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On some Approximation Schemes for Steady Compressible Viscous Flow

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Abstract. This paper continues our development of approximation schemes for steady compressible viscous flow based on an iteration between a Stokes like problem for the velocity and a transport equation for the density, with the aim of improving their suitability for computations. Such schemes seem attractive for computations because they offer a reduction to standard problems for which there is already highly refined software, and because of the guidance that can be drawn from an existence theory based on them. Our objective here is to modify a recent scheme of Heywood and Padula [12], to improve its convergence properties. This scheme improved upon an earlier scheme of Padula [21], [23] through the use of a special "effective pressure" in linking the Stokes and transport problems. However, its convergence is limited for several reasons. Firstly, the steady transport equation itself is only solvable for general velocity fields if they satisfy certain smallness conditions. These conditions are met here by using a rescaled variant of the steady transport equation based on a pseudo time step for the equation of continuity. Another matter limiting the convergence of the scheme in [12] is that the Stokes linearization, which is a linearization about zero, has an inevitably small range of convergence. We replace it here with an Oseen or Newton linearization, either of which has a wider range of convergence, and converges more rapidly. The simplicity of the scheme offered in [12] was conducive to a relatively simple and clearly organized proof of its convergence. The proofs of convergence for the more complicated schemes proposed here are structured along the same lines. They strengthen the theorems of existence and uniqueness in [12] by weakening the smallness conditions that are needed. The expected improvement in the computational performance of the modified schemes has been confirmed by Bause [2], in an ongoing investigation.

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1. Introduction

In this paper we suggest and analyze several modifications to an iteration scheme that was introduced by Heywood and Padula [12] for proving the existence of steady compressible viscous flow, with the aim now of improving its computational efficiency. As a further result, the existence and uniqueness theorems of [12] are reproven under effectively weaker hypotheses. It is expected that many of the ideas given here will prove useful in treating related and more general problems, including nonstationary problems. They can, for example, be combined with the modifications we made in dealing with the superposition of an additional large potential force, in [3]. For simplicity, we have restricted our considerations to isothermal flow. The generalization to other types of barotropic flow is straight forward, as it should also be to the case of an ideal gas with the addition of an energy equation.

The Poisson–Stokes equations (often referred to as the “compressible Navier–Stokes equations”) for an isothermal flow in an open bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, are

$$\begin{aligned} \nabla \cdot (\rho v) &= 0, \\ \rho v \cdot \nabla v - \mu \triangle v &= -\nabla p + (\lambda + \mu) \nabla \nabla \cdot v + \rho f, \\ p &= k\rho, \\ v|_{\partial\Omega} = 0, &\quad \textstyle\int_\Omega \rho \, dx = \overline{\rho}\, |\Omega|. \end{aligned} \tag{1}$$

Here, v denotes the velocity of the fluid, ρ its density, and p its pressure, which are unknowns to be solved for, while f is a prescribed external force density. The mean density $\overline{\rho}$ must also be prescribed, and must be positive. The viscosity coefficients μ and λ are constants satisfying the conditions $\mu > 0$ and $3\lambda + 2\mu \geq 0$. (The latter condition is sometimes replaced by $d\lambda + 2\mu \geq 0$ in mathematical papers, and could be here too.) For isothermal flow, k is a positive constant.

A substantial beginning has been made on the mathematical analysis of this problem and its generalizations, particularly during the last two decades through the works of Padula [21], [22], [23], [24], [25], [26], [27], Beirão da Veiga [4], Valli [30], Farwig [6], Novotny [17], [18], Novotny and Padula [19], Novotny and Pileckas [20], and Heywood and Padula [12], so that the following existence theorem is well known:

Theorem 1.1. *Let Ω be a bounded subdomain of $\mathbb{R}^d$, $d = 2, 3$, with boundary of class $C^{2,1}$. Suppose $f \in W^{1,2}(\Omega)$. Then, if $\|f\|_{1,2}$ is sufficiently small depending on Ω, k, μ, λ and $\overline{\rho}$, there exists a solution v, ρ of problem* (1) *with $v \in W^{3,2}(\Omega) \cap W_0^{1,2}(\Omega)$, $\rho \in W^{2,2}(\Omega)$ and $\inf_\Omega \rho > 0$. It is found within and unique within a certain ball $\|v\|_{3,2} + \|\rho - \overline{\rho}\|_{2,2} \leq R$.*

The paper [12] is based on an iterative scheme that seems to us particularly suggestive of a numerical procedure. It splits the original complicated system into two standard problems for which there are already highly refined methods of discretization; see e.g. [5], [9], [28], [29] regarding the Stokes and Oseen problems, and [14], [15], [16], regarding the transport equation. To describe it, let the problem (1) be first normalized by dividing ρ, p, μ and λ by $\overline{\rho}$ and relabeling, so that $\overline{\rho} = 1$.

Then the perturbation in the density is $\sigma = \rho - 1$, and we may rewrite (1) as

$$\begin{gathered}\nabla \cdot v = -\nabla \cdot (\sigma v), \\ (1+\sigma) v \cdot \nabla v - \mu \triangle v = -\nabla p + (\lambda + \mu) \nabla \nabla \cdot v + (1+\sigma) f, \\ p = k(1+\sigma), \\ v|_{\partial\Omega} = 0, \quad \int_\Omega \sigma dx = 0.\end{gathered} \tag{2}$$

The iteration scheme introduced in [12] is the following:

Scheme HP: *Set* $v_0 = 0$, $\sigma_0 = 0$. *For given* v_n *and* σ_n, $n \geq 0$, *compute* v_{n+1} *and* σ_{n+1} *by carrying out the following steps:*

(i) *Set*

$$g_n = -\nabla \cdot (\sigma_n v_n), \qquad F_n = (1+\sigma_n)\left(f - v_n \cdot \nabla v_n\right). \tag{3}$$

(ii) *Find* v_{n+1}, π_{n+1} *as the solution of the Stokes problem*

$$\begin{gathered}\nabla \cdot v_{n+1} = g_n, \\ -\mu \triangle v_{n+1} = -\nabla \pi_{n+1} + F_n, \\ v_{n+1}|_{\partial\Omega} = 0, \quad \int_\Omega \pi_{n+1} dx = 0.\end{gathered} \tag{4}$$

(iii) *Set*

$$H_{n+1} = \pi_{n+1} - \mu \nabla \cdot v_{n+1}. \tag{5}$$

(iv) *Find* σ_{n+1} *as the solution of the transport equation*

$$k\sigma_{n+1} + (\lambda + 2\mu) \nabla \cdot (\sigma_{n+1} v_{n+1}) = H_{n+1}. \tag{6}$$

From a computational point of view there are several weaknesses in this scheme that must be addressed. The most obvious one is that the Stokes linearization used in problem (4), wherein the nonlinear term is expressed entirely in terms of the preceding iterate and included in the inhomogeneous term, is too crude. It has both a small range and slow rate of convergence. There are similar objections to its use in computing incompressible flow, where experience has shown that it performs poorly. The other point that must be addressed is that the transport equation (6) is not generally solvable if v_{n+1} is large.

One of the modifications we are proposing to the Scheme HP is that the transport equation (6) should be replaced by a more complicated one

$$(\lambda + 2\mu + \epsilon k)\, \sigma_{n+1} + (\lambda + 2\mu) \nabla \cdot (\sigma_{n+1} \epsilon v_{n+1}) = \epsilon H_{n+1} + (\lambda + 2\mu)\, \sigma_n, \tag{7}$$

depending on a pseudo time step ϵ that will be specified later. Clearly the scaled vector field ϵv_{n+1} can be made small by taking ϵ small. We will discuss the derivation of (7) later in this section.

The other modification being proposed is a refinement of the linearization in the momentum equation of problem (4). Among the various ways this might be done, we have chosen three, each quite different from the others, spanning a breadth of possibilities. Either:

(A) *Set* $F_n = (1+\sigma_n)f$ *in* (3) *and replace the momentum equation in* (4) *by*

$$(1+\sigma_n)v_n \cdot \nabla v_{n+1} + \frac{1}{2}\nabla \cdot ((1+\sigma_n)v_n)v_{n+1} - \mu\triangle v_{n+1} = -\nabla\pi_{n+1} + F_n. \quad (8)$$

(B) *Set* $F_n = (1+\sigma_n)f$ *in* (3) *and replace the momentum equation in* (4) *by*

$$P((1+\sigma_n)v_n) \cdot \nabla v_{n+1} - \mu\triangle v_{n+1} = -\nabla\pi_{n+1} + F_n \quad (9)$$

where P *is an* L^2*-projection onto solenoidal functions, defined below.*

(C) *Set* $F_n = (1+\sigma_n)(f + v_n \cdot \nabla v_n)$ *in* (3) *and replace the momentum equation in* (4) *by*

$$(1+\sigma_n)v_n \cdot \nabla v_{n+1} + (1+\sigma_n)v_{n+1} \cdot \nabla v_n - \mu\triangle v_{n+1} = -\nabla\pi_{n+1} + F_n. \quad (10)$$

We shall frequently refer to *problems* (8), (9) and (10), meaning the problems formed by taking these momentum equations together with the supplementary conditions of problem (4), i.e., $\nabla \cdot v_{n+1} = g_n,\ v_{n+1}|_{\partial\Omega} = 0$, and $\int_\Omega \pi_{n+1}dx = 0$. We will refer to the corresponding modifications of the Scheme HP, in which only problem (4) is modified, as Schemes A, B, and C. If, in addition, the transport equation (6) is replaced by (7), we will refer to the resulting schemes as Schemes TA, TB and TC.

The first of the alternative linearizations, (8), is Oseen like, and we refer to it as an Oseen linearization. Notice that the second term on the left can be expected to vanish in the limit, in view of the continuity equation for steady solutions. This second term could be done without, but serves to preserve the energy estimate during iteration, thereby avoiding a smallness condition in the analysis of the scheme, and probably improving its numerical performance. The second alternative is based on another Oseen like linearization; this time the energy preservation during iteration is achieved by use of a certain projection operator. This projection is expected to act as the identity operator on the limiting solution. We have included this second Oseen like scheme because its analogue for the incompressible equations is reported to be useful by Prohl [28], and there are interesting features to its analysis. The third alternative is a Newton linearization. It is likely to perform better than the others because of its quadratic rate of convergence, but its range of convergence is more problematical, the proof of its convergence requiring an additional smallness condition. We have not included compensating terms to preserve energy during its iteration since its rapid rate of convergence makes that less important. Some suggestion (seemingly confirmed in [2]) of the relative merits of the Stokes, Oseen and Newton linearizations may be indicated by solving for the lesser root of the quadratic equation $v^2 - v = f$ using the iterations $-v_{n+1} = f - v_n^2$, or $v_n v_{n+1} - v_{n+1} = f$, or $2v_n v_{n+1} - v_{n+1} = f + v_n^2$, starting with $v_0 = 0$. Notably, the Stokes like iteration only converges for $-1/4 \le f \le 3/4$, while the others converge for all $f \ge -1/4$.

Let us return to the discussion of the transport equation (7). Its derivation is precisely like that given for (6) in [12], provided one substitutes the use of the pseudo time discretized continuity equation

$$\frac{\sigma_{n+1} - \sigma_n}{\epsilon} + \nabla \cdot (\sigma_{n+1} v_{n+1}) = -\nabla \cdot v_{n+1} \,, \tag{11}$$

for that of the steady continuity equation $\nabla \cdot (\sigma_{n+1} v_{n+1}) = -\nabla \cdot v_{n+1}$. Indeed, comparing (4) or one of its proposed alternatives with (2), it follows that if the resulting scheme converges to a solution, $v_n \to v, \sigma_n \to \sigma, \pi_n \to \pi$, then $k\sigma - (\lambda + \mu)\nabla \cdot v = \pi$. This suggests that one might try determining σ_{n+1} from

$$k\sigma_{n+1} - (\lambda + \mu)\nabla \cdot v_{n+1} = \pi_{n+1} \,, \tag{12}$$

without any need of a transport equation. The resulting simplified scheme, however, does not converge, losing regularity for reasons that stem from its disregard of the hyperbolic character of the equation of continuity. Substituting for $\nabla \cdot v_{n+1}$ in (12) its equivalent from (11) results in a transport equation

$$(\lambda + \mu)\frac{\sigma_{n+1} - \sigma_n}{\epsilon} + k\sigma_{n+1} + (\lambda + \mu)\nabla \cdot (\sigma_{n+1} v_{n+1}) = \pi_{n+1} \tag{13}$$

that can be iterated successfully, provided that $(\lambda + \mu)^{-1}$ is small. This is an unnatural condition on the viscosity coefficients which is needed because of a linear dependence of the iterates on their predecessors, coming through the term g_n in problem (4); see [12], pp. 175, 180. Modifying (13), by adding μ times (11) to it, results in the transport equation (7) above that can be iterated without restricting $(\lambda + \mu)^{-1}$. The key reason for this is that the inhomogeneous term H_{n+1} has Laplacian $\triangle H_{n+1} = \nabla \cdot F_n$ independent of g_n; see again [12].

For a treatment of the steady transport equation

$$k\sigma + \nabla \cdot (\sigma v) = H \tag{14}$$

suitable for our purposes see Heywood and Padula [13], which was written as an accompaniment to [12]. An example given at the end of that article shows the necessity of a restriction on v in order to guarantee the solvability of the equation. Both the existence theory given in [13] and the computations of Bause [2] are based on Galerkin approximation.

