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Decay in Time of Incompressible Flows

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Abstract. In this paper we consider the Cauchy problem for incompressible flows governed by the Navier–Stokes or MHD equations. We give a new proof for the time decay of the spatial L_2 norm of the solution, under the assumption that the solution of the heat equation with the same initial data decays. By first showing decay of the first derivatives of the solution, we avoid some technical difficulties of earlier proofs based on Fourier splitting.

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1. Introduction

The aim of this paper is to give a new proof for the decay in time of the spatial L_2 norm of the solutions of the incompressible Navier–Stokes or MHD equations when initial data are given on all space. Up to some limit exponent, the decay rate is the same as for the solution of the (vector) heat equation with the same initial data. The main difference from earlier proofs, which used the Fourier splitting method [8, 11], is that we use a decay estimate of first derivatives in L_2. More precisely, we show that[1]

$$H_1^2(t) := \sum_j \|D_j u(\cdot,t)\|^2 \le C(1+t)^{-1}, \quad D_j = \partial/\partial x_j\,, \tag{1.1}$$

in $N = 3$ space dimensions, and a slightly stronger estimate for $N = 2$. See Theorem 1.3. Once this is established, the decay of

$$H_0^2(t) = \|u(\cdot,t)\|^2 \tag{1.2}$$

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[1] We denote the spatial L_2-norm by $\|u\| = (\int |u(x)|^2 dx)^{1/2}$. Here $|u|$ denotes the Euclidean norm of a vector. Constants C may depend on the initial data u_0 but not on t. They may have different values at different occurrences.

follows from Duhamel's principle, standard heat equation estimates, and Gronwall-type arguments.

To give a more specific outline of the paper, consider the incompressible MHD equations,

$$B_t + v \cdot \nabla B - B \cdot \nabla v = \frac{1}{Rm} \Delta B \tag{1.3}$$

$$v_t + v \cdot \nabla v - SB \cdot \nabla B + \nabla\Big(p + \frac{1}{2} S|B|^2\Big) = \frac{1}{Re} \Delta v \tag{1.4}$$

$$\nabla \cdot B = \nabla \cdot v = 0 \tag{1.5}$$

Here B, v, and p denote the non-dimensionalized magnetic field, fluid velocity, and fluid pressure, respectively. The non-dimensional numbers are the Reynolds number, Re, the magnetic Reynolds number, Rm, and $S = M^2/ReRm$, where M is the Hartman number [8]. For simplicity of presentation, and without restriction, we assume $Re = Rm = S = 1$.

At $t = 0$ we prescribe initial data

$$u(x,0) := \begin{pmatrix} B(x,0) \\ v(x,0) \end{pmatrix} = \begin{pmatrix} B_0(x) \\ v_0(x) \end{pmatrix} := u_0(x)\,. \tag{1.6}$$

We require $u_0 \in H^1$, i.e., $u_0 \in L_2$ and $D_i u_0 \in L_2$, as well as $\nabla \cdot B_0 = \nabla \cdot v_0 = 0$. Under these assumptions the problem (1.3)–(1.6) is known to have a local (in time) solution $u = (B, v)$, which is C^∞ for $0 < t \le T$ and which satisfies $D^\alpha u(\cdot, t) \in L_2$ for all derivatives and $0 < t \le T$. (See, for example, [4, 10] for the development of a local theory of the Navier–Stokes equations and [3] for derivative estimates. The same techniques apply to the MHD system.) Since we are interested in decay for $t \to \infty$, we will always **assume** existence of a C^∞ solution for all $t > 0$. In $N = 2$ space dimensions the existence can be proved (as for the incompressible Navier–Stokes equations), but for $N = 3$ and large initial data the problem remains open.

Our main theorem is the following.

Theorem 1.1. *Consider the MHD system* (1.3)–(1.6) *for* $N = 2$ *or* $N = 3$ *space dimensions under the above assumptions. For the solution of the heat equation*

$$u_t = \Delta u, \quad u(x,0) = u_0(x)\,, \tag{1.7}$$

assume the decay estimate

$$\|e^{\Delta t} u_0\| \le C(1+t)^{-\kappa}, \quad t \ge 0\,, \tag{1.8}$$

for some $\kappa > 0$. *Then the solution* $u = (B, v)$ *of* (1.3)–(1.6) *satisfies*

$$H_0(t) = \|u(\cdot, t)\| \le C(1+t)^{-\gamma}, \quad t \ge 0\,, \tag{1.9}$$

with

$$\gamma = \min\Big\{\kappa, \frac{N}{4} + \frac{1}{2}\Big\}\,.$$

Sufficient conditions for the decay of the solution of the heat equation are well-known. For example, in [8] it is shown that the assumption $u_0 \in H^1 \cap L_1, \nabla \cdot B = \nabla \cdot v = 0$ yields (1.8) with $\kappa = \frac{N}{4} + \frac{1}{2}$. To obtain this result it is important to note that the spatial average of every component of u_0 is zero, a result called Borcher's lemma in [8].

