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Compressible Flow in a Half-Space with Navier Boundary Conditions

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Abstract. We prove the global existence of weak solutions of the Navier–Stokes equations of compressible flow in a half-space with the boundary condition proposed by Navier: the velocity on the boundary is proportional to the tangential component of the stress. This boundary condition allows for the determination of the scalar function in the Helmholtz decomposition of the acceleration density, which in turn is crucial in obtaining pointwise bounds for the density. Initial data and solutions are small in energy-norm with nonnegative densities having arbitrarily large sup-norm. These results generalize previous results for solutions in the whole space and are the first for solutions in this intermediate regularity class in a region with a boundary.

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1. Introduction

We prove the global existence of weak solutions of the Navier–Stokes equations of compressible fluid flow

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0 \\ (\rho u^j)_t + \operatorname{div}(\rho u^j u) + P(\rho)_{x_j} = \mu \Delta u^j + \lambda \operatorname{div} u_{x_j} + \rho f^j \end{cases} \tag{1.1}$$

for x in the half-space $\Omega = \{x \in \mathbb{R}^3 : x_3 > 0\}$ with boundary conditions

$$(u^1(x), u^2(x), u^3(x)) = K(x)(u^1_{x_3}(x), u^2_{x_3}(x), 0),\ x \in \partial\Omega, \tag{1.2}$$

and initial values

$$(\rho, u)|_{t=0} = (\rho_0, u_0). \tag{1.3}$$

Here ρ and $u = (u^1, u^2, u^3)$ are the unknown functions of $x \in \Omega$ and $t \geq 0$, $P = P(\rho)$ is the pressure, f is a given external force, μ and λ are viscosity constants, and

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K is a smooth, positive function. The solutions we obtain are in an "intermediate" regularity class in which densities are bounded and measureable, initial velocities are in L^2, energies are small, but oscillations are arbitrarily large. The results of the present paper are the first establishing the existence of solutions of the Navier–Stokes equations in this intermediate class in a region with a boundary, and generalize and improve upon previous results for solutions in the whole space.

The boundary condition (1.2) was proposed by Navier in [12] and expresses the condition that the velocity on $\partial\Omega$ is proportional to the tangential component of the stress. Observe that, for the half-space case considered here, (1.2) is equivalent to

$$u(x) = -K(x)\omega N(x), \tag{1.4}$$

where ω is the vorticity matrix $\omega^{j,k} = u^j_{x_k} - u^k_{x_j}$ and N is the unit outer normal on $\partial\Omega$. We shall give a complete derivation of the Navier boundary condition for general regions Ω at the end of this introduction.

Specifically, we fix a positive constant reference density $\tilde{\rho}$ and assume that $(\rho_0 - \tilde{\rho}, u_0)$ is small in L^2 and bounded in L^q for some $q > 6$, and that ρ_0 is nonnegative and bounded, with no restrictions on its sup-norm. Our existence result accommodates a wide class of pressures P, including pressures that are not monotone in ρ. The solutions that we obtain may exhibit discontinuities in density and velocity gradient across hypersurfaces in Ω and are consequently much less regular and much more general than those of the small-smooth theories of Matsumura–Nishida [11] and Danchin [5]. On the other hand, our solutions are somewhat more regular than those of the large-weak theories of Lions [10] or Feireisl [6]–[7] in which size restrictions are eliminated altogether, but certain restrictions are imposed on P. The present work generalizes and improves upon earlier results of Hoff [8]–[9] in several significant ways: these are the first results for intermediate-class solutions with these, or any boundary conditions, the restriction on the L^∞ norm of $\rho_0 - \tilde{\rho}$ has been elimininated, nonmonotone pressures are allowed, and various improvements in the analysis allow for weaker restrictions on q, μ, and λ. We also note that, for the solutions considered here, if ρ is bounded below away from zero initially, then it remains so for all time. Thus vacuum states cannot occur if none are present initially. We shall describe the most interesting features of the analysis below, especially the role of the boundary condition (1.2), following the statement of our main theorem.

We now give a precise formulation of our results. First, we say that (ρ, u) is a weak solution of (1.1)–(1.3) if ρ and u are suitably integrable and if

$$\int_\Omega \rho(x,\cdot)\varphi(x,\cdot)dx\Big|_{t_1}^{t_2} = \int_{t_1}^{t_2}\int_\Omega (\rho\varphi_t + \rho u\cdot\nabla\varphi)dxdt \tag{1.5}$$

for all times $t_2 \geq t_1 \geq 0$ and all $\varphi \in C^1(\overline{\Omega}\times[t_1,t_2])$ with supp $\varphi(\cdot,t)$ contained in a fixed compact set for $t \in [t_1, t_2]$, and

$$\begin{aligned}
&\int_\Omega (\rho u)(x,\cdot)\cdot\varphi(x,\cdot)dx\Big|_{t_1}^{t_1} - \int_{t_1}^{t_2}\int_\Omega \left[\rho u\cdot\varphi_t + \rho(\nabla\varphi u)\cdot u + P(\rho)\mathrm{div}\varphi\right] dxdt \\
&\quad = -\mu\int_{t_1}^{t_2}\int_{\partial\Omega} K^{-1}u\cdot\varphi dS_x dt \\
&\qquad - \int_{t_1}^{t_2}\int_\Omega \left[\mu u^j_{x_k}\varphi^j_{x_k} + \lambda(\mathrm{div}u)(\mathrm{div}\varphi)\right] dxdt \\
&\qquad + \int_{t_1}^{t_2}\int_\Omega \rho f\cdot\varphi dxdt
\end{aligned} \tag{1.6}$$

for all times $t_2 \geq t_1 \geq 0$ and all $\varphi = (\varphi^1, \varphi^2, \varphi^3)$, where each φ^j is just as in (1.5) and $\varphi\cdot N = 0$ on $\partial\Omega$. (Summation over repeated indices is understood throughout the paper.)

Concerning the pressure P, we fix $\tilde\rho$ and $\overline\rho$ satisfying $0 < \tilde\rho < \overline\rho$ and assume that

$$\begin{cases} P \in C^2([0,\overline\rho\,]) \\ P(0) = 0 \\ P'(\tilde\rho) > 0 \\ (\rho-\tilde\rho)[P(\rho)-P(\tilde\rho)] > 0, \quad \rho\neq\tilde\rho,\ \rho\in[0,\overline\rho\,]. \end{cases} \tag{1.7}$$

It follows that, if G is the potential energy density, defined by

$$G(\rho) = \rho\int_{\tilde\rho}^{\rho} \frac{P(s)-P(\tilde\rho)}{s^2}\, ds, \tag{1.8}$$

then for any $g \in C^2([0,\overline\rho])$ for which $g(\tilde\rho) = g'(\tilde\rho) = 0$, there is a constant C such that $|g(\rho)| \leq CG(\rho)$, $\rho\in[0,\overline\rho]$.

The viscosity constants λ and μ are assumed to satisfy

$$\mu > 0, \qquad 0 < \lambda < 5\mu/4. \tag{1.9}$$

It follows that there is a $q > 6$, which will be fixed throughout, such that

$$\frac{\mu}{\lambda} > \frac{(q-2)^2}{4(q-1)}\,. \tag{1.10}$$

We assume that

$$\begin{cases} K \in (W^{2,\infty}\cap W^{1,3})(\mathbb{R}^2), \\ K(x) \geq \underline{K} > 0 \end{cases} \tag{1.11}$$

for some positive constant $\underline{K}$, and we measure the sizes of the initial data and the external force by

$$C_0 = \int_\Omega \left[\tfrac12\rho_0|u_0|^2 + G(\rho_0)\right] dx, \tag{1.12}$$

$$C_f = \sup_{t\geq 0} |f(\cdot,t)|_{L^2} + \int_0^\infty \left(|f(\cdot,t)|_{L^2} + \sigma^7 |\nabla f(\cdot,t)|_{L^4}^2 \right) dt \\ + \int_0^\infty \int_\Omega \left(|f|^2 + \sigma^5 |f_t|^2 \right) dxdt, \tag{1.13}$$

where $\sigma(t) = \min\{1,t\}$, and

$$M_q = \int_\Omega \rho_0 |u_0|^q + \sup_{t\geq 0} |f(\cdot,t)|_{L^q} + \int_0^\infty \int_\Omega |f|^q dxdt, \tag{1.14}$$

where q is as above in (1.10).

We recall the definition of the vorticity matrix $\omega^{j,k} = u^j_{x_k} - u^k_{x_j}$, and for a given solution (ρ, u) we define the function

$$F = (\lambda + \mu)\mathrm{div}\, u - P(\rho) + P(\tilde{\rho}). \tag{1.15}$$

The important roles of ω and F will discussed below following the statement of Theorem 1.1. We also define the convective derivative $\frac{d}{dt}$ by $\frac{dw}{dt} = \dot{w} = w_t + \nabla w \cdot u$ for functions $w(x,t)$.

