J. math. fluid mech. 11 (2009) 348–382
1422-6928/09/030348-35

DOI 10.1007/s00021-007-0262-x

Journal of Mathematical Fluid Mechanics

Pointwise Asymptotic Stability of Steady Fluid Motions

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Communicated by M. Padula

To G. Paolo Galdi on his 60th birthday

Abstract. We study pointwise asymptotic stability of steady incompressible viscous fluids. The region of the motion is bounded. Our results of stability are based on the maximum modulus theorem that we prove for solutions of the Navier–Stokes equations. The asymptotic stability is based on a variational formulation. Since the region of the motion is bounded, the time decay is of exponential type. Of course suitable assumptions are made about the smallness of the size of the uniform norm of the perturbations at the initial data. With no restrictions, we are able only to prove an existence theorem of the perturbation local in time.

Mathematics Subject Classification (2000). 35Q30, 76D03, 35B40.

Keywords. Navier–Stokes equations, asymptotic stability, maximum modulus theorem.

1. Introduction

In this paper we study the stability of steady solutions of the Navier–Stokes system. We consider perturbations to the kinetic field of the unperturbed motion $(v, \tilde{\pi})$. It is known (cf. [3, 8]) that the perturbation (u, π) satisfies the following initial boundary value problem:

$$\begin{gathered} u_t + u \cdot \nabla u + v \cdot \nabla u + u \cdot \nabla v + \nabla \pi = \frac{1}{R} \Delta u, \\ \nabla \cdot u = 0 \quad \text{in } \Omega \times (0, T), \\ u(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T), \quad u(x,0) = u_0(x) \quad \text{in } \Omega. \end{gathered} \tag{1.1}$$

The symbol u_t denotes $\dfrac{\partial}{\partial t} u$ and, for any pair of vectors (a, b), by $a \cdot \nabla b$ we mean the term $(a \cdot \nabla) b$. By R we indicate the Reynolds number associated to the unperturbed motion $(v, \tilde{\pi})$ and by u_0 the initial value of the perturbation. The domain $\Omega \subset \mathbb{R}^3$ is assumed bounded and $C^{2,\overline{\alpha}}$-smooth ($\overline{\alpha} \in (0, 1)$). Usually, the nonlinear stability of a steady motion is studied with respect the L^2-norm of the perturbations and the motion is said stable in energy. A very interesting approach to studying the energy stability is the one based on a variational formulation. It was introduced by Serrin in [21] and developed by several authors [6, 7, 8, 18, 28]. The advantage of such a formulation essentially consists in the possibility of determining a critical Reynolds number R_c. If $R < R_c$, then the steady motion is unconditionally stable in L^2-norm and asymptotically stable also. The condition $R < R_c$ means that the result of stability is related to a family of motions, that is, any motion with Reynolds number R ($< R_c$) is stable. However, as stressed in [3], from a physical view point it is interesting to evaluate the stability of a motion with respect to the uniform norm also. A coherent way to attack the question is to consider at the initial instant a continuous distribution of velocity of the perturbation and to assume that its maximum modulus value is finite.[1] Then, one establishes the evolution of the perturbation. *A priori* no other requirement of regularity is plausible. Of course, the above assumption implies that the perturbation at the initial instant is in $L^2(\Omega)$, that is, it has finite energy. However the energy stability does not imply pointwise stability. Actually we have the following implication: the energy stability implies attractivity of the basic motion $(v, \tilde{\pi})$ with respect to the uniform norm (cf. [19]). That is there exists an instant $T_0 = T_0(|u_\circ|_2, R, \Omega)$ such that $u(x,t) \in C(\overline{\Omega})$ and $|u(x,t)| \le C(|u_0|_2, R, w, \Omega)e^{-\gamma t}$ for any $t \ge T_0$. If we assume that $u_0(x) \in C(\overline{\Omega}) \cap L^2(\Omega)$, even if we make an assumption of smallness as $\max_{\overline{\Omega}} |u_0(x)| + |u_0|_2 << \varepsilon$, we do not know if the L^2-theory ensures that $u(x,t)$ exists as a classical solution (for definition see Section 2) for $t > 0$ and, in particular, if $|u(x,t)| < \infty$ for any $(x,t) \in \Omega \times [0, T_0)$. If we assume $u_0(x) \in C(\overline{\Omega}) \cap L^3(\Omega)$, then we have a quite analogous statement in a neighborhood of $t = 0$.

The above considerations lead us to the conclusion that the pointwise stability is still an open problem. In this regard, it is also quite natural to inquire if the pointwise stability can be formulated by means of a variational formulation. Since we are going to work with classical solutions of problem (1.1), we are not able to give a variational formulation on the perturbations (u, π). Actually, we approach the question giving the variational formulation for the solutions of the *adjoint problem*, used to evaluate the solutions of system (1.1) with respect to the uniform norm. Therefore, we define, in a way quite analogous to that of energy stability, the variational formulation for the solutions of the adjoint problem. The consequence is the possibility of defining a bound ($\mathscr{R}_c$) for the Reynolds numbers, which ensures the conditional pointwise asymptotic stability of the solution $(v, \tilde{\pi})$. Since the domain Ω is bounded, then we can prove that the decay of the perturbation is of exponential type. Moreover, the initial data is not subject to assumptions of regularity which, from both the point views, physical and mathematical respectively,

[1] Concerning the problem of the *minimal requirements* on the data for the well posedness and qualitative properties see the recent paper [5] (and the references therein) also, where, in a different context from this note, the above questions are studied in connection with fluid steady motions in a exterior domain.

appear unessential to the well posedness of the problem. Without restrictions on the size of the data, we obtain a theorem of existence local in time. Finally, the bound $\mathscr{R}_c$ is just the critical Reynolds number of the energy stability.

In connection with the above considerations about the variational formulation for the solutions of the adjoint problem, the study of the linearized Navier–Stokes system obtained in paper [23] is also fundamental for the proof of our results. Moreover, the starting point for the existence of our classical solution is the paper [24] on Stokes system. A suitable development and coupling of the above results lead to establish a maximum modulus theorem (see Remark 2.1) for solutions of the Stokes and Navier–Stokes system.

The results of this paper were communicated in [15]. When the paper was completed, Professor V. A. Solonnikov kindly communicated to the author his paper, [27], concerning the maximum modulus theorem for solutions of the Stokes and Navier–Stokes system. The results of [27] are stated in the case of Ω bounded or exterior domain whose boundary $\partial\Omega$ is $C^{2,\alpha}$-smooth and connected; the theorem of existence of the solutions of the Navier–Stokes system is local in time. The technique of the proofs is quite different.

The paper essentially articulates in three parts, each one supporting the next one. Actually, in each of them we establish a maximum modulus theorem and the asymptotic behavior in time of the solutions, respectively, of the Stokes problem, the linearized Navier–Stokes problem and the nonlinear Navier–Stokes system (1.1), which represents our main result.

2. Some preliminaries and statement of the main result

Let $g(x,t)$ be a function defined on $\Omega \times (0,T)$ $(T \leq +\infty)$, for any multi-index $h = (h_1, h_2, h_3)$ and for any $k \in \mathbb{N} \cup \{0\}$, we denote by $D^{h,k}_{x,t}$ the derivatives $\frac{\partial^{|h|+k} g(x,t)}{\partial x_1^{h_1} \partial x_2^{h_2} \partial x_n^{h_3} \partial t^k}$, $|h| = h_1 + h_2 + h_3$. Sometimes, when there is no danger of confusion, we replace $D^h_x g$ by $D^h g$. Let $m \in \mathbb{N} \cup \{0\}$. The symbol $C^m(\overline{\Omega})$, $m \in \mathbb{N}$, denotes the Banach space (endowed with the natural norm) of all functions g which are bounded and uniformly continuous on Ω, together with all their partial derivatives $D^h g$ of order $|h| \leq m$. The norm in $C^m(\overline{\Omega})$ is denoted by $|\cdot|_m$. We denote by $C^{m,\alpha}(\overline{\Omega})$, $\alpha \in (0,1)$, the vector subspace of $C^0(\overline{\Omega})$, consisting of all functions g such that, for any

$$|h| = m, [D^h g]_{\alpha,x} = \sup_{\substack{x,y \in \Omega \\ x \neq y}} \frac{|D^h g(x) - D^h g(y)|}{|x-y|^\alpha} < \infty.$$

Moreover, we denote by $C^m(a,b;X)$ the Banach space (endowed with the natural norm) of all functions bounded and continuous on $(a,b) \subseteq \mathbb{R}$ with value in a Banach space X, together with all derivatives D^k, $k \leq m$. We denote by $C^{m,\beta}(a,b;X)$, $\beta \in (0,1)$, the vector subspace of $C^m(a,b;X)$, consisting of all functions g such that

$$[D_t^m g]_{\beta,t} = \sup_{\substack{\overline{t},\overline{\overline{t}}\in(a,b)\\ \overline{t}\neq\overline{\overline{t}}}} \frac{|D_t^m g(\overline{t}) - D_t^m g(\overline{\overline{t}})|_X}{|\overline{t}-\overline{\overline{t}}|^\beta} < \infty.$$

The norm in $C^{m,\alpha}(\overline{\Omega})$ is defined in the following way:

$$|g|_{m,\alpha,\Omega} = |g|_0 + \sum_{|h|=m} [D^{|h|} g]_{\alpha,x}.$$

The norm in $C^{m,\beta}(a,b;X)$ is defined in the following way:

$$|g|_{m,\beta,(a,b)} = \sup_{(a,b)} |g(t)|_X + [D_t^m g]_{\beta,t}.$$

We denote by $\mathcal{H}^\alpha(\Omega\times(0,T))$ the class of all functions $g(x,t)$ such that $g(x,t)$ is α-Hölder continuous with respect to x and $\frac{\alpha}{2}$-Hölder continuous with respect to t and

$$|g|_\alpha = \sup_{\Omega\times(0,T)} |g(x,t)| + [g]_{\alpha,x,t} < \infty,$$

with

$$[g]_{\alpha,x,t} = \sup_{\overline{x}\neq\overline{\overline{x}},\, t} \frac{|g(\overline{x},t)-g(\overline{\overline{x}},t)|}{|\overline{x}-\overline{\overline{x}}|^\alpha} + \sup_{\overline{t}\neq\overline{\overline{t}},\, x} \frac{|g(x,\overline{t})-g(x,\overline{\overline{t}})|}{|\overline{t}-\overline{\overline{t}}|^{\frac{\alpha}{2}}}.$$

For any $\alpha\in(0,1)$, $|g|_\alpha$ is a norm in $\mathcal{H}^\alpha(\Omega\times(0,T))$. By the symbol $\mathscr{C}_0(\Omega)$ we denote the set $\{\phi(x)\in C_0^\infty(\Omega)$ with $\nabla\cdot\phi=0\}$. By the symbol $\mathscr{C}_{|0}(\overline{\Omega})$ we denote the completion of $\mathscr{C}_0(\Omega)$ with respect to the norm of $C(\overline{\Omega})$ (the symbol $_{|0}$ means that any function has null trace on $\partial\Omega$). Let us consider

$$\widehat{\mathscr{C}}_{|0}(\overline{\Omega}) = \big\{u(x)\in C(\overline{\Omega}) : \nabla\cdot u(x)=0, \text{ in weak form, and } u(\xi)_{|\partial\Omega}=0\big\}.$$

Of course, *a priori*, it is $\mathscr{C}_{|0}(\overline{\Omega})\subseteq\widehat{\mathscr{C}}_{|0}(\overline{\Omega})$; in Section 3 we prove that they coincide when the domain Ω is $C^{1,\alpha}$ smooth. Finally, we denote by $J^p(\Omega)$ and $J^{1,p}(\Omega)$ the completion of $\mathscr{C}_0(\Omega)$ with respect to the norm of $L^p(\Omega)$ and $W^{1,p}(\Omega)$, respectively. The norms in J^p and $J^{1,p}$ are indicated by $\|\cdot\|_p$ and $\|\cdot\|_{1,p}$, respectively. By $P_p\cdot$ we denote the projector from $L^p(\Omega)$ onto $J^p(\Omega)$. If there is no danger of confusion we denote P_p simply by P. We set $H^\alpha(\Omega) := C^{0,\alpha}(\overline{\Omega})$ Following [23] (Section 6), we introduce the H^α-subspaces $G^\alpha(\Omega)$ and $J^\alpha(\Omega)$. The space $G^\alpha(\Omega)$ is the subspace of vectors from $H^\alpha(\Omega)$ having the form $u=\nabla\varphi$, where $\varphi\in C^{1,\alpha}(\Omega)$. The space $J^\alpha(\Omega)$ is the subspace of vectors from $H^\alpha(\Omega)$ satisfying the condition $u\cdot n=0$ on $\partial\Omega$ and $\nabla\cdot u=0$ in weak form. Since Ω is bounded, then $H^\alpha(\Omega)\cap L^p(\Omega)=H^\alpha(\Omega)$, hence $H^\alpha(\Omega)=J^\alpha(\Omega)\oplus G^\alpha(\Omega)$. We define P_α as the projector from $H^\alpha(\Omega)$ onto $J^\alpha(\Omega)$.

Definition 2.1. A pair (u,π) is said to be a classical solution of system (1.1) if, for some $\alpha\in(0,\overline{\alpha})$ and for any $\eta\in(0,T)$, $u\in C(0,T;\mathscr{C}_{|0}(\overline{\Omega}))\cap C^{0,\frac{\alpha}{2}}(\eta,T;\,C^{2,\alpha}(\overline{\Omega}))$,

$u_t \in C^{0,\frac{\alpha}{2}}(\eta, T;\, C^{0,\alpha}(\overline{\Omega}))$, $\pi \in C^{0,\frac{\alpha}{2}}(\eta, T; C^{1,\alpha}(\overline{\Omega}))$, (u, π) satisfies system (1.1) in $\Omega \times (0, T)$ and, for any $x_0 \in \overline{\Omega}$, $\lim\limits_{(x,t)\to(x_0,0)} u(x, t) = u_0(x_0)$.

Remark 2.1. By *maximum modulus theorem* for solutions of system (1.1), we mean the result that, under the only assumption of continuous initial data, ensures the existence of a classical solution, which is defined for any $t > 0$ and verifies the following estimate:

$$\max_{\overline{\Omega}\times(0,+\infty)} |u(x, t)| \leq c \max_{\overline{\Omega}} |u_0(x)|,$$

for some $c \geq 1$, independent of $u_0(x)$.

Definition 2.2. A pair (u, π) is said to be a p-regular solution of system (1.1) if, for some $p \in (1, \infty)$ and for any $\eta \in (0, T)$, $u(x, t) \in C(0, T; J^p(\Omega)) \cap L^p(\eta, T; W^{2,p}(\Omega) \cap J^{1,p}(\Omega))$, $\nabla\pi(x, t)$, $u_t(x, t) \in L^p(\eta, T; L^p(\Omega))$, (u, π) satisfies system (1.1) *a.e.* in $\Omega \times (0, T)$ and $\lim\limits_{t\to 0^+} |u(t) - u_0|_p = 0$.[2]

Remark 2.2. In the sequel we also consider the Hopf–Leray weak solutions of system (1.1). Since the definition is well known, for the sake of brevity, we omit it and we refer to [10, 20] for details. The above definitions are stated for solutions of (1.1). Obviously the same can be stated for solutions of the Navier–Stokes system and for the *linearized forms*. In Sections 3 and 4 we tacitly assume that the definitions 2.1 and 2.2 are meant for the Stokes system and the linearized of (1.1) system as well.

Definition 2.3. A solution $(v, \tilde{\pi})$ of the Navier–Stokes system is said to be *stable* with respect to the metric d_X (X metric space) if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that if $u_0 \in X$ with $d_X(u_0) < \delta$, then $u(x, t)$, suitable solution of (1.1),[3] is defined in X for any $t > 0$, it is unique and $d_X(u(t)) < \varepsilon$ for any $t > 0$. A solution $(v,\tilde{\pi})$ is said *asymptotically stable* if it is stable and $\lim\limits_{t\to\infty} d_X(u(t)) = 0$. Finally, if $\delta = \infty$, then both the stability and the asymptotic stability are said unconditional.

