Density-Dependent Incompressible Fluids
in Bounded Domains

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Abstract. This paper is devoted to the study of the initial value problem for density dependent incompressible viscous fluids in a bounded domain of $\mathbb{R}^N$ ($N \geq 2$) with $C^{2+\epsilon}$ boundary. Homogeneous Dirichlet boundary conditions are prescribed on the velocity. Initial data are almost critical in term of regularity: the initial density is in $W^{1,q}$ for some $q > N$, and the initial velocity has ϵ fractional derivatives in L^r for some $r > N$ and ϵ arbitrarily small. Assuming in addition that the initial density is bounded away from 0, we prove existence and uniqueness on a short time interval. This result is shown to be global in dimension $N = 2$ regardless of the size of the data, or in dimension $N \geq 3$ if the initial velocity is small.

Similar qualitative results were obtained earlier in dimension $N = 2, 3$ by O. Ladyzhenskaya and V. Solonnikov in [18] for initial densities in $W^{1,\infty}$ and initial velocities in $W^{2-\frac{2}{q},q}$ with $q > N$.

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Introduction

This paper is devoted to the study of flows of density dependent incompressible viscous fluids in connected bounded domains Ω of $\mathbb{R}^N$.

The system of PDE's associated to such flows reads:

$$\begin{cases} \partial_t \rho + \operatorname{div} \rho u = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \mu \Delta u + \nabla \Pi = \rho f, \\ \operatorname{div} u = 0, \\ (\rho, u)_{|t=0} = (\rho_0, u_0). \end{cases} \tag{1}$$

Above, $\rho = \rho(t, x) \in \mathbb{R}^+$ denotes the density, $u = u(t, x) \in \mathbb{R}^N$, the velocity field. The parameter $\mu > 0$ stands for the viscosity. The term $\nabla \Pi$ (namely the gradient of the pressure) may be seen as the Lagrange multiplier associated to the

constraint div $u = 0$. In addition, the velocity u is assumed to satisfy homogeneous Dirichlet boundary conditions, namely $u_{|\partial\Omega} \equiv 0$. For the sake of simplicity, it is assumed throughout the paper that $\partial\Omega$ is $C^{2+\epsilon}$ for some $\epsilon > 0$.

Given initial conditions (ρ_0, u_0) (with u_0 satisfying the Dirichlet boundary conditions), and an external force f, we address the question of finding a unique solution to (1) for small or large time.

This question has been studied by a number of authors. Roughly, two different approaches may be distinguished. The oldest one relies on the following formal inequality

$$\|(\sqrt{\rho}u)(t)\|_{L^2}^2 + 2\mu \int_0^t \|\nabla u(\tau)\|_{L^2}^2 \, d\tau \leq \|\sqrt{\rho_0}u_0\|_{L^2}^2 + 2 \int_0^t \int_\Omega \rho(\tau)f(\tau) \cdot u(\tau) \, d\tau. \tag{2}$$

for solutions (ρ, u, Π) of (1).

Using (2) and the fact that the density is advected by the flow of u so that the L^p norms of ρ are (at least formally) conserved during the evolution, it is then possible to use compactness methods to prove the existence of global weak solutions.

This approach has been introduced by J. Leray in 1934 in the homogeneous case (i.e. $\rho \equiv 1$) and no external force. The reader is referred to [19] for more details.

The non-homogeneous equations (1) have been considered in the sixties and seventies by the Russian school (see e.g. [3] and the references therein) and more recently by P.-L. Lions in [20] and B. Desjardins in [7] and [9]. Compare to the homogeneous case, the two main difficulties that one has to face are:

- the control of regions of vacuum,
- the fact that the pressure cannot be eliminated by projecting the momentum equation over the set of solenoidal vector-fields.

Let us mention in passing that in dimension $N = 2$, one can further use a quasi-conservation law for the H^1 norm of the velocity and get global H^1 solutions.

In both cases however, the problem of uniqueness has not been solved.

On the other hand, for smooth enough data and no external force, the question of finding unique strong solutions has been successfully solved by O. Ladyzhenskaya and V. Solonnikov in [18]. They proved:

Theorem 0.1. *Let $q > N$ and $N = 2, 3$. Let $\rho_0 \in C^1(\overline{\Omega})$ satisfy $\inf_{x\in\Omega} \rho_0(x) > 0$ and let u_0 be a solenoidal vector-field with coefficients in $W^{2-\frac{2}{q},q}$ and vanishing on $\partial\Omega$. There exists a positive time T such that (1) has a unique solution $(\rho, u, \nabla\Pi)$ with*

$$\rho \in L^\infty(0, T; C^1), \quad u \in C([0, T]; W^{2-\frac{2}{q},q}) \quad and \quad \partial_t u, \nabla^2 u, \nabla\Pi \in L^q(0, T \times \Omega).$$

If $N = 3$ and $\|u_0\|_{W^{2-\frac{2}{q},q}}$ is sufficiently small, or if $N = 2$ then the solution is global and belongs to the spaces described above for all $T > 0$.

Similar qualitative results have been obtained by H. Okamoto in the framework of Sobolev spaces: the initial density is assumed to be non-negative and the initial velocity belongs to the fractional domain[1] $D(A^\eta)$ with $\eta \in (N/4, 1)$ where A is the Stokes operator in L^2.

In the present work, we aim at finding a class of data (ρ_0, u_0, f) as large as possible for which Ladyzhenskaya and Solonnikov's result remains true.

In order to guess what the limit regularity for the data should be, let us briefly review a few standard results in the homogeneous case $\rho \equiv 1$. System (1) then reduces to the celebrated incompressible Navier–Stokes equations:

$$\begin{cases} \partial_t u + \operatorname{div}(u \otimes u) - \mu \Delta u + \nabla \Pi = f, \\ \operatorname{div} u = 0, \\ u_{|t=0} = u_0. \end{cases} \tag{3}$$

In the case of a (smooth) bounded domain Ω with no external force, it has been stated by Y. Giga in [13] that (3) has a unique local solution for data u_0 in the space

$$X^r \stackrel{\text{def}}{=} \{z \in L^r(\Omega)^N \mid \operatorname{div} z = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad z \cdot n = 0 \quad \text{on} \quad \partial\Omega\}$$

whenever $r \geq N$.

As far as existence and uniqueness is obtained from contracting mapping arguments, the exponent $r = N$ seems to be optimal. This is closely linked to the invariance (for all $\lambda > 0$) of the space $L^N(\mathbb{R}^N)$ by the transform

$$u_0(x) \mapsto u_0^\lambda(x) \stackrel{\text{def}}{=} \lambda u_0(\lambda x)$$

and to the fact that if u is the solution of (3) corresponding to the data u_0 then $(t, x) \mapsto \lambda u(\lambda^2 t, \lambda x)$ is the solution associated to u_0^λ.

Similar scaling considerations are relevant in the density-dependent case (see e.g. [5] for more explanations) and induce us to consider data (ρ_0, u_0) in a *critical* space whose norm is invariant by the transformation

$$\rho_0(x) \mapsto \rho_0(\lambda x), \qquad u_0(x) \mapsto \lambda u_0(\lambda x).$$

As a matter of fact, in [5] and [6], we stated local and global existence results for (1) in the whole space $\mathbb{R}^N$ or in the torus $\mathbb{T}^N$ for data having critical or almost critical Sobolev regularity. There, our proofs rely on Fourier analysis so that they cannot be easily carried out in bounded domains.

In the present work, we aim at generalizing Giga's result to *non-homogeneous* fluids. According to the above scaling considerations, choosing (ρ_0, u_0) in $W^{1,N} \times L^N$ seems to be an appropriate choice. This has to be compared with the assumptions of Theorem 0.1: there u_0 has to be in $W^{2-\frac{2}{q}, q}$ with $q > N$.

Whether existence of strong unique solutions may be proved under such assumptions is open. The fact that $W^{1,N}$ fails to be embedded in L^∞ is one of the

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[1] Roughly, it amounts to asking u_0 to have 2η derivatives in L^2.

reasons why. We shall see however that, if we make slightly stronger assumptions in terms of integrability and regularity then all the results of Theorem 0.1 remain true.

Our paper is structured as follows. In the first section, we state our main local and global existence results. In the next section, we introduce some notation and functional spaces. Section 3 is devoted to the proof of *a priori* estimates for the linearized system (1). Here we get estimates for a non-homogeneous non-stationary Stokes equation, interesting for their own sake. In Part 4, we study uniqueness and stability for (1) whereas the proof of local existence is postponed to Section 5. In the next section, we prove global existence for small initial velocity and non-vanishing density in dimension $N \geq 2$ whereas Section 7 deals with global existence for large data in dimension $N = 2$. Some technical estimates are postponed in appendix.

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1. Main results

Let us first define the functional spaces in which existence is going to be shown:

Definition 1.1. For $T > 0$ and $1 < p, q, r < +\infty$, we denote by $E_T^{p,q,r}$ the set of triplets (ρ, u, Π) such that

$$u \in C([0,T]; D_{A_r}^{1-\frac{1}{p},p}) \cap L^p(0,T; W^{2,r} \cap W_0^{1,r}), \quad \partial_t u \in L^p(0,T; L^r) \text{ and } \operatorname{div} u = 0,$$

$$\rho \in C(0,T; W^{1,q}), \quad \Pi \in L^p(0,T; W^{1,r}) \quad \text{and} \quad \int_\Omega \Pi \, dx = 0.$$

If $q = +\infty$, we agree that ρ belongs to $L^\infty(0,T; W^{1,\infty}) \cap C([0,T] \times \Omega)$ instead of $C([0,T]; W^{1,\infty})$.

The corresponding norm is denoted by $\| \cdot \|_{E_T^{p,q,r}}$.

The space $D_{A_r}^{1-\frac{1}{p},p}$ stands for some fractional domain of the Stokes operator in L^r (see the definition in Section 2.3). Roughly, the vector-fields of $D_{A_r}^{1-\frac{1}{p},p}$ have $2 - \frac{2}{p}$ derivatives in L^r, are divergence-free and vanish on $\partial\Omega$.

Our main local existence result reads:

Theorem 1.2. *Let Ω be a bounded domain with $C^{2+\epsilon}$ boundary. Let $\check{\rho} > 0$, $q \in (N, \infty]$, $1 < p < \infty$ and $r \in (N, q] \cap \mathbb{R}$. Let $\rho_0 \in W^{1,q}$ be bounded away from zero by $\check{\rho}$, $u_0 \in D_{A_r}^{1-\frac{1}{p},p}$ and $f \in L^p(0, T_0; L^r)$ for some $T_0 > 0$. There exists a $T \in (0, T_0]$ such that system (1) has a unique solution (ρ, u, Π) in $E_T^{p,q,r}$ with besides $0 < \check{\rho} \leq \rho$ on $[0,T] \times \Omega$.*

Remark 1.3. 1. The time T of local existence may be bounded by below in terms of the norms of the initial data, and of Ω, μ, $\check{\rho}$, p, r, q and N. The reader is referred to Proposition 6.4 for more details.

2. One can also prove a result of continuity with respect to the data (see Remark 4.3).

For small initial velocity u_0 and external force f but *no restriction on the size of ρ_0*, we actually have *global* existence in any dimension $N \geq 2$:

Theorem 1.4. *Let u_0, ρ_0 and f satisfy the assumptions of Theorem 1.2. Let $\gamma > 0$. There exists a constant η depending on p, q, r, N, γ, $\check{\rho}$, Ω but independent of μ, and $\delta > 0$ depending only on p, q, r and N, such that if*

$$\|u_0\|_{D_{A_r}^{1-\frac{1}{p},p}} + \mu^{\frac{1}{p}-1}\|f\|_{L^p(\mathbb{R}^+;L^r)} + \int_0^\infty e^{\mu\gamma t}\|f(t)\|_{L^2}\,dt \leq \frac{\eta\mu}{\left(1 + \|\rho_0\|_{W^{1,q}}\right)^\delta}$$

then (1) has a unique global solution which belongs to $E_T^{p;q,r}$ for all $T \geq 0$.

Besides, denoting by λ_1 the first eigenvalue of the Dirichlet–Laplace operator in Ω, and $\kappa \overset{\text{def}}{=} \min(\gamma, \lambda_1/\|\rho_0\|_{L^\infty})$, we have the following inequalities for all $t \in \mathbb{R}^+$:

$$\left\|\sqrt{\rho(t)}u(t)\right\|_{L^2} \leq e^{-\kappa\mu t}\left(\|\sqrt{\rho_0}u_0\|_{L^2} + \int_0^t e^{\mu\gamma\tau}\|\sqrt{\rho}f\|_{L^2}\,d\tau\right)$$

and, for some K depending only on $\|\rho_0\|_{W^{1,q}}$, p, q, r, N, μ, γ, $\check{\rho}$ and Ω,

$$\|(\rho, u, \Pi)\|_{E_t^{p,q,r}} \leq K\left(\|u_0\|_{D_{A_r}^{1-\frac{1}{p},p}} + \|f\|_{L_t^p(L^r)} + \int_0^t e^{\mu\gamma t}\|f(\tau)\|_{L^2}\,d\tau\right).$$

It turns out that in dimension $N = 2$, global existence of smooth solutions holds true for large data with non-vanishing density, a result which has to be compared with what we have in the homogeneous case.

Theorem 1.5. *Let $1 < p < \infty$, $2 < r < \infty$ and $q \in [r, \infty]$. Assume that $N = 2$, that $\rho_0 \in W^{1,q}$ is bounded away from 0, that $u_0 \in D_{A_r}^{1-\frac{1}{p},p}$ and that $f \in L_{loc}^p(\mathbb{R}^+; L^r) \cap L_{loc}^2(\mathbb{R}^+; L^2)$. Then system (1) has a unique global solution which belongs to $E_T^{p,q,r}$ for all $T > 0$.*

Remark 1.6. For the sake of simplicity, we restricted ourselves to the study of fluids in *bounded* domains. However, we expect that all the results pertaining to *local* existence may be proved for domains in which the Stokes operator in L^r satisfies condition $(\mathcal{H}_2)$ described in Section B of the appendix. This property is known to be true if Ω is the complementary of a bounded smooth domain for instance (see [15]).

2. Notations and functional spaces

2.1. General notation

Throughout the paper, C stands for a "harmless" constant whose exact meaning depends on the context. Given a set of parameters $S = \{\alpha_1, \cdots, \alpha_k\}$, the notation $C = C_S = C(S)$ means that C depends only on $\alpha_1, \cdots, \alpha_k$. Sometimes, we make use of $A \lesssim B$ in place of $A \leq CB$ and $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

Let Ω be a bounded Lipschitz domain of $\mathbb{R}^N$. Then $\partial\Omega$ denotes the boundary of Ω, and n stands for the outer unit normal at the boundary. We denote by $\delta(\Omega)$ the diameter of Ω and by $|\Omega|$, its Lebesgue measure. The notation $\sigma(\Omega)$ stands for the "dimensionless" open set

$$\sigma(\Omega) = \left\{ \frac{x}{\delta(\Omega)} \,\middle|\, x \in \Omega \right\}.$$

Hence, when denoting $C = C_{\sigma(\Omega)}$, it is understood that the constant C *does not* change under dilation of Ω, i.e. depends only on the shape of Ω.

Let $1 \leq p \leq \infty$ and K be a measurable subset of $\mathbb{R}^N$. The notation $L^p(K)$ (or L^p if no ambiguity) stands for the set of measurable functions on K with values in $\mathbb{R}$ and bounded L^p norm. A similar notation is used for vector-valued functions. More generally, if X is a Banach space and $f = (f_1, \cdots, f_k)$ is such that $f_i \in X$ for each $i \in \{1, \cdots, k\}$, we define

$$\|f\|_X = \|(f_1, \cdots, f_k)\|_X \overset{\text{def}}{=} \sum_{i=1}^{k} \|f_i\|_X.$$

For $r \in [1, +\infty]$, we denote by $L^r(0, T; X)$ the set of Bochner measurable X-valued time dependent functions f such that $t \mapsto \|f\|_X$ belongs to $L^r(0, T)$. The corresponding Lebesgue norm is denoted by $\|\cdot\|_{L^r_T(X)}$ and the conjugate exponent of r (i.e. $r/(r-1)$) is denoted by r'. If I is an interval of $\mathbb{R}$, the notation $C(I; X)$ (resp. $C_b(I; X)$) stands for the set of continuous (resp. continuous and bounded) functions of $\mathcal{F}(I; X)$.

If k is an integer, we denote by $W^{k,q}(\Omega)$ (or $W^{k,q}$) the set of L^q functions whose derivatives up to order k belong to L^q. If I is an interval of $\mathbb{R}$ and X, a Banach space, the notation $W^{1,p}(I; X)$ stands for the set of $L^p(I; X)$ functions whose first time derivative also belongs to $L^p(I; X)$.

We shall also make use of trace spaces $W^{s,q}(\partial\Omega)$ (see their definition in e.g. [1], [11] or [16]).

2.2. Basic properties of the Stokes operator

For $1 < q < \infty$, let X^q be the completion in L^q of the set of solenoidal vector-fields with coefficients in $C_0^\infty(\Omega)$. It is well known (see e.g. [23]) that for C^1 domains,

$$X^q = \left\{ u \in (L^q(\Omega))^N \mid \operatorname{div} u = 0 \quad \text{and} \quad u \cdot n = 0 \quad \text{on} \quad \Omega \right\},$$

and that any vector-field with coefficients in L^q has a *Helmholtz decomposition*:

Lemma 2.1. *Let Ω be a bounded domain of $\mathbb{R}^N$ with C^1 boundary. For all $f \in L^q(\Omega)^N$, there exists a unique couple (f_0, P) with*

$$f = f_0 + \nabla P, \quad f_0 \in X^q, \quad P \in L^q_{loc}(\Omega), \quad \nabla P \in L^q(\Omega) \quad and \quad \int_\Omega P \, dx = 0.$$

Besides the map $f : \begin{cases} L^q \longrightarrow X^q \times L^q \\ f \longmapsto (f_0, \nabla P) \end{cases}$ is continuous.

We denote by $P_q : f \mapsto f_0$ the projector from L^q onto X^q introduced above. We further define (according to [12]) the Stokes operator on L^q:

$$A_q = -P_q \Delta \quad \text{with domain} \quad D(A_q) = W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega) \cap X^q. \qquad (4)$$

2.3. Fractional domains for the Stokes operator

Let us first give the formal definition of the (homogeneous) fractional domains of the Stokes operator in L^q.

Definition 2.2. Let $1 < q < \infty$. For $\alpha \in (0,1)$ and $s \in (1, \infty)$, we set

$$\|u\|_{\dot{D}^{\alpha,s}_{A_q}} \stackrel{\text{def}}{=} \left(\int_0^{+\infty} \left\| t^{1-\alpha} A_q e^{-tA_q} u \right\|^s_{L^q} \frac{dt}{t} \right)^{\frac{1}{s}},$$

where e^{-tA_q} stands for the semi-group associated to A_q. We then define the homogeneous fractional domain $\dot{D}^{\alpha,s}_{A_q}$ as the completion of $D(A_q)$ under $\|u\|_{\dot{D}^{\alpha,s}_{A_q}}$.

The above definition may be made rigorous if A_q generates a bounded analytic semi-group. According to [15], this latter property is known to be true if conditions $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ of Section B in the appendix are fulfilled.

As $1 < q < \infty$, the space $X^q(\Omega)$ is ζ-convex (even if Ω is not bounded or/and not smooth, see [15] page 81) so that $(\mathcal{H}_1)$ always holds. On the other hand, if Ω is a bounded domain with $C^{2+\epsilon}$ boundary, condition $(\mathcal{H}_2)$ is also fulfilled (see [12] and [15]).

Besides, the Stokes operator has the so-called maximal regularity property (see Theorem B.4 in the Appendix and Theorem 3.2 below).

Remark 2.3. 1. Let $\dot{D}(A_q)$ be the completion of $D(A_q)$ in X^q under the norm $\|Au\|_{L^q}$. One can show that $\dot{D}^{\alpha,s}_{A_q}$ agrees with $(X, \dot{D}(A_q))_{\alpha,s}$.

2. One can also define non-homogeneous fractional domains $D_{A_q}^{\alpha,s}$ as the completion of $D(A_q)$ under the following norm:

$$\|u\|_{D_{A_q}^{\alpha,s}} \overset{\text{def}}{=} \|u\|_{L^q} + \left(\int_0^{+\infty} \left\| t^{1-\alpha} A_q e^{-tA_q} u \right\|_{L^q}^s \frac{dt}{t} \right)^{\frac{1}{s}}.$$

Of course, $D_{A_q}^{\alpha,s}$ agrees with $(X, D(A_q))_{\alpha,s}$.

As from now on we shall consider only bounded domains, the following result will be very useful:

Proposition 2.4. *Let Ω be a Lipschitz bounded domain of $\mathbb{R}^N$. There exists a constant $C = C_{N,r,\sigma(\Omega)}$ such that*

$$\|u\|_{W^{2,q}} \overset{\text{def}}{=} \left\| \nabla^2 u \right\|_{L^q} + \delta(\Omega)^{-1} \left\| \nabla u \right\|_{L^q} + \delta(\Omega)^{-2} \left\| u \right\|_{L^q} \leq C \left\| \nabla^2 u \right\|_{L^q}.$$

whenever $u \in W^{2,q} \cap W_0^{1,q}(\Omega)$.

Proof. One just has to notice that Poincaré inequality applies to u (as u vanishes on $\partial\Omega$), and that Poincaré–Wirtinger inequality applies to ∇u (as $\int_\Omega \nabla u \, dx = 0$). $\square$

Hence, for bounded domains, the non-homogeneous fractional spaces $D_{A_q}^{\alpha,s}$ coincide with the homogeneous ones. Therefore, we shall not make the distinction between the two of them from now on.

Let us now explain how the fractional domains of A_q may be identified with Besov spaces $B_{q,s}^\beta$.

Proposition 2.5. *Let $\alpha \in (0,1)$ and $1 < q, s < \infty$. Let $\overset{\circ}{B}_{q,s}^\beta$ be the completion of $C_0^\infty(\Omega)$ in $B_{q,s}^\beta$. Then we have*

$$\overset{\circ}{B}_{q,s}^{2\alpha} \cap X^q \hookrightarrow D_{A_q}^{\alpha,s} \hookrightarrow B_{q,s}^{2\alpha} \cap X^q.$$

Besides, the three sets are the same (with equivalent norms) provided $2\alpha \leq 1/q$.

Proof. Combining Remark 2.3 and (4), we can write

$$D_{A_q}^{\alpha,s} = (X^q, X^q \cap W_0^{1,q} \cap W^{2,q})_{\alpha,s} = X^q \cap (L^q, W_0^{1,q} \cap W^{2,q})_{\alpha,s}.$$

Because $W_0^{2,q} \hookrightarrow W_0^{1,q} \cap W^{2,q} \hookrightarrow W^{2,q}$, we have

$$\overset{\circ}{B}_{q,s}^{2\alpha} = (L^q, W_0^{2,q})_{\alpha,s} \hookrightarrow (L^q, W_0^{1,q} \cap W^{2,q})_{\alpha,s} \hookrightarrow (L^q, W^{2,q})_{\alpha,s} = B_{q,s}^{2\alpha}$$

which yields the desired chain of embeddings (see e.g. [22] for the proof of the first and last equality). Moreover, if $2\alpha \leq 1/q$, the spaces $\overset{\circ}{B}_{q,s}^{2\alpha}$ and $B_{q,s}^{2\alpha}$ are the same (see e.g. [22], page 83). $\square$

3. The linearized equations

This section is devoted to the study of the linearized system (1).

