J. math. fluid mech. 11 (2009) 407–414
1422-6928/09/030407-8

DOI 10.1007/s00021-007-0264-8

Journal of Mathematical Fluid Mechanics

A Note on the Exterior Two-Dimensional Steady-State Navier–Stokes Problem

Antonio Russo

Communicated by G. P. Galdi

Abstract. We consider the stationary motion of a viscous incompressible fluid in a two-dimensional exterior domain; we prove that the problem has a solution for small values of the flux of the boundary datum through the boundary.

Mathematics Subject Classification (2000). 35Q30, 76D03, 76D05.

Keywords. Stokes system, steady-state Navier–Stokes problem, two-dimensional exterior domains.

1. Introduction

The steady-state problem for the Navier–Stokes equations is to find a solution $(\boldsymbol{u}, p)$ of the system[1] [4]

$$\begin{aligned} \nu\Delta\boldsymbol{u} - \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \boldsymbol{f} &= \nabla p && \text{in } \Omega, \\ \operatorname{div}\boldsymbol{u} &= 0 && \text{in } \Omega, \\ \boldsymbol{u} &= \boldsymbol{a} && \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $\boldsymbol{u}$ is the velocity, p the pressure, ν the kinematical viscosity coefficient, $\boldsymbol{f}$ the body force and $\boldsymbol{a}$ the boundary datum. It is well known that if Ω is the exterior domain

$$\Omega = \mathbb{R}^2 \setminus \overline{\Omega}_0, \tag{2}$$

where $\partial\Omega_0$ is connected and Lipschitz, $\boldsymbol{a} \in W^{1/2,2}(\partial\Omega)$, $\boldsymbol{f} \in D_0^{-1,2}(\Omega)$ and

$$\int_{\partial\Omega} \boldsymbol{a}\cdot\boldsymbol{n} = 0,$$

[1] For the relevant definitions and properties of system (1) we quote [4]. Unless we don't specify the symbols, we shall use the notation in [4]. If V is a function space, $V_\sigma = \{\chi \in V : \operatorname{div}\chi = 0\}$; $\mathcal{H}^1(\mathbb{R}^2)$ denotes the Hardy space on $\mathbb{R}^2$. As is always possible, we assume that Ω_0 contains the unit disk.

with $\boldsymbol{n}$ outward unit normal to $\partial\Omega$, then system (1) has a weak solution with a finite Dirichlet integral for every value of ν (see, e.g., [4], [8]). The main purpose of this article is to show that the above result continues to hold under the weaker assumption

$$\nu > \frac{\xi|\Phi|}{2\pi},$$

where

$$\xi = \sup_{\|\boldsymbol{w}\|_{D^{1,2}_\sigma(\mathbb{R}^2)}=1} \left| \int_{\mathbb{R}^2} \log|x| \operatorname{div}(\boldsymbol{w}\cdot\nabla\boldsymbol{w}) \right|, \qquad \Phi = \int_{\partial\Omega} \boldsymbol{a}\cdot\boldsymbol{n}.$$

2. Some auxiliary results

Consider the Stokes problem

$$\begin{aligned} \nu\Delta\boldsymbol{u} + \boldsymbol{f} &= \nabla p \quad \text{in } \Omega,\\ \operatorname{div}\boldsymbol{u} &= 0 \qquad \text{in } \Omega,\\ \boldsymbol{u} &= \boldsymbol{a} \qquad \text{on } \partial\Omega. \end{aligned} \tag{3}$$

The following results are well known.

Theorem 1. *Let Ω be the Lipschitz bounded domain*

$$\Omega = \Omega_1 \setminus \overline{\Omega}_0, \quad \overline{\Omega}_0 \subset \Omega_1. \tag{4}$$

If $\boldsymbol{f} \in D_0^{-1,2}(\Omega)$ and $\boldsymbol{a} \in W^{1/2,2}(\partial\Omega)$ satisfies

$$\int_{\partial\Omega} \boldsymbol{a}\cdot\boldsymbol{n} = 0, \tag{5}$$

then system (3) has a weak solution $\boldsymbol{h} \in W^{1,2}_\sigma(\Omega)$, expressed by

$$\boldsymbol{h}(x) = \boldsymbol{v}(x) - \frac{\Phi x}{2\pi|x|^2}, \tag{6}$$

and

$$\int_{\partial\Omega_0} \boldsymbol{v}\cdot\boldsymbol{n} = \int_{\partial\Omega_1} \boldsymbol{v}\cdot\boldsymbol{n} = 0. \tag{7}$$