Our main results are formulated as new proofs of Theorem 1.1 based on the convergence of the new schemes. Although we state these new results as before, simply requiring that the data be "sufficiently small", the conditions of smallness that enter the proofs appear to be much milder than in [12].

If the data are small, it is natural to expect that the compressible flow problem (1) should have only one solution. Yet that remains an open question, as does the related matter of obtaining a general a priori bound on all possible solutions. The uniqueness asserted in Theorem 1.1 is local in that it applies only to solutions that might lie within a certain ball $\|v\|_{3,2} + \|\sigma\|_{2,2} \leq R$. Each of the iterative schemes we consider has an associated uniqueness theorem, based on the same splitting of problem (2) into part problems and the same inequalities and values of B and R as used in the proof of convergence. We give the complete proof of this only for Scheme A.

If the norm $\|f\|_{1,2}$ of a given force is bounded by the numbers B corresponding to two of our schemes, then the two corresponding solutions must coincide because they are both bounded by the larger of the two corresponding values of R, and hence both subject to a common uniqueness theorem. Thus we can conclude that all of the solutions we construct using our various iteration schemes are the same, provided only that the hypotheses are met under which they are proven to converge. That conclusion would not be possible using only the uniqueness theorem associated with one scheme, say that of [12].

We think an analogue of Theorem 1.2 can be proven in the topology $W^{2,p}(\Omega) \times W^{1,p}(\Omega)$, for $p > d$. It is interesting to reflect, though, on the apparent difficulty of identifying two solutions constructed by different schemes, by using their associated uniqueness theorems, when these theorems are expressed in different topologies.

It does not seem worthwhile to make a rigorous comparison of the numbers B and R that appear in the existence and uniqueness theorems associated with each of our schemes, since sharp constants are not known for many of the inequalities that are used in the proofs. However, it seems clear from the proofs, and borne out in numerical experiments, that those for the new schemes all improve upon the corresponding ones for the Scheme HP.

The modifications to the momentum equation and the transport equation both add to the complexity of the convergence proofs. For simplicity and to minimize redundancy we have chosen to establish first the convergence of the schemes that result from the modifications (A), (B), and (C) to the momentum equation, while retaining the simpler transport equation (6), giving full details only for (A). The new transport equation (7) is dealt with in detail only once, in our final result for the Scheme TA. Schemes TB and TC could be treated similarly. Our main results, proven in Lemmas 4.1, 4.2, Remark 4.3 and Lemmas 5.1, 6.1 and 6.3 are summarized in the following theorem.

Theorem 1.2. *Theorem 1.1 can be reproven using any of the Schemes A, B, C or TA. For each scheme there are numbers B and R such that if $\|f\|_{1,2} \le B$, then the iterates satisfy $\|v_n\|_{3,2} + \|\sigma_n\|_{2,2} \le R$ and converge in $W^{2,2}(\Omega) \times W^{1,2}(\Omega)$, to a solution $v \in W^{3,2}(\Omega) \cap W_0^{1,2}(\Omega)$, $\sigma \in W^{2,2}(\Omega)$, with normalized density $\rho = 1 + \sigma$ satisfying $\inf_\Omega \rho > 0$. This solution is unique among all solutions satisfying $\|v\|_{3,2} + \|\sigma\|_{2,2} \le R$. For the scheme TA, the existence of a suitable pseudo time step size ϵ is asserted along with the numbers B and R.*

Concerning Scheme TA, it will be seen in Lemma 6.1 that there exists a choice of B and R independent of ϵ, for which ϵ may be set at a seemingly optimal value depending on R. This is the largest value of ϵ for which we can prove convergence. For smaller values of ϵ, we need to take also smaller values of B and R. This seems counter intuitive, and suggests that our analysis may not be optimal.

The assumption about the boundary $\partial\Omega$, that $\partial\Omega \in C^{2,1}$, is stronger than needed. We require that the Stokes problem (29) and the Neumann problem for the Laplacian (39) have solutions $(v_{n+1}, \pi_{n+1}) \in W^{3,2}(\Omega) \times W^{2,2}(\Omega)$ and $\psi \in W^{3,2}(\Omega)$, respectively, for right sides in $W^{1,2}(\Omega)$. Also, the theory of the transport equation given in [13] utilized the solvability in $W^{2,2}(\Omega)$ of the homogeneous Dirichlet problem for the Laplacian, when the right side is in $L^2(\Omega)$. These conditions are certainly all satisfied if $\partial\Omega \in C^{2,1}$, but are also satisfied by certain other classes of domains; see, e.g., [10].

The plan for the paper is as follows. We begin by considering Schemes A, B and C. In §2 we provide and recall some regularity results for Oseen like problems, and for the transport equation, in the course of tracing through each step of the iteration process. In §3 we complete the proof that the iterates of Schemes A, B and C are well defined by showing their boundedness. Their convergence to a solution is proven in §4. In §5 the uniqueness of the solutions just constructed is proven within their balls of existence. Finally, in §6, these matters are carried over to Scheme TA, involving the modified transport equation.

Our notation is standard. Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded domain. The Lebesgue and Sobolev spaces $L^p(\Omega)$, $W^{m,p}(\Omega)$, $W_0^{m,p}(\Omega)$, for integers m, are defined as usual. Further, $W^{-m,p'}(\Omega)$ is the space that is dual to $W_0^{m,p}(\Omega)$, $1/p + 1/p' = 1$. These spaces are endowed with the standard norms. L^2-norms are denoted simply by $\|\cdot\|$, while all other norms are distinguished by subscripts. For example, $\|\cdot\|_p$ denotes the norm for $L^p(\Omega)$, and $\|\cdot\|_{m,p}$ the norm for $W^{m,p}(\Omega)$. We employ the notation $\langle\cdot,\cdot\rangle$ for the inner product in $L^2(\Omega)$. $L_0^2(\Omega)$ is the subspace of $L^2(\Omega)$ consisting of functions with vanishing spatial averages. We do not distinguish through notation between scalar and vector valued functions, or function spaces, or norms. We will also need the spaces

$$J_0(\Omega) = \{v \in L^2(\Omega) \mid \nabla \cdot v = 0 \text{ and } v_{|\partial\Omega} \cdot n = 0, \text{ weakly}\},$$

$$J_1(\Omega) = \{v \in W_0^{1,2}(\Omega) \mid \nabla \cdot v = 0\},$$

which are completions of $\mathcal{D}(\Omega) = \{v \in C_0^\infty(\Omega) \mid \nabla \cdot v = 0\}$ with respect to the norms in $L^2(\Omega)$ and $W^{1,2}(\Omega)$, respectively. Finally, the L^2-projection onto $J_0(\Omega)$ will be denoted by P.

2. Regularity lemmas

To prove Theorem 1.2, we will show that the iteration $v_n, \sigma_n \to v_{n+1}, \sigma_{n+1}$ remains bounded in $W^{3,2}(\Omega) \times W^{2,2}(\Omega)$, and is a contraction in $W^{2,2}(\Omega) \times W^{1,2}(\Omega)$. This will be based on some estimates for subsidiary problems that are provided below in Lemmas 2.1 to 2.11. It is tacitly assumed in these lemmas that the boundary has the regularity assumed in Theorem 1.2. The numbered constants $c_0, \dots, c_9$ and the generic constant c which are introduced in these lemmas depend at most on Ω, k, λ and μ in the normalized problem (2), *and on the iteration scheme under consideration.* We emphasize this last dependency because a casual reading of

Lemma 3.1, particularly, may suggest that the constants B and R are the same for Schemes A and B, which they are not. Rather we have organized the proofs so that parts of them can be done in common, with a minimum of redundancy. The reason for numbering some of the constants is for clarity in the proof of Lemma 3.1, to dispel any question of circular reasoning. Only the estimates with numbered constants are needed in §3. In §4 and §5, all the constants are treated as generic.

Lemmas 2.1 to 2.11 will be stated in the course of tracing through the regularity obtained at each intermediate step of our iterative procedure, to be sure that the next step is well defined. To that end, we will take the conditions

$$v_n \in W^{3,2}(\Omega) \cap W_0^{1,2}(\Omega), \ \ \sigma_n \in W^{2,2}(\Omega), \ \ \nabla \cdot (\sigma_n v_n) \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega) \tag{15}$$

as induction hypotheses, and show that they are preserved by our iterative procedure, provided that the norms $\|v_n\|_{3,2}$ remain bounded by a constant c_5 introduced in Lemma 2.9.

In the next section we will add several estimates to the forgoing induction hypotheses and prove the boundedness of the iterates, including the bound $\|v_n\|_{3,2} \leq c_5$ that is assumed in this section.

Let us begin now by noting that the conditions (15) are satisfied by $v_0 = 0$ and $\sigma_0 = 0$. Taking the conditions (15) as induction hypotheses, they are easily seen to guarantee that the functions g_n and F_n defined by (3) satisfy

$$g_n \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega), \quad F_n \in W^{1,2}(\Omega).$$

Here, in verifying that $F_n \in W^{1,2}(\Omega)$, we use the inequality

$$\|\sigma f\|_{1,2} \leq c_0 \|\sigma\|_{2,2} \, \|f\|_{1,2}, \qquad \text{for } \sigma \in W^{2,2}(\Omega), \ f \in W^{1,2}(\Omega), \tag{16}$$

which holds in virtue of the Sobolev embedding inequalities with a constant c_0 that depends only on Ω.

In order to simplify the notation we introduce the trilinearform

$$b(\phi, \psi, \chi) = \tfrac{1}{2}\langle \phi \cdot \nabla \psi, \chi \rangle - \tfrac{1}{2}\langle \phi \cdot \nabla \chi, \psi \rangle, \quad \phi, \psi, \chi \in W^{1,2}(\Omega),$$

which is skew-symmetric with respect to ψ and χ, i.e. $b(\phi, \psi, \chi) = -b(\phi, \chi, \psi)$.

Lemma 2.1. *Let $v_n \in W^{2,2}(\Omega)$, $\sigma_n \in W^{2,2}(\Omega)$, $g_n \in L_0^2(\Omega)$ and $F_n \in W^{-1,2}(\Omega)$ be given. Then, there exists a unique solution $v_{n+1} \in W_0^{1,2}(\Omega)$, $\pi_{n+1} \in L_0^2(\Omega)$ of*

$$\langle \nabla \cdot v_{n+1}, \chi \rangle = \langle g_n, \chi \rangle \quad \forall \, \chi \in L^2(\Omega),$$

$$b((1+\sigma_n)v_n, v_{n+1}, \phi) + \mu \langle \nabla v_{n+1}, \nabla \phi \rangle = \langle \pi_{n+1}, \nabla \cdot \phi \rangle + \langle F_n, \phi \rangle \quad \forall \, \phi \in W_0^{1,2}(\Omega). \tag{17}$$

Moreover, v_{n+1} and π_{n+1} satisfy the estimates

$$\|v_{n+1}\|_{1,2} \leq c(\|F_n\|_{-1,2} + A_n \|g_n\|), \tag{18}$$

$$\|\pi_{n+1}\| \leq c(A_n \|v_{n+1}\|_{1,2} + \|F_n\|_{-1,2}), \tag{19}$$

where $A_n = (1 + \|\sigma_n\|_\infty)\|v_n\|_{1,2} + 1$.

Proof. We look for a solution v_{n+1} of the form

$$v_{n+1} = w_{n+1} + V_{n+1},$$

where $w_{n+1} \in J_1(\Omega)$ and $V_{n+1} \in W_0^{1,2}(\Omega)$ solves (see [7, p. 135])

$$\nabla \cdot V_{n+1} = g_n, \qquad \|\nabla V_{n+1}\| \le c\|g_n\|. \tag{20}$$

We then conclude that w_{n+1} has to satisfy

$$\mathcal{A}(w_{n+1}, \phi) = G(\phi), \tag{21}$$

for all $\phi \in J_1(\Omega)$, where

$$\begin{aligned} \mathcal{A}(w_{n+1}, \phi) &= b((1+\sigma_n)v_n, w_{n+1}, \phi) + \mu\langle \nabla w_{n+1}, \nabla\phi\rangle, \\ G(\phi) &= \langle F_n, \phi\rangle - b((1+\sigma_n)v_n, V_{n+1}, \phi) - \mu\langle\nabla V_{n+1}, \nabla\phi\rangle. \end{aligned} \tag{22}$$

Since $b(\cdot,\cdot,\cdot)$ is skew-symmetric with respect to the last two arguments, we have

$$\mathcal{A}(\phi,\phi) = \mu\|\nabla\phi\|^2$$

for all $\phi \in W_0^{1,2}(\Omega)$, where $\mu > 0$ by assumption (see §1). By the generalized Hölder inequality and Sobolev embedding inequalities we deduce that $\mathcal{A}(\cdot,\cdot)$ is bounded on $W_0^{1,2}(\Omega)$. Further, these inequalities imply that

$$|G(\phi)| \le c\big(\|F_n\|_{-1,2} + (1+\|\sigma_n\|_\infty)\|\nabla v_n\|\,\|\nabla V_{n+1}\| + \|\nabla V_{n+1}\|\big)\|\nabla\phi\|$$

for all $\phi \in W_0^{1,2}(\Omega)$. Hence, the existence of $w_{n+1} \in J_1(\Omega)$, satisfying (21), (22), follows from the theorem of Lax–Milgram by usual arguments. By (20) we obtain

$$\|w_{n+1}\|_{1,2} \le c\big(\|F_n\|_{-1,2} + \big((1+\|\sigma_n\|_\infty)\|v_n\|_{1,2} + 1\big)\|g_n\|\big). \tag{23}$$

Together (23) and (20) imply that $v_{n+1} \in W_0^{1,2}(\Omega)$ and prove the assertion (18).

