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An Example of Finite-time Singularities in the 3d Euler Equations

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Abstract. Let $\Omega = \mathbb{R}^3 \setminus \bar{B}_1(0)$ be the exterior of the closed unit ball. Consider the self-similar Euler system

$$\alpha u + \beta y \cdot \nabla u + u \cdot \nabla u + \nabla p = 0, \quad \operatorname{div} u = 0 \quad \text{in} \quad \Omega.$$

Setting $\alpha = \beta = 1/2$ gives the limiting case of Leray's self-similar Navier–Stokes equations. Assuming smoothness and smallness of the boundary data on $\partial\Omega$, we prove that this system has a unique solution $(u, p) \in \mathcal{C}^1(\Omega; \mathbb{R}^3 \times \mathbb{R})$, vanishing at infinity, precisely

$$u(y) \downarrow 0 \ \text{ as } \ |y| \uparrow \infty, \ \text{ with } \ u = \mathcal{O}(|y|^{-1}), \ \ \nabla u = \mathcal{O}(|y|^{-2}).$$

The self-similarity transformation is $v(x,t) = u(y)/(t^* - t)^\alpha$, $y = x/(t^* - t)^\beta$, where $v(x,t)$ is a solution to the Euler equations. The existence of smooth function $u(y)$ implies that the solution $v(x,t)$ blows up at (x^*, t^*), $x^* = 0$, $t^* < +\infty$. This isolated singularity has bounded energy with unbounded L^2−norm of $\operatorname{curl} v$.

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1. Introduction

The problem of global regularity of the Euler equations is profound. The equations in three space-dimensions describe an ideal, incompressible fluid:

$$\frac{\partial v}{\partial t} + v \cdot \nabla v = -\nabla \pi, \quad \operatorname{div} v = 0 \quad \text{in } \ \mathbb{R}^3 \times (0, T], \tag{1.1}$$

where $v(x,t) = (v_1, v_2, v_3)$ denotes the velocity field, π the pressure scalar. Given $\mathcal{C}^\infty$ divergence-free initial data $v(x, 0) = v_0$, $v_0 \in L^2(\mathbb{R}^3)$, it is an open question whether $v(x,t)$ is regular at $T = +\infty$. In this paper, we shall construct an analytical example, which shows that the smooth Euler solution of (1.1) can break down at an isolated point (x^*, t^*), $x^* = 0$, $t^* < +\infty$.

We shall be using the self-similarity transformation, first suggested by J. Leray [L] for the Navier–Stokes equations in 1930's. The following form was defined in [H] for the Euler equations:

$$v(x,t) = \frac{u(y)}{(t^* - t)^\alpha}, \quad y = \frac{x}{(t^* - t)^\beta} \in \mathbb{R}^3, \quad 0 < t < t^* < \infty, \tag{1.2}$$

$$\beta \in [2/5, 1] \qquad \text{and} \qquad \alpha + \beta = 1, \qquad \alpha, \beta \ > 0. \tag{1.3}$$

Under (1.1)–(1.3), $u(y)$ satisfies the system

$$\alpha u + \beta y \cdot \nabla u + u \cdot \nabla u + \nabla p = 0, \qquad \operatorname{div} u = 0. \tag{1.4}$$

If $\alpha = \beta = 1/2$, then (1.4) is the limiting case of Leray's self-similar Navier–Stokes equations. This is the only case that will primarily interest us. If a solution $u \not\equiv 0$ is found, then a Euler singularity is developed via (1.2). Note that the similarity blowup at $t = t^*$ is consistent with the theorems of [BKM], [CFM]. Moreover numerical studies have indicated localised singular structures with Leray's scaling [GMG], [K], [P]. Naturally a question has arisen: Could a self-similar singularity exist only locally in space and time?

In this work, we give an affirmative answer to the above question. We will prove that a non-trivial solution $u(y)$ to (1.4) exists in an exterior domain $\Omega = \mathbb{R}^3 \backslash \bar{B}_1(0)$; hence by the very construction, the solution $v(x,t)$ to (1.1) develops a singularity at a point (x^*, t^*) (see Remark 1.1 for a informal note). The starting point of the construction is an existence result: adopting a technical device due to [A] and [Mol] for the steady Euler system, the author obtained in [H] that in a bounded domain for $\alpha = \beta$, (1.4) has a unique solution, depending continuously on the boundary data. To extend this result to the exterior domain, the present paper will loosely follow the lines of [F], which treated the steady (elliptic) Navier–Stokes equations.