Theorem 2. *Let Ω be the Lipschitz exterior domain defined by (2). If $\boldsymbol{f} \in D_0^{-1,2}(\Omega)$ and $\boldsymbol{a} \in W^{1/2,2}(\partial\Omega)$, then system (3) has a weak solution $\boldsymbol{h} \in D^{1,2}_\sigma(\Omega)$, expressed by (6) with*

$$\int_{\partial\Omega} \boldsymbol{v}\cdot\boldsymbol{n} = 0.$$

Lemma 1. *The linear functional*

$$\phi \in \mathcal{H}^1 \to \int_{\mathbb{R}^2} \phi(x)\log|x|$$

is continuous.

For a proof of Lemma 1 see, e.g., [11] p. 82.

Lemma 2. [3] *If* $\boldsymbol{w} \in D^{1,2}_\sigma(\mathbb{R}^2)$, *then* $\operatorname{div}(\boldsymbol{w} \cdot \nabla \boldsymbol{w}) \in \mathcal{H}^1$.

From the above lemmas it easily follows that

$$\xi = \sup_{\|\boldsymbol{w}\|_{D^{1,2}_\sigma(\mathbb{R}^2)}=1} \left| \int_{\mathbb{R}^2} \log|x| \operatorname{div}(\boldsymbol{w} \cdot \nabla \boldsymbol{w}) \right| < +\infty. \tag{8}$$

Theorem 3. *Let* Ω *be the Lipschitz bounded domain defined by* (4), *let* $\boldsymbol{f} \in D_0^{-1,2}(\Omega)$ *and let* $\boldsymbol{a} \in W^{1/2,2}(\partial\Omega)$ *satisfy* (5). *If*

$$\nu > \frac{\xi|\Phi|}{2\pi}, \tag{9}$$

where ξ *is defined by* (8), *then system* (1) *has a weak solution* $\boldsymbol{u} \in W^{1,2}_\sigma(\Omega)$.

Proof. Theorem 3 with a different constant ξ is well known [2], [4]. We give a proof for the sake of completeness.

Let us look for a solution to system (1) expressed by

$$\boldsymbol{u} = \boldsymbol{h} + \boldsymbol{w}, \tag{10}$$

with $\boldsymbol{h}$ given by (6) and $\boldsymbol{w} \in W^{1,2}_{0,\sigma}(\Omega)$ solution to the equation

$$\nu \int_\Omega \nabla\boldsymbol{\varphi} \cdot \nabla \boldsymbol{w} = \int_\Omega (\boldsymbol{h} + \boldsymbol{w}) \cdot \nabla\boldsymbol{\varphi} \cdot (\boldsymbol{h} + \boldsymbol{w}), \quad \forall \boldsymbol{\varphi} \in W^{1,2}_{0,\sigma}(\Omega). \tag{11}$$

By a classical argument (the Leray–Schauder fixed point theorem, see, e.g., [8], or H. Fujita's technique, see, e.g., [4]) this aim will be achieved if we show that all the solutions to (11) have Dirichlet integrals bounded uniformly for $\nu \in [\nu_0, \bar{\nu}]$, $\nu_0 > \xi|\Phi|/2\pi$, *i.e.* there is a positive constant c_0 such that, for every pair $(\nu, \boldsymbol{w})$ with $\boldsymbol{w}$ solution to (11) and $\nu \in [\nu_0, \bar{\nu}]$,

$$\int_\Omega (\nabla \boldsymbol{w})^2 \le c_0. \tag{12}$$

To this end we follow a classical reasoning which goes back to J. Leray [9] (see also [2] and [4], Ch. VIII p. 58). If (12) is not true, then we can find two sequences $\{\nu_k\}_{k\in\mathbb{N}}$ in $[\nu_0, \bar{\nu}]$ and $\{\boldsymbol{w}_k\}_{k\in\mathbb{N}}$ in $W^{1,2}_{0,\sigma}(\Omega)$, solutions to (11) such that

$$\lim_{k\to+\infty} \nu_k = \nu \in [\nu_0, \bar{\nu}], \quad \lim_{k\to+\infty} J_k^2 = \lim_{k\to+\infty} \int_\Omega (\nabla \boldsymbol{w}_k)^2 = +\infty.$$

