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Integral Equations and Operator Theory

On Rate of Convergence for Universality Limits

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Abstract. Given a probability measure μ on the unit circle $\mathbb{T}$, consider the reproducing kernel $k_{\mu,n}(z_1, z_2)$ in the space of polynomials of degree at most $n-1$ with the $L^2(\mu)$–inner product. Let $u, v \in \mathbb{C}$. It is known that under mild assumptions on μ near $\zeta \in \mathbb{T}$, the ratio $k_{\mu,n}(\zeta e^{u/n}, \zeta e^{v/n})/k_{\mu,n}(\zeta, \zeta)$ converges to a universal limit $S(u, v)$ as $n \to \infty$. We give an estimate for the rate of this convergence for measures μ with finite logarithmic integral.

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1. Introduction

Consider a probability measure μ on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : \ |z| = 1\}$ of the complex plane, $\mathbb{C}$. Assume that the support of μ is an infinite subset of $\mathbb{T}$, so that monomials z^k, $k \geqslant 0$, are linearly independent in $L^2(\mu)$. For each integer $n \geqslant 1$, the set of polynomials of degree at most $n-1$,

$$\mathcal{P}_n = \mathrm{span}\{z^k,\ k = 0, \dots, n-1\},$$

can be viewed as the n-dimensional Hilbert space of analytic functions with respect to $L^2(\mu)$-inner product. Denote by $k_{\mu,n}(z_1, z_2)$ the reproducing kernel at a point $z_2 \in \mathbb{C}$ in this space, i.e., $k_{\mu,n}(\cdot, z_2) \in \mathcal{P}_n$ and

$$(p, k_{\mu,n}(\cdot, z_2))_{L^2(\mu)} = p(z_2), \quad p \in \mathcal{P}_n.$$

If $\mu = m$ is the Lebesgue measure on $\mathbb{T}$ normalized by $m(\mathbb{T}) = 1$, the reproducing kernel has the following form:

$$k_{m,n}(z_1, z_2) = \frac{1 - \overline{z_2}^n z_1^n}{1 - \overline{z_2} z_1}.$$

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One might check that if $z_1 = \zeta e^{u/n}$, $z_2 = \zeta e^{v/n}$ for some $\zeta \in \mathbb{T}$ and $u, v \in \mathbb{C}$ (equivalently, z_1, z_2 are at a distance comparable to $1/n$ from ζ), then

$$\frac{k_{m,n}(z_1, z_2)}{k_{m,n}(\zeta, \zeta)} = \frac{1 - \overline{z_2}^n z_1^n}{n(1 - \overline{z_2} z_1)} = \frac{e^{u+\bar{v}} - 1}{u + \bar{v}} + O\left(\frac{1}{n}\right),$$

where the remainder is uniform in (u, v) on compact subsets of $\mathbb{C} \times \mathbb{C}$. Such kind of behaviour of reproducing kernels is universal: under mild assumptions on a measure μ near $\zeta \in \mathbb{T}$, we have

$$\frac{k_{\mu,n}(\zeta e^{u/n}, \zeta e^{v/n})}{k_{\mu,n}(\zeta, \zeta)} \to \frac{e^{u+\bar{v}} - 1}{u + \bar{v}}, \quad n \to +\infty,$$

uniformly in (u, v) on compact subsets of $\mathbb{C} \times \mathbb{C}$. Universality of the limiting behaviour of ratios of reproducing kernels attracted major attention in recent years. Several essentially different approaches were developed. Let us mention some of them. First papers dealt with real analytic weights and used the Riemann-Hilbert method, see, e.g., Deift [6] or Kuijlaars and Vanlessen [9]. Lubinsky [10] found a way of reducing a wide class of universality problems to the study of asymptotic behaviour of $k_{\mu,n}(z, z)$, $z \in \mathbb{T}$. The latter asymptotic behaviour has been previously identified for general measures of Szegő class by Máté et al. [12]. Global Szegő condition has been weakened to the local one by Findley [8]. Another approach, also pioneered by Lubinsky [11], is based on compactness of normal families of entire functions and properties of $\frac{\sin x}{x}$ kernel. An overview of this approach and further results can be found in Simon [15] and Totik [16]. Recently, Eichinger et al. [7] found yet another approach to universality based on spectral theory of canonical Hamiltonian systems. While this approach gives extremely general results (even the local Szegő condition can be omitted), it also involves a compactness argument as an essential element of the proof. Most of mentioned papers deal with measures on subsets of the real line due to motivation in the theory of random matrices. However, even in the simplified setting of measures on the unit circle, estimates for the rate of convergence

$$\frac{k_{\mu,n}(z_1, z_2)}{k_{\mu,n}(\zeta, \zeta)} \to \frac{e^{u+\bar{v}} - 1}{u + \bar{v}} \tag{1.1}$$

are missing in the literature. In fact, the rate of convergence in (1.1) is not known even in the case where μ is an absolutely continuous measure on $\mathbb{T}$ with a smooth non-vanishing weight w. Indeed, compactness arguments, widely used for proving universality, cannot give bounds for the rate of convergence.

As an additional motivation of this work, we mention that Poltoratski [13] recently used universality in the proof of convergence of certain nonlinear Fourier transform (NLFT), and a subsequent development of this area, e.g., bounds for NLFT maximal operators, will require estimates for the convergence of universality limits.