We now consider the functional

$$\mathcal{F}(\phi) = \mathcal{A}(w_{n+1}, \phi) - G(\phi), \tag{24}$$

which is linear and bounded in $\phi \in W_0^{1,2}(\Omega)$. By (21), $\mathcal{F}(\phi)$ vanishes when $\phi \in J_1(\Omega)$. Therefore, there exists (see [7, p. 170]) some $\pi_{n+1} \in L_0^2(\Omega)$ with

$$\mathcal{F}(\phi) = \langle \pi_{n+1}, \nabla\cdot\phi\rangle \quad \forall\, \phi \in W_0^{1,2}(\Omega). \tag{25}$$

The proof of existence of functions $v_{n+1} \in W_0^{1,2}(\Omega)$ and $\pi_{n+1} \in L_0^2(\Omega)$ satisfying (17) and (18) is thus established. To prove (19), we consider the problem

$$\nabla\cdot\Psi = \pi_{n+1}, \quad \Psi \in W_0^{1,2}(\Omega), \quad \|\Psi\|_{1,2} \le c\|\pi_{n+1}\|. \tag{26}$$

Since $\pi_{n+1} \in L_0^2(\Omega)$, problem (26) is solvable (see [7, p. 135]). From (26), (25), (24) and (22) we then deduce that

$$\begin{aligned} \|\pi_{n+1}\|^2 &= \langle\pi_{n+1}, \nabla\cdot\Psi\rangle = \mathcal{F}(\Psi) \\ &= b((1+\sigma_n)v_n, v_{n+1}, \Psi) + \mu\langle\nabla v_{n+1}, \nabla\Psi\rangle - \langle F_n, \Psi\rangle \\ &\le c\big(((1+\|\sigma_n\|_\infty)\|v_n\|_{1,2} + 1)\|v_{n+1}\|_{1,2} + \|F_n\|_{-1,2}\big)\|\Psi\|_{1,2}. \end{aligned} \tag{27}$$

Together, (27) and (26) prove (19). Since the equations are linear, the uniqueness of v_{n+1} and π_{n+1} follows directly from (18) and (19) by standard arguments. □

The next two lemmas provide estimates for problem (8) of **Scheme A**.

Lemma 2.2. *Let $v_n \in W^{2,2}(\Omega)$, $\sigma_n \in W^{2,2}(\Omega)$, $g_n \in W^{1,2}(\Omega)$ and $F_n \in L^2(\Omega)$ be given. Then, there exists a unique strong solution $v_{n+1} \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$, $\pi_{n+1} \in W^{1,2}(\Omega) \cap L_0^2(\Omega)$ of problem (8). Moreover, v_{n+1} and π_{n+1} satisfy*

$$\|v_{n+1}\|_{2,2} + \|\pi_{n+1}\|_{1,2} \leq c(B_n\|F_n\| + B_n^2\|g_n\|_{1,2}), \tag{28}$$

where $B_n = (1 + \|\sigma_n\|_{2,2})\|v_n\|_{2,2} + 1$.

Proof. First, Lemma 2.1 ensures that the equations (17) have a unique solution v_{n+1}, π_{n+1} in $W_0^{1,2}(\Omega) \times L_0^2(\Omega)$. Next, employing integration by parts, we note that problem (17) is equivalent to the weak form of the Stokes system

$$\begin{gathered} \nabla \cdot v_{n+1} = \widetilde{G}, \quad -\mu \triangle v_{n+1} + \nabla \pi_{n+1} = \widetilde{F} \quad \text{ in } \Omega, \\ v_{n+1}|_{\partial\Omega} = 0, \quad \textstyle\int_\Omega \pi_{n+1}\, dx = 0. \end{gathered} \tag{29}$$

with right sides

$$\widetilde{G} = g_n, \qquad \widetilde{F} = F_n - (1+\sigma_n)v_n \cdot \nabla v_{n+1} - \tfrac{1}{2}\nabla \cdot \big((1+\sigma_n)v_n\big)v_{n+1}. \tag{30}$$

Under the assumptions of Lemma 2.2 we have $\widetilde{G} \in W^{1,2}(\Omega)$. Further, by the generalized Hölder inequality and Sobolev embedding inequalities it follows that

$$\|\widetilde{F}\| \leq c\big(\|F_n\| + (1+\|\sigma_n\|_\infty)\|v_n\|_{2,2}\|v_{n+1}\|_{1,2} + \|\sigma_n\|_{2,2}\|v_n\|_{1,2}\|v_{n+1}\|_{1,2}\big), \tag{31}$$

which implies $\widetilde{F} \in L^2(\Omega)$. Therefore, the usual regularity results (see [1], Theorem 4) for the Stokes system (29) imply that $v_{n+1} \in W^{2,2}(\Omega)$, $\pi_{n+1} \in W^{1,2}(\Omega)$ and

$$\|v_{n+1}\|_{2,2} + \|\pi_{n+1}\|_{1,2} \leq c(\|\widetilde{F}\| + \|\widetilde{G}\|_{1,2}). \tag{32}$$

Together, the inequalities (32), (31) and (18) prove the assertion (28). This completes the proof of Lemma 2.2. □

Lemma 2.3. *Let $v_n \in W^{2,2}(\Omega)$, $\sigma_n \in W^{2,2}(\Omega)$, $g_n \in W^{2,2}(\Omega)$ and $F_n \in W^{1,2}(\Omega)$ be given. Then, there exists a unique solution $v_{n+1} \in W^{3,2}(\Omega) \cap W_0^{1,2}(\Omega)$, $\pi_{n+1} \in W^{2,2}(\Omega) \cap L_0^2(\Omega)$ of problem (8). Moreover, v_{n+1} and π_{n+1} satisfy*

$$\|v_{n+1}\|_{3,2} + \|\pi_{n+1}\|_{2,2} \leq c_1(B_n^2\|F_n\|_{1,2} + B_n^3\|g_n\|_{2,2}), \tag{33}$$

where $B_n = (1 + \|\sigma_n\|_{2,2})\|v_n\|_{2,2} + 1$.

Proof. Let $v_{n+1} \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$, $\pi_{n+1} \in W^{1,2}(\Omega) \cap L_0^2(\Omega)$ be the unique solution of problem (8) according to Lemma 2.2. Let $\widetilde{G}$ and $\widetilde{F}$ be defined by (30).

By assumption, there holds $\widetilde{G} \in W^{2,2}(\Omega)$. Further, by the Hölder inequality and Sobolev embedding inequalities we have

$$\|\nabla \widetilde{F}\| \leq c\big(\|F_n\|_{1,2} + (1 + \|\sigma_n\|_{2,2})\|v_n\|_{2,2}\|v_{n+1}\|_{2,2}\big). \tag{34}$$

Together with (31), this shows $\widetilde{F} \in W^{1,2}(\Omega)$. Now, by exactly the same argument as in the proof of Lemma 2.2 we conclude that $v_{n+1} \in W^{3,2}(\Omega)$, $\pi_{n+1} \in W^{2,2}(\Omega)$ and

$$\|v_{n+1}\|_{3,2} + \|\pi_{n+1}\|_{2,2} \leq c(\|\widetilde{F}\|_{1,2} + \|\widetilde{G}\|_{2,2}). \tag{35}$$

Combining (35) with (34) and (28) proves the assertion (33). □

We now give similar estimates for problem (9) of **Scheme B.** It needs to be kept in mind, as mentioned at the beginning of this section, that the constant c_1 in the estimate (33) of Lemma 2.4 is different than that of Lemma 2.3.

Lemma 2.4. *Let $v_n \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$, $\sigma_n \in W^{2,2}(\Omega)$, $g_n \in W^{2,2}(\Omega)$ and $F_n \in W^{1,2}(\Omega)$ be given. Then, there exists a unique solution $v_{n+1} \in W^{3,2}(\Omega) \cap W_0^{1,2}(\Omega)$, $\pi_{n+1} \in W^{2,2}(\Omega) \cap L_0^2(\Omega)$ of problem* (9). *Moreover, v_{n+1} and π_{n+1} satisfy the estimates* (28) *and* (33).

Proof. Suppose, for a moment, that we can show $P((1+\sigma_n)v_n) \in W^{2,2}(\Omega)$. Then it will follow that

$$\langle P((1+\sigma_n)v_n) \cdot \nabla \phi, \phi \rangle = \frac{1}{2} \int_\Omega P((1+\sigma_n)v_n) \cdot \nabla(\phi \cdot \phi)\, dx = 0 \tag{36}$$

for all $\phi \in W_0^{1,2}(\Omega)$, since $P((1+\sigma_n)v_n)$ is weakly divergence free by the definition of P. Using (36), the assertions of Lemma 2.4 can be proven in almost exactly the same way as in Lemma 2.3, beginning with analogues of Lemmas 2.1 and 2.2. Thus, it only remains to show that $P((1+\sigma_n)v_n) \in W^{2,2}(\Omega)$ under the assumptions of this lemma.

First, we note that

$$\nabla \cdot ((1+\sigma_n)v_n) \in W^{1,2}(\Omega), \quad v_n \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega), \quad \sigma_n v_n \in W_0^{1,2}(\Omega). \tag{37}$$

By assumption, the first two inclusions are obvious. Further, $\sigma_n v_n$ is the product of a function belonging to $W^{1,2}(\Omega)$, even to $W^{2,2}(\Omega)$, with a function belonging to $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. The product of such functions must belong to $W_0^{1,2}(\Omega)$. Indeed, given any $\phi \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ and any $\psi \in W^{1,2}(\Omega)$, we can approximate ψ in $W^{1,2}(\Omega)$ by functions $\psi_n \in C^1(\overline{\Omega})$. The products $\psi_n\phi$ belong to $W_0^{1,2}(\Omega)$, since they belong to $W^{1,2}(\Omega) \cap C(\overline{\Omega})$ and vanish on $\partial\Omega$. Since $\psi_n\phi \to \psi\phi$ in $W^{1,2}(\Omega)$, it follows that $\psi\phi \in W_0^{1,2}(\Omega)$.

Next, we recall that (see, e.g., [7, p. 107])

$$P((1+\sigma_n)v_n) = (1+\sigma_n)v_n - \nabla\psi, \tag{38}$$

where ψ is a solution of the Neumann problem

$$\triangle\psi = \nabla\cdot((1+\sigma_n)v_n) \quad \text{in } \Omega, \qquad \frac{\partial\psi}{\partial\nu} = (1+\sigma_n)v_n\cdot\nu = 0 \quad \text{on } \partial\Omega. \tag{39}$$

Since $\int_\Omega \nabla\cdot((1+\sigma_n)v_n)\,dx = \int_{\partial\Omega}(1+\sigma_n)v_n\cdot\nu\,dx = 0$, well-known results imply that problem (39) has a unique solution $\psi \in W^{3,2}(\Omega)\cap L_0^2(\Omega)$ satisfying

$$\|\psi\|_{2+i,2} \le c\|\nabla\cdot((1+\sigma_n)v_n)\|_{i,2} \tag{40}$$

for $i \in \{0,1\}$. Together, (38) and (40) imply that $P((1+\sigma_n)v_n) \in W^{2,2}(\Omega)$ and

$$\begin{aligned} \|P((1+\sigma_n)v_n)\|_{1+i,2} &\le c(1+\|\sigma_n\|_{2,2})\|v_n\|_{1+i,2}\,, \\ \|P((1+\sigma_n)v_n)\|_{1+i,2} &\le c(1+\|\sigma_n\|_{1+i,2})\|v_n\|_{2,2}\,, \end{aligned} \tag{41}$$

for $i \in \{0,1\}$. This completes the proof of Lemma 2.4. □

Finally, we give analogous estimates for problem (10) of **Scheme C**.

Lemma 2.5. *Let $v_n \in W^{2,2}(\Omega)$, $\sigma_n \in W^{2,2}(\Omega)$, $g_n \in W^{2,2}(\Omega)$ and $F_n \in W^{1,2}(\Omega)$ be given. Suppose $c_2(1+\|\sigma_n\|_{2,2})\|v_n\|_{1,2} \le \alpha\mu$ for some fixed $\alpha \in (0,1)$, where c_2 is defined in the proof below. Then, there exists a unique solution $v_{n+1} \in W^{3,2}(\Omega)\cap W_0^{1,2}(\Omega)$, $\pi_{n+1} \in W^{2,2}(\Omega)\cap L_0^2(\Omega)$ of problem* (10). *Moreover, v_{n+1} and π_{n+1} satisfy the estimates* (28) *and* (33).

Proof. The existence of a unique weak solution of problem (10) can be proven by standard Galerkin approximation; see, e.g., [8, pp. 14–19] or [11, pp. 650–652]. In order to understand the smallness condition imposed on v_n, σ_n we do this very briefly below. The improved regularity together with the estimates (28) and (33) can then be established in exactly the same way as for the problems (8) and (9).