Finally, for functions $v : A \subseteq \overline{\Omega} \to \mathbb{R}^m$ and for $\alpha \in (0,1]$ we define the Hölder norm

$$\langle v \rangle_A^\alpha = \sup_{\substack{x_1,x_2 \in A \\ x_1 \neq x_2}} \frac{|v(x_2) - v(x_1)|}{|x_2 - x_1|^\alpha} \, ;$$

and for $v : A \times [t_1,t_2] \to \mathbb{R}^m$ and $\alpha, \beta \in (0,1]$,

$$\langle v \rangle_{A\times[t_1,t_2]}^{\alpha,\beta} = \sup \frac{|v(x_2,t_2) - v(x_1,t_1)|}{|x_2 - x_1|^\alpha + |t_2 - t_1|^\beta} \, ,$$

the sup being taken over distinct pairs (x_1,t_1), $(x_2,t_2) \in A \times [t_1,t_2]$.

The following is then the main result of this paper:

Theorem 1.1. *Let Ω be either $\mathbb{R}^3$ or the upper half-space in $\mathbb{R}^3$, and let the hypotheses and notations in* (1.7)–(1.10)*, and in the half-space case* (1.11)*, be in force. Then given a positive number M (not necessarily small) and given $\overline{\rho}_1 \in (\tilde{\rho}, \overline{\rho})$, there are positive numbers ε and C depending on $\tilde{\rho}, \overline{\rho}_1, \overline{\rho}, P, \lambda, \mu, q, M$, and in the half-space case K, and there is a universal positive constant θ, such that, given initial data (ρ_0, u_0) and external force f satisfying*

$$\begin{cases} 0 \leq \inf_\Omega \rho_0 \leq \sup_\Omega \rho_0 \leq \overline{\rho}_1, \\ C_0 + C_f \leq \varepsilon, \\ M_q \leq M, \end{cases} \tag{1.16}$$

where C_0, C_f, and M_q are as above in (1.12)–(1.14)*, the initial–boundary problem* (1.1)–(1.3) *has a global weak solution* (ρ, u) *in the sense of* (1.5)–(1.6) *satisfying:*

$$C^{-1} \inf \rho_0 \leq \rho \leq \overline{\rho} \quad a.e., \tag{1.17}$$

$$\begin{cases} \rho - \tilde{\rho} \in C([0,\infty); H^{-1}(\Omega)), \\ \rho u \in C([0,\infty); \widetilde{H}^1(\Omega)^*), \\ \nabla u \in L^2(\Omega \times [0,\infty)), \end{cases} \tag{1.18}$$

with $\rho = \rho_0$ *and* $\rho u = \rho_0 u_0$ *at* $t = 0$*. Here* $\widetilde{H}^1(\Omega)^*$ *is the dual of the space* $\tilde{H}^1(\Omega) = \{\varphi \in H^1(\Omega)^3 : \varphi \cdot N = 0 \text{ on } \partial\Omega\}$*. In addition,*

$$\langle u \rangle^{1/2,1/4}_{\overline{\Omega}\times[\tau,\infty)},\ \langle F, \omega \rangle^{1/2,1/8}_{\overline{\Omega}\times[\tau,\infty)} \leq C(\tau)(C_0 + C_f)^\theta \tag{1.19}$$

for all $\tau > 0$*, where* $C(\tau)$ *depends additionally on* τ*,*

$$u(x,t) = -K(x)\omega N(x) \quad \text{pointwise for } x \in \partial\Omega,\ t > 0, \tag{1.20}$$

and

$$\begin{aligned} &\sup_{t>0} \int_\Omega \left[\tfrac{1}{2}\rho(x,t)|u(x,t)|^2 + |\rho(x,t) - \tilde{\rho}|^2 + \sigma(t)|\nabla u(x,t)|^2\right] dx \\ &\qquad + \int_0^\infty \int_\Omega \left[|\nabla u|^2 + \sigma \sum_{j=1}^3 \left((\rho u^j)_t + \operatorname{div}(\rho u^j u)\right)^2 + \sigma^3 |\nabla \dot{u}|^2\right] dx dt \\ &\leq C(C_0 + C_f)^\theta, \end{aligned} \tag{1.21}$$

where $\sigma = \min\{1, t\}$*. Finally, in the case that* $\inf \rho_0 > 0$*, the term* $\int_0^\infty \int_\Omega \sigma |\dot{u}|^2 dx dt$ *may be included on the left side of* (1.21)*.*

The proof of Theorem 1.1 consists in the derivation of *a priori* bounds specific to this system for smooth solutions corresponding to mollified initial data, and the application of these bounds in extracting limiting weak solutions as the mollifying parameter goes to zero. Specifically, in Section 2 we fix a smooth, local-in-time solution for which $0 < \rho < \bar{\rho}$, and we obtain bounds for the functionals

$$A_1(T) = \sup_{0 \leq t \leq T} \sigma(t) \int_\Omega |\nabla u(x,t)|^2 dx + \int_0^T \int_\Omega \sigma \rho |\dot{u}|^2 dx dt$$

and

$$A_2(T) = \sup_{0 \leq t \leq T} \sigma(t)^3 \int \rho(x,t)|\dot{u}(x,t)|^2 dx + \int_0^T \int_\Omega \sigma^3 |\nabla \dot{u}|^2 dx dt$$

of the form $A_1 + A_2 \leq C(C_0 + C_f)^\theta$. Then in Section 3 we prove the converse, that ρ remains in a compact subset of $[0, \bar{\rho})$ for as long as $A_1 + A_2 \leq C(C_0 + C_f)^\theta$. The smallness hypothesis then enables us to close these estimates, which are then applied in Section 4 to show that the solution (ρ, u) of Theorem 1.1 can be obtained in the limit as the mollifying parameter goes to zero. Notice that these *a priori* bounds give no more than L^∞ and L^2 control of the approximate densities, insufficient to conclude more than weak convergence. On the other hand, an argument of Feireisl [7] based on the renormalizability of the mass equation in (1.1) can be applied to show that the approximate densities converge strongly, and therefore that the limit of the approximate solutions is indeed the desired weak solution.

We now give a somewhat more detailed description of these *a priori* bounds and the important role played by the boundary condition (1.2). We begin with the derivation of pointwise bounds for ρ under the assumption that bounds for A_1 and A_2 have already been obtained. With some benefit of hindsight we fix a curve $x(t)$ satisfying $\dot{x}(t) = u(x(t), t)$ and substitute the definition (1.15) of the function F into the mass equation $\dfrac{d}{dt}\rho(x(t), t) = -\rho \operatorname{div} u$ to obtain

$$(\lambda + \mu)\frac{d}{dt}\log \rho(x(t), t)) + [P(\rho(x(t), t) - P(\tilde{\rho})] = -F. \tag{1.22}$$

The brackets on the left here is positive when ρ is large and negative when ρ is close to zero, and so is dissipative at critical values. Pointwise bounds for ρ will therefore follow from pointwise bounds for F, and these must somehow be derived from estimates for A_1 and A_2.

Before proceeding, we remark that a careful application of the standard Rankine–Hugoniot condition to (1.1) shows that discontinuities in ρ, $P(\rho)$, and ∇u across hypersurfaces in Ω can be expected to persist for all time, but that the functions F and ω should be relatively smooth in positive time, reflecting a cancellation of singularities (see the introduction to [8], for example). The precise statement is the result in (1.19) of Theorem 1.1, which is one indication of the important roles of F and ω. These two distinguished variables also arise in the Helmholtz decomposition of the acceleration density $\rho\dot{u}$: adding and subtracting terms, we can rewrite the momentum equation in (1.1) in the form

$$\rho\dot{u}^j = F_{x_j} + \mu\omega^{j,k}_{x_k} + \rho f^j. \tag{1.23}$$

In the case that $\Omega = \mathbb{R}^3$, and with sufficient decay at infinity, the first two terms on the right here are orthogonal in L^2. Thus L^2 estimates for $\rho\dot{u}$, which we are anticipating in the definitions of A_1 and A_2, immediately imply L^2 bounds for ∇F and $\nabla\omega$. This orthogonality is lost when $\partial\Omega \neq \phi$, however, because boundary integrals arise in the computation of the inner product. Stated differently, the decomposition (1.23) implies that

$$\Delta F = \operatorname{div}(\rho\dot{u} - \rho f), \tag{1.24}$$

which, in the absence of a boundary condition for F, determines F only up to a harmonic function, enabling interior estimates at best. The no-slip boundary condition $u = 0$ on $\partial\Omega$ does not seem to be of use here, but the Navier boundary condition (1.2) does indeed provide the required boundary information for F: for the half-space case under consideration here, (1.2) implies that $u^3 = 0$, hence $\dot{u}^3 = 0$ on $\partial\Omega$, and therefore by (1.23) and (1.4) that

$$F_{x_3} = \mu \left[(K^{-1}u^1)_{x_1} + (K^{-1}u^2)_{x_2} \right] - \rho f^3 \tag{1.25}$$

on $\partial\Omega$. This together with (1.24) is then sufficient to determine F on Ω. In particular, F can be represented via the Neumann–Green's function for Ω in terms of the functions on the right sides of (1.24) and (1.25), and these can be estimated in terms of A_1 and A_2. Pointwise bounds for F can then be deduced from this representation, and these can then be applied in (1.22) to yield the required pointwise bounds for ρ.