Setting

$$\tilde{\boldsymbol{w}}_k = \frac{\boldsymbol{w}_k}{J_k}, \tag{13}$$

from (11) we have

$$\frac{\nu_k}{J_k}\int_\Omega \nabla\boldsymbol{\varphi}\cdot\nabla\tilde{\boldsymbol{w}}_k = \int_\Omega \tilde{\boldsymbol{w}}_k\cdot\nabla\boldsymbol{\varphi}\cdot\tilde{\boldsymbol{w}}_k + \frac{1}{J_k}\int_\Omega \boldsymbol{h}\cdot\nabla\boldsymbol{\varphi}\cdot\tilde{\boldsymbol{w}}_k + \frac{1}{J_k}\int_\Omega \tilde{\boldsymbol{w}}_k\cdot\nabla\boldsymbol{\varphi}\cdot\boldsymbol{h} + \frac{1}{J_k^2}\int_\Omega \boldsymbol{h}\cdot\nabla\boldsymbol{\varphi}\cdot\boldsymbol{h}, \tag{14}$$

for all $\boldsymbol{\varphi}\in W^{1,2}_{0,\sigma}(\Omega)$. Since $\|\nabla\tilde{\boldsymbol{w}}_k\|_{L^2(\Omega)} = 1$, from $\{\tilde{\boldsymbol{w}}_k\}_{k\in\mathbb{N}}$ we can extract a subsequence which converges weakly in $W^{1,2}(\Omega)$ and strongly in $L^q(\Omega)$, $q\in[1,+\infty)$, to a field $\tilde{\boldsymbol{w}}\in W^{1,2}_{\sigma,0}(\Omega)$, with $\|\nabla\tilde{\boldsymbol{w}}\|_{L^2(\Omega)}\le 1$. Therefore, letting $k\to+\infty$ in (14) yields

$$\int_\Omega \tilde{\boldsymbol{w}}\cdot\nabla\tilde{\boldsymbol{w}}\cdot\boldsymbol{\varphi} = 0, \quad \forall\boldsymbol{\varphi}\in W^{1,2}_{0,\sigma}(\Omega).$$

Hence it follows that $\tilde{\boldsymbol{w}}$ is a weak solution to equations

$$\tilde{\boldsymbol{w}}\cdot\nabla\tilde{\boldsymbol{w}} + \nabla\tilde{Q} = 0 \quad \text{in } \Omega, \tag{15}$$

for some pressure field $\tilde{Q}\in W^{1,q}(\Omega)$, $q\in[1,2)$, constant on $\partial\Omega_0$ and $\partial\Omega_1$, say $\tilde{Q}_0$ on $\partial\Omega_0$ and $\tilde{Q}_1$ on $\partial\Omega_1$ [7].

Choosing $\boldsymbol{\varphi} = \boldsymbol{w}_k$ in (14) and letting $k\to+\infty$, we get

$$\nu = \int_\Omega \tilde{\boldsymbol{w}}\cdot\nabla\tilde{\boldsymbol{w}}\cdot\boldsymbol{h} = \frac{\Phi}{2\pi}\int_\Omega \log|x|\,\mathrm{div}(\tilde{\boldsymbol{w}}\cdot\nabla\tilde{\boldsymbol{w}}) + \int_\Omega \tilde{\boldsymbol{w}}\cdot\nabla\tilde{\boldsymbol{w}}\cdot\boldsymbol{v}.$$