In the introduction we have stressed that the results of asymptotic stability are deduced via a variational formulation of the same kind of the one given for the asymptotic energy stability. The approach to the energy stability is formally the following one. Multiplying equation $(1.1)_1$ by u and integrating by parts on Ω we obtain the relation

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_2^2 = -\frac{1}{R}\|\nabla u(t)\|_2^2 - \int\limits_{\Omega} u \cdot D \cdot u dx,\ \forall t > 0, \tag{2.1}$$

[2] In the sequel by a regular solution we mean a 2-regular solution.
[3] That is, a solution satisfying the conditions stated in the above definitions.

that can be written as

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_2^2 = \|\nabla u(t)\|_2^2\bigg(-\frac{1}{R} - \frac{1}{\|\nabla u(t)\|_2{}^2}\int\limits_\Omega u\cdot D\cdot u dx\bigg),\ \forall t>0. \tag{2.2}$$

Therefore, if we assume

$$\max_{u\in J^{1,2}(\Omega)} -\frac{1}{\|\nabla u\|_2{}^2}\int\limits_\Omega u\cdot D\cdot u dx = \frac{1}{R_c} < \frac{1}{R}, \tag{2.3}$$

equation (2.2) implies the so called *energy inequality*

$$\|u(t)\|_2^2 + 2\bigg(\frac{1}{R} - \frac{1}{R_c}\bigg)\int\limits_s^t \|\nabla u(\tau)\|_2^2 d\tau \le \|u(s)\|_2^2,\ \forall t\ge s \text{ and } s\ge 0. \tag{2.4}$$

Under suitable assumptions on $\|u(t)\|_2$, the above relation is sufficient to study asymptotic energy stability. It is natural to give the following definition.

Definition 2.4. Let $v\in C^1(\overline{\Omega})$ and D be the symmetric part of ∇v. For any $\phi\in J^{1,2}(\Omega)$ we define the functional

$$\mathscr{F}(\phi) = -\frac{1}{\|\nabla\phi\|_2^2}\int\limits_\Omega \phi\cdot D\cdot\phi dx.$$

We set

$$\frac{1}{R_c} = \sup_{\phi\in J^{1,2}(\Omega)} \mathscr{F}(\phi).$$

If R_c is finite, then R_c is called *critical Reynolds number*.

Remark 2.3. Definition 2.4 is well posed. Indeed as proved in [18] (see [7] also) the functional $\mathscr{F}$ attains a maximum in $J^{1,2}(\Omega)$, provided that Ω is bounded and $v\in C^1(\overline{\Omega})$. Of course the assumption on v can be relaxed to a weaker one; however, this is not a relevant issue in the present analysis.

Now we are in a position to state our main result:

Theorem 2.1. *Let us assume $v\in C^{1,\alpha}(\overline{\Omega})$ in system (1.1). There exist two positive numbers $\mathscr{R}_c$ and μ such that if $R<\mathscr{R}_c$ and $u_0\in\mathscr{C}_{|0}(\overline{\Omega})$ with $|u_0|_0<\mu^{-1}$, then system (1.1) admits a unique classical solution (u,π), defined for any $t>0$, and*

$$|u(t)|_0 \le c(|u_0|_0)e^{-\overline{\gamma}t}, t>0. \tag{2.5}$$

where $c(|u_0|_0) = \dfrac{\mu^{\frac12}|u_0|_0}{1+(1-\mu|u_0|_0)^{\frac12}}$; $\overline{\gamma} = \frac{5}{8}\gamma$, *where γ is the constant of the Poincaré inequality.*

Remark 2.4. The positive number $\mathscr{R}_c$ in Theorem 2.1 is just the one given in Definition 2.4. $\mathscr{R}_c$ is not determined in the set of classical solutions of problem (1.1), but in the set of the regular solutions of the *adjoint* problem of the linearized system of the perturbation. It is clear (see also the above formal considerations which lead either to the *energy inequality* or to the definition of the *critical Reynolds number*) that the energy stability of the unperturbed motion, via a variational formulation, is unrelated to the nonlinear or linear character of the system of the perturbation. This aspect is fundamental for the developing our construction.

Remark 2.5. In the proof of Theorem 2.1 the existence of a solution is proved by means of the convergence of the series $\sum_{h=0}^{\infty}(c\,\mathfrak{U}_{\min})^h$, where c is a suitable constant depending on $R, \mathscr{R}_c, v, \Omega$ and $\mathfrak{U}_{\min}$ is the smallest real root of the equation $c\mathfrak{U}^2 - \mathfrak{U} + c\mathfrak{U}_0 = 0$ ($\mathfrak{U}_0 = |u_0|_0$). Therefore the choice of μ must be compatible with the convergence of the above series and the existence of the real root $\mathfrak{U}_{\min}$.

Remark 2.6. Theorem 2.1 ensures the pointwise asymptotic stability of the unperturbed motion $(v, \tilde{\pi})$. Actually it proves a maximum modulus theorem for solutions of system (1.1). Of course such a theorem holds for the Navier–Stokes system also. In [22], Serrin conjectures a result of pointwise stability for small Reynolds number. Theorem 2.1 gives a positive answer to the Serrin's conjecture. As a consequence, our result makes satisfied the assumption of the result stated in [22] about the existence of time-periodic solutions of the Navier–Stokes equations.

Theorem 2.2. *Let us assume* $v \in C^{1,\alpha}(\overline{\Omega})$ *in system* (1.1). *Let* $u_0 \in \mathscr{C}_{|0}(\overline{\Omega})$. *Then, system* (1.1) *admits a unique classical solution* (u, π) *in* $\Omega \times (0, T)$*, for some* $T > 0$.

Remark 2.7. We would like to emphasize that, on the one hand, the physical character of the stability problem, and, on the other hand, the limits of the regularity of steady solutions of the Navier–Stokes system for an arbitrary dimension n, have led us to consider only the three dimensional problem. However, one can prove that, by our technique, Theorem 2.1 holds for any $n \geq 2$.

Remark 2.8. One can prove the maximum modulus theorem also by another approach. Indeed, one can prove the existence of a classical solution, satisfying an estimate of the maximum modulus on some finite interval $(0, T)$, and of a n-regular solution for any $t > 0$. Since the two solutions coincide on $(0, T)$, one has proved the existence of a global solution (u, π) of system (1.1), which in particular satisfies the estimate

$$\begin{aligned} &|u(t)|_0 \leq c|u_0|_0 && \text{for any } t \in (0, T), \\ &|u(t)|_0 \leq \overline{c}t^{-\frac{n}{2p}}|u_0|_0 && \text{for any } t \geq T, \end{aligned} \tag{2.6}$$

where the first equation in (2.6) holds thanks to the local estimate on $(0,T)$ for $u(x,t)$ as classical solution and the second one in (2.6) holds in virtue of the semigroup property for $u(x,t)$ as n-regular solution evaluated for $t \geq T$. Actually, without restriction on $|u_0|_0$, *a priori*, there is the undesirable factor that T depends on $|u_0|_0$; as a consequence constant c in (2.6) is not an uniform constant with respect to $|u_0|_0$, hence it is not the estimate of the maximum modulus theorem. Thus we must restrict the size of $|u_0|_0$ in such a way that T is uniform with respect to u_0. Then another restriction is needed on the size of $|u_0|_n$ just to prove the global existence. Moreover, as far as we known, making use of an n-regular solution, the asymptotic stability of the unperturbed motion w is not connected with a variational formulation ([9]). The latter considerations, in accord with the aims of the stability theory, lead us to prefer our proof: we give one condition on the size of u_0, that is just $|u_0|_0$ *small* with respect some parameters, and we give the asymptotic stability via a variational formulation, which seems to be more interesting. Finally, apart from the requirement $R < \mathscr{R}_c$, we do not require smallness of the size of the norms involving v. The connection between the size of the norms of v and the perturbations is just related with the smallness of the quantity of $|u_0|_0$, which, as assumption, is needed for $v = 0$ also.

3. The Stokes problem with initial data in $\mathscr{C}_{|0}(\overline{\Omega})$

Let us consider the initial boundary value problem for the Stokes system:

$$\begin{aligned} w_t(x,t) - \Delta w(x,t) &= -\nabla p(x,t) + F(x,t), \\ \nabla \cdot w(x,t) &= 0 \text{ in } \Omega \times (0,T), \\ w(x,t) &= 0 \text{ on } \partial\Omega \times (0,T), \quad w(x,0) = w_0(x). \end{aligned} \tag{3.1}$$

The aim of this section is just to prove a maximum modulus theorem for solutions of system (3.1):

Theorem 3.1. *Let $w_0(x) \in \mathscr{C}_{|0}(\overline{\Omega})$ and $F = 0$ in system* (3.1)*. Then, there exists a unique classical solution of problem* (3.1) *such that*

$$|w(t)|_0 \leq c|w_0|_0 \text{ for any } t > 0, \tag{3.2}$$

with c independent of w_0.

We start with some auxiliary results.

Lemma 3.1. *Let Ω be a $C^{1,\alpha}$ smooth domain in $\mathbb{R}^n, n \geq 2$. Then, $\mathscr{C}_0(\Omega)$ is dense in $\widehat{\mathscr{C}_{|0}}(\overline{\Omega})$. Hence $\widehat{\mathscr{C}_{|0}}(\overline{\Omega}) = \mathscr{C}_{|0}(\overline{\Omega})$.*

Proof. Let $u(x) \in \widehat{\mathscr{C}_{|0}}(\overline{\Omega})$ and

$$\hat{u}(x) = \begin{cases} u(x), & \text{if } x \in \overline{\Omega}, \\ 0, & \text{if } x \in \mathbb{R}^n - \Omega. \end{cases}$$

For $\delta > 0$, we consider $J_\delta(u)(x) = \int_{\mathbb{R}^n} J_\delta(x-y)\hat{u}(y)dy$. Of course $J_\delta(u)(x) \in \mathscr{C}_0(\mathbb{R}^n)$ and if $\delta \to 0$, then $J_\delta(u)(x) \to u(x)$ in $C(\overline{\Omega})$. We have $\int_{\partial\Omega} J_\delta(u)(\xi) \cdot n d\sigma = 0$. Moreover, by the uniform convergence of $J_\delta(U)(x)$ to $u(x)$ and since $u(x) = 0$ on $\partial\Omega$, for any $\varepsilon > 0$, there exists $\bar{\delta} > 0$ such that for any $\delta \in (0, \bar{\delta})$

$$|J_\delta(u)(x)| < \varepsilon \text{ uniformly in } x \in \partial\Omega. \tag{3.3}$$

For any $\delta > 0$, let us consider the pair (U_δ, P_δ) which is the smooth solution of

$$\Delta U_\delta - \nabla P_\delta = 0,\ \nabla \cdot U_\delta = 0 \text{ in } \Omega, \quad U_\delta(x) = J_\delta(u)(x) \text{ for any } x \in \partial\Omega. \tag{3.4}$$

For any $p > 1$, $J_\delta(u)(x) \in W^{1-\frac{1}{p},p}(\partial\Omega) \cap C(\partial\Omega)$. Assume $p > n$. For the existence of the solution $(U_\delta, P_\delta) \in W^{1,p}(\Omega) \cap C^k(\Omega)$ we refer the reader to the well know paper [1] (see also [4]). Moreover, by virtue of the maximum modulus theorem proved in [14], there exists a constant M, independent of δ, such that

$$|U_\delta(x)| \le M \max_{\partial\Omega} |J_\delta(u)(x)|. \tag{3.5}$$

We set $u_\delta(x) = J_\delta(u)(x) - U_\delta(x)$. For any $\delta > 0$, u_δ belongs to $J^{1,p}(\Omega) \cap C(\overline{\Omega})$. If $\delta \to 0$, then $u_\delta \to u$ in $C(\overline{\Omega})$. Indeed, by estimates (3.3)–(3.5) and by the uniform convergence of $J_\delta(u)$ to u in $\overline{\Omega}$, for a given $\varepsilon > 0$ we deduce the existence of $\bar{\delta} > 0$ such that

$$\begin{aligned} |u - u_\delta| &\le |u - J_\delta(u)| + |U_\delta| \le \varepsilon + M \max_{\partial\Omega} |J_\delta(u)| \\ &\le \varepsilon(1 + M) \text{ for any } \delta > \bar{\delta} \text{ uniformly in } x \in \overline{\Omega}. \end{aligned}$$

The arbitrariness of ε proves the convergence. Since for any $\delta > 0, u_\delta \in J^{1,p}(\Omega)$ with $p > n$, there exists a sequence $\{u_\delta^n\} \subset \mathscr{C}_0(\Omega)$ converging to u_δ in $J^{1,p}(\Omega)$ and, by the Sobolev imbedding, in $C(\overline{\Omega})$. Therefore for any $\varepsilon > 0$ we have

$$|u(x) - u_\delta^n(x)| \le |u(x) - u_\delta(x)| + |u_\delta(x) - u_\delta^n(x)| < \varepsilon(2 + M)$$

uniformly in $x \in \Omega$, provided we first choose δ sufficiently small and then n sufficiently large. The lemma is proved.

Remark 3.1. In the previous proof it is tacitly assumed that Ω is bounded. However the proof works in any domain for which the maximum modulus theorem holds coupled with problem (3.4).

Lemma 3.2. *Let Ω be a bounded or an exterior C^2 smooth domain of $\mathbb{R}^n$, $n \ge 2$. Suppose $q \in [r_1, \infty]$, and $r, r_1 \in (1, \infty)$. Let $\psi \in L^{r_1}(\Omega)$ and $P\Delta\psi \in L^r(\Omega)$. Then, the following interpolation inequality holds:*

$$\|\psi\|_q \le c\|P\Delta\psi\|_r^a \|\psi\|_{r_1}^{1-a}, \tag{3.6}$$

with c independent of ψ, provided that ψ has zero trace on $\partial\Omega$, and the following dimensional balance is verified:

$$\frac{1}{q} = a\left(\frac{1}{r} - \frac{2}{n}\right) + (1-a)\frac{1}{r_1}\,.$$

Proof. See [13] Theorem 2.1.

We prove some further lemmas.

Lemma 3.3. *Let $w_0 \in C^{2,\alpha} \cap \mathscr{C}_{|0}(\Omega)$ and $F \in C^{0,\frac{\alpha}{2}}(0,T;C^{0,\alpha}(\overline{\Omega}))$ in system* (3.1). *Then there exists a unique classical solution (w,p) such that*

$$|D^2 w|_\alpha + |w_t|_\alpha + |\nabla p|_\alpha + |p|_0 \le c\big(|F|_\alpha + |w_0|_{2,\alpha,\Omega} + \sup_{(0,T)} |w(t)|_0\big), \tag{3.7}$$

with c independent of T, w_0 and F, provided that

$$P_\alpha(F(x,0) + \Delta w_0(x)) = 0 \text{ on } \partial\Omega. \tag{3.8}$$

Proof. The lemma is a special case of Theorem 9.1 proved in [23].

Lemma 3.4. *Let (w,p) be a classical solution of system* (3.1) *with $F = 0$ and $w_0 \in \mathscr{C}_0(\Omega)$. Then, there exists a constant c independent of T and (w,p), such that*

$$|w(t)|_0 \le c|w_0|_0 \quad \text{for any } t \in (0,T). \tag{3.9}$$

Proof. The result is stated and partially proved by Solonnikov in [24]. The complete proof is achieved by Solonnikov in the papers [25, 26].

Remark 3.2. The result of the lemma is an estimate of maximum modulus of a classical solution of the initial boundary value problem of the Stokes system. However it is not a maximum modulus theorem, since the existence of a classical solution with a initial data $w_0 \in \mathscr{C}_{|0}(\overline{\Omega})$ is not ensured. The proof of the maximum modulus theorem is just the object of Theorem 3.1. The result of the theorem cannot be achieved by a simple coupling of Lemma 3.3 and Lemma 3.4. Indeed we have to modify the estimates of Lemma 3.3 from estimates obtained on the cylinder $\Omega \times (0,T)$ to pointwise estimates. Recently, in [26] Solonnikov has proved a maximum modulus theorem. Here, we propose another proof for the sake of completeness.