3.1. The transport equation

The following result is quite standard (as a matter of fact, it is a straightforward generalization of the one presented in [18]):

Proposition 3.1. *Let Ω be a Lipschitz domain of $\mathbb{R}^N$ and $v \in L^1(0, T; \mathrm{Lip})$ be a solenoidal vector-field such that $v \cdot n = 0$ on $\partial\Omega$. Let $a_0 \in W^{1,q}$ with $q \in [1, +\infty]$. Then equation*

$$\begin{cases} \partial_t a + v \cdot \nabla a = 0, \\ \rho_{|t=0} = a_0, \end{cases}$$

has a unique solution in $L^\infty(0, T; W^{1,\infty}) \cap C([0, T]; \cap_{r<\infty} W^{1,r})$ if $q = \infty$ and in $C([0, T]; W^{1,q})$ if $q < \infty$.
    Besides, the following estimate holds true:

$$\forall t \in [0, T], \ \|a(t)\|_{W^{1,q}} \leq e^{\int_0^t \|\nabla v(\tau)\|_{L^\infty} \, d\tau} \|a_0\|_{W^{1,q}}.$$

If in addition a belongs to L^p for some $p \in [1, +\infty]$ then

$$\forall t \in [0, T], \ \|a(t)\|_{L^p} = \|a_0\|_{L^p}.$$

3.2. Estimates for non-stationary Stokes equations

This section is devoted to the proof of estimates for the following linear system:

$$\begin{cases} \partial_t v - \mu \Delta v + \nabla \Pi = f, \qquad \int_\Omega \Pi = 0, \\ \operatorname{div} v = \tau, \\ v_{|t=0} = v_0, \quad v_{|\partial\Omega} = 0. \end{cases} \tag{5}$$

Throughout this section and unless otherwise specified, Ω is a $C^{2+\epsilon}$ bounded domain.

3.2.1. The case of solenoidal vector-fields

We first focus on the non-stationary Stokes equation (5) with $\tau = 0$. Our main statement reads:

Theorem 3.2. *Let Ω be a $C^{2+\epsilon}$ bounded domain of $\mathbb{R}^N$ and $1 < q, s < \infty$. Assume that $u_0 \in D_{A_q}^{1-\frac{1}{s}, s}$ and $f \in L^s(\mathbb{R}^+; L^q)$. Then system*

$$\begin{cases} \partial_t u - \mu \Delta u + \nabla \Pi = f, \qquad \int_\Omega \Pi \, dx = 0, \\ \operatorname{div} u = 0, \qquad u_{|\partial\Omega} = 0, \\ u_{|t=0} = u_0, \end{cases} \tag{6}$$

has a unique solution (u, Π) satisfying the following inequality for all $T \geq 0$:

$$\mu^{\frac{1}{s'}} \|u(T)\|_{D_{A_q}^{\frac{1}{s'},s}} + \left(\int_0^T \left\| \left(\nabla\Pi, \mu\nabla^2 u, \partial_t u \right) \right\|_{L^q}^s dt \right)^{\frac{1}{s}}$$

$$\leq C \left(\mu^{\frac{1}{s'}} \|u_0\|_{D_{A_q}^{\frac{1}{s'},s}} + \left(\int_0^T \|f(t)\|_{L^q}^s dt \right)^{\frac{1}{s}} \right)$$

with $C = C(q, s, N, \sigma(\Omega))$ and $1/s' = 1 - 1/s$.

Proof. Using the change of function $u(t, x) = \mu v(\mu t, x)$, $\Pi(t, x) = \mu^2 P(\mu t, x)$ and $f(t, x) = \mu^2 g(\mu t, x)$ enables us to consider only the case $\mu = 1$.

Now, as under our assumptions on Ω and q, conditions $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ of Section B in the appendix are fulfilled, Theorem B.4 may be applied.

Moreover, for $u \in D(A_q)$, we have (see Prop. 1.4 in [14])

$$\left\| \nabla^2 u \right\|_{L^q} \leq C_{q,N,\sigma(\Omega)} \|A_q u\|_{L^q},$$

and, according to Lemma 2.1,

$$\nabla\Pi = \Delta u + A_q u,$$

which completes the proof. $\square$

3.2.2. The general case

Let us now treat the general case $\operatorname{div} v = \tau$. Maximal regularity estimates for (6) will be obtained by solving first the following stationary Stokes problem:

$$\begin{cases} -\Delta v + \nabla P = f, & \int_\Omega P \, dx = 0, \\ \operatorname{div} v = \tau, \\ v_{|\partial\Omega} = 0, \end{cases} \tag{7}$$

then a problem of type (6) for which Theorem 3.2 applies.

For system (7), we have the following result.

Proposition 3.3. *Let Ω be a C^2 bounded domain and $1 < q < \infty$. Let $f \in L^q(\Omega)$ and $\tau \in W^{1,q}(\Omega)$ with $\int_\Omega \tau \, dx = 0$. Then system (7) has a unique solution (v, P) in $W^{2,q}(\Omega)^N \times W^{1,q}(\Omega)$. Moreover, there exists $C = C_{q,N,\sigma(\Omega)}$ such that*

$$\left\| \nabla^2 v \right\|_{L^q} + \|\nabla P\|_{L^q} \leq C_{q,N,\sigma(\Omega)} \left(\|f\|_{L^q} + \|\nabla\tau\|_{L^q} \right).$$

Proof. See [11] p. 226, and Exercise 6.2. Using scaling arguments enables us to show that the constant depends only on the shape of Ω. $\square$

It turns out that estimates for $\|v\|_{L^q}$ will be also needed.

Proposition 3.4. *Let Ω be a C^2 bounded domain and $1 < q < \infty$. Let Ω' be a Lipschitz open subdomain of $\overline{\Omega}$, star-shaped with respect to some ball $B \subset \Omega'$ centered at x_0 and of diameter $d > 0$. Denote $\chi = \delta(\Omega')/d$ the distortion parameter of Ω' with respect to B, and*

$$c = \frac{1}{\delta(\Omega')} \operatorname{ess\,inf}_{x \in \partial\Omega'} \left(n' \cdot (x - x_0) \right)$$

where n' stands for the outer unit normal on $\partial\Omega'$.

Assume that v solves (7) with $f = 0$ and $\tau = \tau_0 + \operatorname{div} R$, and that τ_0 and R are supported in $\overline{\Omega}'$. The following inequalities hold true whenever $1 < r < \infty$:

$$\|v\|_{L^r(\Omega)} \leq C_{r,N,\sigma(\Omega)} \left(\delta(\Omega') \chi^{\frac{N-1}{r'}} \|\tau_0\|_{L^r(\Omega)} + \|R\|_{L^r(\Omega)} + \|R \cdot n\|_{W^{-\frac{1}{r},r}(\partial\Omega)} \right), \quad (8)$$

$$\|v\|_{L^r(\Omega)} \leq C_{r,N} \left(\delta(\Omega') \chi^{\frac{N-1}{r'}} \|\tau_0\|_{L^r(\Omega)} + \|R\|_{L^r(\Omega)} + \delta(\Omega')^{\frac{1}{r}} \frac{\chi^{\frac{N-1}{r'}}}{c^{\frac{1}{r'}}} \|R \cdot n\|_{L^r(\partial\Omega)} \right). \quad (9)$$

Proof. We have

$$\|v\|_{L^r} = \sup_{\|F\|_{L^{r'}} = 1} \int_\Omega v \cdot F \, dx.$$

Fix a function F in $(L^{r'}(\Omega))^N$. Proposition 3.3 provides a solution (w, Q) to

$$\begin{cases} -\Delta w - \nabla Q = F, & \int_B Q \, dx = 0, \\ \operatorname{div} w = 0, \\ w_{|\partial\Omega} = 0. \end{cases}$$

Combining integrations by parts and equation (7) yields

$$\int_\Omega v \cdot F \, dx = \int_\Omega (\tau_0 Q - R \cdot \nabla Q) \, dx + \int_{\partial\Omega} Q \, R \cdot n \, d\sigma.$$

Therefore, taking advantage of the assumptions on the supports of τ_0 and R,

$$\int_\Omega v \cdot F \, dx \leq \|Q\|_{L^{r'}(\Omega')} \|\tau_0\|_{L^r(\Omega')} + \|\nabla Q\|_{L^{r'}(\Omega')} \|R\|_{L^r(\Omega')}$$

$$+ \begin{cases} \|R \cdot n\|_{W^{-\frac{1}{r},r}(\partial\Omega)} \|Q\|_{W^{\frac{1}{r},r'}(\partial\Omega)}, \\ \|R \cdot n\|_{L^r(\partial\Omega)} \|Q\|_{L^{r'}(\partial\Omega \cap \overline{\Omega}')}. \end{cases}$$

As $\int_B Q = 0$, Poincaré–Wirtinger inequality (97) yields

$$\|Q\|_{L^{r'}(\Omega')} \leq C_N \chi^{\frac{N-1}{r'}} \delta(\Omega') \|\nabla Q\|_{L^{r'}(\Omega')}.$$

Proof of (8). By making use of standard trace theorems, we have $\|Q\|_{W^{\frac{1}{r},r'}(\partial\Omega)} \leq C_{r,N,\sigma(\Omega)} \|Q\|_{W^{1,r'}(\Omega)}$. On the other hand, according to Proposition 3.3, we have

$$\|\nabla Q\|_{L^{r'}(\Omega)} \leq C_{r,N,\sigma(\Omega)} \|F\|_{L^{r'}(\Omega)}, \quad (10)$$

hence inequality (8).

Proof of (9). In order to bound the term $\|Q\|_{L^{r'}(\partial\Omega\cap\overline{\Omega}')}$, it suffices to bound $\|Q\|_{L^{r'}(\partial\Omega')}$.

According to inequality (3.3) page 43 in [11], we have

$$c\delta(\Omega')\|Q\|_{L^{r'}(\partial\Omega')}^{r'} \leq N\|Q\|_{L^{r'}(\Omega')}^{r'} + r'\delta(\Omega')\|Q\|_{L^{r'}(\Omega')}^{r'-1}\|\nabla Q\|_{L^{r'}(\Omega')}.$$

Hence, combining Poincaré–Wirtinger inequality (97) and Hölder inequality,

$$
\begin{aligned}
c^{\frac{1}{r'}}\|Q\|_{L^{r'}(\partial\Omega')} &\leq C_{N,r}\delta(\Omega')^{\frac{1}{r}}\left(\chi^{\frac{N-1}{r'}}\|\nabla Q\|_{L^{r'}(\Omega')} + \chi^{\frac{N-1}{rr'}}\|\nabla Q\|_{L^{r'}(\Omega')}\right), \\
&\leq C_{N,r}\chi^{\frac{N-1}{r'}}\delta(\Omega')^{\frac{1}{r}}\|\nabla Q\|_{L^{r'}(\Omega')}.
\end{aligned}
$$

Using again (10) completes the proof of the proposition. $\qquad\square$

Proposition 3.5. *Let* $1 < p, r < \infty$ *and* $\tau \in L^p(0,T;W^{1,r}(\Omega))\cap W^{1,p}(0,T;W^{-1,r})$ *satisfy*

$$\tau(0,\cdot) \equiv 0, \quad \int_\Omega \tau\,dx = 0 \quad and \quad \partial_t\tau = \tau_0 + \operatorname{div} R$$

with $R, \tau_0 \in L^p(0,T;L^r(\Omega))$, $R\cdot n$ *in* $L^p(0,T;L^r(\partial\Omega))$ *and* $\operatorname{Supp}\tau_0(t,\cdot)\cup\operatorname{Supp}R(t,\cdot) \subset \overline{\Omega}'$ *for all* $t \in (0,T)$ *with* Ω' *satisfying the assumptions of Proposition 3.4.*
The non-stationary Stokes system

$$
\begin{cases}
\partial_t v - \Delta v + \nabla\Pi = 0, & \int_\Omega \Pi\,dx = 0, \\
\operatorname{div} v = \tau, & \\
v_{|\partial\Omega} = 0, & v_{|t=0} = 0,
\end{cases}
\tag{11}
$$

has a unique solution (v,Π) *with*

$$v \in L^p(0,T;W^{2,r}) \cap W^{1,p}(0,T;L^r) \quad and \quad \Pi \in L^p(0,T;W^{1,r}).$$

Moreover, the following inequality holds true:

$$
\|(\partial_t v, \nabla^2 v, \nabla\Pi)\|_{L_T^p(L^r(\Omega))} \leq C_{r,p,N,\sigma(\Omega)}\Big(\delta(\Omega')\chi^{\frac{N-1}{r'}}\|\tau_0\|_{L_T^p(L^r(\Omega))}
$$

$$
+ \|R\|_{L_T^p(L^r(\Omega))} + \delta(\Omega')^{\frac{1}{r}}\frac{\chi^{\frac{N-1}{r'}}}{c^{\frac{1}{r'}}}\|R\cdot n\|_{L_T^p(L^r(\partial\Omega))} + \|\nabla\tau\|_{L_T^p(L^r(\Omega))}\Big).
$$

Proof. For fixed t, let $(v_1(t,\cdot),\Pi_1(t,\cdot))$ be the solution to

$$
\begin{cases}
-\Delta v_1 + \nabla\Pi_1 = 0, & \int_\Omega \Pi_1\,dx = 0, \\
\operatorname{div} v_1 = \tau(t,\cdot), & \\
(v_1)_{|\partial\Omega} = 0. &
\end{cases}
$$

Remark that $(\partial_t v_1, \nabla\partial_t\Pi_1)$ satisfies the stationary Stokes system (7). Hence,

according to Proposition 3.4, we have

$$\|\partial_t v_1\|_{L^r(\Omega)} \le C_{r,N}\left(\delta(\Omega')\chi^{\frac{N-1}{r'}}\|\tau_0\|_{L^r(\Omega)} + \|R\|_{L^r(\Omega)} + \delta(\Omega')^{\frac{1}{r}}\frac{\chi^{\frac{N-1}{r'}}}{c^{\frac{1}{r'}}}\|R\cdot n\|_{L^r(\partial\Omega)}\right). \tag{12}$$

Next, define v_2 as the solution to

$$\begin{cases} \partial_t v_2 - \Delta v_2 + \nabla\Pi_2 = -\partial_t v_1, & \int_\Omega \Pi_2\,dx = 0, \\ \operatorname{div} v_2 = 0, \\ (v_2)_{|\partial\Omega} = 0, & (v_2)_{|t=0} = 0. \end{cases}$$

Note that $\tau(0,\cdot) \equiv 0$ implies $v_1(0,\cdot) \equiv 0$, hence $v \overset{\text{def}}{=} v_1 + v_2$ solves (11).

According to Theorem 3.2, we have

$$\|v_2(T)\|_{D_{A_r}^{1-\frac{1}{p},p}} + \|(\partial_t v_2, \nabla^2 v_2, \nabla\Pi_2)\|_{L_T^p(L^r(\Omega))} \le C_{r,p,N,\sigma(\Omega)}\|\partial_t v_1\|_{L_T^p(L^r(\Omega))}.$$

Hence, using (12) shows that (v_2, Π_2) satisfies the wanted inequality.

Now, according to Proposition 3.3, we have

$$\|(\nabla^2 v_1, \nabla\Pi_1)\|_{L^r(\Omega)} \le C_{r,N,\sigma(\Omega)}\|\nabla\tau\|_{L^r(\Omega)}.$$

The proof of Proposition 3.5 is thus complete. $\square$

We can now easily solve the general non-stationary Stokes system (5). First, using a suitable change of function (see the proof of Th. 3.2), it suffices to consider the case $\mu = 1$. Then, using the decomposition $(v, \nabla\Pi) = (v_1, \nabla\Pi_1) + (v_2, \nabla\Pi_2)$ with

$$\begin{cases} \partial_t v_1 - \Delta v_1 + \nabla\Pi_1 = f, \\ \operatorname{div} v_1 = 0, \quad \int_\Omega \Pi_1 = 0, \\ (v_1)_{|t=0} = v_0, \quad (v_1)_{|\partial\Omega} = 0. \end{cases} \quad\text{and}\quad \begin{cases} \partial_t v_2 - \Delta v_2 + \nabla\Pi_2 = 0, \\ \operatorname{div} v_2 = \tau, \quad \int_\Omega \Pi_2 = 0, \\ (v_2)_{|t=0} = 0, \quad (v_2)_{|\partial\Omega} = 0, \end{cases}$$

Theorem 3.2 and Proposition 3.5 yield the following:

Theorem 3.6. *Let $1 < p, r < \infty$, and Ω be a $C^{2+\epsilon}$ bounded domain of $\mathbb{R}^N$. Let $\Omega' \subset \overline{\Omega}$ be open and star-shaped with respect to some ball of diameter $d > 0$. Let $\tau \in L^p(0,T;W^{1,r})$ satisfy $\tau(0,\cdot) \equiv 0$,*

$$\int_\Omega \tau\,dx = 0, \quad \partial_t\tau = \tau_0 + \operatorname{div} R \quad\text{and}\quad \forall t \in (0,T),\ \operatorname{Supp}\tau_0(t,\cdot) \cap \operatorname{Supp} R(t,\cdot) \subset \overline{\Omega}'$$

with R and τ_0 in $L^p(0,T;L^r(\Omega))$ and $R \cdot n$ in $L^p(0,T;L^r(\partial\Omega))$. Let $v_0 \in D_{A_r}^{1-\frac{1}{p},p}$ and $f \in L^p(0,T;L^r(\Omega))$. Then problem (5) has a unique solution (v,Π) such that

$$v \in L^p(0,T;W^{2,r}) \cap W^{1,p}(0,T;L^r) \quad\text{and}\quad \Pi \in L^p(0,T;W^{1,r}).$$

Besides the following estimate holds true with $C = C_{r,p,N,\sigma(\Omega)}$:

$$\|(\partial_t v, \mu \nabla^2 v, \nabla \Pi)\|_{L^p_T(L^r(\Omega))} \leq C\Big(\mu^{1-\frac{1}{p}}\|v_0\|_{D^{1-\frac{1}{p},p}_{A_r}} + \|f\|_{L^p_T(L^r(\Omega))} + \|R\|_{L^p_T(L^r(\Omega))}$$

$$+\mu\|\nabla\tau\|_{L^p_T(L^r(\Omega))} + \chi^{\frac{N-1}{r'}}\delta(\Omega')\|\tau_0\|_{L^p_T(L^r(\Omega))} + \delta(\Omega')^{\frac{1}{r}}\frac{\chi^{\frac{N-1}{r'}}}{c^{\frac{1}{r'}}}\|R{\cdot}n\|_{L^p_T(L^r(\partial\Omega))}\Big)$$

with c defined as in Proposition 3.4.

3.3. The linearized momentum equation

This section is devoted to the study of the following linear system:

$$\begin{cases} \rho\partial_t u - \mu\Delta u + \nabla\Pi = f, & \int_\Omega \Pi \, dx = 0, \\ \operatorname{div} u = 0, \\ u_{|t=0} = u_0, & u_{|\partial\Omega} = 0. \end{cases} \tag{13}$$

Our main existence theorem reads:

Theorem 3.7. *Let Ω be a $C^{2+\epsilon}$ bounded domain, $1 < p, r < \infty$ and $q \in (N, +\infty]$ such that $q \geq r$. Let $u_0 \in D^{1-\frac{1}{p},p}_{A_r}$ and $f \in L^p(0,T;L^r)$. Assume that the density ρ satisfies*

$$\forall (t,x) \in [0,T] \times \Omega, \ 0 < \check{\rho} \leq \rho(t,x) \leq \widehat{\rho} < \infty$$

and that for some $\beta \in (0,1]$,

$$\rho \in L^\infty\Big(0,T;W^{1,q}(\Omega)\Big) \cap C^\beta\Big([0,T];L^\infty(\Omega)\Big).$$

Then equation (13) has a unique solution (u,Π) such that

$$u \in C([0,T];D^{1-\frac{1}{p},p}_{A_r}) \cap L^p(0,T;W^{2,r} \cap W^{1,r}_0), \quad \Pi \in L^p(0,T;W^{1,r})$$

$$\text{and} \quad \partial_t u \in L^r(0,T;L^p).$$

Besides, there exists $C = C(N,p,q,r,\sigma(\Omega))$ such that, denoting

$$\delta_+ \overset{\text{def}}{=} \max\Big(0, \frac{N}{q} - \frac{N}{r'}\Big),$$

$$\widetilde{\varsigma} = \max\Big(0, \frac{N}{p} - \frac{N}{r}\Big)\max\Big(\frac{r'}{N'}, \frac{q}{q-N}\Big),$$

$$r^* \overset{\text{def}}{=} \max\Big(\frac{6 + 2\widetilde{\sigma}}{1 - \delta_+}, 6 + \frac{4}{r-1} + 2r'(\widetilde{\varsigma} + \delta_+)\Big),$$

$$\mathcal{B}_\rho(t) \overset{\text{def}}{=} 1 + \delta(\Omega)\Big(\check{\rho}^{-1}\|\nabla\rho\|_{L^\infty_t(L^q)}\Big)^{\frac{q}{q-N}},$$

$$M_\beta(t) \overset{\text{def}}{=} \sup_{\substack{x\in\Omega \\ \tau,\tau'\in[0,t],\,\tau\neq\tau'}} \frac{|\rho(\tau,x)-\rho(\tau',x)|}{\check\rho|\tau-\tau'|^\beta},$$

$$\eta_\rho \overset{\text{def}}{=} \widehat\rho/\check\rho, \quad \mathcal{C}_\rho(t) \overset{\text{def}}{=} \eta_\rho^{2r'}(\mathcal{B}_\rho(t))^{r^*} + \widehat\rho\frac{\delta(\Omega)^2}{\mu}\eta_\rho^{1+\frac{1}{\beta}}(\mathcal{B}_\rho(t))^{(\frac{1}{\beta}+1)(2+\widetilde\varsigma+\delta_+)}(M_\beta(t))^{\frac{1}{\beta}},$$

we have the following inequalities for all $t\in[0,T]$:

$$\mu\|u\|_{L_t^p(W^{2,r})} + \check\rho\|\partial_t u\|_{L_T^p(L^r)} + \|\Pi\|_{L_t^p(W^{1,r})} + \check\rho^{\frac{1}{p}}\mu^{\frac{1}{p'}}\|u(t)\|_{D_{A_r}^{1-\frac{1}{p},p}}$$

$$\leq C\eta_\rho^3 \mathcal{B}_\rho^{2+\widetilde\varsigma+\delta_+}(t)\, e^{\frac{C\mu t \mathcal{C}_\rho(t)}{\check\rho\delta(\Omega)^2}}\left(\|f\|_{L_t^p(L^r)} + \check\rho^{\frac{1}{p}}\mu^{\frac{1}{p'}}\|u_0\|_{D_{A_r}^{1-\frac{1}{p},p}}\right). \quad (14)$$

$$\|(\check\rho\partial_t u, \mu\nabla^2 u, \nabla\Pi)\|_{L_t^p(L^r)} + \check\rho^{\frac{1}{p}}\mu^{\frac{1}{p'}}\|u(t)\|_{D_{A_r}^{1-\frac{1}{p},p}}$$

$$\leq C\left(\eta_\rho^4 \mathcal{B}_\rho^{2+\widetilde\varsigma+\delta_+}(t)\left(\|f\|_{L_t^p(L^r)} + \check\rho^{\frac{1}{p}}\mu^{\frac{1}{p'}}\|u_0\|_{D_{A_r}^{1-\frac{1}{p},p}}\right) + \frac{\eta_\rho\mu}{\delta(\Omega)^2}\mathcal{C}_\rho(t)\|u\|_{L_t^p(L^r)}\right). \quad (15)$$

The basic idea is that when ρ is close to a positive constant $\overline\rho$, Theorem 3.2 provides us with the desired estimates. Indeed, it is only a matter of rewriting system (13) as

$$\begin{cases} \overline\rho\partial_t u - \mu\Delta u + \nabla\Pi = f + (\overline\rho-\rho)\partial_t u, \qquad \int_\Omega \Pi\,dx = 0, \\ \operatorname{div} u = 0, \\ u_{|t=0} = u_0, \quad u_{|\partial\Omega} = 0. \end{cases}$$

Now, if $\|\rho-\overline\rho\|_{L^\infty}$ is *small*, the term $\|(\overline\rho-\rho)\partial_t u\|_{L_t^p(L^r)}$ may be absorbed by the left-hand side of the inequality given in Theorem 3.2.