It is necessary to state growth conditions for a bounded solution u(y). The first one is that at infinity, u should be smooth and small (see for instance, [Mof])

$$u = \mathcal{O}(|y|^{-1}), \quad \nabla u = \mathcal{O}(|y|^{-2}), \qquad \text{as} \qquad |y| \uparrow \infty. \tag{1.5}$$

Let initial $\mathcal{C}^\infty$ divergence-free data $v(x,0) = v_0(x) \in \mathbb{R}^3$ be sufficiently localised. The second condition (cf. Theorem 1.1 [Sh]) requires that $\forall\, 0 < t \le t^*$:

$$\int_{\mathbb{R}^3} |v(x,t)|^2 \, dx \le \|v_0\|^2_{L^2(\mathbb{R}^3)}. \tag{1.6}$$

Let $\Omega = \mathbb{R}^3 \setminus \bar{B}_1(0)$ be the exterior of the closed unit ball as above. We shall prove in Section 2 that $\nabla u \in L^2(\Omega)$ [Corollary 2E], by assuming small data on

$\partial\Omega$, and making appropriate assumptions on the outer boundary. In Section 3, a solution (u,p) in Ω is constructed as the limit of a sequence of solutions of bounded domains [Theorem 3C], with u and ∇u tending to limits at ∞. Uniqueness is obtained if the solution is small [Theorem 3D]. In the final section we show that the properties (1.5) and (1.6) are true asymptotically, and clarify that the solution $v(x,t)$ to (1.1) becomes unbounded at the space-time point (x^*,t^*).

Remark 1.1. The essence of our construction is as follows. We first solve (1.4) for $u(y)$ in Ω_n, where Ω_n is the region bounded by spheres $\partial B_1(0)$ and $\partial B_n(0)$, with n being sufficiently large. For fixed t, the set Ω_n transforms by (1.2) into $\widehat{\Omega}(t,n) = \{x \in \mathbb{R}^3 \setminus \{0\} : \sqrt{t^*-t} \le |x| < n\sqrt{t^*-t}\}$. Note that the inner front $\partial\widehat{\Omega}_t$ is shrinking inwards to the origin with scaling $\sqrt{t^*-t}$. For all $0 < t < t^*$, one can assume that the solution $v(x,t)$ as in (1.2) is smooth in $\widehat{\Omega}(t,n)$, and it can be extended to a weak solution of (1.1) in $\widehat{\Omega}^c(t,n)$, the complement of $\widehat{\Omega}(t,n)$ in $\mathbb{R}^3$. This assertion is based on a very recent paper [Ma], which has shown that Leray's backward self-similar solutions satisfy all regularity criteria except the slow decay as in (1.5), with a well-defined profile at the singular time.

Now consider $t \uparrow t^*$. We look at $\widehat{\Omega}(t,n)$ and choose n such that the outer radius $n\sqrt{t^*-t} = \mathcal{O}(|x_0|)$, $|x_0|$ being a constant depending on initially localised data v_0 (for similar arguments, see §5 in [P]). Then in $y-$space, the outer radius of Ω_n is $\mathcal{O}(|x_0|/\sqrt{t^*-t})$, and $u(y)$ constructed above with growth $|y|^{-1}$ is smooth in a neighborhood of infinity. Hence in the limit $t = t^*$, the inner self-similar solution $v(x,t) = u(y)/\sqrt{t^*-t}$ blows up at $x^* = 0$ as $\mathcal{O}(1/\sqrt{t^*-t})$, while at any remaining point bounded away from x^*, the solution is finite. Thus the example gives an isolated Euler singularity.

Remark 1.2. A novelty of the present work is that it concerns the fluid in $\Omega = \mathbb{R}^3 \setminus \bar{B}_1(0)$, in contrast to others which are concerned with Leray's transformation in the entire $\mathbb{R}^3$ (see [NRŠ] for the Navier–Stokes equations, and [Ch] for the Euler equations). The physical intuition for this "partial self-similarity" is that the bending and twisting of vortex lines towards (x^*,t^*) is local in nature. Another feature of the result is related to the energy. An important discovery by [Sh] is that weak solutions to (1.1) can have decreasing energy, which is compatible with the solution here (cf. Lemma 4B). We should also mention that a recent result [CFL] excludes certain hydrodynamic singularities, however we do not think their definition of "regular tube" is applicable in our case.

2. Preliminary estimates

We start by estimating a solution to (1.4) in a bounded domain. Let $\Omega \subset \mathbb{R}^3$ be an open set. Denote by x, y, z generic points in $\mathbb{R}^3$, by $W^{m,p}(\Omega)$ standard Sobolev spaces, and by $W^{\ell,p}(\partial\Omega)$ trace spaces. The usual summation will be used.

Let Ω_n be the region bounded by two smooth surfaces: $\partial\Omega \equiv \partial B_1(0)$, and $\partial B_n \equiv \partial B_n(0)$, where n is a sufficiently large radius of the ball $B_n(0)$. We shall consider an existence result in this domain. In the previous work (Theorem 1.3 [H]), the reference flow $\bar{u}$ was not specified; for the exterior problem, it is necessary to set $\bar{u} = \beta x$. Without going through all the details again, we state

