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Spectral Element Discretization of the Circular Driven Cavity. Part IV: The Navier–Stokes Equations

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Abstract. This paper deals with the spectral element discretization of the Navier–Stokes equations in a disk with discontinuous boundary data, which is known as the driven cavity problem. The numerical treatment does not involve any regularization of these data. Relying on a variational formulation in the primitive variables of velocity and pressure, we describe a discretization of these equations and derive error estimates in appropriate weighted Sobolev spaces. We propose an algorithm to solve the nonlinear discrete system and present numerical experiments to verify its efficiency.

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1. Introduction

The term "driven cavity problem" is usually employed to describe the Navier–Stokes equations in a bounded domain, when the tangential component of the velocity is constant on part of the boundary and zero on its complement. We consider these equations when the domain $\tilde{\Omega}$ is the unit disk in $\mathbb{R}^2$ and its boundary $\partial\tilde{\Omega}$ is divided into two connected components $\tilde{\Gamma}_0$ and $\tilde{\Gamma}_1$:

$$\begin{cases} -\Delta \boldsymbol{u} + \frac{1}{\nu}\,\boldsymbol{u}\cdot\nabla\boldsymbol{u} + \mathbf{grad}\; p = \boldsymbol{f} & \text{in } \tilde{\Omega}, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \tilde{\Omega}, \\ \boldsymbol{u}\cdot\tilde{\boldsymbol{n}} = 0 & \text{on } \partial\tilde{\Omega}, \\ \boldsymbol{u}\cdot\tilde{\boldsymbol{\tau}} = g_0 & \text{on } \tilde{\Gamma}_0, \\ \boldsymbol{u}\cdot\tilde{\boldsymbol{\tau}} = g_1 & \text{on } \tilde{\Gamma}_1, \end{cases} \tag{1.1}$$

in the slightly more general case where the data are a density of body forces $\boldsymbol{f}$ and two sufficiently regular functions g_0 and g_1. We are particularly interested in the case where these functions do not coincide at the common endpoints of $\tilde{\Gamma}_0$ and $\tilde{\Gamma}_1$. The unknowns in this problem are the velocity $\boldsymbol{u}$ and the pressure p, while the viscosity ν is a fixed positive parameter. The discretization of problems set in the same domain and with the same possibly discontinuous boundary data has been analyzed for several systems of linear equations, say the Laplace equation [6], the bilaplacian equation [1] and the Stokes problem [2]. However, handling the nonlinear terms in (1.1) leads to different types of difficulties.

The analysis of this system as for the linear one relies on a weighted variational formulation, where two different weights appear. Indeed, we use polar coordinates in the disk, which gives rise to the weight equal to the distance to the center of the disk. Moreover, since the solution of the problem cannot be sought in a standard Hilbertian Sobolev space in the case of discontinuous boundary data, we introduce another weight, equal to a fixed positive power of the distance to the boundary. So, in a first step we must prove the existence of a solution of the variational problem associated with system (1.1). The arguments for this proof are the same as in [12] where the existence in more usual (unweighted but non Hilbertian) Sobolev spaces is established.

In a second step, we only consider the case where the data g_0 and g_1 are different constants. The analysis of the spectral discretization of the Navier–Stokes equations in a disk with continuous data is performed in [5, Chap. XII], however the type of discretization that we study here is rather different. We propose a mortar spectral element discretization of problem (1.1), built from its variational formulation by the Galerkin method with numerical integration. Indeed, the mortar element method [8] seems ideally suited for handling discontinuities of the data since it relies on a nonconforming approximation of the variational spaces on a partition of the domain (the continuity through the interface is just enforced in a weak way). We choose discrete spaces of velocities and pressures such that the discrete velocity is (nearly) exactly divergence-free, which prevents us from deriving an optimal inf-sup condition on the pressure (we refer to [7, Chap. V] and [5, Chap. X] for the explanation of this contradiction). The numerical analysis of the corresponding discrete problem relies on the properties of the analogous discretization of the Stokes problem, as studied in [2], together with the discrete implicit function theorem [9], and leads to optimal error estimates on the velocity. Even if the convergence order is weak for discontinuous boundary data, this convergence property was apparently unknown before, since most often a regularization of the data is used in the discretization of the problem.

Finally, we describe the algorithm that is used for the implementation of the nonlinear problem. The main idea is that the mortar matching conditions can be handled thanks to the introduction of a Lagrange multiplier, according to an idea of [3]. An iterative algorithm is added to treat the nonlinear term. We present numerical experiments in the standard case of the driven cavity problem, i.e. with the data g_0 equal to zero and g_1 equal to 1. The convergence results are in good agreement with the theoretical ones, at least for a large enough viscosity ν.

An outline of the paper is as follows.

• In Section 2, we describe the weighted spaces that are needed for the analysis of problem (1.1). Next, we write the variational formulation of the problem in these spaces and prove the existence of a solution.

- Section 3 is devoted to the description of the mortar discrete problem.
- In Section 4, we prove the existence of a solution of this problem in a neighbourhood of a solution of problem (1.1) which is nonsingular in the sense of [9]. We also derive error estimates between the exact and discrete velocities that are of the same order as for the (linear) Stokes problem.
- In Section 5, we describe the algorithm for handling the nonlinear term and present some numerical results.

2. The continuous Navier–Stokes equations

The first idea consists in translating equations (1.1) in polar coordinates, as performed in [5, Chap. I], [6] and [1] for other types of problems or formulations. Indeed, we observe that the change of variables

$$x = r\,\cos\theta, \quad y = r\,\sin\theta,$$

maps the disk $\tilde{\Omega}$ onto the rectangle $\Omega = [0,1[\times[0,2\pi[$, while the sectors $\tilde{\Omega}_0$ and $\tilde{\Omega}_1$ such that $\partial\tilde{\Omega}_0 \cap \partial\tilde{\Omega}$ and $\partial\tilde{\Omega}_1 \cap \partial\tilde{\Omega}$ coincide with $\tilde{\Gamma}_0$ and $\tilde{\Gamma}_1$, are mapped onto the rectangles

$$\Omega_0 = [0,1[\times]0,\theta_0[\quad \text{and} \quad \Omega_1 = [0,1[\times]\theta_0, 2\pi[,$$

for a fixed angle θ_0, $0 < \theta_0 < 2\pi$. We denote by Γ_0, Γ_1, Γ_{01} and Γ_{10} the images of $\tilde{\Gamma}_0$, $\tilde{\Gamma}_1$ and of the two segments in $\partial\tilde{\Omega}_0 \cap \partial\tilde{\Omega}_1$. This notation is illustrated in Figure 1.

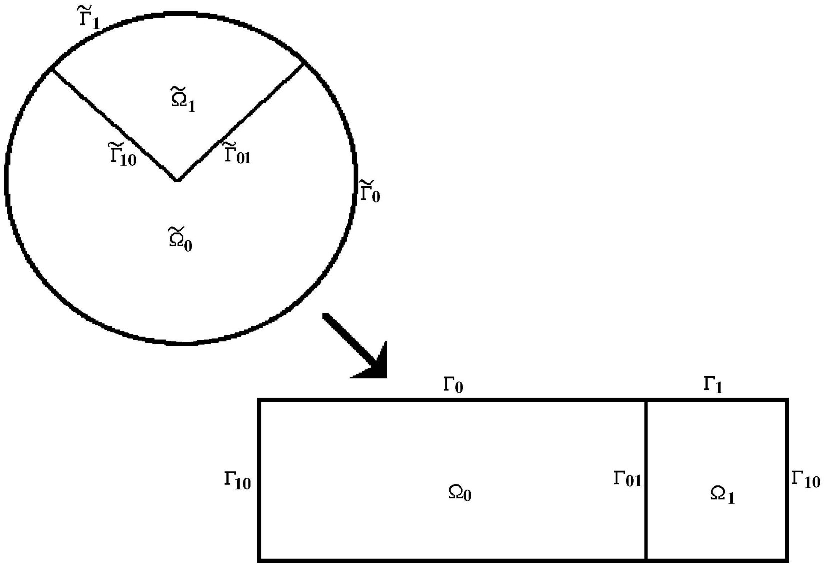

FIG. 1.

On Ω, we introduce the following differential operators in polar coordinates, defined on scalar functions, first the scalar Laplace operator

$$\Delta_r = \partial_r^2 + r^{-1}\,\partial_r + r^{-2}\,\partial_\theta^2,$$

next the vector-valued gradient and curl

$$\mathbf{grad}_r = \begin{pmatrix} \partial_r \\ r^{-1}\,\partial_\theta \end{pmatrix}, \qquad \mathbf{curl}_r = \begin{pmatrix} r^{-1}\,\partial_\theta \\ -\partial_r \end{pmatrix},$$

and finally the divergence operator defined on vector-valued functions $\boldsymbol{v} = (v_r, v_\theta)$ by

$$\mathrm{div}_r \boldsymbol{v} = \partial_r v_r + r^{-1}\,v_r + r^{-1}\,\partial_\theta v_\theta.$$

Let us denote by u_r and u_θ, resp. by f_r and f_θ, the radial and angular components of the velocity $\boldsymbol{u}$, resp. of the data $\boldsymbol{f}$. As described in [5, §IX.2], the Navier–Stokes equations (1.1) then write

$$\begin{cases} -\partial_r^2 u_r - r^{-1}\,\partial_r u_r + r^{-2}\,u_r - r^{-2}\,\partial_\theta^2 u_r + 2r^{-2}\,\partial_\theta u_\theta & \\ \quad +\frac{1}{\nu}\left(u_r\,\partial_r u_r + r^{-1}\,u_\theta\,\partial_\theta u_r - r^{-1}\,u_\theta^2\right) + \partial_r p = f_r & \text{in } \Omega, \\ -\partial_r^2 u_\theta - r^{-1}\partial_r u_\theta + r^{-2} u_\theta - r^{-2}\partial_\theta^2 u_\theta - 2r^{-2}\partial_\theta u_r & \\ \quad +\frac{1}{\nu}\left(u_r \partial_r u_\theta + r^{-1} u_\theta \partial_\theta u_\theta + r^{-1} u_r\,u_\theta\right) + r^{-1}\,\partial_\theta p = f_\theta & \text{in } \Omega, \\ \partial_r u_r + r^{-1}\,u_r + r^{-1}\,\partial_\theta u_\theta = 0 & \text{in } \Omega, \\ (u_r\,, u_\theta) = (0, g_0) & \text{on } \Gamma_0, \\ (u_r\,, u_\theta) = (0, g_1) & \text{on } \Gamma_1, \\ (u_r, u_\theta)(\cdot, 0) = (u_r, u_\theta)(\cdot, 2\pi) & \text{on } \Gamma_{10}, \\ (\partial_\theta u_r, \partial_\theta u_\theta)(\cdot, 0) = (\partial_\theta u_r, \partial_\theta u_\theta)(\cdot, 2\pi) & \text{on } \Gamma_{10}. \end{cases} \tag{2.1}$$

2.1. The weighted spaces

From now on, we fix a real number α, $0 < \alpha < 1$. According to the ideas developed in [6] and [1], weighted Sobolev spaces are needed to handle the possible discontinuity of the boundary data in (1.1), hence to write the variational formulation of problem (2.1). So, on the interval $\mathcal{I} =]0,1[$, for any real number p, $1 < p < +\infty$, and for any separable Banach space E with norm $\|\cdot\|_E$, we consider the spaces

$$L^p_\alpha(\mathcal{I}; E) = \left\{ v : \mathcal{I} \to E \text{ measurable; } \int_0^1 \|v(r)\|_E^p\, r(1-r)^\alpha\, dr < +\infty \right\}.$$

We denote by $L^p_\alpha(\mathcal{I})$ the space $L^p_\alpha(\mathcal{I}; \mathbb{R})$, and by $L^p_\alpha(\Omega)$ the tensorized space $L^p_\alpha\big(\mathcal{I}; L^p(0, 2n)\big)$. All these spaces are provided with natural norms.

We introduce the spaces of vector functions

$$\mathbf{X}^1_\alpha(\Omega) = \big\{ \boldsymbol{v} = (v_r, v_\theta) \in L^2_\alpha(\Omega)^2;\ (\partial_r v_r, \partial_r v_\theta) \in L^2_\alpha(\Omega)^2 \\ \text{and } \big(r^{-1}\,(v_r + \partial_\theta v_\theta), r^{-1}\,(\partial_\theta v_r - v_\theta)\big) \in L^2_\alpha(\Omega)^2 \big\}.$$

Owing to this definition, it is proven in [5, §II.3] that sufficiently smooth functions $\boldsymbol{v}$ in $\mathbf{X}^1_\alpha(\Omega)$ satisfy

$$(v_r + \partial_\theta v_\theta)(0,\theta) = (\partial_\theta v_r - v_\theta)(0,\theta) = 0, \quad 0 \le \theta < 2\pi,$$

and that these nullity properties are the correct ones for the velocity $\boldsymbol{u} = (u_r, u_\theta)$ when it is smooth enough. We also introduce two subspaces of $\mathbf{X}^1_\alpha(\Omega)$, namely

$$\mathbf{X}^1_{\alpha\sharp}(\Omega) = \big\{\boldsymbol{v} \in \mathbf{X}^1_\alpha(\Omega);\, \boldsymbol{v}(\cdot,0) = \boldsymbol{v}(\cdot,2\pi) \text{ on }]0,1[\big\},$$
$$\mathbf{X}^1_{\alpha 0}(\Omega) = \big\{\boldsymbol{v} \in \mathbf{X}^1_{\alpha\sharp}(\Omega);\ \boldsymbol{v} = 0 \text{ on } \overline{\Gamma}_0 \cup \overline{\Gamma}_1\big\}.$$

These spaces are provided with the norm

$$\|\boldsymbol{v}\|_{\mathbf{X}^1_\alpha(\Omega)} = \Big(\|v_r\|^2_{L^2_\alpha(\Omega)} + \|v_\theta\|^2_{L^2_\alpha(\Omega)} + \|\partial_r v_r\|^2_{L^2_\alpha(\Omega)} + \|\partial_r v_\theta\|^2_{L^2_\alpha(\Omega)} + \|r^{-1}(v_r + \partial_\theta v_\theta)\|^2_{L^2_\alpha(\Omega)} + \|r^{-1}(\partial_\theta v_r - v_\theta)\|^2_{L^2_\alpha(\Omega)}\Big)^{\frac12},$$

and the seminorm

$$|\boldsymbol{v}|_{\mathbf{X}^1_\alpha(\Omega)} = \Big(\|\partial_r v_r\|^2_{L^2_\alpha(\Omega)} + \|\partial_r v_\theta\|^2_{L^2_\alpha(\Omega)} + \|r^{-1}(v_r + \partial_\theta v_\theta)\|^2_{L^2_\alpha(\Omega)} + \|r^{-1}(\partial_\theta v_r - v_\theta)\|^2_{L^2_\alpha(\Omega)}\Big)^{\frac12}.$$

We also need the following spaces

$$M_{1\alpha} = \Big\{q \in L^2_\alpha(\Omega);\ \int_\Omega q\, r(1-r)^\alpha\, dr\, d\theta = 0\Big\}.$$
$$M_{2\alpha} = \Big\{q \in L^2_\alpha(\Omega);\ \int_\Omega q\, r\, dr\, d\theta = 0\Big\}.$$

Note that, since α is < 1, both $M_{1\alpha}$ and $M_{2\alpha}$ are closed subspaces of $L^2_\alpha(\Omega)$.

2.2. The variational formulation

The variational formulation of problem (2.1) now reads:

Find a pair $(\boldsymbol{u} = (u_r, u_\theta), p)$ *in* $\mathbf{X}^1_{\alpha\sharp}(\Omega) \times M_{1\alpha}$, *with*

$$(u_r\,, u_\theta) = (0, g_0) \quad \text{on} \quad \Gamma_0 \qquad \textit{and} \qquad (u_r\,, u_\theta) = (0, g_1) \quad \text{on} \quad \Gamma_1, \tag{2.2}$$

such that

$$\forall \boldsymbol{v} = (v_r, v_\theta) \in \mathbf{X}^1_{\alpha 0}(\Omega), \quad a_\alpha(\boldsymbol{u}, \boldsymbol{v}) + b_{1\alpha}(\boldsymbol{v}, p) + C_\alpha(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \big(f_r(r,\theta) v_r(r,\theta) + f_\theta(r,\theta) v_\theta(r,\theta)\big)\, r(1-r)^\alpha\, dr\, d\theta, \tag{2.3}$$
$$\forall q \in M_{2\alpha}, \quad b_{2\alpha}(\boldsymbol{u}, q) = 0,$$

where the bilinear forms $a_\alpha(\cdot,\cdot)$, $b_{1\alpha}(\cdot,\cdot)$ and $b_{2\alpha}(\cdot,\cdot)$ are defined respectively by

$$a_\alpha(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \big(\partial_r u_r\,\partial_r(v_r(1-r)^\alpha) + \partial_r u_\theta\,\partial_r(v_\theta(1-r)^\alpha)\big)\,r\,dr\,d\theta$$
$$+ \int_\Omega \big((u_r+\partial_\theta u_\theta)(v_r+\partial_\theta v_\theta) + (u_\theta-\partial_\theta u_r)(v_\theta-\partial_\theta v_r)\big)\,r^{-1}(1-r)^\alpha\,dr\,d\theta,$$

$$b_{1\alpha}(\boldsymbol{v},q) = -\int_\Omega \big(\partial_r(v_r\,(1-r)^\alpha) + r^{-1}v_r(1-r)^\alpha + r^{-1}\,\partial_\theta v_\theta\,(1-r)^\alpha\big)\,q\,r\,dr\,d\theta,$$
$$b_{2\alpha}(\boldsymbol{v},q) = -\int_\Omega (\partial_r v_r + r^{-1}\,v_r + r^{-1}\,\partial_\theta v_\theta)\,q\,r(1-r)^\alpha\,dr\,d\theta,$$

while the trilinear form $C_\alpha(\cdot;\cdot,\cdot)$ is defined by

$$C_\alpha(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v}) = \frac{1}{\nu}\int_\Omega w_r\,(\partial_r u_r\,v_r + \partial_r u_\theta\,v_\theta)\,r(1-r)^\alpha\,dr\,d\theta$$
$$+ \frac{1}{\nu}\int_\Omega w_\theta\,\big((\partial_\theta u_r - u_\theta)v_r + (\partial_\theta u_\theta + u_r)v_\theta\big)\,(1-r)^\alpha\,dr\,d\theta.$$

Problems (2.1) and (2.2)–(2.3) are equivalent in the following sense:

- any solution $(\boldsymbol{u},p)$ of (2.1) which belongs to $\mathbf{X}^1_{\alpha\sharp}(\Omega)\times M_{1\alpha}$ is a solution of (2.2)–(2.3),
- any solution of (2.2)–(2.3) is a solution of (2.1) when the partial differential equations are taken in the distribution sense.