The proof is structured as follows. First, we restrict ourselves to *null* initial data and prove a priori estimates for (u,Π) under the assumption that ρ is *independent* of t. Next, we prove similar estimates in the case of *time-dependent* densities. These estimates will enable us to prove Theorem 3.7 in the case $u_0 \equiv 0$.

The last step of the proof is devoted to general initial data $u_0 \in D_{A_r}^{1-\frac{1}{p},p}$.

3.3.1. Existence of solutions for null initial data

a) A priori estimates in the case of a density independent of the time
   The main result of this part is the following:

Proposition 3.8. *Let* Ω, α, β, p, q, r *be as in Theorem 3.7. Let* $f \in L^p(0,T;L^r)$. *Let* ρ *be* independent *of* t, *satisfy* $\check\rho \overset{\text{def}}{=} \inf_{x\in\Omega}\rho(x) > 0$ *and belong to* $W^{1,q}(\Omega)$.

There exists $C = C(p, q, r, N, \sigma(\Omega))$ such that the following estimates hold true:

$$\|(\check{\rho}\partial_t u, \mu\nabla^2 u, \nabla\Pi)\|_{L_T^p(L^r)}$$

$$\leq C\left(\mathcal{B}_\rho^{N\varsigma}\|f\|_{L_T^p(L^r)} + \left(\mathcal{B}_\rho - 1\right)^{2+2N\varsigma\max\left(1, \frac{r'}{N'}\right)} \frac{\mu\|u\|_{L_T^p(L^r)}}{\delta(\Omega)^2}\right.$$

$$\left. + \left(\mathcal{B}_\rho - 1\right)^{1+N\varsigma\max\left(\frac{r'}{N'}, \frac{q}{q-N}\right)} \frac{\|\Pi\|_{L_T^p(L^r)}}{\delta(\Omega)}\right), \qquad (16)$$

$$\|(\check{\rho}\partial_t u, \mu\nabla^2 u, \nabla\Pi)\|_{L_T^p(L^r)} \leq C\left(\eta_\rho \mathcal{B}_\rho^{2+\widetilde{\varsigma}+\delta_+}\|f\|_{L_T^p(L^r)}\right.$$

$$\left. + \frac{\mu}{\delta(\Omega)^2}\eta_\rho^{2r'}\mathcal{B}_\rho^{r^*-1}(\mathcal{B}_\rho - 1)\|u\|_{L_T^p(L^r)}\right), \quad (17)$$

with $\eta_\rho \overset{\text{def}}{=} \widehat{\rho}/\check{\rho}$, $\delta_+ \overset{\text{def}}{=} \max\left(0, \frac{N}{q} - \frac{N}{r'}\right)$, $\mathcal{B}_\rho \overset{\text{def}}{=} 1 + \delta(\Omega)\left(\check{\rho}^{-1}\|\nabla\rho\|_{L^q}\right)^{\frac{q}{q-N}}$, $\varsigma \overset{\text{def}}{=} \max(0, \frac{1}{p} - \frac{1}{r})$, $\widetilde{\varsigma} \overset{\text{def}}{=} N\varsigma\max\left(\frac{r'}{N'}, \frac{q}{q-N}\right)$, and $r^* \overset{\text{def}}{=} \max\left(\frac{6+2\widetilde{\sigma}}{1-\delta_+}, 6+\frac{4}{r-1} + 2r'(\widetilde{\varsigma}+\delta_+)\right)$.

Proof. The proof is based on the old argument by O. Ladyzhenskaya and V. Solonnikov in [18]. As explained above, the key idea is that the inequality to be proved is a mere consequence of Theorem 3.2 if ρ is "almost" a constant.

On the other hand, by virtue of Sobolev embeddings, ρ belongs to C^α with $\alpha \overset{\text{def}}{=} 1 - \frac{N}{q}$ so that it does not vary much on small subdomains of Ω. Hence we introduce a convenient partition of unity in the x variable and use Theorem 3.6 in order to control the solution on each subdomain. Of course, one has to be careful that the constants appearing in each local inequality are harmless, a detail which has been passed over in silence in [18].

First step: Local estimates. Let us first notice that one can rule out the case $\|\rho\|_{\dot{C}^\alpha} \leq \kappa\check{\rho}\delta(\Omega)^{-\alpha}$ for κ suitably small constant. Indeed, rewriting the momentum equation as follows:

$$\check{\rho}\partial_t u - \mu\Delta u + \nabla\Pi = f + \left(\frac{\check{\rho} - \rho}{\check{\rho}}\right)\check{\rho}\partial_t u,$$

and using that $\|\rho - \check{\rho}\|_{L^\infty} \leq \delta(\Omega)^\alpha\|\rho\|_{\dot{C}^\alpha}$, Theorem 3.2 obviously yields the desired estimates.

Let us assume from now on that

$$\|\rho\|_{\dot{C}^\alpha} > \kappa\check{\rho}\delta(\Omega)^{-\alpha}. \qquad (18)$$

Let $(\Omega_k)_{1\leq k\leq K}$ be a covering of Ω by connected open sets with C^2 boundaries, finite multiplicity $m = m(N)$ and diameter less than some $\lambda \in (0, \delta(\Omega))$ to be fixed hereafter. As Ω is C^2 we can assume in addition that the parameters χ and c (see Proposition 3.4) associated to each subdomain $\overline{\Omega} \cap \Omega_k$ are bounded and

bounded away from zero independently of k. Consider a subordinate partition of unity $(\varphi_k)_{1\leq k\leq K}$ of class C^2 such that

1. $\mathrm{Supp}\,\varphi_k \subset \Omega_k$,
2. $\sum_k \varphi_k \equiv 1$ on $\overline{\Omega}$,
3. $0 \leq \varphi_k \leq 1$,
4. $\|\nabla^\alpha \varphi_k\|_{L^\infty} \leq C_\alpha |\lambda|^{-|\alpha|}$ for $|\alpha| \leq 2$,
5. $K \approx (\delta(\Omega)\lambda^{-1})^N$ and the number K' of domains Ω_k intersecting $\partial\Omega$ is of order $(\delta(\Omega)\lambda^{-1})^{N-1}$.

Let $f_k \stackrel{\mathrm{def}}{=} \varphi_k f$, $\Pi_k \stackrel{\mathrm{def}}{=} \varphi_k \Pi$ and $u_k \stackrel{\mathrm{def}}{=} \varphi_k u$. Further denote by x_k a point of $\overline{\Omega_k \cap \Omega}$ where the minimum of ρ is attained, and $\rho_k = \rho(x_k)$, $\mu_k = \mu/\rho_k$. Obviously (u_k, Π_k) satisfies

$$
\begin{cases}
\partial_t u_k - \mu_k \Delta u_k + \nabla\left(\dfrac{\Pi_k}{\rho_k}\right) = \underbrace{\dfrac{f_k}{\rho_k} + \left(\dfrac{\rho_k - \rho}{\rho_k}\right)\varphi_k \partial_t u - \mu_k u \Delta\varphi_k - 2\mu_k \nabla\varphi_k \cdot \nabla u + \dfrac{\Pi}{\rho_k}\nabla\varphi_k}_{g_k}, \\[2mm]
\mathrm{div}\, u_k = \underbrace{u \cdot \nabla\varphi_k}_{\tau}, \\[2mm]
u_k(0) = 0, \\
(u_k)_{|\partial\Omega} = 0.
\end{cases}
$$

Of course, as $\mathrm{div}\,u = 0$ and u vanishes on $\partial\Omega$, one has $\int_\Omega \mathrm{div}\,u_k\,dx = 0$. Note also that $u_{|t=0} = 0$ entails $\mathrm{div}\,u_k(0,\cdot) \equiv 0$. On the other hand, using the summation convention on the repeated indices, we have

$$
\partial_t \mathrm{div}\, u_k = \underbrace{\dfrac{f}{\rho}\cdot\nabla\varphi_k + \Pi\partial_i\left(\dfrac{\partial_i\varphi_k}{\rho}\right) - \partial_j u^i \partial_j\left(\dfrac{\mu}{\rho}\partial_i\varphi_k\right)}_{\tau_0} + \mathrm{div}\underbrace{\left(\dfrac{\mu}{\rho}\partial_i\varphi_k \partial_j u^i - \dfrac{\Pi}{\rho}\partial_i\varphi_k\right)}_{R}.
$$

From now on, let C denote a constant depending only on q, r, N, α and $\sigma(\Omega)$. We also use the symbol "$\lesssim$" introduced in Section 2.

Since $\tau(t,\cdot)$ and $R(t,\cdot)$ are supported in $\Omega_k \cap \overline{\Omega}$, applying Theorem 3.6 and using the properties of φ_k gives

$$
\left\| \partial_t u_k, \mu_k \nabla^2 u_k, \nabla\left(\dfrac{\Pi_k}{\rho_k}\right) \right\|_{L^p_T(L^r(\Omega))} \lesssim \|g_k\|_{L^p_T(L^r(\Omega))} + \mu_k \|\nabla(u\cdot\nabla\varphi_k)\|_{L^p_T(L^r(\Omega))}
$$
$$
+ \lambda\|\tau_0\|_{L^p_T(L^r(\Omega))} + \|R\|_{L^p_T(L^r(\Omega))} + \lambda^{\frac{1}{r}}\|R\cdot n\|_{L^p_T(L^r(\partial\Omega))}. \quad (19)
$$

Second step: The global estimate

As $\rho \in C^\alpha$, we have

$$
\|\rho_k - \rho\|_{L^\infty(\Omega_k \cap \Omega)} \leq \lambda^\alpha \|\rho\|_{\dot{C}^\alpha}.
$$

Hence, taking advantage of the properties of $(\varphi_k)_{1\leq k\leq K}$, one gets

$$\|g_k\|_{L^p_T(L^r(\Omega))}$$
$$\lesssim \mu_k\Big(\lambda^{-1}\|\nabla u\|_{L^p_T(L^r(\Omega_k))} + \lambda^{-2}\|u\|_{L^p_T(L^r(\Omega_k))}\Big)$$
$$+\rho_k^{-1}\Big(\|f\|_{L^p_T(L^r(\Omega_k))} + \lambda^\alpha\|\partial_t u_k\|_{L^p_T(L^r(\Omega))}\|\rho\|_{\dot C^\alpha} + \lambda^{-1}\|\Pi\|_{L^p_T(L^r(\Omega_k))}\Big). \quad (20)$$

Clearly, $\mu_k\|\nabla\tau\|_{L^p_T(L^r(\Omega))} = \mu_k\|\nabla(u\cdot\nabla\varphi_k)\|_{L^p_T(L^r(\Omega))}$ may be bounded by the first two terms of the right-hand side of (20). As for τ_0, we have

$$\lambda\|\tau_0\|_{L^p_T(L^r(\Omega))}$$
$$\lesssim \mu_k\Big(\lambda^{-1}\|\nabla u\|_{L^p_T(L^r(\Omega_k))} + \|\nabla u\cdot\nabla\log\rho\|_{L^p_T(L^r(\Omega_k))}\Big)$$
$$+\rho_k^{-1}\Big(\|f\|_{L^p_T(L^r(\Omega_k))} + \lambda^{-1}\|\Pi\|_{L^p_T(L^r(\Omega_k))} + \|\Pi\nabla\log\rho\|_{L^p_T(L^r(\Omega_k))}\Big). \quad (21)$$

For $\|R\|_{L^p_T(L^r(\Omega))}$, easy computations yield

$$\|R\|_{L^p_T(L^r(\Omega))} \lesssim \lambda^{-1}\Big(\mu_k\|\nabla u\|_{L^p_T(L^r(\Omega_k))} + \rho_k^{-1}\|\Pi\|_{L^p_T(L^r(\Omega_k))}\Big). \quad (22)$$

Finally, we have

$$\lambda\|R\cdot n\|_{L^p_T(L^r(\partial\Omega))} \lesssim \mu_k\|\nabla u\|_{L^p_T(L^r(\overline\Omega_k\cap\partial\Omega))} + \rho_k^{-1}\|\Pi\|_{L^p_T(L^r(\overline\Omega_k\cap\partial\Omega))}. \quad (23)$$

Plugging inequalities (20) to (23) in (19) yields

$$\|\rho_k\partial_t u_k, \mu\nabla^2 u_k, \nabla\Pi_k\|_{L^p_T(L^r(\Omega))} \lesssim \|f\|_{L^p_T(L^r(\Omega_k))} + \mu\lambda^{-2}\|u\|_{L^p_T(L^r(\Omega_k))}$$
$$+\mu\lambda^{-1}\|\nabla u\|_{L^p_T(L^r(\Omega_k))} + \lambda^{-1}\|\Pi\|_{L^p_T(L^r(\Omega_k))} + \|\Pi\nabla\log\rho\|_{L^p_T(L^r(\Omega_k))}$$
$$+\mu\|\nabla u\cdot\nabla\log\rho\|_{L^p_T(L^r(\Omega_k))} \qquad + \lambda^{-\frac{1}{r'}}\mu\|\nabla u\|_{L^p_T(L^r(\overline\Omega_k\cap\partial\Omega))}$$
$$+\lambda^{-\frac{1}{r'}}\|\Pi\|_{L^p_T(L^r(\overline\Omega_k\cap\partial\Omega))} + \lambda^\alpha\|\rho\|_{\dot C^\alpha}\|\partial_t u_k\|_{L^p_T(L^r(\Omega))}. \quad (24)$$

On the one hand, by setting $\lambda = \kappa\check\rho^{\frac{1}{\alpha}}\|\rho\|_{\dot C^\alpha}^{\frac{1}{\alpha}}$ with κ small enough – a choice which is allowed thanks to (18), the last term in the right-hand side may be absorbed by the left-hand side. On the other hand, since the covering has finite multiplicity m, it is clear that we have

$$\forall z \in L^p(0,T;\Omega), \ \sum_{k=1}^{K}\|z\|_{L^p_T(L^r(\Omega_k))}^p \leq mK^{\max(0,1-\frac{p}{r})}\|z\|_{L^p_T(L^r(\Omega))}^p.$$

Hence, raising both sides of inequality (19) to the power p then summing on k, we eventually get

$$\|\check\rho\partial_t u, \mu\nabla^2 u, \nabla\Pi\|_{L^p_T(L^r(\Omega))}$$
$$\lesssim (\delta(\Omega)\lambda^{-1})^{N_\varsigma}\Big(\|+\|_{L^p_T(L^r(\Omega))}\mu\lambda^{-2}\|u\|_{L^p_T(L^r(\Omega))}$$
$$+\mu\lambda^{-1}\|\nabla u\|_{L^p_T(L^r(\Omega))} + \lambda^{-1}\|\Pi\|_{L^p_T(L^r(\Omega))} + \mu\|\nabla u\cdot\nabla\log\rho\|_{L^p_T(L^r(\Omega))}$$

$$+\|\Pi\nabla\log\rho\|_{L_T^p(L^r(\Omega))}\Big)$$

$$+(\delta(\Omega)\lambda^{-1})^{(N-1)\varsigma}\lambda^{-\frac{1}{r'}}\Big(\mu\|\nabla u\|_{L_T^p(L^r(\partial\Omega))}+\|\Pi\|_{L_T^p(L^r(\partial\Omega))}\Big). \tag{25}$$

Standard interpolation and trace theorems enable us to simplify the right-hand side. Indeed, from [1] page 75 and obvious scaling considerations, we have for $\eta\leq 1$ and $C=C_{r,N,\sigma(\Omega)}$,

$$\delta(\Omega)\|\nabla u\|_{L^r(\Omega)}\leq C\Big(\eta^{-1}\|u\|_{L^r(\Omega)}+\eta\,\delta(\Omega)^2\|\nabla^2 u\|_{L^r(\Omega)}\Big) \tag{26}$$

and, according to inequality (3.3) page 43 in [11], we have

$$\delta(\Omega)^{\frac{1}{r}}\|\Pi\|_{L^r(\partial\Omega)}\leq C\Big(\widetilde{\eta}^{-\frac{1}{r}}\|\Pi\|_{L^r(\Omega)}+\widetilde{\eta}^{\frac{1}{r'}}\delta(\Omega)\|\nabla\Pi\|_{L^r(\Omega)}\Big). \tag{27}$$

On the other hand, replacing Π with ∇u in (27) and using (26), we also get

$$\delta(\Omega)^{1+\frac{1}{r}}\|\nabla u\|_{L^r(\partial\Omega)}\leq C\Big(\widetilde{\eta}^{-\frac{1}{r}-1}\|u\|_{L^r(\Omega)}+\widetilde{\eta}^{\frac{1}{r'}}\delta(\Omega)^2\|\nabla^2 u\|_{L^r(\Omega)}\Big). \tag{28}$$

Moreover, combining Hölder, Gagliardo–Nirenberg, Poincaré–Wirtinger and Young inequalities (here we use that $r\leq q$ and $q>N$), we get for all positive ϵ,

$$\|\Pi\nabla\log\rho\|_{L^r(\Omega)}\leq C\Big(\|\nabla\log\rho\|_{L^q}\|\Pi\|_{L^r}^{1-\frac{N}{q}}\|\nabla\Pi\|_{L^r}^{\frac{N}{q}}\Big),$$

$$\leq\epsilon\|\nabla\Pi\|_{L^r}+C\epsilon^{-\frac{N}{q-N}}\|\nabla\log\rho\|_{L^q}^{\frac{q}{q-N}}\|\Pi\|_{L^r}.$$

Similarly, since $u\in W^{2,q}\cap W_0^{1,r}$, combining Gagliardo–Nirenberg, Poincaré–Wirtinger and Young inequalities yields

$$\|\nabla u\cdot\nabla\log\rho\|_{L^r(\Omega)}\leq\epsilon\left\|\nabla^2 u\right\|_{L^r}+C\epsilon^{\frac{N-q}{N+q}}\|\nabla\log\rho\|_{L^q}^{\frac{2q}{q-N}}\|u\|_{L^r}.$$

Choose $\epsilon=\kappa'(\lambda\delta(\Omega)^{-1})^{N\varsigma}$, $\eta=\kappa'(\lambda\delta(\Omega)^{-1})^{1+N\varsigma}$ and $\widetilde{\eta}=\kappa'(\lambda\delta(\Omega)^{-1})^{1+r'(N-1)\varsigma}$ with $\kappa\ll 1$. Inserting the above inequalities in (25) and reminding that $\lambda=\kappa\check{\rho}^{\frac{1}{\alpha}}\|\rho\|_{\dot{C}^\alpha}^{\frac{1}{\alpha}}$, we end up with inequality 16.

Third step: estimates for the pressure

Estimating the pressure lies on a duality argument. For technical reasons however, the proof is slightly different depending on $r\geq q'$ or $r<q'$.

• *Case $r\geq q'$.* Since the pressure has null mean, we have

$$\|\Pi\|_{L^r}=\sup_{\substack{\|h\|_{L^{r'}}\leq 1\\ \int_\Omega h\,dx=0}}\int_\Omega\Pi h\,dx. \tag{29}$$

As $\log\rho$ belongs to $W^{1,q}$ with $q>N$ and $q\geq r'$, and as ρ is bounded by above and by below, Proposition C.1 insures that the following Neumann problem

$$\begin{cases}\operatorname{div}(\rho^{-1}\nabla v)=h, & \int_\Omega v\,dx=0,\\ \partial_n v_{|\partial\Omega}=0.\end{cases}$$

has a unique solution v in $W^{2,r'}(\Omega)$ with besides

$$\|\nabla^2 v\|_{L^{r'}(\Omega)} \lesssim \widehat{\rho} \mathcal{B}_\rho^2 \|h\|_{L^{r'}} \quad \text{and} \quad \|\nabla v\|_{L^{r'}(\Omega)} \lesssim \widehat{\rho} \delta(\Omega) \mathcal{B}_\rho \|h\|_{L^{r'}}. \qquad (30)$$

On the other hand, integrating by parts and using the definition of v and $\nabla\Pi$ yields

$$\int_\Omega \Pi h\, dx = -\int_\Omega \rho^{-1} \nabla\Pi \cdot \nabla v\, dx = \int_\Omega \nabla v \cdot \left(\partial_t u - \mu\rho^{-1}\Delta u - \rho^{-1} f\right) dx.$$

Hence, denoting $\check{\mu} \overset{\text{def}}{=} \mu/\check{\rho}$, integrating by parts once again and using the summation convention for repeated indices,

$$\int_\Omega \Pi h\, dx = \mu \int_\Omega \partial_j u^i \partial_j\left(\frac{\partial_i v}{\rho}\right) dx - \mu \int_{\partial\Omega} \frac{\partial_j u^i \partial_i v}{\rho} n_j d\sigma - \int_\Omega \frac{\nabla v \cdot f}{\rho}\, dx,$$
$$\leq \check{\mu}\|\nabla u\|_{L^r(\Omega)}\|\nabla^2 v\|_{L^{r'}(\Omega)} + \check{\mu}\|\nabla\log\rho\|_{L^q(\Omega)}\|^t\nabla u \cdot \nabla v\|_{L^{q'}(\Omega)}$$
$$+ \check{\mu}\|\nabla u\|_{L^r(\partial\Omega)}\|\nabla v\|_{L^{r'}(\partial\Omega)} + \check{\rho}^{-1}\|f\|_{L^r(\Omega)}\|\nabla v\|_{L^{r'}(\Omega)}. \qquad (31)$$

Let us now bound the terms in the right-hand side of (31). Taking advantage of (30), we easily get

$$\check{\mu}\|\nabla u\|_{L^r}\|\nabla^2 v\|_{L^{r'}} \lesssim \mu\eta_\rho \mathcal{B}_\rho^2 \|\nabla u\|_{L^r}\|h\|_{L^{r'}}. \qquad (32)$$

Next, as $q' \leq r$, Hölder inequality yields

$$\|^t\nabla u \cdot \nabla v\|_{L^{q'}} \leq \|\nabla u\|_{L^r}\|\nabla v\|_{L^s}$$

with s satisfying $1/s = 1/r' - 1/q$.