We now outline the proof of Theorem 1.1 and the remaining parts of the paper. First, the following energy equation, which is well-known, can be shown through integration by parts.

Theorem 1.2. *Let $u = (B, v)$ denote the solution of (1.3)–(1.6) where $S = Re = Rm = 1$ and let $H_j(t)$ be defined by (1.1), (1.2). Then we have*

$$\frac{1}{2}\frac{d}{dt}H_0^2(t) + H_1^2(t) = 0, \quad t \geq 0\,, \tag{1.10}$$

thus

$$H_0(t) \leq H_0(s) \quad \text{for} \quad t \geq s \tag{1.11}$$

and

$$\int_0^\infty H_1^2(t)\,dt \leq \frac{1}{2}\,\|u_0\|^2\,. \tag{1.12}$$

In addition, by considering $(d/dt)H_1^2(t)$, it is not difficult to prove the following decay estimate for $H_1^2(t)$.

Theorem 1.3. *Under the assumptions of Theorem 1.2 we have*

$$(1+t)H_1^2(t) \leq C \quad \text{for} \quad N = 2, 3 \tag{1.13}$$

and

$$\lim_{t\to\infty} tH_1^2(t) = 0 \quad \text{for} \quad N = 2\,. \tag{1.14}$$

A proof will be given in Section 2.

As above, denote the solution operator of the (vector) heat equation by $e^{\Delta t}$. We write

$$u(t) = e^{\Delta t}u_0 + \int_0^t e^{\Delta(t-s)}Q(s)\,ds \tag{1.15}$$

where Q contains the nonlinear terms in (1.3), (1.4). Let $\hat{Q}(k,t)$ denote the Fourier transform of $Q(x,t)$. Then the structure of the nonlinearity Q yields the following two estimates, for any time $s \geq 0$ and any wave vector k:

$$|\hat{Q}(k,s)| \leq CH_0(s)H_1(s) \tag{1.16}$$

and

$$|\hat{Q}(k,s)| \leq C|k|H_0^2(s)\,. \tag{1.17}$$

For details, see Section 2.

Using Parseval's relation, it is straightforward to show the following estimate for the solution of the heat equation:

Lemma 1.1. *Let $\hat{u}_0 \in L_\infty(\mathbb{R}^N)$ and $|k|^{-1}\hat{u}_0 \in L_\infty(\mathbb{R}^N)$. Then we have*[2]

$$\|e^{\Delta t}u_0\| \leq Ct^{-N/4}|\hat{u}_0|_\infty \tag{1.18}$$

and

$$\|e^{\Delta t}u_0\| \leq Ct^{-\frac{N}{4}-\frac{1}{2}}|\hat{w}_0|_\infty \quad \text{with} \quad \hat{w}_0(k) = |k|^{-1}\hat{u}_0(k)\,. \tag{1.19}$$

This auxiliary result is also shown in Section 2.

Now assume decay for the solution of the heat equation with initial data u_0, i.e., assume (1.8). Using (1.15),

$$\|u(t)\| \leq C(1+t)^{-\kappa} + \int_0^t \|e^{\Delta(t-s)}Q(s)\|\,ds\,. \tag{1.20}$$

Lemma 1.1 and the estimates (1.16), (1.17) yield the bounds

$$\|e^{\Delta(t-s)}Q(s)\| \leq C(t-s)^{-N/4}H_0(s)H_1(s) \tag{1.21}$$

and

$$\|e^{\Delta(t-s)}Q(s)\| \leq C(t-s)^{-\frac{N}{4}-\frac{1}{2}}H_0^2(s) \tag{1.22}$$

for the integrand in (1.20). Furthermore, Theorem 1.3 provides a decay estimate for $H_1(s)$, and therefore we obtain two integral inequalities for the function $H_0(t)$. If $N = 3$, for example, then (1.20), (1.21), and (1.13) yield

$$H_0(t) \leq C(1+t)^{-\kappa} + C\int_0^t (t-s)^{-3/4}(1+s)^{-1/2}H_0(s)\,ds\,. \tag{1.23}$$

It is now not difficult to complete the proof of Theorem 1.1 using the stated estimates and Gronwall-type arguments. The details are given in Section 3. Finally, in Section 4 we briefly discuss extensions of the results and the (minor) simplifications of the proof that are possible if one is only interested in the Navier–Stokes equations.