In contrast to most variational formulations, the linear part of the problem (i.e. with $\frac{1}{\nu}$ equal to zero) is not of standard saddle-point type, however its well-posedness is proven in [2, Thm 2.6] by using the arguments in [11] and [4, §2]. Moreover, we introduce the kernels, for $i=1$ and 2,

$$\mathbf{V}_{i\alpha} = \big\{\boldsymbol{v}\in\mathbf{X}^1_{\alpha0}(\Omega);\ \forall q\in M_{i\alpha},\ b_{i\alpha}(\boldsymbol{v},q)=0\big\}.$$

It is readily checked that

$$\mathbf{V}_{1\alpha} = \big\{\boldsymbol{v}\in\mathbf{X}^1_{\alpha0}(\Omega);\ \mathrm{div}_r(\boldsymbol{v}\,(1-r)^\alpha)=0\ \text{ in }\ \Omega\big\},$$
$$\mathbf{V}_{2\alpha} = \big\{\boldsymbol{v}\in\mathbf{X}^1_{\alpha0}(\Omega);\ \mathrm{div}_r\boldsymbol{v}=0\ \text{ in }\ \Omega\big\}. \tag{2.4}$$

Let us also introduce the subspace

$$\mathbf{V}^\sharp_{2\alpha} = \big\{\boldsymbol{v}\in\mathbf{X}^1_{\alpha\sharp}(\Omega);\ \mathrm{div}_r\boldsymbol{v}=0\ \text{ in }\ \Omega\big\}.$$

Then, problem (2.2)–(2.3) admits the reduced formulation

Find $\boldsymbol{u}$ in $\mathbf{V}^\sharp_{2\alpha}$ satisfying (2.2) *and such that*

$$\forall\boldsymbol{v}=(v_r,v_\theta)\in\mathbf{V}_{1\alpha},\quad a_\alpha(\boldsymbol{u},\boldsymbol{v}) + C_\alpha(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v})$$
$$= \int_\Omega \big(f_r(r,\theta)v_r(r,\theta) + f_\theta(r,\theta)v_\theta(r,\theta)\big)\,r(1-r)^\alpha\,dr\,d\theta. \tag{2.5}$$

2.3. Some properties of the bilinear and trilinear forms

The continuity of the three bilinear forms $a_\alpha(\cdot,\cdot)$, $b_{1\alpha}(\cdot,\cdot)$ and $b_{2\alpha}(\cdot,\cdot)$ on $\mathbf{X}^1_\alpha(\Omega)\times\mathbf{X}^1_{\alpha 0}(\Omega)$, $\mathbf{X}^1_{\alpha 0}(\Omega)\times L^2_\alpha(\Omega)$ and $\mathbf{X}^1_\alpha(\Omega)\times L^2_\alpha(\Omega)$, respectively, is derived in [2, Lemma 2.1] from Hardy type inequalities. Moreover, the following inf-sup conditions are derived in [2, Lemmas 2.2 & 2.3]: there exist positive constants β_i, $i=1$ and 2, such that

$$\forall q \in M_{i\alpha}, \qquad \sup_{\boldsymbol{v}\in\mathbf{X}^1_{\alpha 0}(\Omega)} \frac{b_{i\alpha}(\boldsymbol{v},q)}{\|\boldsymbol{v}\|_{\mathbf{X}^1_\alpha(\Omega)}} \geq \beta_i \, \|q\|_{L^2_\alpha(\Omega)}. \tag{2.6}$$

From this property, it is readily checked that problems (2.2)–(2.3) and (2.2)–(2.5) are equivalent in the following sense:

- for any solution $(\boldsymbol{u},p)$ of problem (2.2)–(2.3), the velocity $\boldsymbol{u}$ is a solution of problem (2.2)–(2.5),
- for any solution $\boldsymbol{u}$ of problem (2.2)–(2.5), there exists a unique pressure p in $M_{1\alpha}$ such that $(\boldsymbol{u},p)$ is a solution of problem (2.2)–(2.3).

The form $a(\cdot,\cdot)$ is elliptic on $\mathbf{X}^1_{\alpha 0}(\Omega)$, however this is not sufficient for the analysis of problem (2.2)–(2.5). So, we recall from [2, Proposition 2.4] the next property. Let T_α be the operator defined as follows: any function $\boldsymbol{u}$ in $\mathbf{V}_{2\alpha}$ is divergence-free, hence there exists a function ψ, unique up to an additive constant, such that $\boldsymbol{u}=\mathbf{curl}_r\psi$; thus, we fix the constant such that ψ vanishes at $r=1$ and take $T_\alpha\boldsymbol{u}$ equal to $(1-r)^{-\alpha}\,\mathbf{curl}_r\big(\psi\,(1-r)^\alpha\big)$.

Lemma 2.1. *The operator T_α is an isomorphism from $\mathbf{V}_{2\alpha}$ onto $\mathbf{V}_{1\alpha}$. Moreover, there exists a real number α_0, $\frac{1}{2}<\alpha_0<1$, and a positive constant γ only depending on α such that, for all α, $0\leq\alpha<\alpha_0$, the following property holds*

$$\forall \boldsymbol{u}\in\mathbf{V}_{2\alpha}, \quad a_\alpha(\boldsymbol{u},T_\alpha\boldsymbol{u}) \geq \gamma\,\|\boldsymbol{u}\|^2_{\mathbf{X}^1_\alpha(\Omega)}. \tag{2.7}$$

From now on, we work with an $\alpha\leq\alpha_0$, where $\alpha_0>\frac{1}{2}$ was introduced in the previous lemma. Indeed, when this condition holds, by combining all the previous arguments, we derive that, in the linear case (i.e. for $\frac{1}{\nu}=0$), problem (2.2)–(2.3) has a unique solution $(\boldsymbol{u},p)$. So, we must now investigate the nonlinear term. We begin with a lemma which is proved in [5, §IX.2] for instance.

Lemma 2.2. *The space $\mathbf{X}^1_\alpha(\Omega)$ is imbedded in $L^4_\alpha(\Omega)^2$ with a continuous and compact imbedding.*

We now state the properties of the form $C_\alpha(\cdot;\cdot,\cdot)$. The continuity property is a simple consequence of Hölder's inequality.

Lemma 2.3. *There exists a constant c independent of ν such that the following* *continuity property holds*

$$\begin{aligned}&\forall \boldsymbol{w} \in L^4_\alpha(\Omega)^2,\ \forall \boldsymbol{u} \in \mathbf{X}^1_\alpha(\Omega),\ \forall \boldsymbol{v} \in L^4_\alpha(\Omega)^2,\\ &\qquad |\, C_\alpha(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v})\,| \le \frac{c}{\nu}\, \|\boldsymbol{w}\|_{L^4_\alpha(\Omega)^2}\, \|\boldsymbol{u}\|_{\mathbf{X}^1_\alpha(\Omega)}\, \|\boldsymbol{v}\|_{L^4_\alpha(\Omega)^2}.\end{aligned} \tag{2.8}$$

Unfortunately, the form $C_\alpha(\cdot;\cdot,\cdot)$ does not satisfy the same antisymmetry properties as for the unweighted formulation of the Navier–Stokes equations, but only the restricted one

$$\forall \boldsymbol{v} \in \mathbf{V}_{1\alpha}, \forall \boldsymbol{u} \in \mathbf{X}^1_{\alpha\sharp}(\Omega), \quad C_\alpha(\boldsymbol{v}; \boldsymbol{u}, \boldsymbol{u}) = 0, \tag{2.9}$$

which is not sufficient in what follows. So, we need a further property that is stated in the next lemma.

Lemma 2.4. *For any α, $0 \le \alpha \le \alpha_0$, there exists a constant λ such that the following property holds*

$$\forall \boldsymbol{w} \in \mathbf{V}_{2\alpha}, \forall \boldsymbol{u} \in \mathbf{V}_{2\alpha}, \quad |\, C_\alpha(\boldsymbol{w}; \boldsymbol{u}, T_\alpha \boldsymbol{u})\,| \le \lambda\, \frac{\alpha}{\nu}\, \|\boldsymbol{w}\|_{\mathbf{X}^1_\alpha(\Omega)} \|\boldsymbol{u}\|^2_{\mathbf{X}^1_\alpha(\Omega)}. \tag{2.10}$$

Proof. We derive from (2.9) that

$$C_\alpha(\boldsymbol{w}; \boldsymbol{u}, T_\alpha \boldsymbol{u}) = C_\alpha(\boldsymbol{w} - T_\alpha \boldsymbol{w}; \boldsymbol{u}, T_\alpha \boldsymbol{u}) - C_\alpha(T_\alpha \boldsymbol{w}; \boldsymbol{u}, \boldsymbol{u} - T_\alpha \boldsymbol{u}),$$

whence, from Lemmas 2.2 and 2.3,

$$\begin{aligned}&|\, C_\alpha(\boldsymbol{w}; \boldsymbol{u}, T_\alpha \boldsymbol{u})\,|\\ &\le \frac{c}{\nu}\, \|\boldsymbol{u}\|_{\mathbf{X}^1_\alpha(\Omega)} \big(\|\boldsymbol{w} - T_\alpha \boldsymbol{w}\|_{\mathbf{X}^1_\alpha(\Omega)} \|T_\alpha \boldsymbol{u}\|_{\mathbf{X}^1_\alpha(\Omega)} + \|T_\alpha \boldsymbol{w}\|_{\mathbf{X}^1_\alpha(\Omega)} \|\boldsymbol{u} - T_\alpha \boldsymbol{u}\|_{\mathbf{X}^1_\alpha(\Omega)} \big).\end{aligned}$$

Moreover, it follows from the definition of T_α that, for any function $\boldsymbol{u} = \mathbf{curl}_r \psi$, $T_\alpha \boldsymbol{u}$ is equal to $\boldsymbol{u} - \alpha\, \boldsymbol{z}(\boldsymbol{u})$, with $z_r(\boldsymbol{u})$ equal to zero and $z_\theta(\boldsymbol{u})$ equal to $\psi\,(1-r)^{-1}$. Thanks to standard Hardy inequalities, it can also be checked [2, Lemma 2.7] that there exists a constant c only depending on α_0 (but not on α) such that

$$\|\boldsymbol{z}(\boldsymbol{u})\|_{\mathbf{X}^1_\alpha(\Omega)} \le c\, \|\boldsymbol{u}\|_{\mathbf{X}^1_\alpha(\Omega)}.$$

Combining all this yields the desired result.

2.4. Existence and uniqueness for homogeneous boundary conditions

In a first step, we need a density result.

Lemma 2.5. *The space $\mathbf{X}^1_{\alpha\sharp}(\Omega)$ is separable.*

Proof. This separability property is an easy consequence of the density of the space $\mathcal{D}_\sharp(\overline{\Omega})$ of infinitely differentiable and periodic functions on Ω in $\mathbf{X}^1_{\alpha\sharp}(\Omega)$.

Any function $\boldsymbol{v}$ in $\mathbf{X}^1_{\alpha\sharp}(\Omega)$ admits the Fourier expansion $\sum_{k\in\mathbb{Z}} \boldsymbol{v}^k\, e^{ik\theta}$, and it is readily checked that the series $(\boldsymbol{v}_K)_K$ of truncated Fourier approximations $\boldsymbol{v}_K = \sum_{k=-K}^{K} \boldsymbol{v}^k\, e^{ik\theta}$ converges to $\boldsymbol{v}$ in $\mathbf{X}^1_{\alpha\sharp}(\Omega)$. Moreover, the Fourier coefficients $\boldsymbol{v}^k$ of $\boldsymbol{v}$ belong to appropriate Sobolev spaces on $[0,1]$, identified in [5 §II.3.b], and proving the density of $\mathcal{D}([0,1])$ in these spaces follows from standard arguments.

By using Brouwer's fixed point theorem, we are now in a position to prove the existence result. However, we first consider problem (2.2)–(2.5).

Proposition 2.6. *In the case of homogeneous boundary conditions $g_0 = g_1 = 0$ and for any data (f_r, f_θ) in the dual space of $\mathbf{X}^1_{\alpha_0 0}(\Omega)$, there exists a real number α_*, $0 < \alpha_* < \alpha_0$, such that, for any α, $0 \le \alpha \le \alpha_*$, problem* (2.2)*–*(2.5) *has a solution $\boldsymbol{u}$ in $V_{2\alpha}$.*

Proof. We derive from Lemma 2.5 that there exists an increasing sequence of finite-dimensional subspaces $\mathbf{V}^n_\alpha$, $n \ge 0$, of $\mathbf{V}_{2\alpha}$ such that $\cup_{n\ge 0}\mathbf{V}^n_\alpha$ is dense in $\mathbf{V}_{2\alpha}$. The proof is then performed in two steps.

Step 1) We define a mapping Φ from $\mathbf{V}_{2\alpha}$ into its dual space and, by restriction, from each $\mathbf{V}^n_\alpha$ into its dual space by

$$\begin{aligned}\langle \Phi(\boldsymbol{u}), \boldsymbol{w}\rangle &= a_\alpha(\boldsymbol{u}, T_\alpha \boldsymbol{w}) + C_\alpha(\boldsymbol{u}; \boldsymbol{u}, T_\alpha \boldsymbol{w}) \\ &\quad - \int_\Omega \big(f_r(r,\theta)(T_\alpha \boldsymbol{w})_r(r,\theta) + f_\theta(r,\theta)(T_\alpha \boldsymbol{w})_\theta(r,\theta)\big)\, r(1-r)^\alpha\, dr\, d\theta,\end{aligned}$$

where the last integral can be replaced by a duality product when necessary. The continuity of this mapping follows from Lemmas 2.1 to 2.3. Moreover, we deduce from Lemmas 2.1 and 2.4 that, for all $\boldsymbol{u}$ in $\mathbf{V}_{2\alpha}$,

$$\langle \Phi(\boldsymbol{u}), \boldsymbol{u}\rangle \ge \gamma\, \|\boldsymbol{u}\|^2_{\mathbf{X}^1_\alpha(\Omega)} - \lambda\, \frac{\alpha}{\nu}\, \|\boldsymbol{u}\|^3_{\mathbf{X}^1_\alpha(\Omega)} - c\, \|\boldsymbol{f}\|_{\mathbf{X}^1_\alpha(\Omega)'} \|\boldsymbol{u}\|_{\mathbf{X}^1_\alpha(\Omega)}$$

(it can be checked that the constant γ in (2.7) depends continuously on α, so for a while we denote by γ the minimal value of this constant for α running through $[0, \alpha_0]$; similarly, λ stands for the maximal value of the constant in (2.10) for α running through the same interval). It is readily checked that, if α is $\ge \alpha'$, $\mathbf{X}^1_{\alpha' 0}(\Omega)$ is imbedded in $\mathbf{X}^1_{\alpha 0}(\Omega)$ and moreover that, for smooth enough functions $\boldsymbol{v}$, the mapping: $\alpha \mapsto \|\boldsymbol{v}\|_{\mathbf{X}^1_\alpha(\Omega)}$ is decreasing. So, the mapping: $\alpha \mapsto \|\boldsymbol{f}\|_{\mathbf{X}^1_{\alpha 0}(\Omega)'}$ is increasing, and the previous estimate leads to

$$\langle \Phi(\boldsymbol{u}), \boldsymbol{u}\rangle \ge \gamma\, \|\boldsymbol{u}\|^2_{\mathbf{X}^1_\alpha(\Omega)} - \lambda\, \frac{\alpha}{\nu}\, \|\boldsymbol{u}\|^3_{\mathbf{X}^1_\alpha(\Omega)} - c\, \|\boldsymbol{f}\|_{\mathbf{X}^1_{\alpha_0 0}(\Omega)'} \|\boldsymbol{u}\|_{\mathbf{X}^1_\alpha(\Omega)}.$$

Let us now fix an α^* such that

$$\gamma^2 - 4c\lambda\, \frac{\alpha^*}{\nu}\, \|\boldsymbol{f}\|_{\mathbf{X}^1_{\alpha_0 0}(\Omega)'} > 0,$$

and, for each $\alpha \le \alpha^*$, take μ_α such that

$$\mu_\alpha = \frac{\nu\, \gamma}{2\lambda\, \alpha}.$$

Thus, for each n, $\langle \Phi(\boldsymbol{u}_n), \boldsymbol{u}_n \rangle$ is nonnegative on the sphere of $\mathbf{V}_\alpha^n$ with radius μ_α. So, applying Brouwer's fixed point theorem [10, Chap. IV, Cor. 1.1] yields that there exists at least a function $\boldsymbol{u}_n$, with $\|\boldsymbol{u}_n\|_{\mathbf{X}_\alpha^1(\Omega)} \leq \mu_\alpha$, such that $\Phi(\boldsymbol{u}_n) = 0$.