Proposition 2A. *Let $\Omega_n \subset \mathbb{R}^3$ be the domain described above, with boundary $\partial\Omega_n = \partial\Omega \cup \partial B_n$ of class $\mathcal{C}^3$. Suppose that $f \in W^{2,4}(\partial\Omega;\mathbb{R})$ and $g \in W^{3,4}(\partial\Omega;\mathbb{R})$, and that these functions are small in appropriate norms. Then there exists a unique, stable solution pair $(u,p) \in W^{3,4}(\Omega_n;\mathbb{R}^3 \times \mathbb{R})$ of (1.4), satisfying the boundary conditions:*

$$\left.\begin{aligned} \sigma\cdot u|_{\partial\Omega} &= 0;\\ \sigma\cdot \operatorname{curl} u|_{\partial\Omega} &= f(x), \quad \int_{\partial\Omega} f\,dx = 0;\\ ((u+\beta x)\times \operatorname{curl} u)_\tau|_{\partial\Omega} &= \nabla_\tau\, g(x); \end{aligned}\right\} \tag{2.1}$$

where σ is the unit outward normal, and τ the tangential. Furthermore there is a constant $\gamma > 0$ depending on f, g, and Ω_n, such that $\|u\|_{W^{3,4}} \le \gamma$.

Proof. See [A], [Mol], and [H]. □

Remark 2.1. By Sobolev's embedding theorem, $W^{3,4}(\Omega_n) \hookrightarrow \mathcal{C}^2(\bar{\Omega}_n)$, hence the above result has higher regularity than the $W^{3,2}$ solution in [H], which implies a $\mathcal{C}^1$ solution. Seeking a solution in $W^{3,4}$, as was done in [Mol] for the $W^{2,4}$ solution, the $\mathcal{C}^2$ regularity is achieved when more regularity of the data is assumed. In what follows, data on $\partial\Omega$ and solutions to (1.4) in Ω_n will be assumed to be sufficiently smooth.

Remark 2.2. When viewing (2.1) conceptually, one should keep in mind that the unit sphere $\partial\Omega$, though "stationary" in the Leray frame, is itself moving inwards according to (1.2). The boundary condition $(2.1)_1$ means that the normal velocity of the fluid on the unit sphere is equal to that of the sphere. There is non-zero normal vorticity $f(x)$ on the boundary, so $(2.1)_2$ is sufficient to exclude the case $u = \nabla\Phi$, where Φ is a arbitrary harmonic function. Eq. $(2.1)_3$ can equivalently be written as $(|u|^2/2 + p(x) + \beta x\cdot u)|_{\partial\Omega} = g(x)$; as we see, the pressure $p(x)$ is defined on $\partial\Omega$ by data g.

To apply the proposition, we decompose the velocity gradient matrix into

$$\nabla u = S + A, \qquad \text{with} \quad \text{vorticity} \;\; \omega := \operatorname{curl} u, \tag{2.2}$$

where the rate of strain $S = (s_{ij})$ is the symmetric part of ∇u, and $A = (a_{ij})$ the anti-symmetric part. Here $a_{ij} = -\frac{1}{2}\sum_{k=1}^{3} \epsilon_{ijk}\omega_k$, where ϵ_{ijk} is the usual permutation symbol. We may rewrite $|\nabla u|^2$ as

$$|\nabla u|^2 := \sum_{i,j=1}^{3} \Big(\frac{\partial u_i}{\partial x_j}\Big)^2 \equiv \sum_{i,j=1}^{3} s_{ij}^2 + \sum_{i=1}^{3} \frac{\omega_i^2}{2} \tag{2.3}$$

Let us first examine possible global relations between the rate of strain and vorticity.

Lemma 2B. *Let $w(x)$ be a solution to* (1.4) *as in Proposition 2A, and s_{ij}, ω_i as defined above. Suppose that $w(x)|_{\partial B_n} = u_0$, where u_0 is an assigned small constant vector. Then the difference between $\frac{1}{2}\|\omega\|^2_{L^2(\Omega_n)}$ and $\|s_{ij}\|^2_{L^2(\Omega_n)}$ is a constant depending on f, u_0, and $\partial\Omega$ only.*

Proof. Put $u(x) = w(x) - u_0$, so that $u|_{\partial\Omega} = w|_{\partial\Omega} - u_0$, $u|_{\partial B_n} = 0$, $\nabla u = \nabla w$. Equation (1.4) then becomes

$$\alpha(u + u_0) + \beta x \cdot \nabla u + u \cdot \nabla u + u_0 \cdot \nabla u + \nabla p = 0, \qquad \operatorname{div} u = 0. \tag{2.4}$$

By differentiating (2.4), we obtain a Poisson equation for the pressure: $-\triangle p = \operatorname{div}(u \cdot \nabla u) = \sum_{i,j} s_{ij}^2 - \sum_i \omega_i^2/2$, where the Neumann boundary conditions for p can be determined by considering the normal component of (2.4) and using (2.1). Integrating this equation yields

$$\int_{\Omega_n} \sum_{i=1}^{3} \frac{\omega_i^2}{2}\, dx - \int_{\Omega_n} \sum_{i,j=1}^{3} s_{ij}^2\, dx = \kappa(f, u_0, \partial\Omega), \tag{2.5}$$

where we have used $\int_{\partial\Omega} \sigma \cdot \nabla p = \kappa$, and $\int_{\partial B_n} \sigma \cdot \nabla p = 0$ given $\int_{\partial B_n} \sigma \cdot u_0 = 0$, $u|_{\partial B_n} = 0$, and $\operatorname{div} u = 0$. Clearly, κ is finite. □

