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On the Regularity Conditions of Suitable Weak Solutions of the 3D Navier–Stokes Equations

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Abstract. Let v and ω be the velocity and the vorticity of the a suitable weak solution of the 3D Navier–Stokes equations in a space-time domain containing $z_0=(x_0,t_0)$, and let $Q_{z_0,r}=B_{x_0,r}\times(t_0-r^2,t_0)$ be a parabolic cylinder in the domain. We show that if either $v\times\frac{\omega}{|\omega|}\in L^{\gamma,\alpha}_{x,t}(Q_{z_0,r})$ with $\frac{3}{\gamma}+\frac{2}{\alpha}\le 1$, or $\omega\times\frac{v}{|v|}\in L^{\gamma,\alpha}_{x,t}(Q_{z_0,r})$ with $\frac{3}{\gamma}+\frac{2}{\alpha}\le 2$, where $L^{\gamma,\alpha}_{x,t}$ denotes the Serrin type of class, then z_0 is a regular point for v. This refines previous local regularity criteria for the suitable weak solutions.

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1. Introduction

The Navier–Stokes equations in a domain $\Omega\in\mathbb{R}^3$ are the following.

$$(\mathrm{NS})\begin{cases}\dfrac{\partial v}{\partial t}+(v\cdot\nabla)v=-\nabla p+\Delta v, & (x,t)\in\Omega\times(0,T)\\ \operatorname{div} v=0, & (x,t)\in\Omega\times(0,T)\\ v(x,0)=v_0(x), & x\in\Omega\end{cases}$$

where $v=(v_1,v_2,v_3)$, $v_j=v_j(x,t)$, $j=1,2,3$, is the velocity of the flow, $p=p(x,t)$ is the scalar pressure, and v_0 is the given initial velocity satisfying div $v_0=0$. The global in time existence of a smooth solution to the system (NS) is an outstanding open problem in mathematics, and is chosen as one of the seven millennium problems by Clay Institute. One traditional approach to the problem is to prove global in time existence of weak solutions, and then prove their regularity. A notion of weak solution of (NS) was introduced, and its global in time existence in $\mathbb{R}^3$ was proved by Leray in [18]. Later, Hopf proved existence of weak solution in a bounded domain in [14]. After that there are numerous conditional regularity

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results on the weak solutions, imposing integrability conditions on the velocity or the vorticity, which guarantee regularity of the weak solutions (see e.g. [25, 21, 23, 17, 10, 26, 1, 2, 3, 28, 9, 13, 15, 16, 20, 5, 6]). For the local analysis of the regularity properties of weak solutions Caffarelli–Kohn–Nirenberg introduced the notion of suitable weak solutions and proved its partial regularity as well as global in time existence ([4]). A refined definition of suitable weak solutions, using a stronger condition for pressure, which we adopt here, was introduced by Lin in [19]. Let $Q_T = \Omega \times (0,T)$. For a point $z = (x,t) \in Q_T$, we denote below

$$B_{x,r} = \{y \in \mathbb{R}^3 : |y - x| < r\}, \quad Q_{z,r} = B_{x,r} \times (t - r^2, t).$$

We also define the space-time norm,

$$\|v\|_{L^{\gamma,\alpha}_{x,t}(Q_T)} := \left\| \|v(\cdot,t)\|_{L^\gamma_x(\Omega)} \right\|_{L^\alpha_t(0,T)}, \ 1 \le \alpha, \gamma \le \infty.$$

Definition 1. A pair (v,p) of measurable functions is a suitable weak solution of (NS) if the following conditions are satisfied:

(i) $v \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W^{1,2}(\Omega)), \quad p \in L^{\frac{3}{2}}(Q_T)$.

(ii) The following integral identity holds

$$\int_{Q_T} [-v \cdot \partial_t \varphi + (v\cdot\nabla)v \cdot \varphi + \nabla v : \nabla\varphi]\, dxdt = \int_\Omega v_0 \cdot \varphi(x,0)dx$$

for all vector test functions $\varphi \in [C_0^\infty(\Omega \times [0,T))]^3$.