For the above purposes, we prove the following.

Lemma 3.5. *Let $w_0 \in J^p(\Omega) \cap L^r(\Omega), p \in (1,\infty)$, $r \in \{1,p\}$, and $F = 0$ in system* (3.1). *Then there exists a unique p-regular solution (w,p) of system* (3.1).

Moreover, for $k = 0, 1$,

$$\|D_t^k w(t)\|_q \le c\|w_0\|_r t^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-k},$$
$$\|\nabla D_t^k w(t)\|_{\hat q} \le c\|w_0\|_r t^{-\frac{1}{2}-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-k}, \tag{3.10}$$

with $q \in [p, \infty]$ if $r = p$ and $q \in (1, \infty]$ if $r = 1$; with $\hat q \in [p, n]$ if $r = p$ and $\hat q \in (1, n]$ if $r = 1$. Moreover, if $\hat q > n$

$$\|\nabla D_t^k w(t)\|_{\hat q} \le c\|w_0\|_r \begin{cases} t^{-\frac{1}{2}-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{\hat q}\right)-k}, & \text{if } t \in (0,1], \\ t^{-\frac{n}{2r}-k}, & \text{if } t \ge 1. \end{cases} \tag{3.11}$$

The constant c is independent of w_0. In particular, if $w_0 \in \mathscr{C}_0(\Omega)$, then (w, p) is a classical solution; moreover

$$\begin{aligned}&\left|D^2 w(\overline{x},\overline{t}) - D^2 w(\overline{\overline{x}},\overline{\overline{t}})\right| + \left|w_t(\overline{x},\overline{t}) - w_t(\overline{\overline{x}},\overline{\overline{t}})\right| + \left|\nabla p(\overline{x},\overline{t}) - \nabla p(\overline{\overline{x}},\overline{\overline{t}})\right| \\ &\quad \le c(p)H(t_0)\left(|\overline{x} - \overline{\overline{x}}| + |\overline{t} - \overline{\overline{t}}|^{\frac{1}{2}}\right)^{\alpha} |w_0|_0^{1-\alpha}\left(\|w_0\|_p^{\alpha} + |w_0|_0^{\alpha}\right),\end{aligned} \tag{3.12}$$

where $t_\circ = \min\{\overline{t}, \overline{\overline{t}}\}$, $H(t_0)$ is a function depending on t_0 in such a way that $H(t_\circ) \to \infty$ for $t_\circ \to 0$, $c(p)^{-1} \to 0$ for $p \to \infty$; $c(p)$ and $H(t_0)$ are independent of w_0.

Proof. The existence, uniqueness and estimate $(3.10)_1$ and $(3.10)_2$ for $k = 0$ can be found in [16]. By the same technique employed in [16], one completes the proof of (3.10) and proves (3.11) also. Let us prove (3.12). Let $\overline{\overline{t}} \ge \overline{t} > \frac{\eta}{2}$, for some $\eta > 0$. Let us consider a smooth function $\zeta(t) \in [0, 1]$, with $\zeta(t) = 0$ for $t \le \frac{\overline{t}}{2}$ and $\zeta(t) = 1$ for $t \ge \overline{t}$ and $|\zeta'(t)| \le ct^{-1}$. Multiplying equation (3.1) by $\zeta(t)$, a simple computation gives

$$W_t - \Delta W = -\nabla P - \zeta' w, \qquad \nabla \cdot W = 0 \text{ in } \Omega \times (0, T),$$
$$W = 0 \text{ on } \partial\Omega \times (0, T), \quad W(x, 0) = 0,$$

where $(W, P) = \zeta(w, p)$. The term $w\zeta' \in C^{0,\frac{\alpha}{2}}(0, T; \mathscr{C}_{|0}(\overline{\Omega}) \cap C^{0,\alpha}(\overline{\Omega}))$. Therefore by virtue of Lemma 3.3 we deduce the existence of a unique solution (W, P) such that

$$|D^2 W|_\alpha + |W_t|_\alpha + |\nabla p|_\alpha \le c(|w\zeta'|_\alpha + |W(t)|_0). \tag{3.13}$$

Of course, by virtue of Lemmas 3.3–3.4, we have

$$|W(x, t)| + |w(x, t)\zeta'(t)| \le c\Big(1 + \frac{1}{t}\Big)|w_0|_0 \quad \text{for any } (x, t) \in \Omega \times (0, T). \tag{3.14}$$

Applying the Sobolev imbedding theorem and the Gagliardo–Nirenberg inequality (see [17], and also [2]), we estimate the spatial Hölder seminorm:

$$[w\zeta']_{\alpha,x} \le c|\zeta'||\nabla w|_p^a |w|_0^{1-a}, \text{ with } a = \alpha p/(p - n),$$

for some $p > n$ such that $1 - \frac{n}{p} \ge \alpha$. Moreover, by coupling estimates (3.9) and (3.11), we find

$$[w\zeta']_{\alpha,x} \le c\overline{t}^{-1-\frac{a}{2}}\|w_0\|_p^a |w_0|_0^{1-a}. \tag{3.15}$$

Analogously, with respect to time, we have

$$\begin{aligned}[][w\zeta']_{\frac{\alpha}{2},t} &\le |w(x,\overline{\overline{s}})|\frac{|\zeta'(\overline{s})-\zeta'(\overline{\overline{s}})|}{|\overline{s}-\overline{\overline{s}}|^{\frac{\alpha}{2}}}+|\zeta'(\overline{s})|\frac{|w(x,\overline{s})-w(x,\overline{\overline{s}})|}{|\overline{s}-\overline{\overline{s}}|^{\frac{\alpha}{2}}}\\ &\le c\overline{t}^{-1}|w_t(s)|_0^{\frac{\alpha}{2}}|w(s)|_0^{1-\frac{\alpha}{2}}+c\overline{t}^{-1-\frac{\alpha}{2}}|w(\overline{\overline{s}})|_0\\ &\le c\overline{t}^{-1-(2+\frac{n}{p})\frac{\alpha}{4}}\|w_0\|_p^{\frac{\alpha}{2}}|w_0|_0^{1-\frac{\alpha}{2}}+c\overline{t}^{-1-\frac{\alpha}{2}}|w_0|_0. \qquad (3.16)\end{aligned}$$

Taking into account the definition of ζ, for $\overline{\overline{t}}>\overline{t}$ we obtain

$$\begin{aligned}\frac{|D^2w(\overline{x},\overline{t})-D^2w(\overline{\overline{x}},\overline{\overline{t}})|}{(|\overline{x}-\overline{\overline{x}}|+|\overline{t}-\overline{\overline{t}}|^{\frac{1}{2}})^{\alpha}}+\frac{|w_t(\overline{x},\overline{t})-w_t(\overline{\overline{x}},\overline{\overline{t}})|}{(|\overline{x}-\overline{\overline{x}}|+|\overline{t}-\overline{\overline{t}}|^{\frac{1}{2}})^{\alpha}}+\frac{|\nabla p(\overline{x},\overline{t})-\nabla p(\overline{\overline{x}},\overline{\overline{t}})|}{(|\overline{x}-\overline{\overline{x}}|+|\overline{t}-\overline{\overline{t}}|^{\frac{1}{2}})^{\alpha}}&\\ \le |D^2W|_\alpha+|W_t|_\alpha+|\nabla p|_\alpha,&\end{aligned}$$

hence the result follows from estimates (3.13)–(3.16). The lemma is completely proved.

Now we are in a position to prove Theorem 3.1.

Proof of Theorem 3.1. Since $w_0\in\mathscr{C}_{|0}(\overline{\Omega})$, by virtue of Lemma 3.1, there exists a sequence $\{w_0^n\}\subset\mathscr{C}_0(\Omega)$ converging to w_0 in $C(\overline{\Omega})$. By virtue of Lemma 3.5, for any $n\in\mathbb{N}$, we can consider the classical solution (w^n,p^n) of problem (3.1) assuming initial data w_0^n. Moreover, any element of the sequence satisfies estimates (3.9) and (3.12). Taking into account the linearity of the Stokes problem, from estimates (3.9) and (3.12) we easily obtain the estimates ($\eta>0$)

$$|w^n(t)-w^m(t)|_0\le c|w_0^n-w_0^m|_0 \text{ for any } t\in(0,T), \qquad (3.17)$$

$$\begin{aligned}&[D^2w^n-D^2w^m]_{\alpha,x,t}+[w_t^n-w_t^m]_{\alpha,x,t}+[\nabla p^n-\nabla p^m]_{\alpha,x,t}\\ &\quad\le c(p)H(t_\circ)|w_0^n-w_0^m|_0^{1-\alpha}\left(\|w_0^n-w_0^m\|_p^\alpha+|w_0^n-w_0^m|_0^\alpha\right), \qquad (3.18)\end{aligned}$$

provided that $\alpha\le 1-\frac{3}{p}$, $t_\circ=\min\{\overline{t},\overline{\overline{t}}\}$, for any $\overline{t},\overline{\overline{t}}\in(\eta,T)$. By making use of the interpolation inequalities, from estimates (3.17)–(3.18) we deduce that w^n is, for any $\eta>0$, a Cauchy sequence in $C(0,T;\mathscr{C}_{|0}(\overline{\Omega}))\cap C^{0,\frac{\alpha}{2}}(\eta,T;C^{2,\alpha}(\overline{\Omega}))\cap C^{1,\frac{\alpha}{2}}(\eta,T;C^{0,\alpha}(\overline{\Omega}))$, and $p^n\in C^{0,\frac{\alpha}{2}}(\eta,T;C^{1,\alpha}(\overline{\Omega}))$. It is immediate to prove that $\lim_{t\to0}|w(t)-w_0|_0=0$. As far as the uniqueness is concerned, we multiply equation $(3.1)_1$ by $\phi(x,t-\tau)$, $\tau\in(0,t)$, where $(\phi(x,s),\pi(x,s))$ is another solution of problem (3.1) with $\phi_0\in\mathscr{C}_0(\Omega)$. An integration by parts on $\Omega\times(\eta,t)$ furnishes the relation

$$(w(t),\phi_0)=(w(\eta),\phi(t-\eta)).$$

Since for the uniqueness we assume $w_0=0$, in the limit for $\eta\to0$ we can deduce $w(x,t)=0$. The proof of the theorem is completed.

We conclude this section with some asymptotic estimates of the solutions of the Stokes problem. Thanks to the Poincaré inequality, we can prove an asymptotic behavior of exponential type with respect to the time variable.

Lemma 3.6. *Let (w, p) be the solution given in Theorem 3.1, then,*

$$|w(t)|_0 \le c(\Omega) e^{-\gamma(t-s)} |w(s)|_0,\ t \ge s \ge 0. \tag{3.19}$$

Proof. It is known that the energy differential equation,

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|_2^2 + \frac{1}{R}\|\nabla w(t)\|_2^2 = 0,$$

and the Poincaré inequality, $\|w(t)\|_2 \le \frac{1}{\sqrt{\gamma}}\|\nabla w(t)\|_2$, imply

$$\|w(s)\|_2 \le e^{-\gamma(s-\tau)}\|w(\tau)\|_2,\ s \ge \tau \ge 0. \tag{3.20}$$

Since, see [16],

$$|w(t)|_0 = \|w(t)\|_\infty \le c(t-s)^{-\frac{3}{4}}\|w(s)\|_2, \tag{3.21}$$

estimate (3.20), evaluated for $s = t - \frac{1}{2}$ and $\tau = 0$, and estimate (3.21), evaluated for t and $s = t - \frac{1}{2}$, imply

$$|w(t)|_0 \le ce^{-\gamma t}\|w_0\|_2 \le c(\text{meas}(\Omega))^{\frac{1}{2}} e^{-\gamma t}|w_0|_0,\ t \ge 1. \tag{3.22}$$

Estimates (3.2) and (3.22) imply (3.19). Theorem 3.1 allows to define the resolving operator of the Stokes problem on $\mathscr{C}_{|0}(\overline{\Omega})$. The semigroup property holds and we have in particular

$$|w(t)|_0 \le c(\Omega) e^{-\gamma(t-s)} |w(s)|_0,\ t \ge s \ge 0. \tag{3.23}$$

The lemma is completely proved.

4. The linearized perturbation system with initial data in $\mathscr{C}_{|0}(\overline{\Omega})$

In this section we give some results concerning the linearized system of the perturbation:

$$\begin{gathered} u_t^0 - \frac{1}{R}\Delta u^0 = -\nabla p^0 - \widetilde{v}\cdot\nabla u^0 - u^0\cdot A + F, \\ \nabla\cdot u^0 = 0,\ \text{in } \Omega\times(0,T), \\ u^0(x,t) = 0,\ \text{on } \partial\Omega\times(0,T),\ u^0(x,0) = u_\circ(x) \in \mathscr{C}_0(\Omega), \end{gathered} \tag{4.1}$$

where $\widetilde{v}$ stands for v or $-v$ and A for ∇v or ∇v^T, indifferently. These detailed positions are necessary because we refer to system (4.1) both as the linearized system of the perturbation and as its adjoint. In what follows we make the following convention: the pair (ϕ, π) denotes a solution of the adjoint problem; the pair (u^0, p^0) a solution of the linearized problem. The study of system (4.1) is crucial for our aims. Indeed, from a variational approach to the L^2 stability of the unperturbed motion $(v, \tilde{p})$, whose perturbations are governed by the linearized system (4.1), we can deduce by duality the asymptotic pointwise stability of the motion $(v, \tilde{p})$ itself. However the main result of this section is the following

Theorem 4.1. *Assume $v \in C^{0,\alpha}(\overline{\Omega})$, $A \in C^{0,\alpha}(\overline{\Omega})$ and $R < \mathscr{R}_c$ in system (4.1). Let $u_0 \in \mathscr{C}_{|0}(\overline{\Omega})$. Then, there exists a unique classical solution of system (4.1) such that*

$$|u^0(x,t)| \le c|u_0|_0 e^{-\overline{\gamma}t}, \text{ for any } (x,t) \in \Omega \times (0,T), \tag{4.2}$$

with c independent of u_0.

Remark 4.1. It is quite natural to ask why the maximum modulus theorem for solutions to the Stokes problem is not seen as special case of Theorem 4.1, obtained by setting $\widetilde{v} = 0$ and $A = 0$ in system (4.1). As it will be clear from the proof, the first step is the result of Theorem 3.1, so we have preferred to separate the two results completely.

The proof of Theorem 4.1 is achieved through several intermediate results. We start with an existence result for solutions of problem (4.1). By virtue of the results concerning the linearized Navier–Stokes system obtained by Solonnikov in [23], we can state

Theorem 4.2. *Let us assume $v \in C^{0,\alpha}(\overline{\Omega})$ and $A \in C^{0,\alpha}(\Omega)$ in system (4.1). Let $\phi_0 \in C^{2,\alpha}(\Omega) \cap \mathscr{C}_{|0}(\Omega)$ and $F \in C^{0,\frac{\alpha}{2}}(0,T; C^{0,\alpha}(\Omega))$. Then, there exists a unique classical solution (ϕ, π) of system (4.1), such that*

$$|D^2\phi|_\alpha + |\phi_t|_\alpha + |\nabla\pi|_\alpha + |\pi|_0 \le c\big(|F|_\alpha + |\phi_0|_{2,\alpha,\Omega} + \sup_{(0,T)} |\phi(t)|_0\big), \tag{4.3}$$

provided that

$$P_\alpha\bigg(F(x,0) + \frac{1}{R}\Delta\phi_0(x) - \widetilde{v} \cdot \nabla\phi_0(x) - \phi_0(x) \cdot A\bigg) = 0 \text{ on } \partial\Omega.$$

Proof. See Theorem 9.1 proved in [23].