We also remark on a different point in the analysis in which F and ω play a crucial role. Certain higher order terms arise in the derivation of bounds for A_1 and A_2, resulting from nonlinearities in the momentum equation in (1.1). One such term is $\int\int |\nabla u|^4 dxdt$. Now, $H^2 \subset W^{1,4}$, so that an estimate for $\|u(\cdot,t)\|_{H^2}$ would suffice here. But as we indicated above, ∇u can be discontinuous across hypersurfaces in Ω, in which case $u(\cdot,t) \notin H^2$. On the other hand, adding and subtracting terms, we can write

$$\begin{aligned}(\mu+\lambda)\Delta u^j &= [(\mu+\lambda)\mathrm{div}\, u - P(\rho)]_{x_j} + (\mu+\lambda)(u^j_{x_k} - u^k_{x_j})_{x_k} + (P(\rho) - \tilde{P})_{x_j} \\ &= F_{x_j} + (\mu+\lambda)\omega^{j,k}_{x_k} + (P - \tilde{P})_{x_j} \ .\end{aligned}$$

The velocity u therefore satisfies a Poisson equation with a Robin-type boundary condition (1.2). Standard elliptic theory therefore applies to show that, for fixed $t > 0$,

$$\|\nabla u\|_{L^4} \leq C \left[\|F\|_{L^4} + \|\omega\|_{L^4} + \|P - \tilde{P}\|_{L^4} \right].$$

Bounds for the terms on the right are then available because $F, \omega \in H^1 \subset L^4$ via (1.24) and (1.25), and $\rho - \tilde{\rho} \in L^2 \cap L^\infty$.

We conclude this introduction with an elementary derivation of an explicit form of the Navier boundary condition (1.2) for general regions Ω. First recall that in the derivation of standard fluid models such as (1.1) (see Batchelor [2], for example), the fluid on one side of a given surface through a given point is regarded as exerting a force on the fluid on the other side given by the integral over the surface of the "stress" with respect to surface area. Stress therefore has the units of pressure, and in the Navier–Stokes equations is taken to be σN, where N is the unit normal to the surface at the given point and σ is the 3×3 matrix

$$\sigma^{j,k} = \mu(u^j_{x_k} + u^k_{x_j}) + (\lambda\, \mathrm{div}\, u - P(\rho))\delta^{j,k}. \tag{1.26}$$

The boundary condition proposed by Navier [12] is that the velocity at a point on $\partial\Omega$ should be proportional to the tangential component of the stress at that point; that is, that

$$u = -K(\sigma N)_{\tan} = -\mu K((\nabla u + \nabla u^t)N)_{\tan}, \tag{1.27}$$

where the proportionality factor K may vary across $\partial\Omega$. In particular, $u \cdot N = 0$ on $\partial\Omega$.

To obtain a more explicit and useable expression for the Navier boundary condition, we suppose that the boundary is given locally by $\partial\Omega = \{x : g(x) = 0\}$, so that $N(x) = \nabla g(x)/|\nabla g(x)|$ is the unit outer normal. We let $x = x(s)$ be a curve on $\partial\Omega$ and differentiate the relation $\nabla g(x(s)) \cdot u(x(s), t) = 0$ to obtain that the tangent vector $T = \dot{x}$ satisfies

$$(\nabla u^t N) \cdot T = -(g'' u) \cdot T/|\nabla g|.$$

We now fix a point $x \in \partial\Omega$ and let $\{T_1, T_2\}$ be an orthonormal basis for the tangent space to $\partial\Omega$ at x. Then at x,

$$\begin{aligned}(\nabla u^t N)_{\tan} &= \sum_{j=1,2} ((\nabla u^t N) \cdot T_j)T_j \\ &= -\sum_{j=1,2} ((g'' u) \cdot T_j)T_j/|\nabla g| \equiv Au\end{aligned}$$

where A is a matrix-valued function of $x \in \partial\Omega$ determined solely by $\partial\Omega$. Then from (1.26),

$$\begin{aligned}(\sigma N)_{\tan} &= \mu(\nabla u - \nabla u^t)N + 2\mu(\nabla u^t N)_{\tan} \\ &= \mu\omega N + 2\mu A u,\end{aligned}$$

since $(\omega N) \cdot N = 0$, which implies that $(\omega N)_{\tan} = \omega N$. The Navier boundary condition (1.27) then becomes

$$u = -\mu K(I + 2\mu K A)^{-1}\omega N. \tag{1.28}$$

Observe that when $\partial\Omega$ is flat, $A = 0$ and (1.28) reduces to (1.4) with the constant μ subsumed into the definition of K.

The boundary condition (1.2) for the flat-space case has been applied in a number of problems, usually for incompressible flows; see Arbogast and Lehr [1], Beavers and Joseph [3], Caflisch and Rubinstein [4], and Saffman [13], for example. We also remark that the no-slip boundary condition $u = 0$ on $\partial\Omega$ may be regarded as the singular limit as $K \to 0$ of the Navier boundary condition (1.28). The analysis of the present paper is insufficient to justify this limit, but it is likely that the large-weak solutions discussed in Lions [10] and Feireisl [6]–[7] with no-slip boundary conditions are indeed these limits for certain pressures.

2. Energy estimates

In this section we derive *a priori* bounds for smooth, local-in-time solutions of (1.1)–(1.2) whose densities are strictly positive and bounded. (The existence of such solutions will be established later in Proposition 3.2.) We thus fix a smooth solution (ρ, u) of (1.1)–(1.2) on $\Omega \times [0, T]$ for some time $T > 0$, with smooth initial data (ρ_0, u_0) and external force f satisfying all the hypotheses of Theorem 1.1, and with $\rho(x,t) \le \bar{\rho}$. We let C and θ be generic positive constants as described in Theorem 1.1 and we write $C = C(\overline{\rho})$ to emphasize the assumption that $\rho \le \overline{\rho}$.

In Proposition 2.1 below we state the standard L^2 energy estimate for (ρ, u) and we establish a bound for ρu in L^q. In Lemma 2.2 we derive preliminary L^2 estimates for ∇u and $\rho \dot{u}$ reflecting the parabolic smoothing in the second equation in (1.1). Higher order terms occur in these estimates, and to control these we need certain technical embedding results, given in Lemma 2.3. These are then applied in Proposition 2.4 to complete the regularity estimates for u.

Proposition 2.1. *There is a positive constant $C = C(\overline{\rho})$ as described in Theorem 1.1 such that, if (ρ, u) is a smooth solution of* (1.1)–(1.2) *on $\overline{\Omega} \times [0, T]$ as described above with $0 < \rho(x,t) \le \overline{\rho}$, then*

$$\begin{aligned}\sup_{0\le t\le T} \int_\Omega \left[\tfrac{1}{2}\rho(x,t)|u(x,t)|^2 + G(\rho(x,t))\right] dx \\ + \int_0^T \int_\Omega |\nabla u|^2 dx dt + \int_0^T \int_{\partial\Omega} |u|^2 dS_x dt \\ \le C(\overline{\rho})(C_0 + C_f),\end{aligned} \tag{2.1}$$

where G is as in (1.8)*. Also, there is a positive constant $C = C(\overline{\rho}, T)$ such that*

$$\begin{aligned}\sup_{0\le t\le T} \int_\Omega \rho(x,t)|u(x,t)|^q dx + \int_0^T \int_\Omega |u|^{q-2}|\nabla u|^2 dx dt + \int_0^T \int_{\partial\Omega} |u|^q dS_x dt \\ \le C(\overline{\rho}, T)(C_0 + C_f + M_q),\end{aligned} \tag{2.2}$$

where q is as in (1.10)*.*

Proof. To prove (2.1) we multiply the first equation in (1.1) by $G'(\rho)$ and the second by u^j and integrate, applying the boundary condition (1.2). The computation is standard, and the details are omitted.

We shall sketch the proof of (2.2), which is similar. First, from the second equation in (1.1) we obtain that

$$\begin{aligned}\rho\Big[(|u|^q)_t + (\nabla|u|^q)\cdot u\Big] + q|u|^{q-2}u\cdot\nabla P + \mu q|u|^{q-2}|\nabla u|^2 + \lambda q|u|^{q-2}(\operatorname{div} u)^2 \\ = q|u|^{q-2}\left[\tfrac{1}{2}\mu\Delta|u|^2 + \lambda \operatorname{div}((\operatorname{div} u)u) + \rho u\cdot f\right].\end{aligned}$$

Adding the equation $|u|^q(\rho_t + \operatorname{div}(\rho u)) = 0$, integrating, and rearranging, we then obtain

$$\begin{aligned}
&\int_\Omega \rho|u|^q dx\Big|_0^t + \int_0^t \int_\Omega \Big\{ q|u|^{q-2}\big[\mu|\nabla u|^2 + \lambda(\operatorname{div} u)^2 + \mu(q-2)\big|\nabla|u|\big|^2\big] \\
&\qquad\qquad + q\lambda(\nabla|u|^{q-2})\cdot u \operatorname{div} u\Big\} dxds \\
&= \int_0^t \int_{\partial\Omega} \tfrac{1}{2}\mu q|u|^{q-2}(\nabla|u|^2)\cdot N dS_x ds \\
&\qquad\qquad + \int_0^t \int_\Omega \Big[q \operatorname{div}(|u|^{q-2}u)P + q|u|^{q-2}\rho u\cdot f\Big] dxds.
\end{aligned} \tag{2.3}$$