Hence, since by (7)

$$\int_\Omega \tilde{\boldsymbol{w}}\cdot\nabla\tilde{\boldsymbol{w}}\cdot\boldsymbol{v} = -\int_\Omega \boldsymbol{v}\cdot\nabla\tilde{Q} = -\tilde{Q}_0\int_{\partial\Omega_0}\boldsymbol{v}\cdot\boldsymbol{n} - \tilde{Q}_1\int_{\partial\Omega_1}\boldsymbol{v}\cdot\boldsymbol{n} = 0,$$

it follows

$$\nu = \frac{\Phi}{2\pi}\int_{\mathbb{R}^2}\log|x|\,\mathrm{div}(\tilde{\boldsymbol{w}}\cdot\nabla\tilde{\boldsymbol{w}}). \tag{16}$$

Therefore, taking into account Lemmas 1, 2, (16) implies

$$\nu \le \frac{\xi\Phi}{2\pi},$$

which contradicts hypothesis (9). Then, we conclude that (12) holds. Hence it follows that there is a field $\boldsymbol{w}\in W^{1,2}_{0,\sigma}$ such that (10) is a weak solution to system (1). □

3. An existence theorem in exterior domains

We are now in a position to prove our main result.

Theorem 4. *Let Ω be the exterior domain defined by* (2)*, let $\boldsymbol{f}\in D^{-1,2}_0(\Omega)$ and let $\boldsymbol{a}\in W^{1/2,2}(\partial\Omega)$. If*

$$\nu > \frac{\xi|\Phi|}{2\pi}, \tag{17}$$

where ξ is defined by (8)*, then system* (1) *has a weak solution $\boldsymbol{u}\in D^{1,2}_\sigma(\Omega)$.*

Proof. Set $T_R = S_{2R} \setminus S_R$, $\Omega_R = \Omega \cap S_R$, with $\overline{\Omega}_0 \subset S_R = \{x : |x| < R\}$. Let $\boldsymbol{v}$ be the field appearing in (6) and let g be a C^∞ cut-off function, vanishing outside $S_{2\bar{R}}$ and equal to 1 in $S_{\bar{R}}$, with $\bar{R}$ fixed positive constant. Let $\boldsymbol{\psi} \in W_0^{1,2}(T_{\bar{R}})$ be a solution to the problem (see, e.g., [4] Ch. III)

$$\begin{aligned} \operatorname{div} \boldsymbol{\psi} + \operatorname{div}(g\boldsymbol{v}) = 0 \quad \text{in } T_{\bar{R}}, \\ \|\nabla \boldsymbol{\psi}\|_{L^2(T_{\bar{R}})} \le c \|\operatorname{div}(g\boldsymbol{v})\|_{L^2(T_{\bar{R}})} \end{aligned}$$

and set

$$\boldsymbol{\gamma} = \boldsymbol{\zeta} - \frac{\Phi x}{2\pi |x|^2}, \qquad \boldsymbol{\zeta} = \begin{cases} \boldsymbol{v}, & \text{in } \Omega_{\bar{R}}, \\ \boldsymbol{\psi} + g\boldsymbol{v}, & \text{in } T_{\bar{R}}, \\ \mathbf{0}, & \text{in } \mathbb{R}^2 \setminus \Omega_{2\bar{R}}. \end{cases} \tag{18}$$

Let $\{R_k\}_{k\in\mathbb{N}}$ be an increasing and divergent sequence in $(0, +\infty)$, with $S_{R_1} \supset \Omega_0$. By Theorem 3 the equation

$$\nu \int_{\Omega_{R_k}} \nabla \boldsymbol{\varphi} \cdot \nabla \boldsymbol{w} = \int_{\Omega_{R_k}} (\boldsymbol{\gamma} + \boldsymbol{w}) \cdot \nabla \boldsymbol{\varphi} \cdot (\boldsymbol{\gamma} + \boldsymbol{w}) - \nu \int_{\Omega_{R_k}} \nabla \boldsymbol{\gamma} \cdot \nabla \boldsymbol{\varphi} + \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle, \tag{19}$$