Theorem 4.3. *Let us assume $v \in \mathscr{C}_{|0}(\overline{\Omega})$ and $A \in C^0(\Omega)$ in system (4.1). Let $\phi_0 \in J^p(\Omega)$ and $F \in L^p(0,T; L^p(\Omega))$. Then, there exists a unique regular solution (ϕ, π) of system (4.1). Moreover, assuming $F = 0$ the following estimates hold*

$$\begin{aligned} &q \in [p,\infty],\ p > 1, \quad \|\phi(t)\|_q \le \overline{c}t^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} e^{\overline{\gamma}_0 t}\|\phi_0\|_p, \quad t > 0,\\ &q \in [p,\infty],\ p > 1, \quad \|\nabla\phi(t)\|_q \le \overline{c}t^{-\frac{1}{2}-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} e^{\overline{\gamma}_0 t}\|\phi_0\|_p,\ t > 0, \end{aligned} \tag{4.4}$$

with constants $\overline{c}$ and $\overline{\gamma}_0$ independent of ϕ_0.

Proof. See Theorem 4.2 and Theorem 5.1 proved in [23]. Actually, Theorem 4.2 in [23] is proved assuming the data ϕ_0 more regular. However the linear character of the system and the estimates given in Theorem 5.1 of [23] allow us to extend the results to the case of $\phi_0 \in J^p(\Omega)$, as stated in our theorem.

In the above estimates (4.4) the constants $\overline{c}$ and $\overline{\gamma}_0$ *a priori* depend on q and p. We can solve this question, partially at least, with the following.

Lemma 4.1. *Let $q > r$ and $r \in [1, q)$. Then estimate* (4.4) *can be improved to the following*

$$\begin{aligned} &q \in (r, \infty],\ r \geq 1, \qquad \|\phi(t)\|_q \leq ct^{-\frac{3}{2}\left(\frac{1}{r}-\frac{1}{q}\right)} e^{\gamma_0 t} \|\phi_0\|_r, \qquad t > 0,\\ &q \in (r, \infty],\ r \geq 1, \quad \|\nabla\phi(t)\|_q \leq ct^{-\frac{1}{2}-\frac{3}{2}\left(\frac{1}{r}-\frac{1}{q}\right)} e^{\gamma_0 t} \|\phi_0\|_r,\ t > 0, \end{aligned} \tag{4.5}$$

where c and γ_0 are independent of r and ϕ_0.

Proof. Assume that (4.4) holds for some constants $\overline{c}$ and $\overline{\gamma}_0$ *a priori* depending on q and r. Then we will establish the existence of a constants c and γ_0 such that (4.4) holds with $\overline{c}, \overline{\gamma}_0$ replaced by c, γ_0, which are independent of r, provided that $q > r$. To this end, consider two solutions of problem (4.1) with $F = 0 : (w, p)$, with $w_0 \in L^r(\Omega)$, $\widetilde{v} = v$ and $A = \nabla v$ and $(h, \tilde{p})$, with initial data $h_0 \in \mathscr{C}_0(\Omega)$, $\widetilde{v} = -v$ and $A = \nabla v^T$. For a fixed $t > 0$, we set $\hat{h}(x, \tau) = h(x, t - \tau)$, for any $\tau \in [0, t]$. Multiplying $(4.1)_1$ by $\hat{h}$, and integrating by parts gives:

$$(w(t), h_0) = (w_0, h(t)). \tag{4.6}$$

Applying to the right-hand side of the last relation the Hölder inequality and the classical L^p convexity inequality, we obtain

$$\begin{aligned} |(w(t), h_0)| \leq \|w_0\|_r \|h(t)\|_{r'} \leq \|w_0\|_r \|h(t)\|_\infty^\theta \|h(t)\|_{q'}^{1-\theta},\\ \text{with } q' = \frac{q}{q-1},\ \theta = \frac{1}{r}\frac{q-r}{q-1} \ \text{ for any } q > r. \end{aligned} \tag{4.7}$$

The right-hand side can be estimated by (4.4) applied to $\|h(t)\|_{q'}$ and $\|h(t)\|_\infty$. Hence from (4.7) and (4.4) we deduce

$$|(w(t), h_0)| \leq e^{(\theta\overline{\gamma}_0(q',\infty)+(1-\theta)\overline{\gamma}_0(q',q'))t} (\overline{c}(q', \infty))^\theta (\overline{c}(q', q'))^{1-\theta} t^{-\frac{3}{2}(\frac{1}{r}-\frac{1}{q})} \|w_0\|_r \|h_0\|_{q'},$$
$$\text{for any } h_0 \in \mathscr{C}_0(\Omega),$$

which implies

$$|w(t)|_q \leq e^{(\theta\overline{\gamma}_0(q',\infty)+(1-\theta)\overline{\gamma}_0(q',q'))t} (\overline{c}(q', \infty))^\theta (\overline{c}(q', q'))^{1-\theta} t^{-\frac{3}{2}(\frac{1}{r}-\frac{1}{q})} \|w_0\|_r,$$
$$\text{for } q > r \geq 1, t > 0.$$

Therefore, setting

$$c = \max_{\theta \in [0,1]} (\overline{c}(q', \infty))^\theta (\overline{c}(q', q'))^{1-\theta} \ \text{ and } \ \gamma = \max_{\theta \in [0,1]} \theta\overline{\gamma}_0(q', \infty) + (1-\theta)\overline{\gamma}_0(q', q')),$$

c and γ are independent of r, which proves the estimate for $(4.5)_1$. The semigroup property

$$\|\nabla D_t^k w(t)\|_{\hat q} \le c\|w(t/2)\|_{\hat q} e^{\gamma_0(\hat q,\hat q)\frac{t}{2}} \begin{cases} t^{-\frac12 - k}, \ t\in(0,1), \ \hat q > r, \ t>0, \ \hat q \in (r,n], \\ t^{-\frac{n}{2\hat q}-k}, \ t>1, \ \hat q \ge n, \end{cases}$$

and estimate $(4.5)_1$ imply the latter of (4.5). The lemma is proved.

Remark 4.2. The result of the above lemma was given as a remark (Remark 3.1) in [13] for solutions of the Stokes problem. The result is fundamental for our aims just to prove pointwise estimate.

Lemma 4.2. *Let* (ϕ,π) *be the solution of system* (4.1) *whose existence is ensured by Theorem 4.3. Then, there exist a critical Reynolds number* $\mathscr{R}_c$ *and a constant* c *independent of* ϕ_0 *such that*

$$\begin{aligned} \|\phi(t)\|_q &\le c_1(q,t)\|\phi_0\|_2 e^{-\gamma(q,2)\left(\frac1R - \frac{1}{\mathscr{R}_c}\right)t}, \ t>0, \ q\in[2,\infty]; \\ \|\nabla\phi(t)\|_2 &\le c_2(t)\|\phi_0\|_2 e^{-\frac34\gamma\left(\frac1R - \frac{1}{\mathscr{R}_c}\right)t}, \qquad t>0; \\ \|\phi_t(t)\|_2 &\le c_3(t)\|\phi_0\|_2 e^{-\frac12\gamma\left(\frac1R - \frac{1}{\mathscr{R}_c}\right)t}, \qquad t>0; \\ \|P\Delta\phi(t)\|_2 &\le c_4(t)\|\phi_0\|_2 e^{-\frac12\gamma\left(\frac1R - \frac{1}{\mathscr{R}_c}\right)t}, \qquad t>0; \end{aligned} \tag{4.8}$$

where $\gamma(q,2) = \left(\frac58 + \frac34\frac1q\right)\gamma$, γ *is the constant of the Poincaré inequality,* $c_3(t) = c(t^{-1} + c(v)t^{-\frac12})$, $c_4(t) = c_3(t) + c_3^{\frac12}(t)c(v)$; $c_1(q,t) = c_4(t)^{\frac32\left(\frac12 - \frac1q\right)}$, $c_2(t) = c_3(t)^{\frac12}$; *in* $c_3(t)$, c *is just a numerical constant and* $c(v) = \|v\|_\infty + c_s\|\nabla v\|_3$, *where* c_s *is the Sobolev constant. Moreover, the following inequalities hold*

$$\|\phi_{tt}(t)\|_2 \le c_3(t)^2\|\phi_0\|_2 \ \text{ and } \ \|\nabla\phi_t(t)\| \le c_3(t)^{\frac32}\|\phi_0\|_2, \ t>0. \tag{4.9}$$

Proof. In order to prove $(4.8)_1$, the first step is just the energy inequality of solutions to problem (4.1), which we deduce employing the variational formulation given in Definition 2.4. Hence, multiplying $(4.1)_1$ by ϕ and integrating by parts furnishes

$$\frac12\frac{d}{dt}\|\phi(t)\|_2^2 = -\frac1R\|\nabla\phi(t)\|_2^2 - \int_\Omega \phi\cdot D\cdot\phi\, dx, \ t>0.$$

Since $\mathscr{F}(\phi) \le \frac{1}{\mathscr{R}_c}$ and $R < \mathscr{R}_c$, we deduce

$$\frac12\frac{d}{dt}\|\phi(t)\|_2^2 + \left(\frac1R - \frac{1}{\mathscr{R}_c}\right)\|\nabla\phi(t)\|_2^2 \le 0, \ t>0. \tag{4.10}$$

A first implication is the following inequality

$$\|\phi(t)\|_2^2 + 2\left(\frac1R - \frac{1}{\mathscr{R}_c}\right)\int_s^t \|\nabla\phi(\tau)\|_2^2 d\tau \le \|\phi(s)\|_2^2 \text{ for any } t\ge s \text{ and } s\ge 0. \tag{4.11}$$

Employing the Poincaré inequality, from the differential inequality (4.10) we obtain the asymptotic behavior

$$\|\phi(t)\|_2 \le e^{-\gamma\left(\frac{1}{R}-\frac{1}{\mathscr{R}_c}\right)(t-s)}\|\phi(s)\|_2 \text{ for any } t \ge s \text{ and } s \ge 0. \tag{4.12}$$

We prove $(4.8)_3$ in several steps. The first one is very easy. Differentiating equation $(4.1)_1$ with respect to t, one obtains, by quite analogous arguments, the following estimate

$$\|\phi_t(t)\|_2^2 \le e^{-2\gamma\left(\frac{1}{R}-\frac{1}{\mathscr{R}_c}\right)(t-s)}\|\phi_s(s)\|_2^2 \text{ for any } t \ge s \text{ and } s > 0. \tag{4.13}$$

Now, we try an estimate for the right-hand side of (4.13) in terms of $\|\phi_0\|_2$. Multiplying $(4.1)_1$ by ϕ_t and integrating on Ω we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla\phi(t)\|_2^2 + \|\phi_t(t)\|_2^2 = -(v\cdot\nabla\phi,\phi_t) - (\phi\cdot A(v),\phi_t).$$

Applying the Hölder inequality we can estimate the right-hand side in the following way:

$$|(v\cdot\nabla\phi,\phi_t) + (\phi\cdot A(v),\phi_t)| \le (\|v\|_\infty\|\nabla\phi\|_2 + \|\phi\|_6\|\nabla v\|_3)\|\phi_t\|_2;$$

applying the Sobolev inequality we obtain

$$|(v\cdot\nabla\phi,\phi_t) + (\phi\cdot A(v),\phi_t)| \le \frac{1}{2}(\|v\|_\infty + c_s\|\nabla v\|_3)^2\|\nabla\phi\|_2^2 + \frac{1}{2}\|\phi_t\|_2^2. \tag{4.14}$$

Hence, we can write

$$\frac{d}{dt}\|\nabla\phi(t)\|_2^2 + \|\phi_t(t)\|_2^2 \le (\|v\|_\infty + c_s\|\nabla v\|_3)^2\|\nabla\phi\|_2^2.$$

Let $\sigma \in (0,\frac{t}{2})$. Multiplying by $(\tau-\sigma)$ and integrating on $(\sigma,\frac{t}{2})$, we have

$$\begin{aligned}\left(\frac{t}{2}-\sigma\right)\left\|\nabla\phi\left(\frac{t}{2}\right)\right\|_2^2 &+ \int_\sigma^{\frac{t}{2}}(\tau-\sigma)\|\phi_\tau(\tau)\|_2^2 d\tau \\ &\le \left(\|v\|_\infty + c_s\|\nabla v\|_3\right)^2\int_\sigma^{\frac{t}{2}}(\tau-\sigma)\|\nabla\phi(\tau)\|_2^2 d\tau + \int_\sigma^{\frac{t}{2}}\|\nabla\phi(\tau)\|_2^2 d\tau. \end{aligned} \tag{4.15}$$

Multiplying (4.13) by $(\tau-\sigma)$ and integrating with respect to $\tau \in (\sigma,\frac{t}{2})$, by a simple computation we deduce

$$\frac{1}{2}\left(\frac{t}{2}-\sigma\right)^2\|\phi_t(t)\|_2^2 \le e^{-\gamma t\left(\frac{1}{R}-\frac{1}{\mathscr{R}_c}\right)}\int_\sigma^{\frac{t}{2}}(\tau-\sigma)\|\phi_\tau(\tau)\|_2^2 ds,\ t > 2\sigma.$$

We estimate the right-hand side of the last inequality by (4.15), hence

$$\|\phi_t(t)\|_2^2 \le 8\frac{e^{-\gamma t\left(\frac{1}{R}-\frac{1}{\mathscr{R}_c}\right)}}{(t-2\sigma)^2}\left[(\|v\|_\infty + c_s\|\nabla v\|_3)^2\int_\sigma^{\frac{t}{2}}(\tau-\sigma)\|\nabla\phi(\tau)\|_2^2 d\tau + \int_\sigma^{\frac{t}{2}}\|\nabla\phi(\tau)\|_2^2 d\tau\right].$$

Now, applying estimate (4.11), we can deduce

$$\|\phi_t(t)\|_2 \leq c\frac{e^{-\gamma\frac{t}{2}\left(\frac{1}{R}-\frac{1}{\mathscr{R}_c}\right)}}{(t-2\sigma)^{\frac{1}{2}}}\Big[(\|v\|_\infty + c_s\|\nabla v\|_3) + (t-2\sigma)^{-\frac{1}{2}}\Big]\|\phi(\sigma)\|_2, \qquad (4.16)$$

where c is just a numerical constant. The last estimate proves $(4.8)_3$ provided that $\sigma = \frac{t}{4}$. We set $c_3(t) = c\left[t^{-\frac{1}{2}}(\|v\|_\infty + c_s\|\nabla v\|_3) + t^{-1}\right]$. The differential inequality (4.10) also implies the following one

$$\left(\frac{1}{R} - \frac{1}{R_c}\right)\|\nabla\phi(t)\|_2^2 \leq \|\phi(t)\|_2\|\phi_t(t)\|_2. \qquad (4.17)$$

Taking into account $(4.8)_1$ and $(4.8)_3$, we easily obtain $(4.8)_2$. Multiplying equation $(4.1)_1$ by $P\Delta\phi$, integrating on Ω and applying the Hölder inequality, as made in estimate (4.14), we arrive at

$$\|P\Delta\phi(t)\|_2^2 \leq \|\phi_t(t)\|_2^2 + (\|v\|_\infty + c_s\|\nabla v\|_3)^2\|\nabla\phi(t)\|_2^2.$$

Hence, by virtue of estimates $(4.8)_2$ and $(4.8)_3$ we obtain $(4.8)_4$. Now we complete the proof of estimate $(4.8)_1$ for $q \geq 2$. To this end we recall the Sobolev inequality of Lemma 3.2. We apply inequality (3.6) with $r = r_1 = 2$ to the L^q-norm of ϕ. Hence the result follows from (4.12) and $(4.8)_4$. Since v is steady, by differentiating equation $(4.1)_1$ and by quite analogous arguments to those employed to obtain (4.16), we can deduce the following estimates:

$$\|\phi_{tt}(t)\|_2 \leq c\frac{e^{-\gamma\frac{t}{2}\left(\frac{1}{R}-\frac{1}{\mathscr{R}_c}\right)}}{(t-2\sigma)^{\frac{1}{2}}}\Big[(\|v\|_\infty + c_s\|\nabla v\|_3) + (t-2\sigma)^{-\frac{1}{2}}\Big]\|\phi_\sigma(\sigma)\|_2. \qquad (4.18)$$

Choosing $\sigma = \frac{t}{2}$ in (4.18), estimate $(4.8)_3$ implies $(4.9)_1$. In the same way as we showed estimate (4.17), we can prove

$$\left(\frac{1}{R} - \frac{1}{\mathscr{R}_c}\right)\|\nabla\phi_t(t)\|_2^2 \leq \|\phi_t(t)\|_2\|\phi_{tt}(t)\|_2,$$

hence, $(4.9)_2$ easily follows from (4.8).