As, moreover, $q > N$, Gagliardo–Nirenberg inequality combined with (30) yields

$$\|\nabla v\|_{L^s} \lesssim \|\nabla v\|_{L^{r'}}^{1-\frac{N}{q}} \left(\|\nabla^2 v\|_{L^{r'}} + \delta(\Omega)^{-1}\|\nabla v\|_{L^{r'}}\right)^{\frac{N}{q}},$$
$$\lesssim \delta(\Omega)^{-\frac{N}{q}}\|\nabla v\|_{L^{r'}} + \|\nabla v\|_{L^{r'}}^{1-\frac{N}{q}}\|\nabla^2 v\|_{L^{r'}}^{\frac{N}{q}},$$
$$\lesssim \widehat{\rho}\delta(\Omega)^{1-\frac{N}{q}}\mathcal{B}_\rho^{1+\frac{N}{q}}\|h\|_{L^{r'}},$$

whence

$$\|^t\nabla u \cdot \nabla v\|_{L^{q'}} \lesssim \widehat{\rho}\delta(\Omega)^{1-\frac{N}{q}}\mathcal{B}_\rho^{1+\frac{N}{q}}\|\nabla u\|_{L^r}\|h\|_{L^{r'}}.$$

Note that as $\delta(\Omega)^{1-\frac{N}{q}}\|\nabla\log\rho\|_{L^q} \leq \mathcal{B}_q^{1-\frac{N}{q}}$, we actually have

$$\check{\mu}\|\nabla\log\rho\|_{L^q}\|^t\nabla u \cdot \nabla v\|_{L^{q'}} \lesssim \mu\eta_\rho\mathcal{B}_\rho^2\|\nabla u\|_{L^r}\|h\|_{L^{r'}}. \qquad (33)$$

In order to bound the last term in the right-hand side of (31), we first remark that trace estimates and (30) yield:

$$\|\nabla v\|_{L^{r'}(\partial\Omega)} \lesssim \|\nabla v\|_{L^{r'}(\Omega)}^{\frac{1}{r}}\left(\|\nabla^2 v\|_{L^{r'}(\Omega)} + \delta(\Omega)^{-1}\|\nabla v\|_{L^{r'}(\Omega)}\right)^{\frac{1}{r'}},$$
$$\lesssim \widehat{\rho}\mathcal{B}_\rho^{1+\frac{1}{r'}}\delta(\Omega)^{\frac{1}{r}}\|h\|_{L^{r'}(\Omega)}.$$

Hence

$$\breve{\mu}\|\nabla u\|_{L^r(\partial\Omega)}\|\nabla v\|_{L^{r'}(\partial\Omega)}\lesssim\mu\eta_\rho\mathcal{B}_\rho^{1+\frac{1}{r'}}\delta(\Omega)^{\frac{1}{r}}\|\nabla u\|_{L^r(\partial\Omega)}\|h\|_{L^{r'}(\Omega)}. \tag{34}$$

Therefore, plugging (32), (33), (34) in (31) and reminding of (29), we conclude that

$$\|\Pi\|_{L^r(\Omega)}\lesssim\eta_\rho\mathcal{B}_\rho\delta(\Omega)\bigg(\|f\|_{L^r(\Omega)}+\mu\mathcal{B}_\rho\frac{\|\nabla u\|_{L^r(\Omega)}}{\delta(\Omega)}+\mu\left(\frac{\mathcal{B}_\rho}{\delta(\Omega)}\right)^{\frac{1}{r'}}\|\nabla u\|_{L^r(\partial\Omega)}\bigg). \tag{35}$$

We now claim that inequality (35) entails estimates for $\|(\breve{\rho}\partial_t u,\mu\nabla^2 u,\nabla\Pi)\|_{L^p_T(L^r)}$ involving only the data and $\|u\|_{L^p_T(L^r)}$. Indeed, inequalities (28) and (26) give us for suitably small ϵ and η:

$$\|\nabla u\|_{L^r(\Omega)}\lesssim\eta^{-1}\delta(\Omega)^{-1}\|u\|_{L^r(\Omega)}+\eta\delta(\Omega)\|\nabla^2 u\|_{L^r(\Omega)},$$

$$\|\nabla u\|_{L^r(\partial\Omega)}\lesssim\left(\epsilon\delta(\Omega)\right)^{-1-\frac{1}{r}}\|u\|_{L^r(\Omega)}+\left(\epsilon\delta(\Omega)\right)^{1-\frac{1}{r}}\|\nabla^2 u\|_{L^r(\Omega)}. \tag{36}$$

Hence, plugging the above inequalities in (35),

$$\frac{\|\Pi\|_{L^r(\Omega)}}{\delta(\Omega)}\lesssim\eta_\rho\mathcal{B}_\rho\bigg(\|f\|_{L^r(\Omega)}+\frac{\mu\|u\|_{L^r(\Omega)}}{\delta(\Omega)^2}\Big(\mathcal{B}_\rho\eta^{-1}+\mathcal{B}_\rho^{\frac{1}{r'}}\epsilon^{-1-\frac{1}{r}}\Big)$$

$$+\mu\|\nabla^2 u\|_{L^r(\Omega)}\Big(\mathcal{B}_\rho\eta+\mathcal{B}_\rho^{\frac{1}{r'}}\epsilon^{\frac{1}{r'}}\Big)\bigg).$$

Let κ be a suitably small positive constant depending only on N, p, q and r. Inserting the above inequality with $\eta=\kappa\eta_\rho^{-1}\mathcal{B}_\rho^{-3-\widetilde{\varsigma}}$ and $\epsilon=\kappa\eta_\rho^{-r'}\mathcal{B}_\rho^{-2r'-1-r'\widetilde{\varsigma}}$ in (16) yields inequality (17) in the case $r\geq q'$.

• *Case $1<r<q'$.* Note that the condition $q\geq r'$ was needed to get (30). We would like to avoid this additional assumption (otherwise we will run into troubles when proving uniqueness in dimension 2...)

Let us first use the fact that

$$\|\Pi\|_{L^r(\Omega)}\leq\delta(\Omega)^{\frac{N}{r}-\frac{N}{q'}}\|\Pi\|_{L^{q'}(\Omega)}. \tag{37}$$

Now, the same duality argument as above will enable us to bound $\|\Pi\|_{L^{q'}(\Omega)}$. Indeed, we have

$$\|\Pi\|_{L^{q'}}=\sup_{\substack{\|h\|_{L^q}\leq 1\\\int_\Omega h\,dx=0}}\int_\Omega\Pi h\,dx. \tag{38}$$

According to Proposition C.1,[2] the Neumann problem

$$\begin{cases}\mathrm{div}(\rho^{-1}\nabla v)=h, & \int_\Omega v\,dx=0,\\\partial_n v_{|\partial\Omega}=0\end{cases}$$

has a unique solution v in $W^{2,q}(\Omega)$ with besides

$$\|\nabla^2 v\|_{L^q(\Omega)}\lesssim\widehat{\rho}\mathcal{B}_\rho^2\|h\|_{L^q(\Omega)}\quad\text{and}\quad\|\nabla v\|_{L^q(\Omega)}\lesssim\widehat{\rho}\delta(\Omega)\mathcal{B}_\rho\|h\|_{L^q(\Omega)}. \tag{39}$$

---

[2] Remark that since $1<r<q'$ we must have q finite.

Mimicking the proof of (31), one gets

$$\int_\Omega \Pi h\, dx \le \check{\mu}\|\nabla u\|_{L^{q'}(\Omega)}\|\nabla^2 v\|_{L^q(\Omega)} + \check{\mu}\|\nabla\log\rho\|_{L^q(\Omega)}\|\nabla u\|_{L^{q'}(\Omega)}\|\nabla v\|_{L^\infty(\Omega)}$$
$$+ \check{\mu}\|\nabla u\|_{L^r(\partial\Omega)}\|\nabla v\|_{L^{r'}(\partial\Omega)} + \check{\rho}^{-1}\|f\|_{L^r(\Omega)}\|\nabla v\|_{L^{r'}(\Omega)}. \qquad (40)$$

Taking advantage of (39), we get

$$\check{\mu}\|\nabla u\|_{L^{q'}(\Omega)}\|\nabla^2 v\|_{L^q(\Omega)} \lesssim \mu\eta_\rho \mathcal{B}_\rho^2 \|\nabla u\|_{L^{q'}(\Omega)}\|h\|_{L^q(\Omega)}. \qquad (41)$$

According to Gagliardo–Nirenberg inequality, we have

$$\|\nabla v\|_{L^\infty(\Omega)} \lesssim \|\nabla v\|_{L^q(\Omega)}^{1-\frac{N}{q}}\Big(\|\nabla^2 v\|_{L^q(\Omega)} + \delta(\Omega)^{-1}\|\nabla v\|_{L^q(\Omega)}\Big)^{\frac{N}{q}},$$

so that, by virtue of (39),

$$\check{\mu}\|\nabla\log\rho\|_{L^q(\Omega)}\|\nabla u\|_{L^{q'}(\Omega)}\|\nabla v\|_{L^\infty(\Omega)} \lesssim \mu\eta_\rho \mathcal{B}_\rho^2\|\nabla u\|_{L^{q'}(\Omega)}\|h\|_{L^q(\Omega)}. \qquad (42)$$

According to inequality (3.3) page 43 in [11], we have

$$\|\nabla v\|_{L^{r'}(\partial\Omega)} \lesssim \delta(\Omega)^{-\frac{1}{r'}}\|\nabla v\|_{L^{r'}(\Omega)} + \|\nabla v\|_{L^{q'(r'-1)}(\Omega)}^{\frac{1}{r}}\|\nabla^2 v\|_{L^q(\Omega)}^{\frac{1}{r'}}.$$

On the other hand, according to Gagliardo–Nirenberg inequality, we have

$$\|\nabla v\|_{L^{q'(r'-1)}(\Omega)} \lesssim \|\nabla v\|_{L^q(\Omega)}^{1-r\left(\frac{N}{q}-\frac{N}{r'}\right)}\Big(\|\nabla^2 v\|_{L^q(\Omega)} + \delta(\Omega)^{-1}\|\nabla v\|_{L^q(\Omega)}\Big)^{r\left(\frac{N}{q}-\frac{N}{r'}\right)},$$

$$\|\nabla v\|_{L^{r'}(\Omega)} \lesssim \|\nabla v\|_{L^q(\Omega)}^{1-\frac{N}{q}+\frac{N}{r'}}\Big(\|\nabla^2 v\|_{L^q(\Omega)} + \delta(\Omega)^{-1}\|\nabla v\|_{L^q(\Omega)}\Big)^{\frac{N}{q}-\frac{N}{r'}}, \qquad (43)$$

hence, taking advantage of (39),

$$\|\nabla v\|_{L^{r'}(\partial\Omega)} \lesssim \widehat{\rho}\,\delta(\Omega)^{\frac{1}{r}-\frac{N}{q}+\frac{N}{r'}}\mathcal{B}_\rho^{1+\frac{1}{r'}+\frac{N}{q}-\frac{N}{r'}}\|h\|_{L^q(\Omega)},$$

whence

$$\check{\mu}\|\nabla u\|_{L^r(\partial\Omega)}\|\nabla v\|_{L^{r'}(\partial\Omega)} \lesssim \mu\eta_\rho \delta(\Omega)^{\frac{1}{r}-\frac{N}{q}+\frac{N}{r'}}\mathcal{B}_\rho^{1+\frac{1}{r'}+\frac{N}{q}-\frac{N}{r'}}\|\nabla u\|_{L^r(\partial\Omega)}\|h\|_{L^q(\Omega)}. \qquad (44)$$

Combining (43) and (39), we get

$$\check{\rho}^{-1}\|f\|_{L^r(\Omega)}\|\nabla v\|_{L^{r'}(\Omega)} \lesssim \eta_\rho \mathcal{A}_\rho^{1+\frac{N}{q}-\frac{N}{r'}}\delta(\Omega)^{1-\frac{N}{q}+\frac{N}{r'}}\|f\|_{L^r(\Omega)}\|h\|_{L^q(\Omega)}. \qquad (45)$$

Therefore, denoting $\delta \stackrel{\text{def}}{=} \frac{N}{q} - \frac{N}{r'}$ and plugging (41), (42), (44) and (45) in (40), and taking advantage of (37), we conclude that

$$\|\Pi\|_{L^r(\Omega)} \lesssim \eta_\rho \mathcal{B}_\rho^{1+\delta}\delta(\Omega)\|f\|_{L^r(\Omega)}$$
$$+ \mu\eta_\rho \mathcal{B}_\rho^2\left(\delta(\Omega)^\delta\|\nabla u\|_{L^{q'}(\Omega)} + \mathcal{B}_\rho^{\delta-\frac{1}{r}}\delta(\Omega)^{\frac{1}{r}}\|\nabla u\|_{L^r(\partial\Omega)}\right).$$

Combining inequality (36) and the following Gagliardo–Nirenberg inequality:

$$\forall \eta \in (0,1),\ \delta(\Omega)^{1+\delta}\|\nabla u\|_{L^{q'}(\Omega)} \lesssim \eta^{\frac{1}{2}-\frac{\delta}{2}}\delta(\Omega)^2\|\nabla^2 u\|_{L^r(\Omega)} + \eta^{-\frac{1}{2}-\frac{\delta}{2}}\|u\|_{L^r(\Omega)},$$

(which stems from the fact that $[L^r, W^{2,r}]_{\frac{\delta}{2}+\frac{1}{2}} \hookrightarrow W^{1,q'}$), one can further get for all $\epsilon, \eta \in (0,1)$,

$$\|\Pi\|_{L^r(\Omega)} \lesssim \eta_\rho \mathcal{B}_\rho^{1+\delta} \delta(\Omega) \|f\|_{L^r(\Omega)} + \mu \eta_\rho \mathcal{B}_\rho \left(\delta(\Omega) \left(\eta^{\frac{1}{2}-\frac{\delta}{2}} \mathcal{B}_\rho + \epsilon^{\frac{1}{r'}} \mathcal{B}_\rho^{\frac{1}{r'}+\delta} \right) \|\nabla^2 u\|_{L^r(\Omega)} \right.$$

$$\left. + \delta(\Omega)^{-1} \left(\eta^{-\frac{1}{2}-\frac{\delta}{2}} \mathcal{B}_\rho + \epsilon^{-1-\frac{1}{r}} \mathcal{B}_\rho^{\frac{1}{r'}+\delta} \right) \|u\|_{L^r(\Omega)} \right).$$

Choose $\epsilon = \kappa \left(\eta_\rho \mathcal{B}_\rho^{2+\frac{1}{r'}+\delta+\widetilde{\varsigma}} \right)^{-r'}$ and $\eta = \kappa \left(\eta_\rho \mathcal{B}_\rho^{3+\widetilde{\varsigma}} \right)^{\frac{2}{\delta-1}}$ for a suitably small constant κ and insert the above estimate in inequality (16). We end up with[3]

$$\|(\check{\rho}\partial_t u, \mu\nabla^2 u, \nabla\Pi)\|_{L^p_T(L^r)} \lesssim \eta_\rho \mathcal{B}_\rho^{2+\delta+\widetilde{\sigma}} \|f\|_{L^p_T(L^r)}$$
$$+ \mu \delta(\Omega)^{-2} \eta_\rho^{2r'} \left(\mathcal{B}_\rho - 1 \right) \mathcal{B}_\rho^{\max\left(\frac{5+2\widetilde{\sigma}+\delta}{1-\delta}, 5+2r'\delta+\frac{4}{r-1}+2\widetilde{\varsigma} \right)} \|u\|_{L^p_T(L^r)}, \quad (46)$$

which is inequality (17) in the case $1 < r < q'$. $\qquad\square$

b) A priori estimates for (13) with time-dependent density

In this section, we aim at generalizing Proposition 3.8 to the case of a time-dependent density. We have the following proposition:

Proposition 3.9. *Let* $1 < p, r < \infty$ *and* $f \in L^p(0,T;L^r)$. *Assume that* ρ *satisfies the assumptions of Theorem 3.7 and that* (u, Π) *solves* (13) *with null initial data. Let* δ_+, $\widetilde{\varsigma}$ *and* r^* *be defined as in the statement of Proposition 3.8 and denote*

$$\mathcal{B}_\rho(t) \stackrel{\text{def}}{=} 1 + \delta(\Omega) \left(\check{\rho}^{-1} \|\nabla\rho\|_{L^\infty_t(L^q)} \right)^{\frac{q}{q-N}}, \quad M_\beta(t) \stackrel{\text{def}}{=} \sup_{\substack{x \in \Omega \\ \tau, \tau' \in [0,t], \tau \neq \tau'}} \frac{|\rho(\tau, x) - \rho(\tau', x)|}{\check{\rho}|\tau - \tau'|^\beta},$$

$$\mathcal{C}_\rho(t) \stackrel{\text{def}}{=} \eta_\rho^{2r'} (\mathcal{B}_\rho(t))^{r^*} + \widehat{\rho} \frac{\delta(\Omega)^2}{\mu} \eta_\rho^{1+\frac{1}{\beta}} (\mathcal{B}_\rho(t))^{(\frac{1}{\beta}+1)(2+\widetilde{\varsigma}+\delta_+)} (M_\beta(t))^{\frac{1}{\beta}}.$$

There exists $C = C(\alpha, \beta, q, r, N, \sigma(\Omega))$ *such that for all* $t \in [0,T]$ *we have*

$$\|(\check{\rho}\partial_t u, \mu\nabla^2 u, \nabla\Pi)\|_{L^p_t(L^r)} \leq C \left(\eta_\rho \mathcal{B}_\rho^{2+\widetilde{\varsigma}+\delta_+}(t) \|f\|_{L^p_t(L^r)} + \frac{\mu}{\delta(\Omega)^2} \mathcal{C}_\rho(t) \|u\|_{L^p_t(L^r)} \right).$$
$$(47)$$

Proof. Let (u, Π) solve (13) with null initial data.

1. *Estimates on a small time interval.* Proposition 3.8 provides us with a priori estimates on a small interval even if ρ depends on t. Indeed, (u, Π) satisfies

$$\begin{cases} \rho(0)\partial_t u - \mu\Delta u + \nabla\Pi = f + (\rho(0) - \rho)\partial_t u, \\ \text{div } u = 0, \qquad \int_\Omega \Pi \, dx = 0, \\ u_{|t=0} = 0, \qquad u_{|\partial\Omega} = 0, \end{cases}$$

---

[3] Note that since $q > N$, we have $\frac{1+\delta}{1-\delta} \leq \frac{r+1}{r-1}$.

so that Proposition 3.8 applies with density $\rho(0)$. From it, we get

$$\|(\check{\rho}\partial_t u, \mu\nabla^2 u, \nabla\Pi)\|_{L^p_t(L^r)}$$
$$\leq C\eta_{\rho(0)}\mathcal{B}^{2+\widetilde{\varsigma}+\delta_+}_{\rho(0)}\left(\|f\|_{L^p_t(L^r)} + \|\rho(t)-\rho(0)\|_{L^\infty}\|\partial_t u\|_{L^p_t(L^r)}\right)$$
$$+ \frac{C\mu}{\delta(\Omega)^2}\eta^{2r'}_{\rho(0)}\mathcal{B}^{r^*}_{\rho(0)}\|u\|_{L^p_t(L^r)}$$

so that denoting

$$\tau \stackrel{\text{def}}{=} \min\left(T, \left(2C\eta_\rho\mathcal{B}^{2+\widetilde{\varsigma}+\delta_+}_\rho(T)M_\beta(T)\right)^{-\frac{1}{\beta}}\right)$$

and using the Hölder continuity of ρ with respect to t, we end up with

$$\|(\check{\rho}\partial_t u, \mu\nabla^2 u, \nabla\Pi)\|_{L^p_t(L^r)} \leq 2C\eta_\rho\left(\mathcal{B}^{2+\widetilde{\varsigma}+\delta_+}_\rho(t)\|f\|_{L^p_t(L^r)} + \frac{\mu\eta^{2r'}_\rho}{\delta(\Omega)^2}\mathcal{B}^{r^*}_\rho(t)\|u\|_{L^p_t(L^r)}\right)$$

whenever t belongs to $[0,\tau]$. Of course, we used above the fact that $\mathcal{B}_\rho(t) \geq \mathcal{B}_{\rho(0)}$.

2. *Estimates on $[0,T]$.* Estimates on the whole interval $[0,T]$ may be proved by introducing a partition of unity with respect to the t variable so that one can proceed as in the previous step.

Of course, it suffices to prove (47) for $t = T$ and we can assume that $T > \tau$. Let us introduce a partition of unity $(\psi_k)_{k\in\mathbb{N}}$ of $\mathbb{R}^+$ such that

- Supp $\psi_0 \subset [0,\tau]$ and $\psi_0 \equiv 1$ in a neighborhood of 0,
- For $k \geq 1$, Supp $\psi_k \subset [\frac{k}{2}\tau, \frac{k}{2}\tau + \tau]$ and $\|\partial_t\psi_k\|_{L^\infty} \leq \frac{C}{\tau}$.

Denoting $u_k \stackrel{\text{def}}{=} \psi_k u$, $\Pi_k \stackrel{\text{def}}{=} \psi_k\Pi$ and $f_k \stackrel{\text{def}}{=} \psi_k f$, we have

$$\begin{cases} \rho\partial_t u_k - \mu\Delta u_k + \nabla\Pi_k = f_k + u\rho\partial_t\psi_k, \\ \operatorname{div} u_k = 0, \qquad \int_\Omega \Pi_k\,dx = 0, \\ u_k(\frac{k}{2}\tau) = 0, \qquad u_{k|\partial\Omega} = 0. \end{cases}$$

For $t \geq k\tau/2$, we have

$$\|\rho u\partial_t\psi_k\|_{L^p(\frac{k}{2}\tau,t;L^r)} \leq \widehat{\rho}\|u\|_{L^p(\frac{k}{2}\tau,t;L^r)}\|\partial_t\psi_k\|_{L^\infty},$$
$$\leq C\widehat{\rho}\left(\eta_\rho\mathcal{B}^{2+\widetilde{\varsigma}+\delta_+}_\rho(T)M_\beta(T)\right)^{\frac{1}{\beta}}\|u\|_{L^p(\frac{k}{2}\tau,t;L^r)},$$

hence, according to the first step of the proof,

$$\|(\check{\rho}\partial_t u_k, \mu\nabla^2 u_k, \nabla\Pi_k)\|_{L^p(\frac{k}{2}\tau,t;L^r)} \lesssim \eta_\rho\mathcal{B}^{2+\widetilde{\varsigma}+\delta_+}_\rho(t)\|f\|_{L^p(\frac{k}{2}\tau,t;L^r)}$$

$$+ \frac{\mu\eta^{2r'}_\rho}{\delta(\Omega)^2}\mathcal{B}^{r^*}_\rho(t)\|u_k\|_{L^p(\frac{k}{2}\tau,t;L^r)} + \widehat{\rho}\eta_\rho\mathcal{B}^{2+\widetilde{\varsigma}+\delta_+}_\rho(t)\left(\eta_\rho\mathcal{B}^{2+\widetilde{\sigma}+\delta_+}_\rho(T)M_\beta(T)\right)^{\frac{1}{\beta}}\|u\|_{L^p(\frac{k}{2}\tau,t;L^r)}\right)$$

whenever t belongs to $I_k \stackrel{\text{def}}{=} [\frac{k}{2}\tau, \frac{k}{2}\tau + \tau]$.