The integrand in the boundary integral on the right here is

$$-\mu q|u|^{q-2}(u^1u^1_{x_3} + u^2u^2_{x_3}) = -\mu qK(x)^{-1}|u|^q \le 0$$

by (1.2). Next, since $\big|\nabla|u|\big| \le |\nabla u|$, we can bound the integrand in the double integral on the left side of (2.3) from below by

$$\begin{aligned}
&q|u|^{q-2}\Big[\mu|\nabla u|^2 + \lambda(\operatorname{div} u)^2 + \mu(q-2)|\nabla|u||^2 - \lambda(q-2)\big|\nabla|u|\big|\big|\operatorname{div} u\big|\Big] \\
&= q|u|^{q-2}\Big[\mu|\nabla u|^2 + \lambda\big(\operatorname{div} u - \tfrac{1}{2}(q-2)\big|\nabla|u|\big|\big)^2\Big] \\
&\qquad + q|u|^{q-2}\Big[\mu(q-2) - \tfrac{1}{4}\lambda(q-2)^2\Big]\big|\nabla|u|\big|^2 \\
&\ge q|u|^{q-2}\Big[\mu(q-1) - \tfrac{1}{4}\lambda(q-2)^2\Big]|\nabla u|^2 \\
&\ge C^{-1}|u|^{q-2}|\nabla u|^2
\end{aligned}$$

by the hypothesis (1.10). Bounds for the remaining two terms on the right side of (2.3) are similar, and (2.2) then follows from (2.3). □

The following lemma contains preliminary versions of L^2 bounds for ∇u and $\rho\dot{u}$:

Lemma 2.2. *There is a constant $C = C(\overline{\rho})$ as described in the statement of Theorem 1.1 such that, if (ρ, u) is a smooth solution of (1.1)–(1.2) as in Proposition 2.1, then*

$$\begin{aligned}
&\sup_{0\le t\le T} \sigma(t)\int_\Omega |\nabla u(x,t)|^2 dx + \int_0^T \int_\Omega \sigma\rho|\dot{u}|^2 dxdt \\
&\le C(\overline{\rho})\Bigg[C_0 + C_f + \int_0^T \int_\Omega \sigma(|u|^2|\nabla u| + |u||\nabla u|^2) dxdt \\
&\qquad + \sum\Big|\int_0^T \int_\Omega \sigma u^{j_1}_{x_{k_1}} u^{j_2}_{x_{k_2}} u^{j_3}_{x_{k_3}} dxdt\Big|\Bigg],
\end{aligned} \tag{2.4}$$

where $\sigma(t) = \min\{1, t\}$ and the (finite) sum on the right is over all combinations of indices; and

$$\begin{aligned} &\sup_{0\le t\le T} \sigma(t)^3 \int \rho(x,t)|\dot{u}(x,t)|^2 dx + \int_0^T \int_\Omega \sigma^3 |\nabla \dot{u}|^2 dx dt \\ &\le C(\bar{\rho})\left[C_0 + C_f + A_1(T)\right] \\ &\qquad + C(\bar{\rho}) \int_0^T \int_\Omega \sigma^3 \left[|u|^4 + |\nabla u|^4 + |\dot{u}||\nabla u||u| + |\dot{u}||\nabla u|^2\right] dx dt, \end{aligned} \tag{2.5}$$

where $A_1(T)$ is the left side of (2.4).

Proof. The proofs are nearly the same as those of (2.9) and (2.12) in [8], except that boundary effects occur here. We therefore restrict attention to representative boundary integrals arising in what are otherwise routine but somewhat technical estimates. For example, to prove (2.4), we multiply the second equation in (1.1) by $\sigma \dot{u}^j$ and integrate over $\Omega \times [0, t]$. This results in the following boundary integral on the right:

$$\int_0^t \int_{\partial\Omega} \sigma \left[(\tilde{P} - P)\dot{u} \cdot N + \mu u^j_{x_k} \dot{u}^j N^k + \lambda(\operatorname{div} u)\dot{u} \cdot N\right] dS_x ds.$$

The condition $u \cdot N = 0$ on Ω implies that the first and third terms here vanish, and by (1.2) the second term may be written

$$\begin{aligned} &-\mu \int_0^t \int_{\partial\Omega} \sigma u^j_{x_3} \dot{u}^j dS_x ds = -\mu \int_0^t \int_{\partial\Omega} \sigma K^{-1} u^j \left(u^j_t + u^j_{x_k} u^k\right) dS_x ds \\ &= -\tfrac{1}{2}\mu\sigma(t) \int_{\partial\Omega} K^{-1} |u(x,t)|^2 dS_x - \mu \int_0^t \int_{\partial\Omega} \sigma K^{-1} u^j u^k u^j_{x_k} dS_x ds. \end{aligned}$$

The first term on the right here is nonpositive, and for the second term we apply the fact that, for $h \in (C^1 \cap W^{1,1})(\overline{\Omega})$,

$$\int_{\partial\Omega} h(x) dS_x = \int_{\Omega\cap\{0\le x_3\le 1\}} [h(x) + (x_3 - 1)h_{x_3}(x)] dx. \tag{2.6}$$

Since $j, k \in \{1, 2\}$ in the term in question, we can apply (2.6) and integrate by parts in the x_1 and x_2 directions to obtain the bound

$$\int_0^t \int_\Omega \sigma \left(|u|^2 |\nabla u| + |u|\, |\nabla u|^2\right) dx ds,$$

which is included on the right side of (2.4).

To prove (2.5) we take the convective derivative in the second equation in (1.1), multiply by $\sigma^3\dot{u}^j$, and integrate. This results in the following boundary integral on the right:

$$\int_0^t \int_{\partial\Omega} \sigma^3 \Big[(P-\tilde{P})(\nabla\dot{u}u)\cdot N + \mu\dot{u}\cdot(\nabla u_t N) - \mu(\nabla\dot{u}u)\cdot(\nabla u N) \\ - (\lambda+\mu)(\operatorname{div} u)(\nabla\dot{u}u)\cdot N\Big] dS_x ds, \tag{2.7}$$

where the relation $u\cdot N = -u^3 = 0$ on $\partial\Omega$ has already been applied. A short computation shows that $(\nabla\dot{u}u)\cdot N = 0$ on $\partial\Omega$, so that the first and last terms in (2.7) vanish, and we can apply (1.2) to write the second term as

$$\begin{aligned} -\mu\int_0^t\int_{\partial\Omega}\sigma^3\dot{u}^j u^j_{x_3t} &= -\mu\int_0^t\int_{\partial\Omega}\sigma^3 K^{-1}\dot{u}^j u^j_t \\ &= -\mu\int_0^t\int_{\partial\Omega}\sigma^3K^{-1}|\dot{u}|^2 dS_x ds + \mu\int_0^t\int_{\partial\Omega}\sigma^3K^{-1}\dot{u}^j u^j_{x_k}u^k dS_x ds. \end{aligned} \tag{2.8}$$

The first term here is nonpositive, and for the second we apply (2.6) and the fact that $j,k\in\{1,2\}$, allowing for integration by parts in the x_1 and x_2 directions, to obtain the bound

$$C\int_0^t\int_\Omega \sigma^3\left[|u||\nabla u||\dot{u}| + |u||\nabla u||\nabla\dot{u}| + |\nabla u|^2|\dot{u}|\right] dx ds,$$

which in turn is bounded by terms on the right side of (2.5), modulo a small multiple of $\int\int\sigma^3|\nabla\dot{u}|^2 dxds$, which can be absorbed into the left side. The third term in (2.7) is handled in a similar way. □

The following auxiliary estimates will be applied to bound the higher order terms occurring on the right sides of (2.4) and (2.5):

Lemma 2.3. *There is a constant $C = C(\bar{\rho})$ such that, if (ρ,u) is a smooth solution of (1.1)–(1.2) on $\overline{\Omega}\times[0,T]$ as in Proposition 2.1, then for $0\le t\le T$,*

(a) $\displaystyle\int |u|^p dx$

$$\le C(\overline{\rho})\Big[(C_0+C_f)^{(6-p)/4}\Big(\int|\nabla u|^2dx\Big)^{(3p-6)/4} + (C_0+C_f)^{(6-p)/6}\Big(\int|\nabla u|^2dx\Big)^{p/2}\Big],$$

$2\le p\le 6$,

(b) $\displaystyle\int|\nabla u|^p dx \le \int\Big(|F|^p + |\omega|^p + |P-\tilde{P}|^p + |u|^p\Big)dx,\ 1<p<\infty,$

(c) $\displaystyle\int(|\nabla F|^p + |\nabla\omega|^p)\,dx \le \int(|\rho\dot{u}|^p + |\nabla u|^p + |u\nabla K|^p + |f|^p)\,dx,\ 1<p<\infty,$

(d) $\int(|F|^p+|\omega|^p)dx \le C(\overline{\rho})\Big[\Big(\int \rho|\dot{u}|^2dx\Big)^{(3p-6)/4}\Big(\int(|\nabla u|^2+|P-\tilde{P}|^2)dx\Big)^{(6-p)/4}$
$+\Big(\int(|\nabla u|^2+|P-\tilde{P}|^2+|f|^2)dx\Big)^{p/2}\Big]$, $2\le p\le 6$,

where all functions are evaluated at time t and all integrals are over Ω. Also, for $0\le t_1\le t_2\le T$, $r\ge 2$, and $s\ge 0$,

(e) $\displaystyle\int_{t_1}^{t_2}\int_\Omega \sigma(t)^s|\rho(x,t)-\tilde{\rho}|^r dxdt \le C(\overline{\rho})\left[\int_{t_1}^{t_2}\int_\Omega \sigma(t)^s|F(x,t)|^r dxdt + C_0 + C_f\right]$,

where $\sigma=\min\{1,t\}$.