By use of (2.5), we will show that $\|\nabla u\|_{L^2(\Omega_n)}$ is controllable in

Lemma 2C. *Let $u(x)$ be a solution as in Lemma 2B. Then $\int_{\Omega_n} |\nabla u|^2$ is bounded above by a positive constant that is independent of the size of Ω_n.*

Proof. Computing the curl of (2.4) yields the vorticity equation

$$\omega + \beta x \cdot \nabla\omega + (u_0 + u) \cdot \nabla\omega - \omega \cdot \nabla u = 0, \qquad \operatorname{div}\omega = 0.$$

Taking the inner product with ω, we have

$$|\omega|^2 + \operatorname{div}[x \mid \omega|^2] + 2\operatorname{div}[(u + u_0)|\omega|^2] = 4\omega \cdot \operatorname{div}[\omega \otimes u],$$

where we used $\beta\omega \cdot (x \cdot \nabla\omega) = (\beta/2)\mathrm{div}\,[x|\omega|^2] - (3\beta/2)|\omega|^2$, $\beta = 1/2$. Integrating by parts and applying the boundary conditions, we arrive at

$$\int_{\Omega_n} |\omega|^2 + n\int_{\partial B_n} |\omega|^2 = \int_{\partial\Omega} |\omega|^2 + 4fu\cdot\omega - 4\int_{\Omega_n} u\cdot(\omega\cdot\nabla)\omega,$$

with f as defined in (2.1). By vector identities, $(\omega \cdot \nabla)\omega = \omega \times \triangle u + \nabla|\omega|^2/2$. So $4\int_{\Omega_n} u\cdot(\omega\cdot\nabla)\omega = 2\int_{\Omega_n} \mathrm{div}\,(u|\omega|^2) = 2\int_{\partial\Omega} \sigma\cdot u|\omega|^2$. Proposition 2A asserts that all the above surface integrals on $\partial\Omega$ are finite. Hence we have found a constant $c(f, u_0, \partial\Omega) > 0$ such that

$$\int_{\Omega_n} |\omega|^2 + n\int_{\partial B_n} |\omega|^2 \le c.$$

This means that each of the two terms on the left-hand side of inequality is bounded above. From (2.5), we deduce that $\int_{\Omega_n}\sum_{i,j} s_{ij}^2$ is bounded.

In view of (2.3), we have shown that all first order derivatives of u are bounded in the L^2-norm, the bound being independent of the radius n of Ω_n. □

Lemma 2D. *Let $\Omega = \mathbb{R}^3 \setminus \bar{B}_1(0)$ be an exterior domain, with $\partial\Omega = \partial B_1(0)$. Let $\Omega_n = \Omega \cap B_n(0)$ for every large integer n. Denote by $\{u_n(x)\}$ a sequence of solutions to* (1.4) *in Ω_n under the hypotheses of Lemma 2B, and by $\{\nabla u_n\}$ the sequence of first order derivatives of u_n. Let K be any compact subdomain of Ω. Then both $\{u_n\}$ and $\{\nabla u_n\}$ are uniformly bounded and equicontinuous on K.*

Proof. Taking curl of the vorticity equation, by similar arguments to these used in the above lemma we compute $\int_K |\mathrm{curl}\,\omega_n|^2 \le c$ for any n, where $c > 0$ depends only on the assigned data and $\partial\Omega$. This together with Lemma 2C implies that $\|u_n\|_{W^{1,6}(K)} \le c$. The imbedding $W^{1,6}(K) \hookrightarrow L^\infty(K)$ shows $\{u_n\}$ has a uniform bound. To see the uniform boundedness of $\{\nabla u_n\}$, we take curl of the curl of the vorticity equation, which gives an expression for $\mathrm{curl}^2\omega_n$. Integrating $|\mathrm{curl}^2\omega_n|^2$ by use of the boundary conditions, and using $\int_K |\mathrm{D}^3 u_n|^2 = \int_K |\mathrm{curl}^2\omega_n|^2$, we obtain $\int_K |\mathrm{D}^3 u_n|^2 \le c$. These results mean that $\|\nabla u_n\|_{W^{2,2}(K)} \le c$. The imbedding of $W^{2,2}$ into L^∞ implies that $\{\nabla u_n\}$ is uniformly bounded on K. In consequence of these facts, we conclude that the pressure function, $\{\nabla p_n\}$, is also bounded uniformly on K.