Of course, u_k and Π_k vanish outside I_k so that performing a summation on $k \in \{0,\cdots,K\}$ (with K such that $K\tau \leq T < (K+1)\tau$), we obtain inequality (47). $\qquad\square$

c) Existence and uniqueness for (13) with null initial data

We here want to prove the following result, which is a particular case of Theorem 3.7:

Proposition 3.10. *Let f and ρ satisfy the assumptions of Theorem 3.7. Then equation* (13) *with null initial data has a unique solution* (u, Π) *such that*

$$u \in C([0,T]; D_{A_r}^{1-\frac{1}{p},p}) \cap L^p(0,T; W^{2,r} \cap W_0^{1,r}), \quad \Pi \in L^p(0,T; W^{1,r})$$
$$and \ \ \partial_t u \in L^p(0,T; L^r).$$

Besides, there exists $C = C(\alpha, q, r, N, \sigma(\Omega))$ such that the following estimates hold true:

$$\|(\check{\rho}\partial_t u, \mu\nabla^2 u, \nabla\Pi)\|_{L_t^p(L^r)} \leq C\eta_\rho \mathcal{B}_\rho^{2+\widetilde{\varsigma}+\delta_+}(t) e^{\frac{C\mu t \mathcal{C}_\rho(t)}{\check{\rho}\delta(\Omega)^2}} \|f\|_{L_t^p(L^r)},$$

$$\mu^{\frac{1}{p'}} \check{\rho}^{\frac{1}{p}} \|u(t)\|_{D_{A_r}^{1-\frac{1}{p},p}} \leq C\eta_\rho^2 \mathcal{B}_\rho^{2+\widetilde{\varsigma}+\delta_+}(t) e^{\frac{C\mu t \mathcal{C}_\rho(t)}{\check{\rho}\delta(\Omega)^2}} \|f\|_{L_t^p(L^r)}.$$

where η_ρ, $\mathcal{C}_\rho$ and $\mathcal{B}_\rho$ have been defined in the statement of Proposition 3.9, and $t \in [0,T]$.

Proof. Let us first remark that estimate (47) enables us to get an a priori estimate for $\|(\check{\rho}\partial_t u, \mu\nabla^2 u, \nabla\Pi)\|_{L_t^p(L^r)}$ involving only ρ and f. Indeed, let us recall the following inequality (which holds true for smooth functions):

$$\frac{d}{dt} \|u(t)\|_{L^r} \leq \|\partial_t u\|_{L^r}. \tag{48}$$

Taking advantage of (48), (47) and of an appropriate smoothing of the function u, the following formal computations may be made rigorous for all $\epsilon > 0$:

$$\|u(t)\|_{L^r}^p = p \int_0^t \|u(\tau)\|_{L^r}^{p-1} \frac{d}{dt} \|u(\tau)\|_{L^r} \, d\tau,$$

$$\leq (p-1)\epsilon \int_0^t \|u(\tau)\|_{L^r}^p \, d\tau + \epsilon^{1-p} \int_0^t \|\partial_t u(\tau)\|_{L^r}^p \, d\tau,$$

$$\leq \left((p-1)\epsilon + C\epsilon^{1-p}\left(\frac{\mu\mathcal{C}_\rho(t)}{\check{\rho}\delta(\Omega)^2}\right)^p\right) \int_0^t \|u\|_{L^r}^p \, d\tau$$

$$+ C\epsilon^{1-p}\left(\frac{\eta_\rho}{\check{\rho}}\mathcal{B}_\rho^{2+\widetilde{\varsigma}+\delta_+}\right)^p \int_0^t \|f\|_{L^r}^p \, d\tau.$$

Choosing $\epsilon = \left(\frac{C}{p-1}\right)^{\frac{1}{p}} \frac{\mu\mathcal{C}_\rho}{\check{\rho}\delta(\Omega)^2}$ and applying Gronwall lemma yields for some con-

stant $C = C_{N,r,q,\sigma(\Omega)}$,

$$\|u(t)\|_{L^r} \leq C \left(\frac{\mu \mathcal{C}_\rho(t)}{\delta(\Omega)^2}\right)^{\frac{1}{p}-1} \frac{\eta_\rho \mathcal{B}_\rho^{2+\widetilde{\varsigma}+\delta_+}(t)}{\check{\rho}^{\frac{1}{p}}} e^{\frac{C\mu\mathcal{C}_\rho(t)t}{\check{\rho}\delta(\Omega)^2}} \|f\|_{L_t^p(L^r)}, \quad (49)$$

$$\frac{\mu}{\delta(\Omega)^2} \|u\|_{L_t^p(L^r)} \leq C \left(\frac{\eta_\rho \mathcal{B}_\rho^{2+\widetilde{\varsigma}+\delta_+}(t)}{\mathcal{C}_\rho(t)}\right) e^{\frac{C\mu\mathcal{C}_\rho(t)t}{\check{\rho}\delta(\Omega)^2}} \|f\|_{L_t^p(L^r)}. \quad (50)$$

Plugging inequality (50) in (47), we conclude that

$$\|(\check{\rho}\partial_t u, \mu\nabla^2 u, \nabla\Pi)\|_{L_t^p(L^r)} \leq C\eta_\rho \mathcal{B}_\rho^{2+\widetilde{\varsigma}+\delta_+}(t) e^{\frac{C\mu\mathcal{C}_\rho(t)t}{\check{\rho}\delta(\Omega)^2}} \|f\|_{L_t^p(L^r)}. \quad (51)$$

In order to prove estimates for u in $L^\infty(0, T; D_{A_r}^{1-\frac{1}{p},p})$, we use the fact that (u, Π) satisfies

$$\begin{cases} \check{\rho}\partial_t u - \mu\Delta u + \nabla\Pi = f + (\check{\rho} - \rho)\partial_t u, \\ \operatorname{div} u = 0, \qquad \int \Pi\, dx = 0, \\ u_{|t=0} = 0, \quad u_{\partial\Omega} = 0, \end{cases}$$

hence, according to Theorem 3.2,

$$\mu^{\frac{1}{p'}}\check{\rho}^{\frac{1}{p}} \|u(t)\|_{D_{A_r}^{1-\frac{1}{p},p}} \leq C\left(\|f\|_{L_t^p(L^r)} + \eta_\rho\|\check{\rho}\partial_t u\|_{L_t^p(L^r)}\right).$$

Inserting (51) in the above inequality, we conclude that

$$\mu^{\frac{1}{p'}}\check{\rho}^{\frac{1}{p}} \|u(t)\|_{D_{A_r}^{1-\frac{1}{p},p}} \leq C\eta_\rho^2 \mathcal{B}_\rho^{2+\widetilde{\varsigma}+\delta_+}(t) e^{\frac{C\mu\mathcal{C}_\rho(t)T}{\check{\rho}\delta(\Omega)^2}} \|f\|_{L_t^p(L^r)}. \quad (52)$$

Now, it is easy to prove the existence of a solution for (13) with null initial data. Indeed, in the case $p = r$ and $q = +\infty$, it has been proved by O. Ladyzhenskaya and V. Solonnikov in [18]. By making use of a standard mollifying process for smoothing out f and ρ, and of estimates (51), (52) and Remark 2.4, one can prove that under our assumptions, there exists a solution with the required regularity properties. The details are left to the reader.

Uniqueness obviously stems from estimates (51) and (52). $\qquad\square$

3.3.2. General initial data

Let us now consider the initial value problem (13) for general initial data u_0 in $D_{A_r}^{1-\frac{1}{p},p}$.

We claim that it may be reduced to the case $u_0 \equiv 0$ by splitting the unknown solution (u, Π) into the sum of a solution of the standard non-stationary Stokes system (5) with u_0 as initial data and f as external force, and a solution of (13) with null initial data.

More precisely, let (w, Q) and (v, P) be the solutions of

$$\begin{cases} \check{\rho}\partial_t w - \mu\Delta w + \nabla Q = f, \\ \operatorname{div} w = 0, \qquad \int_\Omega Q\, dx = 0, \\ w_{|t=0} = u_0, \quad w_{|\partial\Omega} = 0 \end{cases} \quad \text{and} \quad \begin{cases} \rho\partial_t v - \mu\Delta v + \nabla P = (\check{\rho} - \rho)\partial_t w, \\ \operatorname{div} v = 0, \qquad \int_\Omega P\, dx = 0, \\ v_{|t=0} = 0, \quad v_{|\partial\Omega} = 0. \end{cases}$$

The existence of (w, Q) is insured by Theorem 3.2 while Proposition 3.10 provides a solution for the system on the right. On the other hand $(u, \Pi) \overset{\text{def}}{=} (v + w, P + Q)$ is clearly a solution to (13) with initial data u_0.

As uniqueness stems from estimate (14), we are left with the proof of (14) and (15).

• *Estimates for w:* According to Theorem 3.6, we have for $t \in [0, T]$:

$$\|(\check{\rho}\partial_t w, \mu\nabla^2 w, \nabla Q)\|_{L^p_t(L^r)} + \check{\rho}^{\frac{1}{p}}\mu^{\frac{1}{p'}}\|w(t)\|_{D_{A_r}^{1-\frac{1}{p},p}} \lesssim \|f\|_{L^p_t(L^r)} + \check{\rho}^{\frac{1}{p}}\mu^{\frac{1}{p'}}\|u_0\|_{D_{A_r}^{1-\frac{1}{p},p}}.$$
(53)

• *Estimates for v:* According to Proposition 3.10, we have

$$\|(\check{\rho}\partial_t v, \mu\nabla^2 v, \nabla P)\|_{L^p_t(L^r)} + \check{\rho}^{\frac{1}{p}}\mu^{\frac{1}{p'}}\|v(t)\|_{D_{A_r}^{\frac{1}{p'},p}}$$
$$\lesssim C\eta_\rho^2 \mathcal{B}_\rho^{2+\widetilde{\varsigma}+\delta_+}(t)\, e^{\frac{C\mu\mathcal{C}_\rho(t)t}{\check{\rho}\delta(\Omega)^2}} \left\|1 - \frac{\rho}{\check{\rho}}\right\|_{L^\infty} \|\check{\rho}\partial_t w\|_{L^p_T(L^r)},$$

hence, plugging (53) in the above inequality, we get

$$\|(\check{\rho}\partial_t v, \mu\nabla^2 v, \nabla P)\|_{L^p_t(L^r)} + \check{\rho}^{\frac{1}{p}}\mu^{\frac{1}{p'}}\|v(t)\|_{D_{A_r}^{1-\frac{1}{p},p}}$$
$$\leq C\eta_\rho^3 \mathcal{B}_\rho^{2+\widetilde{\varsigma}+\delta_+}(t)\, e^{\frac{C\mu\mathcal{C}_\rho(t)t}{\check{\rho}\delta(\Omega)^2}} \left(\check{\rho}^{\frac{1}{p}}\mu^{\frac{1}{p'}}\|u_0\|_{\dot{D}_{A_r}^{1-\frac{1}{p},p}} + \|f\|_{L^p_T(L^r)}\right). \quad (54)$$

• *Estimates for u:* Adding inequalities (53) and (54), and using Proposition 2.4 yields (14). In order to prove (15), we apply estimate (47) to (v, P). We get

$$\|(\check{\rho}\partial_t v, \mu\nabla^2 v, \nabla P)\|_{L^p_t(L^r)} \lesssim \eta_\rho \mathcal{B}_\rho^{2+\widetilde{\varsigma}+\delta_+}(t)\left(\check{\rho}^{\frac{1}{p}}\mu^{\frac{1}{p'}}\|u_0\|_{D_{A_r}^{\frac{1}{p'},p}} + \|f\|_{L^p_t(L^r)}\right)$$
$$+ \frac{\mathcal{C}_\rho(t)\mu}{\delta(\Omega)^2}\|v\|_{L^p_t(L^r)}.$$

Now, we have

$$\|v\|_{L^p_t(L^r)} \leq \|u\|_{L^p_t(L^r)} + \|w\|_{L^p_t(L^r)}$$

and, according to Proposition 2.4,

$$\|w\|_{L^p_t(L^r)} \leq C\delta(\Omega)^2\|\nabla^2 w\|_{L^p_t(L^r)}.$$

Hence, by virtue of (53),

$$\|(\check{\rho}\partial_t u, \mu\nabla^2 u, \nabla\Pi)\|_{L^p_t(L^r)}$$
$$\lesssim \eta_\rho \mathcal{B}_\rho^{2+\widetilde{\varsigma}+\delta_+}(t)\left(\|f\|_{L^p_t(L^r)} + \mu^{\frac{1}{p'}}\check{\rho}^{\frac{1}{p}}\|u_0\|_{D_{A_r}^{1-\frac{1}{p},p}}\right) + \frac{\mathcal{C}_\rho(t)\mu}{\delta(\Omega)^2}\|u\|_{L^p_t(L^r)}. \quad (55)$$

Now, as the system satisfied by (u, Π) may be rewritten

$$\begin{cases} \check{\rho}\partial_t u - \mu\Delta u + \nabla\Pi = f + (\check{\rho} - \rho)\partial_t u, \\ \operatorname{div} u = 0, \qquad \int_\Omega \Pi\, dx = 0, \\ u_{|t=0} = u_0, \quad u_{|\partial\Omega} = 0. \end{cases}$$

Theorem 3.2 insures that

$$\|u(t)\|_{D_{A_r}^{1-\frac{1}{p},p}} \lesssim \|u_0\|_{D_{A_r}^{1-\frac{1}{p},p}} + \check{\rho}^{-\frac{1}{p}}\mu^{-\frac{1}{p'}}\left(\|f\|_{L_t^p(L^r)} + \left\|1 - \frac{\rho}{\check{\rho}}\right\|_{L^\infty}\|\check{\rho}\partial_t u\|_{L_t^p(L^r)}\right).$$

Using (55), we thus conclude that $\|u(t)\|_{D_{A_r}^{1-\frac{1}{p},p}}$ is bounded by the right-hand side of (15). $\qquad\square$

4. Uniqueness

Before tackling the problem of uniqueness, let us state two interpolation inequalities.

Lemma 4.1. *Let* $1 < p, q, r, s < +\infty$ *satisfy* $0 < \frac{p}{2} - \frac{Np}{2r} < 1$ *and* $\frac{1}{s} = \frac{1}{r} + \frac{1}{q}$. *The following inequalities hold true:*

$$\|\nabla f\|_{L_T^p(L^\infty)} \leq CT^{\frac{1}{2}-\frac{N}{2r}}\|f\|_{L_T^\infty(D_{A_r}^{1-\frac{1}{p},p})}^{1-\theta}\|f\|_{L_T^p(W^{2,r})}^{\theta},$$

$$\|\nabla f\|_{L_T^p(L^q)} \leq CT^{\frac{1}{2}-\frac{N}{2r}}\|f\|_{L_T^\infty(D_{A_s}^{1-\frac{1}{p},p})}^{1-\theta}\|f\|_{L_T^p(W^{2,s})}^{\theta},$$

for some constant C *depending only on* Ω, N, p, q *and* r *and* $(1-\theta)/p = 1/2 - N/2r$.

Proof. The proof is based on the use of embeddings and interpolation results which may be found in [22].

For proving the first inequality, we use that $(B_{\infty,\infty}^{1-\frac{2}{p}-\frac{N}{r}}, B_{\infty,\infty}^{1-\frac{N}{r}})_{\theta,1} = B_{\infty,1}^0$ (with θ defined as in the statement of the lemma) and that $B_{\infty,1}^0 \hookrightarrow L^\infty$ so that

$$\|\nabla f\|_{L^\infty} \lesssim \|\nabla f\|_{B_{\infty,\infty}^{1-\frac{N}{r}}}^{\theta}\|\nabla f\|_{B_{\infty,\infty}^{1-\frac{2}{p}-\frac{N}{r}}}^{1-\theta}. \tag{56}$$

Remark that $D_{A_r}^{1-\frac{1}{p},p} \hookrightarrow B_{r,p}^{2-\frac{2}{p}} \hookrightarrow B_{\infty,\infty}^{2-\frac{2}{p}-\frac{N}{r}}$ (see Prop. 2.5). Hence, according to (56) and because $W^{1,r} \hookrightarrow B_{\infty,\infty}^{1-\frac{N}{r}}$, we have

$$\|\nabla f\|_{L_T^p(L^\infty)} \leq C\left(\int_0^T \|\nabla f\|_{B_{\infty,\infty}^{1-\frac{N}{r}}}^{p\theta}\|\nabla f\|_{B_{\infty,\infty}^{1-\frac{2}{p}-\frac{N}{r}}}^{p(1-\theta)} dt\right)^{\frac{1}{p}},$$

$$\leq C\left(\int_0^T \|f\|_{W^{2,r}}^{p\theta}\|f\|_{D_{A_r}^{1-\frac{1}{p},p}}^{p(1-\theta)} dt\right)^{\frac{1}{p}},$$

$$\leq CT^{\frac{1}{2}-\frac{N}{2r}}\|f\|_{L_T^\infty(D_{A_r}^{1-\frac{1}{p},p})}^{1-\theta}\|f\|_{L_T^p(W^{2,r})}^{\theta}.$$

The proof of the second inequality is based on the fact that

$$B_{q,1}^0 = (B_{q,p}^{1-\frac{2}{p}-\frac{N}{r}}, B_{q,q}^{1-\frac{N}{r}})_{\theta,1} \hookrightarrow L^q$$

and that $W^{1,s} \hookrightarrow B_{q,q}^{1-\frac{N}{r}}$ whereas $D_{A_s}^{1-\frac{1}{p},p} \hookrightarrow B_{q,p}^{1-\frac{2}{p}-\frac{N}{r}}$.

Then going along the lines of the proof of the first inequality yields the desired result. $\qquad\square$

Proposition 4.2. *Let $p \in (1,\infty)$, $r \in (N,+\infty)$ and $q \in [r,+\infty]$ with besides $q > N$. Denote $s \overset{\text{def}}{=} rq/(r+q)$ ($s = r$ if $q = +\infty$). Let $(\rho^1, u^1, \nabla\Pi^1)$ and $(\rho^2, u^2, \nabla\Pi^2)$ be two solutions of (1) with the same data $\rho_0 \in W^{1,q}$, $u_0 \in D_{A_r}^{1-\frac{1}{p},p}$ and $f \in L^p(0,T;L^r)$. Assume that (u^1, Π^1) and (u^2, Π^2) belong to*

$$\left(C([0,T]; D_{A_s}^{1-\frac{1}{p},p}) \cap W^{1,p}(0,T;L^s) \cap L^p(0,T;W^{2,s}) \right) \times L^p(0,T;W^{1,s})$$

and that, in addition, $\forall (t,x) \in [0,T] \times \Omega$, $0 < \check\rho \le \rho^i(t,x)$ for $i = 1,2$,

$$\rho^1 \in L^\infty(0,T;L^q), \quad \nabla u^1 \in L^p(0,T;L^\infty), \quad \partial_t u^1 + u^1 \cdot \nabla u^1 \in L^p(0,T;L^r),$$

$\rho^2 \in L^\infty(0,T;W^{1,q}) \cap C^\beta([0,T];L^\infty)$ *for some $\beta \in (0,1)$ and $u^2 \in L^\infty(0,T;L^r)$. Then $(\rho^1, u^1, \nabla\Pi^1) \equiv (\rho^2, u^2, \nabla\Pi^2)$ on $[0,T] \times \Omega$.*

Proof. Let $\delta\rho \overset{\text{def}}{=} \rho^2 - \rho^1$, $\delta u \overset{\text{def}}{=} u^2 - u^1$, $\delta\Pi \overset{\text{def}}{=} \Pi^2 - \Pi^1$. Remark that $(\delta\rho, \delta u, \delta\Pi)$ satisfies the following system:

$$\begin{cases} \partial_t \delta\rho + u^1 \cdot \nabla\delta\rho = -\delta u \cdot \nabla\rho^2, \\ \rho^2 \partial_t \delta u + \nabla\delta\Pi - \mu\Delta\delta u = f\delta\rho - \delta\rho\, \partial_t u^1 - \delta\rho\, u^1 \cdot \nabla u^1 - \rho^2 u^2 \cdot \nabla\delta u - \rho^2 \delta u \cdot \nabla u^1, \\ \operatorname{div} \delta u = 0, \qquad \int_\Omega \delta\Pi = 0, \\ (\delta\rho, \delta u)_{|t=0} = (0,0), \qquad \delta u_{|\partial\Omega} = 0. \end{cases}$$

(57)

Obviously, we have for all $t \in [0,T]$,

$$\|\delta\rho(t)\|_{L^q} \le \int_0^t \left\| \nabla\rho^2(\tau) \right\|_{L^q} \|\delta u(\tau)\|_{L^\infty}\, d\tau. \tag{58}$$

On the other hand, Theorem 3.7 yields for some 'constant' C_T depending on T, N, q, r, p, μ, $\check\rho$ and on the norm of ρ^2 in $L^\infty(0,T;W^{1,q}) \cap C^\beta(0,T;L^\infty)$ and all $t \in [0,T]$,

$$\|\delta u\|_{L_t^p(W^{2,s})} + \|\delta\Pi\|_{L_t^p(W^{1,s})} + \|\delta u\|_{L_t^\infty(D_{A_s}^{\frac{1}{p'},p})}$$
$$\le C_T \Big(\|\delta\rho(f - \partial_t u^1 - u^1 \cdot \nabla u^1)\|_{L_t^p(L^s)}$$
$$+ \|\rho^2 u^2 \cdot \nabla\delta u\|_{L_t^p(L^s)} + \|\rho^2 \delta u \cdot \nabla u^1\|_{L_t^p(L^s)} \Big). \tag{59}$$

Thanks to Hölder inequality, we get, up to a change of C_T,

$$\|\delta u\|_{L_t^p(W^{2,s})} + \|\delta\Pi\|_{L_t^p(W^{1,s})} + \|\delta u\|_{L_t^\infty(D_{A_s}^{\frac{1}{p'},p})}$$
$$\le C_T \|\delta\rho\|_{L_t^\infty(L^q)} \|f - \partial_t u^1 - u^1 \cdot \nabla u^1\|_{L_t^p(L^r)}$$
$$+ C_T \Big(\|u^2\|_{L_t^\infty(L^r)} \|\nabla\delta u\|_{L_t^p(L^q)} + \|\nabla u^1\|_{L_t^p(L^\infty)} \|\delta u\|_{L_t^\infty(L^s)} \Big). \tag{60}$$

If $1/2 - N/2r < 1/p$, Lemma 4.1 yields

$$\|\nabla \delta u\|_{L_t^p(L^q)} \lesssim t^{\frac{1}{2} - \frac{N}{2r}} \left(\|\delta u\|_{L_t^p(W^{2,s})} + \|\delta u\|_{L_t^\infty(D_{A_s}^{1-\frac{1}{p},p})} \right).$$

If $1/2 - N/2r > 1/p$, we have $D_{A_s}^{1-\frac{1}{p},p} \hookrightarrow W^{1,q}$ so that the above inequality holds with $t^{\frac{1}{p}}$. The limit case $1/2 - N/2r = 1/p$ may be handled by noticing that we also have $\|\rho^2 u^2 \cdot \nabla \delta u\|_{L_t^p(L^s)} \leq \widehat{\rho} \|u^2\|_{L_t^\infty(L^{r^+})} \|\nabla \delta u\|_{L_t^p(L^{q^-})}$ with r^+ (resp. q^-) slightly greater (resp. smaller) than r (resp. q) and by using the embeddings $D_{A_s}^{1-\frac{1}{p},p} \hookrightarrow W^{1,q^-}$ and $D_{A_s}^{1-\frac{1}{p},p} \hookrightarrow L^{r^+}$. We eventually get $\|\rho^2 u^2 \cdot \nabla \delta u\|_{L_t^p(L^s)} \lesssim t^{\frac{1}{p}} \|u^2\|_{L_t^\infty(D_{A_s}^{\frac{1}{p'},p})} \|\delta u\|_{L_t^\infty(D_{A_s}^{\frac{1}{p'},p})}$.