(iii) The pair (v,p) satisfies the local energy inequality,

$$\int_\Omega |v(x,t)|^2 \phi(x,t)dx + 2\int_0^t \int_\Omega |\nabla v(x,\tau)|^2 \phi(x,\tau)dxd\tau$$

$$\le \int_0^t \int_\Omega \left(|v|^2 (\partial_t \phi + \Delta\phi) + (|v|^2 + 2p)v \cdot \nabla\phi \right) dxd\tau$$

for almost all $t \in (0,T)$ and for all nonnegative scalar test function $\phi \in C_0^\infty(Q_T)$.

We say that a weak solution v is regular at z, if v is bounded in $Q_{z,r}$ for some $r > 0$. Such point z is called a regular point. A point in Q_T, which is not regular, is called a singular point. Caffarelli–Kohn–Nirenberg showed that the one dimensional Hausdorff measure of the set $\mathcal{S}$ of possible interior singular points of suitable weak solutions is zero ([4]), which refines the previous results due to Scheffer ([24]).

In this paper our aim is to obtain refined versions of regularity conditions for velocity and vorticity for suitable weak solutions, incorporating the directions of each vector field as well as the magnitudes. Our conditions are not directly on the velocity or vorticity, but on the orthogonal component of velocity to vorticity direction, or on the orthogonal component of vorticity to velocity direction. The associated integral norms are scaling invariant. Below we use extended definitions the direction fields $\omega(x,t)/|\omega(x,t)|$ and $v(x,t)/|v(x,t)|$, which are set to zero whenever $\omega(x,t)=0$ or $v(x,t)=0$ respectively.

Theorem 1.1. *Let $z_0=(x_0,t_0)\in Q_T$ with $\bar{Q}_{z_0,r}\subset Q_T$, and let (v,p) be a suitable weak solution of* (NS) *in Q_T with the vorticity $\omega=\operatorname{curl} v$, where the derivatives are in the sense of distribution. Suppose v and ω satisfy one of the following conditions:*

(i) *There exists an absolute constant ε_0 such that*

$$\left\| v\times\frac{\omega}{|\omega|}\right\|_{L^{3,\infty}_{x,t}(Q_{z_0,r})}\le\varepsilon_0. \tag{1.1}$$

(ii) *There exist $\gamma\in(3,\infty]$ and $\alpha\in[2,\infty)$ with $3/\gamma+2/\alpha\le 1$ such that*

$$v\times\frac{\omega}{|\omega|}\in L^{\gamma,\alpha}_{x,t}(Q_{z_0,r}). \tag{1.2}$$

(iii) *There exist $\gamma\in(3/2,\infty]$ and $\alpha\in[1,\infty)$ with $3/\gamma+2/\alpha\le 2$ such that*

$$\omega\times\frac{v}{|v|}\in L^{\gamma,\alpha}_{x,t}(Q_{z_0,r}). \tag{1.3}$$

Then, z_0 is a regular point.

Remark 1.1. We say v is a Beltrami flow in $Q_{z_0,r}$ if $v\times\omega=0$ in $Q_{z_0,r}$. In the study of physics of turbulent flows the Beltrami structure has important roles (see e.g. [8, 22] and the references therein). The condition that $v\times\frac{\omega}{|\omega|}$ or $\omega\times\frac{v}{|v|}$ is controllable in a space-time region implies intuitively that the weak solutions are not far from the Beltrami flows in that region in an appropriate sense, and the above theorem says that this implies regularity of the flows in that region.

2. Proof of Theorem 1.1

Before starting our proof we recall previous results concerning the notion of *an epoch of possible irregularity* of the weak solution of the Navier–Stokes equations. It is known that for weak solutions there exists a set $E\subset I=[0,T]$ such that E is closed, of 1/2-dimensional Hausdorff measure zero, and solutions are regular in $I\backslash E$ ([18, 12, 11]). Moreover, the set E can be written as $I\backslash\cup_{i\in\mathcal{J}}I_i$, where set $\mathcal{J}$ is at most countable, and $I_i=(\alpha_i,\beta_i)$ are disjoint open intervals in $[0,T]$. Following [12], we call the instant time β_i an epoch of possible irregularity. We recall a fact proved by Neustupa and Penel in [20] on the epoch of possible irregularity for suitable weak solutions.