Discussion. The decay results in this paper assume classical, i.e., sufficiently differentiable solutions. In particular, in the proof of Theorem 1.3, we use second space derivatives of the solution. In contrast, Wiegner's important paper [11] directly shows decay results for *weak* solutions of the Navier–Stokes equations. Using the results of our paper, we can indirectly also obtain decay results for suitably constructed weak solutions, namely weak solutions that satisfy a generalized energy inequality, because these weak solutions become classical after a finite time for $N = 3$. See [9]. (For $N = 2$ all weak solutions are classical for $t > 0$.) The result for $N = 3$ follows from the estimates

$$\|u\|_{L_3}^2 \leq C\|u\|\,\|Du\| \tag{1.24}$$

[2] By $|\cdot|_\infty$ we denote the supremum norm.

and $\int_0^\infty \|Du\|^2\,dt < \infty$. The integral bound implies that there is a time t_0 for which $\|Du(t_0)\|$ is small. Then (1.24) implies $u(\cdot,t_0)$ to be small in L_3. By a result of Kato [2], the solution starting with the data $u(\cdot,t_0)$ at $t=t_0$ is strong and classical for $t>t_0$ and the generalized energy inequality implies that the weak and the classical solutions agree for $t>t_0$. In this indirect way, our results also imply decay for weak solutions. The same arguments apply to the MHD system.

2. Proof of auxiliary results

Proof of Theorem 1.3. Define

$$H_2^2(t) = \sum_{i,j} \|D_iD_ju(\cdot,t)\|^2, \quad t>0\,,$$

to measure the second derivatives of u in L_2. Through integration by parts,

$$\frac{d}{dt}\,H_1^2 \le C|u|_\infty H_1H_2 - 2H_2^2, \quad t>0\,. \tag{2.1}$$

First let $N=3$ and recall the Gagliardo–Nirenberg inequalities ([1, 5])

$$|u|_\infty \le CH_0^{1/4}H_2^{3/4}, \quad H_1^2 \le CH_0H_2\,. \tag{2.2}$$

The last estimate yields

$$H_2^{-1/4} \le CH_0^{1/4}H_1^{-1/2}\,.$$

Therefore, using (2.1),

$$\begin{aligned}\frac{d}{dt}\,H_1^2 &\le C|u|_\infty H_1H_2 - 2H_2^2\\ &\le CH_0^{1/4}H_1H_2^{7/4} - 2H_2^2\\ &= 2H_2^2(CH_0^{1/4}H_1H_2^{-1/4}-1)\\ &\le 2H_2^2\Big(CH_0^{1/2}H_1^{1/2}-1\Big)\end{aligned} \tag{2.3}$$

We also claim that for any $\varepsilon>0$ there is a time $t_\varepsilon>0$ with

$$H_0^{1/2}(t_\varepsilon)H_1^{1/2}(t_\varepsilon) \le \varepsilon\,.$$

For suppose that t_ε does not exist. Then, for all $t\ge 0$,

$$\|u_0\|^2H_1^2(t) \ge H_0^2(t)H_1^2(t) \ge \varepsilon^4 > 0\,,$$

but the lower bound

$$H_1^2(t) \ge \varepsilon^4\|u_0\|^{-2} > 0$$

contradicts finiteness of the integral, $\int_0^\infty H_1^2\,dt < \infty$. See (1.12). Therefore, if $C>0$ is the constant in (2.3) and $\varepsilon = 1/C$, then we have $(d/dt)H_1^2(t) \le 0$ for $t \ge t_\varepsilon$. Thus, for all $t>t_\varepsilon$,

$$(t-t_\varepsilon)H_1^2(t) \le \int_{t_\varepsilon}^t H_1^2\,ds \le C_1 < \infty\,,$$

which proves $tH_1^2(t) \le C$. Therefore, since we have assumed $u_0 \in H^1$, the estimate (1.13) follows for $N = 3$.