Remark 4.3. With the exception of constant γ, all remaining constants appearing in estimates (4.8) are independent of the assumption of Ω bounded. Moreover, all estimates hold because $R < \mathscr{R}_c$. Apart from the assumption on the Reynolds number, we would like to stress that no requirement of smallness is made for the norms involving v. If $v = 0$ estimates (4.8) become those of the solutions of the Stokes problem. Finally, in estimates $(4.8)_{2,3,4}$ the coefficients of the exponential decay are not sharp. However, we are not interested in finding best constants.

Lemma 4.3. *In system* (4.1)*, assume* $v \in \mathscr{C}_{|0}(\overline{\Omega})$*,* $A \in C^0(\overline{\Omega})$ *and* $R < \mathscr{R}_c$ *and let* $\varphi_0 \in J^2(\Omega)$*. Then, there exists a unique regular solution* (φ, ϖ) *of problem* (4.1)*. Moreover,* $\varphi(x,t)$ *satisfies estimates* (4.8).

Proof. Since φ_0 can be approximated by a sequence $\{\phi^n\} \subset \mathscr{C}_0(\Omega)$, we can establish the existence of a sequence of solutions (ϕ^n, π^n), whose elements satisfy estimates (4.8). Taking into account the linearity of the problem, one deduces

$$\begin{aligned}
\|\phi^n(t) - \phi(t)^m\|_q &\le c_1(q,t)\|\phi_0^n - \phi_0^m\|_2 e^{-\gamma(q,2)\left(\frac{1}{R}-\frac{1}{\mathscr{R}_c}\right)t}, \ t>0, \ q\in[2,\infty];\\
\|\nabla(\phi^n(t) - \phi^m(t))\|_2 &\le c_2(t)\|\phi_0^n - \phi_0^m\|_2 e^{-\frac{3}{4}\gamma\left(\frac{1}{R}-\frac{1}{\mathscr{R}_c}\right)t}, \qquad t>0;\\
\|\phi_t^n(t) - \phi_t^m(t)\|_2 &\le c_3(t)\|\phi_0^n - \phi_0^m\|_2 e^{-\frac{1}{2}\gamma\left(\frac{1}{R}-\frac{1}{\mathscr{R}_c}\right)t}, \qquad t>0;\\
\|P\Delta(\phi^n(t) - \phi^m(t))\|_2 &\le c_4(t)\|\phi_0^n - \phi_0^m\|_2 e^{-\frac{1}{2}\gamma\left(\frac{1}{R}-\frac{1}{\mathscr{R}_c}\right)t}, \qquad t>0.
\end{aligned}$$

Hence, the sequence is a Cauchy sequence and the limit is a regular solution, which satisfies estimates (4.1). The uniqueness part is immediate, and hence it will be omitted.

Lemma 4.4. *Let $(\psi, \tilde{\pi})$ be the regular solution of system (4.1) corresponding to $\psi_0 \in \mathscr{C}_0(\Omega)$ and whose existence is ensured by Theorem 4.3. Then, there exists a constant c such that*

$$\begin{aligned}
\|\psi(t)\|_2 &\le cc_1(2,t/2)c_1(p',t/2)\|\psi_0\|_p e^{-\gamma(2,p)\left(\frac{1}{R}-\frac{1}{\mathscr{R}_c}\right)\frac{t}{2}}, \ \ t>0,\\
\|\nabla\psi(t)\|_2 &\le cc_1(p',t/2)c_2(t)\|\psi_0\|_p e^{-(\frac{3}{4}\gamma+\gamma(2,p))\left(\frac{1}{R}-\frac{1}{\mathscr{R}_c}\right)\frac{t}{2}}, \ \ t>0,
\end{aligned} \tag{4.19}$$

provided that $p \in [1,2]$ and $q \in [2,\infty]$. The constant c is independent of ψ_0, p; $\gamma(2,p) = \left(\frac{11}{8} - \frac{3}{4}\frac{1}{p}\right)\gamma$.

Proof. In system (4.1) we assume $\psi \cdot A(v) = \psi \cdot \nabla v$. Let (ψ, π) be the solution corresponding to ψ_0. Likewise, we denote by $(\varphi, \widehat{\pi})$ the solution of system (4.1) with $\varphi \cdot A(v) = \varphi \cdot \nabla v^T$ and corresponding to φ_0. We multiply equation $(4.1)_1$ corresponding to $(\psi, \widehat{\pi})$ by $\varphi(t-\tau, x)$, $\tau(0,t)$. Integrating by parts on $\Omega \times (0,t)$ we obtain, for any $t > 0$,

$$|(\psi(t), \varphi_0)| = |(\psi_0, \varphi(t))| \le \|\psi_0\|_p \|\varphi(t)\|_{p'}.$$

Since $p \in [1,2]$, then $p' \in [2,\infty]$ and, by virtue of $(4.8)_1$, we have

$$|(\psi(t), \varphi_0)| \le c_1(p', t/2)\|\psi_0\|_p \|\varphi_0\|_2 e^{-\gamma(p',2)\left(\frac{1}{R}-\frac{1}{\mathscr{R}_c}\right)t}, \ t>0.$$

Hence we have the estimate

$$\|\psi(t)\|_2 \le \|\psi_0\|_p e^{-\gamma(2,p)\left(\frac{1}{R}-\frac{1}{\mathscr{R}_c}\right)t}, \ t>0. \tag{4.20}$$

Estimate $(4.19)_2$ is an easy consequence of $(4.8)_2$ evaluated on the interval $\left(\frac{t}{2}, t\right)$ and $(4.19)_1$ evaluated on the interval $\left(0, \frac{t}{2}\right)$.

Remark 4.4. It is important to emphasize that $\gamma(2,p)$ is a continuous function on $[1,p]$, its minimum value is $\frac{5}{8}$ and it is assumed in $p = 1$; hence the minimum exponent of $(4.19)_2$ is $\frac{11}{16}$.

Lemma 4.5. *Let $u_0 \in \mathscr{C}_0(\Omega)$ and $F = 0$ in system (4.1). Assume $R < \mathscr{R}_c$. Then, there exists a unique classical solution (u^0, p^0) of system (4.1) such that*

$$|u^0(t)|_0 \le c|u_0|_0 e^{-\overline{\gamma}\left(\frac{1}{R}-\frac{1}{\mathscr{R}_c}\right)t},\ t > 0; \tag{4.21}$$

with $\overline{\gamma} = \frac{5}{8}\gamma$, where γ is the constant of the Poincaré inequality, and c independent of u_0.

Proof. The existence and uniqueness are ensured by Theorem 4.2. For $t \ge 1$ estimate (4.21) follows from $(4.8)_1$ and from the inequality $\|u_0\|_2 \le (\text{meas}(\Omega))^{\frac{1}{2}}|u_0|_0$. Thus we prove (4.21) for $t \in (0,1)$. Let us consider $u^0(x,t) = U^0 + U^1$ and $p^0 = P^0 + P^1$, where

$$\begin{gathered} U^0_t - \frac{1}{R}\Delta U^0 = -\nabla P^0, \nabla \cdot U^0 = 0,\ \text{in } \Omega, \\ U^0 = 0 \text{ on } \partial\Omega \times (0,T),\ U^0(x,0) = u_0(x) \text{ in } \Omega; \end{gathered} \tag{4.22}$$

and

$$\begin{gathered} U^1_t - \frac{1}{R}\Delta U^1 = -\nabla P^1 - U^1 \cdot \nabla v - v \cdot \nabla U^1 - U^0 \cdot \nabla v - v \cdot \nabla U^0, \\ \nabla \cdot U^1 = 0,\ \text{in } \Omega, \\ U^1 = 0 \text{ on } \partial\Omega \times (0,T),\ U^1(x,0) = 0 \text{ in } \Omega. \end{gathered} \tag{4.23}$$

Since Lemma 3.4 proves that $|U^0(t)|_0 \le c|u_0|_0$, we must prove the estimate just for $U^1(x,t)$. We multiply equation $(4.23)_1$ by $\phi(x, t-\tau), \tau \in (0,t), t \in (0,1)$ and $\phi(x,s)$ solution of system (4.1) with $v = -v$ and $\phi \cdot A = \phi \cdot \nabla v^T$. Integrating by parts on $\Omega \times (0,t)$, we obtain

$$(U^1(t), \phi_0) = \int_0^t [(v \cdot \nabla\phi, U^0) + (U^0 \cdot \nabla\phi, v)]d\tau = I_1(t) + I_2(t). \tag{4.24}$$

Applying the Hölder inequality, estimate (3.2) and estimate (4.5), we deduce

$$\begin{aligned} |I_1(t) + I_2(t)| &\le c\|v\|_{q'}|u_0|_0 \int_0^t \|\nabla\phi(t-\tau)\|_q d\tau \\ &\le c\|v\|_{q'}|u_0|_0\|\phi_0\|_r \int_0^t (t-\tau)^{-\frac{1}{2}-\frac{3}{2}\left(\frac{1}{r}-\frac{1}{q}\right)} d\tau, \end{aligned}$$

where $r \in (1,q)$ and $q < \frac{3}{2}$. Employing the above estimates for the right-hand side of (4.24), taking into account that the estimates hold for any $\phi_o \in \mathscr{C}_0(\Omega)$, we obtain

$$\|U^1(t)\|_{r'} \le c\|v\|_{q'}|u_0|_0,\ \text{where } r' = \frac{r}{r-1}.$$

By virtue of Lemma 4.1, the last estimate holds for any $r \in (1, q)$ and with a constant c which can be chosen independent of r. Therefore making the limit for $r' \to \infty$ we deduce

$$|U^1(t)|_0 = \|U^1(t)\|_\infty \le c\|v\|_{q'}|u_0|_0,$$

which completes the proof.

Remark 4.5. The above lemma gives an estimate of the maximum modulus of a classical solution $u^0(x, t)$ of the initial boundary value problem of the linearized system of the perturbation. The estimates of the lemma have the unsatisfactory feature that the constant c depends on the measure of Ω. Hence one cannot extend the proof to the case of unbounded domains. Therefore, we would like to emphasize that the estimate of the maximum modulus of a solution $u^0(x, t)$ holds in the case of the initial boundary value problem in an exterior domain also. In fact, we could show the validity of the estimate, with a constant c independent of meas(Ω), but depending on t in an increasing fashion. We do not give the proof of this property here, because by such an estimate, even with an assumption of small data, we would not able to prove, in an exterior domain, existence of classical solutions of system (1.1) defined for any $t > 0$.

Let us consider the Stokes problem:

$$\Delta h - \nabla g = q,\ \nabla \cdot h = 0,\ \text{in } \Omega,\ h = 0 \text{ on } \partial\Omega.$$

We recall the following well known results holding in a bounded domain.

Lemma 4.6. *Let $q \in L^2(\Omega)$. Then, there exists a unique regular solution (h, g) of the Stokes problem such that*

$$\|h\|_{2,2} + \|\nabla g\|_2 \le c\|q\|_2, \tag{4.25}$$

with c independent of q.

Lemma 4.7. *Let $u_0 \in \mathscr{C}_0(\Omega)$ and let $F = 0$ in system (4.1). Then, there exists a unique classical solution (u^0, p^0) of system (4.1) such that,*

$$|u^0(x, t)| \le c|u_0|_0 \text{ for any } t > 0, \tag{4.26}$$

with a constant c independent of u_0. Moreover, let $\eta > 0$. Then, for any $\overline{t}, \overline{\overline{t}} \in (\eta, T)$, $\overline{x}, \overline{\overline{x}} \in \overline{\Omega}$, the following inequality holds

$$\begin{aligned} &|D^2u^0(\overline{x},\overline{t}) - D^2u^0(\overline{\overline{x}},\overline{\overline{t}})| + |u^0_t(\overline{x},\overline{t}) - u^0_t(\overline{\overline{x}},\overline{\overline{t}})| + |\nabla p^0(\overline{x},\overline{t}) - \nabla p^0(\overline{\overline{x}},\overline{\overline{t}})| \\ &\qquad \le c(p)H(t_\circ)(|\overline{x} - \overline{\overline{x}}| + |\overline{t} - \overline{\overline{t}}|^{\frac{1}{2}})^\beta |u_0|_0^{1-\beta}\big(\|u_0\|_p^\beta + |u_0|_0^\beta\big), \end{aligned} \tag{4.27}$$

where $\beta \in (0, 1)$, $t_\circ = \min\{\overline{t}, \overline{\overline{t}}\}$, $c(p)^{-1} \to 0$ as $p \to \infty$, and $H(t_\circ) \to \infty$ as $t_\circ \to 0$. Finally, $c(p)$ and $H(t_0)$ are independent of u_0.

Proof. The existence and uniqueness of (u^0, p^0) are given in Theorem 4.2. Estimate (4.26) is a trivial consequence of Lemma 4.5. To prove (4.27), first of all we prove the following Hölder estimates for the solution u^0:

$$|u^0(\overline{x}, t) - u^0(\overline{\overline{x}}, t)| \le c|\overline{x} - \overline{\overline{x}}|^{\frac{1}{2}} \|u_0\|_2 t^{-\gamma_3}, \tag{4.28}$$

$$|u^0(x, \overline{t}) - u^0(x, \overline{\overline{t}})| \le c|\overline{t} - \overline{\overline{t}}|^{\frac{1}{4}} \|u_0\|_2 \overline{t}^{-\gamma_4}, \tag{4.29}$$

where c is independent of u_0 and γ_3, γ_4 are suitable positive exponents and we have assumed $\overline{\overline{t}} > \overline{t} > \frac{\eta}{2} > 0$. This assumption is considered in the sequel also; the case $\overline{\overline{t}} < \overline{t}$ is quite analogous. Since $u_0 \in \mathscr{C}_0(\Omega)$, by assuming $p = 2$, we have that (u^0, p^0) is also a regular solution. From the estimates of Lemma 4.2 and of Lemma 4.7 we have in particular

$$\|u^0(t)\|_{2,2} \le c(\|u^0_t(t)\|_2 + \|v \cdot \nabla u^0(t)\|_2 + \|u^0(t) \cdot \nabla v\|_2),$$
$$\|u^0_t(t)\|_{2,2} \le c(\|u^0_{tt}(t)\|_2 + \|v \cdot \nabla u^0_t(t)\|_2 + \|u^0_t(t) \cdot \nabla v\|_2).$$

Employing the Hölder inequality and the Sobolev embedding theorem we deduce

$$\|u^0(t)\|_{2,2} \le c\|u^0_t(t)\|_2 + c(\|v\|_\infty + c_s\|\nabla v\|_3)\|\nabla u^0(t)\|_2$$
$$\|u^0_t(t)\|_{2,2} \le c\|u^0_{tt}(t)\|_2 + c(\|v\|_\infty + c_s\|\nabla v\|_3)\|\nabla u^0_t(t)\|_2.$$

Estimates (4.8) and the Sobolev embedding theorem allow us to deduce estimates (4.28)–(4.29). In order to obtain estimate (4.27) we argue as in the proof of Lemma 3.5 to prove estimate (3.12). Hence, we multiply equation $(4.1)_1$ by $\zeta(t)$, and, by a simple computation, we have

$$\begin{aligned} U^0_t - \Delta U^0 &= -\nabla P^0 - v \cdot \nabla U^0 - A(v) \cdot U^0 + \zeta'(t) u^0, \\ \nabla \cdot U^0 &= 0, \text{ in } \Omega \times (0, T), \\ u^0 &= 0 \text{ on } \partial\Omega \times (0, T),\ U^0(x, 0) = 0. \end{aligned} \tag{4.30}$$

Theorem 4.2 ensures the validity of following estimate

$$|D^2 U^0|_\alpha + |U^0_t|_\alpha + |\nabla P^0|_\alpha + |P^0|_0 \le c\big(|\zeta' u^0|_\alpha + \sup_{(0,T)} |U^0(t)|_0\big). \tag{4.31}$$

Taking into account estimate (4.26) and estimates (4.28)–(4.29), the right-hand side of (4.31) can be just evaluated by the same arguments employed for solutions of the Stokes problem, and (4.27) follows.