Lemma 2.1. *Let $z_0=(x_0,t_0)\in Q_T$. Suppose v is a suitable weak solution of* (NS) *in Q_T and t_0 be an epoch of possible irregularity. Then there exist positive numbers τ, r_1, and r_2 with $r_1<r_2$ such that the followings are satisfied:*

(a) *τ is sufficiently small so that t_0 is only one epoch of possible irregularity in time interval $[t_0-\tau,t_0]$.*
(b) *The closure of $B_{x_0,r_2}\times(t_0-\tau,t_0)$ is contained in Q_T, i.e. $\overline{B_{x_0,r_2}}\times[t_0-\tau,t_0]\subset Q_T$.*
(c) *$\left((\overline{B_{x_0,r_2}}-B_{x_0,r_1})\times[t_0-\tau,t_0]\right)\cap\mathcal{S}=\phi$, where $\mathcal{S}$ is the set of possible singular points of v.*
(d) *v, v_t, and p are, together with all their space derivatives, continuous on $(\overline{B_{x_0,r_2}}-B_{x_0,r_1})\times[t_0-\tau,t_0]$.*

Next we recall the following result proved in [4], a corollary of which will be used in the proof of our main theorem.

Proposition 2.1. *There exists an absolute constant $\varepsilon_1>0$ with the following property. If (v,p) is a suitable weak solution of* (NS) *near z_0 and if*

$$\limsup_{\rho\to0+}\frac{1}{\rho}\int_{Q_{z_0,\rho}}|\nabla v|^2dxdt\le\varepsilon_1, \tag{2.1}$$

then z_0 is a regular point.

As an immediate corollary we have the following local regularity criterion, which is a local version of the one obtained in [2].

Corollary 2.1. *If (v,p) is a suitable weak solution of* (NS) *near z_0, and if either*

$$\|\nabla v\|_{L^{\frac{3}{2},\infty}_{x,t}(Q_{z_0,r})}\le\varepsilon_1,$$

where ε_1 is the constant in Proposition 2.1, or there exist $\gamma\in(3/2,\infty]$ and $\alpha\in[2,\infty)$ with $3/\gamma+2/\alpha\le2$ such that

$$\nabla v\in L^{\gamma,\alpha}_{x,t}(Q_{z_0,r}),$$

then, z_0 is a regular point.

Proof. We observe that

$$\limsup_{\rho\to0+}\frac{1}{\rho}\int_{Q_{z_0,\rho}}|\nabla v|^2\,dxdt\le\limsup_{\rho\to0+}\rho^{2(2-\frac{3}{\gamma}-\frac{2}{\alpha})}\|\nabla v\|_{L^{\gamma,\alpha}_{x,t}(Q_{z_0,\rho})}$$
$$\begin{cases}=0, & \text{if }\gamma>3/2\text{ and }3/\gamma+2/\alpha\le2,\\ \le\varepsilon_1, & \text{if }\gamma=3/2,\alpha=\infty\end{cases}$$

by the Hölder inequality. Then the conclusion is immediate by Proposition 2.1.□

Proof of Theorem 1.1. We first assume that t_0 is an epoch of possible irregularity for v in $Q_{z_0,r}$. Suppose that $0<r_1<r_2<r$, and $r^2<\tau$ are the positive numbers in Lemma 2.1. Below, we denote $B_1=B_{x_0,r_1}$ and $B_2=B_{x_0,r_2}$. Following [20], we choose a cut-off function $\varphi \in C_0^\infty(B_2)$ such that $\varphi = 1$ on B_1, and set $u = \varphi v - V$, where $V \in C_0^2(B_2 \backslash \overline{B_1})$ satisfies div $V = v \cdot \nabla\varphi$. In particular, all the spatial derivatives of V and $\frac{\partial V}{\partial t}$ are smooth. Using the well-known form of the Navier–Stokes equations,

$$\frac{\partial v}{\partial t} - v \times \omega = -\nabla\left(p + \frac{1}{2}|v|^2\right) + \Delta v,$$

one can check easily that u satisfies the following equations:

$$\frac{\partial u}{\partial t} - \varphi v \times \omega = h - \nabla\left(\varphi\Big(p + \frac{1}{2}|v|^2\Big)\right) + \Delta u, \quad \text{div } u = 0, \tag{2.2}$$

where we set

$$h = -\frac{\partial V}{\partial t} + \left(p + \frac{1}{2}|v|^2\right)\nabla\varphi - v\Delta\varphi - 2(\nabla\varphi\cdot\nabla)v + \Delta V.$$

We observe that $h(\cdot,t)$ is supported on $\left(\overline{B_2} \setminus B_1\right)$ for each $t \in [t_0 - \tau, t_0)$, which is sufficiently smooth in the region. Operating D on (2.2), and taking $L^2(B_{x_0,r_2})$ inner product it by Du, we obtain, after integration by part