To establish that $\{u_n\}$ is equicontinuous on K, observe that under the condition (2.1), $u_n(x)$ can be represented in terms of the Biot–Savart law:

$$u_n(x) = \frac{1}{4\pi}\int_K \frac{x-y}{|x-y|^3} \times \omega_n(y)\,dy. \tag{2.6}$$

Strictly speaking, there is a boundary term above if $\sigma \cdot \omega_n|_{\partial\Omega} \ne 0$. With the understanding that the vorticity field is to be extended on the surface, (2.6) is valid (cf. §2.4 in [Sa]). Let $x_1, x_2 \in K$, and denote by m the uniform bound for $\omega_n(x)$ on the set K. It is only necessary to consider when x_1 is close to x_2. For simplicity, set $x_1 = x$, $x_2 = 0$, and so $|x_1 - x_2| = |x| = \delta > 0$, δ being sufficiently small. From (2.6) we have

$$|u_n(x) - u_n(0)| \leq \frac{m}{4\pi} \int_K \left| \frac{x}{|x-y|^3} + \frac{y}{|y|^3} - \frac{y}{|x-y|^3} \right| dy,$$

which splits into $\int_{|y|\leq\delta/2} \big| \cdot \big| + \int_{|y|>\delta/2,\, |x-y|\leq\delta/2} \big| \cdot \big| + \int_{\substack{|y|>\delta/2 \\ |x-y|>\delta/2}} \big| \cdot \big|$. Evaluating each of the above integrals, we find, for example,

$$\int_{|y|\leq\delta/2} \left| \frac{y}{|y|^3} \right| = \mathcal{O}(|x|), \quad \int_{\substack{|y|>\delta/2 \\ |x-y|>\delta/2}} \left| \frac{x}{|x-y|^3} \right| + \left| \frac{|x-y|^3 - |y|^3}{|y|^2 |x-y|^3} \right| = \mathcal{O}(|x| \ln |x|),$$

where we used $\Big| |x-y|^3 - |y|^3 \Big| \leq |x|(|x-y|^2 + |y||x-y| + |y|^2)$. This convinces us that there is a constant $c > 0$ such that

$$|u_n(x_1) - u_n(x_2)| \leq c|x_1 - x_2| \cdot \big| \ln |x_1 - x_2| \big|.$$

Since $x \ln x \downarrow 0$ as $x \downarrow 0$, the equicontinuity follows.

Next we wish to show that $\{\nabla p_n\}$ is equicontinuous on K. Recall that the pressure satisfies the Poisson equation (see the proof of Lemma 2B), so it can be represented by the explicit formula

$$\nabla p_n(x) = \frac{1}{4\pi} \int_K \frac{x-y}{|x-y|^3} F_n(y)\, dy + \phi(x), \qquad F_n = \operatorname{div}(u_n \cdot \nabla u_n), \tag{2.7}$$

where ϕ is a uniformly continuous function on the inner boundary and vanishes sufficiently fast from the boundary, and $F_n \in \mathcal{C}(K)$ is uniformly bounded on K. The same reasoning as above ensures that $\{\nabla p_n\}$ is equicontinuous on K.

Finally we turn to the equicontinuity of $\{\nabla u_n\}$. For brevity, we write $u^1 = u_n(x_1)$, $\nabla u^1 = \nabla u_n(x_1)$, and so on. Because u^1 and u^2 are solutions to (1.4), we have $\alpha(u^1 - u^2) + \beta(x_1 \cdot \nabla u^1 - x_2 \cdot \nabla u^2) + (u^1 \cdot \nabla u^1 - u^2 \cdot \nabla u^2) + (\nabla p^1 - \nabla p^2) = 0$, rearranged to give the matrix product

$$[V]\,[X] = [Y],$$

where $V = (v_{ij}) = \nabla u^1 - \nabla u^2$, $X = \beta x_1 + u^1$, $Y = \beta(x_2 - x_1) \cdot \nabla u^2 + (u^2 - u^1) \cdot (\nabla u^2 + \alpha) + (\nabla p^2 - \nabla p^1)$. Note that $Y \neq 0$, otherwise it would imply a trivial constant solution that is impossible given the smooth data on $\partial\Omega$. Computing $\det[V]$ by use of (2.2), we find $\det[V] \neq 0$ if a constant solution is excluded, in particular $0 < c_1 \leq |\det[V]|$, c_1 being the uniform lower bound of the determinant. So V is invertible as it is a linear operator, hence $X = V^{-1}Y$, that is, $|X| \leq$

$\|V^{-1}\|\,|Y|$. We then estimate $\|V\| \le 3m$, $\|V^{-1}\| \le 6m^2/|\det V|$, where m is the uniform bound for $\{\nabla u_n\}$. By $I = VV^{-1}$, we obtain $1 \le \|V\|\|V^{-1}\| \le 3m \cdot 6m^2/c_1$, implying $\|V^{-1}\| \le C\|V\|^{-1}$, $C = 18m^3/c_1$. This produces an inequality

$$\|V\| \le \frac{C\,|Y|}{|X|}. \tag{2.8}$$

On the right of (2.8), we know that in the numerator Y, $\alpha = \beta = 1/2$ and $|\nabla u^2|$ is bounded above by an absolute constant. In the denominator X, we observe $|\beta x_1| \ge 1/2$; since the solution is guaranteed to be small by Proposition 2A, we may set $|u^1| \le 1/4$ everywhere, so $|X| > 1/4$. Note $|v_{ij}| \le \|V\|$ for all i, j. Combining these with (2.8), we see there is a constant $c > 0$ so that

$$\big|\nabla u_n(x_1) - \nabla u_n(x_2)\big| \le c\Big(|x_1 - x_2| + |u_n(x_1) - u_n(x_2)| + |\nabla p_n(x_1) - \nabla p_n(x_2)|\Big).$$