Proof of Theorem 4.1. The proof is quite analogous to the one given for Theorem 3.1 concerning the solutions of the Stokes problem. We just recall the main steps. Since $u_0(x) \in \mathscr{C}_{|0}(\overline{\Omega})$, there exists a sequence $\{u^n_0\} \subset \mathscr{C}_0(\Omega)$ converging to u_0 in $C(\overline{\Omega})$. By virtue of the linearity of system (4.1) and Lemma 4.5 and Lemma 4.7 we can show the following estimates

$$|u^{0n}(x, t) - u^{0m}(x, t)| \le c|u^{0n} - u^{0m}|_0 e^{-\gamma t};$$

for any $\eta > 0$ and $\overline{t}, \overline{\overline{t}} \in (\eta, T)$, $\overline{x}, \overline{\overline{x}} \in \Omega$,

$$\begin{aligned}&\left|D^2u^{0n}(\overline{x},\overline{t})-D^2u^{0m}(\overline{\overline{x}},\overline{\overline{t}})\right|+\left|u^{0n}_t(\overline{x},\overline{t})-u^{0m}_t(\overline{\overline{x}},\overline{\overline{t}})\right|+\left|\nabla p^{0n}(\overline{x},\overline{t})-\nabla p^{0m}(\overline{\overline{x}},\overline{\overline{t}})\right|\\ &\quad\leq c(p)H(t_\circ)\big(\left|\overline{x}-\overline{\overline{x}}\right|+\left|\overline{t}-\overline{\overline{t}}\right|^{\frac{1}{2}}\big)^{\beta}\left|u^n_0-u^m_0\right|_0^{1-\beta}\big(\|u^n_0-u^m_0\|^{\beta}_p+\left|u^n_0-u^m_0\right|^{\beta}_0\big),\end{aligned}$$

for some β and $t_\circ = \min\{\overline{t}, \overline{\overline{t}}\}$. The above estimates ensure existence. As far as uniqueness is concerned, assume that, in system (4.1), $u_0(x) = 0$ and multiply system (4.1) by $\phi(x, t-\tau)$, with $(\phi(x, s), \pi(x, s))$ solution of the adjoint system to (4.1). Integrating by parts on $(\overline{t}, t) \times \Omega$, we obtain the relation

$$(u^0(t), \phi_0) = (u(\overline{t}), \phi(t - \overline{t})).$$

By the arbitrariness of $\overline{t} > 0$ and by the continuity of functions $u(x, t)$ and $\phi(x, s)$, letting $\overline{t} \to 0$, we arrive at $(u^0(t), \phi_0) = 0$. Assuming $\phi_0 = u^0(t)$ we deduce the uniqueness. Finally estimate (4.2) is a consequence of the uniform convergence and of estimate (4.21).

5. Problem (1.1) with initial data in $\mathscr{C}_{|0}(\overline{\Omega})$

Proof of Theorem 2.1. To prove Theorem 2.1 we follow an idea given in [12] for the nonlinear stationary Navier–Stokes system. At first, we establish the existence of a weak solution u (weak in the sense of Leray–Hopf) to problem (1.1) such that $u \in C(0, T; \mathscr{C}_{|0}(\overline{\Omega}))$. For the latter task it is required that the size of the uniform norm of the data u_0 is sufficiently small. Subsequently, we prove that such weak solution is a classical solution. The result of existence is obtained by applying the method of successive approximations. We set $U^0 = u^0$, $P^0 = p^0$ where (u^0, p^0) is the solution to system (4.1) with $F = 0$. For $n \in \mathbb{N}$, we set $u^n = U^0 + U^n$, $\pi^n = P^0 + P^n$, where the pair (U^n, P^n) is the solution of the problem

$$\begin{aligned}U^n_t - \Delta U^n &= -\nabla P^n - v \cdot \nabla U^n - U^n \cdot \nabla v - u^{n-1} \cdot \nabla u^{n-1},\\ \nabla \cdot U^n &= 0, \text{ in } \Omega \times (0, T);\\ U^n &= 0 \text{ on } \partial\Omega \times (0, T),\ U^n(x, 0) = 0,\end{aligned} \tag{5.1}$$

In system (5.1) the coefficients v and ∇v are the same as in system (1.1). It then follows that the pair (u^n, π^n) is a solution to the following problem

$$\begin{aligned}u^n_t + u^{n-1} \cdot \nabla u^{n-1} + (v \cdot \nabla)u^n + (u^n \cdot \nabla)v + \nabla\pi^n &= \frac{1}{R}\Delta u^n,\\ \nabla \cdot u^n = 0, \quad \text{in } \Omega \times (0, T),&\\ u^n(x, t) = 0, \text{ on } \partial\Omega \times (0, T),\ u^n(x, 0) = u_0(x) \text{ in } \Omega.&\end{aligned} \tag{5.2}$$

The existence of the approximates $u^n(x, t)$

The pair (U^0, P^0) exists by virtue of Theorem 4.8 and it is a classical solution. Moreover, assuming that $u^{n-1} \in C(0, T; \mathscr{C}_{|0}(\overline{\Omega})) \cap L^2(0, T; J^{1,2}(\Omega))$, then $u^{n-1} \cdot$

$\nabla u^{n-1} \in L^2(0,T;L^2(\Omega))$. Hence the pair (U^n,P^n) exists by virtue of Theorem 4.3 and it is a regular solution of (5.1). This last property implies that $u^n(x,t)$ is a regular solution of system (5.2). However, assuming that, for any $\eta > 0$, $u^{n-1} \in C^{0,\frac{\alpha}{2}}(\eta,T;C^{1,\alpha}(\overline{\Omega}))$, we can prove that (U^n,P^n) is a classical solution on (η,T).[4] Let $\overline{\eta} > 0$. Since (U^n,P^n) is a regular solution, the Sobolev embedding theorem implies that, for any $\eta_1 \in (0,\overline{\eta})$, $U^n \in L^p(\eta_1,T;J^p(\Omega))$, for $p \leq 4$. Of course, by the arbitrariness of $\eta > 0$ in our assumption, $u^{n-1}\cdot\nabla u^{n-1} \in L^p(\eta_1,T;L^p(\Omega))$, for any $p \leq 4$ also. Let us consider a smooth function $h_1(t)$, defined for $t \geq \eta_1$, with $h_1(\eta_1) = 0$ and $h_1(t) > 0$, for $t > \eta_1$. The pair $(\widetilde{U}^n,\widetilde{P}^n) = h_1(t)(U^n(x,t),P^n(x,t))$ is a regular solution to the following initial boundary value problem:

$$\begin{gathered}\widetilde{U}^n_t - \Delta\widetilde{U}^n + \nabla\widetilde{P}^n = -v\cdot\nabla\widetilde{U}^n - \widetilde{U}^n\cdot\nabla v - h_1(t)u^{n-1}\cdot\nabla u^{n-1} + h_1'(t)U^n,\\ \nabla\cdot\widetilde{U}^n = 0, \text{ in } \Omega\times(\eta_1,T),\\ \widetilde{U}^n = 0 \text{ on } \partial\Omega\times(\eta_1,T),\ \widetilde{U}^n(x,\eta_1) = 0.\end{gathered} \tag{5.3}$$

For $p = 4$, by virtue of Theorem 4.3, system (5.3) admits a unique p-regular solution. By uniqueness, $(\widetilde{U}^n,\widetilde{P}^n)$ is a p-regular solution also. Taking into account that $\widetilde{U}^n = 0$ for $t \in (0,\eta_1)$, by imbedding we deduce that $\widetilde{U}^n \in \mathcal{H}^\alpha(\Omega\times(0,T))$ for any $\alpha \in (0,\frac{3}{4}]$. Let $\eta_2 \in (\eta_1,\overline{\eta})$. Let us consider a smooth function $h_2(t)$, defined for $t \geq \eta_2$, with $h_2(\eta_2) = 0$ and $h_2(t) > 0$, for $t > \eta_2$. The pair $(\widehat{U}^n,\widehat{P}^n) = h_2(t)(U^n(x,t),P^n(x,t))$ is a regular solution to the following initial boundary value problem:

$$\begin{gathered}\widehat{U}^n_t - \Delta\widehat{U}^n + \nabla\widehat{P}^n = -v\cdot\nabla\widehat{U}^n - \widehat{U}^n\cdot\nabla v - h_2(t)u^{n-1}\cdot\nabla u^{n-1} + h_2'(t)U^n,\\ \nabla\cdot\widehat{U}^n = 0, \text{ in } \Omega\times(\eta_2,T),\\ \widehat{U}^n = 0 \text{ on } \partial\Omega\times(\eta_2,T),\ \widehat{U}^n(x,\eta_2) = 0.\end{gathered} \tag{5.4}$$

Taking into account that $h_2(\eta_2) = 0$ and that $U^n(x,t) \in C(0,T;\mathscr{C}_{|0}(\overline{\Omega})) \cap \mathcal{H}^\alpha(\Omega\times(0,T))$, the data of the IBVP of system (5.4), that is $F = h_2u^{n-1}\cdot\nabla u^{n-1} + h_2'U^n$ and $\widehat{U}^n(x,\eta_2) = 0$, satisfy the compatibility condition required in Theorem 4.2. Hence there exists a unique classical solution of system (5.4) and, by uniqueness, it coincides with the pair $(\widehat{U}^n,\widehat{P}^n)$. This proves that the pair (U^n,P^n) is a classical solution in $(\overline{\eta},T)\times\Omega$. As a consequence we can state that (u^n,π^n) is a classical solution in $(\overline{\eta},T)\times\Omega$. The arbitrariness of $\overline{\eta}$ completes the proof of the result.

The sequence $\{u^n(x,t)\} \subset C(0,T;\mathscr{C}_{|0}(\overline{\Omega}))$

Since U^0 is a classical solution, then $U^0 \in C(0,T;\mathscr{C}_{|0}(\overline{\Omega}))$. Now let us prove that $u^{n-1} \in C(0,T;\mathscr{C}_{|0}(\overline{\Omega}))$ implies that $u^n \in C(0,T;\mathscr{C}_{|0}(\overline{\Omega}))$. Since $u^n = U^0 + U^n$, we just have to prove that $U^n(x,t)$ is uniformly continuous on $[0,T)\times\overline{\Omega}$. Let

[4] Here and in the sequel, proving the proprieties of the classical solutions for regular solutions actually we employ the technique of the structure theorem applied by Leray for a weak solution [11].

$t_2, t_1 \in (0,T)$ and assume $t_2 > t_1$. Since $U^n(x,\tau) \in C(0,T; J^2(\Omega))$, by a simple computation one obtains the following relations:

$$(U^n(t_2), \phi_0) = -\int_0^{t_2} \big(u^{n-1}(t_2-\tau)\cdot\nabla\phi(\tau), u^{n-1}(t_2-\tau)\big)d\tau,$$

$$(U^n(t_1), \phi_0) = -\int_0^{t_1} \big(u^{n-1}(t_1-\tau)\cdot\nabla\phi(\tau), u^{n-1}(t_1-\tau)\big)d\tau,$$

where $\phi(x,\tau)$ is the regular solution of the problem (4.1) corresponding to $\phi_0 \in \mathscr{C}_0(\Omega)$. Therefore

$$\begin{aligned}
&|(U^n(t_2) - U^n(t_1), \phi_0)| \\
&\leq \int_0^{t_1} |(u^{n-1}(t_2-\tau)\cdot\nabla\phi(\tau), u^{n-1}(t_2-\tau) - u^{n-1}(t_1-\tau))|d\tau \\
&\quad + \int_0^{t_1} |((u^{n-1}(t_2-\tau) - u^{n-1}(t_1-\tau))\cdot\nabla\phi(\tau), u^{n-1}(t_1-\tau))|d\tau \\
&\quad + \int_{t_1}^{t_2} |(u^{n-1}(t_2-\tau)\cdot\nabla\phi(\tau), u^{n-1}(t_2-\tau))|d\tau = I_1 + I_2 + I_3.
\end{aligned} \tag{5.5}$$

We estimate I_i, for $i = 1,2,3$. Employing estimate $(4.5)_2$ and estimate $(4.19)_2$ for I_1 and I_2 respectively, we obtain

$$\begin{aligned}
I_1 + I_2 &\leq \max_{\overline{\Omega}\times[0,T]} |u^{n-1}(x,t_2-\tau) - u^{n-1}(x,t_1-\tau)| \max_{\overline{\Omega}\times[0,T]} |u^{n-1}(x,s)| \cdot \\
&\quad \cdot c(\Omega)\left[\int_0^1 |\nabla\phi(\tau)|_p d\tau + \int_1^T |\nabla\phi(\tau)|_2 d\tau\right] \\
&\leq \max_{\overline{\Omega}\times[0,T]} |u^{n-1}(x,t_2-\tau) - u^{n-1}(x,t_1-\tau)| \max_{\overline{\Omega}\times[0,T]} |u^{n-1}(x,s)| \cdot \\
&\quad \cdot c(\Omega,R,v)|\phi_0|_r \left[\int_0^1 \tau^{-\frac12-\frac32\left(\frac1r-\frac1p\right)}d\tau + \int_1^T e^{-\frac{11}{16}\left(\frac1R-\frac1{\mathscr{R}}\right)\tau}\tau^{-\frac12-\frac32\left(\frac1r-\frac12\right)}d\tau\right];
\end{aligned}$$

in the above estimate we have assumed $p < \frac32$ and $r \in (1,p)$. As far as I_3 is concerned, by means of $(4.5)_2$ again, we have

$$I_3 \leq \max_{\overline{\Omega}\times[0,T]} |u^{n-1}(x,s)|^2 c(\Omega)\int_{t_1}^{t_2} |\nabla\phi(\tau)|_p d\tau$$

$$\leq \max_{\overline{\Omega}\times[0,T]} |u^{n-1}(x,s)|^2 c(\Omega,R,v) e^{\gamma_0 T} |\phi_0|_r \Big(t_2^{\frac{1}{2}-\frac{3}{2}\left(\frac{1}{r}-\frac{1}{p}\right)} - t_1^{\frac{1}{2}-\frac{3}{2}\left(\frac{1}{r}-\frac{1}{p}\right)} \Big),$$

where we have employed the assumptions $p < \frac{3}{2}$ and $r \in (1,p)$ again. We estimate the right-hand side of (5.5) by the estimates obtained for I_i, $i = 1,2,3$. The arbitrariness of ϕ_0 implies that

$$\begin{aligned}
&\|U^n(t_2) - U^n(t_1)\|_{r'} \\
&\leq c(\Omega,R,v) \max_{\overline{\Omega}\times[0,T]} |u^{n-1}(x,t_2-\tau) - u^{n-1}(x,t_1-\tau)| \max_{\overline{\Omega}\times[0,T]} |u^{n-1}(x,s)| \\
&\quad + \max_{\overline{\Omega}\times[0,T]} |u^{n-1}(x,s)|^2 c(\Omega,R,v) e^{\gamma_0 T} \Big(t_2^{\frac{1}{2}-\frac{3}{2}\left(\frac{1}{r}-\frac{1}{p}\right)} - t_1^{\frac{1}{2}-\frac{3}{2}\left(\frac{1}{r}-\frac{1}{p}\right)} \Big).
\end{aligned}$$

By virtue of Lemma 4.1, the right-hand side of this latter inequality is independent of r, and, therefore, in the limit $r' \to \infty$ we deduce that

$$\begin{aligned}
&|U^n(t_2) - U^n(t_1)|_0 \\
&\leq c(\Omega,R,v) \max_{\overline{\Omega}\times[0,T]} |u^{n-1}(x,t_2-\tau) - u^{n-1}(x,t_1-\tau)| \cdot \max_{\overline{\Omega}\times[0,T]} |u^{n-1}(x,s)| \\
&\quad + \max_{\overline{\Omega}\times[0,T]} |u^{n-1}(x,s)|^2 c(\Omega,R,v) e^{\gamma_0 T} \Big(t_2^{\frac{1}{2}-\frac{3}{2}\left(\frac{1}{r}-\frac{1}{p}\right)} - t_1^{\frac{1}{2}-\frac{3}{2}\left(\frac{1}{r}-\frac{1}{p}\right)} \Big),
\end{aligned}$$

which proves that $U^n(x,t)$ is uniformly continuous in (x,t). As a consequence, we have that the limit as $t \to 0$ exists in the uniform norm. Since in the L^2 norm the limit of $U^n(x,t)$ is zero, we have $\lim_{(x,t)\to(x_0,0)} U^n(x,t) = 0$. The above results for U^n allow us to state that (u^n, π^n) is a classical solution of problem (5.2) in $\Omega \times (0,T)$ and, for any $T > 0$, $u_n \in C(0,T;\mathscr{C}_{|0}(\overline{\Omega}))$. The proof then follows from the fact that $n \in \mathbb{N}$ is arbitrary.