Next let $N = 2$. Instead of (2.2) we have in 2D

$$|u|_\infty \le C H_0^{1/2} H_2^{1/2} \le C H_2^{1/2},$$

which yields

$$\frac{d}{dt} H_1^2 \le C H_1 H_2^{3/2} - 2H_2^2\,.$$

Using the exponents $\alpha = 4, \beta = 4/3$ (satisfying $\alpha^{-1} + \beta^{-1} = 1$) we have by Young's inequality

$$H_1 H_2^{3/2} \le C_\varepsilon H_1^4 + \varepsilon H_2^2\,,$$

and therefore

$$\frac{d}{dt} H_1^2 \le C H_1^4\,.$$

Since $\int_0^\infty H_1^2\,dt < \infty$ we obtain the bound

$$H_1^2(t) \le C H_1^2(s) \quad \text{for all} \quad 0 \le s \le t\,,$$

or, with $\gamma = 1/C > 0$,

$$H_1^2(s) \ge \gamma H_1^2(t) \quad \text{for all} \quad 0 \le s \le t\,.$$

Now suppose that (1.14) does not hold. Then there is a sequence $t_n \to \infty$ so that

$$t_n H_1^2(t_n) \ge \delta > 0 \quad \text{for all} \quad n$$

and we may assume $t_{n+1} \ge 2t_n$. Then we obtain

$$\begin{aligned} \int_{t_n}^{t_{n+1}} H_1^2(s)\,ds &\ge \gamma(t_{n+1} - t_n) H_1^2(t_{n+1}) \\ &\ge \gamma\delta(t_{n+1} - t_n)/t_{n+1} \\ &= \gamma\delta(1 - t_n/t_{n+1}) \\ &\ge \gamma\delta/2 \end{aligned}$$

This contradicts $\int_0^\infty H_1^2\,dt < \infty$, and (1.14) is proved.

Proof of (1.16) *and* (1.17). Note that Q contains nonlinear terms $v \cdot \nabla B$ etc. and ∇p. The components of the nonlinear terms $v \cdot \nabla B$ etc. are sums of terms $u_i D_j u_l$. Clearly,

$$\begin{aligned} |(u_i D_j u_l)\hat{}(k,s)| &\le \int |u(x,s)||D_j u(x,s)|\,dx \\ &\le H_0(s) H_1(s) \end{aligned}$$

Also, by (1.4) and (1.5), Δp equals a sum of terms $D_i(u_j D_l u_m)$, which yields

$$|(\nabla p)\hat{}| \le C|(uDu)\hat{}| \le C H_0 H_1\,.$$

This proves (1.16). The estimate (1.17) follows similarly if one uses that

$$v \cdot \nabla B_i = \sum_j D_j(v_j B_i)\,,$$

i.e., the nonlinearities can be written as sums of derivative terms, $D_j(u_i u_l)$.

Proof of Lemma 1.1. By Parseval's relation,

$$\begin{aligned} \|e^{\Delta t} u_0\|^2 &= \int e^{-2|k|^2 t} |\hat{u}_0(k)|^2\, dk \\ &\le \omega_N |\hat{u}_0|^2_\infty \int_0^\infty r^{N-1} e^{-2r^2 t}\, dr \\ &= \omega_N |\hat{u}_0|^2_\infty\, t^{-N/2} \int_0^\infty \rho^{N-1} e^{-2\rho^2}\, d\rho\,. \end{aligned}$$

Here $\omega_2 = 2\pi, \omega_3 = 4\pi$, and we have used the substitution $r = t^{-1/2}\rho$. This proves (1.18). The estimate (1.19) follows in the same way.

3. Proof of Theorem 1.1

We will use the following Gronwall-type lemma. Results of this type are well-known, of course, but a proof is included for completeness.

Lemma 3.1. *Let $y(t), t \ge 0$, denote a real, nonnegative, continuous function satisfying*

$$y(t) \le C(1+t)^{-\kappa} + C \int_0^t (t-s)^{-\alpha} (1+s)^{-\beta} y(s)\, ds\,. \tag{3.1}$$

Then $y(t)(1+t)^\kappa$ is bounded provided that

$$0 < \kappa \le \alpha < 1 < \alpha + \beta\,. \tag{3.2}$$

Proof. Set

$$E(t) = y(t)(1+t)^\kappa, \quad E_{\max}(t) = \max_{0 \le s \le t} E(s) \tag{3.3}$$

and multiply (3.1) by $(1+t)^\kappa$ to obtain

$$E(t) \le C + C E_{\max}(t)(1+t)^\kappa \int_0^t (t-s)^{-\alpha} (1+s)^{-\beta-\kappa}\, ds\,.$$

A) Assume $\kappa < \alpha$. We will show below that the factor multiplying $E_{\max}(t)$ tends to 0 as $t \to \infty$. Therefore, there exists t_1 with

$$E(t) \le C + \frac{1}{2} E_{\max}(t), \quad t \ge t_1\,.$$

Since $E(t)$ is bounded for $0 \le t \le t_1$ we have

$$E(t) \le C_1 + \frac{1}{2}\, E_{\max}(t), \quad t \ge 0\,,$$

and therefore,

$$E_{\max}(t) \le C_1 + \frac{1}{2} E_{\max}(t), \quad t \ge 0\,.$$

This implies $E(t) \le E_{\max}(t) \le 2C_1$, i.e., $E(t)$ is bounded.