Let $\epsilon > 0$ be given. In conjunction with the equicontinuity of $\{u_n\}$ and $\{\nabla p_n\}$, one can choose a $\delta > 0$, s.t. $|x_1 - x_2| < \delta \implies |\nabla u_n(x_1) - \nabla u_n(x_2)| < \epsilon$. This is true for all $x_1, x_2 \in K$ and for all n. □

Corollary 2E. *Let $\Omega \subset \mathbb{R}^3$ be the exterior domain as above, $\{u_n\}$ and $\{\nabla u_n\}$ be as defined in Lemma 2D. Then up to subsequences*

$$u_n(x) \to u(x), \quad \nabla u_n(x) \to \nabla u(x), \quad u, \nabla u \in \mathcal{C}(\Omega).$$

Moreover, there is a constant $M(f, u_0, \partial\Omega) > 0$ such that

$$\int_\Omega |\nabla u|^2\,dx \le M.$$

Proof. By the Arzelá–Ascoli theorem, there are subsequences of $\{u_n\}$ and $\{\nabla u_n\}$ converging uniformly on compact subdomains of Ω to limit functions $u(x)$, $\nabla u(x)$, respectively. $\|\nabla u\|^2_{L^2(\Omega)} \le M$ is an immediate consequence of Lemma 2C. □

3. Existence and uniqueness

We begin by showing the sequence of solutions converges to a solution of (1.4) in the unbounded domain Ω, and the solution is continuous at infinity. Uniqueness is obtained for smallness of data.

Lemma 3A. *Let $\Omega = \mathbb{R}^3 \setminus \bar{B}_1(0)$, $\Omega_n = \Omega \cap B_n(0)$, and $\{u_n(x)\}$, $\{\nabla u_n(x)\}$ be as defined in Lemma 2D. Then $\{u_n\}$ converges, together with its derivatives of first order, uniformly on compact subdomains of Ω to a solution $u(x)$ of* (1.4).

Proof. By a diagonal procedure, we can select from $\{u_n\}$ a subsequence $\{u_{n_k}\}$ such that $u_{n_k} \to u$, together with $\nabla u_{n_k} \to h$, uniformly on any compact subdomain of Ω, and so u and h are continuous at each $x \in \Omega$. Because the $\{u_n\}$ are differentiable and the $\{\nabla u_n\}$ converges uniformly on compact subdomains of Ω to the continuous function h, it follows that $u(x)$ is differentiable, and for all its derivatives of first order, we have $\nabla u(x) = h(x) = \lim_{n_k \to \infty} \nabla u_{n_k}(x)\ \forall x \in \Omega$.

Now for a fixed n, let u_n be a solution to (1.4) in Ω_n:

$$\alpha u_n + \beta x \cdot \nabla u_n + u_n \cdot \nabla u_n + \nabla p_n = 0, \qquad \operatorname{div} u_n = 0,$$

where p_n is a solution to the Poisson equation: $- \triangle p_n = \operatorname{div}(u_n \cdot \nabla u_n)$ (cf. Lemma 2B). Let $n \to \infty$, so that $\Omega_n \to \Omega$. Passing to the limit by Lemma 2D, we arrive at $\alpha u_n \to \alpha u$, $\beta x \cdot \nabla u_n \to \beta x \cdot \nabla u$, $u_n \cdot \nabla u_n \to u \cdot \nabla u$, $\nabla p_n \to \nabla p$, and $\operatorname{div} u_n \to \operatorname{div} u = 0$, uniformly on any compact subdomain of Ω. Thus $u(x)$ solves (1.4) at every x in Ω. □

Lemma 3B. *Let u be a solution to* (1.4) *as constructed in Lemma 3A for the exterior problem. Define $u|_\infty = 0$, and $\nabla u|_\infty = 0$. Then both $u(x)$ and $\nabla u(x)$ vanish at infinity.*

Proof. Lemma 3A allows one to write

$$u(x) = \frac{1}{4\pi} \int_\Omega \frac{x-y}{|x-y|^3} \times \omega(y)\, dy, \qquad \omega = \operatorname{curl} u. \tag{3.1}$$

Applying Cauchy–Schwarz's inequality to (3.1) to a ball $B_1(x)$ gives

$$\int_{B_1(x)} \left| \frac{x-y}{|x-y|^3} \times \omega(y) \right| dy \le c \|\nabla u\|_{L^2(B_1(x))} \|\, |x-y|^{-2} \|_{L^2(B_1(x))}.$$