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|Du\|_{L^2}^2 + \|D^2u\|_{L^2}^2 &= -(\varphi v \times \omega, D^2u)_{L^2} - (D^2u, h)\\ &\le |(\varphi v \times \omega, D^2u)_{L^2}| + \frac{1}{8}\|D^2u\|_{L^2}^2 + C\|h\|_{L^2}^2, \end{aligned}\tag{2.3}$$

where (and below) we used simplified notation for the L^p-norm in B_2,

$$\|f\|_{L^p} = \|f\|_{L^p(B_2)}, \quad p \in [1,\infty],$$

unless other domain is specified. Let us set $\xi = \omega/|\omega|$. We estimate the nonlinear term as follows:

$$\begin{aligned}|(\varphi v \times \omega, D^2u)_{L^2}| &\le \int_{B_2} |v\times\xi||\varphi\omega||D^2u|dx\\ &\le \int_{B_2} |v\times\xi||\varphi Dv||D^2u|dx\\ &= \int_{B_2} |v\times\xi||Du - v\nabla\varphi + DV||D^2u|dx\\ &\le \int_{B_2} |v\times\xi||Du||D^2u|dx + \int_{B_2} |v\times\xi||g||D^2u|dx\\ &= I_1 + I_2,\end{aligned}\tag{2.4}$$

where we set $g = v\nabla\varphi - DV$. Since g is a smooth function supported on $(B_2 \setminus \bar{B}_1) \times (t_0 - \tau, t_0]$, we estimate I_2 simply as

$$I_2 \le \|g\|_{L^\infty}\|v\|_{L^2}\|D^2u\|_{L^2} \le C\|v\|_{L^2}^2 + \frac{1}{4}\|D^2u\|_{L^2}^2. \tag{2.5}$$

We first assume the condition (i) of Theorem 1.1 holds true. In this case we estimate

$$I_1 \le \|v\times\xi\|_{L^3}\|Du\|_{L^6}\|D^2u\|_{L^2} \le C\|v\times\xi\|_{L^3}\|D^2u\|_{L^2}^2. \tag{2.6}$$

Combining estimates (2.3)–(2.6) together, we have

$$\begin{aligned}\frac{d}{dt}\|Du\|_{L^2}^2+\|D^2u\|_{L^2}^2 &\le C_1\|v\times\xi\|_{L^3}\|D^2u\|_{L^2}^2+C\|h\|_{L^2}^2+C\|v\|_{L^2}^2\\ &\le C_1\varepsilon_0\|D^2u\|_{L^2}^2+C\|h\|_{L^2}^2+C\|v\|_{L^2}^2 \qquad (2.7)\end{aligned}$$

for $t\in(t_0-r_2^2,t_0]$, and for an absolute constant C_1. If $C_1\varepsilon_0<1$, then integrating (2.7) in time over $[t_0-r_2^2,t_0]$, we can obtain

$$\begin{aligned}\sup_{t_0-r_2^2<t<t_0}\|Du(\cdot,t)\|_{L^2}^2 &\le \|Du(\cdot,t_0-r_2^2)\|_{L^2}^2+C\int_{t_0-r_2^2}^{t_0}\|v\|_{L^2}^2dt\\ &\quad +C\int_{t_0-r_2^2}^{t_0}\|h\|_{L^2}^2dt<\infty.\end{aligned}$$

Hence, $Du\in L^{2,\infty}_{x,t}(Q_{z_0,r_2})$, and therefore $Dv\in L^{2,\infty}_{x,t}(Q_{z_0,r_1})$. Applying Corollary 2.1, we conclude that z_0 is a regular point. Next, we assume that the condition (ii) of Theorem 1.1 holds true, and estimate

$$\begin{aligned}I_1 &\le \|v\times\xi\|_{L^\gamma}\|Du\|_{L^{\frac{2\gamma}{\gamma-2}}}\|D^2u\|_{L^2}\\ &\le C\|v\times\xi\|_{L^\gamma}\|Du\|^{1-\frac{3}{\gamma}}\|D^2u\|_{L^2}^{1+\frac{3}{\gamma}}\\ &\le C\|v\times\xi\|_{L^\gamma}\|Du\|^{1-\frac{3}{\gamma}}\|D^2u\|_{L^2}^{1+\frac{3}{\gamma}}\\ &\le C\|v\times\xi\|_{L^\gamma}^{\frac{2\gamma}{\gamma-3}}\|Du\|_{L^2}^2+\frac{1}{4}\|D^2u\|_{L^2}^2, \qquad (2.8)\end{aligned}$$