Step 2) The sequence $(\boldsymbol{u}_n)_{n\geq 0}$ of these solutions is bounded in $\mathbf{X}_\alpha^1(\Omega)$, so there exists a subsequence, still denoted by $(\boldsymbol{u}_n)_{n\geq 0}$ for simplicity, which converges to a function $\boldsymbol{u}$ weakly in $\mathbf{X}_\alpha^1(\Omega)$ and, from the compactness property stated in Lemma 2.2, also converges to $\boldsymbol{u}$ strongly in $L_\alpha^4(\Omega)^2$. Since the sequence $(\mathbf{V}_\alpha^n)_n$ is increasing, we have for all integers n_0 and $n \geq n_0$,

$$\forall \boldsymbol{w}_{n_0} \in \mathbf{V}_\alpha^{n_0}, \quad \langle \Phi(\boldsymbol{u}_n), \boldsymbol{w}_{n_0} \rangle = 0.$$

Thanks to Lemma 2.3, letting n tend to $+\infty$ gives

$$\forall \boldsymbol{w}_{n_0} \in \mathbf{V}_\alpha^{n_0}, \quad \langle \Phi(\boldsymbol{u}), \boldsymbol{w}_{n_0} \rangle = 0.$$

Finally it follows from the density of $\cup_{n\geq 0}\mathbf{V}_\alpha^n$ in $\mathbf{V}_{2\alpha}$ that $\boldsymbol{u}$ is a solution of problem (2.2)–(2.5).

Theorem 2.7. *In the case of homogeneous boundary conditions $g_0 = g_1 = 0$, for any data (f_r, f_θ) in $\mathbf{X}_{\alpha_0 0}^1(\Omega)'$, problem* (2.2)–(2.3) *has a solution $(\boldsymbol{u}, p)$ in $\mathbf{X}_{\alpha 0}^1(\Omega) \times M_{1\alpha}$.*

Proof. When α is $\leq \alpha^*$, the existence of a solution u of problem (2.2)–(2.5) is established in Proposition 2.6 and then the existence of a function p such that $(\boldsymbol{u}, p)$ is a solution of problem (2.2)–(2.3) is an easy consequence of the inf-sup condition (2.6) for $i = 1$, see [10, Chap. I, Lemma 4.1]. When α is $> \alpha^*$, the solution $(\boldsymbol{u}^*, p^*)$ of problem (2.2)–(2.3) for $\alpha = \alpha^*$ is a solution of problem (2.1). Moreover, it follows from the imbedding of $\mathbf{X}_{\alpha*0}^1$ into $\mathbf{X}_{\alpha 0}^1$ for all $\alpha \geq \alpha^*$ and also of $L_{\alpha^*}^2(\Omega)$ into $L_\alpha^2(\Omega)$ that this solution, up to an additive constant on the pressure (depending on α), is also a solution of problem (2.2)–(2.5) for all $\alpha > \alpha^*$.

As is usual in the case of the Navier–Stokes equations, the uniqueness result requires some further assumptions.

Theorem 2.8. *In the case of homogeneous boundary conditions $g_0 = g_1 = 0$ and for $0 \leq \alpha \leq \alpha_0$, there exists a constant c_0 such that there is at most one solution $(\boldsymbol{u}, p)$ of problem* (2.2)–(2.3) *in $\mathbf{X}_{\alpha 0}^1(\Omega) \times M_{1\alpha}$ which satisfies*

$$\|\boldsymbol{u}\|_{X_\alpha^1(\Omega)} \leq \frac{\gamma\,\nu}{\lambda\,\alpha + c_0}. \tag{2.11}$$

Proof. Let $(\boldsymbol{u}_1, p_1)$ and $(\boldsymbol{u}_2, p_2)$ be two solutions of problem (2.2)–(2.3) such that both $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ satisfy (2.11). Thus, we have

$$\forall \boldsymbol{v} \in \mathbf{V}_{1\alpha}, \quad a_\alpha(\boldsymbol{u}_1 - \boldsymbol{u}_2, \boldsymbol{v}) = -C_\alpha(\boldsymbol{u}_1; \boldsymbol{u}_1, \boldsymbol{v}) + C_\alpha(\boldsymbol{u}_2; \boldsymbol{u}_2, \boldsymbol{v}).$$

Next, we set $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$ and we take $\boldsymbol{v}$ equal to $T_\alpha \boldsymbol{u}$ in the previous line. Thanks to Lemma 2.1, this yields

$$\gamma \|\boldsymbol{u}\|^2_{\mathbf{X}^1_\alpha(\Omega)} \le -C_\alpha(\boldsymbol{u}_1; \boldsymbol{u}, T_\alpha \boldsymbol{u}) - C_\alpha(\boldsymbol{u}; \boldsymbol{u}_2, T_\alpha \boldsymbol{u}).$$

Using Lemma 2.4 for the first term and Lemma 2.3 for the second term gives

$$\gamma \|\boldsymbol{u}\|^2_{\mathbf{X}^1_\alpha(\Omega)} \le \|\boldsymbol{u}\|^2_{\mathbf{X}^1_\alpha(\Omega)} \big(\lambda \frac{\alpha}{\nu} \|\boldsymbol{u}_1\|_{\mathbf{X}^1_\alpha(\Omega)} + \frac{c}{\nu} \|\boldsymbol{u}_2\|_{\mathbf{X}^1_\alpha(\Omega)}\big).$$

For an appropriate constant c_0 in (2.11), this yields that the function $\boldsymbol{u}$ satisfies

$$\|\boldsymbol{u}\|^2_{\mathbf{X}^1_\alpha(\Omega)} \le \mu \|\boldsymbol{u}\|^2_{\mathbf{X}^1_\alpha(\Omega)},$$

for a constant $\mu < 1$, hence is zero. So, $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ coincide. Finally, using the inf-sup condition (2.6) yields that p_1 and p_2 also coincide, whence the result.

However, this uniqueness result is rather restrictive, limited to solutions of small magnitude. We therefore intend to handle the case of possibly non unique solutions for the discrete problem.

2.5. Existence and uniqueness for nonhomogeneous boundary conditions

As is standard practice of the Navier–Stokes equations, the existence result in this case is derived from an analogue of Hopf's lemma that we now prove. Let g denote the function equal to g_0 on Γ_0 and to g_1 on Γ_1. We recall from [2, Thm 2.6] that, for $\alpha \le \alpha_0$ and for any data g in $H^{\frac{1-\alpha}{2}}(\overline{\Gamma}_0 \cup \overline{\Gamma}_1)$, the Stokes problem (i.e. problem (2.1) with $\frac{1}{\nu} = 0$) with right-hand side $\boldsymbol{f} = \boldsymbol{0}$, has a unique solution $(\tilde{\boldsymbol{u}}, \tilde{p})$. The velocity $\tilde{\boldsymbol{u}}$ satisfies

$$\operatorname{div}_r \tilde{\boldsymbol{u}} = 0 \quad \text{in } \Omega \qquad \text{and} \qquad \tilde{\boldsymbol{u}} = (0, g) \quad \text{on } \overline{\Gamma}_0 \cup \overline{\Gamma}_1, \tag{2.12}$$

together with the estimate

$$\|\tilde{\boldsymbol{u}}\|_{\mathbf{X}^1_\alpha(\Omega)} \le c \, \|g\|_{H^{\frac{1-\alpha}{2}}(\overline{\Gamma}_0 \cup \overline{\Gamma}_1)}. \tag{2.13}$$

The following lemma states the appropriate version of Hopf's lemma for the present situation.

Lemma 2.9. *For any α, $0 \le \alpha \le \alpha_0$, for any data g in $H^{\frac{1-\alpha}{2}}(\overline{\Gamma}_0 \cup \overline{\Gamma}_1)$ and for any positive real number ε, there exists a function $\boldsymbol{u}_\varepsilon$ in $\mathbf{V}^\sharp_{2\alpha}$ such that $\boldsymbol{u}_\varepsilon$ is equal to $(0, g)$ on $\overline{\Gamma}_0 \cup \overline{\Gamma}_1$ and satisfies*

$$\|\boldsymbol{u}_\varepsilon\|_{\mathbf{X}^1_\alpha(\Omega)} \le c(\varepsilon) \, \|g\|_{H^{\frac{1-\alpha}{2}}(\overline{\Gamma}_0 \cup \overline{\Gamma}_1)}, \qquad \|\boldsymbol{u}_\varepsilon\|_{L^4_\alpha(\Omega)^2} \le \varepsilon \, \|g\|_{H^{\frac{1-\alpha}{2}}(\overline{\Gamma}_0 \cup \overline{\Gamma}_1)}, \tag{2.14}$$

where the constant $c(\varepsilon)$ possibly depends on ε.

Proof. This result is rather standard, so we only sketch the proof. For any $\eta > 0$, there exists [10, Chap. IV, Lemma 2.4] a function θ_η of class $\mathcal{C}^2$ on $[0,1]$ such that

$$\theta_\eta(r) = \begin{cases} 1 & \text{if } r \geq 1 - e^{-\frac{2}{\eta}}, \\ 0 & \text{if } r \leq 1 - 2e^{-\frac{1}{\eta}}, \end{cases} \qquad \text{and} \qquad \sup_{r \in [0,1]} |\theta'_\eta(r)| \leq \frac{\eta}{1-r}.$$

Now, since the function $\tilde{\boldsymbol{u}}$ introduced in (2.12) is divergence-free, there exists a function ψ in $L^2_\alpha(\Omega)$ such that $\tilde{\boldsymbol{u}} = \mathbf{curl}_r\, \psi$. The idea is to choose $\boldsymbol{u}_\varepsilon = \mathbf{curl}_r\,(\theta_\eta\, \psi)$, so that $\boldsymbol{u}_\varepsilon$ is divergence-free and is equal to $(0, g)$ on $\overline{\Gamma}_0 \cup \overline{\Gamma}_1$. Moreover it has a compact support in a neighbourhood of $\overline{\Gamma}_0 \cup \overline{\Gamma}_1$. So standard arguments [10, Chap. IV, Lemma 2.3], [12, Lemme 1.4] lead to estimate (2.14) for an appropriate choice of η.

Theorem 2.10. *For any data (f_r, f_θ) in $\mathbf{X}^1_{\alpha_0 0}(\Omega)'$ and (g_0, g_1) in $H^{\frac{1}{2}}(\Gamma_0) \times H^{\frac{1}{2}}(\Gamma_1)$, problem (2.2)–(2.3) has a solution $(\boldsymbol{u}, p)$ in $\mathbf{X}^1_{\alpha\sharp}(\Omega) \times M_{1\alpha}$.*

The proof is very similar to that of Proposition 2.6, so we only sketch it.

Proof. For some $\varepsilon > 0$, we set: $\boldsymbol{u}_0 = \boldsymbol{u} - \boldsymbol{u}_\varepsilon$, where the function $\boldsymbol{u}_\varepsilon$ is introduced in Lemma 2.9. Thus the function $\boldsymbol{u}_0$ belongs to $\mathbf{V}_{2\alpha}$ and is a solution of

$$\forall \boldsymbol{v} = (v_r, v_\theta) \in \mathbf{V}_{1\alpha}, \quad a_\alpha(\boldsymbol{u}_0, \boldsymbol{v}) + C_\alpha(\boldsymbol{u}_0; \boldsymbol{u}_0, \boldsymbol{v}) + C_\alpha(\boldsymbol{u}_0; \boldsymbol{u}_\varepsilon, \boldsymbol{v}) + C_\alpha(\boldsymbol{u}_\varepsilon; \boldsymbol{u}_0, \boldsymbol{v})$$
$$= \int_\Omega \big(f_r(r,\theta) v_r(r,\theta) + f_\theta(r,\theta) v_\theta(r,\theta)\big)\, r(1-r)^\alpha \, dr\, d\theta - C_\alpha(\boldsymbol{u}_\varepsilon; \boldsymbol{u}_\varepsilon, \boldsymbol{v}).$$

So, for the same sequence $(\mathbf{V}^n_\alpha)_{n \geq 0}$ as in the proof of Proposition 2.6, we define the mapping Φ_0 from $\mathbf{V}_{2\alpha}$ into its dual space and from each $\mathbf{V}^n_\alpha$ into its dual space by

$$\begin{aligned} \langle \Phi_0(\boldsymbol{u}_0), \boldsymbol{w} \rangle = {} & a_\alpha(\boldsymbol{u}_0, T_\alpha \boldsymbol{w}) + C_\alpha(\boldsymbol{u}_0; \boldsymbol{u}_0, T_\alpha \boldsymbol{w}) + C_\alpha(\boldsymbol{u}_0; \boldsymbol{u}_\varepsilon, T_\alpha \boldsymbol{w}) \\ & + C_\alpha(\boldsymbol{u}_\varepsilon; \boldsymbol{u}_0, T_\alpha \boldsymbol{w}) \\ & - \int_\Omega \big(f_r(r,\theta)(T_\alpha \boldsymbol{w})_r v_r(r,\theta) \\ & + f_\theta(r,\theta)(T_\alpha \boldsymbol{w})_\theta(r,\theta)\big)\, r(1-r)^\alpha \, dr\, d\theta \\ & - C_\alpha(\boldsymbol{u}_\varepsilon; \boldsymbol{u}_\varepsilon, T_\alpha \boldsymbol{w}). \end{aligned}$$

By combining Lemma 2.9 with Lemmas 2.3 and 2.4 and the modified continuity property (which is derived by integration by parts)

$$\begin{aligned} & \forall \boldsymbol{w} \in \mathbf{X}^1_{\alpha 0}(\Omega),\ \forall \boldsymbol{u} \in L^4_\alpha(\Omega)^2,\ \forall \boldsymbol{v} \in \mathbf{X}^1_{\alpha\sharp}(\Omega), \\ & \qquad |\, C_\alpha(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v})\,| \leq \frac{c}{\nu}\, \|\boldsymbol{w}\|_{\mathbf{X}^1_\alpha(\Omega)}\, \|\boldsymbol{u}\|_{L^4_\alpha(\Omega)^2}\, \|\boldsymbol{v}\|_{\mathbf{X}^1_\alpha(\Omega)}, \end{aligned} \tag{2.15}$$

we obtain

$$\langle \Phi_0(\boldsymbol{u}_0), \boldsymbol{u}_0 \rangle \geq \gamma_0\, \|\boldsymbol{u}_0\|^2_{\mathbf{X}^1_\alpha(\Omega)} - \lambda\, \frac{\alpha}{\nu}\, \|\boldsymbol{u}_0\|^3_{\mathbf{X}^1_\alpha(\Omega)} - F_\varepsilon\, \|\boldsymbol{u}_0\|_{\mathbf{X}^1_\alpha(\Omega)},$$

where γ_0 is equal to $\gamma - \frac{2c\,\varepsilon}{\nu}$ and the quantity F_ε only depends on the norms of $\boldsymbol{f}$ in $\mathbf{X}^1_{\alpha_0 0}(\Omega)'$, g_0 in $H^{\frac{1}{2}}(\Gamma_0)$ and g_1 in $H^{\frac{1}{2}}(\Gamma_1)$ and also of ε. So, taking ε small enough and using the same arguments as in the proofs of Proposition 2.6 and Theorem 2.7 give the existence of a solution $(\boldsymbol{u}, p)$.

We skip the proof of the following result, since it relies on exactly the same arguments as for Theorem 2.8, combined with the modified continuity property

$$\forall \boldsymbol{w} \in \mathbf{V}_{2\alpha}^{\sharp}, \forall \boldsymbol{u} \in \mathbf{V}_{2\alpha}, \quad |\, C_\alpha(\boldsymbol{w}; \boldsymbol{u}, T_\alpha \boldsymbol{u})\,| \leq \lambda\, \frac{\alpha}{\nu}\, \|\boldsymbol{w}\|_{\mathbf{X}_\alpha^1(\Omega)} \|\boldsymbol{u}\|_{\mathbf{X}_\alpha^1(\Omega)}^2. \tag{2.16}$$

Theorem 2.11. *For $0 \leq \alpha \leq \alpha_0$, there exists a constant c_0 such that there is at most one solution $(\boldsymbol{u}, p)$ of problem (2.2)–(2.3) in $\mathbf{X}_{\alpha\sharp}^1(\Omega) \times M_{1\alpha}$ which satisfies (2.11).*

To conclude, we introduce the scale of Sobolev spaces $Y_\alpha^s(\Omega)$ as follows: when s is a positive integer m,

$$Y_\alpha^m(\Omega) = \big\{ v \in L_\alpha^2(\Omega);\ \partial_r^m v \in L_\alpha^2(\Omega) \,\text{and}\, r^{-m}\, \partial_\theta^m v \in L_\alpha^2(\Omega) \big\}; \tag{2.17}$$

When s is positive but is not an integer, the space $Y_\alpha^s(\Omega)$ is defined by appropriate Hilbertian interpolation between $Y_\alpha^{m+1}(\Omega)$ and $Y_\alpha^m(\Omega)$, with m equal to the integral part of s. Then it can be noted that, when the data f_r and f_θ are smooth enough, say in $L^2(\Omega)$ and the pair (g_0, g_1) belongs to $H^{\frac{1}{2}}(\Gamma_0) \times H^{\frac{1}{2}}(\Gamma_1)$, any solution $(\boldsymbol{u}, p)$ of problem (2.2)–(2.3) belongs to $Y_\alpha^s(\Omega)^2 \times Y_\alpha^{s-1}(\Omega)$ for all $s < 1 + \frac{\alpha}{2}$ (this is derived by a bootstrap argument, relying on the analogous property for the Stokes problem, see [2, Prop. 2.7]).