In view of Corollary 2E, $\|\nabla u\|_{L^2(B_1(x))} \| \downarrow 0$ as $|x| \uparrow \infty$, consequently

$$\lim_{|x| \uparrow \infty} u(x) = 0.$$

To show that ∇u vanishes at infinity, differentiating (3.1) yields a formula:

$$\nabla u(x) = P.V. \frac{1}{4\pi} \int_\Omega \nabla_x \frac{x-y}{|x-y|^3} \times \omega(y)\, dy, \tag{3.2}$$

where *P.V.* stands for the principal value. Applying the same argument as above to this integral, we conclude

$$\lim_{|x| \uparrow \infty} \nabla u(x) = 0.$$

This proves that $u(x)$ and $\nabla u(x)$ tend to 0 as $|x| \uparrow \infty$. □

Remark 3.1. Solving for the pressure function from the Poisson equation and using the explicit formula (2.7), one can show that $p(x) \downarrow 0$, $\nabla p(x) \downarrow 0$, as $|x| \uparrow \infty$. Collecting theses results, we have

Theorem 3C (Existence)**.** *Let $\Omega = \mathbb{R}^3 \setminus \bar{B}_1(0)$. On the boundary let data be given as defined in Lemma 2B. Then there exists a solution $(u, p) \in \mathcal{C}^1(\Omega; \mathbb{R}^3 \times \mathbb{R})$ to* (1.4)*, which assumes the prescribed data on $\partial\Omega$ and vanishes at infinity*

$$\lim_{|x|\uparrow\infty} u(x) = 0, \qquad \lim_{|x|\uparrow\infty} \nabla u(x) = 0.$$

Furthermore, the solution satisfies the estimate

$$\|u\|_{\mathcal{C}(\bar{\Omega})} \leq c(f, g). \tag{3.3}$$

Proof. The existence follows directly from Lemmas 3A and 3B, and Remark 3.1. Since $u(x)$ is continuous in $\bar{\Omega}$ and tends to zero at infinity, we deduce (3.3). □

Theorem 3D (Uniqueness)**.** *Let (u_1, p_1) be a solution to* (1.4) *as in Theorem 3C. Assume that* (i) $\sup_{x\in\Omega} |\nabla u_1(x)|$ *is suitably small;* (ii) $|u_1(x)| \leq c|x|^{-1}$*, and $|p_1(x)| \leq c|x|^{-2}$. Then (u_1, p_1) is the only solution under the same set of data.*

Proof. Write $m := \sup |\nabla u_1(x)|, x \in \Omega$. Let (u_2, p_2) be another solution, with the same data as that of (u_1, p_1). Set $U = u_1 - u_2$, $P = p_1 - p_2$, where $U(x)$ satisfies

$$\begin{cases} \alpha U + \beta x \cdot \nabla U = U \cdot \nabla U - U \cdot \nabla u_1 - u_1 \cdot \nabla U - \nabla P, \\ \operatorname{div} U = 0, \quad U|_{\partial\Omega} = 0, \quad \lim_{|x|\uparrow\infty} U(x) = 0. \end{cases}$$

Define $r = |x - x_0|$, x_0 being an arbitrary fixed point in Ω, and $r > r_0 = 1$. Taking the inner product with $r^{-2}U$, one gets

$$[\alpha|U|^2 + \beta U \cdot x \cdot \nabla U]r^{-2} = [U \cdot U \cdot \nabla U - U \cdot U \cdot \nabla u_1 - U \cdot u_1 \cdot \nabla U - U \cdot \nabla P]r^{-2}.$$

On the right-hand side, integrating the third term by parts and applying the boundary conditions and (ii), we find

$$\int_\Omega r^{-2} U \cdot (u_1 \cdot \nabla U) \leq r_0^{-2} \int_\Omega \operatorname{div}(u_1|U|^2/2) = r_0^{-2} \int_{\partial\Omega\cup S_\infty} u_1 \cdot n|U|^2/2 = 0.$$

Similarly, $\int_\Omega r^{-2}U \cdot (U \cdot \nabla U) = 0$, and $\int_\Omega r^{-2}U \cdot \nabla P = 0$. As a result we have

$$\frac{\beta}{2} \int_\Omega r^{-2}|U|^2 = -\int_\Omega r^{-2}\, U \cdot (U \cdot \nabla u_1) \leq m \int_\Omega r^{-2}\, |U|^2. \tag{3.4}$$

We know that $\|U/r\|_{L^2(\Omega)} \leq c$, a consequence of Corollary 2E, and that $\beta = 1/2$. Clearly, if in (3.4) m is smaller than $1/4$, then one can only conclude $U \equiv 0$. Hence $u_1 \equiv u_2$, and so $p_1 \equiv p_2$. □

Remark 3.2. It can be shown that the solution (u, p) is stable, in the sense it has continuous dependence on the boundary data (cf. Theorem 1.2 [H]). We will see in the next section that the assumption (ii) is valid.