It remains to show that

$$(1+t)^{\kappa} \int_0^t (t-s)^{-\alpha}(1+s)^{-\beta-\kappa}\, ds$$

tends to 0 as $t \to \infty$. Split the integral into $I_1 + I_2$ where

$$\begin{aligned} I_1 &= \int_0^{t/2} (t-s)^{-\alpha}(1+s)^{-\beta-\kappa}\, ds \\ &\le Ct^{-\alpha} \int_0^t (1+s)^{-\beta-\kappa}\, ds \\ &\le Ct^{-\alpha} \cdot \begin{cases} 1 & \text{if } \beta+\kappa > 1 \\ \ln(e+t) & \text{if } \beta+\kappa = 1 \\ (1+t)^{1-\beta-\kappa} & \text{if } \beta+\kappa < 1 \end{cases} \end{aligned}$$

Thus $(1+t)^{\kappa} I_1 \to 0$ as $t \to \infty$ since

$$\kappa - \alpha < 0 \quad \text{and} \quad 1 - \alpha - \beta < 0\,.$$

Also,

$$\begin{aligned} I_2 &= \int_{t/2}^{t} (t-s)^{-\alpha}(1+s)^{-\beta-\kappa}\, ds \\ &\le C(1+t)^{-\beta-\kappa}\, t^{1-\alpha}\,. \end{aligned}$$

Therefore, $(1+t)^{\kappa} I_2 \to 0$ as $t \to \infty$ since $1-\alpha-\beta < 0$.

B) Assume $\kappa = \alpha$. Choose $\delta > 0$ so small that $\delta - \alpha - \beta < -1$. We may replace κ in (3.1) by $\alpha - \delta$ and obtain from part A) of the proof that

$$y(t) \le C(1+t)^{-\alpha+\delta}, \quad C = C_\delta\,.$$

Using this bound in the integral in (3.1), we have

$$\int_0^t (t-s)^{-\alpha}(1+s)^{-\beta} y(s)\, ds \le C \int_0^t (t-s)^{-\alpha}(1+s)^{\delta-\alpha-\beta}\, ds\,.$$

Since $\delta - \alpha - \beta < -1$ we have

$$\int_0^{t/2} (t-s)^{-\alpha}(1+s)^{\delta-\alpha-\beta}\, ds \le Ct^{-\alpha}$$

and

$$\int_{t/2}^{t}(t-s)^{-\alpha}(1+s)^{\delta-\alpha-\beta}\,ds \;\le\; C(1+t)^{\delta-\alpha-\beta}\,t^{1-\alpha}$$
$$\le\; Ct^{-\alpha}\,.$$

Therefore (3.1) yields $y(t)\le C(1+t)^{-\alpha}$, completing the proof of the lemma. □

Proof of Theorem 1.1, $N=3$. We must show that $H_0(t)=\|u(t)\|\le C(1+t)^{-\kappa}$ for $0<\kappa\le\frac{5}{4}$. As noted in the introduction (see (1.23)), we have

$$H_0(t)\le C(1+t)^{-\kappa}+C\int_0^t(t-s)^{-3/4}(1+s)^{-1/2}H_0(s)\,ds\,.$$

A) Let $0<\kappa\le\frac{3}{4}$. We can apply Lemma 3.1 and obtain $H_0(t)\le C(1+t)^{-\kappa}$.

B) Let $\frac{3}{4}<\kappa\le\frac{5}{4}$. By A) we know that $H_0(t)\le C(1+t)^{-3/4}$. Also, using (1.20), (1.19), and (1.17),

$$H_0(t)\le C(1+t)^{-\kappa}+C\int_0^{t/2}(t-s)^{-5/4}H_0^2(s)ds+C\int_{t/2}^{t}(t-s)^{-3/4}(1+s)^{-1/2}H_0(s)ds.$$

The first integral can be bounded as follows,

$$\begin{aligned}I_1\;&\le\;\int_0^{t/2}(t-s)^{-5/4}(1+s)^{-3/2}\,ds\\&\le\;Ct^{-5/4}\int_0^t(1+s)^{-3/2}ds\\&\le\;Ct^{-5/4}\end{aligned}$$