Proof. We first apply the standard embedding

$$\int_{\mathbb{R}^2} w^6dx \le C\left(\int_{\mathbb{R}^2} w^4dx\right)\left(\int_{\mathbb{R}^2}|\nabla w|^2dx\right),$$

which holds for $w\in H^1(\mathbb{R}^2)$ (Ziemer [14], Theorem 2.4.1) to functions $w:\Omega\to\mathbb{R}$ for fixed x_3, then integrate with respect to $x_3\in[0,\infty)$ and interpolate the L^4 norm between L^2 and L^6 to obtain

$$\int_\Omega |w|^pdx \le C\left(\int_\Omega |w|^2dx\right)^{(6-p)/4}\left(\int_\Omega|\nabla w|^2dx\right)^{(3p-6)/4} \tag{2.9}$$

for $p=6$. The same result then holds for $p\in[2,6]$ by interpolation.

To prove (a), we write

$$\tilde{\rho}\int_\Omega |u|^2dx \le \int_\Omega \rho|u|^2dx + \left(\int_\Omega|\rho-\tilde{\rho}|^2dx\right)^{1/2}\left(\int|u|^4dx\right)^{1/2}$$

and apply (2.9) and Proposition 2.1 to obtain that

$$\int_\Omega |u|^2dx \le C(\overline{\rho})\left[(C_0+C_f)+(C_0+C_f)^{2/3}\int_\Omega|\nabla u|^2dx\right]. \tag{2.10}$$

We now apply (2.9) to u and substitute (2.10) on the right to complete the proof of (a).

To prove (b) we observe that u satisfies the elliptic boundary value problem

$$\begin{aligned}(\mu+\lambda)\Delta u^j &= [(\mu+\lambda)\mathrm{div}\,u-P(\rho)]_{x_j}+(\mu+\lambda)(u^j_{x_k}-u^k_{x_j})_{x_k}+(P(\rho)-\tilde{P})_{x_j}\\ &= F_{x_j}+(\mu+\lambda)\omega^{j,k}_{x_k}+(P-\tilde{P})_{x_j}\ ,\end{aligned}$$
$$\begin{cases} u^1_{x_3} &= K^{-1}u^1,\\ u^2_{x_3} &= K^{-1}u^2,\\ u^3 &= 0, \qquad x\in\partial\Omega.\end{cases}$$

The bounds in (b) then follow from standard elliptic theory.

To prove (c) we compute from the second equation in (1.1) that

$$\mu\Delta\omega^{j,k} = (\rho\dot{u}^j)_{x_k} - (\rho\dot{u}^k)_{x_j} + (\rho f^k)_{x_j} - (\rho f^j)_{x_k}.$$

Thus if $H \equiv \omega^{1,3} - K^{-1}u^1$, for example, then $H = 0$ on $\partial\Omega$ by (1.2), and

$$\mu\Delta H = (\rho\dot{u}^j)_{x_k} - (\rho\dot{u}^k)_{x_j} + (\rho f^j)_{x_k} - (\rho f^k)_{x_j} - \mu\Delta(K^{-1}u^1)$$

in Ω. Standard elliptic theory again yields a bound for $\|\nabla H\|_{L^p}$ for $1 < p < \infty$ and therefore for $\omega^{1,3}$:

$$\|\nabla\omega^{1,3}\|_{L^p(\Omega)} \le C(\overline{\rho})\big[\|\rho\dot{u}\|_{L^p} + \|f\|_{L^p} + \|\nabla u\|_{L^p} + \|u\|_{L^p}\big],\ 1 < p < \infty\ . \tag{2.11}$$

A similar argument applies to $\omega^{2,3}$. To derive a bound for $\omega^{1,2}$, we differentiate the $j = 1$ equation in (1.1) with respect to x_2, then reverse the indices and subtract. The result is that

$$\mu\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)\omega^{1,2} = (\rho\dot{u}^1)_{x_2} - (\rho\dot{u}^2)_{x_1} + \mu(\omega^{2,3}_{x_3,x_1} - \omega^{1,3}_{x_3,x_2}) + (\rho f^1)_{x_2} - (\rho f^2)_{x_1}.$$

This gives a bound for $\|\nabla_{x_1,x_2}\omega^{1,2}(\cdot,\cdot,x_3)\|_{L^p(\mathbb{R}^2)}$, and integrating this bound with respect to x_3 and applying (2.11), we obtain that $\nabla_{x_1,x_2}\omega^{1,2}$ is also bounded by the right side of (2.11), as is $\omega^{1,2}_{x_3}$, since

$$\omega^{1,2}_{x_3} = \omega^{1,3}_{x_2} - \omega^{2,3}_{x_1}.$$

This proves the bound in (c) for ω, and the bound for ∇F then follows from the decomposition (1.23).

Part (d) follows from part (c) with $p = 2$, the bounds in Proposition 2.1, and (2.9). To prove (e), we multiply the mass equation in (1.1) by $r\sigma(t)^{s-1}|\rho-\tilde{\rho}|^{r-1}\mathrm{sgn}(\rho-\tilde{\rho})$ and substitute the definition (1.15) for $\operatorname{div} u$ in terms of F. The result is that

$$\begin{aligned}&[D_t + \mathrm{div}(u\cdot)]\,(\sigma^s|\rho-\tilde{\rho}|^r) + (\lambda+\mu)^{-1}\sigma^s\,[(r-1)\rho+\tilde{\rho}]\,|\rho-\tilde{\rho}|^{r-1}|P-\tilde{P}|\\ &\quad = s\sigma^{s-1}\sigma_t|\rho-\tilde{\rho}|^r - (\lambda+\mu)^{-1}\sigma^s\,[(r-1)\rho+\tilde{\rho}]\,\mathrm{sgn}(\rho-\tilde{\rho})|\rho-\tilde{\rho}|^{r-1}F.\end{aligned}$$

Integrating, we then obtain

$$\begin{aligned}&\int_\Omega \sigma^s|\rho-\tilde{\rho}|dx\Big|_{t_1}^{t_2} + C^{-1}\int_{t_1}^{t_2}\int_\Omega \sigma^s|\rho-\tilde{\rho}|^r dxdt\\ &\quad \le C(\overline{\rho})\left[\int_{t_1}^{\max\{t_1,\sigma(t_2)\}}\int_\Omega \sigma_t|\rho-\tilde{\rho}|^r dxdt + \int_{t_1}^{t_2}\int_\Omega \sigma^s|F|^r dxdt\right].\end{aligned}$$

The bound in (e) then follows from (d) and the results of Proposition 2.1. □

We now apply the auxiliary estimates of Lemma 2.3 to close the bounds in Lemma 2.2:

Proposition 2.4. *There exist constants $C = C(\overline{\rho})$ and θ as described in Theorem 1.1 such that, if (ρ, u) is a smooth solution of (1.1)–(1.2) on $\overline{\Omega} \times [0, T]$ as in Proposition 2.1 with $\rho \leq \overline{\rho}$, then*

$$\begin{aligned}
\sup_{0\leq t\leq T} \int_\Omega &\left(\tfrac{1}{2}\rho|u|^2 + |\rho - \tilde{\rho}|^2 + \sigma|\nabla u|^2 + \sigma^3\rho|\dot{u}|^2\right)dx \\
&+ \int_0^T \int_\Omega \left(|\nabla u|^2 + \sigma|\rho\dot{u}|^2 + \sigma^3|\nabla\dot{u}|^2\right)dxdt \\
&\leq C(\overline{\rho})(C_0 + C_f)^\theta,
\end{aligned} \tag{2.12}$$

where $\sigma(t) = \min\{1, t\}$. In particular, $C(\overline{\rho})$ is independent of T, and θ is a universal positive constant.