for all $\boldsymbol{\varphi} \in W_{0,\sigma}^{1,2}(\Omega)$, has a solution $\boldsymbol{w}_k \in W_{0,\sigma}^{1,2}(\Omega_{R_k})$. Extend each field $\boldsymbol{w}_k$ onto $\mathbb{R}^2$ by setting $\boldsymbol{w}_k = \mathbf{0}$ outside Ω_{R_k}. Let us show that every solution to (19) has Dirichlet integral uniformly bounded with respect to R_k. Once again we follow a contradiction argument. Assume that a sequence $\{\boldsymbol{w}_k\}_{k\in\mathbb{N}}$ of solutions to (19) exists such that $\lim_{k\to\infty} \|\nabla \boldsymbol{w}_k\|_{L^2(\Omega)} = +\infty$. Then the field $\tilde{\boldsymbol{w}}_k$ defined by (13) satisfies the relation

$$\begin{aligned} \frac{\nu}{J_k} \int_\Omega \nabla \boldsymbol{\varphi} \cdot \nabla \tilde{\boldsymbol{w}}_k = \int_\Omega \tilde{\boldsymbol{w}}_k \cdot \nabla \boldsymbol{\varphi} \cdot \tilde{\boldsymbol{w}}_k + \frac{1}{J_k} \int_\Omega \boldsymbol{\gamma} \cdot \nabla \boldsymbol{\varphi} \cdot \tilde{\boldsymbol{w}}_k + \frac{1}{J_k} \int_\Omega \tilde{\boldsymbol{w}}_k \cdot \nabla \boldsymbol{\varphi} \cdot \boldsymbol{\gamma} \\ + \frac{1}{J_k^2} \int_\Omega (\boldsymbol{\gamma} \cdot \nabla \boldsymbol{\varphi} \cdot \boldsymbol{\gamma} - \nu \nabla \boldsymbol{\gamma} \cdot \nabla \boldsymbol{\varphi}) + \frac{1}{J_k^2} \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle, \end{aligned} \tag{20}$$

for all $\boldsymbol{\varphi} \in W_{0,\sigma}^{1,2}(\Omega_{R_k})$. Since $\|\nabla \tilde{\boldsymbol{w}}_k\|_{L^2(\Omega)} = 1$, from $\{\tilde{\boldsymbol{w}}_k\}_{k\in\mathbb{N}}$ we can extract a subsequence we denote by the same symbol which converges strongly in $L^q_{\mathrm{loc}}(\Omega)$, for all $q \in [1, +\infty)$, and weakly in $D_0^{1,2}(\Omega)$ to a field $\tilde{\boldsymbol{w}} \in D_{0,\sigma}^{1,2}(\Omega)$. Therefore, letting $k \to +\infty$ in (20), we see that the field $\tilde{\boldsymbol{w}}$ is a weak solution to equations (15). Choosing $\boldsymbol{\varphi} = \tilde{\boldsymbol{w}}_k$ in (20) yields

$$\nu = \int_\Omega \tilde{\boldsymbol{w}}_k \cdot \nabla \tilde{\boldsymbol{w}}_k \cdot \boldsymbol{\gamma} + \frac{1}{J_k} \int_\Omega (\boldsymbol{\gamma} \cdot \nabla \tilde{\boldsymbol{w}}_k \cdot \boldsymbol{\gamma} - \nu \nabla \boldsymbol{\gamma} \cdot \nabla \tilde{\boldsymbol{w}}_k) + \frac{1}{J_k} \langle \boldsymbol{f}, \tilde{\boldsymbol{w}}_k \rangle. \tag{21}$$

Since by Lemmas 1, 2

$$\left| \int_\Omega \tilde{\boldsymbol{w}}_k \cdot \nabla \tilde{\boldsymbol{w}}_k \cdot \nabla \log |x| \right| = \left| \int_{\mathbb{R}^2} \log |x| \operatorname{div}(\tilde{\boldsymbol{w}}_k \cdot \nabla \tilde{\boldsymbol{w}}_k) \right| \le \xi,$$

(21) yields

$$\nu - \frac{\xi |\Phi|}{2\pi} \le \int_\Omega \tilde{\boldsymbol{w}}_k \cdot \nabla \tilde{\boldsymbol{w}}_k \cdot \boldsymbol{\zeta} + \frac{1}{J_k} \int_\Omega (\boldsymbol{\gamma} \cdot \nabla \tilde{\boldsymbol{w}}_k \cdot \boldsymbol{\gamma} - \nu \nabla \boldsymbol{\gamma} \cdot \nabla \tilde{\boldsymbol{w}}_k) + \frac{1}{J_k} \langle \boldsymbol{f}, \tilde{\boldsymbol{w}}_k \rangle.$$