Estimates for the convergence of the sequence $\{u^n\}$

Here, unless the contrary is explicitly stated, we denote by c a generic constant whose value is independent of the initial data $u_0(x)$ and it is uniform with respect to n and (x,t). We set $\mathfrak{U}_0 = |u_0|_0$ and, for any $n \in \mathbb{N}_0$, $\mathfrak{U}^n(T) = \max_{[0,T]} |u^n(t)|_0 e^{\overline{\gamma}\left(\frac{1}{R}-\frac{1}{\mathscr{R}_c}\right)t}$.

Let us prove the following estimates:

a) If $\mathfrak{U}^{n-1}(T) < +\infty$, then, there exists $c \geq 1$ such that $\mathfrak{U}^n(T) \leq c\,\mathfrak{U}_0 + c\,(\mathfrak{U}^{n-1}(T))^2$, for any $n \in N$;

b) $\mathfrak{U}^n(T) \leq \mathfrak{U}_{\min}$, for any $n \in \mathbb{N}$ and for any $T > 0$, provided that $\mathfrak{U}_0 \leq (2c)^{-2}$, where $\mathfrak{U}_{\min}$ is the least root of the quadratic equation $c\,\mathfrak{U}^2 - \mathfrak{U} + c\,\mathfrak{U}_0 = 0$.

c)

$$\begin{aligned}
&\|u^n(t)\|_2^2 + \left(\frac{1}{R} - \frac{1}{\mathscr{R}_c}\right) \int_s^t \|\nabla u^n(\tau)\|_2^2 d\tau \\
&\leq \|u^n(s)\|_2^2 + c(\mathfrak{U}_{\min})^4, \text{ for any } t > s,\ s \geq 0 \text{ and } n. \qquad (5.6)
\end{aligned}$$

Estimate **a)** for $n=0$ is true, thanks to estimate (4.2): $\mathfrak{U}^0(T)\le c\mathfrak{U}_0$. Now, let $n\ge 1$. We multiply equation $(5.1)_1$ by $\phi(x,s)$, solution on $\Omega\times(0,t)$ of the adjoint of system (4.1) of the solution $u^0(x,t)$. By an integration by parts we get, for all $t>1$,

$$\begin{aligned}(U^n(t),\phi_0)&=\int\limits_0^t\big(u^{n-1}(\tau)\cdot\nabla\phi(t-\tau),u^{n-1}(\tau)\big)d\tau\\&=\int\limits_0^{t-1}(u^{n-1}(\tau)\cdot\nabla\phi(t-\tau),u^{n-1}(\tau))d\tau+\int\limits_{t-1}^t\big(u^{n-1}(\tau)\cdot\nabla\phi(t-\tau),u^{n-1}(\tau)\big)d\tau\\&=I_1(t)+I_2(t).\end{aligned}\tag{5.7}$$

Employing estimate $(4.19)_2$, taking into account Remark 4.4, we easily obtain

$$\begin{aligned}|I_1(t)|&\le(\mathfrak{U}^{n-1}(t))^2|\Omega|^{\frac12}\int\limits_0^{t-1}|\nabla\phi(t-\tau)|_2e^{-2\overline{\gamma}\left(\frac1R-\frac1{\mathscr{R}_c}\right)\tau}d\tau\\&\le c(\mathfrak{U}^{n-1}(t))^2|\Omega|^{\frac12}|\phi_0|_re^{-\overline{\gamma}\left(\frac1R-\frac1{\mathscr{R}_c}\right)t}\int\limits_0^{t-1}e^{-\overline{\gamma}\left(\frac1R-\frac1{\mathscr{R}_c}\right)\tau}c_1(r',(t-\tau)/2)c_2(t-\tau)d\tau,\end{aligned}$$

which, for a suitable constant c, implies

$$e^{\overline{\gamma}\left(\frac1R-\frac1{\mathscr{R}_c}\right)t}|I_1(t)|\le c(\mathfrak{U}^{n-1}(t))^2|\phi_0|_r\quad\text{for any } r\in(1,2].$$

Let us consider $I_2(t)$. For some $q\in(r,\frac32)$ and for any $r\in(1,q)$, by the Hölder inequality and by estimate $(4.5)_2$, we deduce

$$\begin{aligned}|I_2(t)|&\le(\mathfrak{U}^{n-1}(t))^2|\Omega|^{\frac1{q'}}\int\limits_{t-1}^t|\nabla\phi(t-\tau)|_qe^{-2\overline{\gamma}\left(\frac1R-\frac1{\mathscr{R}_c}\right)\tau}d\tau\\&\le c(\mathfrak{U}^{n-1}(t))^2|\Omega|^{\frac1{q'}}|\phi_0|_re^{-\overline{\gamma}\left(\frac1R-\frac1{\mathscr{R}_c}\right)t}\int\limits_{t-1}^t(t-\tau)^{-\frac12-\frac32\left(\frac1r-\frac1q\right)}d\tau\\&\le c(\mathfrak{U}^{n-1}(t))^2|\Omega|^{\frac1{q'}}|\phi_0|_re^{-\overline{\gamma}\left(\frac1R-\frac1{\mathscr{R}_c}\right)t}.\end{aligned}$$

In the above estimate, we have chosen q in such a way that $\sup\limits_{r\in(1,q)}\Big[\frac12+\frac32\big(\frac1r-\frac1q\big)\Big]=\frac34$. In the case of $t\in(0,1)$ it is sufficient to argue as in the case of the estimate of the term $I_1(t)$. Hence by means of a suitable constant c we obtain

$$|I_2(t)|\le c(\mathfrak{U}^{n-1}(t))^2|\phi_0|_re^{-\overline{\gamma}\left(\frac1R-\frac1{\mathscr{R}_c}\right)t},\ \text{uniformly in } t\ge1;$$

$$\Big|\int_0^t (u^{n-1}(\tau)\cdot\nabla\phi(t-\tau),u^{n-1}(\tau))d\tau\Big| \le c(\mathfrak{U}^{n-1}(t))^2|\phi_0|_r e^{-\overline{\gamma}\left(\frac{1}{R}-\frac{1}{\mathscr{R}_c}\right)t}, \quad \text{uniformly in } \ t\in(0,1). \tag{5.8}$$

Applying the estimates obtained for $I_1(t)$ and (5.8) to the right-hand side of (5.7), we deduce, uniformly in $t>0$,

$$e^{\overline{\gamma}\left(\frac{1}{R}-\frac{1}{\mathscr{R}_c}\right)t}|U^n(t)|_{r'}\le c(\mathfrak{U}^{n-1})^2, \text{ for any } r\in(1,q). \tag{5.9}$$

As already remarked, since from (4.2) $U^0(x,t)$ satisfies the estimate $\mathfrak{U}^0(T)\le c\,\mathfrak{U}_0$, taking into account that $u^n=U^0+U^n$, by a simple computation we achieve estimate **a)**.

Estimate **b)** is a consequence of **a)** and Lemma 10.2 of [23].

Finally we prove **c)**. We multiply equation $(5.7)_1$ by u^n. By integrating by parts on $\Omega\times(s,t)$, $s>0$, we find

$$\begin{aligned}&\frac{1}{2}\|u^n(t)\|_2^2+\frac{1}{R}\int_s^t\|\nabla u^n(\tau)\|_2^2d\tau\\ &=\frac{1}{2}\|u^n(s)\|_2^2-\int_s^t(u^n(\tau)\cdot\nabla v,u^n(\tau))d\tau+\int_s^t(u^{n-1}\otimes u^{n-1}(\tau),\nabla u^n(\tau))d\tau.\end{aligned} \tag{5.10}$$

Applying the variational formulation to the first integral term on the right-hand side of (5.10) we obtain the differential inequality

$$\|u^n(t)\|_2^2+2\Big(\frac{1}{R}-\frac{1}{\mathscr{R}_c}\Big)\int_s^t\|\nabla u^n(\tau)\|_2^2d\tau\le\|u^n(s)\|_2^2+2\int_s^t(u^{n-1}\otimes u^{n-1}(\tau),\nabla u^n(\tau))d\tau.$$

Taking into account estimate **b)** and applying the Hölder inequality, we have

$$\begin{aligned}&\|u^n(t)\|_2^2+2\left(\frac{1}{R}-\frac{1}{\mathscr{R}_c}\right)\int_s^t\|\nabla u^n(\tau)\|_2^2d\tau\\ &\le\|u^n(s)\|_2^2+c(\mathfrak{U}_{\min})^2\int_s^t e^{-2\overline{\gamma}\left(\frac{1}{R}-\frac{1}{\mathscr{R}_c}\right)\tau}\|\nabla u^n(\tau)\|_2d\tau.\end{aligned}$$

Applying the Cauchy inequality and taking into account the uniform continuity of $u^n(x,s)$, we deduce, in the limit of $s\to0$, the energy relation

$$\|u^n(t)\|_2^2+\left(\frac{1}{R}-\frac{1}{\mathscr{R}_c}\right)\int_0^t\|\nabla u^n(\tau)\|_2^2d\tau\le\|u^n(0)\|_2^2+c(\mathfrak{U}_{\min})^4 \text{ uniformly in } t \text{ and } n,$$

which proves relation **c)**.

The convergence of $\{u^n\}$

By $\overline{c}$ we denote a constant, whose value is independent of the initial data $u_0(x)$ and is uniform with respect to n and (x,t). Let us prove that $\{u^n(x,t)\}$ converges in $C(0,T;\mathscr{C}_{|0}(\overline{\Omega}))\cap L^2(0,T;J^{1,2}(\Omega))$. To this end we set

$$u^n(x,t)=\sum_{h=1}^{n}[u^h(x,t)-u^{h-1}(x,t)]+u^0(x,t)=\sum_{h=1}^{n}w^h(x,t)+u^0(x,t). \quad (5.11)$$

Of course, for any h, function w^h is a solution of the following problem

$$\begin{gathered} w^h_t+v\cdot\nabla w^h+w^h\cdot\nabla v+\nabla\pi^n-\frac{1}{R}\Delta w^h \\ =-w^{h-1}\cdot\nabla u^{h-1}-u^{h-2}\cdot\nabla w^{h-1}, \\ \nabla\cdot w^h=0 \ \text{ in } \Omega\times(0,T), \quad w^h(x,t)=0 \ \text{ on } \partial\Omega\times(0,T), \ w^h(x,0)=0 \text{ in } \Omega. \end{gathered} \quad (5.12)$$

We set $w^0=u^0$. Let us prove that

$$\begin{gathered} |w^{h-1}(t)|_0 e^{\overline{\gamma}\left(\frac{1}{R}-\frac{1}{\mathscr{R}_c}\right)t}\le(\overline{c}\mathfrak{U}_{\min})^h \text{ implies} \\ |w^h(t)|_0 e^{\overline{\gamma}\left(\frac{1}{R}-\frac{1}{\mathscr{R}_c}\right)t}\le(\overline{c}\mathfrak{U}_{\min})^{h+1} \text{ for any } h\in\mathbb{N}. \end{gathered}$$

We multiply equation $(5.12)_1$ by $\phi(x,t-\tau)$. By an integration by parts on $\Omega\times(0,t)$ we get

$$\begin{aligned} (w^h(t),\phi_0)=&\int_0^{t-1}(w^{h-1}\otimes u^{h-1}(\tau)+u^{h-2}\otimes w^{h-1}(\tau),\nabla\phi(t-\tau))d\tau \\ &+\int_{t-1}^{t}(w^{h-1}\otimes u^{h-1}(\tau)+u^{h-2}\otimes w^{h-1}(\tau),\nabla\phi(t-\tau))d\tau \\ &=I_1^h(t)+I_2^h(t). \end{aligned} \quad (5.13)$$

We estimate the right-hand side in quite an analogous way as we did for estimating I_1 and I_2 in relation (5.7). Taking into account items **a)**–**b)** and applying the Hölder inequality, we obtain

$$|I_1^h(t)|\le|\Omega|^{\frac{1}{2}}c\overline{c}^h(\mathfrak{U}_{\min})^{h+1}\int_0^{t-1}e^{-2\overline{\gamma}\left(\frac{1}{R}-\frac{1}{\mathscr{R}}\right)\tau}|\nabla\phi(t-\tau)|_2 d\tau.$$

Applying $(4.19)_2$, we deduce

$$\begin{aligned} |I_1^h(t)|&\le|\Omega|^{\frac{1}{2}}c\overline{c}^h(\mathfrak{U}_{\min})^{h+1}\int_0^{t-1}e^{-2\overline{\gamma}\left(\frac{1}{R}-\frac{1}{\mathscr{R}}\right)\tau}|\nabla\phi(t-\tau)|_2 d\tau \\ &\le|\Omega|^{\frac{1}{2}}e^{-\overline{\gamma}\left(\frac{1}{R}-\frac{1}{\mathscr{R}}\right)t}c\overline{c}^h(\mathfrak{U}_{\min})^{h+1}|\phi_0|_r \text{ for any } r\in(2,1). \end{aligned} \quad (5.14)$$

As far as $I_2^h(t)$ is concerned, applying the Hölder inequality we have the estimate

$$|I_2^h(t)| \le |\Omega|^{\frac{1}{q'}} c\overline{c}^h (\mathfrak{U}_{\min})^{h+1} \int\limits_{t-1}^{t} e^{-2\overline{\gamma}\left(\frac{1}{R}-\frac{1}{\mathscr{R}}\right)\tau} |\nabla\phi(t-\tau)|_q d\tau$$
$$\le |\Omega|^{\frac{1}{q'}} c\overline{c}^h (\mathfrak{U}_{\min})^{h+1} e^{-\overline{\gamma}\left(\frac{1}{R}-\frac{1}{\mathscr{R}}\right)t} \int\limits_{t-1}^{t} |\nabla\phi(t-\tau)|_q d\tau.$$

By virtue of estimate $(4.5)_2$ we prove

$$|I_2^h(t)| \le |\Omega|^{\frac{1}{q'}} c\overline{c}h (\mathfrak{U}_{\min})^{h+1} e^{-\gamma\left(\frac{1}{R}-\frac{1}{\mathscr{R}}\right)t} \int\limits_{t-1}^{t} |\nabla\phi(t-\tau)|_q d\tau$$
$$\le |\Omega|^{\frac{1}{q'}} c\overline{c}^h (\mathfrak{U}_{\min})^{h+1} \|\phi_0\|_r e^{-\overline{\gamma}\left(\frac{1}{R}-\frac{1}{\mathscr{R}}\right)t} \int\limits_{t-1}^{t} \tau^{-\frac{1}{2}-\frac{3}{2}\left(\frac{1}{r}-\frac{1}{q}\right)} d\tau$$
$$\le \frac{c(q)}{\widetilde{\gamma}(r,q)} c\overline{c}^h (\mathfrak{U}_{\min})^{h+1} |\Omega|^{\frac{1}{q'}} \|\phi_0\|_r e^{-\overline{\gamma}\left(\frac{1}{R}-\frac{1}{\mathscr{R}}\right)t}, \text{ for any } r\in(1,q), \tag{5.15}$$

provided that $q \in (1, \frac{3}{2})$. Estimating the right-hand side of (5.13) by employing (5.14)–(5.15), taking into account the arbitrariness of ϕ_0, uniformly in $h \in \mathbb{N}$ and $t > 0$, we obtain

$$|w^h(t)|_{r'} \le \frac{c(q)}{\widetilde{\gamma}(r,q)} c\overline{c}^h (\mathfrak{U}_{\min})^{h+1} |\Omega|^{\frac{1}{q'}} e^{-\overline{\gamma}\left(\frac{1}{R}-\frac{1}{\mathscr{R}}\right)t}, \text{ for any } r\in(1,q).$$