where we used the interpolation inequality,

$$\|Du\|_{L^{\frac{2\gamma}{\gamma-2}}}\le C\|Du\|_{L^2}^{1-\frac{3}{\gamma}}\|D^2u\|_{L^2}^{\frac{3}{\gamma}}$$

for $3<\gamma\le\infty$. Combining (2.8) and (2.5) with (2.3), we obtain

$$\frac{d}{dt}\|Du\|_{L^2}^2+\|D^2u\|_{L^2}^2\le C\|v\times\xi\|_{L^\gamma}^{\frac{2\gamma}{\gamma-3}}\|Du\|_{L^2}^2+C\|v\|_{L^2}^2+C\|h\|_{L^2}^2. \qquad (2.9)$$

By Gronwall's lemma we have

$$\begin{aligned}&\|Du(\cdot,t_0)\|_{L^2}^2+\nu\int_{t_0-r_2^2}^{t_0}\|D^2u(\cdot,t)\|_{L^2}^2dt\\ &\le \|Du(\cdot,t_0-r_2^2)\|_{L^2}^2\exp\left(C\int_{t_0-r_2^2}^{t_0}\|v\times\xi(\cdot,t)\|_{L^\gamma}^{\frac{2\gamma}{\gamma-3}}dt\right)\\ &\quad +C\int_{t_0-r_2^2}^{t_0}\|h(\cdot,t)\|_{L^2}^2dt+C\int_{t_0-r_2^2}^{t_0}\|v(\cdot,t)\|_{L^2}^2dt. \qquad (2.10)\end{aligned}$$

Since $v\times\xi\in L^{\gamma,\alpha}_{x,t}(Q_{z_0,r_2})$ with $3/\gamma+2/\alpha\le 1$ and $\gamma>3$, we estimate

$$\int_{t_0-r_2^2}^{t_0}\|v\times\xi(\cdot,t)\|_{L^\gamma}^{\frac{2\gamma}{\gamma-3}}dt\le\|v\times\xi\|_{L^{\gamma,\alpha}_{x,t}(B_2\times(t_0-r_2^2,t_0))}^{\frac{\gamma}{\gamma-3}}r_2^{\frac{2\gamma}{\gamma-3}(1-\frac{3}{\gamma}-\frac{2}{\alpha})}<\infty. \qquad (2.11)$$

From (2.10), (2.11) we find that $Du \in L^{2,\infty}_{x,t}(Q_{z_0,r_2})$, and hence $Dv \in L^{2,\infty}_{x,t}(Q_{z_0,r_1})$. Similarly to the previous case, we conclude that z_0 is a regular point for v.

Now we assume (iii) of the theorem holds true, and set $\eta = v/|v|$. We multiply $u|u|$ on the first equation of (2.2), and integrate over B_2 to obtain

$$\begin{aligned}
&\frac{1}{3}\frac{d}{dt}\int_{B_2} |u|^3 dx + \frac{2}{3}\int_{B_2} |\nabla |u|^{\frac{3}{2}}|^2 dx \\
&= \int_{B_2} \varphi v \times \omega \cdot u|u| dx + \int_{B_2} |u|(u\cdot\nabla)\left[\varphi\Big(p + \frac{1}{2}|v|^2\Big)\right] dx + \int_{B_2} h \cdot u|u| dx \\
&\le \int_{B_2} |u|^2 |u+V||\omega \times \eta| dx + \int_{B_2} |u|^2 \left|\nabla\left[\varphi\Big(p+\frac{1}{2}|v|^2\Big)\right]\right| dx + \int_{B_2} |h||u|^2 dx \\
&:= J_1 + J_2 + J_3, \qquad (2.12)
\end{aligned}$$

where we used the fact $\varphi v = u + V$. Using Hölder's, Gagliardo–Nirenberg's, and Young's inequalities we estimate as follows.