Hence, letting $k \to +\infty$ and taking into account that

$$|\langle \boldsymbol{f}, \tilde{\boldsymbol{w}}_k \rangle| \le c\|\boldsymbol{f}\|_{D^{-1,2}(\Omega)} \|\nabla \tilde{\boldsymbol{w}}_k\|_{L^2(\Omega)} \le c\|\boldsymbol{f}\|_{D^{-1,2}(\Omega)},$$

$$\left| \int_\Omega (\boldsymbol{\gamma} \cdot \nabla \tilde{\boldsymbol{w}}_k \cdot \boldsymbol{\gamma} - \nabla \boldsymbol{\gamma} \cdot \nabla \tilde{\boldsymbol{w}}_k) \right| \le c \int_\Omega \frac{|\nabla \tilde{\boldsymbol{w}}_k|}{|x|^2} \le c \left\{ \int_\Omega \frac{1}{|x|^4} \int_\Omega (\nabla \tilde{\boldsymbol{w}}_k)^2 \right\}^{1/2} \le c,$$

it follows

$$\nu - \frac{\xi|\Phi|}{2\pi} \le \int_\Omega \tilde{\boldsymbol{w}} \cdot \nabla \tilde{\boldsymbol{w}} \cdot \boldsymbol{\zeta}. \tag{22}$$

Since $\tilde{Q}$ is constant (say $\tilde{Q}_0$) on $\partial\Omega$, taking into account (15) we have

$$\int_\Omega \tilde{\boldsymbol{w}} \cdot \nabla \tilde{\boldsymbol{w}} \cdot \boldsymbol{\zeta} = -\int_\Omega \boldsymbol{\zeta} \cdot \nabla \tilde{Q} = -\tilde{Q}_0 \int_{\partial\Omega} \boldsymbol{\zeta} \cdot \boldsymbol{n} = 0. \tag{23}$$

Therefore, under hypothesis (17), (22) and (23) are incompatible. Hence it follows that the sequence $\{\tilde{\boldsymbol{w}}_k\}_{k\in\mathbb{N}}$ is uniformly bounded in $D_0^{1,2}(\Omega)$ so that from it we can extract a subsequence, we denote by the same symbol, which converges weakly in $D_0^{1,2}(\Omega)$ to a field $\boldsymbol{w} \in D_{0,\sigma}^{1,2}(\Omega)$ a standard argument shows to be a solution to equation (19) (see, e.g., [10], Ch. 3). □

One the most intriguing and difficult question related to the above solution is the knowledge of its behavior at infinity (see [1], [5] ,[6]). Our last result, which is suggested by a recent one of G. P. Galdi [5] (Theorem 3.2), gives a little contribution to this problem.

Theorem 5. *Let Ω be an exterior Lipschitz domain, symmetric with respect to the reference axes. Let $\boldsymbol{f} = (f_1, f_2) \in D_0^{-1,2}(\Omega)$ and let $\boldsymbol{a} = (a_1, a_2) \in W^{1/2,2}(\partial\Omega)$ satisfy*

$$\begin{aligned} f_1(x_1, x_2) &= -f_1(-x_1, x_2) = f_1(x_1, -x_2), \\ f_2(x_1, x_2) &= f_2(-x_1, x_2) = -f_2(x_1, -x_2), \\ a_1(x_1, x_2) &= -a_1(-x_1, x_2) = a_1(x_1, -x_2), \\ a_2(x_1, x_2) &= a_2(-x_1, x_2) = -a_2(x_1, -x_2). \end{aligned} \tag{24}$$

If (17) holds, then the Navier–Stokes problem has a weak solution vanishing at infinity in the following sense

$$\lim_{R\to+\infty} \int_0^{2\pi} \boldsymbol{u}^2(R, \theta) = \boldsymbol{0}. \tag{25}$$

Moreover, if $\boldsymbol{f}$ has a compact support, then

$$\lim_{|x|\to+\infty} \boldsymbol{u}(x) = \boldsymbol{0} \tag{26}$$

uniformly.