4. The isolated self-similar singularity

In this section, we shall denote by x, y the original and the self-similar variables, respectively. By definition, $x^* = 0$ and $t^* < +\infty$. Denote a space-time set by $\widehat{\Omega}(t,n) = \{x \in \mathbb{R}^3 \setminus \{0\} : \sqrt{t^* - t} \leq |x| < n\sqrt{t^* - t}\}$, where $0 < t < t^*$, and n is sufficiently large. The first lemma illustrates that $u(y)$ has decay rate (1.5).

Lemma 4A (Growth condition)**.** *Let $\Omega = \mathbb{R}^3 \setminus \bar{B}_1(0)$ be the exterior domain, and $u(y)$ be a solution to* (1.4) *as in Theorem 3C. Then there exists a class of solutions $u \in L^4(\Omega)$ and $\nabla u \in L^2(\Omega)$ such that*

$$u = \mathcal{O}(|y|^{-1}), \qquad \nabla u = \mathcal{O}(|y|^{-2}), \qquad \textit{as} \ \ |y| \uparrow \infty.$$

Proof. We first observe the following fact. Theorem 3C states that u is continuously differentiable in Ω, with the properties $\int_\Omega |\nabla u|^2 \, dy < +\infty$, and $\lim_{|y| \uparrow \infty} u(y) = 0$. Writing $S_r(0)$ for a sphere centered at the origin with radius $r = |y|$, one gets

$$\lim_{r \uparrow \infty} \frac{1}{r} \int_{S_r(0)} |u|^2 \, ds = 0 \tag{4.1}$$

(see Lemma 3.3 [F]). Then by the symmetries of similarity transformation, we deduce from (4.1) that $|u| \leq c|y|^{-1}$. (One could also use the methods in [Ga] to prove the growth.) It further follows that $|\nabla u| \leq c|y|^{-2}$ by Euler's theorem on homogeneous functions of degree -1. We have also correspondingly from (2.7) that $p(y) = \mathcal{O}(|y|^{-2})$ as $|y| \uparrow \infty$. □

Lemma 4B (Bounded energy)**.** *Let initial data $v(x, 0) = v_0$ be $\mathcal{C}^\infty$ divergence-free, and $v_0 \in L^2(\mathbb{R}^3)$. Suppose that $n\sqrt{t^* - t} = \mathcal{O}(|x_0|)$, $|x_0|$ being a number depending on v_0 only. Let $v(x,t) = u(y)/\sqrt{t^* - t}$ be a solution to* (1.1) *on the set $\widehat{\Omega}(t,n)$, where $u(y)$ is as in Lemma 4A. Then the energy inequality* (1.6) *holds.*

Proof. To see (1.6) is satisfied, it suffice to show the solution $v(x,t)$ has bounded energy on $\widehat{\Omega}(t,n)$. Let Ω_n be any subset of the above exterior domain Ω. For each

$0 < t < t^*$, $\int_{\widehat{\Omega}(t,n)} |v(x,t)|^2 \, dx = (t^* - t)^{1/2} \int_{\Omega_n} |u(y)|^2 \, dy$. It is clear that at fixed n, the energy associated with the self-similar solution is decreasing.

Now let $t \uparrow t^*(n \to \infty)$. Consider

$$\lim_{t \uparrow t^*} \int_{\widehat{\Omega}(t,\infty)} |v(x,t)|^2 \, dx = \lim_{t \uparrow t^*} (t^* - t)^{1/2} \int_{\Omega} |u(y)|^2 \, dy. \tag{4.2}$$

In the light of Lemma 4A, $\|u\|^2_{L^2(\Omega)}$ grows as $\mathcal{O}(|y|)$. By the above hypothesis, we have $\mathcal{O}(|y|) = \mathcal{O}(|x_0|/\sqrt{t^* - t})$. Hence the asymptotics show that the limit on the right of (4.2) exists (see also a similar proof in §4 [Ma]). □

Corollary 4C. *Let $v(x,t)$ be as in Lemma 4B. Then in* (1.1), $T = t^* < +\infty$. *The singularity has finite energy, and it grows at $x^* = 0$ as*

$$\begin{aligned}
&\textit{velocity} && \limsup_{t \uparrow t^*} |v|_{L^\infty} = \infty, && |v|_{L^\infty} \sim (|t^* - t|^{-1/2});\\
&\textit{vorticity} && \limsup_{t \uparrow t^*} |\operatorname{curl} v|_{L^\infty} = \infty, && |\operatorname{curl} v|_{L^\infty} \sim (|t^* - t|^{-1});\\
&\textit{enstrophy} && \limsup_{t \uparrow t^*} \|\operatorname{curl} v\|^2_{L^2(\mathbb{R}^3)} = \limsup_{t \uparrow t^*} (t^* - t)^{-1/2} \|\operatorname{curl} u\|^2_{L^2(\Omega)} = \infty.
\end{aligned}$$

Proof. The blowup rates are computed from (1.2) and Corollary 2E. The singularity forms at (x^*, t^*), so its Hausdorff dimension is zero. □

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