Proof. The proof consists in applying the bounds in Lemma 2.3 to estimate the terms on the right sides of (2.4) and (2.5). We give the details for the term $\iint \sigma^3|\nabla u|^4$ under the assumption that the bound in (2.4) has already been closed, that is, that the left side of (2.4) has been shown to be bounded by the right side of (2.12). First, by Lemma 2.2(b),

$$\iint \sigma^3|\nabla u|^4 \leq C(\overline{\rho}) \iint \sigma^3[F^4 + |\omega|^4 + |u|^4 + (\rho - \tilde{\rho})^4], \tag{2.13}$$

and by (2.9), Proposition 2.1, Lemma 2.3(a)(c), and our assumption about (2.4),

$$\begin{aligned}
\iint \sigma^3 F^4 &\leq C \int \sigma^3 \left(\int F^2 dx\right)^{1/2} \left(\int |\nabla F|^2 dx\right)^{3/2} dt \\
&\leq C \sup_t \left(\sigma \int F^2 dx\right)^{1/2} \sup_t \left(\sigma^3 \int |\nabla F|^2 dx\right)^{1/2} \iint \sigma|\nabla F|^2 dxdt \\
&\leq C(\overline{\rho})(C_0 + C_f)^\theta \sup_t \left(\sigma^3 \int \left(\rho|\dot{u}|^2 + |\nabla u|^2 + |u|^2 + |f|^2\right) dx\right) \\
&\qquad \times \iint \sigma^3\left(\rho|\dot{u}|^2 + |\nabla u|^2 + |u|^2 + |f|^2\right)dxdt \\
&\leq C(\overline{\rho})(C_0 + C_f)^\theta \left[1 + A_2(T)^{1/2}\right],
\end{aligned} \tag{2.14}$$

where A_2 is the left side of (2.5). The vorticity term in (2.13) is handled in a similar way, and the same bound for $\iint \sigma^3(\rho - \tilde{\rho})^4$ is immediate from Lemma 2.3(e) and

(2.14), and for $\iint \sigma^3|u|^4$ from Lemma 2.3(a) and our presumed bound for the left side of (2.4). We thus obtain from (2.13) that

$$\int_0^T \int_\Omega \sigma^3|\nabla u|^4 dxdt \leq C(\overline{\rho})(C_0 + C_f)^\theta \left[1 + A_2(T)^{1/2}\right].$$

Treating the other terms on the right side of (2.5) in a similar way, we conclude that $A_2(T) \leq C(\overline{\rho})(C_0 + C_f)^\theta \left[1 + A_2(T)^{1/2}\right]$, which proves the required bound for $A_2(T)$. Combining this with the corresponding bound for the left side of (2.4) and the result of Proposition 2.1, we then obtain (2.12). □

3. Pointwise bounds for the density

In this section we derive *a priori* pointwise bounds for the density ρ, where (ρ, u) is the smooth solution discussed in Section 2. These pointwise bounds are proved in Proposition 3.1 below and are then applied in Proposition 3.2 to show that (ρ, u) can be extended as a smooth solution for all time.

Proposition 3.1. *Given numbers $0 < \underline{\rho}_2 < \underline{\rho}_1 < \tilde{\rho} < \overline{\rho}_1 < \overline{\rho}_2 < \overline{\rho}$, there is an $\varepsilon > 0$ such that, if (ρ, u) is a smooth solution of (1.1)–(1.2) as described at the beginning of section 2 with $C_0 + C_f \leq \varepsilon$, $0 < \rho_0(x) \leq \overline{\rho}_1$, and $0 < \rho(x,t) \leq \overline{\rho}$ for $x \in \Omega$ and $t \in [0,T]$, then $0 < \rho(x,t) \leq \overline{\rho}_2$ for $(x,t) \in \overline{\Omega} \times [0,T]$. Similarly if $\rho_0(x) \geq \underline{\rho}_1$ for all x, then $\rho(x,t) \geq \underline{\rho}_2$ for all x and t.*

Proof. We fix a curve $x(t)$ satisfying $\dot{x}(t) = u(x(t),t)$ and substitute the definition (1.15) of F into the mass equation $\frac{d}{dt}\rho(x(t),t) = -\rho \operatorname{div} u$ to obtain

$$(\lambda + \mu)\frac{d}{dt}\log(\rho(x(t),t)) + P(\rho(x(t),t)) - P(\tilde{\rho}) = -F. \tag{3.1}$$

We shall replace F here by its representation in terms of quantitites for which bounds are known from Proposition 2.4. To do this we first observe from (1.23) that

$$\Delta F = \operatorname{div}(\rho\dot{u} - \rho f), \tag{3.2}$$

and for $x \in \partial\Omega$,

$$0 = \rho\dot{u}^3 = F_{x_3} + \mu\left(\omega^{3,1}_{x_1} + \omega^{3,2}_{x_2}\right) + \rho f^3$$

so that, by (1.2),

$$F_{x_3} = \mu\left[(K^{-1}u^1)_{x_1} + (K^{-1}u^2)_{x_2}\right] - \rho f^3,\ x \in \partial\Omega. \tag{3.3}$$

Letting g denote the Neumann–Green's function for the Laplace operator on Ω and noting that $F(\cdot,t) \in L^2(\Omega)$, we then obtain that, for $y \in \Omega$,

$$\begin{aligned} F(y,t) = &- \int_\Omega g_{x_j}(x,y)(\rho\dot{u}^j - \rho f^j)(x,t)dx \\ &+ \mu \int_{\partial\Omega} g(x,y)\left[(K^{-1}u^1)_{x_1} + (K^{-1}u^2)_{x_2}\right](x,t)dS_x. \end{aligned} \tag{3.4}$$

We write $\rho\dot{u}^j = (\rho u^j)_t + \operatorname{div}(\rho u^j u)$ and substitute (3.4) into (3.1) to obtain that

$$\begin{aligned} &(\lambda+\mu)\frac{d}{dt}\log\rho(x(t),t) + [P(\rho(x(t),t)) - P(\tilde\rho)] \\ &= \frac{d}{dt}\int_\Omega g_{x_j}(x,x(t))(\rho u^j)(x,t)dx \\ &- \int_\Omega \Big\{ \left[g_{x_jy_k}(x,x(t))u^k(x(t),t) + g_{x_jx_k}(x,x(t))u^k(x,t)\right](\rho u^j)(x,t) \\ &\qquad\qquad + g_{x_j}(x,x(t))(\rho f^j)(x,t)\Big\} dx \\ &- \mu\int_{\partial\Omega} g(x,x(t))\left[(K^{-1}u^1)_{x_1} + (K^{-1}u^2)_{x_2}\right] dS_x. \end{aligned} \tag{3.5}$$

As we shall see, the pressure term on the left here is dissipative, so that most of the proof consists in obtaining bounds for the terms on the right. We examine two representative terms in detail. First, applying (2.6), we can write the boundary integral in (3.5) as

$$-\mu\int_{\{0\le x_3\le 1\}}[g(K^{-1}u^1)_{x_1} + g(K^{-1}u^2)_{x_2}]dx + O\left(\int_\Omega |\nabla_x g|(|u||\nabla K| + |\nabla u|)dx\right). \tag{3.6}$$

Taking $p \in (3,6]$, we can bound the last term here by

$$\begin{aligned} &\left(\int_{B_1^c(x(t))} + \int_{B_1(x(t))}\right)\frac{|\nabla u(x,t)|dx}{|x-x(t)|^2} \\ &\le C\left[|\nabla u(\cdot,t)|_{L^2} + |\nabla u(\cdot,t)|_{L^p}\right] \\ &\le C\left[|\nabla u(\cdot,t)|_{L^2} + |F(\cdot,t)|_{L^p} + |\omega(\cdot,t)|_{L^p} + |u(\cdot,t)|_{L^p} + |P(\cdot,t)-\tilde P|_{L^p}\right] \end{aligned}$$

by Lemma 2.3(b). Applying Lemma 2.3(a)(d) and estimating the other terms in (3.6) in a similar way, we obtain the following bound for the expression in (3.6):

$$\begin{aligned} C(\overline{\rho})\Bigg[&(C_0+C_f)^\theta + \left(\int|\nabla u|^2dx\right)^{1/2} \\ &+ \left(\int\rho|\dot u|^2dx\right)^{(3p-6)/4p}\left(\int|\nabla u|^2dx\right)^{(6-p)/4}\Bigg] \end{aligned} \tag{3.7}$$

for some fixed $p \in (3, 6]$.

Next we bound the D^2g terms on the right side of (3.5). First recall that $g = g^1 + g^2$ where $g^1(x, y) = \Gamma(|x - y|)$ and Γ is the fundamental solution of the Laplace operator on $\mathbb{R}^3$, and $g^2(x, y) = \Gamma(|x - \overline{y}|)$ where $\overline{y}$ is the reflected point $(y_1, y_2, -y_3)$. The g^1 contribution to the D^2g terms in (3.5) is therefore bounded by

$$\begin{aligned}\left|\int_\Omega \Gamma_{x_j x_k}(|x - x(t)|)\left[u^k(x(t), t) - u^k(x, t)\right](\rho u^j)(x, t)dx\right| \\ \le C\langle u(\cdot, t)\rangle^\beta \int_\Omega |x - x(t)|^{\beta-3}|(\rho u)(x, t)|dx,\end{aligned}$$

where β is to be chosen. The g^2 contribution is the same for $k \ne 3$, but with $x(t)$ replaced by $\overline{x}(t)$, and so satisfies the same bound, since $|x - x(t)| \le |x - \overline{x}(t)|$. For $k = 3$ we instead apply the boundary condition $u^3 = 0$ on $\partial\Omega$ to get

$$|u^3(x, t)| \le \langle u(\cdot, t)\rangle^\beta |x_3|^\beta \le \langle u(\cdot, t)\rangle^\beta |x - \overline{x}(t)|^\beta,$$

and similarly for $u^3(x(t), t)$. The D^2g terms in (3.5) are thus bounded by

$$C\langle u(\cdot, t)\rangle^\beta \left(\int_{B_1(x(t))} + \int_{B_1^c(x(t))}\right)|x - x(t)|^{\beta-3}|(\rho u)(x, t)|dx.$$