Analogous to the proof of Lemma 2.1, we look for a weak solution v_{n+1} of (10) of the form $v_{n+1} = w_{n+1} + V_{n+1}$, where $w_{n+1} \in J_1(\Omega)$, and $V_{n+1} \in W_0^{1,2}(\Omega)$ is a solution of problem (20). We then conclude that w_{n+1} must satisfy

$$\mathcal{B}(w_{n+1},\phi) + \mu\langle\nabla w_{n+1},\nabla\phi\rangle = \mathcal{G}(\phi), \tag{42}$$

for all $\phi \in J_1(\Omega)$, where

$$\mathcal{B}(w_{n+1},\phi) = \langle(1+\sigma_n)v_n\cdot\nabla w_{n+1},\phi\rangle + \langle(1+\sigma_n)w_{n+1}\cdot\nabla v_n,\phi\rangle\,,$$

$$\mathcal{G}(\phi) = \langle F_n,\phi\rangle - \mathcal{B}(V_{n+1},\phi) - \mu\langle\nabla V_{n+1},\nabla\phi\rangle\,.$$

In order to show the existence of a solution w_{n+1} of (42), we denote by $\{\psi_k\}$ a denumerable set of solenoidal functions of $\mathcal{D}(\Omega)$ whose linear hull is dense in $J_1(\Omega)$. We normalize them so $\langle\psi_j,\psi_k\rangle = \delta_{jk}$. For each $m = 1,2,\ldots$ we then look for an approximate solution $w_{n+1}^m = \sum_{k=1}^m \xi_{km}\psi_k$ that satisfies, for $k = 1,\ldots,m$, the equations

$$\mathcal{B}(w_{n+1}^m,\psi_k) + \mu\langle\nabla w_{n+1}^m,\nabla\psi_k\rangle = \mathcal{G}(\psi_k). \tag{43}$$

Relation (43) represents a system of linear equations in the m unknowns ξ_{km}, $k = 1, \dots, m$. Multiplication of (43) by ξ_{km} and summation gives

$$\mu\|\nabla w_{n+1}^m\|^2 = \mathcal{G}(w_{n+1}^m) - \mathcal{B}(w_{n+1}^m, w_{n+1}^m). \tag{44}$$

By Hölder's inequality, Sobolev embedding inequalities and the relation (20) we find

$$\begin{aligned} |\mathcal{B}(w_{n+1}^m, w_{n+1}^m)| &\le c_2(1 + \|\sigma_n\|_{2,2})\|v_n\|_{1,2}\|\nabla w_{n+1}^m\|^2, \\ |\mathcal{G}(w_{n+1}^m)| &\le c\big(\|F_n\|_{-1,2} + (1 + \|\sigma_n\|_{2,2})\|v_n\|_{1,2}\|g_n\| + \mu\|g_n\|\big)\|\nabla w_{n+1}^m\|. \end{aligned} \tag{45}$$

Since $\mu - c_2(1 + \|\sigma_n\|_{2,2})\|v_n\|_{1,2} \ge (1 - \alpha)\mu > 0$ by assumption, we conclude (see, e.g., [7, VIII Lemma 3.2]) from (44) and (45) that problem (43) admits a solution for all $m \in \mathbb{N}$. Moreover, the sequence $\{w_{n+1}^m\}$ is uniformly bounded in $W_0^{1,2}(\Omega)$, and there exists a subsequence, again denoted by $\{w_{n+1}^m\}$, and a field $w_{n+1} \in W_0^{1,2}(\Omega)$ such that w_{n+1}^m converges weakly in $W_0^{1,2}(\Omega)$ to w_{n+1}. By compactness we choose this subsequence to be one for which w_{n+1}^m converges strongly in $L^2(\Omega)$ to w_{n+1}. Therefore, passing to the limit $m \to \infty$ in (43) and recalling the density of $\{\psi_k\}$ in $J_1(\Omega)$ we may conclude that w_{n+1} satisfies (42) for all $\phi \in J_1(\Omega)$. The existence of $\pi_{n+1} \in L_0^2(\Omega)$ such that v_{n+1}, π_{n+1} is a weak solution of problem (10) then follows by the same arguments as in the proof of Lemma 2.1. □

Next we give estimates for the effective pressure H_{n+1} on the right of (5), first for **Schemes A and B**.

Lemma 2.6. *Suppose $v_n \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$, $\sigma_n \in W^{2,2}(\Omega)$, $g_n \in W^{2,2}(\Omega)$ and $F_n \in W^{1,2}(\Omega)$ are given. Let $v_{n+1} \in W^{3,2}(\Omega)$, $\pi_{n+1} \in W^{2,2}(\Omega)$ be the unique solution of problem* (8) *obtained in Lemma 2.3, or of problem* (9) *obtained in Lemma 2.4. Then, H_{n+1} defined by* (5) *belongs to $W^{2,2}(\Omega)$ and satisfies*

$$\begin{aligned} \|H_{n+1}\|_{1,2} &\le c\big(\|F_n\| + \big((1 + \|\sigma_n\|_{2,2})\|v_n\|_{1,2} + 1\big)\|v_{n+1}\|_{2,2}\big), \\ \|H_{n+1}\|_{2,2} &\le c_3\big(\|F_n\|_{1,2} + \big((1 + \|\sigma_n\|_{2,2})\|v_n\|_{2,2} + 1\big)\|v_{n+1}\|_{3,2}\big), \\ \|\triangle H_{n+1}\| &\le c_4\big(\|F_n\|_{1,2} + (1 + \|\sigma_n\|_{2,2})\|v_n\|_{2,2}\|v_{n+1}\|_{2,2}\big), \\ \|\triangle H_{n+1}\|_{-1,2} &\le c\big(\|F_n\| + (1 + \|\sigma_n\|_{2,2})\|v_n\|_{2,2}\|v_{n+1}\|_{1,2}\big). \end{aligned} \tag{46}$$

Proof. We give, first, a detailed proof for problem (8) in Scheme A. From inequality (19) and equation (5) we have

$$\|H_{n+1}\| \le c\big(\|F_n\|_{-1,2} + \big((1 + \|\sigma\|_\infty)\|v_n\|_{1,2} + 1\big)\|v_{n+1}\|_{1,2}\big). \tag{47}$$

Since $-\triangle v = \nabla \times \nabla \times v - \nabla\nabla \cdot v$, the second equation of (8) together with (5) implies that

$$(1 + \sigma_n)v_n \cdot \nabla v_{n+1} + \tfrac{1}{2}\nabla \cdot \big((1 + \sigma_n)v_n\big)v_{n+1} + \mu\nabla \times \nabla \times v_{n+1} + \nabla H_{n+1} = F_n. \tag{48}$$

Therefore, we deduce by Hölder's inequality and Sobolev embedding inequalities,

$$\|\nabla H_{n+1}\| \le c\big(\|F_n\| + \|v_{n+1}\|_{2,2} + (1+\|\sigma_n\|_{2,2})\|v_n\|_{1,2}\|v_{n+1}\|_{2,2}\big) \tag{49}$$

as well as

$$\|\nabla^2 H_{n+1}\| \le c\big(\|F_n\|_{1,2} + \|v_{n+1}\|_{3,2} + (1+\|\sigma_n\|_{2,2})\|v_n\|_{2,2}\|v_{n+1}\|_{2,2}\big). \tag{50}$$

Thus, in virtue of (47) to (50) we have proven the first two inequalities of (46). Since $\nabla \cdot \nabla \times w = 0$ for any function $w \in W^{2,2}(\Omega)$, the identity (48) yields

$$\|\triangle H_{n+1}\| \le \|F_n\|_{1,2} + c(1+\|\sigma_n\|_{2,2})\|v_n\|_{2,2}\|v_{n+1}\|_{2,2},$$

which proves the third estimate of (46). Finally, using the definition of $\|\cdot\|_{-1,2}$, we obtain

$$\|\triangle H_{n+1}\|_{-1,2} \le c\big(\|F_n\| + (1+\|\sigma_n\|_{2,2})\|v_n\|_{2,2}\|v_{n+1}\|_{1,2}\big).$$

This completes the proof for Scheme A. Recalling (41), the lemma is proved in almost exactly the same way for Scheme B. □

The estimates (46) are proven similarly also for **Scheme C,** except that a smallness condition is required:

Lemma 2.7. *In addition to the assumptions of Lemma 2.6 suppose that* $c_2(1+\|\sigma_n\|_{2,2})\|v_n\|_{1,2} \le \alpha\mu$ *is satisfied with* α *as in Lemma 2.5. Let* $v_{n+1} \in W^{3,2}(\Omega)$, $\pi_{n+1} \in W^{2,2}(\Omega)$ *be the unique solution of problem* (10) *obtained in Lemma 2.5. Then,* H_{n+1} *defined by* (5) *belongs to* $W^{2,2}(\Omega)$ *and satisfies the estimates* (46).

Analyzing the constants in the proof of [13, Theorem 9] carefully, we obtain:

Lemma 2.8. *Consider the transport equation*

$$k\sigma + \gamma\nabla\cdot(\sigma v) = H. \tag{51}$$

Suppose that $v \in W^{3,2}(\Omega)$, $v\cdot\nu|_{\partial\Omega} = 0$ *and* $k > \frac{\gamma}{2}\|\nabla\cdot v\|_\infty + 2\gamma\|Dv\|_\infty$, *where* $Dv = \frac{1}{2}\big(\nabla v + (\nabla v)^T\big)$ *and* $\|Dv\|_\infty = \sup_\Omega |\big(\sum_{i,j}(Dv)^2_{i,j}\big)^{1/2}|$. *Then, for any prescribed right side* $H \in W^{2,2}(\Omega)$, *there exists a unique solution* $\sigma \in W^{2,2}(\Omega)$ *of equation* (51) *and it satisfies the estimates*

$$\|\sigma\|_{1+i,2} \le \mathcal{C}\mathcal{D}\|H\|_{1+i,2}, \quad i \in \{0,1\},$$

$$\|\triangle\sigma\|_{-i,2} \le \mathcal{D}\|\triangle H\|_{-i,2} + \mathcal{C}\mathcal{D}^2\|v\|_{3,2}\|H\|_{2-i,2}, \quad i \in \{0,1\},$$

with $\mathcal{D} = \big(k - \frac{\gamma}{2}\|\nabla\cdot v\|_\infty - 2\gamma\|Dv\|_\infty\big)^{-1}$ *and some constants* $\mathcal{C} = \mathcal{C}(\Omega,\gamma,\|v\|_{3,2},\mathcal{D})$ *that can be bounded above (monotonically) in terms of Sobolev constants for* Ω, γ, $\|v\|_{3,2}$ *and the constant* $\mathcal{D}$.

The reader who checks these constants carefully will find more precisely that $\mathcal{C} = c(1 + \|v\|_{3,2} + \mathcal{D}\|v\|_{3,2})$, where c depends on Sobolev constants for Ω and γ.

However, only the monotonicity of the dependence of $\mathcal{C}$ on $\mathcal{D}$ and $\|v\|_{3,2}$ is needed in what follows.

Applying now Lemma 2.8 to the transport equation (6) gives:

Lemma 2.9. *There exists a positive constant c_5, such that if*

$$\|v_{n+1}\|_{3,2} \le c_5, \tag{52}$$

then for any prescribed right side $H_{n+1} \in W^{2,2}(\Omega)$, there exists a unique solution $\sigma_{n+1} \in W^{2,2}(\Omega)$ of the transport equation (6)*, and it satisfies*

$$\begin{aligned}
\|\sigma_{n+1}\|_{1,2} &\le c\|H_{n+1}\|_{1,2},\\
\|\sigma_{n+1}\|_{2,2} &\le c_6\|H_{n+1}\|_{2,2},\\
\|\triangle\sigma_{n+1}\| &\le c\|\triangle H_{n+1}\| + c\|v_{n+1}\|_{3,2}\|H_{n+1}\|_{2,2},\\
\|\triangle\sigma_{n+1}\|_{-1,2} &\le c\|\triangle H_{n+1}\|_{-1,2} + c\|v_{n+1}\|_{3,2}\|H_{n+1}\|_{1,2}.
\end{aligned} \tag{53}$$

Together, equation (6) *and the third of these inequalities imply that*

$$\|\triangle\nabla\cdot(\sigma_{n+1}v_{n+1})\| \le c_7\|\triangle H_{n+1}\| + c_8\|v_{n+1}\|_{3,2}\|H_{n+1}\|_{2,2}. \tag{54}$$

From the second of the estimates (53) and the transport equation (6), it follows that $\nabla\cdot(\sigma_{n+1}v_{n+1}) \in W^{2,2}(\Omega)$. Thus, as long as the values of $\|v_{n+1}\|_{3,2}$ remain less than c_5, the sequence of functions v_{n+1}, σ_{n+1} will continue to be well defined, and will satisfy the first two of the induction hypotheses (15), and the first inclusion of the third hypothesis in (15).

Further, the induction hypothesis $\nabla\cdot(\sigma_n v_n) \in W_0^{1,2}(\Omega)$ together with the condition $\nabla\cdot v_{n+1} = -\nabla\cdot(\sigma_n v_n)$ in the Oseen like problems (8), (9) and (10), respectively, implies that $\nabla\cdot v_{n+1} \in W_0^{1,2}(\Omega)$. Thus all hypotheses of the following lemma are satisfied. It guarantees that the second inclusion of the third hypothesis in (15) is also preserved under iteration.