Remark. Each Fourier coefficient with respect to θ of the velocity belongs to a weighted space on the interval $\mathcal{I} =]0, 1[$ and is a solution of a one-dimensional variational problem. For instance, the Fourier coefficients of order ± 1 of the velocity belong to the spaces

$$\mathbf{H}_{\alpha\pm}^1(\mathcal{I}) = \big\{ \boldsymbol{\varphi} = (\varphi_r, \varphi_\theta) \in L_\alpha^2(\mathcal{I})^2;\ \boldsymbol{\varphi}' \in L_\alpha^2(\mathcal{I})^2 \text{ and } r^{-1}(\varphi_r + (\pm i)\varphi_\theta) \in L_\alpha^2(\mathcal{I}) \big\}, \tag{2.18}$$

and the test functions of the corresponding one-dimensional problem run through the subspace $\mathbf{H}_{\alpha\pm 0}^1(\mathcal{I})$ of $\mathbf{H}_{\alpha\pm}^1(\mathcal{I})$ consisting of all functions vanishing at $r = 1$.

3. Description of the discrete problem

As already explained for the Stokes problem [2], writing down the discrete problem is rather complex, for the following reason: Smooth functions in $\mathbf{X}_{\alpha\sharp}^1(\Omega)$ do not have a null trace on the line $r = 0$. It can however be checked [5, Thm II.3.6] that all their Fourier coefficients with respect to θ vanish on this axis, except the coefficients of order ± 1. On the contrary, the intersection of $\mathbf{X}_{\alpha\sharp}^1(\Omega)$ with piecewise polynomial functions is made of functions vanishing at $r = 0$, so that this space provides a very poor approximation of the Fourier coefficients of order

± 1 of functions in $\mathbf{X}^1_{\alpha\sharp}(\Omega)$. A different approximation of these Fourier coefficients must therefore be introduced.

When going back to the exact solution $(\boldsymbol{u}, p)$, we observe that it admits the expansion

$$\begin{aligned} \boldsymbol{u} = \tilde{\boldsymbol{u}} + \boldsymbol{u}^\diamond, \quad \text{with} \quad \boldsymbol{u}^\diamond = \boldsymbol{u}^-\, e^{-i\theta} + \boldsymbol{u}^+\, e^{+i\theta}, \\ p = \tilde{p} + p^\diamond, \quad \text{with} \quad p^\diamond = p^-\, e^{-i\theta} + p^+\, e^{i\theta}, \end{aligned} \tag{3.1}$$

where $\boldsymbol{u}^\pm$ and $p^\pm$ denote the Fourier coefficients of order ± 1 of $\boldsymbol{u}$ and p. Each pair $(\boldsymbol{u}^\pm, p^\pm)$ is the solution of a one-dimensional Stokes problem on $[0, 1[$, while $(\tilde{\boldsymbol{u}}, \tilde{p})$ is a solution of a problem of type (2.1) with modified data and a nonlinear term. Consequently, the discrete problem is a system of equations of different types: two one-dimensional problems in order to approximate the Fourier coefficients of order ± 1, and a problem on Ω to approximate the remaining part of the solution.

We first introduce the basic tools for the discretization, namely the discrete spaces and the corresponding quadrature formulas. Next, we describe the discrete problem in the linear case (i.e. for $\frac{1}{\nu} = 0$). Finally, we present the discrete problem corresponding to the full system (2.1).

In all that follows, the discretization parameter N is a fixed positive integer ≥ 2. From now on and for simplicity, we only handle the case we are interested in, i.e. we assume that g_0 is zero and g_1 is a fixed constant ξ. Thus, condition (2.2) is replaced by

$$(u_r\,, u_\theta) = (0, 0) \quad \text{on} \quad \Gamma_0 \qquad \textit{and} \qquad (u_r\,, u_\theta) = (0, \xi) \quad \text{on} \quad \Gamma_1. \tag{3.2}$$

3.1. Discrete spaces and quadrature formulas

We first introduce the following polynomial spaces, for any $n \geq 0$:

(i) For any interval Λ, $\mathbb{P}_n(\Lambda)$ stands for the space of polynomials with one variable and degree $\leq n$. More specifically, $\mathbb{P}_n(\Gamma_0)$ and $\mathbb{P}_n(\Gamma_1)$ denote the spaces of polynomials with degree $\leq n$ with respect to θ on Γ_0 and Γ_1, respectively, while $\mathbb{P}_n(\Gamma_{01})$ and $\mathbb{P}_n(\Gamma_{10})$ denote the spaces of polynomials with degree $\leq n$ with respect to r on Γ_{01} and Γ_{10}, respectively.

(ii) $\mathbb{P}_n(\Omega_0)$ and $\mathbb{P}_n(\Omega_1)$ denote the spaces of polynomials with degree $\leq n$ with respect to r and θ in Ω_0 and Ω_1, respectively.

(iii) $\mathbb{P}^*_n(\Omega_\ell)$ is the space of polynomials in $\mathbb{P}_n(\Omega_\ell)$ that vanish at $r = 0$, $\ell = 1, 2$.

Finally, for any pair (m, n) of nonnegative integers, we introduce the spaces $\mathbb{P}_{m,n}(\Omega_\ell)$, $\ell = 0$ and 1, of polynomials on Ω_ℓ with degree $\leq m$ with respect to r and $\leq n$ with respect to θ. Note also that, for simplicity, we use the same notation for real and complex valued polynomials (these last ones are needed for the approximation of the Fourier coefficients of order ± 1).

As usual, for any pair (α, β) of real numbers > -1, $(J_n^{\alpha,\beta})_n$ stands for the family of Jacobi polynomials: each polynomial $J_n^{\alpha,\beta}$ is orthogonal to the other ones on $]-1, 1[$ for the measure $(1-\zeta)^\alpha (1+\zeta)^\beta\, d\zeta$. The Legendre polynomials

$J_n^{0,0}$ are denoted by L_n. In the r-direction, we use the "weighted" Gauss–Radau quadrature formula:

$$\forall \Phi \in \mathbb{P}_{2N-2}(-1,1), \qquad \int_{-1}^{1} \Phi(\zeta)\,(1-\zeta)^\alpha (1+\zeta)\,d\zeta = \sum_{i=1}^{N} \Phi(\zeta_i^\alpha)\,\omega_i^\alpha,$$

where the nodes ζ_i^α, $1 \le i \le N$, are the zeros of the polynomial $(1-\zeta)\,J_{N-1}^{\alpha+1,1}$ in increasing order (we refer to [6, Appendix A] for its complete analysis and the values of the weights ω_i^α which are positive). In the θ-direction, we employ the usual Gauss–Lobatto formula: we recall that

$$\forall \Phi \in \mathbb{P}_{2N-1}(-1,1), \qquad \int_{-1}^{1} \Phi(\zeta)\,d\zeta = \sum_{j=0}^{N} \Phi(\xi_j)\,\rho_j,$$

where the ξ_j, $0 \le j \le N$, are the zeros of $(1-\zeta^2)\,L_N'$ in increasing order and the corresponding weights ρ_j are positive. We therefore set

$$r_i^\alpha = \frac{1+\zeta_i^\alpha}{2}, \qquad \theta_{0j} = \frac{\theta_0}{2}(\xi_j+1) \quad \text{and} \quad \theta_{1j} = \frac{2\pi-\theta_0}{2}\,\xi_j + \frac{2\pi+\theta_0}{2}.$$

For the one-dimensional problems set on the interval $\mathcal{I} =]0,1[$, we only use the Gauss–Radau quadrature formula in the r-direction. So we define the discrete product on all continuous functions φ and ψ on $[0,1]$ by

$$\big(\varphi,\psi\big)_N^\diamond = 2^{-\alpha-2} \sum_{i=1}^{N} \varphi(r_i^\alpha)\overline{\psi}(r_i^\alpha)\,\omega_i^\alpha.$$

The interpolation operator at the nodes r_i^α, with values in $\mathbb{P}_{N-1}(\mathcal{I})$, is denoted by i_{N-1}. For the two-dimensional problem, the discrete scalar product is defined on all functions that are continuous in both $\overline{\Omega}_0$ and $\overline{\Omega}_1$ by

$$\begin{aligned}\big(u,v\big)_N = 2^{-\alpha-3}\Big(&\theta_0 \sum_{i=1}^{N}\sum_{j=0}^{N} u(r_i^\alpha,\theta_{0j})\overline{v}(r_i^\alpha,\theta_{0j})\,\omega_i^\alpha\,\rho_j \\ &+ (2\pi-\theta_0)\sum_{i=1}^{N}\sum_{j=0}^{N} u(r_i^\alpha,\theta_{1j})\overline{v}(r_i^\alpha,\theta_{1j})\omega_i^\alpha\,\rho_j\Big).\end{aligned} \tag{3.3}$$

For $\ell = 0$ and 1, we also introduce the Lagrange interpolation operators $j_N^{(\ell)}$, at all nodes $\theta_{\ell j}$, $0 \le j \le N$, with values in $\mathbb{P}_N(\Gamma_\ell)$, and $\mathcal{I}_N^{(\ell)}$, at all nodes $(r_i,\theta_{\ell j})$, $1 \le i \le N$, $0 \le j \le N$, with values in $\mathbb{P}_{N-1,N}(\Omega_\ell)$.

For the one-dimensional problems, the discrete spaces are made of polynomials. We introduce the space $\mathbf{X}_N^\pm$ of all polynomials $\boldsymbol{\varphi}_N = (\varphi_{rN},\varphi_{\theta N})$ in $\mathbb{P}_N(0,1)^2$ such that the term $\varphi_{rN} + (\pm i)\varphi_{\theta N}$ vanishes at $r=0$. We also consider the subspace $\mathbf{X}_N^{\pm 0}$ of polynomials in $\mathbf{X}_N^\pm$ vanishing at $r=1$ and the space $M_N^\diamond$ equal to $\mathbb{P}_{N-1}(0,1)$.

For the two-dimensional problem, the choice of the discrete space of velocities is that of the mortar element method [8] in the simpler case of a conforming decomposition but with weights. As in [6], we define the space $Y_N(\Omega)$ to be the space of all functions v_N such that:

(i) the restriction of v_N to each Ω_ℓ, $\ell = 1, 2$, belongs to $\mathbb{P}_N^*(\Omega_\ell)$,

(ii) the following matching condition holds on each $\Gamma_{k\ell}$, $k\ell = 01$ and 10:

$$\forall \psi \in \mathbb{P}_{N-2}(\Gamma_{k\ell}), \qquad \int_{\Gamma_{k\ell}} (v_{N\,|\Omega_0} - v_{N\,|\Omega_1})(r)\, \psi(r)\, r(1-r)^\alpha\, dr = 0. \tag{3.4}$$

Next, we set:

$$\mathbf{X}_N = Y_N(\Omega) \times Y_N(\Omega) \quad \text{and} \quad \mathbf{X}_N^0 = \big\{ \boldsymbol{v}_N \in \mathbf{X}_N;\ \boldsymbol{v}_N = \mathbf{0} \text{ on } \overline{\Gamma}_0 \cup \overline{\Gamma}_1 \big\}.$$

Remark. It is proven in [6, §3] that, due to the matching condition (3.4), the jump of each function v_N in $Y_N(\Omega)$ on each $\Gamma_{k\ell}$ can be written as

$$(v_{N\,|\Omega_0} - v_{N\,|\Omega_1})(r) = \alpha_{k\ell}\, (J_N^{\alpha,1} + \frac{N+1}{N}\, J_{N-1}^{\alpha,1})(2r-1). \tag{3.5}$$

This leads to the following observation:

• The functions in $Y_N(\Omega)$ and $\mathbf{X}_N$ are not necessarily continuous at the intersection of $\overline{\Gamma}_0$ and $\overline{\Gamma}_1$, which is suitable for handling discontinuous boundary data.

• However, the functions in $\mathbf{X}_N^0$ are continuous across Γ_{01} and across Γ_{10}, so that $\mathbf{X}_N^0$ is contained in $\mathbf{X}_\alpha^1(\Omega)$.

For these reasons, the method is said to be *semi-conforming*.

Finally, we choose the discrete space which approximates $M_{1\alpha}$ and $M_{2\alpha}$ as equal to

$$M_N = \big\{ q_N \in L^2_\alpha(\Omega);\ q_{N\,|\Omega_\ell} \in \mathbb{P}_{N-1,N}(\Omega_\ell),\ \ell = 0, 1 \big\}. \tag{3.6}$$

To conclude, we introduce very accurate polynomial approximations $e_N^\pm$ of $e^{\pm i\theta}$ of degree $\le N$ with respect to θ which preserve the periodicity of $e^{\pm i\theta}$ up to the order 2. Since the functions $e^{\pm i\theta}$ are analytic, the distance between these functions and their approximations $e_N^\pm$ in $H^1(0, 2\pi)$ behaves like $c\, e^{-c'N}$ and is always neglected in what follows.

3.2. The discrete Stokes problem

We first define the Fourier coefficients of order ± 1 of the boundary data in (3.2):

$$g_N^- = \frac{\xi}{2i\pi}\, (1 - e^{i\theta_0}), \qquad g_N^+ = -\frac{\xi}{2i\pi}\, (1 - e^{-i\theta_0}). \tag{3.7}$$

Next, we set:

$$\tilde{g}_{N0}(\theta) = -g_N^-\, e_N^- - g_N^+\, e_N^+, \qquad \tilde{g}_{N1} = \xi - g_N^-\, e_N^- - g_N^+\, e_N^+. \tag{3.8}$$

For the sake of generality, in order to compute the Fourier coefficients of $\boldsymbol{f}$, we use the Gauss–Lobatto quadrature formula. This gives

$$\begin{aligned}\boldsymbol{f}_N^-(r) &= \frac{1}{4\pi}\bigg(\theta_0 \sum_{j=0}^{N} \boldsymbol{f}(r,\theta_{0j})\, e^{i\theta_{0j}}\,\rho_j + (2\pi-\theta_0)\sum_{j=0}^{N} \boldsymbol{f}(r,\theta_{1j})\, e^{i\theta_{1j}}\,\rho_j\bigg),\\ \boldsymbol{f}_N^+(r) &= \frac{1}{4\pi}\bigg(\theta_0 \sum_{j=0}^{N} \boldsymbol{f}(r,\theta_{0j})\, e^{-i\theta_{0j}}\,\rho_j + (2\pi-\theta_0)\sum_{j=0}^{N} \boldsymbol{f}(r,\theta_{1j})\, e^{-i\theta_{1j}}\,\rho_j\bigg).\end{aligned} \tag{3.9}$$

Next, we set:

$$\tilde{\boldsymbol{f}}_N(r,\theta) = \boldsymbol{f}(r,\theta) - \boldsymbol{f}_N^-\, e_N^- - \boldsymbol{f}_N^+\, e_N^+. \tag{3.10}$$

In the next step, we first introduce the bilinear forms corresponding to the one-dimensional problems

$$\begin{aligned} a_{N\pm}(\boldsymbol{\psi}_N,\boldsymbol{\varphi}_N) &= \big(\psi_{rN}', (1-r)^{-\alpha}(\varphi_{rN}\,(1-r)^{\alpha})'\big)_N^{\diamond}\\ &\quad + \big(\psi_{\theta N}', (1-r)^{-\alpha}\,(\varphi_{\theta N}(1-r)^{\alpha})'\big)_N^{\diamond}\\ &\quad + 2\big(r^{-1}(\psi_{rN} + (\pm i)\psi_{\theta N}), r^{-1}(\varphi_{rN}+(\pm i)\varphi_{\theta N})\big)_N^{\diamond},\\ b_{1N\pm}(\boldsymbol{\varphi}_N,\chi_N) &= -\big(\chi_N, (1-r)^{-\alpha}\,(\varphi_{rN}\,(1-r)^{\alpha})' + r^{-1}\,\varphi_{rN} + r^{-1}\,(\pm i)\varphi_{\theta N}\big)_N^{\diamond},\\ b_{2N\pm}(\boldsymbol{\varphi}_N,\chi_N) &= -\big(\chi_N, \varphi_{rN}' + r^{-1}\,\varphi_{rN} + r^{-1}(\pm i)\varphi_{\theta N}\big)_N^{\diamond}.\end{aligned}$$

Then the one-dimensional discrete problems can be written as

Find a pair $(\boldsymbol{u}_N^{\pm} = (u_{rN}^{\pm}, u_{\theta N}^{\pm}), p_N^{\pm})$ *in* $\mathbf{X}_N^{\pm} \times M_N^{\diamond}$, *with*

$$u_{rN}^{\pm}(1) = 0, \qquad u_{\theta N}^{\pm}(1) = g_N^{\pm}, \tag{3.11}$$

such that

$$\begin{aligned} &\forall \boldsymbol{\varphi}_N = (\varphi_{rN},\varphi_{\theta N}) \in \mathbf{X}_N^{\pm 0}, \quad a_{N\pm}(\boldsymbol{u}_N^{\pm},\boldsymbol{\varphi}_N) + b_{1N\pm}(\boldsymbol{\varphi}_N, p_N^{\pm})\\ &\qquad\qquad\qquad\qquad = \big(f_{rN}^{\pm},\varphi_{rN}\big)_N^{\diamond} + \big(f_{\theta N}^{\pm},\varphi_{\theta N}\big)_N^{\diamond},\\ &\forall \chi_N \in M_N^{\diamond}, \quad \overline{b}_{2N\pm}(\boldsymbol{u}_N^{\pm},\chi_N) = 0.\end{aligned} \tag{3.12}$$

Remark. The one-dimensional problems proposed in [2, Appendix B] are slightly different since an unweighted formulation ($\alpha = 0$) is considered there. This comes from the fact that these problems are not coupled with the two-dimensional one for the Stokes problem but are coupled for the Navier–Stokes equations as it appears later on.

