\mathbb{R}^2} p\ \nabla\cdot\varphi\ dxdt, \end{aligned} \tag{4.7}$$

since $\nabla\cdot\varphi \in C_0((0,\infty);\mathcal{H}^1)$. Recall that $\mathcal{H}^1$ denotes the Hardy space. Then, by (4.5), (4.7) and the fact that (u_ϵ, p_ϵ) satisfies (E), we see that (u,p) satisfies (E) in $\mathcal{D}'((0,\infty)\times\mathbb{R}^2)$. □

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