Lemma 2.10. *The conditions $v_{n+1} \in W^{3,2}(\Omega) \cap W_0^{1,2}(\Omega)$, $\sigma_{n+1} \in W^{2,2}(\Omega)$ and $\nabla\cdot v_{n+1} \in W_0^{1,2}(\Omega)$ imply that $\nabla\cdot(\sigma_{n+1}v_{n+1}) \in W_0^{1,2}(\Omega)$.*

Proof. This is easily proved writing

$$\nabla\cdot(\sigma_{n+1}v_{n+1}) = \sigma_{n+1}\nabla\cdot v_{n+1} + v_{n+1}\cdot\nabla\sigma_{n+1}$$

and noting that the right side is a sum of terms, each of which is the product of a function belonging to $W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega)$ with a function belonging to $W^{1,2}(\Omega)$. The product of such functions belongs to $W_0^{1,2}(\Omega)$ which was already shown in the proof of Lemma 2.4. □

This completes the induction argument for the preservation of the conditions (15), modulo the hypothesis $\|v_{n+1}\|_{3,2} \le c_5$ of Lemma 2.9, which will be established in the next section along with other estimates. There we will also need:

Lemma 2.11. *The solutions v_{n+1}, σ_{n+1} of Lemmas 2.3, 2.4 or 2.5, respectively, and 2.9 satisfy*

$$\|\nabla \cdot (\sigma_{n+1} v_{n+1})\|_{2,2} \le c_9 \|\triangle \nabla \cdot (\sigma_{n+1} v_{n+1})\|. \tag{55}$$

Proof. This is a standard elliptic regularity estimate for $\nabla \cdot (\sigma_{n+1} v_{n+1})$ considered as the solution φ of a Dirichlet problem $\triangle \varphi = \psi \equiv \triangle \nabla \cdot (\sigma_{n+1} v_{n+1})$, $\varphi|_{\partial\Omega} = 0$. We know that $\psi \in L^2(\Omega)$ by Lemma 2.9 and that $\varphi|_{\partial\Omega} = 0$ by Lemma 2.10. □

3. Bounds for the iterates

The following lemma completes the proof that the iterates of the Schemes A and B are well defined and bounded. The proof is exactly the same for either scheme, except that the constants $c_0, \dots, c_9$ of §2 depend upon the particular scheme under consideration. In the course of proving it we introduce further constants $b_1, \dots, b_{11}$, which depend only on Ω, k, λ, and μ in the normalized problem (2), and on the constants $c_0, \dots, c_9$ of §2.

Lemma 3.1. *If R, L, B are chosen to satisfy*

$$R \le R^* = \min\{1, c_5/(9c_1 b_1 + 27c_1), 1/(2b_4), b_{10}, b_{11}\}, \quad L = 2b_6 R^2, \quad B = R^2, \tag{56}$$

and if $\|f\|_{1,2} \le B$, then the iterates v_n, σ_n of the Schemes A and B defined in §1 satisfy the hypothesis (52) *of Lemma 2.9 and the estimates*

$$\|v_n\|_{3,2} + \|\sigma_n\|_{2,2} \le R, \qquad \|\nabla \cdot (\sigma_n v_n)\|_{2,2} \le L \tag{57}$$

as well as the previously considered induction hypotheses (15)*.*

Proof. We take (57) along with (15) as induction hypotheses. During the course of the proof, we will assume a number of restrictions on the size of L, namely that

$$L \le \min\{1, R, R/(2b_5), 1/(2b_7)\}. \tag{58}$$

At the end of the proof, it will be shown that these restrictions are satisfied by L as defined in (56).

By (3), (16), (56), (57) we have

$$\begin{gathered} \|F_n\|_{1,2} \le (1 + c_0 \|\sigma_n\|_{2,2}) \|f\|_{1,2} \le (1 + c_0 R) \|f\|_{1,2} \le b_1 R^2, \\ \|g_n\|_{2,2} = \|\nabla \cdot (\sigma_n v_n)\|_{2,2} \le L, \end{gathered} \tag{59}$$

where $b_1 = 1 + c_0$. Therefore, the estimate (33), from Lemmas 2.3 and 2.4, yields

$$\|v_{n+1}\|_{3,2} \le 9c_1 \|F_n\|_{1,2} + 27c_1 \|g_n\|_{2,2} \le 9c_1 b_1 R^2 + 27 c_1 L. \tag{60}$$

Using the second of the estimates (46), together with (56), (57), (59) and (60), gives

$$\begin{aligned}\|H_{n+1}\|_{2,2} &\le c_3(\|F_n\|_{1,2} + 3\|v_{n+1}\|_{3,2}) \\ &\le c_3(b_1R^2 + 27c_1b_1R^2 + 81c_1L) = b_2R^2 + b_3L,\end{aligned} \tag{61}$$

where $b_2 = c_3(1 + 27c_1)b_1$ and $b_3 = 81c_1c_3$. The hypothesis of Lemma 2.9, that $\|v_{n+1}\|_{3,2} \le c_5$, is satisfied in virtue of (60) and the restrictions $L \le R$, $R \le 1$, and $R \le c_5/(9c_1b_1 + 27c_1)$. Therefore, from the second of the estimates (53), we get

$$\|\sigma_{n+1}\|_{2,2} \le c_6b_2R^2 + c_6b_3L. \tag{62}$$

Combining (60) and (62), we have

$$\|v_{n+1}\|_{3,2} + \|\sigma_{n+1}\|_{2,2} \le b_4R^2 + b_5L, \tag{63}$$

where $b_4 = 9c_1b_1 + c_6b_2$ and $b_5 = 27c_1 + c_6b_3$. Therefore, v_{n+1}, σ_{n+1} satisfy the first of the induction estimates (57) in virtue of the restrictions $L \le R/(2b_5)$ and $R \le 1/(2b_4)$.

The second of the estimates (57) is proved by using first Lemma 2.11, then (54), then the third of the estimates (46) combined with (57) and (59), along with (61) and (63), and finally the restrictions $R \le 1$, $L \le 1$ and $L \le R$ to obtain

$$\begin{aligned}\|\nabla\cdot(\sigma_{n+1}v_{n+1})\|_{2,2} &\le c_9\|\triangle\nabla\cdot(\sigma_{n+1}v_{n+1})\| \\ &\le c_9c_7\|\triangle H_{n+1}\| + c_9c_8\|v_{n+1}\|_{3,2}\|H_{n+1}\|_{2,2} \\ &\le c_9c_7c_4b_1R^2 + 2\,c_9c_7c_4R(b_4R^2 + b_5L) + c_9c_8(b_4R^2 + b_5L)(b_2R^2 + b_3L) \\ &\le b_6R^2 + b_7L^2\end{aligned}$$

with suitable definitions of b_6 and b_7. In virtue of the restriction $L \le 1/(2b_7)$, we have $b_6R^2 + b_7L^2 \le L$, confirming that v_{n+1}, σ_{n+1} satisfy (57), provided that

$$2b_6R^2 \le L. \tag{64}$$

This last requirement is consistent with our preceding restrictions, since those relating L to R have been linear. Indeed, the restrictions used so far are $R \le R_1 \equiv \min\{1, c_5/(9c_1b_1 + 27c_1), 1/(2b_4)\}$ and $L \le b_8$ and $L \le b_9R$, where $b_8 = \min\{1, 1/(2b_7)\}$ and $b_9 = \min\{1, 1/(2b_5)\}$. The curve $2b_6R^2 = L$, bounding the region (64) in the R, L-plane, intersects $L = b_8$ at $R = b_{10} \equiv \sqrt{b_8/(2b_6)}$, and intersects $L = b_9R$ at $R = b_{11} \equiv b_9/(2b_6)$. These then present two additional restrictions, namely $R \le b_{10}$ and $R \le b_{11}$. If $R \le R^* = \min\{1, c_5/(9c_1b_1 + 27c_1), 1/(2b_4), b_{10}, b_{11}\}$ and $L = 2b_6R^2$, then L will satisfy all of the restrictions that have been put upon it.

Therefore, we have reached the desired conclusion that every iterate, either of Scheme A or of Scheme B, will satisfy (57), provided $\|f\|_{1,2} \le B \equiv R^2$. This completes the proof. □

Remark 3.2. In the case of Scheme C we obtain instead of the first inequality in (59) the estimate

$$\|F_n\|_{1,2} \le (1 + c_0\|\sigma_n\|_{2,2})(\|f\|_{1,2} + c_{10}\|v_n\|_{2,2}^2),$$

which then implies $\|F_n\|_{1,2} \le b_1 R^2$ with $b_1 = 2(1+c_0)c_{10}$ if $\|f\|_{1,2} \le B = c_{10}R^2$ is supposed. The rest of the proof of Lemma 3.1 also holds for Scheme C. In virtue of (57) and the restriction $R \le 1$, the solvability condition $c_2(1 + \|\sigma_n\|_{2,2})\|v_n\|_{1,2} \le \alpha\mu$ of Lemma 2.5 can now be rewritten as $R \le \alpha\mu/(2c_2)$, which then has to be included in the definition of R^*. Hence, the iterates of Scheme C also satisfy the hypotheses (15) and the estimates (57). This ensures that they are well defined and bounded.

4. Convergence of the iterates

It remains to prove the convergence of the iterative schemes. First, we consider **Scheme A**. We denote by v'_{n+1}, π'_{n+1}, g'_{n+1}, F'_{n+1}, H'_{n+1} and σ'_{n+1} the differences $v'_{n+1} \equiv v_{n+1} - v_n$ etc.. For $n \ge 1$, they satisfy the Oseen like problem

$$\begin{gathered} \nabla \cdot v'_{n+1} = g'_n, \\ (1+\sigma_n)v_n \cdot \nabla v'_{n+1} + \tfrac{1}{2}\nabla \cdot ((1+\sigma_n)v_n)v'_{n+1} - \mu\triangle v'_{n+1} = -\nabla\pi'_{n+1} + \mathcal{F}'_n, \\ v'_{n+1}|_{\partial\Omega} = 0, \quad \textstyle\int_\Omega \pi'_{n+1}\,dx = 0 \end{gathered} \tag{65}$$

with right sides g'_n and $\mathcal{F}'_n$ defined by

$$\begin{aligned} g'_n &= -\nabla \cdot (\sigma_n v_n) + \nabla \cdot (\sigma_{n-1}v_{n-1}), \\ \mathcal{F}'_n &= \sigma'_n f - (1+\sigma_n)v'_n \cdot \nabla v_n - \sigma'_n v_{n-1} \cdot \nabla v_n \\ &\quad -\tfrac{1}{2}\nabla \cdot ((1+\sigma_n)v'_n)v_n - \tfrac{1}{2}\nabla \cdot (\sigma'_n v_{n-1})v_n. \end{aligned} \tag{66}$$

Further, they satisfy the linear equation (5) as well as the transport equation

$$k\sigma'_{n+1} + (\lambda + 2\mu)\nabla \cdot (\sigma'_{n+1}v_{n+1}) = \mathcal{H}'_{n+1}, \tag{67}$$

with right side $\mathcal{H}'_{n+1}$ defined by

$$\mathcal{H}'_{n+1} = H'_{n+1} - (\lambda + 2\mu)\nabla \cdot (\sigma_n v'_{n+1}). \tag{68}$$

Hence, $v'_{n+1}, \pi'_{n+1}, H'_{n+1}$ and σ'_{n+1} satisfy the estimates of Lemmas 2.2, 2.6 and 2.9, with $\mathcal{F}'_n$ replacing F_n in Lemmas 2.2 and 2.6 and $\mathcal{H}'_{n+1}$ replacing H'_{n+1} in Lemma 2.9.

For convenience, let us also recall from Lemma 3.1, that if $R \le R^*$, and if

$$\|f\|_{1,2} \le B = R^2, \tag{69}$$

then, for all $n \ge 0$ we have

$$\|v_n\|_{3,2} + \|\sigma_n\|_{2,2} \le R. \tag{70}$$

Lemma 4.1. *If R and f satisfy the hypotheses of Lemma 3.1 for Scheme A, then*

$$\|v'_{n+1}\|_{2,2} + \|\sigma'_{n+1}\|_{1,2} \leq cR\big(\|v'_n\|_{2,2} + \|\sigma'_n\|_{1,2} + \|v'_{n-1}\|_{2,2} + \|\sigma'_{n-1}\|_{1,2}\big), \quad (71)$$

for all $n \geq 1$, where c is a constant that depends only on Ω, k, λ and μ, and where we have set $v'_0 \equiv 0$ and $\sigma'_0 \equiv 0$. Under the additional assumption that $cR < 1/2$, the inequality (71) implies the geometric convergence of the iterates v_n, σ_n of Scheme A in $W^{2,2}(\Omega) \times W^{1,2}(\Omega)$. The limit v, σ belongs to $W^{3,2}(\Omega) \times W^{2,2}(\Omega)$ and satisfies all of the equations and conditions of problem (2).