$$\begin{aligned}
J_1 &\le \big\||u|^2\big\|_{L^{\frac{3\gamma}{2(\gamma-1)}}} \|u+V\|_{L^{3\gamma}\gamma-1}\|\omega\times\eta\|_{L^\gamma} \\
&\le \|u\|^2_{L^{\frac{3\gamma}{\gamma-1}}}\|u\|_{L^{3\gamma}\gamma-1}\|\omega\times\eta\|_{L^\gamma} + C\|u\|^2_{L^{\frac{3\gamma}{\gamma-1}}}\|\omega\times\eta\|_{L^\gamma} \\
&:= J_1^a + J_1^b, \qquad (2.13)
\end{aligned}$$

where we estimate J_1^a and J_1^b;

$$\begin{aligned}
J_1^a &\le C\|u\|_{L^3}^{\frac{3(2\gamma-3)}{2\gamma}} \|\nabla|u|^{\frac{3}{2}}\|_{L^2}^{\frac{3}{\gamma}} \|\omega\times\eta\|_{L^\gamma} \\
&\le C\|\omega\times\eta\|_{L^\gamma}^{\frac{2\gamma}{2\gamma-3}}\|u\|_{L^3}^3 + \frac{1}{12}\|\nabla|u|^{\frac{3}{2}}\|_{L^2}^2, \qquad (2.14)
\end{aligned}$$

and

$$\begin{aligned}
J_1^b &\le C\|u\|_{L^3}^{\frac{2(2\gamma-3)}{2\gamma}} \|\nabla|u|^{\frac{3}{2}}\|_{L^2}^{\frac{2}{\gamma}} \|\omega\times\eta\|_{L^\gamma} \\
&\le C\|\omega\times\eta\|_{L^\gamma}^{\frac{2\gamma}{2\gamma-3}}\|u\|_{L^3}^3 + \frac{1}{12}\|\nabla|u|^{\frac{3}{2}}\|_{L^2}^2 + C\|\omega\times\eta\|_{L^\gamma}^{\frac{2\gamma}{2\gamma-3}}. \qquad (2.15)
\end{aligned}$$

In order to estimate J_2 we need a preliminary elliptic estimate as follows. We take operation of $\operatorname{div}(\cdot)$ on the first equation of (2.2) to have

$$\Delta\left[\varphi\Big(p+\frac{1}{2}|v|^2\Big)\right] = \operatorname{div}[\varphi v\times\omega] + \operatorname{div} h,$$

which can be extended to the equation on the whole domain. Hence, by the gradient estimate for the elliptic operator Δ, we obtain

$$\left\|\nabla\left[\varphi\Big(p+\frac{1}{2}|v|^2\Big)\right]\right\|_{L^p} \le C_p\|\varphi v\times\omega\|_{L^p} + C_p\|h\|_{L^p}, \quad 1<p<\infty.$$

Using this fact, we estimate J_2 as follows.

$$\begin{aligned} J_2 \;&\le\; \big\| |u|^2 \big\|_{L^{\frac{3\gamma}{2(\gamma-1)}}} \left\| \nabla \left[\varphi\Big(p+\frac{1}{2}|v|^2\Big)\right] \right\|_{L^{\frac{3\gamma}{\gamma+2}}} \\ &\le\; C\|u\|^2_{L^{\frac{3\gamma}{\gamma-1}}} \|\varphi v\times\omega\|_{L^{\frac{3\gamma}{\gamma+2}}} + C\|u\|^2_{L^{\frac{3\gamma}{\gamma-1}}} \|h\|_{L^{\frac{3\gamma}{\gamma+2}}} \\ &:=\; J_2^a + J_2^b. \end{aligned} \tag{2.16}$$

We estimate J_2^a and J_2^b separately as follows.

$$\begin{aligned} J_2^a \;&\le C\|u\|^2_{L^{\frac{3\gamma}{\gamma-1}}} \|u+V\|_{L^{\frac{3\gamma}{\gamma-1}}} \|\omega\times\eta\|_{L^\gamma} \\ &\le C\|u\|^2_{L^{\frac{3\gamma}{\gamma-1}}} \|u\|_{L^{\frac{3\gamma}{\gamma-1}}} \|\omega\times\eta\|_{L^\gamma} + CC\|u\|^2_{L^{\frac{3\gamma}{\gamma-1}}} \|\omega\times\eta\|_{L^\gamma} \\ &\le \text{(following the similar estimate of } J_1) \\ &\le C\|\omega\times\eta\|_{L^\gamma}^{\frac{2\gamma}{2\gamma-3}} \|u\|_{L^3}^3 + \frac{1}{12}\|\nabla|u|^{\frac{3}{2}}\|_{L^2}^2 + C\|\omega\times\eta\|_{L^\gamma}^{\frac{2\gamma}{2\gamma-3}}. \end{aligned} \tag{2.17}$$