We also consider the forms associated with the two-dimensional problem

$$\begin{aligned} a_N(\boldsymbol{u}_N,\boldsymbol{v}_N) &= \big(\partial_r u_{rN}, (1-r)^{-\alpha}\,\partial_r(v_{rN}(1-r)^{\alpha})\big)_N\\ &\quad + \big(\partial_r u_{\theta N}, (1-r)^{-\alpha}\,\partial_r(v_{\theta N}(1-r)^{\alpha})\big)_N\\ &\quad + \big(r^{-1}\,(u_{rN} + \partial_\theta u_{\theta N}), r^{-1}\,(v_{rN} + \partial_\theta v_{\theta N})\big)_N\\ &\quad + \big(r^{-1}\,(u_{\theta N} - \partial_\theta u_{rN}), r^{-1}\,(v_{\theta N} - \partial_\theta v_{rN})\big)_N,\end{aligned}$$

$$b_{1N}(\boldsymbol{v}_N, q_N) = -\big((1-r)^{-\alpha}\,\partial_r(v_{rN}\,(1-r)^{\alpha}) + r^{-1}\,v_{rN} + r^{-1}\,\partial_\theta v_{\theta N}, q_N\big)_N,$$
$$b_{2N}(\boldsymbol{u}_N, q_N) = -\big(\partial_r u_{rN} + r^{-1}\,u_{rN} + r^{-1}\,\partial_\theta u_{\theta_N}, q_N\big)_N.$$

The discrete problem can now be written as

Find a pair $(\tilde{\boldsymbol{u}}_N = (\tilde{u}_{rN}, \tilde{u}_{\theta N}), \tilde{p}_N)$ *in* $\mathbf{X}_N \times M_N$, *with*

$$(\tilde{u}_{rN}\,, \tilde{u}_{\theta N}) = (0, \tilde{g}_{N0}) \quad \text{on} \quad \Gamma_0 \quad \textit{and} \quad (\tilde{u}_{rN}\,, \tilde{u}_{\theta N}) = (0, \tilde{g}_{N1}) \quad \text{on} \quad \Gamma_1, \tag{3.13}$$

such that

$$\begin{aligned} &\forall \boldsymbol{v}_N = (v_{rN}, v_{\theta N}) \in \mathbf{X}_N^0, \quad a_N(\tilde{\boldsymbol{u}}_N, \boldsymbol{v}_N) + b_{1N}(\boldsymbol{v}_N, \tilde{p}_N) \\ &\qquad\qquad\qquad\qquad = (\tilde{f}_{rN}, v_{rN})_N + (\tilde{f}_{\theta N}, v_{\theta N})_N, \\ &\forall q_N \in M_N, \quad b_{2N}(\tilde{\boldsymbol{u}}_N, q_N) = 0. \end{aligned} \tag{3.14}$$

The global problem reads

Find a pair $(\boldsymbol{u}_N, p_N)$ *such that*

$$\begin{aligned} \boldsymbol{u}_N = \tilde{\boldsymbol{u}}_N + \boldsymbol{u}_N^\diamond, \quad \text{with} \quad \boldsymbol{u}_N^\diamond = \boldsymbol{u}_N^-\, e_N^- + \boldsymbol{u}_N^+\, e_N^+, \\ p_N = \tilde{p}_N + p_N^\diamond, \quad \text{with} \quad p_N^\diamond = p_N^-\, e_N^- + p_N^+\, e_N^+, \end{aligned} \tag{3.15}$$

where each $(\boldsymbol{u}_N^\pm, p_N^\pm)$ *is a solution of problem* (3.11)–(3.12) *and* $(\tilde{\boldsymbol{u}}_N, \tilde{p}_N)$ *is a solution of problem* (3.13)–(3.14).

Remark. It follows from the choice of the discrete spaces $M_N^\diamond$ and M_N that the velocity $\boldsymbol{u}_N$ in (3.15) is nearly divergence-free, in the sense that the function

$$\tilde{\boldsymbol{u}}_N + \boldsymbol{u}_N^-\, e^{-i\theta} + \boldsymbol{u}_N^+\, e^{i\theta}, \tag{3.16}$$

is divergence-free. However, the space M_N contains spurious modes on the pressure (which are identified in [2, Lemma 3.1]), so that there is no uniqueness of the pressure $\tilde{p}_N$ in the solution $(\tilde{\boldsymbol{u}}_N, \tilde{p}_N)$ of problem (3.13)–(3.14). In fact, we are more interested in the convergence of the discrete velocity $\boldsymbol{u}_N$ and in the possibility of recovering an exactly divergence-free velocity. Moreover, an accurate approximation of the pressure can be computed from $\boldsymbol{u}_N$ in a postprocessing step.

3.3. The discrete Navier–Stokes problem

We first introduce the discrete trilinear form which approximates $C_\alpha(\cdot;\cdot,\cdot)$. For smooth enough functions $\boldsymbol{w}$, $\boldsymbol{u}$ and $\boldsymbol{v}$, this is defined as

$$\begin{aligned} C_N(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v}) = \frac{1}{\nu}\Big(&\big(w_r\,\partial_r u_r, v_r\big)_N + \big(w_r\,\partial_r u_\theta, v_\theta\big)_N \\ &+ \big(r^{-1}\,w_\theta\,(\partial_\theta u_r - u_\theta), v_r\big)_N + \big(r^{-1}\,w_\theta\,(\partial_\theta u_\theta + u_r), v_\theta\big)_N\Big). \end{aligned}$$

However, we need to treat separately the Fourier coefficients of order ± 1 of the nonlinear term. This leads to considering the forms defined on sufficiently smooth functions $\boldsymbol{u}$ and $\boldsymbol{w}$ on Ω and continuous functions $\boldsymbol{\varphi}$ on $[0,1]$

$$c_N^-(\boldsymbol{w};\boldsymbol{u},\boldsymbol{\varphi}) = \frac{1}{2\pi}\,C_N(\boldsymbol{w};\boldsymbol{u},\boldsymbol{\varphi}\,e^{-i\theta}), \qquad c_N^+(\boldsymbol{w};\boldsymbol{u},\boldsymbol{\varphi}) = \frac{1}{2\pi}\,C_N(\boldsymbol{w};\boldsymbol{u},\boldsymbol{\varphi}\,e^{i\theta}).$$

The discrete global problem now reads

Find a pair $(\boldsymbol{u}_N, p_N)$ *in* $\mathbf{X}_N \times M_N$ *satisfying* (3.15), *where each* $(\boldsymbol{u}_N^\pm, p_N^\pm)$ *is a solution in* $\mathbf{X}_N^\pm \times M_N^\diamond$ *of* (3.11) *and*

$$\begin{aligned}
&\forall \boldsymbol{\varphi}_N = (\varphi_{rN}, \varphi_{\theta N}) \in \mathbf{X}_N^{\pm 0},\\
&a_{N\pm}(\boldsymbol{u}_N^\pm, \boldsymbol{\varphi}_N) + b_{1N\pm}(\boldsymbol{\varphi}_N, p_N^\pm) + c_N^\pm(\tilde{\boldsymbol{u}}_N; \boldsymbol{u}_N^\diamond, \boldsymbol{\varphi}_N) + c_N^\pm(\boldsymbol{u}_N^\diamond; \tilde{\boldsymbol{u}}_N, \boldsymbol{\varphi}_N)\\
&\qquad\qquad = \big(f_{rN}^\pm, \varphi_{rN}\big)_N^\diamond + \big(f_{\theta N}^\pm, \varphi_{\theta N}\big)_N^\diamond - c_N^\pm(\tilde{\boldsymbol{u}}_N; \tilde{\boldsymbol{u}}_N, \boldsymbol{\varphi}_N),\\
&\forall \chi_N \in M_N^\diamond, \quad \overline{b_{2N\pm}}(\boldsymbol{u}_N^\pm, \chi_N) = 0,
\end{aligned} \tag{3.17}$$

and $(\tilde{\boldsymbol{u}}_N, \tilde{p}_N)$ *is a solution in* $\mathbf{X}_N \times M_N$, *of* (3.13) *and*

$$\begin{aligned}
&\forall \boldsymbol{v}_N = (v_{rN}, v_{\theta N}) \in \mathbf{X}_N^0,\\
&\quad a_N(\tilde{\boldsymbol{u}}_N, \boldsymbol{v}_N) + b_{1N}(\boldsymbol{v}_N, \tilde{p}_N) + C_N(\tilde{\boldsymbol{u}}_N; \tilde{\boldsymbol{u}}_N, \boldsymbol{v}_N)\\
&\qquad\qquad + C_N(\boldsymbol{u}_N^\diamond; \tilde{\boldsymbol{u}}_N, \boldsymbol{v}_N) + C_N(\tilde{\boldsymbol{u}}_N; \boldsymbol{u}_N^\diamond, \boldsymbol{v}_N)\\
&\qquad\qquad = (\tilde{f}_{rN}, v_{rN})_N + (\tilde{f}_{\theta N}, v_{\theta N})_N - C_N(\boldsymbol{u}_N^\diamond; \boldsymbol{u}_N^\diamond, \boldsymbol{v}_N),\\
&\forall q_N \in M_N, \quad b_{2N}(\tilde{\boldsymbol{u}}_N, q_N) = 0.
\end{aligned} \tag{3.18}$$

Remark. It can be observed that, using the decomposition (3.1), the following property holds

$$\forall \boldsymbol{\varphi} \in L_0^4(0,1), \quad C_\alpha(\boldsymbol{u}^\diamond; \boldsymbol{u}^\diamond, \boldsymbol{\varphi}\,e^{\pm i\theta}) = 0. \tag{3.19}$$

For this reason, we have suppressed the term of type $C_N(\boldsymbol{u}_N^\diamond; \boldsymbol{u}_N^\diamond, \boldsymbol{\varphi}_N\,e^{\pm i\theta})$ in the discrete problem. As a consequence, problem (3.11)–(3.17) is linear.

For simplicity, problem (3.15)–(3.11)–(3.17)–(3.13)–(3.18) will be called problem (3.74).

4. Numerical analysis of the discrete problem

The aim of this section is to prove that problem (3.74) admits at least one solution and that this solution converges to an appropriate solution of problem (3.2)–(2.3) when N tends to ∞. To derive this result, we first write a new formulation of both the continuous and the discrete problems.

4.1. A new formulation

Let $\mathcal{S}$ denote the Stokes operator: $(\boldsymbol{f}, \xi) \mapsto \boldsymbol{u}$, where the velocity $\boldsymbol{u}$ is the solution in $\mathbf{V}^{\sharp}_{2\alpha}$ of (3.2) and

$$\begin{aligned}
&\forall \boldsymbol{v} = (v_r, v_\theta) \in \mathbf{V}_{1\alpha},\\
&a_\alpha(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \big(f_r(r,\theta) v_r(r,\theta) + f_\theta(r,\theta) v_\theta(r,\theta)\big) r(1-r)^\alpha \, dr\, d\theta.
\end{aligned} \tag{4.1}$$

It can be noted from Lemma 2.1 that this operator is well-defined and continuous from $\mathbf{X}^1_{\alpha 0}(\Omega)' \times \mathbb{R}$ into $\mathbf{X}^1_{\alpha\sharp}(\Omega)$. Moreover the following property is obvious: the velocity $\boldsymbol{u}$ is a solution of problem (3.2)–(2.5) if and only if it satisfies

$$\boldsymbol{u} + \mathcal{S} \begin{pmatrix} \mathbf{F}(\boldsymbol{u}) \\ -\xi \end{pmatrix} = 0, \tag{4.2}$$

where the function $\mathbf{F}$ is defined from $\mathbf{X}^1_{\alpha\sharp}(\Omega)$ into $\mathbf{X}^1_{\alpha 0}(\Omega)'$ by

$$\begin{aligned}
&\forall \boldsymbol{v} \in \mathbf{X}^1_{\alpha 0}(\Omega),\\
&\quad \langle F(\boldsymbol{u}), \boldsymbol{v} \rangle = C_\alpha(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}) - \int_\Omega \big(f_r(r,\theta) v_r(r,\theta) + f_\theta(r,\theta) v_\theta(r,\theta)\big) r(1-r)^\alpha \, dr\, d\theta.
\end{aligned} \tag{4.3}$$

We also introduce the operator $\mathcal{K}$ which associates with any function $\boldsymbol{f}$ in $L^2(\Omega)$ the triplet $(\boldsymbol{f}^-, \boldsymbol{f}^+, \tilde{\boldsymbol{f}})$ made of its Fourier coefficients $\boldsymbol{f}^\pm$ of order ± 1 and of the remaining part $\tilde{\boldsymbol{f}}$

$$\tilde{\boldsymbol{f}}(r,\theta) = \boldsymbol{f}(r,\theta) - \boldsymbol{f}^-(r)\, e^{-i\theta} - \boldsymbol{f}^+(r)\, e^{i\theta}. \tag{4.4}$$

The space $\mathbf{H}^1_{\alpha-}(\mathcal{I}) \times \mathbf{H}^1_{\alpha+}(\mathcal{I}) \times \mathbf{X}^1_{\alpha\sharp}(\Omega)$ (see (2.18) for the definition of the $\mathbf{H}^1_{\alpha\pm}(\mathcal{I})$) is also denoted by $\mathcal{X}$, and its subspace made of triplets vanishing at $r = 1$ by $\mathcal{X}_0$. Indeed, problem (4.1) can equivalently be written as a system consisting of two one-dimensional problems with data $\boldsymbol{f}^\pm$ and one two-dimensional problem with data $\tilde{\boldsymbol{f}}$.

Similarly, let $\mathcal{S}_N$ denote the operator: $(\boldsymbol{f}, \xi) \mapsto \boldsymbol{u}_N$, where $\boldsymbol{u}_N$ satisfies (3.15) and where each $(\boldsymbol{u}^\pm_N, p^\pm_N)$ is a solution in $\mathbf{X}^\pm_N \times M^\diamond_N$ of (3.11) and

$$\begin{aligned}
&\forall \boldsymbol{\varphi}_N = (\varphi_{rN}, \varphi_{\theta N}) \in \mathbf{X}^{\pm 0}_N,\\
&\qquad a_{N\pm}(\boldsymbol{u}^\pm_N, \boldsymbol{\varphi}_N) + b_{1N\pm}(\boldsymbol{\varphi}_N, p^\pm_N)\\
&\qquad\qquad = \int_0^1 \big(f^\pm_r(r)\varphi_{rN}(r) + f^\pm_\theta(r)\varphi_{\theta N}(r,\theta)\big)\, r\, dr,\\
&\forall \chi_N \in M^\diamond_N, \quad \overline{b}_{2N\pm}(\boldsymbol{u}^\pm_N, \chi_N) = 0,
\end{aligned} \tag{4.5}$$

while $(\tilde{\boldsymbol{u}}_N, \tilde{p}_N)$ is a solution in $\mathbf{X}_N \times M_N$ of (3.13) and

$$\begin{aligned}
&\forall \boldsymbol{v}_N = (v_{rN}, v_{\theta N}) \in \mathbf{X}_N^0, \\
&\quad a_N(\tilde{\boldsymbol{u}}_N, \boldsymbol{v}_N) + b_{1N}(\boldsymbol{v}_N, \tilde{p}_N) \\
&\qquad = \int_\Omega \big(\tilde{f}_r(r,\theta) v_{rN}(r,\theta) + \tilde{f}_\theta(r,\theta) v_{\theta N}(r,\theta)\big)\, r(1-r)^\alpha \, dr \, d\theta, \\
&\forall q_N \in M_N, \quad b_{2N}(\tilde{\boldsymbol{u}}_N, q_N) = 0,
\end{aligned} \tag{4.6}$$

where the triplet $(\boldsymbol{f}^-, \boldsymbol{f}^+, \tilde{\boldsymbol{f}})$ is equal to $\mathcal{K}\boldsymbol{f}$. Note that, up to the integrals on the right-hand side, this definition is rather similar to problems (3.11)–(3.12)–(3.13)–(3.14).

We also denote by $\mathcal{K}_N$ the operator which associates with any continuous function $\boldsymbol{f}$ on Ω the triplet $(\boldsymbol{f}_N^-, \boldsymbol{f}_N^+, \tilde{\boldsymbol{f}}_N)$ defined in (3.9) and (3.10), and by $\mathcal{X}_N^0$ the product space $\mathbf{X}_N^{-0} \times \mathbf{X}_N^{+0} \times \mathbf{X}_N^0$. Then problem (3.74) admits the equivalent formulation

$$\boldsymbol{u}_N + \mathcal{S}_N \begin{pmatrix} \mathbf{F}_N(\boldsymbol{u}_N) \\ -\xi \end{pmatrix} = 0, \tag{4.7}$$

where the function $\mathbf{F}_N$ is defined in the following way: if $\mathbf{F}_N^\pm$ and $\widetilde{\mathbf{F}}_N$ stand for the components of $\mathcal{K}\mathbf{F}_N$,

$$\begin{aligned}
\langle \mathbf{F}_N^\pm(\boldsymbol{u}_N), \boldsymbol{\varphi}_N \rangle &= c_N^\pm(\tilde{\boldsymbol{u}}_N; \tilde{\boldsymbol{u}}_N, \boldsymbol{\varphi}_N) + c_N^\pm(\tilde{\boldsymbol{u}}_N; \boldsymbol{u}_N^\diamond, \boldsymbol{\varphi}_N) \\
&\quad + c_N^\pm(\boldsymbol{u}_N^\diamond; \tilde{\boldsymbol{u}}_N, \boldsymbol{\varphi}_N) - \big(f_{rN}^\pm, \varphi_{rN}\big)_N^\diamond - \big(f_{\theta N}^\pm, \varphi_{\theta N}\big)_N^\diamond,
\end{aligned} \tag{4.8}$$

$$\begin{aligned}
\langle \widetilde{\mathbf{F}}_N(\boldsymbol{u}_N), \tilde{\boldsymbol{v}}_N \rangle &= C_N(\boldsymbol{u}_N^\diamond; \boldsymbol{u}_N^\diamond, \boldsymbol{v}_N) + C_N(\tilde{\boldsymbol{u}}_N; \tilde{\boldsymbol{u}}_N, \boldsymbol{v}_N) + C_N(\boldsymbol{u}_N^\diamond; \tilde{\boldsymbol{u}}_N, \boldsymbol{v}_N) \\
&\quad + C_N(\tilde{\boldsymbol{u}}_N; \boldsymbol{u}_N^\diamond, \boldsymbol{v}_N) - (\tilde{f}_{rN}, v_{rN})_N - (\tilde{f}_{\theta N}, v_{\theta N})_N.
\end{aligned} \tag{4.9}$$

This last formulation does not only possess the advantage of brevity but is also the appropriate one for applying the implicit function theorem of [9]. This application requires a further assumption that we now state.