Proof. From Lemmas 2.2 and 2.9, we have

$$\begin{aligned} \|v'_{n+1}\|_{2,2} &\leq cB_n\|\mathcal{F}'_n\| + cB_n^2\|g'_n\|_{1,2}, \\ \|\sigma'_{n+1}\|_{1,2} &\leq c\|\mathcal{H}'_{n+1}\|_{1,2} \end{aligned} \quad (72)$$

for $n \geq 1$, where $B_n = (1+\|\sigma_n\|_{2,2})\|v_n\|_{2,2}+1$. To prove (71) we need to bound the terms on the right sides of (72) by expressions of the form $cR\big(\|v'_n\|_{2,2} + \|\sigma'_n\|_{1,2} + \|v'_{n-1}\|_{2,2} + \|\sigma'_{n-1}\|_{1,2}\big)$. In the sequel we suppose that $R \leq 1$.

Using (69) and (70), we obtain

$$\begin{aligned} \|\mathcal{F}'_n\| \leq c\big(&\|\sigma'_n\|_{1,2}\|f\|_{1,2} + (1+\|\sigma_n\|_{2,2})\|v_n\|_{2,2}\|v'_n\|_{1,2} \\ &+\|\sigma'_n\|_{1,2}\|v_{n-1}\|_{1,2}\|v_n\|_{2,2} + \|\sigma_n\|_{2,2}\|v_n\|_{1,2}\|v'_n\|_{1,2} \\ &+(1+\|\sigma_n\|_{2,2})\|v_n\|_{2,2}\|v'_n\|_{1,2} + \|\sigma'_n\|_{1,2}\|v_{n-1}\|_{2,2}\|v_n\|_{2,2} \\ &+\|\sigma'_n\|_{1,2}\|v_{n-1}\|_{2,2}\|v_n\|_{1,2}\big) \\ \leq & cR(\|v'_n\|_{1,2} + \|\sigma'_n\|_{1,2}). \end{aligned} \quad (73)$$

In estimating g'_n, we need to treat separately the case $n = 1$. Since $v_0 = 0$, and $v'_1 = v_1$, it follows from (66) and (70) that

$$\|g'_1\|_{1,2} = \|\nabla \cdot (\sigma_1 v_1)\|_{1,2} \leq c\|\sigma_1\|_{2,2}\|v_1\|_{2,2} \leq cR\|v'_1\|_{2,2}, \quad (74)$$

which provides a suitable estimate for g'_1 on the right side of (72).

To estimate g'_n for $n \geq 2$, we take the difference between the transport equation (6) for σ_n and for σ_{n-1} to get

$$g'_n = \frac{k}{\lambda + 2\mu}\sigma'_n - \frac{1}{\lambda + 2\mu}H'_n. \quad (75)$$

Arguing as in the proof of Lemma 2.11, then using (75) and finally the fourth of the estimates (46) along with (73) and (70), we find

$$\begin{aligned} \|g'_n\|_{1,2} &\leq c\|\triangle g'_n\|_{-1,2} \leq c\|\triangle\sigma'_n\|_{-1,2} + c\|\triangle H'_n\|_{-1,2} \\ &\leq c\|\triangle\sigma'_n\|_{-1,2} + c\|\mathcal{F}'_{n-1}\| + c(1+\|\sigma_{n-1}\|_{2,2})\|v_{n-1}\|_{2,2}\|v'_n\|_{1,2} \\ &\leq c\|\triangle\sigma'_n\|_{-1,2} + cR(\|v'_{n-1}\|_{1,2} + \|\sigma'_{n-1}\|_{1,2} + \|v'_n\|_{1,2}). \end{aligned} \quad (76)$$

The term $\|\triangle\sigma'_n\|_{-1,2}$ needs further consideration. From Lemma 2.9 we deduce

$$\|\triangle\sigma'_n\|_{-1,2} \leq c\|\triangle\mathcal{H}'_n\|_{-1,2} + c\|v_n\|_{3,2}\|\mathcal{H}'_n\|_{1,2}. \quad (77)$$

Substituting (68) into (77) yields

$$\begin{aligned}\|\triangle\sigma_n'\|_{-1,2} \le c\|\triangle H_n'\|_{-1,2} + c\|\triangle\nabla\cdot(\sigma_{n-1}v_n')\|_{-1,2}\\ +c\|v_n\|_{3,2}\|H_n'\|_{1,2} + c\|v_n\|_{3,2}\|\nabla\cdot(\sigma_{n-1}v_n')\|_{1,2}.\end{aligned}$$

By Lemma 2.6, the inequality $\|\nabla\cdot\varphi\|_{-1,2} \le \|\varphi\|$ and (70) we find

$$\begin{aligned}\|\triangle\sigma_n'\|_{-1,2} &\le c\big(\|\mathcal{F}_{n-1}'\| + (1+\|\sigma_{n-1}\|_{2,2})\|v_{n-1}\|_{2,2}\|v_n'\|_{1,2}\big)\\ &\quad +c\|\sigma_{n-1}\|_{2,2}\|v_n'\|_{2,2} + c\|v_n\|_{3,2}\big(\|\mathcal{F}_{n-1}'\|\\ &\quad +((1+\|\sigma_{n-1}\|_{2,2})\|v_{n-1}\|_{1,2}+1)\|v_n'\|_{2,2}\big)\\ &\quad +c\|v_n\|_{3,2}\|\sigma_{n-1}\|_{2,2}\|v_n'\|_{2,2}\\ &\le c\|\mathcal{F}_{n-1}'\| + cR\|v_n'\|_{2,2}.\end{aligned} \tag{78}$$

Substituting (78) into (76) and using (73) gives the desired estimate

$$\|g_n'\|_{1,2} \le cR(\|v_{n-1}'\|_{1,2} + \|\sigma_{n-1}'\|_{1,2} + \|v_n'\|_{2,2}). \tag{79}$$

We now obtain the estimate

$$\|v_{n+1}'\|_{2,2} \le cR(\|v_n'\|_{2,2} + \|\sigma_n'\|_{1,2} + \|v_{n-1}'\|_{1,2} + \|\sigma_{n-1}'\|_{1,2}), \tag{80}$$

by substituting (73) and (79) into the first of the inequalities (72) and noting (70) and the restriction $R \le 1$.

It remains to estimate $\mathcal{H}_{n+1}'$ on the right side of (72). Recalling its definition (68), using Lemma 2.6 and finally (73), (70) and (80), we get

$$\begin{aligned}\|\mathcal{H}_{n+1}'\|_{1,2} &\le \|H_{n+1}'\|_{1,2} + c\|\nabla\cdot(\sigma_n v_{n+1}')\|_{1,2}\\ &\le c\big(\|\mathcal{F}_n'\| + ((1+\|\sigma_n\|_{2,2})\|v_n\|_{1,2}+1)\|v_{n+1}'\|_{2,2}\big) + c\|\sigma_n\|_{2,2}\|v_{n+1}'\|_{2,2}\\ &\le c\|v_{n+1}'\|_{2,2} + cR(\|v_n'\|_{1,2} + \|\sigma_n'\|_{1,2})\\ &\le cR(\|v_n'\|_{2,2} + \|\sigma_n'\|_{1,2} + \|v_{n-1}'\|_{1,2} + \|\sigma_{n-1}'\|_{1,2}).\end{aligned} \tag{81}$$

Together, (80), (72) and (81) imply the assertion (71).

The inequality (71), together with the assumption that $cR < 1/2$, implies the geometric convergence of the iterates v_n, σ_n in $W^{2,2}(\Omega) \times W^{1,2}(\Omega)$ to a limit v, σ; see [12], p. 183. Since the iterates are bounded in $W^{3,2}(\Omega) \times W^{2,2}(\Omega)$, their limit must also belong to this space, by a standard argument.

The geometric convergence of $(1+\sigma_n)v_n\cdot\nabla v_{n+1}$ and $\nabla\cdot((1+\sigma_n)v_n)v_{n+1}$ in $L^2(\Omega)$ now follows from

$$\begin{aligned}\|(1+\sigma_n)v_n\cdot\nabla v_{n+1} - (1+\sigma_{n-1})v_{n-1}\cdot\nabla v_n\|&\\ = \|(1+\sigma_n)v_n\cdot\nabla v_{n+1}' + (1+\sigma_n)v_n'\cdot\nabla v_n + \sigma_n' v_{n-1}\cdot\nabla v_n\|&\\ \le c\|v_{n+1}'\|_{1,2} + c\|v_n'\|_{1,2} + c\|\sigma_n'\|_{1,2}&\end{aligned}$$

and

$$\begin{aligned}
&\|\nabla\cdot((1+\sigma_n)v_n)v_{n+1}-\nabla\cdot((1+\sigma_{n-1})v_{n-1})v_n\|\\
&\qquad=\|\nabla\cdot((1+\sigma_n)v_n)v'_{n+1}+\nabla\cdot((1+\sigma_n)v'_n)v_n+\nabla\cdot(\sigma'_n v_{n-1})v_n\|\\
&\qquad\le c\|v'_{n+1}\|_{1,2}+c\|v'_n\|_{1,2}+c\|\sigma'_n\|_{1,2}
\end{aligned}$$

and the convergence of v_n,σ_n in $W^{2,2}(\Omega)\times W^{1,2}(\Omega)$. Further,

$$\|F'_n\|=\|\sigma'_n f\|\le c\|\sigma'_n\|_{1,2}\|f\|_{1,2}$$

implies the geometric convergence of F_n in $L^2(\Omega)$ to some limit F. The convergence of g_n in $W^{1,2}(\Omega)$ to some limit g is a direct consequence of (79) and the convergence of v_n,σ_n. The geometric convergence of H_n in $W^{1,2}(\Omega)$ to some limit H follows from Lemma 2.6 combined with (73) and the convergence of v_n,σ_n. Then, equation (5) yields the convergence of π_n in $W^{1,2}(\Omega)$ to some limit π. Thus passing to the limit in the equations (3), (8), (5) and (6) and setting $p=k(1+\sigma)$, it is easily seen that all of the conditions of problem (2) are satisfied. This completes the proof of Lemma 4.1. □

We now consider **Scheme B**. Instead of (65) and (66) we have

$$\begin{gathered}
\nabla\cdot v'_{n+1}=g'_n,\\
P\big((1+\sigma_n)v_n\big)\cdot\nabla v'_{n+1}-\mu\triangle v'_{n+1}=-\nabla\pi'_{n+1}+\mathcal{F}'_n,\\
v'_{n+1}|_{\partial\Omega}=0,\quad \textstyle\int_\Omega \pi'_{n+1}\,dx=0
\end{gathered}\tag{82}$$

with right sides g'_n and $\mathcal{F}'_n$ defined by

$$\begin{aligned}
g'_n&=-\nabla\cdot(\sigma_n v_n)+\nabla\cdot(\sigma_{n-1}v_{n-1}),\\
\mathcal{F}'_n&=\sigma'_n f-P\big((1+\sigma_n)v'_n\big)\cdot\nabla v_n-P\big(\sigma'_n v_{n-1}\big)\cdot\nabla v_n.
\end{aligned}\tag{83}$$

Further, v'_{n+1}, π'_{n+1}, H'_{n+1} and σ'_{n+1} satisfy the linear equation (5) as well as the transport equation (67) with $\mathcal{H}'_{n+1}$ being defined in (68). Therefore, v'_{n+1}, H'_{n+1} and σ'_{n+1} satisfy the estimates of Lemmas 2.4, 2.6 and 2.9, with $\mathcal{F}'_n$ replacing F_n in Lemmas 2.4 and 2.6 and $\mathcal{H}'_{n+1}$ replacing H'_{n+1} in Lemma 2.9.

Lemma 4.2. *If R and f satisfy the hypotheses of Lemma 3.1 for Scheme B, then*

$$\|v'_{n+1}\|_{2,2}+\|\sigma'_{n+1}\|_{1,2}\le cR\big(\|v'_n\|_{2,2}+\|\sigma'_n\|_{1,2}+\|v'_{n-1}\|_{2,2}+\|\sigma'_{n-1}\|_{1,2}\big),\tag{84}$$

for all $n\ge 1$, where c is a constant that depends only on Ω, k, λ and μ, and where we have set $v'_0\equiv 0$ and $\sigma'_0\equiv 0$. Under the additional assumption that $cR<1/2$, the inequality (84) *implies the geometric convergence of the iterates v_n,σ_n of Scheme B in $W^{2,2}(\Omega)\times W^{1,2}(\Omega)$. The limit v,σ belongs to $W^{3,2}(\Omega)\times W^{2,2}(\Omega)$ and satisfies all of the equations and conditions of problem* (2).