On the other hand, since $N/s < 2$, we have $W^{2,s} \hookrightarrow L^\infty$ so that

$$\|\delta u\|_{L_t^p(L^\infty)} \lesssim \|\delta u\|_{L_t^p(W^{2,s})}.$$

Hence, denoting

$$X(t) \overset{\text{def}}{=} \|\delta\rho(t)\|_{L_t^\infty(L^q)} + \|\delta u\|_{L_t^p(W^{2,s})} + \|\delta\Pi\|_{L_t^p(W^{1,s})} + \|\delta u\|_{L_t^\infty(D_{A_s}^{1-\frac{1}{p},p})}$$

and coming back to (58, 60), we eventually gather

$$X(t) \leq C_T \Big\{ \|f - \partial_t u^1 - u^1 \cdot \nabla u^1\|_{L_t^p(L^r)} + t^{\min(\frac{1}{p}, \frac{1}{2} - \frac{N}{2r})} \|u^2\|_{L_t^\infty(L^r \cap D_{A_s}^{\frac{1}{p'},p})}$$
$$+ \|\nabla u^1\|_{L_t^p(L^\infty)} + \|\nabla\rho^2\|_{L_t^{p'}(L^q)} \Big\} X(t). \quad (61)$$

Now, choosing η so small as the term between brackets is less than $1/2$ for $t = \eta$ enables us to conclude that $X \equiv 0$ on $[0, \eta]$. As the constant C_T does not depend on η, a standard induction argument yields uniqueness on the whole interval $[0, T]$. $\qquad\square$

Remark 4.3. Going along the lines of the proof of Proposition 4.2, one can easily prove that if (ρ_1, u_1, Π_1) and (ρ_2, u_2, Π_2) solve (1) with different initial data (ρ_0^1, u_0^1) and (ρ_0^2, u_0^2), and external forces f_1 and f_2, and satisfy the assumptions of Proposition 4.2 then the following estimate holds true on $[0, T]$ (with obvious notations):

$$\|\delta\rho(t)\|_{L^q} + \|\delta u\|_{L_t^p(W^{2,s})} + \|\delta\Pi\|_{L_t^p(W^{1,s})} + \|\delta u(t)\|_{D_{A_s}^{\frac{1}{p'},p}}$$
$$\leq C_T \left(\|\delta\rho_0\|_{L^q} + \|\delta u_0\|_{D_{A_s}^{\frac{1}{p'},p}} + \|\delta f\|_{L_t^\infty(L^s)} \right).$$

Combining with Theorem 1.2, we conclude that for small enough T, the map $(\rho_0, u_0, f) \mapsto (\rho, u, \Pi)$ is Lipschitz continuous from bounded sets of $W^{1,q} \times D_{A_r}^{\frac{1}{p'},p} \times L^p(0, T; L^r)^N$ to

$$C([0, T]; L^q) \times \left(W^{1,p}(0, T; L^s)^N \cap L^p(0, T; W^{2,s})^N \cap C([0, T]; D_{A_s}^{\frac{1}{p'},p}) \right) \times L^p(0, T; W^{1,s}).$$

5. Existence on a small time interval

This section is devoted to the proof of Theorem 1.2.

First step: construction of approximate solutions. We initialize the construction of approximate solutions by prescribing $\rho^0 \overset{\text{def}}{=} \rho_0$ and $u^0 \overset{\text{def}}{=} u_0$. Given (ρ^n, u^n), Propositions 3.1 and 3.7 enable us to define ρ^{n+1} as the (global) solution of the transport equation

$$\partial_t \rho^{n+1} + u^n \cdot \nabla \rho^{n+1} = 0, \qquad \rho^{n+1}_{|t=0} = \rho_0 \tag{62}$$

and (u^{n+1}, Π^{n+1}) as the (global) solution of

$$\begin{cases} \rho^{n+1}\partial_t u^{n+1} - \mu\Delta u^{n+1} + \nabla\Pi^{n+1} = \rho^{n+1}f - \rho^{n+1}u^n \cdot \nabla u^n, & \int_\Omega \Pi^{n+1}\, dx = 0, \\ \operatorname{div} u^{n+1} = 0, \\ u^{n+1}_{|t=0} = u_0, \qquad u^{n+1}_{|\partial\Omega} = 0. \end{cases} \tag{63}$$

Arguing by induction yields $(\rho^n, u^n, \Pi^n) \in E^{p,q,r}_T$ for all positive T.

Second step: uniform bounds for some small fixed T. We aim at finding a positive time T independent of n for which $(\rho^n, u^n, \Pi^n)_{n\in\mathbb{N}}$ is uniformly bounded in the space $E^{p,q,r}_T$.
   Applying Proposition 3.1 to (62) yields

$$\|\rho^{n+1}(t)\|_{W^{1,q}} \leq \|\rho_0\|_{W^{1,q}} e^{\int_0^t \|\nabla u^n(\tau)\|_{L^\infty}\, d\tau}. \tag{64}$$

In addition, for all time t, we have

$$\min_{x\in\bar\Omega} \rho^n(t,x) = \check{\rho} \overset{\text{def}}{=} \min_{x\in\bar\Omega}\rho_0(x) \quad \text{and} \quad \max_{x\in\bar\Omega} \rho^n(t,x) = \widehat{\rho} \overset{\text{def}}{=} \max_{x\in\bar\Omega}\rho_0(x). \tag{65}$$

Therefore, if one can prove that ρ^{n+1} belongs to $C^\beta(0,T;L^\infty)$ for some positive β then applying Theorem 3.7 to system (63) yields

$$\|\partial_t u^{n+1}\|_{L^p_t(L^r)} + \|u^{n+1}\|_{L^p_t(W^{2,r})} + \|\Pi^{n+1}\|_{L^p_t(W^{1,r})} + \|u^{n+1}\|_{L^\infty_t(D^{1-\frac{1}{p},p}_{A_r})}$$

$$\leq C\Big(\|u_0\|_{D^{1-\frac{1}{p},p}_{A_r}} + \|f\|_{L^p_t(L^r)} + \|u^n \cdot \nabla u^n\|_{L^p_t(L^r)}\Big)$$

$$\times e^{Ct\big(1+\|\rho^{n+1}\|_{L^\infty_t(W^{1,q})}\big)^{\gamma_1}\big(1+\|\rho^{n+1}\|^{\frac{1}{\beta}}_{\dot{C}^\beta_t(L^\infty)}\big)} \tag{66}$$

for some positive exponent γ_1 depending only on N, p, q, r and β, and a constant C depending only on p, q, r, N, $\check{\rho}$, $\widehat{\rho}$, Ω, μ and β.
   Remark that interpolating between $L^\infty(0,T;W^{1,q})$ and $W^{1,\infty}(0,T;L^s)$ (with $s \overset{\text{def}}{=} rq/(r+q)$) shows that ρ^{n+1} belongs to $C^\beta(0,T;L^\infty)$ whenever $\beta \in \left(0, \frac{1-\frac{N}{q}}{1+\frac{N}{r}}\right)$ and that

$$\|\rho^{n+1}\|_{\dot{C}^\beta_t(L^\infty)} \lesssim \|\rho^{n+1}\|_{L^\infty_t(W^{1,q})} + \|\partial_t \rho^{n+1}\|_{L^\infty_t(L^s)}. \tag{67}$$

Now, we have $\partial_t \rho^{n+1} = -u^n \cdot \nabla \rho^{n+1}$ so that $\partial_t \rho^{n+1} \in L^\infty_{loc}(\mathbb{R}^+; L^s)$, and for $t \geq 0$,

$$\|\partial_t \rho^{n+1}\|_{L^\infty_t(L^s)} \leq \|u^n\|_{L^\infty_t(L^r)} \|\nabla \rho^{n+1}\|_{L^\infty_t(L^q)}.$$

Hence, inserting the above inequalities in (66), we gather

$$\|\partial_t u^{n+1}\|_{L^p_t(L^r)} + \|u^{n+1}\|_{L^p_t(W^{2,r})} + \|\Pi^{n+1}\|_{L^p_t(W^{1,r})} + \|u^{n+1}\|_{L^\infty_t(D^{1-\frac{1}{p},p}_{A_r})}$$

$$\leq C \Big(\|u_0\|_{D^{1-\frac{1}{p},p}_{A_r}} + \|f\|_{L^p_t(L^r)}$$

$$+ \|u^n\|_{L^\infty_t(L^r)} \|\nabla u^n\|_{L^p_t(L^\infty)} \Big) e^{Ct(1+\|\rho^{n+1}\|_{L^\infty_t(W^{1,q})})^\gamma (1+\|u^n\|_{L^\infty_t(L^r)})^\delta}. \tag{68}$$

for some positive exponents γ and δ depending only on N, p, q and r.

Fix a (large) reference time T_0 and define

$$U^n(t) \stackrel{\text{def}}{=} \|u^n\|_{L^\infty_t(D^{1-\frac{1}{p},p}_{A_r})} + \|u^n\|_{L^p_t(W^{2,r})},$$

$U_0 \stackrel{\text{def}}{=} \|u_0\|_{D^{1-\frac{1}{p},p}_{A_r}} + \|f\|_{L^p_{T_0}(L^r)}$, $P^n(t) \stackrel{\text{def}}{=} \|\rho^n\|_{L^\infty_t(W^{1,q})}$ and $P_0(t) \stackrel{\text{def}}{=} \|\rho_0\|_{W^{1,q}}$.

To simplify, assume from now on that $p/2 - Np/2r < 1$ so that Lemma 4.1 may be applied to ∇u^n. We get[4]

$$\|\nabla u^n\|_{L^p_t(L^\infty)} \leq t^{\frac{1}{2}-\frac{N}{2r}} \|u^n\|^\theta_{L^p_t(W^{2,r})} \|u^n\|^{1-\theta}_{L^\infty_t(D^{1-\frac{1}{p},p}_{A_r})} \tag{69}$$

with θ such that $(1-\theta)/p = 1/2 - N/2r$.

Plugging inequality (69) in (64) and (68), we eventually get

$$U^{n+1}(t) \leq C e^{Ct(1+P^{n+1}(t))^\gamma (1+U^n(t))^\delta} \Big(U_0 + t^{\frac{1}{2}-\frac{N}{2r}} (U^n(t))^2 \Big), \tag{70}$$

$$P^{n+1}(t) \leq P_0 e^{Ct^{\frac{1}{p'}+\frac{1}{2}-\frac{N}{2r}} U^n(t)}. \tag{71}$$

Inserting (71) in (70) yields

$$U^{n+1}(t) \leq C e^{Ct(1+U^n(t))^\delta (1+P_0)^\gamma e^{\gamma Ct^{\frac{1}{p'}+\frac{1}{2}-\frac{N}{2r}}}} \Big(U_0 + t^{\frac{1}{2}-\frac{N}{2r}} (U^n(t))^2 \Big)$$

so that, assuming that t is so small as to satisfy

$$\gamma C t^{\frac{1}{p'}+\frac{1}{2}-\frac{N}{2r}} \leq \log 2,$$

we get

$$U^{n+1}(t) \leq C e^{2Ct(1+U^n(t))^\delta (1+P_0)^\gamma} \Big(U_0 + t^{\frac{1}{2}-\frac{N}{2r}} (U^n(t))^2 \Big) \quad \text{and} \quad P^{n+1}(t) \leq 2P_0.$$

---

[4] If $p/2 - Np/2r \geq 1$, we would get $t^{\frac{1}{p}}$ instead of $t^{\frac{1}{2}-\frac{N}{2r}}$ below.

Now, if we assume that $U^n(t) \leq 4CU_0$ on $[0,T]$ with[5]

$$T = \min\left(T_0, \left(\frac{\log 2}{\gamma C}\right)^{\frac{1}{\frac{1}{p'}+\frac{1}{2}-\frac{N}{2r}}}, \left(\frac{1}{16CU_0}\right)^{\frac{1}{\frac{1}{2}-\frac{N}{2r}}}, \frac{\log 2}{2C(1+P_0)^\gamma(1+4CU_0)^\delta}\right),$$

(72)

easy computations show that $U^{n+1}(t) \leq 4CU_0$ on $[0,T]$. Coming back to (71), we conclude that the sequence (ρ^n, u^n, Π^n) is uniformly bounded in $E_T^{p,q,r}$. More precisely, we have for all $t \in [0,T]$:

$$P^n(t) \leq 2P_0 \quad \text{and} \quad U^n(t) \leq 4CU_0. \tag{73}$$

Third step: Convergence of the sequence in small norm. In this part, we aim at proving that $(\rho^n, u^n, \Pi^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the space $E_T^{p,q,s}$ with $s \stackrel{\text{def}}{=} rq/(r+q)$.

Let $\delta u^n \stackrel{\text{def}}{=} u^{n+1} - u^n$, $\delta\Pi^n \stackrel{\text{def}}{=} \Pi^{n+1} - \Pi^n$ and $\delta\rho^n \stackrel{\text{def}}{=} \rho^{n+1} - \rho^n$. Define

$$\delta U^n(t) \stackrel{\text{def}}{=} \|(\partial_t \delta u^n, \nabla^2 \delta u^n, \nabla\delta\Pi^n)\|_{L_t^p(L^s)} + \|\delta u^n\|_{L_t^\infty(D_{A_s}^{1-\frac{1}{p},p})}.$$

The triplet $(\delta\rho^n, \delta u^n, \delta\Pi^n)$ satisfies

$$\begin{cases} \partial_t \delta\rho^n + u^n \cdot \nabla\delta\rho^n = -\delta u^{n-1} \cdot \nabla\rho^n, \\ \delta\rho^n_{|t=0} = 0, \\ \\ \rho^{n+1}\partial_t \delta u^n - \mu\Delta\delta u^n + \nabla\delta\Pi^n = \delta\rho^n(f - \partial_t u^{n+1} - u^n \cdot \nabla u^n) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad - \rho^n(u^n \cdot \nabla\delta u^{n-1} + \delta u^{n-1} \cdot \nabla u^{n-1}), \\ \operatorname{div} \delta u^n = 0, \qquad \int_\Omega \delta\Pi^n \, dx = 0, \\ \delta u^n_{|t=0} = 0, \qquad \delta u^n_{|\partial\Omega} = 0. \end{cases}$$

(74)

Hence, according to Theorem 3.7 and by virtue of (73),

$$\delta U^n(t) \leq C\Big(\|\delta\rho^n(f - \partial_t u^{n+1} - u^n \cdot \nabla u^n)\|_{L_t^p(L^s)}$$
$$\qquad + \|u^n \cdot \nabla\delta u^{n-1}\|_{L_t^p(L^s)} + \|\delta u^{n-1} \cdot \nabla u^{n-1}\|_{L_t^p(L^s)}\Big),$$

$$\leq C\Big(\|\delta\rho^n\|_{L_t^\infty(L^q)}(\|f\|_{L_t^p(L^r)} + \|\partial_t u^{n+1}\|_{L_t^p(L^r)} + \|u^n\|_{L_t^\infty(L^r)}\|\nabla u^n\|_{L_t^p(L^\infty)})$$
$$\qquad + \|u^n\|_{L_t^\infty(L^r)}\|\nabla\delta u^{n-1}\|_{L_t^p(L^q)} + \|\delta u^{n-1}\|_{L_t^\infty(L^s)}\|\nabla u^{n-1}\|_{L_t^p(L^\infty)}\Big)$$

for some constant C depending only on the regularity parameters and on P_0 and U_0.

Of course, in this step, one can assume with no loss of generality that $p(\frac{1}{2} - \frac{N}{2r}) < 1$. Hence, taking advantage of (69) and (73), we eventually get

$$\delta U^n(t) \leq C\Big(\|\delta\rho^n\|_{L_t^\infty(L^q)} + \|\nabla\delta u^{n-1}\|_{L_t^p(L^q)} + t^{\frac{1}{2}-\frac{N}{2r}}\|\delta u^{n-1}\|_{L_t^\infty(L^s)}\Big). \tag{75}$$

---

[5] If $\frac{p}{2} - \frac{Np}{2r} \geq 1$, replace $\frac{1}{2} - \frac{N}{2r}$ (whenever it appears) with $\frac{1}{p}$.

Remark that by virtue of the second inequality of Lemma 4.1, we have

$$\|\nabla\delta u^{n-1}\|_{L_t^p(L^q)} \leq Ct^{\frac{1}{2}-\frac{N}{2r}}\delta U^{n-1}(t).$$

Plugging this latter inequality in (75), we conclude that

$$\delta U^n(t) \leq C\Big(\|\delta\rho^n\|_{L_t^\infty(L^q)} + t^{\frac{1}{2}-\frac{N}{2r}}\delta U^{n-1}(t)\Big). \qquad (76)$$

On the other hand, we obviously have

$$\begin{aligned}
\|\delta\rho^n(t)\|_{L^q} &\leq \int_0^t \big\|(\delta u^{n-1}\cdot\nabla\rho^n)(\tau)\big\|_{L^q}\,d\tau, \\
&\leq t^{\frac{1}{p'}}\|\delta u^{n-1}\|_{L_t^p(L^\infty)}\|\nabla\rho^n\|_{L_t^\infty(L^q)}, \\
&\leq Ct^{\frac{1}{p'}}\|\delta u^{n-1}\|_{L_t^p(L^\infty)}. \qquad (77)
\end{aligned}$$

Since $N/s = N/r + N/q < 2$, the space $W^{2,s}$ is embedded in L^∞. Hence inequality (77) rewrites

$$\|\delta\rho^n(t)\|_{L^q} \leq Ct^{\frac{1}{p'}}\delta U^{n-1}(t).$$

Inserting this latter inequality in (76), we get for $t \in [0,T]$

$$\delta U^n(t) \leq C\Big(t^{\frac{1}{p'}} + t^{\frac{1}{2}-\frac{N}{2r}}\Big)\delta U^{n-1}(t)$$

If we choose an $\eta \in (0,T]$ such that the condition

$$C\Big(\eta^{\frac{1}{p'}} + \eta^{\frac{1}{2}-\frac{N}{2r}}\Big) \leq \frac{1}{2} \qquad (78)$$

is fulfilled, it is now clear that $(\rho^n, u^n, \Pi^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $E_\eta^{p,q,s}$.

Note that the time of existence η (that we shall denote by T from now on) depends (continuously) on the norms of the data, on the lower bound for the density, on the domain and on the regularity parameters.

Fourth step: Checking that the limit is a solution. Let $(\rho, u, \Pi) \in E_T^{p,q,s}$ be the limit of the sequence $(\rho^n, u^n, \Pi^n)_{n\in\mathbb{N}}$.

Passing to the limit in (73) and (65) shows that the density ρ is bounded by below by $\check{\rho}$, and by above by $\widehat{\rho}$, and that $\rho \in L^\infty(0,T;W^{1,q})$, $u \in L^\infty(0,T;D_{A_r}^{\frac{1}{p'},p})\cap L^p(0,T;W^{2,r})$, $\partial_t u \in L^p(0,T;L^r)$ and $\Pi \in L^p(0,T;W^{1,p})$. Combining with the properties of convergence stated in the previous part of the proof, we gather that $(\rho^n, u^n, \Pi^n)_{n\in\mathbb{N}}$ converges to (ρ, u, Π) in $E_\eta^{p,q',r'}$ for all $q' < q$ and $r' < r$, which suffices to pass to the limit in equations (63) and (62). The details are left to the reader.

Last step: Uniqueness and continuity. Since $\rho \in L^\infty\big(0,T;\ W^{1,q}\big)\cap W^{1,\infty}(0,T;L^s)$ implies that ρ belongs to $C^\beta(0,T;L^\infty)$ whenever $\beta \in \Big(0, \frac{1-\frac{N}{q}}{1+\frac{N}{r}}\Big)$, uniqueness is a mere consequence of Proposition 4.2.

Finally, as ρ satisfies a transport equation with data in $W^{1,q}$ and u satisfies

$$\check{\rho}\partial_t u - \mu\Delta u + \nabla\Pi \in L^p(0,T;L^r),$$

Proposition 3.1 and Theorem 3.7 insure that $\rho \in C([0,T];W^{1,q})$ (if $q \neq \infty$) and $u \in C([0,T];D_{A_r}^{1-\frac{1}{p},p})$. $\qquad\square$

6. Global existence for small initial velocities

6.1. An estimate for $\|u\|_{L^2}$

Lemma 6.1. *Let p, q, r satisfy the usual conditions and let $(\rho,u,\Pi) \in E_T^{p,q,r}$ be a solution to (1) on $[0,T] \times \Omega$. Then the following inequality holds true for all $t \in [0,T]$:*

$$\|(\sqrt{\rho}u)(t)\|_{L^2} \leq e^{-\frac{\mu\lambda_1}{\hat{\rho}}t}\left(\|\sqrt{\rho_0}u_0\|_{L^2} + \int_0^t e^{\frac{\mu\lambda_1}{\hat{\rho}}\tau}\|(\sqrt{\rho}f)(\tau)\|_{L^2}\,d\tau\right), \qquad (79)$$

where $\hat{\rho} \overset{\text{def}}{=} \|\rho_0\|_{L^\infty}$ and λ_1 stands for the first eigenvalue of the Dirichlet Laplace operator in Ω.