Assumption A.1. The operator

$$\mathrm{Id} + \mathcal{S} \begin{pmatrix} D\mathbf{F}(\boldsymbol{u}) \\ 0 \end{pmatrix}$$

is an isomorphism of $\mathbf{X}_{\alpha 0}^1(\Omega)$.

Equivalently, Assumption A.1 means that the linearized Navier–Stokes equations at $\boldsymbol{u}$ have a unique solution. This implies the local uniqueness of the solution but is much less restrictive than the conditions for its global uniqueness, see Theorem 2.11.

4.2. Some properties of the discrete operators

We state some properties of the discrete Stokes operator $\mathcal{S}_N$: its stability and its convergence to $\mathcal{S}$.

Lemma 4.1. *The operator $\mathcal{S}_N$ satisfies, for all $\boldsymbol{f}$ in $\mathbf{X}^1_{\alpha 0}(\Omega)'$*

$$\Big\| \mathcal{S}_N \begin{pmatrix} \boldsymbol{f} \\ 0 \end{pmatrix} \Big\|_{\mathbf{X}^1_\alpha(\Omega)} \le c \sup_{\mathcal{W}_N \in \mathcal{X}^0_N} \frac{\langle \mathcal{K}\boldsymbol{f}, \mathcal{W}_N \rangle}{\|\mathcal{W}_N\|_{\mathcal{X}}}. \tag{4.10}$$

Proof. Let $\boldsymbol{u}_N$ stand for the solution $\mathcal{S}_N \begin{pmatrix} \boldsymbol{f} \\ 0 \end{pmatrix}$, and $(\boldsymbol{u}^-_N, \boldsymbol{u}^+_N, \tilde{\boldsymbol{u}}_N)$ denote the triplet $\mathcal{K}_N \boldsymbol{u}_N$. It can be checked [2, §3] that $\tilde{\boldsymbol{u}}_N$ for instance belongs to $\mathbf{X}^0_N \cap V_{2\alpha}$, so that $T_\alpha \tilde{\boldsymbol{u}}_N$ belongs to $\mathbf{X}^0_N \cap V_{1\alpha}$. By taking $\boldsymbol{v}_N$ equal to $T_\alpha \tilde{\boldsymbol{u}}_N$ in (4.6) and using the discrete analogue of (2.7) which is proven in [2, Prop. 3.3], we derive the estimate

$$\|\tilde{\boldsymbol{u}}_N\|_{\mathbf{X}^1_\alpha(\Omega)} \le \sup_{\boldsymbol{v}_N \in \mathbf{X}^0_N} \frac{\langle \tilde{\boldsymbol{f}}, \boldsymbol{v}_N \rangle}{\|\boldsymbol{v}_N\|_{\mathbf{X}^1_\alpha(\Omega)}}.$$

Similar arguments also allow to bound $\|\boldsymbol{u}^\pm_N\|_{\mathbf{H}^1_{\alpha\pm}(\mathcal{I})}$, whence the desired estimate.

Remark. Estimate (4.10) can equivalently be written

$$\Big\| \mathcal{K}_N \mathcal{S}_N \begin{pmatrix} \boldsymbol{f} \\ 0 \end{pmatrix} \Big\|_{\mathcal{X}} \le c \sup_{\mathcal{W}_N \in \mathcal{X}^0_N} \frac{\langle \mathcal{K}\boldsymbol{f}, \mathcal{W}_N \rangle}{\|\mathcal{W}_N\|_{\mathcal{X}}}, \tag{4.11}$$

which means that the stability properties of (3.11)–(4.5) and (3.13)–(4.6) can be stated separately.

In order to state the next property, we introduce the following broken norm (with obvious definitions for the $\|\cdot\|_{\mathbf{X}^1_\alpha(\Omega_\ell)}$ by restriction):

$$\|\boldsymbol{v}\|_{\mathbf{X}^1_\alpha(\Omega)^\diamond} = \Big(\|\boldsymbol{v}\|^2_{\mathbf{X}^1_\alpha(\Omega_0)} + \|\boldsymbol{v}\|^2_{\mathbf{X}^1_\alpha(\Omega_1)} \Big)^{\frac{1}{2}}, \tag{4.12}$$

The convergence property is a consequence of the following error estimate which is proven in [2, Thms 3.6 & 3.9] and requires the Sobolev spaces $Y^s_\alpha(\Omega)$ introduced in Section 2 (see (2.17)).

Lemma 4.2. *For $0 < \alpha \le \alpha_0$, the operator $\mathcal{S}_N$ satisfies, for all $\boldsymbol{f}$ in $\mathbf{X}^1_{\alpha 0}(\Omega)'$ such that $\mathcal{S} \begin{pmatrix} \boldsymbol{f} \\ 0 \end{pmatrix}$ belongs to $Y^s_\alpha(\Omega)^2$, $s \ge 1$,*

$$\Big\| (\mathcal{S} - \mathcal{S}_N) \begin{pmatrix} \boldsymbol{f} \\ 0 \end{pmatrix} \Big\|_{\mathbf{X}^1_\alpha(\Omega)} \le c\, N^{1-s}\, \|\mathcal{S} \begin{pmatrix} \boldsymbol{f} \\ 0 \end{pmatrix} \|_{Y^s_\alpha(\Omega)^2}, \tag{4.13}$$

and, more generally, for all $\boldsymbol{f}$ *in* $\mathbf{X}^1_{\alpha 0}(\Omega)'$ *and* ξ *in* $\mathbb{R}$ *such that* $\mathcal{S}\begin{pmatrix}\boldsymbol{f}\\ \xi\end{pmatrix}$ *belongs to* $Y^s_\alpha(\Omega)^2$*,* $s > 1$*,*

$$\left\|(\mathcal{S}-\mathcal{S}_N)\begin{pmatrix}\boldsymbol{f}\\ \xi\end{pmatrix}\right\|_{\mathbf{X}^1_\alpha(\Omega)^\diamond} \le c\,\sup\{N^{1-s}, N^{\frac{1}{4}-\alpha}\}\,\|\mathcal{S}\begin{pmatrix}\boldsymbol{f}\\ \xi\end{pmatrix}\|_{Y^s_\alpha(\Omega)^2}. \tag{4.14}$$

Combining Lemma 4.1 and (4.13) leads to the following convergence property: for any compact subset $\mathcal{C}$ of $\mathbf{X}^1_{\alpha 0}(\Omega)'$,

$$\lim_{N\to+\infty}\sup_{\boldsymbol{f}\in\mathcal{C}}\left\|(\mathcal{S}-\mathcal{S}_N)\begin{pmatrix}\boldsymbol{f}\\ 0\end{pmatrix}\right\|_{\mathbf{X}^1_\alpha(\Omega)} = 0. \tag{4.15}$$

Otherwise, for nonhomogeneous boundary conditions, the convergence is obtained only for $\frac{1}{4} < \alpha \le \alpha_0$.

4.3. Some properties of the trilinear forms

We also need to check that the norm of the form $C_N(\cdot;\cdot,\cdot)$ and $c^\pm_N(\cdot;\cdot,\cdot)$ on the discrete spaces is bounded independently of N.

Lemma 4.3. *There exists a constant* c *independent of* ν *and* N *such that the following continuity property holds*

$$\begin{aligned}&\forall \boldsymbol{w}_N \in \mathbf{X}_N,\ \forall \boldsymbol{u}_N \in \mathbf{X}_N, \forall \boldsymbol{v}_N \in \mathbf{X}_N,\\ &|\,C_N(\boldsymbol{w}_N;\boldsymbol{u}_N,\boldsymbol{v}_N)\,| \le \frac{c}{\nu}\,\|\boldsymbol{w}_N\|_{L^4_\alpha(\Omega)^2}\,\|\boldsymbol{u}_N\|_{\mathbf{X}^1_\alpha(\Omega)^\diamond}\,\|\boldsymbol{v}_N\|_{L^4_\alpha(\Omega)^2}.\end{aligned} \tag{4.16}$$

Proof. We recall [7, form. (13.20)] the continuity property of the quadrature formula with respect to θ

$$\forall\varphi_N\in\mathbb{P}_N(-1,1), \forall\psi_N(-1,1),\ \sum_{j=0}^{N}\varphi_N(\xi_j)\psi_N(\xi_j)\rho_j \le 3\|\varphi_N\|_{L^2(-1,1)}\|\psi_N\|_{L^2(-1,1)}.$$

So, since the operators $\mathcal{I}^\ell_N$ take their values in $\mathbb{P}_{N-1,N}(\Omega_\ell)$, combining the previous line with the exactness property of the quadrature formula with respect to r yields the following bound for the first term in $C_N(\boldsymbol{w}_N;\boldsymbol{u}_N,\boldsymbol{v}_N)$:

$$\begin{aligned}\frac{1}{\nu}\,|\big(w_{rN}\partial_{rN}u_{rN}, v_{rN}\big)_N| &= \frac{1}{\nu}\sum_{\ell=0}^{1}|\big(\partial_r u_{rN}\,, \mathcal{I}^{(\ell)}_N(w_{rN}\,v_{rN})\big)_N|\\ &\le \frac{3}{\nu}\sum_{\ell=0}^{1}\|\partial_r u_{rN}\|_{L^2_\alpha(\Omega_\ell)}\|\mathcal{I}^{(\ell)}_N(w_{rN}\,v_{rN})\|_{L^2_\alpha(\Omega_\ell)}.\end{aligned}$$

The following property for the operators $j_N^{(\ell)}$

$$\forall \varphi_M \in \mathbb{P}_M(\Gamma_\ell), \qquad \|j_N^{(\ell)} \varphi_M\|_{L^2(\Gamma_\ell)} \leq c \left(1 + \frac{M}{N}\right) \|\varphi_M\|_{L^2(\Gamma_\ell)},$$

is proven in [7, form. (13.28)], and its analogue for the interpolation operator i_{N-1} at the nodes r_i can be derived from [6, Appendix C] by similar arguments

$$\forall \varphi_M \in \mathbb{P}_M(\mathcal{I}), \qquad \|i_{N-1} \varphi_M\|_{L^2_\alpha(\mathcal{I})} \leq c \left(1 + \frac{M}{N}\right) \|\varphi_M\|_{L^2_\alpha(\mathcal{I})}.$$

Since $\mathcal{I}_N^{(\ell)}$ is equal to $i_{N-1} \circ j_N^{(\ell)}$, applying the previous inequalities with $M = 2N$ yields

$$\|\mathcal{I}_N^{(\ell)}(w_{rN}\, v_{rN})\|_{L^2_\alpha(\Omega_\ell)} \leq 9c\, \|w_{rN}\, v_{rN}\|_{L^2_\alpha(\Omega_\ell)} \leq 9c\, \|w_{rN}\|_{L^4_\alpha(\Omega_\ell)} \|v_{rN}\|_{L^4_\alpha(\Omega_\ell)}.$$

So, we derive the continuity property for the first term. Similar arguments applied to the next terms lead to the desired property.

We skip the proof of the analogous result concerning the form $c_N^{\pm}(\cdot;\cdot,\cdot)$, since it is simpler.

Lemma 4.4. *There exists a constant c independent of ν and N such that the following continuity property holds*

$$\begin{aligned} &\forall \boldsymbol{w}_N \in \mathbf{X}_N,\, \forall \boldsymbol{u}_N \in \mathbf{X}_N,\, \forall \boldsymbol{\varphi}_N \in \mathbf{X}_N^{\pm}, \\ &\qquad |\, c_N^{\pm}(\boldsymbol{w}_N; \boldsymbol{u}_N, \boldsymbol{\varphi}_N)\,| \leq \frac{c}{\nu}\, \|\boldsymbol{w}_N\|_{L^4_\alpha(\Omega)^2}\, \|\boldsymbol{u}_N\|_{\mathbf{X}^1_\alpha(\Omega)}\, \|\boldsymbol{\varphi}_N\|_{L^4_\alpha(\mathcal{I})^2}. \end{aligned} \tag{4.17}$$

4.4. Some additional lemmas

From now on, we choose α such that

$$\frac{1}{4} < \alpha \leq \alpha_0. \tag{4.18}$$

We consider a solution $\boldsymbol{u}$ of problem (4.2) which satisfies Assumption A.1 and an approximation $\boldsymbol{u}_N^*$ of it such that $\mathcal{K}_N \boldsymbol{u}_N^*$ belongs to $\mathcal{X}_N$ and which satisfies the following properties

$$\|\boldsymbol{u}_N^*\|_{\mathbf{X}^1_\alpha(\Omega)^\diamond} \leq c\, \|\boldsymbol{u}\|_{\mathbf{X}^1_\alpha(\Omega)} \qquad \text{and} \qquad \lim_{N \to +\infty} \|\boldsymbol{u} - \boldsymbol{u}_N^*\|_{\mathbf{X}^1_\alpha(\Omega)^\diamond} = 0. \tag{4.19}$$

Note also that the definition of the space $\mathcal{K}_N^{-1} \mathcal{X}_N^0$ is obvious.

Lemma 4.5. *If Assumption* A.1 *holds, there exists an integer N_0 such that, for all $N \geq N_0$, the operator*

$$\mathrm{Id} + \mathcal{S}_N \begin{pmatrix} D\mathbf{F}_N(\boldsymbol{u}_N^*) \\ 0 \end{pmatrix}$$

is an isomorphism of $\mathcal{K}_N^{-1}\mathcal{X}_N^0$. *Moreover, the norm of its inverse is bounded above by a constant independent of* N.

Proof. To prove this, we write the identity

$$\begin{aligned}\mathrm{Id} + \mathcal{S}_N \begin{pmatrix} D\mathbf{F}_N(\boldsymbol{u}_N^*) \\ 0 \end{pmatrix} &= \mathrm{Id} + \mathcal{S} \begin{pmatrix} D\mathbf{F}(\boldsymbol{u}) \\ 0 \end{pmatrix} - (\mathcal{S} - \mathcal{S}_N) \begin{pmatrix} D\mathbf{F}(\boldsymbol{u}) \\ 0 \end{pmatrix} \\ &\quad - \mathcal{S}_N \begin{pmatrix} D\mathbf{F}(\boldsymbol{u}) - D\mathbf{F}(\boldsymbol{u}_N^*) \\ 0 \end{pmatrix} - \mathcal{S}_N \begin{pmatrix} D\mathbf{F}(\boldsymbol{u}_N^*) - D\mathbf{F}_N(\boldsymbol{u}_N^*) \\ 0 \end{pmatrix}.\end{aligned} \tag{4.20}$$

Indeed, let $\boldsymbol{w}_N$ be any element such that $\mathcal{K}_N\boldsymbol{w}_N$ belongs to the unit sphere of $\mathcal{X}_N^0$. It follows from Assumption A.1 that, for a constant c_0 independent of N,

$$\Big\| \boldsymbol{w}_N + \mathcal{S} \begin{pmatrix} D\mathbf{F}(\boldsymbol{u}).\boldsymbol{w}_N \\ 0 \end{pmatrix} \Big\|_{\mathbf{X}_\alpha^1(\Omega)} \geq c_0.$$

So it remains to check that the last three terms in (4.20) tend to zero.

1) Denoting $\mathcal{K}D\mathbf{F}(\boldsymbol{u})$ by $(G^-, G^+, \widetilde{G})$, we have for instance

$$\begin{aligned}\langle \widetilde{G}\,\boldsymbol{w}_N, \boldsymbol{v}_N\rangle = C_\alpha(\tilde{\boldsymbol{u}}; \tilde{\boldsymbol{w}}_N, \boldsymbol{v}_N) &+ C_\alpha(\tilde{\boldsymbol{w}}_N; \tilde{\boldsymbol{u}}, \boldsymbol{v}_N) \\ &+ C_\alpha(\boldsymbol{u}^\diamond; \tilde{\boldsymbol{w}}_N, \boldsymbol{v}_N) + C_\alpha(\boldsymbol{w}_N^\diamond; \tilde{\boldsymbol{u}}, \boldsymbol{v}_N) \\ &+ C_\alpha(\tilde{\boldsymbol{u}}; \boldsymbol{w}_N^\diamond, \boldsymbol{v}_N) + C_\alpha(\tilde{\boldsymbol{w}}_N; \boldsymbol{u}^\diamond, \boldsymbol{v}_N).\end{aligned}$$

By combining the continuity properties (2.8) and (2.15) (still valid here since $\boldsymbol{w}_N$ belongs to $\mathbf{X}_{\alpha 0}^1(\Omega)$), and using a similar argument for the terms $G^\pm \cdot \boldsymbol{w}_N$ we obtain

$$\sup_{V_N \in \mathcal{X}_N^0} \frac{\langle \mathcal{K}D\mathbf{F}(\boldsymbol{u}) \cdot \boldsymbol{w}_N, V_N\rangle}{\|V_N\|_{\mathcal{X}}} \leq c\, \|\boldsymbol{u}\|_{\mathbf{X}_\alpha^1(\Omega)} \|\mathcal{K}_N\boldsymbol{w}_N\|_{L_\alpha^4(\mathcal{I})^2 \times L_\alpha^4(\mathcal{I})^2 \times L_\alpha^4(\Omega)^2}.$$

So, by combining (4.15) with the compactness of the imbedding of $\mathbf{X}_\alpha^1(\Omega)$ into $L_\alpha^4(\Omega)^2$ (see Lemma 2.2), we derive that the first of the three terms tends to zero.