The exterior integral here is bounded by $C\int \rho|u|^2 dx \le C(C_0 + C_f)$ and the interior integral by

$$C(\overline{\rho})\langle u(\cdot, t)\rangle^\beta \left(\int_0^1 \tau^{2+r(\beta-3)}d\tau\right)^{1/s}\left(\int \rho|u|^s dx\right)^{1/s}, \tag{3.8}$$

where $s^{-1} + r^{-1} = 1$ and $s \in (3, 6)$. We now choose β, s, and r so that $r < s/(3-\beta)$ and $s \in (3/\beta, q)$, where q is as in (1.10). The first integral in (3.8) is then finite, and the second is bounded by $C(\rho)(C_0 + C_f)^\theta$, by Proposition 2.1 for small time, and by Lemma 2.3(a) and Proposition 2.4 for t away from zero. Observe that this argument requires that $s \le 6$, which implies that $r \ge 6/5$ and $\beta \le 1/2$. We may choose $\beta < 1/2$ and $p = 3/(1 - \beta) \in (3, 6)$, so that, by Lemma 2.3,

$$\begin{aligned}\langle u(\cdot, t)\rangle^\beta &\le C\|\nabla u(\cdot, t)\|_{L^p} \\ &\le C\left[(C_0 + C_f)^\theta + \left(\int|\nabla u|^2\right)^{1/2}\right. \\ &\quad \left. + \left(\int\rho|\dot{u}|^2\right)^{(3p-6)/4p}\left(\int|\nabla u|^2\right)^{(6-p)/4p}\right],\end{aligned} \tag{3.9}$$

where p is the same as in (3.7). Combining these bounds and estimating the other terms in (3.5) in a similar way, we then conclude that

$$
\begin{aligned}
&(\lambda+\mu)\frac{d}{dt}\log\rho(x(t),t)+[P(\rho(x(t)),t)-P(\tilde\rho)]\\
&\quad=\frac{d}{dt}\int_\Omega g_{x_j}(x,x(t))(\rho u^j)(x,t)dx\\
&\qquad+C(\overline{\rho})O\Bigg(\Bigg[(C_0+C_f)^\theta+\left(\int|\nabla u|^2\right)^{1/2}+\left(\int|\nabla u|^2\right)^{3/4}\\
&\qquad\qquad+|f|_{L^2}+|f|_{L^2}^{\alpha}|f|_{L^q}^{1-\alpha}\\
&\qquad\qquad+\left(\int\rho|\dot u|^2\right)^{(3p-6)/4p}\left(\int|\nabla u|^2\right)^{(6-p)/4p}\Bigg]\Bigg)
\end{aligned}
\tag{3.10}
$$

for some fixed $p\in(3,6)$ and $\alpha\in(0,1)$. In addition, a very easy estimate shows that

$$\left|\int_\Omega g_{x_j}(x,x(t))(\rho u^j)(x,t)dx\right|\le C(\overline{\rho})(C_0+C_f)^\theta. \tag{3.11}$$

The pointwise bounds (3.1) are now derived from (3.10) in two steps. First for small time we integrate (3.10) with respect to t, controlling the pressure term by the assumption that $\rho\le\overline{\rho}$ and the first term on the right by the bound in (3.11). Estimates for the other terms are derived from (2.12). For example, if $\tau\le 1$,

$$
\begin{aligned}
&\int_0^\tau\left(\int\rho|\dot u|^2dx\right)^{(3p-6)/4p}\left(\int|\nabla u|^2dx\right)^{(6-p)/4p}dt\\
&\le\left(\int_0^\tau t^{(6-3p)/2p}dt\right)^{1/2}\left(\int_0^\tau\int t\rho|\dot u|^2dxdt\right)^{(3p-6)/4p}\left(\int_0^\tau\int|\nabla u|^2dxdt\right)^{(6-p)/4p}\\
&\le C(\overline{\rho})(C_0+C_f)^\theta,
\end{aligned}
$$

since $p<6$. There is thus a small time τ such that, for $t\le\tau$,

$$\inf\{\log\rho_0(x)\}-C(C_0+C_f)^\theta\le\log\rho(x(t),t)\le\log\overline{\rho}_1+C(C_0+C_f)^\theta,$$

and the assertions in Proposition 3.1 follow immediately for $t\le\tau$.

We shall complete the proof of the upper bound for ρ, the proof of the lower bound being similar. Fix the above $\tau>0$, so that

$$0<\rho(x(t),t)\le\overline{\rho}_1+\tfrac{1}{3}(\bar\rho_2-\bar\rho_1),\ t\in[0,\tau],$$

if ε is small. Now define

$$H(t)=(\lambda+\mu)\log\rho(x(t),t)-\int_\Omega g_{x_j}(x,x(t))(\rho u^j)(x,t)dx \tag{3.12}$$

so that by (3.10) and the results of Proposition 2.4,

$$\frac{dH}{dt} + P(\rho(x(t), t) - P(\tilde{\rho}) = O(C(\overline{\rho})(C_0 + C_f)^{\theta}) \tag{3.13}$$

for $t \geq \tau$. We now apply a standard maximum principle argument: the right side of (3.11) can be made arbitrarily small by taking the constant ε in the statement of Proposition 3.1 sufficiently small, so that $H(t) < (\lambda + \mu) \log[\overline{\rho}_1 + \frac{2}{3}(\overline{\rho_2} - \overline{\rho}_1)]$ on $[0, \tau]$. If $t_0 > \tau$ is the first time that equality occurs then $\rho(x(t_0), t_0) \in [\overline{\rho}_1 + \frac{1}{3}(\overline{\rho_2} - \overline{\rho}_1), \overline{\rho}]$, again by (3.11), so that $P(\rho(x(t_0), t_0)) - P(\tilde{\rho})$ is bounded below by a positive constant, by (1.7). But this contradicts (3.13) because $\frac{dH}{dt}(t_0) \geq 0$ and the right side is small. Thus $H(t) < (\lambda + \mu) \log[\overline{\rho}_1 + \frac{2}{3}(\overline{\rho_2} - \overline{\rho}_1)]$ for all t, and therefore $\rho(x(t), t) \leq \overline{\rho}_2$, again by (3.12) and (3.11), provided that ε is small. □

We can now prove the global existence of smooth solutions of (1.1)–(1.2):

Proposition 3.2. *Assume that the hypotheses and notations of Theorem 1.1 are in force and in addition that $\rho_0 - \tilde{\rho}$, $u_0 \in H^{\infty}(\Omega)$ with $\rho_0(x) > 0$ for all x, that $P \in C^{\infty}([0, \overline{\rho}])$, $K \in H^{\infty}(\mathbb{R}^2)$, and $f \in L^{\infty}([0, \infty); H^{\infty}(\Omega))$. Then there exists $\rho \in C^1(\overline{\Omega} \times [0, \infty))$ and $u \in C^2(\overline{\Omega} \times [0, \infty))$ satisfying* (1.1)–(1.2) *pointwise and for which the conclusions of Propositions 2.4 and 3.1 hold with $T = \infty$.*

Proof. The first step is to obtain a local-in-time solution. This can be done in any of several ways, one of which we sketch. First fix $\eta > 0$ and iterate to solve the system

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = \eta \Delta \rho \\ (\rho u^j)_t + \operatorname{div}(\rho u^j u) + P(\rho)_{x_j} = \mu \Delta u^j + \lambda \operatorname{div} u_{x_j} + \rho f^j \end{cases} \tag{3.14}$$

with the given initial data and with boundary conditions (1.2) together with $\frac{\partial \rho}{\partial n} = 0$. For example, the solution ρ to the problem

$$\begin{cases} \rho_t = \eta \Delta \rho + \operatorname{div} A, & x \in \Omega,\ t > 0 \\[2ex] \dfrac{\partial \rho}{\partial n} = 0, & x \in \partial\Omega,\ t > 0 \end{cases}$$

can be represented via the Neumann–Green's function for Ω in terms of ρ_0 and A, resulting in the bound

$$\|\rho(\cdot, t)\|_{H^k(\Omega)} \leq \|\rho_0\|_{H^k(\Omega)} + C \int_0^t \|A(\cdot, s)\|_{H^k(\Omega)} ds. \tag{3.15}$$

For the unknown u we need to prove the existence of solutions to equations

$$(\rho u^j)_t = \mu \Delta u^j + \lambda \mathrm{div}\, u_{x_j} + \mathrm{div}\, A + B$$

together with the boundary conditions (1.2), where ρ is a given function which is smooth and positive. Standard techniques (finite differences, for example) can be applied to show that a solution u exists and satisfies

$$\|u(\cdot,t)\|_{H^k} \le C\|u_0\|_{H^k} + C\int_0^t (\|A(\cdot,s)\|_{H^k} + \|B(\cdot,s)\|_{H^k})\, ds, \tag{3.16}$$

where the constant depends on properties of ρ. The bounds (3.15)–(3.16) then enable us to solve (3.14) by iteration for small time, obtaining solutions $\rho - \tilde{\rho}$, $u \in H^k$ for large k and up to a positive time which may depend on η. Routine energy estimates similar to but simpler than those of section 2 can then be applied to obtain H^k bounds for u, H^{k-1} bounds for ρ, and time of existence which are *independent of* η. We can then take the limit as $\eta \to 0$ to obtain a smooth solution (ρ, u) of (1.1)–(1.2) for small time, and the *a priori* bounds of Propositions 2.4 and 3.1 then apply to show that this solution can be extended to all time. $\square$

4. Proof of Theorem 1.1

Let (ρ_0, u_0) be as in the statement of Theorem 1.1 and let $j_\delta(x)$ be a standard mollifying kernel of width δ. Define approximate initial data $(\rho_0^\delta, u_0^\delta)$ by

$$\rho_0^\delta = j_\delta * \rho_0 + \delta, \qquad u_0^\delta = j_\delta * u_0.$$

Assuming that similar smooth approximations have been constructed for the system functions P, f, and K, we may then apply Proposition 3.2 to obtain a global smooth solution (ρ^δ, u^δ) of (1.1)–(1.2) satisfying the bounds in Propositions 2.4 and 3.1 with constants which are independent of δ. The proof then consists in showing that these bounds are sufficient to extract a solution (ρ, u) in the limit as $\delta \to 0$.