Proof. Note that ρ is continuous in (t,x), and that $u \in C\left([0,T];\ D_{A_r}^{\frac{1}{p'},p}\right) \cap L^p(0,T;W^{2,r})$ with $r > N \geq 2$ so that $u \in C([0,T];H^\epsilon) \cap L^2(0,T;H^{1+\epsilon})$ for some positive ϵ. This enables us to justify the following computations.

Taking the L^2 scalar product of the momentum equation in (1) with u and performing integrations by parts when necessary, we gather

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}u\|_{L^2}^2 + \mu\|\nabla u\|_{L^2}^2 = \int_\Omega \rho f \cdot u\,dx.$$

Now, by virtue of Poincaré inequality, we have

$$\|\nabla u\|_{L^2}^2 \geq \lambda_1\|u\|_{L^2}^2,$$

hence,

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}u\|_{L^2}^2 + \frac{\mu\lambda_1}{\hat{\rho}}\|\sqrt{\rho}u\|_{L^2}^2 \leq \|\sqrt{\rho}u\|_{L^2}\|\sqrt{\rho}f\|_{L^2}.$$

It is now easy to get (79). $\qquad\square$

6.2. A more explicit lower bound for the existence time

A lower bound for the existence time has already been obtained when proving Theorem 1.2 (see (72) and (78)). It is rather inexplicit though. In this section, we want to take advantage of (79) in order to get a more accurate lower bound.

Let us first clarify what we call a smooth solution:

Definition 6.2. Let $T^* \in (0, +\infty]$ and (ρ, u, Π) be a solution to (1) on $[0, T^*) \times \Omega$ with data (ρ_0, u_0, f). The triplet (ρ, u, Π) is called a smooth solution of (1) on $[0, T^*)$ if it satisfies (1) on $[0, T^*) \times \Omega$ in the weak sense and belongs to $E_T^{p,q,r}$ whenever $T < T^*$. The time T^* is called 'maximal existence time' if (ρ, u, Π) cannot be continued beyond T^* into a smooth solution of (1).

Let us first state a continuation criterion:

Lemma 6.3. *Let ρ_0, u_0 and f satisfy the assumptions of Theorem 1.2 and assume that* (1) *has a smooth solution on a finite time interval $[0, T^*)$ with besides,*

$$\rho \in L^\infty(0, T^*; W^{1,q}), \quad \inf_{t < T^*, \, x \in \Omega} \rho(t, x) > 0 \quad and \quad u \in L^\infty(0, T^*, D_{A_r}^{\frac{1}{p'}, p}).$$

Then (ρ, u, Π) may be continued beyond T^* into a smooth solution of (1).

Proof. Obviously, the existence time given by (72) and (78) has a positive lower bound $\underline{\eta}$ when (ρ_0, u_0, f) remain in a bounded set of

$$W^{1,q} \times D_{A_r}^{\frac{1}{p'}, p} \times L^p(0, T; L^r)^N$$

with in addition $\inf_x \rho_0(x) \geq \check{\rho}$ for a fixed $\check{\rho} > 0$.

Hence system (1) with initial density $\rho(T^* - \underline{\eta}/2)$, initial velocity $u(T^* - \underline{\eta}/2)$ and external force $f(\cdot + T^* - \underline{\eta}/2)$ has a unique smooth solution on $[0, \underline{\eta}]$. This provides us with a continuation of u beyond T^*. $\qquad\square$

Combining Lemmas 6.1 and 6.3 will enable us to get the following result:

Proposition 6.4. *Let ρ_0, u_0 and f satisfy the assumptions of Theorem 1.2 and let (ρ, u, Π) denote the corresponding smooth solution of* (1). *There exists $c = c(p, q, r, \mu, \Omega, \check{\rho})$ such that the maximal existence time T^* for (ρ, u, Π) satisfies*

$$T^* \geq \frac{c}{(1 + \|\rho_0\|_{W^{1,q}})^\gamma \, U_0^\delta(T^*)}$$

for some positive exponents γ and δ depending only on the regularity parameters, and

$$U_0(T^*) \stackrel{\text{def}}{=} \|u_0\|_{D_{A_r}^{\frac{1}{p'}, p}} + \|f\|_{L_{T^*}^p(L^r)}.$$

Proof. Fix a $T < T^*$. We aim at proving that if $T \leq c(1 + \|\rho_0\|_{W^{1,q}})^{-\gamma} U_0^{-\delta}(T)$ for a convenient choice of c, γ and δ then (ρ, u, Π) may be bounded in $E_T^{p,q,r}$ by a function depending only on the data. Then Proposition 6.3 will entail Proposition 6.4.

Let $U(t) \stackrel{\text{def}}{=} \|u\|_{L_t^p(W^{2,r})} + \|\partial_t u\|_{L_t^p(L^r)} + \|u\|_{L_t^\infty(D_{A_r}^{\frac{1}{p'}, p})} + \|\Pi\|_{L_t^p(W^{1,r})}$. Accord-

ing to Theorem 3.7 and (67), we have

$$U(t) \lesssim \mathcal{B}_\rho^2(t) \left(U_0(t) + \|u \cdot \nabla u\|_{L_t^p(L^r)} \right) + \mathcal{C}_\rho(t) \|u\|_{L_t^p(L^r)}. \tag{80}$$

In the definition of $\mathcal{C}_\rho$, we can take e.g. $\beta = \frac{1}{2}\left(\frac{1 - N/q}{1 + N/r}\right)$.

Arguing once again by interpolation, we have for all $\epsilon > 0$

$$\|u\|_{L^r} \lesssim \epsilon \|u\|_{W^{2,r}} + \epsilon^{1 - \frac{1}{\theta}} \|u\|_{L^2} \quad \text{with} \quad \theta \stackrel{\text{def}}{=} \frac{2}{2 + \frac{N}{2} - \frac{N}{r}}. \tag{81}$$

Hence, taking $\epsilon = c\,\mathcal{C}_\rho^{-1}(t)$ with c suitably small and plugging (81) in (80), we get

$$U(t) \lesssim \mathcal{B}_\rho^2(t) \left(U_0(t) + \|u \cdot \nabla u\|_{L_t^p(L^r)} \right) + \mathcal{C}_\rho^{\frac{1}{\theta}}(t) \|u\|_{L_t^p(L^2)}. \tag{82}$$

Note that Lemma 6.1 insures that

$$\|u\|_{L_t^\infty(L^2)} \lesssim U_0(t). \tag{83}$$

For the sake of simplicity, assume from now on that $\frac{p}{2} - \frac{Np}{2r} < 1$ so that Lemma 4.1 may be applied.[6] We then gather that

$$\|u \cdot \nabla u\|_{L_t^p(L^r)} \lesssim t^{\frac{1}{2} - \frac{N}{2r}} U^2(t) \quad \text{and} \quad \|\nabla u\|_{L_t^1(L^\infty)} \lesssim t^{\frac{1}{p'} + \frac{1}{2} - \frac{N}{2r}} U(t) \tag{84}$$

On the other hand, Proposition 3.1 yields

$$\|\rho(t)\|_{W^{1,q}} \le \|\rho_0\|_{W^{1,q}} e^{\int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau} \tag{85}$$

so that according to (84) and to the definition of $\mathcal{B}_\rho$,

$$\mathcal{B}_\rho(t) \le C e^{C t^{\frac{1}{p'} + \frac{1}{2} - \frac{N}{2r}} U(t)} \left(1 + \|\rho_0\|_{W^{1,q}} \right)^{\frac{q}{q - N}}. \tag{86}$$

Arguing like in (67), we get

$$\begin{aligned}
\|\rho\|_{\dot{C}_t^\beta(L^\infty)} &\lesssim \|\rho\|_{L_t^\infty(W^{1,q})} + \|\partial_t \rho\|_{L_t^\infty(L^s)}, \\
&\lesssim \|\rho\|_{L_t^\infty(W^{1,q})} + \|\nabla \rho\|_{L_t^\infty(L^q)} \|u\|_{L_t^\infty(L^r)},
\end{aligned}$$

hence taking advantage of (86), we get for some positive exponents γ_1 and γ_2 depending only on N, p, q, r and β,

$$\mathcal{C}_\rho(t) \le C e^{C t^{\frac{1}{p'} + \frac{1}{2} - \frac{N}{2r}} U(t)} \left((1 + \|\rho_0\|_{W^{1,q}})^{\gamma_1} + (1 + \|\rho_0\|_{W^{1,q}})^{\gamma_2} \|\rho_0\|_{W^{1,q}}^{\frac{1}{\beta}} (1 + U(t))^{\frac{1}{\beta}} \right). \tag{87}$$

Plugging (83,86,87) in (82), we conclude that for some positive exponents δ_1 and δ_2, we have

$$U(t) \le C e^{C t^{\frac{1}{p'} + \frac{1}{2} - \frac{N}{2r}} U(t)} \left(1 + \|\rho_0\|_{W^{1,q}} \right)^{\delta_1} \left(U_0(t) \left(1 + t^{\frac{1}{p}} (1 + U(t))^{\delta_2} \right) + t^{\frac{1}{2} - \frac{N}{2r}} U^2(t) \right). \tag{88}$$

---

[6] If this condition is not satisfied, replace everywhere $\frac{1}{2} - \frac{N}{2r}$ with $\frac{1}{p}$ in the following computations.

Assume that T has been chosen such that

$$U(T) \le 8C \left(1 + \|\rho_0\|_{W^{1,q}} \right)^{\delta_1} U_0(T). \tag{89}$$

Using the continuity of the function $t \mapsto U(t)$, a standard induction argument shows that (89) is satisfied at time $t \le T$ with a strict inequality whenever the following three inequalities are satisfied:

$$\left(1 + 8C(1 + \|\rho_0\|_{W^{1,q}})^{\delta_1} U_0(T) \right)^{\delta_2} t^{\frac{1}{p}} \le 1,$$

$$8C^2 t^{\frac{1}{p'} + \frac{1}{2} - \frac{N}{2r}} (1 + \|\rho_0\|_{W^{1,q}})^{\delta_1} U_0(T) < \log 2,$$

$$8C(1 + \|\rho_0\|_{W^{1,q}})^{\delta_1} t^{\frac{1}{2} - \frac{N}{2r}} U_0(T) \le \frac{1}{2}.$$

Hence Proposition 6.3 enables us to continue the solution beyond T. This completes the proof of Proposition 6.4. □

6.3. The case of a small initial velocity

Proposition 6.4 insures that the existence time of a smooth solution for (1) goes to infinity (for fixed initial density) when u_0 (resp. f) tends to 0 in $D_{A_r}^{\frac{1}{p'},p}$ (resp. $L^p(\mathbb{R}^+; L^r)$).

We here aim at stating that (1) has indeed a global smooth solution if u_0 and f are suitably small. This will give Theorem 1.4.

Let (ρ, u, Π) be the smooth solution given by Theorem 1.2. Before going into the heart of the proof, let us stress the fact that it suffices to prove Theorem 1.4 when the viscosity coefficient $\mu = 1$. Indeed, the following change of functions and variables:

$$\rho(t, x) = \rho'(\mu t, x), \quad u(t, x) = \mu v(\mu t, x),$$

$$\Pi(t, x) = \mu^2 P(\mu t, x) \quad \text{and} \quad f(t, x) = \mu^2 g(\mu t, x)$$

transform system (1) with viscosity μ and data (ρ_0, u_0, f) into system (1) with viscosity μ and data (ρ_0, u_0, g) (this change of variable does not affect Ω).

Denote by T^* the maximal time of existence for (ρ, u, Π). Define the functions U and U_0 as in the previous section and further denote

$$U_{0,2,\gamma}(t) \stackrel{\text{def}}{=} \left\| \sqrt{\rho_0} u_0 \right\|_{L^2} + \int_0^t e^{\gamma \tau} \left\| (\sqrt{\rho} f)(\tau) \right\|_{L^2} d\tau \quad \text{and} \quad U_{0,2}(t) \stackrel{\text{def}}{=} U_{0,2,0}(t).$$

First, applying Lemma 6.1 yields for $t < T^*$,

$$\left\| \sqrt{\rho} u \right\|_{L_t^p(L^2)} \lesssim U_{0,2}(t) \quad \text{and} \quad \left\| (\sqrt{\rho} u)(t) \right\|_{L^2} \le e^{-\kappa t} U_{0,2,\gamma}(t) \tag{90}$$

with $\kappa \stackrel{\text{def}}{=} \min(\gamma, \lambda_1 / \widehat{\rho})$.

Hence, starting from inequality (80), using inequalities (81,90), and the fact that

$$\|u \cdot \nabla u\|_{L_t^p(L^r)} \le \|u\|_{L_t^\infty(L^r)} \|\nabla u\|_{L_t^p(L^\infty)},$$
$$\lesssim \|u\|_{L_t^\infty(D_{A_r}^{\frac{1}{p'},p})} \|u\|_{L_t^p(W^{2,r})},$$

we end up with

$$U(t) \le C\left(\mathcal{B}_\rho^2(t)\left(U_0(t) + U^2(t)\right) + \mathcal{C}_\rho^{\frac{1}{\theta}}(t)U_{0,2}(t) \right) \quad \text{with} \quad \theta \overset{\text{def}}{=} \frac{2}{2 + N/2 - N/r}. \tag{91}$$

Once again, the bounds for $\mathcal{B}_\rho$ and $\mathcal{C}_\rho$ will follow from inequality (85). In contrast with the previous section however, we are going to take advantage of Lemma 6.1 to avoid the appearance of the factor $t^{\frac{1}{p'}+\frac{1}{2}-\frac{N}{2r}}$.

Indeed, denoting $\varphi = (1+\frac{N}{2})/(2+\frac{N}{2}-\frac{N}{r})$ and using once more that $(L^2, W^{2,r})_\varphi \hookrightarrow W^{1,\infty}$, we gather:

$$\int_0^t \|\nabla u(\tau)\|_{L^\infty} \, d\tau \lesssim \int_0^t e^{-\kappa(1-\varphi)\tau} U_{0,2,\gamma}^{1-\varphi}(\tau) \|u(\tau)\|_{W^{2,r}}^\varphi \, d\tau,$$
$$\lesssim U_{0,2,\gamma}^{1-\varphi}(t) U^\varphi(t).$$

Now, bounding $\mathcal{B}_\rho$ and $\mathcal{C}_\rho$ may be done by mimicking the proof of Proposition 6.4 and we eventually conclude that

$$U(t) \le C(1 + \|\rho_0\|_{W^{1,q}})^{\delta_1} e^{CU_{0,2,\gamma}^{1-\varphi}(t)U^\varphi(t)} \left(U_0(t)(1 + U(t))^{\delta_2} + U^2(t) \right) \tag{92}$$

for some positive exponents δ_1 and δ_2 depending only on p, q, r and N.

Fix a positive T and assume that

$$\forall t \in [0, T], \, U(t) \le 8C(1 + \|\rho_0\|_{W^{1,q}})^{\delta_1} U_0(T) \tag{93}$$

If the data are so small as to satisfy

$$CU_{0,2,\gamma}^{1-\varphi}(T)\left(8C(1 + \|\rho_0\|_{W^{1,q}})^{\delta_1} U_0(T)\right)^\varphi \le \log 2 \tag{94}$$

then inequality (92) implies

$$U(t) \le 2C(1 + \|\rho_0\|_{W^{1,q}})^{\delta_1} \left(U_0(t)(1 + U(t))^{\delta_2} + U^2(t) \right).$$

Now, one can be easily convinced that if in addition

$$16C^2(1 + \|\rho_0\|_{W^{1,q}})^{2\delta_1} U_0(T) \le \frac{1}{3} \quad \text{and} \quad \left(1 + 8C(1 + \|\rho_0\|_{W^{1,q}})^{\delta_1} U_0(T)\right)^{\delta_2} \le 2,$$

then (93) is satisfied with the constant $6C$ instead of $8C$. A standard bootstrap argument enables to conclude to Theorem 1.4. $\qquad \square$

7. Global existence in dimension $N = 2$

In dimension $N = 2$, it is well-known that for all $T > 0$, the $L^\infty(0, T; H^1) \cap L^2(0, T; H^2)$ norm of the velocity u may be bounded by the data if u_0 belongs to

H^1 and $f \in L^2_{loc}(\mathbb{R}^+; L^2)$. This fact has been noticed by different authors (see [3], [9] [18] and [20]) and is quite straightforward (at least formally) if the density is bounded away from 0.

The following inequality may be easily stated in the case of a bounded domain Ω (see the proof in [6] in the case $\Omega = \mathbb{R}^2$ or $\mathbb{T}^2$):

Proposition 7.1. *Let v be divergence-free and satisfy $v \cdot n = 0$ on $\partial\Omega$ and let (ρ, u, Π) solve*

$$\begin{cases} \partial_t \rho + v \cdot \nabla \rho = 0, \\ \rho(\partial_t u + v \cdot \nabla u) - \mu \Delta u + \nabla \Pi = \rho f, \\ \operatorname{div} u = 0, \end{cases}$$

for some divergence-free time-dependent vector field v. There exists a universal constant C such that the following a priori estimate holds true:

$$\|\nabla u(t)\|_{L^2}^2 + \int_0^t \left(\frac{\|\sqrt{\rho}\partial_t u\|_{L^2}^2}{\mu} + \frac{\|\nabla\Pi\|_{L^2}^2}{\mu \|\rho_0\|_{L^\infty}} + \mu \frac{\|\nabla^2 u\|_{L^2}^2}{\|\rho_0\|_{L^\infty}} \right) d\tau$$
$$\leq e^{\frac{C\|\rho_0\|_{L^\infty}}{\mu^3} \int_0^t \|\sqrt{\rho}v\|_{L^4}^4 d\tau} \left(\|\nabla u_0\|_{L^2}^2 + C \int_0^t \frac{\|\sqrt{\rho}f\|_{L^2}^2}{\mu} d\tau \right).$$

Gagliardo–Nirenberg inequality enables us to bound $\|\sqrt{\rho}v\|_{L^4}$. From it, we get

$$\frac{\|\rho_0\|_{L^\infty}}{\mu^3} \int_0^t \|\sqrt{\rho}v\|_{L^4}^4 d\tau \leq \frac{C\|\rho_0\|_{L^\infty}^3}{\check{\rho}\mu^4} \left(\mu\|\nabla v\|_{L^2_t(L^2)}^2 \right) \left(\|\sqrt{\rho}v\|_{L^\infty_t(L^2)}^2 \right).$$

Now, if (ρ, u) solves (1), the basic energy inequality (2) is satisfied so that the above inequalities eventually yield

$$\|\nabla u(t)\|_{L^2}^2 + \int_0^t \left(\frac{\|\sqrt{\rho}\partial_t u\|_{L^2}^2}{\mu} + \frac{\|\nabla\Pi\|_{L^2}^2}{\mu \|\rho_0\|_{L^\infty}} + \mu \frac{\|\nabla^2 u\|_{L^2}^2}{\|\rho_0\|_{L^\infty}} \right) d\tau$$
$$\leq e^{\frac{C\|\rho_0\|_{L^\infty}^3}{\check{\rho}\mu^4} \left(\|\sqrt{\rho_0}u_0\|_{L^2}^2 + \int_0^t \|\sqrt{\rho}f\|_{L^2}^2 d\tau \right)^2} \left(\|\nabla u_0\|_{L^2}^2 + C \int_0^t \frac{\|\sqrt{\rho}f\|_{L^2}^2}{\mu} d\tau \right). \quad (95)$$

Let us now turn to the proof of Theorem 1.5. Theorem 1.2 provides us with a local smooth solution (ρ, u, Π). Let T^* denote its maximal existence time.

7.1. The case of smooth data

We first assume that (ρ, u, Π) belongs to $\cap_{T<T^*} E_T^{p,q,r}$ for some $p \geq 2$. Hence in particular $u_0 \in H^1$ and $f \in L^2_{loc}(\mathbb{R}^+; L^2)$ so that Proposition 7.1 applies.

Existence of a global H^1 solution. Taking advantage of the formal inequality given by Proposition 7.1, one can prove that whenever $\rho_0 \in L^\infty$ is bounded away

from vacuum, $u_0 \in H^1$ and $f \in L^2_{loc}(\mathbb{R}^+; L^2)$, system (1) has a global solution $(\widetilde{\rho}, \widetilde{u}, \widetilde{\Pi})$ in

$$L^\infty(\mathbb{R}^+; L^\infty) \times \left(L^\infty_{loc}(\mathbb{R}^+; H^1) \cap W^{1,2}(0, T; L^2) \cap L^2_{loc}(\mathbb{R}^+; H^2) \right)^N \times L^2_{loc}(\mathbb{R}^+; H^1)$$

which besides satisfies the energy inequality (2) and the inequality given in Proposition 7.1 (see e.g. [3]). Note that the question of uniqueness in the above class has remained unsolved. On the other hand, as the density $\widetilde{\rho}$ satisfies

$$\begin{cases} \partial_t \widetilde{\rho} + \widetilde{u} \cdot \nabla \widetilde{\rho} = 0, \\ \widetilde{\rho}_{|t=0} = \rho_0 \in W^{1,q}, \end{cases}$$

with $u \in L^2_{loc}(\mathbb{R}^+; H^2)$, Theorem 1 in [8] insures that $\rho \in C(\mathbb{R}^+; W^{1,q^-})$ for all $q^- < q$.

Weak-strong uniqueness. Therefore, we are now given two solutions for (1) with the same data. The smooth one, (ρ, u, Π) belongs to $E^{p,q,r}_T$ whenever $T < T^*$ whereas the second one $(\widetilde{\rho}, \widetilde{u}, \widetilde{\Pi})$ is global and satisfies for all $T > 0$,

$$(\widetilde{\rho}, \widetilde{u}, \widetilde{\Pi}) \in \cap_{q^- < q} E^{2,q^-,2}_T.$$

Besides, both ρ and $\widetilde{\rho}$ are bounded away from 0.

With no loss of generality, one can assume that $2 > rq/(r+q)$ (indeed, if it is not the case, one can always take smaller $q \geq r > 2$). Hence one can find a $q^- \in (2, q)$ such that both solutions belong to E^{2,q^-,s^-}_T with $s^- \stackrel{\text{def}}{=} rq^-/(r+q^-)$.

Since in addition $\partial_t \widetilde{\rho} \in L^\infty_{loc}(0, T^*; L^{s^-})$, the same interpolation argument as in (67) shows that $\widetilde{\rho} \in C^\beta(0, T^*; L^\infty)$ for some $\beta > 0$. Now, as obviously $\nabla u \in L^p(0, T; L^\infty)$, $\partial_t u + u \cdot \nabla u \in L^p(0, T; L^r)$, and $\widetilde{u} \in L^\infty(0, T; L^r)$ for all $0 < T < T^*$, Proposition 4.2 insures that $(\widetilde{\rho}, \widetilde{u}, \widetilde{\Pi}) \equiv (\rho, u, \Pi)$ on $0, T^* \times \Omega$.