Since $H_0(t)$ is bounded near $t=0$, one obtains

$$H_0(t)\le C(1+t)^{-\kappa}+C\int_{t/2}^{t}(t-s)^{-3/4}(1+s)^{-1/2}H_0(s)\,ds\,.$$

Define $E(t)$ and $E_{\max}(t)$ as in (3.3) with $y(t)=H_0(t)$ and obtain

$$E(t)\le C+CE_{\max}(t)(1+t)^{\kappa}\int_{t/2}^{t}(t-s)^{-3/4}(1+s)^{-\frac{1}{2}-\kappa}\,ds\,.$$

The integral is bounded by $C(1+t)^{-\frac{1}{2}-\kappa}\,t^{1/4}$, and therefore the factor multiplying $E_{\max}(t)$ tends to zero as $t\to\infty$. As in the proof of Lemma 3.1, this yields boundedness of $E(t)$, and the proof of Theorem 1.1 is complete for $N=3$.

Proof of Theorem 1.1, $N=2$. We must show that $H_0(t)=\|u(t)\|\le C(1+t)^{-\kappa}$ for $0<\kappa\le 1$.

A) Assume $0<\kappa<\frac{1}{2}$. By (1.20), (1.18), (1.16), and (1.14) we have,

$$H_0(t)\le C(1+t)^{-\kappa}+C\int_0^t(t-s)^{-1/2}(1+s)^{-1/2}\phi(s)H_0(s)\,ds \tag{3.4}$$

where $\phi(s) \to 0$ as $s \to \infty$. Note that Lemma 3.1 does not apply here since $\alpha = \beta = \frac{1}{2}$ in (3.4), but $\alpha + \beta > 1$ is required in Lemma 3.1. To overcome the difficulty, we will use the fact that $\lim \phi(t) = 0$.

Set

$$E(t) = H_0(t)(1+t)^\kappa, \quad E_{\max}(t) = \max_{0 \le s \le t} E(s) \tag{3.5}$$

and obtain

$$E(t) \le C + CE_{\max}(t)J(t)$$

with

$$J(t) = (1+t)^\kappa \int_0^t (t-s)^{-1/2}(1+s)^{-\frac{1}{2}-\kappa}\phi(s)\,ds\,.$$

Once we have shown that $\lim_{t\to\infty} J(t) = 0$, boundedness of $E(t)$ follows as in the proof of Lemma 3.1. To prove $\lim_{t\to\infty} J(t) = 0$ we consider

$$\begin{aligned} I(t) &= \int_0^t (t-s)^{-1/2}(1+s)^{-\frac{1}{2}-\kappa}\phi(s)\,ds \\ &= \int_0^{T_0} \cdots + \int_{T_0}^{t/2} \cdots + \int_{t/2}^t \cdots \\ &:= I_1(t) + I_2(t) + I_3(t) \end{aligned}$$

For any fixed T_0 and all large t we have $I_1(t) \le C(T_0)(t-T_0)^{-1/2}$, thus

$$(1+t)^\kappa I_1(t) \to 0 \quad \text{as} \quad t \to \infty$$

since $\kappa < \frac{1}{2}$. Also,

$$\begin{aligned} I_2(t) &= \int_{T_0}^{t/2} (t-s)^{-1/2}(1+s)^{-\frac{1}{2}-\kappa}\phi(s)\,ds \\ &\le C\Big(\max_{s\ge T_0}\phi(s)\Big)\, t^{-1/2}\left(1+\frac{t}{2}\right)^{\frac{1}{2}-\kappa}, \end{aligned}$$

thus

$$\sup_{t\ge T_0}\,(1+t)^\kappa\, I_2(t) \le C \max_{s\ge T_0}\phi(s) \le \varepsilon \tag{3.6}$$

if T_0 is sufficiently large. Furthermore,

$$\begin{aligned} I_3(t) &= \int_{t/2}^t (t-s)^{-1/2}(1+s)^{-\frac{1}{2}-\kappa}\phi(s)\,ds \\ &\le C\Big(\max_{s\ge T_0}\phi(s)\Big)\left(1+\frac{t}{2}\right)^{-\frac{1}{2}-\kappa} t^{1/2}\,, \end{aligned}$$

thus an estimate like (3.6) holds with I_2 replaced by I_3. To summarize, we have shown $J(t) \to 0$ as $t \to \infty$, which implies $H_0(t) \le C(1+t)^{-\kappa}$ for $0 < \kappa < \frac{1}{2}$.