First, the uniform pointwise bounds in Proposition 3.1 imply that there is a sequence $\delta_j \to 0$ such that $\rho^{\delta_j}(\cdot,t)$ converges weakly in $L^2(K)$ and strongly in $H^{-1}(K)$ for all compact $K \subseteq \Omega$ and for times t in some countable dense subset of $[0,\infty)$. Also, the bounds in Propositions 2.4 and 3.1 applied in the weak equation (1.5) show that $\{\rho^{\delta_j}\}$ is equicontinuous in $C([0,\infty) : H^{-1}(K))$, so that $\rho^{\delta_j}(\cdot,t)$ converges weakly in $L^2_{loc}(\Omega)$ and strongly in $H^{-1}_{loc}(\Omega)$, say to $\rho(\cdot,t)$, for *all* $t \in [0,\infty)$. We can now apply an argument of Feireisl [7] to show that this convergence is in fact *strong*: $\rho^{\delta_j}(\cdot,t) \to \rho(\cdot,t)$ strongly in $L^2(K)$ for all compact $K \subseteq \Omega$. In particular, $P(\rho^{\delta_j}(\cdot,t)) \to P(\rho(\cdot,t))$ pointwise a.e.

To obtain strong limits of $\{u^\delta\}$, we establish uniform Hölder continuity away from $t = 0$. First a standard imbedding result (Ziemer [14], Remark 2.4.3 and Theorem 2.3.4) shows that, for $t \geq \tau > 0$,

$$\langle u^\delta(\cdot,t)\rangle^{1/2}_{\overline{\Omega}} \leq C\|\nabla u^\delta(\cdot,t)\|_{L^6(\Omega)} \leq C(\tau) \tag{4.1}$$

independently of δ by the bounds in Lemma 2.3 and Proposition 2.4. To prove Hölder continuity in time, we fix $x \in \overline{\Omega}$ and times $t_2 > t_1 \geq \tau > 0$ and let B_R be a ball of radius R centered at x. Then by (4.1) and the bounds in Proposition 2.4,

$$\begin{aligned}
&\left|u^\delta(x,t_2) - u^\delta(x,t_1)\right| \leq |B_R\cap\Omega|^{-1}\int_{B_R\cap\Omega}\left|u^\delta(y,t_2)-u^\delta(y,t_1)\right|dy + C(\tau)R^{1/2}\\
&\leq CR^{-3}\int_{t_1}^{t_2}\int_{B_R\cap\Omega}\left|u^\delta_t(y,t)\right|dydt + C(\tau)R^{1/2}\\
&\leq CR^{-1/2}|t_2-t_1|^{1/2}\left[\int_{t_1}^{t_2}\left(\int|u_t^\delta|^6dy\right)^{1/3}dt\right]^{1/2} + C(\tau)R^{1/2}\\
&\leq CR^{-1/2}|t_2-t_1|^{1/2}\left[\int_{t_1}^{t_2}\int_\Omega\left(|\nabla\dot{u}|^2+|u|^2|\nabla u|^2\right)dydt\right]^{1/2} + C(\tau)R^{1/2}\\
&C(\tau)\left[R^{-1/2}|t_2-t_1|^{1/2}+R^{1/2}\right]\\
&\leq C(\tau)|t_2-t_1|^{1/4},
\end{aligned} \tag{4.2}$$

if $R = |t_2 - t_1|^{1/4}$. We have used here a uniform pointwise bound for $u^\delta(x,t)$ for $t \geq \tau$, which follows easily from the bounds in Proposition 2.4. The Ascoli–Arzela theorem therefore applies to show that there is a further subsequence $\delta_j \to 0$ such that $u^{\delta_j} \to u$ uniformly on compact sets in $\overline{\Omega}\times(0,\infty)$.

It follows easily from the strong convergence $\rho^\delta \to \rho$, $u^\delta \to u$, that the limiting functions (ρ, u) are indeed weak solutions of (1.5) and (1.6) with initial data (ρ_0, u_0). One has only to check that each term in the weak equations for $(\rho^{\delta_j}, u^{\delta_j})$ converges as $\delta_j \to 0$ to the corresponding term with $(\rho^{\delta_j}, u^{\delta_j})$ replaced by (ρ, u). We do note, however, that for the boundary term in (1.6),

$$\int_{\partial\Omega}K^{-1}(x)u^{\delta_j}(x,t)\cdot\varphi(x,t)dS_x \longrightarrow \int_{\partial\Omega}K^{-1}(x)u(x,t)\cdot\varphi(x,t)dS_x$$

for each fixed $t > 0$ by the local uniform convergence $u^{\delta_j}(\cdot,t) \to u(\cdot,t)$. Also,

$$\int_0^t\int_{\partial\Omega}K^{-1}u^{\delta_j}\cdot\varphi dS_xdt \longrightarrow 0 \quad \text{as } t\to 0$$

at a rate which is uniform in δ_j by an argument which is similar to and simpler than that given above beginning in (3.9) for the uniform integrability in time of $\langle u^{\delta_j}(\cdot,t)\rangle^\beta$. It therefore follows that

$$\int_{t_1}^{t_2}\int_{\partial\Omega} K^{-1}u^{\delta_j}\cdot\varphi dS_x dt \longrightarrow \int_{t_1}^{t_2}\int_{\partial\Omega} K^{-1}u\cdot\varphi dS_x dt$$

as $\delta_j \to 0$ for test functions φ and for all times $t_2 \geq t_1 \geq 0$.

We have now proven the existence of the solution (ρ, u) satisfying the weak forms (1.5)–(1.6), the pointwise bounds in (1.17), and the Hölder continuity for u in (1.19). The statements in (1.18) follow immediately from the weak forms (1.5)–(1.6), and the bounds in (1.21) follow from Proposition 2.4. For example, if V is an open set with $\overline{V} \subseteq \Omega \times (0,\infty)$, then the strong convergence of u^{δ_j} to u in V implies that $\dot{u}^{\delta_j} = u_t^{\delta_j} + (\nabla u^{\delta_j})u^{\delta_j} \to \dot{u}$ in $\mathcal{D}'(V)$, so that $\nabla\dot{u}^{\delta_j} \to \nabla\dot{u}$ in $\mathcal{D}'(V)$. The bound

$$\iint_V \sigma(t)^3|\nabla\dot{u}^{\delta_j}(x,t)|^2 dxdt \leq C(C_0 + C_f)^\theta \tag{4.3}$$

in Proposition 2.4 then shows that $\nabla\dot{u}^{\delta_j} \rightharpoonup \nabla\dot{u}$ weakly in $L^2(V)$, so that $\nabla\dot{u}$ satisfies (4.3) as well. Since V is arbitrary, we conclude that

$$\int_0^\infty\int_\Omega \sigma^3|\nabla\dot{u}|^2 dxdt \leq C(C_0 + C_f)^\theta.$$

A similar argument applies to the other terms in (1.21). Note that if $\inf \rho_0 > 0$, then $\iint \sigma|\dot{u}^{\delta_j}|^2 dxdt \leq C(C_0 + C_f)^\theta$, so that $\iint \sigma|\dot{u}|^2 dxdt \leq C(C_0 + C_f)^\theta$, which proves the last assertion in the statement of Theorem 1.1.

To complete the proof we need to establish the Hölder continuity in (1.19) for F and ω, which will then imply the strong form of the boundary condition (1.20) for $t > 0$. To do this we first prove the bound

$$\int \rho^\delta(x,t)|\dot{u}^\delta(x,t)|^6 dx \leq C(\tau) \tag{4.4}$$

for $t \geq \tau > 0$, where $C(\tau)$ is again independent of δ. The proof of (4.4) is straightforward but formidably technical, and parallels the proof of (2.5): we differentiate the second equation in (1.1) to obtain an evolution equation for $|\dot{u}|^6$, then integrate. We omit the details. Taking (4.4) as given, we then apply Lemma 2.3(c) and the bounds in Proposition 2.4 to obtain that

$$\begin{aligned}\langle F^\delta(\cdot,t), \omega^\delta(\cdot,t)\rangle_{\Omega}^{1/2} &\leq C\|\nabla F^\delta(\cdot,t), \nabla\omega^\delta(\cdot,t)\|_{L^6(\Omega)} \\ &\leq C(\tau)\end{aligned}$$

for $t \geq \tau > 0$. The proof of Hölder continuity in time follows by an argument similar to that given above in (4.2) for u^δ, except that the weaker information

$$\int_{t_1}^{t_2}\int_\Omega \left((F_t^\delta)^2 + |\omega_t^\delta|^2\right) dxdt \leq C(\tau)$$

for $t_2 \geq t_1 \geq \tau > 0$ results in an estimate for the slightly weaker seminorm $\langle F^\delta, \omega^\delta\rangle^{1/2,1/8}$. □

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