2) By similar arguments, we have

$$\sup_{V_N \in \mathcal{X}_N^0} \frac{\langle \mathcal{K}(D\mathbf{F}(\boldsymbol{u}) - D\mathbf{F}(\boldsymbol{u}_N^*)) \cdot \boldsymbol{w}_N, V_N\rangle}{\|V_N\|_{\mathcal{X}}} \leq c\, \|\boldsymbol{u} - \boldsymbol{u}_N^*\|_{\mathbf{X}_\alpha^1(\Omega)^\diamond} \|\boldsymbol{w}_N\|_{\mathbf{X}_\alpha^1(\Omega)}.$$

So the convergence to zero of the second of the three terms follows from (4.19).

3) The third term comes from numerical integration, and the idea is the following: the orthogonal projection operators $\Pi_{N'}^\ell$ from $\mathbf{X}_\alpha^1(\Omega_\ell)$ onto $\mathbb{P}_{N'}(\Omega_\ell)$, where N' is equal to the integral part of $\frac{N-1}{2}$ satisfies, for all $\boldsymbol{v}$ in $\mathbf{X}_\alpha^1(\Omega)$,

$$\|\boldsymbol{v} - \Pi_{N'}^\ell \boldsymbol{v}\|_{L_\alpha^4(\Omega_\ell)^2} \leq c\, N^{\frac{1}{4}(\alpha-1)}\, \|\boldsymbol{v}\|_{\mathbf{X}_\alpha^1(\Omega)}$$

(this is derived from [6, Appendix B] thanks to a duality argument), and a similar property holds for the orthogonal projection operators $\pi_{N'}^\pm$ from $\mathbf{H}_{\alpha\pm}^1(\mathcal{I})$ onto $\mathbb{P}_{N'}(\mathcal{I})$. So the idea consists of adding and subtracting these operators applied to $\tilde{\boldsymbol{u}}_N^*$ or $\boldsymbol{u}_N^{*\diamond}$, $\tilde{\boldsymbol{w}}_N$ or $\boldsymbol{w}_N^\diamond$ and $\tilde{\boldsymbol{v}}_N$ ou $\boldsymbol{v}_N^\diamond$ and a similar one to $e_N^\pm$, until the forms

$C_\alpha(\cdot;\cdot,\cdot)$ and $C_N(\cdot;\cdot,\cdot)$ applied to these projections coincide and applying the previous approximation property combined with (4.16). This yields the convergence to zero of the third term.

Lemma 4.6. *For all $\boldsymbol{z}_N$ such that $\mathcal{K}_N\boldsymbol{z}_N$ belongs to $\mathcal{X}_N$ and $\|\boldsymbol{u}_N^* - \boldsymbol{z}_N\|_{\mathbf{X}^1_\alpha(\Omega)^\diamond} \leq \lambda$, the following Lipschitz property holds*

$$\left\| \mathcal{S}_N \begin{pmatrix} D\mathbf{F}_N(\boldsymbol{u}_N^*) \\ 0 \end{pmatrix} - \mathcal{S}_N \begin{pmatrix} D\mathbf{F}_N(\boldsymbol{z}_N) \\ 0 \end{pmatrix} \right\|_{\mathbf{X}^1_\alpha(\Omega)} \leq c\,\lambda. \tag{4.21}$$

Proof. This follows from the fact that the form $C_N(\cdot;\cdot,\cdot)$ is trilinear and continuous, see Lemma 4.4, combined with (4.10).

We now choose $\boldsymbol{u}_N^*$ such that $\mathcal{K}_N\boldsymbol{u}_N^*$ is equal to the image of $\boldsymbol{u}$ by the orthogonal projection operator from $\mathcal{X}$ onto $\mathcal{X}_N$ (thus, it satisfies (4.19)). We set:

$$\varepsilon_N = \left\| \boldsymbol{u}_N^* + \mathcal{S}_N \begin{pmatrix} F_N(\boldsymbol{u}_N^*) \\ -\xi \end{pmatrix} \right\|_{\mathbf{X}^1_\alpha(\Omega)^\diamond}, \tag{4.22}$$

and we check that it tends to zero.

Lemma 4.7. *Let $\boldsymbol{f}$ belong to $Y^\sigma(\Omega)^2$, $\sigma > \frac{3}{2}$, and $\boldsymbol{u}$ be a solution of problem (4.2) in $Y^s_\alpha(\Omega)^2$, $s > 1$. Then, the following estimate holds for a constant $c(\boldsymbol{u})$ depending on $\boldsymbol{u}$ and s but not on N,*

$$\varepsilon_N \leq c(\boldsymbol{u}) \sup\{N^{1-s}, N^{\frac{1}{4}-\alpha}\} + c\,N^{-\sigma}\,\|\boldsymbol{f}\|_{Y^\sigma_\alpha(\Omega)^2}. \tag{4.23}$$

Proof. By subtracting (4.2), we observe that

$$\boldsymbol{u}_N^* + \mathcal{S}_N \begin{pmatrix} F_N(\boldsymbol{u}_N^*) \\ -\xi \end{pmatrix} = -(\boldsymbol{u} - \boldsymbol{u}_N^*) - (\mathcal{S} - \mathcal{S}_N) \begin{pmatrix} \mathbf{F}(\boldsymbol{u}) \\ -\xi \end{pmatrix} - \mathcal{S}_N \begin{pmatrix} \mathbf{F}(\boldsymbol{u}) - \mathbf{F}(\boldsymbol{u}_N^*) \\ 0 \end{pmatrix} - \mathcal{S}_N \begin{pmatrix} \mathbf{F}(\boldsymbol{u}_N^*) - \mathbf{F}_N(\boldsymbol{u}_N^*) \\ 0 \end{pmatrix}.$$

From the choice of $\boldsymbol{u}_N^*$, bounding the first term relies on already known approximation properties, see [2, Prop. 3.8]. To estimate the second term, we use (4.14) together with the regularity property of $\boldsymbol{u}$. The bound for the third one is a consequence of (4.10) together with the continuity of $\mathbf{F}$ and the choice of $\boldsymbol{u}_N^*$. Finally, to evaluate the last term which is due to numerical integration:

1) for the nonlinear part, as in the proof of Lemma 4.6, we add and subtract $\Pi^\ell_{N'}\boldsymbol{u}$ when necessary, in order to bound it by

$$c\,\|\boldsymbol{u}_N^*\|_{\mathbf{X}^1_\alpha(\Omega)^\diamond} \big(\|\boldsymbol{u} - \boldsymbol{u}_N^*\|_{\mathbf{X}^1_\alpha(\Omega)^\diamond} + \sum_{\ell=0}^{1} \|\boldsymbol{u} - \Pi^{(\ell)}_{N'}\boldsymbol{u}\|_{\mathbf{X}^1_\alpha(\Omega_\ell)} \big),$$

2) for the linear part, we add and subtract the orthogonal projection $\boldsymbol{f}^{(\ell)}_{N'}$ of $\boldsymbol{f}$ onto the $\mathbb{P}_{N'}(\Omega_\ell)^2$ for the scalar product of $L^2_\alpha(\Omega_\ell)^2$ and use the interpolation

operator $\mathcal{I}_N^{(\ell)}$, in order to bound it by

$$\sum_{\ell=0}^{1}\big(\|\boldsymbol{f}-\boldsymbol{f}_{N'}^{(\ell)}\|_{L^2_\alpha(\Omega_\ell)^2}+\|\boldsymbol{f}-\mathcal{I}_N^{(\ell)}\boldsymbol{f}\|_{L^2_\alpha(\Omega_\ell)^2}\big).$$

We conclude by using the approximation properties of these operators, see [6, Appendices B & C].

4.5. Existence of a solution and error estimates

From Lemmas 4.5 to 4.7, the assumptions of the implicit function theorem of Brezzi, Rappaz and Raviart [9] are now satisfied. Applying this theorem leads to the main result of this section.

Theorem 4.8. *Assume that* (4.18) *holds, that the data* $\boldsymbol{f}$ *belong to* $Y^\sigma(\Omega)^2$, $\sigma>\frac{3}{2}$, *and that* ξ *is a given constant. Let* $(\boldsymbol{u},p)$ *be a solution of problem* (3.2)–(2.3) *in* $Y^s_\alpha(\Omega)^2$, $s>1$, *which satisfies Assumption* A.1. *Thus, there exists an integer* N_* *and a positive real number* λ_*, *such that, for all integers* $N\geq N_*$, *problem* (3.74) *has a solution* $(\boldsymbol{u}_N,p_N)$ *with unique velocity* $\boldsymbol{u}_N$ *in the ball*

$$\big\{\boldsymbol{w}_N;\mathcal{K}_N\boldsymbol{w}_N\in\mathcal{X}_N \text{ and } \|\boldsymbol{u}-\boldsymbol{w}_N\|_{\mathbf{X}^1_\alpha(\Omega)^\diamond}\leq\lambda_*\big\}. \tag{4.24}$$

Moreover, this solution satisfies for a constant $c(\boldsymbol{u})$ *depending on* $\boldsymbol{u}$ *and* s *but not on* N,

$$\|\boldsymbol{u}-\boldsymbol{u}_N\|_{\mathbf{X}^1_\alpha(\Omega)^\diamond}\leq c(\boldsymbol{u})\,\sup\{N^{1-s},N^{\frac{1}{4}-\alpha}\}+c\,N^{-\sigma}\,\|\boldsymbol{f}\|_{Y^\sigma_\alpha(\Omega)^2}. \tag{4.25}$$

The previous estimate yields the convergence of the family of discrete solutions to the exact one, and this seems to be the first convergence result for this problem without regularization of the boundary data. The convergence order is rather low. However by taking $\alpha=\frac{1}{2}$ and using the regularity properties recalled in Section 2, we can deduce that the error behaves like $N^{-\frac{1}{4}}$. As for the Stokes problem, no error estimate is derived for the pressure (this can be obtained by solving a further problem with the pressure as the only unknown once the discrete velocity is computed), but the velocity $\boldsymbol{u}_N$ has the further property to be nearly exactly divergence-free, in the sense that the restrictions of $\boldsymbol{u}_N$ to both Ω_0 and Ω_1 are exactly divergence-free.

5. Description of the algorithm and numerical experiments

We first present the algorithm that is used to handle the nonlinear terms in problem (3.74). Since the description of the implementation for the Stokes problem is given in [2, §4], we briefly explain the modifications that must be applied to solve the linear systems resulting from the previous algorithm. We conclude with some numerical experiments.

5.1. An algorithm for the nonlinear term

The convergence of Newton's method for solving problem (3.74) iteratively can be derived from [9] for any initial data $\boldsymbol{u}_N^0$ in the ball introduced in (4.24). However it leads to solving simultaneously the two one-dimensional problems and the two-dimensional one, which seems too expensive. We therefore propose a new algorithm where Newton's method is only applied to the two-dimensional problem.

Initialization step: We choose $\boldsymbol{u}_N^0$ to be the solution of the Stokes problem (i.e. with $\frac{1}{\nu} = 0$). Equivalently, the pair $(\boldsymbol{u}_N^0, p_N^0)$ satisfies

$$\begin{aligned} \boldsymbol{u}_N^0 = \tilde{\boldsymbol{u}}_N^0 + \boldsymbol{u}_N^{0\diamond}, \quad \text{with} \quad \boldsymbol{u}_N^{0\diamond} = \boldsymbol{u}_N^{0-}\, e_N^- + \boldsymbol{u}_N^{0+}\, e_N^+, \\ p_N^0 = \tilde{p}_N^0 + p_N^{0\diamond}, \quad \text{with} \quad p_N^{0\diamond} = p_N^{0-}\, e_N^- + p_N^{0+}\, e_N^+, \end{aligned} \tag{5.1}$$

where each $(\boldsymbol{u}_N^{0\pm}, p_N^{0\pm})$ is a solution in $\mathbf{X}_N^\pm \times M_N^\diamond$ of (3.11) and

$$\begin{aligned} &\forall \boldsymbol{\varphi}_N \in \mathbf{X}_N^{\pm 0}, \quad a_{N\pm}(\boldsymbol{u}_N^{0\pm}, \boldsymbol{\varphi}_N) + b_{1N\pm}(\boldsymbol{\varphi}_N, p_N^{0\pm}) = \big(f_{rN}^\pm, \varphi_{rN}\big)_N^\diamond + \big(f_{\theta N}^\pm, \varphi_{\theta N}\big)_N^\diamond, \\ &\forall \chi_N \in M_N^\diamond, \quad \overline{b}_{2N\pm}(\boldsymbol{u}_N^{0\pm}, \chi_N) = 0, \end{aligned} \tag{5.2}$$

and $(\tilde{\boldsymbol{u}}_N^0, \tilde{p}_N^0)$ is a solution in $\mathbf{X}_N \times M_N$ of (3.13) and

$$\begin{aligned} &\forall \boldsymbol{v}_N \in \mathbf{X}_N^0, \quad a_N(\tilde{\boldsymbol{u}}_N^0, \boldsymbol{v}_N) + b_{1N}(\boldsymbol{v}_N, \tilde{p}_N^0) = (\tilde{f}_{rN}, v_{rN})_N + (\tilde{f}_{\theta N}, v_{\theta N})_N, \\ &\forall q_N \in M_N, \quad b_{2N}(\tilde{\boldsymbol{u}}_N^0, q_N) = 0. \end{aligned} \tag{5.3}$$

It can be noted that problems (3.11)–(5.2) and (3.13)–(5.3) are completely independent.

Iteration step: Assuming that, for an integer $k > 0$, the velocity $\boldsymbol{u}_N^{k-1}$ at iteration $k-1$ is known, we compute a solution $(\boldsymbol{u}_N^k, p_N^k)$ of the form

$$\begin{aligned} \boldsymbol{u}_N^k = \tilde{\boldsymbol{u}}_N^k + \boldsymbol{u}_N^{k\diamond}, \quad \text{with} \quad \boldsymbol{u}_N^{k\diamond} = \boldsymbol{u}_N^{k-}\, e_N^- + \boldsymbol{u}_N^{k+}\, e_N^+, \\ p_N^k = \tilde{p}_N^k + p_N^{k\diamond}, \quad \text{with} \quad p_N^{k\diamond} = p_N^{k-}\, e_N^- + p_N^{k+}\, e_N^+. \end{aligned} \tag{5.4}$$

The $(\boldsymbol{u}_N^{k\pm}, p_N^{k\pm})$ are simply a solution of problem (3.17) linearized at $\boldsymbol{u}_N^{k-1}$. They belong to $\mathbf{X}_N^\pm \times M_N^\diamond$, and satisfy (3.11) and

$$\begin{aligned} &\forall \boldsymbol{\varphi}_N \in \mathbf{X}_N^{\pm 0}, \\ &\quad a_{N\pm}(\boldsymbol{u}_N^{k\pm}, \boldsymbol{\varphi}_N) + b_{1N\pm}(\boldsymbol{\varphi}_N, p_N^{k\pm}) + c_N^\pm(\tilde{\boldsymbol{u}}_N^{k-1}; \boldsymbol{u}_N^{k\diamond}, \boldsymbol{\varphi}_N) + c_N^\pm(\boldsymbol{u}_N^{k\diamond}; \tilde{\boldsymbol{u}}_N^{k-1}, \boldsymbol{\varphi}_N) \\ &\qquad = \big(f_{rN}^\pm, \varphi_{rN}\big)_N^\diamond + \big(f_{\theta N}^\pm, \varphi_{\theta N}\big)_N^\diamond - c_N^\pm(\tilde{\boldsymbol{u}}_N^{k-1}; \tilde{\boldsymbol{u}}_N^{k-1}, \boldsymbol{\varphi}_N), \\ &\forall \chi_N \in M_N^\diamond, \quad \overline{b}_{2N\pm}(\boldsymbol{u}_N^{k\pm}, \chi_N) = 0. \end{aligned} \tag{5.5}$$

Newton's algorithm is then applied to compute the pair $(\tilde{\boldsymbol{u}}_N^k, \tilde{p}_N^k)$: it is a solution in $\mathbf{X}_N \times M_N$ of (3.13) and

$$\begin{aligned} \forall \boldsymbol{v}_N \in \mathbf{X}_N^0, \quad & a_N(\tilde{\boldsymbol{u}}_N^k, \boldsymbol{v}_N) + b_{1N}(\boldsymbol{v}_N, \tilde{p}_N^k) \\ & \quad + C_N(\boldsymbol{u}_N^{k\diamond} + \tilde{\boldsymbol{u}}_N^{k-1}; \tilde{\boldsymbol{u}}_N^k, \boldsymbol{v}_N) + C_N(\tilde{\boldsymbol{u}}_N^k; \boldsymbol{u}_N^{k\diamond} + \tilde{\boldsymbol{u}}_N^{k-1}, \boldsymbol{v}_N) \\ & \quad = (\tilde{f}_{rN}, v_{rN})_N + (\tilde{f}_{\theta N}, v_{\theta N})_N w \\ & \qquad - C_N(\boldsymbol{u}_N^{k\diamond}; \boldsymbol{u}_N^{k\diamond}, \boldsymbol{v}_N) + C_N(\tilde{\boldsymbol{u}}_N^{k-1}; \tilde{\boldsymbol{u}}_N^{k-1}, \boldsymbol{v}_N), \\ \forall q_N \in M_N, \quad & b_{2N}(\tilde{\boldsymbol{u}}_N^k, q_N) = 0. \end{aligned} \tag{5.6}$$

Here also problems (3.11)–(5.5) and (3.13)–(5.6) are completely independent.