In the limit of $r' \to \infty$, by suitable meaning of $\overline{c}$, we prove

$$e^{\overline{\gamma}\left(\frac{1}{R}-\frac{1}{\mathscr{R}}\right)t} |w^h(t)|_0 \le (\overline{c}\mathfrak{U}_{\min})^{h+1}, \text{ uniformly in } h \in \mathbb{N}. \tag{5.16}$$

Analogously, let us prove that

$$\|w^{h-1}(t)\|_2 \le (\overline{c}\mathfrak{U}_{\min})^h \text{ implies}$$
$$\|w^h(t)\|_2^2 + \left(\frac{1}{R} - \frac{1}{\mathscr{R}_c}\right) \int\limits_0^t \|\nabla w^h(\tau)\|_2^2 d\tau \le (\overline{c}\mathfrak{U}_{\min})^{2(h+1)},$$
$$\text{for any } t > 0 \text{ and for any } n \in \mathbb{N}.$$

We begin to derive the energy relation for w^h. Multiplying equation $(5.12)_1$ by w^h, and by integrating by parts we find

$$\|w^h(t)\|_2^2 + 2\left(\frac{1}{R} - \frac{1}{\mathscr{R}_c}\right) \int\limits_0^t \|\nabla w^h(\tau)\|_2^2 d\tau$$
$$\le \int\limits_0^t \left|(w^{h-1} \otimes u^{h-1}(\tau), \nabla w^h(\tau)) + (u^{h-2} \otimes w^{h-1}(\tau), \nabla w^h(\tau))\right| d\tau. \tag{5.17}$$

Applying the Hölder inequality, taking into account items **a)-c)** and (5.16), we can easily get the inequality

$$\begin{aligned}&\big|(w^{h-1}\otimes u^{h-1},\nabla w^h)+(u^{h-2}\otimes w^{h-1},\nabla w^h)\big|\\ &\leq\Big(\frac{1}{R}-\frac{1}{\mathscr{R}_c}\Big)^{-1}\|w^{h-1}(t)\|_2^2|u^{h-1}(t)|_0^2+\Big(\frac{1}{R}-\frac{1}{\mathscr{R}_c}\Big)\|\nabla w^h(t)\|_2^2\\ &\leq\frac{(\overline{c}\mathfrak{U}_{\min})^{2h}\mathfrak{U}_{\min}^2}{\Big(\frac{1}{R}-\frac{1}{\mathscr{R}_c}\Big)}e^{-2\overline{\gamma}\left(\frac{1}{R}-\frac{1}{\mathscr{R}_c}\right)t}+\Big(\frac{1}{R}-\frac{1}{\mathscr{R}_c}\Big)\|\nabla w^h(t)\|_2^2.\end{aligned}\tag{5.18}$$

Estimates (5.17)–(5.18), for a suitable constant $\overline{c}$, imply

$$\|w^h(t)\|_2^2+\Big(\frac{1}{R}-\frac{1}{\mathscr{R}_c}\Big)\int_0^t\|\nabla w^h(\tau)\|_2^2d\tau\leq(\overline{c}\mathfrak{U}_{\min})^{2(h+1)},\qquad\text{for any } t>0 \text{ and for any } h\in\mathbb{N}.\tag{5.19}$$

Estimates (5.16) and (5.19) imply that the series (5.11) is convergent in $C(0,T;\mathscr{C}_{|0}(\overline{\Omega}))\cap L^2(0,T;J^{1,2}(\Omega))$, provided that $\overline{c}\mathfrak{U}_{\min}<1$ (actually for a suitable value $\mathfrak{U}_0$). Hence, if we set $\mu=\max\{4c^2,\overline{c}^2\}$, we have proved the bound for the initial data as claimed in the theorem for the existence of the solutions. Indeed the convergence of the series implies that $\{u^n(x,t)\}$ converges in $C(0,T;\mathscr{C}_{|0}(\overline{\Omega}))\cap L^2(0,T;J^{1,2}(\Omega))$. Denoting by $u(x,t)$ the limit of the sequence in $C(0,T;\mathscr{C}_{|0}(\overline{\Omega}))\cap L^2(0,T;J^{1,2}(\Omega))$, then $u(x,t)$ is a weak solution of system (1.1).

The regularity of the weak limit $u(x,t)$

Employing classical results of regularity, of the kind proved by Sather–Serrin, it is possible to associate to $u(x,t)$ a pressure field $\pi(x,t)$ and the pair (u,π) is a regular solution in $\Omega\times(0,T)$ such that, for any $\eta_1>0$, $u\in C^0(0,T,\mathscr{C}_{|0}(\overline{\Omega}))\cap L^2(\eta_1,T;W^{2,2}(\Omega))\cap C(\eta_1,T;J^{1,2}(\Omega))$. This last property by the Sobolev imbedding theorem ensures that $(u\cdot\nabla)u\in L^{\frac{10}{3}}(\eta_1,T;L^{\frac{10}{3}}(\Omega))$. Now to prove that (u,π) is a classical solution is sufficient to argue in the same way as we did in the case of the approximating solutions. We introduce a smooth function $h_1(t)$ such that $h(\eta_1)=0$ and $h_1(t)>0$ for $t>\eta_1$. We set $(\widetilde{u},\widetilde{\pi})=h_1(u,\pi)$. The pair $(\tilde{u},\tilde{\pi})$ is a solution to the system

$$\begin{aligned}\widetilde{u}_t-\Delta\widetilde{u}+\nabla\widetilde{p}&=-v\cdot\nabla\widetilde{u}-\widetilde{u}\cdot\nabla v-u\cdot\nabla\widetilde{u}+h_1'(t)u,\\ \nabla\cdot\widetilde{u}&=0\text{ in }\Omega\times(\eta_1,T),\\ \widetilde{u}&=0\text{ on }\partial\Omega\times(\eta_1,T),\ \widetilde{u}(x,\eta_1)=0.\end{aligned}\tag{5.20}$$

Since $u\cdot\nabla\widetilde{u}+h_1'(t)u\in L^{\frac{10}{3}}(0,T;L^{\frac{10}{3}}(\Omega))$, by virtue of Theorem 4.3 and by the uniqueness property, we deduce that $\widetilde{u}\in L^{\frac{10}{3}}(0,T;W^{2,\frac{10}{3}}(\Omega))$, $\nabla\widetilde{\pi},\widetilde{u}_t\in L^{\frac{10}{3}}(0,T;L^{\frac{10}{3}}(\Omega))$. Furthermore, by the Sobolev imbedding theorem we obtain $\nabla\widetilde{u}\in L^{\frac{32}{9}}(0,T;L^{\frac{32}{9}}(\Omega))$, and, hence, $u\cdot\nabla\widetilde{u}+h_1'(t)u\in L^{\frac{32}{9}}(0,T;L^{\frac{32}{9}}(\Omega))$. By iterating the above argument we establish the existence of $p \in (n, \infty)$ such that $\widetilde{u} \in L^p(0,T; W^{2,p}(\Omega))$, $\nabla\widetilde{\pi}, \widetilde{u}_t \in L^p(0,T; L^p(\Omega))$ and, by imbedding, that, for some $\alpha \in (0,1)$, $\widetilde{u} \in C^{0,\frac{\alpha}{2}}(0,T; \mathscr{C}_{|0}(\overline{\Omega}) \cap C^{1,\alpha}(\overline{\Omega}))$. Of course, for any $\eta > \eta_1$, we have $u \in C^{0,\frac{\alpha}{2}}(\eta, T; \mathscr{C}_{|0}(\overline{\Omega}) \cap C^{1,\alpha}(\overline{\Omega}))$. Introducing a new smooth function $h(t)$ with $h(\eta) = 0$, $h(t) > 0$, for $t > \eta$, setting $(\widehat{u}, \widehat{\pi}) = h(u, \pi)$, we have

$$\begin{aligned} \widehat{u}_t - \Delta\widehat{u} + \nabla\widehat{p} &= -v \cdot \nabla\widehat{u} - \widehat{u} \cdot \nabla v - u \cdot \nabla\widehat{u} + h_1'(t)u, \\ \nabla \cdot \widehat{u} &= 0 \text{ in } \Omega \times (\eta, T); \\ \widehat{u} &= 0 \text{ on } \partial\Omega \times (\eta, T),\ \widehat{u}(x, \eta) = 0. \end{aligned} \tag{5.21}$$

Since $-u\nabla\widehat{u} + h_1'(t)u$ satisfies the compatibility condition expressed in Theorem 4.2, then we can establish the regularity of solution (u, π) in (η, T). The arbitrariness of $\eta_1 > 0$ proves completely that (u, π) is a classical solution.

To end the proof of the theorem, we must show uniqueness. To this end, we refer to the uniqueness part of the proof of Theorem 2.2 below.

Proof of Theorem 2.2. In order to prove the existence of a classical solution, we consider the same sequence $\{u^n(x,t)\}$ determined for the global existence with *small* data. We modify estimates **a)**–**c)** in the subsection "*Estimates for the convergence of the sequence* $\{u^n\}$" to the following ones:

- **d)** $\widehat{\mathfrak{U}}^n(T) \le c\mathfrak{U}^0 + c(\widehat{\mathfrak{U}}^{n-1}(T))^2 e^{\gamma_0 T} T^{\widetilde{\gamma}}$, for any $n \in N$, for some $T > 0$, for some constants γ_0 and $\widetilde{\gamma} > 0$;
- **e)** $\widehat{\mathfrak{U}}^n(T) \le \widehat{\mathfrak{U}}_{\min}$, for any $n \in \mathbb{N}$, provided that $\mathfrak{U}^0 e^{\gamma_0 T} T^{\widetilde{\gamma}} \le (2c)^{-2}$, where $\widehat{\mathfrak{U}}_{\min}$ is the minimal square root of the quadratic equation $ce^{\gamma_0 T} T^{\widetilde{\gamma}} \widehat{\mathfrak{U}}^2 - \widehat{\mathfrak{U}} + c\mathfrak{U}^0 = 0$, $c \ge 1$.
- **f)** = **c)**,

where, for any $n \in \mathbb{N}$, we set $\widehat{\mathfrak{U}}^n(T) = \max\limits_{[0,T]} |u^n(t)|_0$. Estimate **d)** for $n = 0$ is true, thanks to estimate (4.2). Now, let $n \ge 1$. We multiply equation $(5.1)_1$ by $\phi(x, t-\tau)$, solution of the adjoint of system (4.1) of the solution $u^0(x,t)$. An integration by parts gives

$$(U^n(t), \phi_0) = \int_0^t \big(u^{n-1}(\tau) \cdot \nabla\Phi(t-\tau), u^{n-1}(\tau)\big) d\tau = I_1(t). \tag{5.22}$$

Employing the Hölder inequality and estimate $(4.5)_2$, for any $t \in [0,T]$ and for some $q \in (r, \frac{3}{2})$, we easily obtain

$$\begin{aligned} |I_1(t)| &\le (\mathfrak{U}^{n-1})^2 |\Omega|^{\frac{1}{q'}} \int_0^t |\nabla\phi(t-\tau)|_q d\tau \\ &\le c(\mathfrak{U}^{n-1})^2 |\Omega|^{\frac{1}{q'}} |\phi_0|_r e^{\gamma_0 t} \int_0^t \tau^{-\frac{1}{2} - \frac{3}{2}\left(\frac{1}{r} - \frac{1}{q}\right)} d\tau \end{aligned}$$

$$\leq c(q)(\widetilde{\gamma}(r,q))^{-1}(\mathfrak{U}^{n-1})^2|\Omega|^{\frac{1}{q'}}|\phi_0|_r e^{\gamma_0 t}t^{\widetilde{\gamma}(r,q)},$$

with $\widetilde{\gamma}(r,q) = \frac{1}{2} - \frac{3}{2}\left(\frac{1}{r} - \frac{1}{q}\right)$. Therefore, from relation (5.22), taking into account the arbitrariness of ϕ_0 we obtain the inequality

$$\|U^n(t)\|_{r'} \leq c(q)(\widetilde{\gamma}(r,q))^{-1}(\mathfrak{U}^{n-1})^2|\Omega|^{\frac{1}{q'}}e^{\gamma_0 t}t^{\widetilde{\gamma}(r,q)}.$$

Passing to the limit $r \to 1$ in the last inequality, we deduce

$$|U^n(t)|_\infty \leq c(q)(\widetilde{\gamma}(q))^{-1}(\mathfrak{U}^{n-1})^2|\Omega|^{\frac{1}{q'}}e^{\gamma_0 t}t^{\widetilde{\gamma}(q)},$$

with $\widetilde{\gamma}(q) = \frac{1}{2} - \frac{3}{2}\frac{1}{q}$. Taking into account that $|U^0(x,t)| \leq c\mathfrak{U}^0$, for any chosen $q \in (1, \frac{3}{2})$ we obtain relation **d)** for a suitable constant c.

Assuming that for some $T > 0$ $\mathfrak{U}^0 e^{\gamma_0 T}T^{\widetilde{\gamma}} \leq (2c)^{-2}$, then estimate in **e)** is implied by that in **d)** and by Lemma 10.2 of [23].

The estimate in **f)** is deduced in a quite analogous way as the estimate given in **c)**.

Since $ce^{\gamma_0 T}T^{\widetilde{\gamma}} \to 0$ as $T \to 0$, by employing the same arguments used to prove the convergence in the case of *small data,* we can prove the convergence of $\{u^n(x,t)\}$ with respect the norm $C(0,T;\mathscr{C}_{|0}(\overline{\Omega})) \cap L^2(0,T;J^{1,2}(\Omega))$ on some interval $(0,T)$. Indeed in quite analogous way, one proves the following implications:

$$|w^{h-1}(t)|_0 \leq (\overline{c}\mathfrak{U}_{\min}e^{\gamma_0 T}T^{\overline{\gamma}})^h \text{ implies}$$
$$|w^h(t)|_0 \leq (\overline{c}\mathfrak{U}_{\min}e^{\gamma_0 T}T^{\overline{\gamma}})^{h+1}$$
$$\text{for any } t \in [0,T] \text{ and for any } h \in \mathbb{N};$$

$$|w^{h-1}(t)|_2 \leq (\overline{c}\mathfrak{U}_{\min}e^{\gamma_0 T}T^{\overline{\gamma}})^h \text{ implies}$$
$$\|w^h(t)\|_2^2 + \left(\frac{1}{R} - \frac{1}{\mathscr{R}_c}\right)\int_0^t \|\nabla w^h(\tau)\|_2^2 d\tau \leq (\overline{c}\mathfrak{U}_{\min}e^{\gamma_0 T}T^{\overline{\gamma}})^{h+1}$$
$$\text{for any } t \in [0,T] \text{ and for any } h \in \mathbb{N}.$$

Thus, for a suitable $T > 0$, the series (5.11) is convergent in $C(0,T;\mathscr{C}_{|0}(\overline{\Omega})) \cap L^2(0,T;J^{1,2}(\Omega))$. In this way, we have obtained a weak solution $u(x,t)$ belonging to $C(0,T;\mathscr{C}_{|0}(\overline{\Omega})) \cap L^2(0,T;J^{1,2}(\Omega))$. Now, the regularity follows as in subsection "*The regularity of the weak solution* $u(x,t)$." The theorem will be completely proved once we prove the uniqueness (see next section).

The uniqueness of a classical solution

Since a classical solution $u(x,t)$ belongs to $C(0,T;\mathscr{C}_{|0}(\overline{\Omega})) \cap L^2(0,T;J^{1,2}(\Omega))$, then the uniqueness can be deduced as in the case of a weak solution.

Acknowledgement. The author would like to give special thanks to Professor V. A. Solonnikov for bringing paper [27] to his knowledge before its publication as well as to Professor G. Mulone and to Professor S. Rionero for their interesting comments and valuable discussions.

The paper was written under the auspices of GNFM (INdAM) and with the support of a MURST Contract.

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(accepted: October 31, 2006; published Online First: December 20, 2007)