Proof. Under the above assumptions the argument used in the proof of Theorem 3 delivers existence of a weak solution $\boldsymbol{u} \in D^{1,2}_{\sigma}(\Omega)$ enjoying the symmetry properties (24). Therefore, taking into account that by symmetry

$$\int_{T_R} \boldsymbol{u} = 0,$$

by the trace theorem and the Poincaré inequality we have

$$\int_0^{2\pi} \boldsymbol{u}^2(R,\theta) \leq \frac{c}{R^2}\int_{T_R} \boldsymbol{u}^2 + c_1 \int_{T_R} (\nabla \boldsymbol{u})^2 \leq c_2 \int_{T_R} (\nabla \boldsymbol{u})^2,$$

with c, c_1 and c_2 positive constants independent of R. Hence (25) follows; (26) is proved by Lemma 3.10 in [5]. □

Remark 3.1. By the method of this paper it is not difficult to treat problem (1) in the exterior domain

$$\Omega = \mathbb{R}^2 \setminus \bigcup_{i=1}^{m} \overline{\Omega}_i,$$

where Ω_i are m Lipschitz domains with connected boundaries and such that $\overline{\Omega}_i \cap \overline{\Omega}_j = \varnothing$, $i \neq j$. In such a case one must require that

$$\nu > \frac{\xi}{2\pi} \sum_{i=1}^{m} |\Phi_i|,$$

with

$$\Phi_i = \int_{\partial\Omega_i} \boldsymbol{a} \cdot \boldsymbol{n}.$$

References

[1] C. J. Amick, On Leray's problem of steady Navier–Stokes flow past a body, *Acta Math.* **161** (1988), 71–130.

[2] W. Borchers and K. Pileckas, Note on the flux problem for stationary Navier–Stokes equations in domains with multiply connected boundary, *Acta App. Math.* **37** (1994), 21–30.

[3] R. R. Coifman, J. L. Lions, Y. Meier and S. Semmes, Compensated compactness and Hardy spaces, *J. Math. Pures App. IX Sér.* **72** (1993), 247–286.

[4] G. P. Galdi, *An Introduction to the Mathematical Theory of the Navier–Stokes Equations*, vol. I, II, revised edition, Springer Tracts in Natural Philosophy (ed. C. Truesdell), **38**, **39**, Springer-Verlag, Berlin, 1998.

[5] G. P. Galdi, Stationary Navier–Stokes problem in a two-dimensional exterior domain, in: *Stationary partial differential equations* Vol. I, 71–155, Handb. Differ. Equ., North-Holland, Amsterdam, 2004.

[6] D. Gilbarg and H. Weinberger, Asymptotic properties of steady plane solutions of the Navier–Stokes equations with bounded Dirichlet integral, *Ann. Scuola Norm. Sup. Pisa* (4) **5** (1978), 381–404.

[7] L. V. Kapitanskii and K. Pileckas, On spaces of solenoidal vector fields and boundary value problems for the Navier–Stokes equations in domains with noncompact boundaries, *Trudy Mat. Inst. Steklov* **159** (1983), 5–36. English transl.: *Proc. Math. Inst. Steklov*, **159** (1984), 3–34.
[8] O. A. Ladyzhenskaia, *The Mathematical theory of viscous incompressible fluid*, Gordon and Breach, 1969.
[9] J. Leray, Étude de diverses équations intégrales non linéaire et de quelques problèmes que pose l'hydrodynamique, *J. Math. Pures Appl.* **12** (1933), 1–82.
[10] H. Sohr, *The Navier–Stokes equations*, Birkhäuser, Basel, 2001.
[11] M. E. Taylor, *Partial Differential Equations III, Nonlinear Equation*, Springer, Berlin, 1996.

A. Russo
Dipartimento di Matematica
Seconda Università di Napoli
via Vivaldi 43
81100 Caserta
Italy
e-mail: a.russo@unina2.it

(accepted: July 2, 2007; published Online First: December 20, 2007)