Showing that (ρ, u, Π) is global. Assume that the maximal existence time T^* for (ρ, u, Π) is finite. According to the previous step of the proof, $\rho \equiv \widetilde{\rho}$ on $[0, T^*)$ so that in particular

$$\rho \in C_b([0, T^*); W^{1,q^-}) \cap C^\beta_b([0, T^*); L^\infty) \quad \text{for all} \quad q^- < q \quad \text{and some} \quad \beta > 0. \tag{96}$$

Now, if $q > r$, Theorem 3.7 may be applied. From it, we gather that $u \in L^\infty(0, T^*; D^{\frac{1}{p'}, p}_{A_r}) \cap L^p(0, T^*; W^{2,r})$. This entails that $\nabla u \in L^1(0, T^*; L^\infty)$. Hence Proposition 3.1 shows that no loss of integrability for ρ occurs: $\rho \in L^\infty(0, T^*; W^{1,q})$. Then Proposition 6.3 shows that T^* cannot be finite.

In the limit case $q = r$, repeating the above argument for some $r- \in (N, r)$, we get $u \in L^\infty(0, T^*; D^{\frac{1}{p'}, p}_{A_{r^-}}) \cap L^p(0, T^*; W^{2,r^-})$. We thus get $\nabla u \in L^1(0, T^*; L^\infty)$, whence $\rho \in L^\infty(0, T^*; W^{1,q})$. One can now apply Proposition 3.7 with r instead of r^- and conclude as before.

7.2. The case of rough data

We now assume that $\rho_0 \in W^{1,q}$ has a positive lower bound, that $u_0 \in D_{A_r}^{\frac{1}{p'},p}$ and that $f \in L^p(0,T;L^r) \cap L^2_{loc}(0,T;L^2)$.

Let (ρ, u, Π) be the maximal smooth solution given by Theorem 1.2 and denote by T^* the existence time. In order to show that $T^* = +\infty$, we are going to proceed like in [6].

As $u \in L^p_{loc}(0,T^*;W^{2,r})$, one can find some $t_0 \in (0,T^*)$ such that $u(t_0) \in W^{2,r}$. Hence $u(t_0)$ also belongs to H^1.

According to the previous solution, one can find a unique smooth global solution $(\widetilde{\rho}, \widetilde{u}, \widetilde{\Pi})$ for (1) with data $\rho(t_0)$, $u(t_0)$ and $f(\cdot + t_0)$.

On the other hand, the weak-strong uniqueness result proved above does not use the fact that $p \geq 2$. Hence $(\widetilde{\rho}, \widetilde{u}, \widetilde{\Pi})$ is a global smooth continuation of (ρ, u, Π).

$\square$

Appendix

A. Poincaré–Wirtinger type inequalities

Let us first state the Poincaré–Wirtinger inequality in a bounded domain Ω, star-shaped with respect to a convex set $C \subset \Omega$. We aim at giving an estimate of the constant in terms of Ω and C.

Lemma A.1. *For all $p \in [1, +\infty]$, the following inequality holds true:*

$$\left\| f - \overline{f}_C \right\|_{L^p(\Omega)} \leq \left(\frac{|\mathbb{S}^{N-1}| \, \delta(C)\delta(\Omega)^{N-1}}{|C|} \right)^{\frac{1}{p}} \delta(\Omega) \left\| \nabla f \right\|_{L^p(\Omega)}, \tag{97}$$

where $\overline{f}_C$ denotes the average of f over the convex set C and $\mathbb{S}^{N-1}$, the unit sphere in $\mathbb{R}^N$.

Proof. It is inspired by [4], page 104.

Fix a $x \in \Omega$. We have

$$f(x) - \overline{f}_C = \frac{1}{|C|} \int_C \int_0^1 \nabla f((1-t)\widetilde{x} + tx) \cdot (x - \widetilde{x}) \, dt \, d\widetilde{x}, \tag{98}$$

which yields (97) in the case $p = +\infty$.

For proving the inequality in the case $p = 1$, make the change of variables $y = \widetilde{x} + t(x - \widetilde{x})$ and $\rho = (1-t)^{-1}|x - y|$ in (98). We get

$$f(x) - \overline{f}_C = \int_{V_x} \frac{x-y}{|x-y|} \frac{\nabla f(y)}{|x-y|^{N-1}} \left(\int_{|x-y|}^{+\infty} \rho^{N-1} \omega \left(x + \rho \left(\frac{y-x}{|y-x|} \right) \right) d\rho \right) dy, \tag{99}$$

where V_x denotes the convex hull of $\{x\} \cup C$, and $\omega \overset{\text{def}}{=} 1_C/|C|$.

Note that for fixed $y \in V_x$, the integration is actually restricted to the intersection of the half-line $[x, y]$ and C, which is a segment of length less than $\delta(C)$. Besides, as $V_x \subset \Omega$, the integration may be restricted to $\rho < \delta(\Omega)$.

Therefore

$$\int_{|x-y|}^{+\infty} \rho^{N-1} \omega\left(x + \rho\left(\frac{y-x}{|y-x|}\right)\right) d\rho \leq \frac{\delta(C)\delta(\Omega)^{N-1}}{|C|}. \tag{100}$$

Plugging (100) in (99) yields

$$|f(x) - \overline{f}_C| \leq \frac{\delta(C)\delta(\Omega)^{N-1}}{|C|} \left(\int_{V_x} \frac{|\nabla f(y)|}{|x-y|^{N-1}} \, dy\right). \tag{101}$$

Integrating over Ω and using a convolution inequality yields the desired result in the case $p = 1$. Interpolation then entails the general case $p \in [1, +\infty]$. $\qquad\square$

Remark A.2. 1. The first term of the left-hand side of (97) is a measurement of the distortion of Ω with respect to C. If C is a ball, it reduces to $(\delta(\Omega)/\delta(C))^{\frac{N-1}{p}}$.

2. If Ω itself is convex, one can improve the inequality:

$$\left\|f - \overline{f}_\Omega\right\|_{L^p(\Omega)} \leq C_N \delta(\Omega) \left\|\nabla f\right\|_{L^p(\Omega)},$$

with C_N depending only on the dimension N.

Let us mention in passing the following variation on Poincaré–Wirtinger inequality.

Lemma A.3. *Let Ω and C satisfy the above assumptions and let $1 < p < N$. The following inequality holds true:*

$$\left\|f - \overline{f}_C\right\|_{L^{\frac{Np}{N-p}}(\Omega)} \leq C_{p,N} \left(\frac{\delta(C)\delta(\Omega)^{N-1}}{|C|}\right) \left\|\nabla f\right\|_{L^p(\Omega)}. \tag{102}$$

Proof. Starting from inequality (101), the desired inequality easily stems from Hardy–Littlewood–Sobolev inequality. $\qquad\square$

B. Maximal regularity for abstract evolution equations

Let X be a Banach space and A, a non-bounded linear operator in X with domain $D(A)$. We here want to review a few results on the following abstract evolution equation:

$$\begin{cases} \dfrac{d}{dt}u + Au = f \in L^s(0, T; X), \\ u_{|t=0} = u_0 \in X. \end{cases} \tag{L}$$

Following Y. Giga and H. Sohr in [15], we make the following assumptions on X and A:

- $(\mathcal{H}_1)$ X is a ζ-convex Banach space,
- $(\mathcal{H}_2)$ A is a closed nonnegative linear operator in X belonging to $E_K^\theta(X)$ for some $K \geq 1$ and $\theta \geq 0$, namely

  - Both the range and the domain of A are dense in X,
  - The operator $t + A$ is invertible for $t > 0$, and $\sup_{t>0} \|t(t+A)^{-1}\|_X < \infty$,
  - $\forall y \in \mathbb{R}$, $\|A^{iy}\|_{\mathcal{L}(X)} \leq Ke^{\theta|y|}$.

Remark B.1. Under assumption $(\mathcal{H}_2)$ with $\theta \in [0, \pi/2)$, the operator $-A$ generates a bounded analytic semi-group $\left(e^{-tA}\right)_{t \geq 0}$.

Definition B.2. For $\alpha \in (0,1)$ and $s \in (1,\infty)$, set

$$\|u\|_{\dot{D}_A^{\alpha,s}} \stackrel{\text{def}}{=} \left(\int_0^{+\infty} \|t^{1-\alpha} Ae^{-tA}u\|_X^s \frac{dt}{t} \right)^{\frac{1}{s}}.$$

We then define the homogeneous fractional domains $\dot{D}_A^{\alpha,s}$ as the completion of $D(A)$ under $\|u\|_{\dot{D}_A^{\alpha,s}}$.

Remark B.3. 1. Let $\dot{D}(A)$ be the completion of $D(A)$ in X under $\|Au\|_X$. One can show that $\dot{D}_A^{\alpha,s}$ agrees with $(X, \dot{D}(A))_{\alpha,s}$.

2. One can also define non-homogeneous fractional domains $D_A^{\alpha,s}$ as the completion of $D(A)$ under the following norm:

$$\|u\|_{D_A^{\alpha,s}} \stackrel{\text{def}}{=} \|u\|_X + \left(\int_0^{+\infty} \|t^{1-\alpha} Ae^{-tA}u\|_X^s \frac{dt}{t} \right)^{\frac{1}{s}}.$$

And of course, $D_A^{\alpha,s}$ agrees with $(X, D(A))_{\alpha,s}$.

The main result of this section is the following:

Theorem B.4. *Let X and A satisfy assumptions $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ for some $\theta \in \left[0, \frac{\pi}{2}\right)$. Let $s \in (1,\infty)$, $f \in L^s(\mathbb{R}^+; X)$ and $u_0 \in \dot{D}_A^{1-\frac{1}{s},s}$. The abstract evolution problem (L) has a unique solution u in $L^s(\mathbb{R}^+; \dot{D}(A)) \cap C_b(\mathbb{R}^+; \dot{D}_A^{1-\frac{1}{s},s})$ with $\partial_t u \in L^s(\mathbb{R}^+; X)$. Moreover, there exists a constant $C = C(s, \theta, K, X)$ such that the following inequality holds true for all $T \geq 0$:*

$$\left\| \left(\frac{du}{dt}, Au \right) \right\|_{L_T^s(X)} + \|u(T)\|_{\dot{D}_A^{1-\frac{1}{s},s}} \leq C \left(\|u_0\|_{\dot{D}_A^{1-\frac{1}{s},s}} + \|f\|_{L_T^s(X)} \right).$$

Proof. As $-A$ generates a bounded analytic semi-group $\left(e^{-tA}\right)_{t\geq 0}$, the solution u to (L) writes $u = v + w$ with

$$v(t) \stackrel{\text{def}}{=} \int_0^t e^{-(t-\tau)A} f(\tau)\, d\tau \quad \text{and} \quad w(t) \stackrel{\text{def}}{=} e^{-tA} u_0.$$

First step: Maximal regularity for the Duhamel term v. Under assumptions $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$, the operator A has the so-called "maximal regularity" property (see e.g. [15]), namely

$$\frac{d}{dt}v + Av = f, \quad v_{|t=0} = 0,$$

implies

$$\left\|\left(\frac{dv}{dt}, Av\right)\right\|_{L_T^s(X)} \lesssim \|f\|_{L_T^s(X)}.$$

Second step: Additional estimates for v. Straightforward computations yield

$$
\begin{aligned}
\|v(T)\|_{\dot{D}_A^{1-\frac{1}{s},s}} &= \left(\int_0^{+\infty} \|Ae^{-tA}v(T)\|_X^s\, dt\right)^{\frac{1}{s}}, \\
&= \left(\int_0^{+\infty} \left\|\int_0^T Ae^{-(t+T-\tau)A} f(\tau)\, d\tau\right\|_X^s dt\right)^{\frac{1}{s}}, \\
&= \left(\int_T^{+\infty} \left\|\int_0^T Ae^{-(t'-\tau)A} f(\tau) d\tau\right\|_X^s dt'\right)^{\frac{1}{s}} \\
&\leq \left(\int_0^{+\infty} \left\|\int_0^{t'} Ae^{-(t'-\tau)A} f(\tau) 1_{[0,T]}(\tau)\, d\tau\right\|_X^s dt'\right)^{\frac{1}{s}}.
\end{aligned}
$$

Hence, taking advantage of the maximal regularity of the operator A,

$$\|v(T)\|_{\dot{D}_A^{1-\frac{1}{s},s}} \lesssim \|f 1_{[0,T]}\|_{L^s(X)} = \|f\|_{L_T^s(X)}. \tag{103}$$

Third step: Estimates for w. Because $w(T) = e^{-TA} u_0$ and $\left(e^{-tA}\right)_{t\geq 0}$ is a bounded semi-group, we have

$$
\begin{aligned}
\|w(T)\|_{\dot{D}_A^{1-\frac{1}{s},s}} &= \left(\int_0^{+\infty} \|e^{-TA} Ae^{-tA} u_0\|_X^s\, dt\right)^{\frac{1}{s}}, \\
&\leq C\left(\int_0^{+\infty} \|Ae^{-tA} u_0\|_X^s\, dt\right)^{\frac{1}{s}} = C\|u_0\|_{\dot{D}_A^{1-\frac{1}{s},s}}.
\end{aligned}
$$

On the other hand,

$$\|Aw\|_{L_T^s(X)} = \left(\int_0^T \|Ae^{-tA} u_0\|_X^s\, dt\right)^{\frac{1}{s}} \leq \left(\int_0^{+\infty} \|Ae^{-tA} u_0\|_X^s\, dt\right)^{\frac{1}{s}} = \|u_0\|_{\dot{D}_A^{1-\frac{1}{s},s}}.$$

Last step: Time continuity for v and w. Fix a positive T. Let $(f^n)_{n\in\mathbb{N}}$ be a sequence of functions in $C([0,T];X)\cap L^s(0,T;X)$ which tends to f in $L^s(0,T;X)$. Denote $v^n(t) = \int_0^t e^{-(t-\tau)A}f^n(\tau)\,d\tau$. According to the previous estimates, v^n tends to v in $L^\infty(0,T;\dot{D}_A^{1-\frac{1}{s},s})$. On the other hand, f^n belongs to e.g. $L^{2s}(0,T;X)$, hence by maximal regularity, $\partial_t v^n$ is in $L^{2s}(0,T;X)$, and $v^n \in L^\infty(0,T;\dot{D}_A^{1-\frac{1}{2s},2s})$. Interpolation thus yields $v^n \in C([0,T];\dot{D}_A^{1-\frac{1}{s},s})$. As $(v^n)_{n\in\mathbb{N}}$ converges to v in $L^\infty(0,T;\dot{D}_A^{1-\frac{1}{s},s})$, we conclude that $v \in C([0,T];\dot{D}_A^{1-\frac{1}{s},s})$.

Since $D(A)$ is dense in $\dot{D}_A^{1-\frac{1}{s},s}$, one can find a sequence $(u_0^n)_{n\in\mathbb{N}} \in D(A)^{\mathbb{N}}$ tending to u_0 in $\dot{D}_A^{1-\frac{1}{s},s}$. Denoting $w^n(t) = e^{-tA}u_0^n$, the estimate of step three insures that w^n tends to w in $L^\infty(\mathbb{R}^+;\dot{D}_A^{1-\frac{1}{s},s})$.

Using the fact that u_0^n belongs to $D(A)$ we gather that $w^n \in L^\infty(\mathbb{R}^+;\dot{D}_A^{1-\frac{1}{s'},s'})$ and that $\partial_t w^n \in L^{s'}(\mathbb{R}^+;X)$ for all $s' \geq s$. Hence by interpolation, $w^n \in C(\mathbb{R}^+;\dot{D}_A^{1-\frac{1}{s},s})$ which completes the proof. $\qquad\square$

C. An elliptic equation with Neumann boundary conditions

In this section, we state an estimate for the following elliptic problem:

$$\begin{cases} \text{div}(\tau\nabla u) = h, \\ \partial_n u_{|\partial\Omega} = 0, \end{cases} \tag{104}$$

where h satisfies the compatibility condition $\int_\Omega h(x)\,dx = 0$.

Proposition C.1. *Let $r \in (N, +\infty]$ and $q \in (1, r] \cap \mathbb{R}$. Let Ω be a C^2 bounded domain of $\mathbb{R}^N$. Assume that $h \in L^q(\Omega)$ and that $\tau \in W^{1,r}(\Omega)$ satisfies $\check{\tau} \stackrel{\text{def}}{=} \inf_{x\in\Omega}\tau(x) > 0$. Then (104) has a solution $u \in W^{2,q}(\Omega)$ such that*

$$\check{\tau}\left\|\nabla^2 u\right\|_{L^q} \leq C_{N,r,q,\sigma(\Omega)}\|h\|_{L^q}\left(1 + |\Omega|^{\frac{2}{N}}\|\nabla\log\tau\|_{L^r}^{\frac{2r}{r-N}}\right),$$

$$\check{\tau}\left\|\nabla u\right\|_{L^q} \leq C_{N,r,q,\sigma(\Omega)}|\Omega|^{\frac{1}{N}}\|h\|_{L^q}\left(1 + |\Omega|^{\frac{1}{N}}\|\nabla\log\tau\|_{L^r}^{\frac{r}{r-N}}\right).$$

Proof. Arguing by dilation, it suffices to prove the inequality in the case $|\Omega| = 1$.

The existence of a solution for (104) is stated in e.g. [2]. Of course, uniqueness in $W^{2,q}$ holds true up to a constant.

Remark that

$$\Delta u = \tau^{-1}h - \nabla u \cdot \nabla\log\tau,$$

hence, according to e.g. [17] page 105,

$$\left\|\nabla^2 u\right\|_{L^q} \lesssim \left\|\tau^{-1}h\right\|_{L^q} + \|\nabla u \cdot \nabla\log\tau\|_{L^q},$$
$$\lesssim \left\|\tau^{-1}h\right\|_{L^q} + \|\nabla u\|_{L^s}\|\nabla\log\tau\|_{L^r},$$

with $1/r + 1/s = 1/q$.

According to Gagliardo–Nirenberg inequality, we have

$$\|\nabla u\|_{L^s} \lesssim \|u\|_{L^q}^{\frac{1}{2} - \frac{N}{2}(\frac{1}{q} - \frac{1}{s})} \left(\|u\|_{L^q} + \|\nabla^2 u\|_{L^q} \right)^{\frac{1}{2} + \frac{N}{2}(\frac{1}{q} - \frac{1}{s})}.$$

Therefore, thanks to Young inequality

$$\|\nabla^2 u\|_{L^q} \lesssim \left(\|\tau^{-1} h\|_{L^q} + (1 + \|\nabla \log \tau\|_{L^r}^{\frac{2r}{r-N}}) \|u\|_{L^q} \right).$$

We are left with bounding $\|u\|_{L^q}$. Obviously, proving that

$$\check{\tau} \|u\|_{L^q} \lesssim \|h\|_{L^q} \tag{105}$$

yields the desired inequality for $\|\nabla^2 u\|_{L^q}$. Then, arguing by interpolation will give the inequality for $\|\nabla u\|_{L^q}$.

Inequality (105) will be achieved by prescribing appropriate mean for u (in that, we follow [18]). Of course, changing the mean of u amounts to adding constants so that ∇u and $\nabla^2 u$ are unchanged.

Case $q = 2$. Integration by parts yields

$$- \int_\Omega \tau |\nabla u|^2 \, dx = \int_\Omega h u \, dx. \tag{106}$$

Choose u with null mean value. By virtue of Poincaré–Wirtinger inequality, we conclude that

$$\check{\tau} \|u\|_{L^2}^2 \lesssim \check{\tau} \|\nabla u\|_{L^2}^2 \leq \|h\|_{L^2} \|u\|_{L^2}.$$

Case $2 \leq q \leq 2N/(N-2)$. We still prescribe null mean value for u. Gagliardo–Nirenberg inequality thus reduces to

$$\|u\|_{L^q} \lesssim \|\nabla u\|_{L^2}^{\frac{N}{2} - \frac{N}{q}} \|u\|_{L^2}^{1 - \frac{N}{2} + \frac{N}{q}}.$$

Now, by virtue of (106),

$$\sqrt{\check{\tau}} \|\nabla u\|_{L^2} \leq \sqrt{\|h\|_{L^2} \|u\|_{L^2}},$$

hence

$$\|u\|_{L^q} \lesssim \left(\frac{\|h\|_{L^2}}{\check{\tau}} \right)^{\frac{N}{4} - \frac{N}{2q}} \|u\|_{L^2}^{1 + \frac{N}{2q} - \frac{N}{4}}.$$

Hölder inequality enables us to replace L^2 norms with L^q norms in the right-hand side, which completes the proof.

Case $2N/(N-2) < q < +\infty$. Let $\ell = q(N-2)/(2N)$. As the function

$$\kappa \longmapsto \int_\Omega |u + \kappa|^{\ell-1}(u + \kappa) \, dx$$

is continuous and tends to $\pm\infty$ when κ goes to $\pm\infty$, one can find a solution u to (104) such that $v \stackrel{\text{def}}{=} |u|^{\ell-1}u$ has null mean on Ω.

On the other hand, integrating by parts yields

$$\int_\Omega \tau|\nabla v|^2\,dx = -\left(\frac{\ell^2}{2\ell-1}\right)\int_\Omega h|u|^{2\ell-1}\operatorname{sgn} u\,dx,$$

hence, in view of Sobolev embeddings and Hölder inequality,

$$\check\tau\,\|u\|_{L^q}^{2\ell} = \check\tau\,\|v\|_{L^{\frac{2N}{N-2}}}^2 \lesssim \check\tau\,\|\nabla v\|_{L^2}^2 \lesssim \|u\|_{L^q}^{2\ell-1}\,\|h\|_{L^{\frac{q}{q+1-2\ell}}} \le \|u\|_{L^q}^{2\ell-1}\,\|h\|_{L^q}\,.$$

Case $2N/(N+2) \le q \le 2$. Let us choose a solution u with null mean value. We have

$$\|u\|_{L^q} = \sup_{\substack{\|g\|_{L^{q'}} \le 1 \\ \int_\Omega g\,dx = 0}} \int_\Omega ug\,dx. \tag{107}$$

Let v be a solution to

$$\operatorname{div}(\tau\nabla v) = g, \qquad \int_\Omega v\,dx = 0 \qquad \text{and} \qquad \partial_n v_{|\partial\Omega} = 0.$$

As $2 \le q' \le 2N/(N-2)$, we already know that

$$\check\tau\,\|v\|_{L^{q'}} \lesssim \|g\|_{L^{q'}}\,.$$

As of course $\displaystyle\int_\Omega ug\,dx \le \|h\|_{L^q}\,\|v\|_{L^{q'}}$, one obtains the desired inequality.

Case $1 < q \le 2N/(N+2)$. Once again, the desired inequality stems from a duality argument. The solution u is chosen such that $\int_\Omega u\,dx = 0$ so that (107) still holds. Since now $q' > 2N/(N-2)$ the function v such that $\operatorname{div}(\tau\nabla v) = g$ has to be normalized in the following way:

$$\int_\Omega |v|^{\ell-1}v\,dx = 0 \quad \text{with} \quad \ell = q'(N-2)/2N. \qquad\qquad \square$$

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