Proof. Recalling (38) to (41), the assertion (84) can be proven in almost exactly the same way as its analogue (71) in Lemma 4.1. Hence, it only remains to verify that $P((1+\sigma_n)v_n)\cdot\nabla v_{n+1}$ converges to $(1+\sigma)v\cdot\nabla v$ in $L^2(\Omega)$. Passing to the limit in the first equation of problem (9) yields that $\nabla\cdot((1+\sigma)v)=0$ in $L^2(\Omega)$. In virtue of (38) and (39) this implies that $P((1+\sigma)v)=(1+\sigma)v$. Therefore, recalling (41) we get

$$\begin{aligned}
\|P((1+\sigma_n)\,v_n)&\cdot\nabla v_{n+1}-(1+\sigma)v\cdot\nabla v\| \\
&\le \|P((1+\sigma_n)v_n)\cdot\nabla(v_{n+1}-v)+P\big((1+\sigma_n)(v_n-v)\big)\cdot\nabla v\| \\
&\quad +\|P((\sigma_n-\sigma)v)\cdot\nabla v\| \\
&\le c(1+\|\sigma_n\|_{2,2})(\|v_n\|_{2,2}\|v_{n+1}-v\|_{2,2}+\|v\|_{2,2}\|v_n-v\|_{2,2}) \\
&\quad +c\|\sigma_n-\sigma\|_{1,2}\|v\|_{2,2}^2 \\
&\le c(\|v_{n+1}-v\|_{2,2}+\|v_n-v\|_{2,2}+\|\sigma_n-\sigma\|_{1,2}).
\end{aligned}$$

In the last inequality we have made use of

$$\|v\|_{3,2}+\|\sigma\|_{2,2}\le R, \tag{85}$$

which is a consequence of (57). The convergence of v_n,σ_n in $W^{2,2}(\Omega)\times W^{1,2}(\Omega)$ thus implies that $P((1+\sigma_n)v_n)\cdot\nabla v_{n+1}$ does in fact converge to $(1+\sigma)v\cdot\nabla v$ in $L^2(\Omega)$. This completes the proof of Lemma 4.2. □

Remark 4.3. The convergence of Scheme C can be proven in almost exactly the same way as the convergence of the Schemes A and B in Lemmas 4.1 and 4.2.

In order to be physically reasonable, the density $\rho=1+\sigma$ must be everywhere positive, as claimed in Theorem 1.2 and needed in the proof of uniqueness in §5. Up to this point, we have not addressed this matter. However, Lemmas 3.1 and 4.1 and Lemmas 3.1 and 4.2, respectively, allow for the setting of further smallness conditions on the constant R, i.e. on the size of the force. Therefore, we can make the additional assumption that R is sufficiently small that

$$\|\sigma\|_{2,2}\le R \quad \text{implies} \quad \sup_\Omega|\sigma|<1. \tag{86}$$

Then the positivity of the density is ensured by the conclusion (57) of Lemma 3.1, which implies (85). This completes the proof of existence claimed in Theorem 1.2.

5. Uniqueness in the ball of existence

The proof of convergence given in Lemmas 4.1 and 4.2, respectively, suggests an analogous argument for the local uniqueness of the solution that was obtained. The resulting uniqueness theorem is independent of the existence theorem, in that it applies to any solutions satisfying the stated hypotheses. The assumption (86) can be dropped by simply assuming that the solutions under consideration have positive densities.

Lemma 5.1. *If R and f satisfy the hypotheses of Lemmas 3.1 and 4.1 (Scheme A) or Lemmas 3.1 and 4.2 (Scheme B) or analogous hypotheses in the case of Scheme C, respectively, then there can be at most one solution of the problem* (2) *satisfying* $\inf_\Omega(1+\sigma) > 0$ *and*

$$\|v\|_{3,2} + \|\sigma\|_{2,2} \le R. \tag{87}$$

Proof. We only give the proof on the basis of Scheme A, since those for Schemes B and C are almost exactly the same. Now, given one solution v, σ such as that obtained in Lemma 4.1 when (86) is assumed, we set

$$\begin{aligned} g &= -\nabla\cdot(\sigma v), \qquad F = (1+\sigma)f, \\ p &= k(1+\sigma), \qquad \pi = p - k - (\lambda+\mu)\nabla\cdot v. \end{aligned} \tag{88}$$

Then, the Oseen like problem

$$\begin{aligned} \nabla\cdot v &= g, \\ w\cdot\nabla v + \tfrac{1}{2}(\nabla\cdot w)v - \mu\triangle v &= -\nabla\pi + F, \\ v|_{\partial\Omega} = 0, &\quad \textstyle\int_\Omega \pi\,dx = 0 \end{aligned} \tag{89}$$

with $w = (1+\sigma)v$ is satisfied. Also, setting

$$H = \pi - \mu\nabla\cdot v, \tag{90}$$

the transport equation

$$k\sigma + (\lambda+2\mu)\nabla\cdot(\sigma v) = H \tag{91}$$

holds. Although $\nabla\cdot v = g$ implies that $\nabla\cdot w = 0$, we retain the term $(1/2)(\nabla\cdot w)v$ on the left side of problem (89) in order to have the same type of equations as those in Lemma 4.1.

If $\overline{v}, \overline{\sigma}$ is a second solution satisfying the hypotheses of the lemma, we define $\overline{g}, \overline{F}, \overline{p}, \overline{\pi}, \overline{H}$ similarly as in (88) and (90), and then define $v', \sigma', g', F', p', \pi', H'$ by setting $v' = \overline{v} - v$ etc.. The differences v', g', F', π' satisfy

$$\begin{aligned} \nabla\cdot v' &= g', \\ \overline{w}\cdot\nabla v' + \tfrac{1}{2}(\nabla\cdot\overline{w})v' - \mu\triangle v' &= -\nabla\pi' + \mathcal{F}', \\ v'|_{\partial\Omega} = 0, &\quad \textstyle\int_\Omega \pi'\,dx = 0, \end{aligned} \tag{92}$$

where $\overline{w} = (1+\overline{\sigma})\overline{v}$ and

$$\begin{aligned} g' &= -\nabla\cdot(\overline{\sigma}\,\overline{v}) + \nabla\cdot(\sigma v), \\ \mathcal{F}' &= \sigma' f - (1+\overline{\sigma})v'\cdot\nabla v - \sigma' v\cdot\nabla v - \tfrac{1}{2}\nabla\cdot((1+\overline{\sigma})v')v - \tfrac{1}{2}\nabla\cdot(\sigma' v)v. \end{aligned} \tag{93}$$

For H' an analogue of equation (90) holds. Also, σ' satisfies the transport equation

$$k\sigma' + (\lambda + 2\mu)\nabla \cdot (\sigma'\overline{v}) = \mathcal{H}', \tag{94}$$

with right side

$$\mathcal{H}' = H' - (\lambda + 2\mu)\nabla \cdot (\sigma v'). \tag{95}$$

Repeating step by step the arguments of the previous section from (72) to (81), we obtain at the end

$$\|v'\|_{2,2} + \|\sigma'\|_{1,2} \leq 2cR(\|v'\|_{2,2} + \|\sigma'\|_{1,2}).$$

with the constant c defined in (71); for further details see [12, p. 185]. If $cR < 1/2$, then $v' = 0$ and $\sigma' = 0$. This completes the proof of Lemma 5.1. □

6. A relaxed transport equation

The main drawback of the Schemes A, B and C presented in §1 is the requirement (52) that the velocity field v_{n+1} has to be small in order to ensure the solvability of the transport equation (6). We will now show that this restriction can be effectively eliminated through use of the modified transport equation (7). Its formulation was suggested by a pseudo-transient interpretation of the time dependent equation of continuity, as a means of eliminating the condition (52). It can also be viewed as a numerical relaxation technique applied to the steady state equation of continuity.

In what follows we will prove the convergence of the iterative **Scheme TA** obtained by replacing equation (6) in Scheme A with the transport equation (7). Analogues results can be established for the analogous **Schemes TB and TC** in almost exactly the same way. As before, the numbered constants $c_{11}, \dots, c_{18}$ and $b_{12}, \dots, b_{22}$ introduced below and the generic constant c depend at most on Ω, k, λ and μ in the normalized problem (2), and on the particular scheme under consideration. For clarity of reasoning, we use numbered constants in the proof of Lemma 6.1. In Lemma 6.3 all the constants are treated as generic.

Lemma 6.1. *If R, L, B and ϵ are chosen to satisfy*

$$R \leq R^* = \min\{1, 1/(2b_{13}), b_{21}, b_{22}\}, \quad L = 2b_{19}R^2, \quad B = R^2, \tag{96}$$

$$\epsilon = \epsilon^* = \frac{\lambda + 2\mu}{c_{11}b_{12}R}, \tag{97}$$

and if $\|f\|_{1,2} \leq B$, then the iterates v_n, σ_n of Scheme TA satisfy the estimates

$$\|v_n\|_{3,2} + \|\sigma_n\|_{2,2} \leq R, \qquad \|\nabla \cdot (\sigma_n v_n)\|_{2,2} \leq L \tag{98}$$

as well as the induction hypotheses (15). *The smallness assumption about f is now solely due to the preservation of the bounds* (98) *under iteration and no longer dependent solvability requirements for the transport equation* (7).

Proof. We prove that for $\epsilon = \epsilon^*$ and any prescribed $H_{n+1} \in W^{2,2}(\Omega)$ the equation (7) admits a unique solution $\sigma_{n+1} \in W^{2,2}(\Omega)$ without imposing a smallness condition on v_{n+1} and, moreover, that the bounds (98) are preserved. As in the case of Scheme A, it then follows that all of the hypotheses (15) are satisfied. We take (98) along with (15) as induction hypotheses. Again, we will assume some restrictions on the size of L, namely that

$$L \leq \min\{1, R, R/(2b_{14}), 1/(2b_{20})\}. \tag{99}$$

At the end of the proof, it will be shown that these restrictions are satisfied by L as defined in (96).

Together, (60), (98) and (99) imply that

$$\|v_{n+1}\|_{3,2} \leq 9c_1 b_1 R^2 + 27c_1 L \leq b_{12} R. \tag{100}$$

From (61) we get

$$\|H_{n+1}\|_{2,2} \leq b_2 R^2 + b_3 L. \tag{101}$$

Now, we apply Lemma 2.8 to equation (7) with $\epsilon = \epsilon^*$. First, by (100) we have

$$\tfrac{\lambda+2\mu}{2}\|\nabla \cdot v_{n+1}\|_\infty + 2(\lambda + 2\mu)\|Dv_{n+1}\|_\infty \leq c_{11}\|v_{n+1}\|_{3,2} \leq c_{11} b_{12} R$$

with a suitable definition of c_{11}. Recalling (97), we deduce that

$$(\lambda + 2\mu + k\epsilon^*) - \epsilon^*\big(\tfrac{\lambda+2\mu}{2}\|\nabla \cdot v_{n+1}\|_\infty + 2(\lambda + 2\mu)\|Dv_{n+1}\|_\infty\big) \geq k\epsilon^* > 0.$$

Therefore, it follows that

$$\Big((\lambda + 2\mu + k\epsilon^*) - \big(\tfrac{\lambda+2\mu}{2}\|\nabla \cdot (\epsilon^* v_{n+1})\|_\infty + 2(\lambda + 2\mu)\|D(\epsilon^* v_{n+1})\|_\infty\big)\Big)^{-1} \leq \frac{1}{k\epsilon^*}.$$

Hence, according to Lemma 2.8, the modified transport equation (7) has a unique solution $\sigma_{n+1} \in W^{2,2}(\Omega)$. Moreover, in virtue of (100), the previous inequality along with (97), and the restriction $R \leq 1$, the solution σ_{n+1} satisfies the estimates

$$\begin{aligned} \|\sigma_{n+1}\|_{1+i,2} &\leq c_{13}c_{14}\|H_{n+1}\|_{1+i,2} + c_{13}c_{15}R\|\sigma_n\|_{1+i,2}, \\ \|\triangle\sigma_{n+1}\|_{-i,2} &\leq c_{14}\|\triangle H_{n+1}\|_{-i,2} + c_{15}R\|\triangle\sigma_n\|_{-i,2} \\ &\quad + c_{13}c_{14}\|v_{n+1}\|_{3,2}(c_{14}\|H_{n+1}\|_{2-i,2} + c_{15}R\|\sigma_n\|_{2-i,2}) \end{aligned} \tag{102}$$

for $i \in \{0, 1\}$, where the constants are defined by

$$c_{12} = \frac{c_{11}b_{12}}{k(\lambda + 2\mu)}, \quad c_{13} = \mathcal{C}(\Omega, \lambda + 2\mu, b_{12}, c_{12}), \quad c_{14} = \frac{1}{k}, \quad c_{15} = \frac{c_{11}b_{12}}{k}$$

with $\mathcal{C}(\cdot,\cdot,\cdot,\cdot)$ being introduced in Lemma 2.8. From (102), (101) and (98) we get

$$\|\sigma_{n+1}\|_{2,2} \leq c_{13}(c_{14}b_2 + c_{15})R^2 + c_{13}c_{14}b_3 L. \tag{103}$$

Together, (100) and (103) yield

$$\|v_{n+1}\|_{3,2} + \|\sigma_{n+1}\|_{2,2} \leq b_{13}R^2 + b_{14}L, \tag{104}$$

where $b_{13} = 9c_1b_1 + c_{13}(c_{14}b_2 + c_{15})$ and $b_{14} = 27c_1 + c_{13}c_{14}b_3$. Therefore, v_{n+1}, σ_{n+1} satisfy the first of the induction estimates (98) in virtue of the restrictions $L \le R/(2b_{14})$ and $R \le 1/(2b_{13})$.