$$\begin{aligned} J_2^b \;&\le C\|u\|_{L^3}^{\frac{2\gamma-3}{\gamma}} \|\nabla|u|^{\frac{3}{2}}\|_{L^2}^{\frac{2}{\gamma}} \le C\|u\|_{L^3}^{\frac{2\gamma-3}{\gamma-1}} + \frac{1}{12}\|\nabla|u|^{\frac{3}{2}}\|_{L^2}^2 \\ &\le C\|u\|_{L^3}^3 + \frac{1}{12}\|\nabla|u|^{\frac{3}{2}}\|_{L^2}^2 + C. \end{aligned} \tag{2.18}$$

The estimate of J_3 is simple as the following,

$$J_3 \le \|h\|_{L^3}\|u\|_{L^3}^2 \le C\|u\|_{L^3}^3 + C. \tag{2.19}$$

Combining (2.12)–(2.19), and absorbing the terms involving $\|\nabla|u|^{\frac{3}{2}}\|_{L^2}^2$ to the left hand side of (2.12), we obtain

$$\frac{d}{dt}\big(\|u\|_{L^3}^3+1\big) + \|\nabla|u|^{\frac{3}{2}}\|_{L^2}^2 \le C\big(\|\omega\times\eta\|_{L^\gamma}^{\frac{2\gamma}{2\gamma-3}}+1\big)\big(\|u\|_{L^3}^3+1\big), \tag{2.20}$$

from which, after integration over $[t_0-r_2^2, t_0]$, we derive

$$\|u(\cdot,t_0)\|_{L^3}^3 \le \big(\|u(\cdot,t_0-r_2^2)\|_{L^3}^3+1\big)\exp\left(C\int_{t_0-r_2^2}^{t_0} \|\omega\times\eta\|_{L^\gamma}^{\frac{2\gamma}{2\gamma-3}}dt + Cr_2^2\right). \tag{2.21}$$

Since $\omega\times\eta\in L^{\gamma,\alpha}_{x,t}(Q_{z_0,r_2})$ with $3/\gamma+2/\alpha\le 2$ by hypothesis, we have

$$\int_{t_0-r_2^2}^{t_0} \|\omega\times\eta(\cdot,t)\|_{L^\gamma}^{\frac{2\gamma}{2\gamma-3}} dt \le \|\omega\times\eta\|^{\frac{2\gamma}{2\gamma-3}}_{L^{\gamma,\alpha}_{x,t}(B_2\times(t_0-r_2^2,t_0))} r_2^{\frac{4\gamma}{2\gamma-3}(2-\frac{3}{\gamma}-\frac{2}{\alpha})} < \infty. \tag{2.22}$$

From (2.21)–(2.22) we find that $u\in L^{3,\infty}_{x,t}(Q_{z_0,r_2})$, and therefore applying the regularity criterion due to [9], we conclude that z_0 is a regular point.

Next, we assume that z_0 is a singular point for which t_0 is not an epoch of possible irregularity. Then, there exists a time $t^*\in(t_0-r^2,t_0)$ and $0<\tilde r_1<\tilde r_2<r$ such that v is regular on $(B_{x_0,\tilde r_2}\setminus B_{x_0,\tilde r_1})\times[t^*,t_0]$. This is due to that fact that the one dimensional Hausdorff measure of the set of all possible singular space-time points is equal to zero. We claim v is regular on $B_{x_0,\tilde r_1}\times[t^*,t_0]$. Suppose not,

then there exists another time $s \in [t^*, t_0]$ such that the weak solution is regular on $B_{x_0,\tilde{r}_1} \times [t^*, s)$, and singularity occurs at $(y, s) \in B_{x_0,\tilde{r}_1} \times \{s\}$. We can repeat the above argument for the parabolic neighborhoods of (y, s) to conclude that (y, s) is actually a regular point. Hence, there exists no space-time point of singularity in $B_{x_0,\tilde{r}_1} \times [t^*, t_0]$, and we are reduced to the already considered case that t_0 is an epoch of possible irregularity. This completes the proof. □

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