Note the further property of this algorithm: when some norm of $\boldsymbol{u}_N^{k\diamond} - \boldsymbol{u}_N^{k-1\diamond}$, respectively of $\tilde{\boldsymbol{u}}_N^k - \tilde{\boldsymbol{u}}_N^{k-1}$, is larger than a given tolerance, several iterations on problem (3.11)–(5.5), respectively on problem (3.13)–(5.6), can be performed inside the global iteration k.

5.2. Implementational details

We just give some details about the linear systems equivalent to the two-dimensional problems (3.13)–(5.3) and (3.13)–(5.6) since the one-dimensional problems are much simpler (no mortar condition is enforced here).

The mortar matching conditions (3.4) on Γ_{01} and Γ_{10} are enforced via a Lagrange multiplier, according to [3], which leads to a further unknown $\boldsymbol{\lambda}_N^k = (\lambda_{N,01}^k, \lambda_{N,10}^k)$. Thus, the unknowns associated with problem (3.13)–(5.3) for $k = 0$, (3.13)–(5.6) for $k > 0$, are the values

- of $\tilde{\boldsymbol{u}}_N^k$ at the nodes $(r_i, \theta_{\ell j})$, $1 \le i \le N-1$, $0 \le j \le N$, for $\ell = 0$ and 1,
- of $\tilde{p}_N^k$ at the nodes $(r_i, \theta_{\ell j})$, $1 \le i \le N$, $0 \le j \le N$, for $\ell = 0$ and 1,
- of each component of $\boldsymbol{\lambda}_N^k$ at the r_i, $1 \le i \le N-1$.

Let U^k, P^k and Λ^k denote the vectors consisting of the above values. Their dimensions are respectively $4(N-1)(N+1)$, $2N(N+1)$ and $4(N-1)$.

Then, for $k = 0$ and with obvious notation, problem (3.13)–(5.3) is now equivalent to the square linear system

$$\begin{pmatrix} A & B_1 & L \\ B_2^T & 0 & 0 \\ L^T & 0 & 0 \end{pmatrix} \begin{pmatrix} U^0 \\ P^0 \\ \Lambda^0 \end{pmatrix} = \begin{pmatrix} M\,F - A^*\,G \\ -B_2^{*T}\,G \\ -L^{*T}\,G \end{pmatrix}. \tag{5.7}$$

We refer to [2, §4] for the description and basic properties of the matrices A, B_1, B_2, L and M and also of the matrices A^*, B_2^* and L^*, and of the data vectors F and G.

For $k > 0$, problem (3.13)–(5.6) results into the slightly modified system

$$\begin{pmatrix} A & B_1 & L \\ B_2^T & 0 & 0 \\ L^T & 0 & 0 \end{pmatrix} \begin{pmatrix} U^k \\ P^k \\ \Lambda^k \end{pmatrix} + \begin{pmatrix} C_k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} U^k \\ P^k \\ \Lambda^k \end{pmatrix} = \begin{pmatrix} M\,F - A^*\,G - C_k^*\,G + K_k \\ -B_2^{*T}\,G \\ -L^{*T}\,G \end{pmatrix}. \tag{5.8}$$

Note that only the matrices C_k and C_k^* and the vector K_k must be computed at each iteration step.

To describe these last quantities, we need some further notation. We introduce the Lagrange polynomials p_i in $\mathbb{P}_N(0,1)$, $1 \le i \le N$, associated with the nodes r_i and which morevoer vanish at $r = 0$, and also the Lagrange polynomials q_j^ℓ, $0 \le j \le N$, $\ell = 0$ and 1, associated with the nodes θ_j^ℓ. Then, the coefficients of the square matrix C_k of order $4(N-1)(N+1)$ are the

$$C_N(\boldsymbol{u}_N^{k\diamond} + \tilde{\boldsymbol{u}}_N^{k-1}; p_i q_j^\ell\, \boldsymbol{e}, p_{i'} q_{j'}^\ell\, \boldsymbol{e}') + C_N(p_i q_j^\ell\, \boldsymbol{e}; \boldsymbol{u}_N^{k\diamond} + \tilde{\boldsymbol{u}}_N^{k-1}, p_{i'} q_{j'}^\ell\, \boldsymbol{e}'),$$
$$1 \le i, i' \le N-1, 0 \le j, j' \le N, \ell = 0, 1,$$

where both $\boldsymbol{e}$ and $\boldsymbol{e}'$ run through the set of the two vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The coefficients of the matrix C_k^* are the same, with $1 \le i \le N-1$ replaced by $i = N$. The vector K_k is made of the

$$-C_N(\boldsymbol{u}_N^{k\diamond}; \boldsymbol{u}_N^{k\diamond}, p_{i'} q_{j'}^\ell\, \boldsymbol{e}') + C_N(\tilde{\boldsymbol{u}}_N^{k-1}; \tilde{\boldsymbol{u}}_N^{k-1}, p_{i'} q_{j'}^\ell\, \boldsymbol{e}'),$$
$$1 \le i' \le N-1, 0 \le j' \le N, \ell = 0, 1,$$

with $\boldsymbol{e}'$ equal to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Finally, it can be observed that the values of $\tilde{\boldsymbol{u}}_N^{k-1}$ and $\boldsymbol{u}_N^{k\diamond}$ at the points $(r_i, \theta_{\ell j})$ are known or can easily be computed, so that the best way for computing the matrices C_k or C_k^* and the vector K_k is to keep in memory the tensor made by the quantities

$$C_N(p_{i''} q_{j''}^\ell\, \boldsymbol{e}''; p_i q_j^\ell\, \boldsymbol{e}, p_{i'} q_{j'}^\ell\, \boldsymbol{e}'),$$

for the set of parameters

$$1 \le i, i'' \le N, 1 \le i' \le N-1, \quad 0 \le j, j', j'' \le N, \quad \ell = 0, 1,$$

and $\boldsymbol{e}$, $\boldsymbol{e}'$ and $\boldsymbol{e}''$ equal to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. From the definition of the form $C_N(\cdot; \cdot, \cdot)$, they are very easy to compute: for instance, if $\boldsymbol{e}$, $\boldsymbol{e}'$ and $\boldsymbol{e}''$ are all equal to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, we have

$$C_N(p_{i''} q_{j''}^\ell\, \boldsymbol{e}''; p_i q_j^\ell\, \boldsymbol{e}, p_{i'} q_{j'}^\ell\, \boldsymbol{e}') = \frac{2^{-\alpha-3}}{\nu}\, \theta_\ell\, \delta_{i'i''} \delta_{jj'} \delta_{jj''}\, p_i'(\xi_{i'})\, \omega_{i'}^\alpha \rho_j, \tag{5.9}$$

where $\delta_{..}$ denotes the Krönecker symbol and with θ_ℓ equal to θ_0 or $2\pi-\theta_0$ according to whether ℓ is equal to 0 or 1.

Both systems (5.7) and (5.8) are nonsymmetric. Since they result from saddle-point problems, they can be solved by Uzawa's algorithm. However and as in [2], we prefer to solve these systems by the least squares algorithm, the idea being that the space M_N contains spurious modes on the pressure but that the system to solve is simpler when they are not eliminated (see [2] for details).

5.3. Numerical results

We are interested in the real driven cavity case, namely when the data are given by

$$\boldsymbol{f} = \boldsymbol{0}, \qquad g_0 = 0, \qquad g_1 = 1. \tag{5.10}$$

In Figure 2, we present the isovalues of the solution u_{rN} (Fig. 2a) and $u_{\theta N}$ (Fig. 2b) obtained with the following set of parameters

$$\nu = 10^{-1}, \quad \alpha = \frac{1}{2}, \quad N = 17, \quad K = 12,$$

in the case $\theta_0 = \pi$. Next, we use a projection $\boldsymbol{u}_N^*$ of $\boldsymbol{u}_N$ onto exactly divergence-free functions and present in Figure 3 the isovalues of the associated stream-function ψ (i.e. such that $\boldsymbol{u}_N^* = \mathbf{curl}_r\, \psi$).

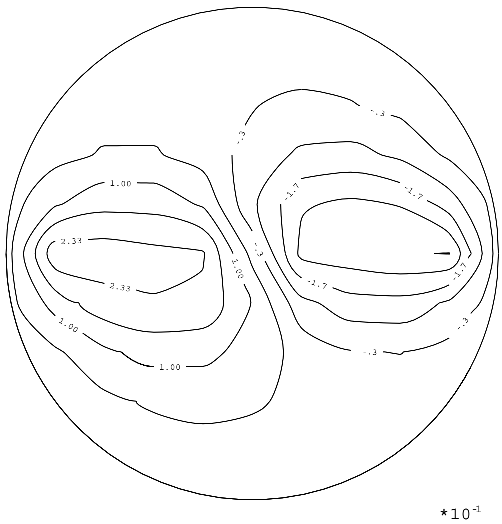

FIG. 2a.

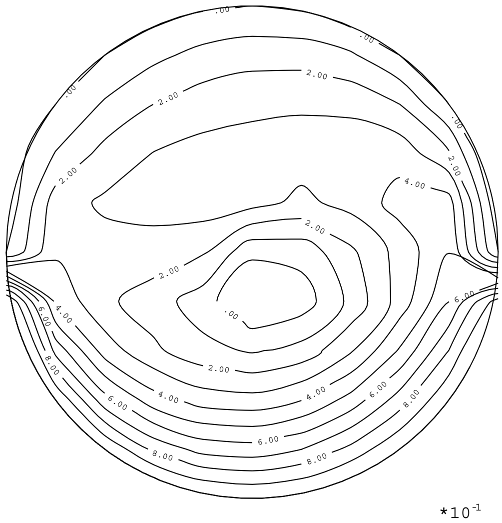

Fig. 2b.

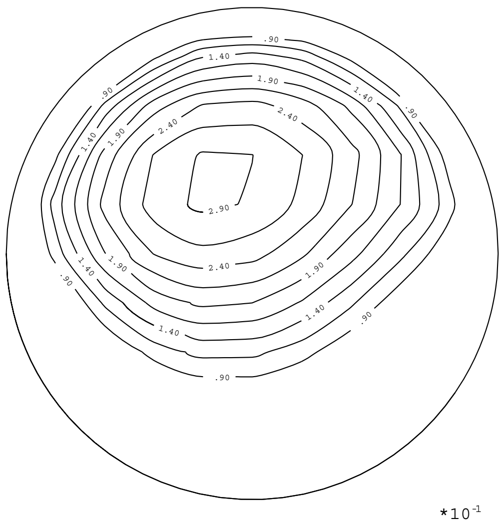

Fig. 3.

In Figure 4, we present the isovalues of the same stream-function ψ, now in the cases $\theta_0 = \frac{\pi}{2}$ (Fig. 4a) and $\theta_0 = \frac{3\pi}{2}$ (Fig. 4b).

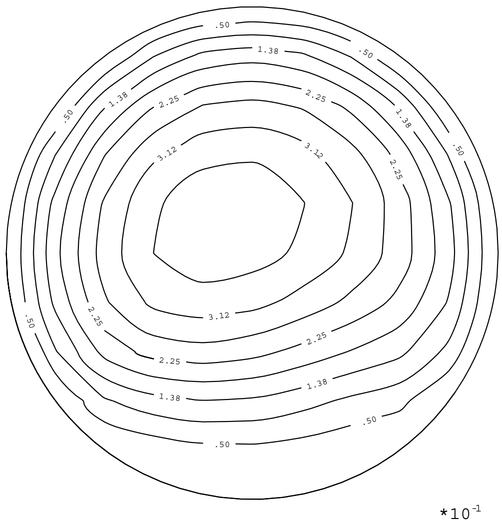

Fig. 4a.

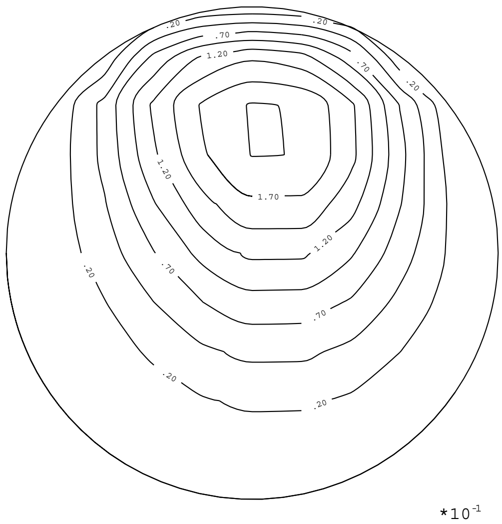

Fig. 4b.

In the following Table 1, we check the efficiency of the algorithm described in Section 5.1. For the following set of parameters

$$\theta_0 = \pi, \quad \nu = 10^{-1}, \quad \alpha = \frac{1}{2}, \quad N = 17,$$

we present the error $E_k = \|\tilde{\boldsymbol{u}}_N^k - \tilde{\boldsymbol{u}}_N^{k-1}\|_{X_\alpha^1(\Omega)^\diamond}$ as a function of k, for k varying from 1 to 12. The results in this table show the fast convergence of the algorithm in this case.

	$k = 2$	$k = 4$	$k = 6$	$k = 8$	$k = 10$	$k = 12$
E_k	0.165×10^{-1}	0.313×10^{-3}	0.558×10^{-5}	0.892×10^{-7}	0.138×10^{-8}	0.211×10^{-10}

Table 1.

We now investigate how the error depends on N, with the set of parameters

$$\theta_0 = \pi, \quad \nu = 10^{-1}, \quad \alpha = \frac{1}{2}, \qquad K = 12.$$

In Figure 5, we present the graph, in logarithmic scales, of the error $\|\tilde{\boldsymbol{u}}_{17}^K - \tilde{\boldsymbol{u}}_N^K\|_{X_\alpha^1(\Omega)^\diamond}$ as a function of N, for $N = 8$, 10, 12, 14, 16. Even though the slope is weak, varying from 0.19 to 0.16 (this is due to the low regularity of the solution), the convergence appears to be in good agreement with the results of the analysis, where the maximal possible slope is 0.25.

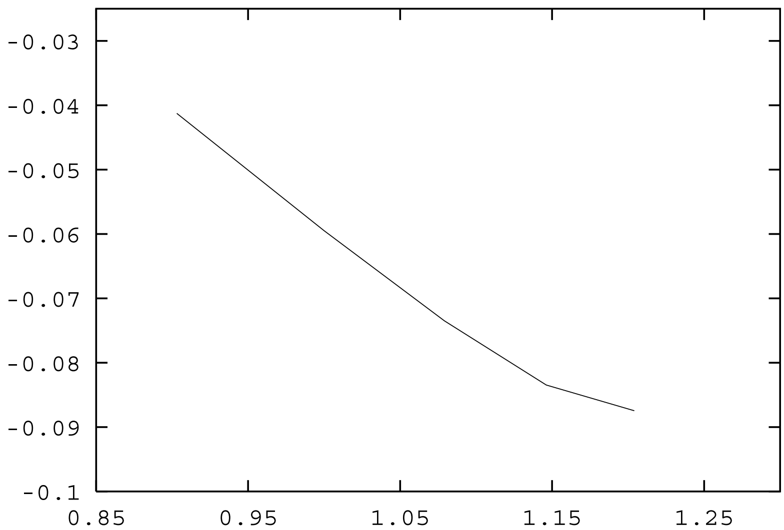

Fig. 5.

Finally, we investigate the behavior of the solution when ν decreases. In Figure 6, for the set of parameters

$$\theta_0 = \pi, \quad \alpha = \frac{1}{2}, \quad N = 17,$$

we present the isovalues of the stream-function associated with the velocity $\boldsymbol{u}_N^K$ for the two values of the viscosity $\nu = 5.10^{-2}$ (Fig. 6a) and $\nu = 2.10^{-2}$ (Fig. 6b). As expected, the number of iterations K which are needed for the error $E_k = \|\tilde{\boldsymbol{u}}_N^k - \tilde{\boldsymbol{u}}_N^{k-1}\|_{X_\alpha^1(\Omega)^\diamond}$ to become smaller than a given tolerance, increases when ν decreases (it is equal to 20 for $\nu = 5.10^{-2}$ and to 40 for $\nu = 2.10^{-2}$). However the results still appear to be correct.

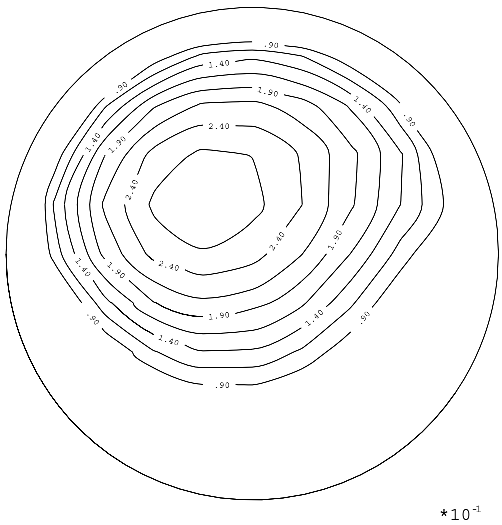

FIG. 6a.

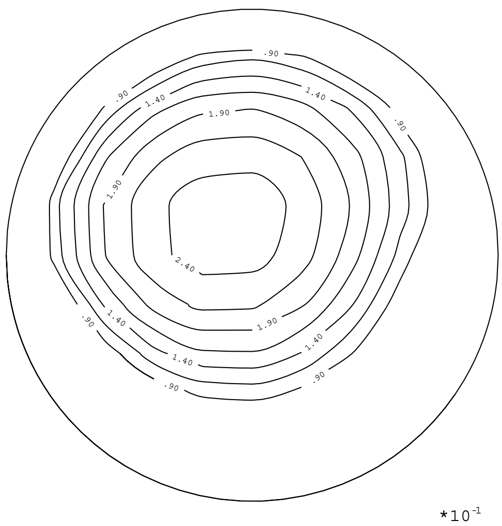

FIG. 6b.

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