B) Let $\frac{1}{2} \le \kappa < 1$. Fix $0 < \gamma < \frac{1}{2}$ with $\gamma + \kappa < 1$ and note that, by part A) of the proof, $H_0(t) \le C(1+t)^{-\gamma}$. By (1.20) and (1.17) we have

$$\begin{aligned} H_0(t) \;&\le\; C(1+t)^{-\kappa} + C\int_0^{t/2} (t-s)^{-1} H_0^2(s)\,ds \\ &\quad + C\int_{t/2}^t (t-s)^{-1/2}(1+s)^{-1/2}\phi(s) H_0(s)\,ds\,. \end{aligned} \tag{3.7}$$

Use the estimate $H_0(s) \le C(1+s)^{-\gamma}$ for one of the H_0-factors in the first integral. If $E(t)$ and $E_{\max}(t)$ are defined as in (3.5), then one obtains

$$\begin{aligned} E(t) \;&\le\; C + CE_{\max}(t)(1+t)^{\kappa}\int_0^{t/2} (t-s)^{-1}(1+s)^{-\gamma-\kappa}\,ds \\ &\quad + CE_{\max}(t)(1+t)^{\kappa}\int_{t/2}^t (t-s)^{-1/2}(1+s)^{-\frac{1}{2}-\kappa}\phi(s)\,ds\,. \end{aligned}$$

We claim that the two factors multiplying $E_{\max}(t)$ tend to zero as $t \to \infty$. In fact,

$$\begin{aligned} I_4(t) \;&:=\; \int_0^{t/2} (t-s)^{-1}(1+s)^{-\gamma-\kappa}\,ds \\ &\le\; Ct^{-1}\Big(1+\frac{t}{2}\Big)^{1-\gamma-\kappa}, \end{aligned}$$

thus $(1+t)^{\kappa} I_4(t) \to 0$ since $\gamma > 0$. Finally,

$$\begin{aligned} I_5(t) \;&:=\; \int_{t/2}^t (t-s)^{-1/2}(1+s)^{-\frac{1}{2}-\kappa}\phi(s)\,ds \\ &\le\; \max_{s\ge t/2}\phi(s)\int_{t/2}^t (t-s)^{-1/2}(1+s)^{-\frac{1}{2}-\kappa}\,ds \\ &\le\; C\max_{s\ge t/2}\phi(s)\Big(1+\frac{t}{2}\Big)^{-\frac{1}{2}-\kappa} t^{1/2}\,, \end{aligned}$$

thus

$$(1+t)^{\kappa} I_5(t) \le C\max_{s\ge t/2}\phi(s) \to 0 \quad \text{as} \quad t\to\infty\,.$$

If t is sufficiently large, then $E(t) \le C + \frac{1}{2}E_{\max}(t)$, and the estimate $H_0(t) \le C(1+t)^{-\kappa}$ follows as above.

C) Let $\kappa = 1$. Fix $\frac{1}{2} < \gamma < 1$ and note that $H_0(t) \le C(1+t)^{-\gamma}$ by part B) of the proof. Therefore, by (3.7),

$$\begin{aligned} H_0(t) \;&\le\; C(1+t)^{-1} + C\int_0^{t/2} (t-s)^{-1}(1+s)^{-2\gamma}\,ds \\ &\quad + C\int_{t/2}^t (t-s)^{-1/2}(1+s)^{-1/2}\phi(s) H_0(s)\,ds\,. \end{aligned} \tag{3.8}$$

The first integral is bounded by $Ct^{-1}\int_0^t (1+s)^{-2\gamma} ds \le C(1+t)^{-1}$, i.e., it decays like the first term on the right-hand side of (3.8). The remaining arguments are the same as in part B) of the proof since the relation $(1+t)^\kappa I_5(t) \to 0$ as $t \to \infty$ also holds for $\kappa = 1$. This completes the proof of Theorem 1.1.

4. Discussion and extensions

1. *Simplification of the proof for 2D Navier–Stokes.* The proof of the crucial decay estimate $tH_1^2(t) \to 0$ as $t \to \infty$ (see Theorem 1.3) is somewhat simpler for the 2D Navier–Stokes equations than for the MHD system: If $\xi = D_1 u_2 - D_2 u_1$ denotes the vorticity, then $\|\xi(t)\| = H_1(t)$ and $(d/dt)H_1^2(t) = -2\|D^2 u\|^2 \le 0$. Therefore,

$$tH_1^2(t) \to 0 \quad \text{as} \quad t \to \infty$$

holds since otherwise there is a sequence $t_j \to \infty$ with

$$t_j H_1^2(t_j) \ge \delta > 0, \quad t_{j+1} \ge 2t_j\,,$$

thus

$$\int_{t_j}^{t_{j+1}} H_1^2(t)\,dt \ge (t_{j+1} - t_j)\delta/t_{j+1} \ge \delta/2\,, \tag{4.1}$$

which contradicts $\int_0^\infty H_1^2 dt < \infty$.