To prove the second of the estimates (98) we first use Lemma 2.11 and then equation (7) with $\epsilon = \epsilon^*$ along with (97) and the restriction $R \le 1$ to obtain

$$\begin{aligned} \|\nabla \cdot (\sigma_{n+1}v_{n+1})\|_{2,2} &\le c_9\epsilon^{*-1}\|\triangle\nabla \cdot (\sigma_{n+1}\epsilon^* v_{n+1})\| \\ &\le \tfrac{c_9}{\lambda+2\mu}\|\triangle H_{n+1}\| + c_9\epsilon^{*-1}\|\triangle\sigma_n\| + \tfrac{c_9}{\lambda+2\mu}(\lambda + 2\mu + k\epsilon^*)\epsilon^{*-1}\|\triangle\sigma_{n+1}\| \\ &\le c_{16}\|\triangle H_{n+1}\| + c_{17}R\|\triangle\sigma_n\| + c_{18}\|\triangle\sigma_{n+1}\| \end{aligned} \tag{105}$$

with $c_{16} = c_9/(\lambda + 2\mu)$, $c_{17} = c_9c_{11}b_{12}/(\lambda + 2\mu)$ and $c_{18} = c_{16}(c_{11}b_{12} + k)$. Next, from (105) and the second of the estimates (102) we get

$$\begin{aligned} \|\nabla \cdot (\sigma_{n+1}v_{n+1})\|_{2,2} \le b_{15}\|\triangle H_{n+1}\| + b_{16}R\|\triangle\sigma_n\| \\ +b_{17}\|v_{n+1}\|_{3,2}\|H_{n+1}\|_{2,2} + b_{18}R\|v_{n+1}\|_{3,2}\|\sigma_n\|_{2,2}, \end{aligned} \tag{106}$$

where $b_{15} = c_{16} + c_{14}c_{18}$, $b_{16} = c_{17} + c_{15}c_{18}$, $b_{17} = c_{13}c_{14}^2c_{18}$, $b_{18} = c_{13}c_{14}c_{15}c_{18}$. From the third of the estimates (46), (59), (98) and the restriction $R \le 1$ it follows that

$$\|\triangle H_{n+1}\| \le c_4b_1R^2 + 2c_4R\|v_{n+1}\|_{2,2}. \tag{107}$$

Finally, combining (106) with (107), (104), (101) and (98) and then using the restrictions $L \le R$ and $R \le 1$, we find

$$\|\nabla \cdot (\sigma_{n+1}v_{n+1})\|_{2,2} \le b_{19}R^2 + b_{20}L^2 \tag{108}$$

with suitable definitions of b_{19} and b_{20}. Thus, in virtue of the restriction $L \le 1/(2b_{20})$, we have $b_{19}R^2 + b_{20}L^2 \le L$, confirming that v_{n+1}, σ_{n+1} satisfy (98), provided that $2b_{19}R^2 \le L$.

Now, by exactly the same arguments as in the proof of Lemma 3.1 we conclude that if $R \le R^* = \min\{1, 1/(2b_{13}), b_{21}, b_{22}\}$, where $b_{21} = \sqrt{\min\{1, 1/(2b_{20})\}/(2b_{19})}$ and $b_{22} = \min\{1, 1/(2b_{14})\}/(2b_{19})$, and $L = 2b_{19}R^2$, then L will satisfy all of the restrictions that have been put upon it. Therefore, we have reached the desired conclusion that every iterate of Scheme TA will satisfy (98), provided $\|f\|_{1,2} \le B = R^2$. We explicitly note that in the proof of Lemma 6.1 no smallness condition has been imposed on v_{n+1} or incorporated in R^*, respectively, in order to ensure the solvability of the transport equation (7). Its solvability is automatically ensured by the choice (97) of ϵ. This completes the proof of Lemma 6.1. □

Remark 6.2. (i) From the numerical point of view the choice (97) of ϵ is not satisfactory since ϵ^* depends on various unknown constants $c_i = c_i(\Omega, \lambda, \mu)$, mainly Sobolev embedding constants. If ϵ is chosen as

$$\epsilon = \epsilon_{n+1} = \frac{1}{\frac{1}{2}\|\nabla \cdot v_{n+1}\|_\infty + 2\|Dv_{n+1}\|_\infty}, \tag{109}$$

then the boundedness (98) and well-posedness of the iterates can be established in almost exactly the same way as for $\epsilon = \epsilon^*$. Now the parameter $\epsilon = \epsilon_{n+1}$ no longer depends on unknown constants. However, the convergence proof given in Lemma 6.3 below for $\epsilon = \epsilon^*$ cannot be done in the same way for $\epsilon = \epsilon_{n+1}$. It remains an open problem to analyze the convergence behavior of the Scheme TA with $\epsilon = \epsilon_{n+1}$. From the numerical point of view such an "adaptive" choice of the relaxation parameter ϵ might be more appropriate and improve the convergence behavior of the iteration scheme.

(ii) Lemmas 6.1 and 6.3 still hold for

$$\epsilon = \epsilon^* \in \Big[\frac{1}{N}\frac{\lambda+2\mu}{c_{11}b_{12}R}, \frac{\lambda+2\mu}{c_{11}b_{12}R}\Big]$$

with arbitrary but fixed $N \in \mathbb{N}$. Then, R^* will additionally depend on N. Hence, since $N \in \mathbb{N}$ is arbitrary, we may chose $\epsilon = \epsilon^* \in \big(0, \frac{\lambda+2\mu}{c_{11}b_{12}R}\big]$. However, R^* tends to zero for increasing N. Therefore, the smallness condition imposed on f will become stricter the smaller ϵ^* is chosen. Since the parameter ϵ has been interpreted as a time step size, one would expect that the smallness condition should become weaker with decreasing ϵ. Our analysis might not be optimal in this regard.

The following lemma proves the convergence of the Scheme TA with $\epsilon = \epsilon^*$.

Lemma 6.3. *If R, ϵ and f satisfy the hypotheses of Lemma 6.1, then*

$$\|v'_{n+1}\|_{2,2} + \|\sigma'_{n+1}\|_{1,2} \le cR\big(\|v'_n\|_{2,2} + \|\sigma'_n\|_{1,2} + \|v'_{n-1}\|_{2,2} + \|\sigma'_{n-1}\|_{1,2}\big), \quad (110)$$

for all $n \ge 1$, where c is a constant that depends only on Ω, k, λ, μ and our choice of scheme, and where we have set $v'_0 \equiv 0$ and $\sigma'_0 \equiv 0$. Under the additional assumption that $cR < 1/2$, the inequality (110) *implies the geometric convergence of the iterates v_n, σ_n of Scheme TA in $W^{2,2}(\Omega) \times W^{1,2}(\Omega)$. The limit v, σ belongs to $W^{3,2}(\Omega) \times W^{2,2}(\Omega)$ and satisfies all of the equations and conditions of problem* (2).

Proof. Let $v'_{n+1}, \pi'_{n+1}, g'_n, \mathcal{F}'_n, H'_{n+1}$, $\mathcal{H}'_{n+1}$ and σ'_{n+1} be defined as in §4. Then, they satisfy problem (65), (66) and the transport equation

$$(\lambda + 2\mu + k\epsilon^*)\sigma'_{n+1} + (\lambda + 2\mu)\nabla \cdot (\sigma'_{n+1}\epsilon^* v_{n+1}) = \epsilon^*\mathcal{H}'_{n+1} + (\lambda + 2\mu)\sigma'_n. \quad (111)$$

Hence, $v'_{n+1}, \pi'_{n+1}, H'_{n+1}$ and σ'_{n+1} satisfy the estimates of Lemmas 2.2 and 2.6 and the inequalities (102), with $\mathcal{F}'_n$ replacing F_n in Lemmas 2.2 and 2.6 and $\mathcal{H}'_{n+1}$ replacing H'_{n+1} in (102).

From Lemma 2.2 and the first inequality in (102) we find

$$\begin{aligned} \|v'_{n+1}\|_{2,2} &\le cB_n\|\mathcal{F}'_n\| + cB_n^2\|g'_n\|_{1,2}, \\ \|\sigma'_{n+1}\|_{1,2} &\le c\|\mathcal{H}'_{n+1}\|_{1,2} + cR\|\sigma'_n\|_{1,2} \end{aligned} \quad (112)$$

for $n \ge 1$, where $B_n = (1 + \|\sigma_n\|_{2,2})\|v_n\|_{2,2} + 1$. As in the proof of Lemma 4.1, we now need to bound the terms on the right sides of (112) by expressions of the form $cR\big(\|v'_n\|_{2,2}+\|\sigma'_n\|_{1,2}+\|v'_{n-1}\|_{2,2}+\|\sigma'_{n-1}\|_{1,2}\big)$. First, by (73) we have

$$\|\mathcal{F}'_n\| \le cR\|v'_n\|_{1,2}+cR^2\|\sigma'_n\|_{1,2}. \tag{113}$$

In estimating g'_n, we again treat separately the case $n=1$. In virtue of (74),

$$\|g'_1\|_{1,2} \le cR\|v'_1\|_{2,2}, \tag{114}$$

which provides a suitable estimate for g'_1 on the right side of (112).

To estimate g'_n for $n\ge 2$, we take the difference between the transport equation (7) with $\epsilon=\epsilon^*$ for σ_n and for σ_{n-1} to obtain

$$g'_n = \frac{\lambda+2\mu+k\epsilon^*}{(\lambda+2\mu)\epsilon^*}\sigma'_n - \frac{1}{\lambda+2\mu}H'_n - \frac{1}{\epsilon^*}\sigma'_{n-1}. \tag{115}$$

Arguing as in the proof of Lemma 2.11 and then using (115) along with (97) and the restriction $R\le 1$ it follows that

$$\|g'_n\|_{1,2} \le c\|\triangle g'_n\|_{-1,2} \le c\|\triangle\sigma'_n\|_{-1,2}+c\|\triangle H'_n\|_{-1,2}+cR\|\triangle\sigma'_{n-1}\|_{-1,2}.$$

Applying now the fourth of the estimates (46) along with (113), (98) and the restriction $R\le 1$, and recalling the inequality $\|\nabla\cdot\varphi\|_{-1,2}\le\|\varphi\|$, gives

$$\|g'_n\|_{1,2} \le c\|\triangle\sigma'_n\|_{-1,2}+cR\|v'_{n-1}\|_{1,2}+cR\|\sigma'_{n-1}\|_{1,2}+cR\|v'_n\|_{1,2}. \tag{116}$$

The term $\|\triangle\sigma'_n\|_{-1,2}$ needs further consideration. The second inequality in (102), with $\mathcal{H}'_n$ replacing H'_n, yields

$$\begin{aligned}\|\triangle\sigma'_n\|_{-1,2} \le{}& c\|\triangle H'_n\|_{-1,2}+c\|\triangle\nabla\cdot(\sigma_{n-1}v'_n)\|_{-1,2}+cR\|\triangle\sigma'_{n-1}\|_{-1,2}\\ &+c\|v_n\|_{3,2}\|H'_n\|_{1,2}+c\|v_n\|_{3,2}\|\nabla\cdot(\sigma_{n-1}v'_n)\|_{1,2}+cR\|v_n\|_{3,2}\|\sigma'_{n-1}\|_{1,2}.\end{aligned}$$

Finally, by Lemma 2.6, the first estimate in (98), and the restriction $R\le 1$, we find

$$\begin{aligned}\|\triangle\sigma'_n\|_{-1,2} \le{}& c\big(\|\mathcal{F}'_{n-1}\|+R\|v'_n\|_{1,2}\big)+c\|\sigma_{n-1}\|_{2,2}\|v'_n\|_{2,2}+cR\|\sigma'_{n-1}\|_{1,2}\\ &+c\|v_n\|_{3,2}(\|\mathcal{F}'_{n-1}\|+\|v'_n\|_{2,2})+c\|v_n\|_{3,2}\|\sigma_{n-1}\|_{2,2}\|v'_n\|_{2,2}\\ \le{}& c\|\mathcal{F}'_{n-1}\|+cR\|v'_n\|_{2,2}+cR\|\sigma'_{n-1}\|_{1,2}.\end{aligned} \tag{117}$$

Thus, combining (116), (117), (113) and using the restriction $R\le 1$ yields

$$\|g'_n\|_{1,2} \le cR\|v'_n\|_{2,2}+cR\|v'_{n-1}\|_{1,2}+cR\|\sigma'_{n-1}\|_{1,2}. \tag{118}$$

Together, (112), (113), (114), (118) and the restriction $B_n\le 3$ imply that

$$\|v'_{n+1}\|_{1,2} \le cR(\|v'_n\|_{2,2}+\|\sigma'_n\|_{1,2}+\|v'_{n-1}\|_{1,2}+\|\sigma'_{n-1}\|_{1,2}). \tag{119}$$

It remains to estimate $\mathcal{H}'_{n+1}$ in (112). Arguing as in (81), we have

$$\begin{aligned}\|\mathcal{H}'_{n+1}\|_{1,2} &\le c\|v'_{n+1}\|_{2,2}+cR\|v'_n\|_{2,2}+cR^2\|\sigma'_n\|_{1,2}\\ &\le cR\big(\|v'_n\|_{2,2}+\|\sigma'_n\|_{1,2}+\|v'_{n-1}\|_{1,2}+\|\sigma'_{n-1}\|_{1,2}\big).\end{aligned} \tag{120}$$

Combining the estimates (119), (112), (120) proves the inequality (110). By the same arguments as in the proof of Lemma 4.1 we now obtain the convergence of

v_n, σ_n in $W^{2,2}(\Omega) \times W^{1,2}(\Omega)$ to a solution v, σ of problem (2) with $v \in W^{3,2}(\Omega)$ and $\sigma \in W^{2,2}(\Omega)$. $\square$

As we have already seen in Lemma 5.1, the proof of the local uniqueness of the solution obtained by the iteration procedure can be based on exactly the same arguments and estimates as used in the existence proof. Therefore, there is no need to repeat the uniqueness proof again.

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