2. *Estimates of derivatives.* The decay estimates for $H_1^2(t)$ stated in Theorem 1.3 are usually not optimal. We presented the theorem only as a starting point for our proof of decay of $H_0(t) = \|u(\cdot,t)\|$. Once Theorems 1.1 and 1.3 are shown, one can improve the decay estimate for $H_1(t)$ and can inductively derive decay estimates for all derivatives. Such a process is carried out in [7] for the Navier–Stokes equations. See also [6, 9] for other approaches to decay estimates for derivatives of the solution of the Navier–Stokes equations. The basic result is as follows: Let $D^\alpha = D_1^{\alpha_1} \dots D_N^{\alpha_N}$ denote any space derivative, thus

$$D^\alpha u(t) = D^\alpha e^{\Delta t} u_0 + \int_0^t D^\alpha e^{\Delta(t-s)} Q(s)\,ds\,. \tag{4.2}$$

As in our main theorem, we assume $\|e^{\Delta t} u_0\| \le C(1+t)^{-\kappa}$ for the solution of the heat equation. Then derivatives decay faster,

$$\|D^\alpha e^{\Delta t} u_0\| \le C_1 t^{-\kappa - \frac{|\alpha|}{2}}$$

as can be seen from the Fourier representations of $e^{\Delta t}u_0$ and $D^\alpha e^{\Delta t} u_0$. (Decay of solutions of the heat equation can be characterized in terms of the behavior of the Fourier transform of the initial function at zero wave vector, $k = 0$. See, for example, [8].) Then, using L_q–L_r estimates for the solution of the heat equation together with Gagliardo–Nirenberg inequalities one obtains inductively from (4.2) that

$$\|D^\alpha u(t)\| \le C_2 t^{-\gamma - \frac{|\alpha|}{2}}\,,$$

where $\gamma = \min\{\kappa, \frac{N}{4} + \frac{1}{2}\}$, as in Theorem 1.1.

We note that these decay estimates for the derivatives allow us to estimate the difference between $u(\cdot, t)$ and the solution of the vector heat equation with initial data u_0. By Duhamel's principle,

$$u(t) - e^{\Delta t} u_0 = \int_0^t e^{\Delta(t-s)} Q(s)\, ds \tag{4.3}$$

where $Q(s)$ can be estimated in terms of $H_0(s)$ and $H_1(s)$. In a generic situation, and assuming $u_0 \in L_1 \cap H^1$, the L_2 norm of the integral in (4.3) decays at the same rate as $\|e^{\Delta t} u_0\|$, namely like $t^{-\frac{N}{4}-\frac{1}{2}}$.

3. *Other parabolic systems.* Theorem 1.1 and its proof can be extended to other parabolic systems. To give sufficient conditions, let $f_j : \mathbb{R}^n \to \mathbb{R}^n, 1 \le j \le N$, denote smooth vector fields with symmetric Jacobian $A_j(u) = D_u f_j(u)$ and assume $|f_j(u)| \le C|u|^2$. Consider the parabolic system

$$u_t + \sum_j A_j(u) D_j u + D_j(A_j(u)u) = Pu \tag{4.4}$$

where the first order terms have antisymmetric form and the linear constant coefficient operator P satisfies:

$$P = \sum_j R_j D_j + \sum_{i,j} V_{ij} D_i D_j,$$

$$\hat{P}(k) + \hat{P}^*(k) \le -c|k|^2 I, \quad c > 0.$$

Because of

$$\int u^T D_j(A_j(u)u)\, dx = -\int (D_j u^T) A_j(u) u\, dx$$

and $A_j = A_j^T$, the nonlinear terms in (4.4) do not change the energy, i.e., the energy equation (1.10) holds. The estimates (2.1) as well as (1.16) and (1.17) are also easily shown and the analogue of Lemma 1.1 holds for P. Here Q denotes the nonlinear terms in (4.4). Note that $A_j(u) D_j u = D_j f_j(u)$, which is used to show (1.17). Once the estimates (1.10), (1.16), (1.17), and (2.1) are derived, the result of Theorem 1.1 follows by the proof given in the paper.

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