In this paper, we estimate the rate of convergence in (1.1) for probability measures on the unit circle with finite logarithmic integral. For this we use an entropy function of a measure—a powerful instrument that recently found several applications in inverse problems [2,3], scattering theory [5], and orthogonal polynomials [1,4]. To be precise, let μ be a probability measure on $\mathbb{T}$, and let $\mu = w\,dm + \mu_{\mathbf{s}}$ be its Radon-Nikodym decomposition into the absolutely continuous and singular parts. The measure μ is said to belong to the Szegő class $\mathrm{Sz}(\mathbb{T})$ if its logarithmic integral is finite:

$$\int_{\mathbb{T}} \log w\,dm > -\infty.$$

Since $\log x \leqslant x$ for all $x > 0$, the latter condition is equivalent to $\log w \in L^1(m)$. Let $\mathbb{D} = \{z \in \mathbb{C} : \ |z| < 1\}$ denote the open unit disk. For a measure $\mu \in \mathrm{Sz}(\mathbb{T})$, its entropy function is given by

$$\mathcal{K}_\mu(z) = \log\left(\int_{\mathbb{T}} \frac{1-|z|^2}{|1-\bar{\xi}z|^2}\,d\mu(\xi)\right) - \int_{\mathbb{T}} \log w(\xi)\frac{1-|z|^2}{|1-\bar{\xi}z|^2}\,dm(\xi), \quad z \in \mathbb{D}.$$

The function $\mathcal{K}_\mu$ is nonnegative in $\mathbb{D}$ by Jensen's inequality. Moreover, for m-almost all $\zeta \in \mathbb{T}$, we have $\mathcal{K}_\mu(z) \to 0$ as z non-tangentially approaches ζ. This follows from well-known properties of the Poisson kernel: we have

$$\lim_{r\to 1}\int_{\mathbb{T}} \frac{1-r^2}{|1-\bar{\xi}r\zeta|^2}\,d\mu_{\mathbf{s}}(\xi) = 0 \tag{1.2}$$

for m-almost all $\zeta \in \mathbb{T}$, and

$$\lim_{r\to 1}\int_{\mathbb{T}} w(\xi)\frac{1-r^2}{|1-\bar{\xi}r\zeta|^2}\,dm(\xi) \to w(\zeta), \tag{1.3}$$

$$\lim_{r\to 1}\int_{\mathbb{T}} \log w(\xi)\frac{1-r^2}{|1-\bar{\xi}r\zeta|^2}\,dm(\xi) \to \log w(\zeta), \tag{1.4}$$

at each Lebesgue point ζ of functions $w, \log w \in L^1(m)$. In case (1.2)–(1.4) are satisfied, we have $\mathcal{K}_\mu(r\zeta) \to 0$ as $r \to 1$. Let $B(\zeta, r) = \{z \in \mathbb{C} : \ |z-\zeta| \leqslant r\}$. The following theorem is the main result of the paper.

Theorem 1.1. *Let $\mu \in \mathrm{Sz}(\mathbb{T})$, $A \geqslant 1$, $n \geqslant 10A$, $\zeta \in \mathbb{T}$. There exists $\varepsilon_0 > 0$ depending only on A, such that if $z_{1,2} \in B(\zeta, A/n)$ and $\mathcal{K}_\mu(\rho\zeta) \leqslant \varepsilon_0$ for all $\rho \in [1-A/n, 1)$, then*

$$\left|\frac{k_{\mu,n}(z_1,z_2)}{k_{\mu,n}(\zeta,\zeta)} - \frac{1-\overline{z_2}^n z_1^n}{n(1-\overline{z_2}z_1)}\right| \leqslant ce^{4A}\sup_{\rho\in[1-\delta,1)}\sqrt{\mathcal{K}_\mu(\rho\zeta)}, \tag{1.5}$$

where $\delta = \max\limits_{k=1,2}|z_k - \zeta|$, and the constant $c > 0$ is absolute.

Note that $\rho \in [1-A/n, 1)$ in Theorem 1.1 tends to 1 as $n \to \infty$, therefore, the right hand side of (1.5) tends to zero for all ζ satisfying (1.2)–(1.4). This gives a nontrivial bound for the rate of convergence in (1.1) for Lebesgue almost all $\zeta \in \mathbb{T}$. If μ has some regularity in a neighbourhood of $\zeta \in \mathbb{T}$, its entropy function can be explicitly estimated. For functions $f, g > 0$, we use notation $f \lesssim g$ (resp., $f \gtrsim g$) if $f \leqslant cg$ (resp., $f \leqslant cg$) for some constant c, and $f \sim g$ if both relations $f \lesssim g$ and $f \gtrsim g$ are satisfied.

Theorem 1.2. *Let $\mu = w\,dm$ be an absolutely continuous probability measure in $\mathrm{Sz}(\mathbb{T})$ such that w is positive and continuous in a neighbourhood $I \subset \mathbb{T}$*

of $\zeta \in \mathbb{T}$. Assume, moreover, that $|w(\xi) - w(\zeta)| \sim |\xi - \zeta|^s$ for all $\xi \in I$ and some $s > 0$. Then we have

$$\mathcal{K}_\mu(\rho\zeta) \sim \begin{cases} 1-\rho, & s \in (1/2, +\infty), \\ (1-\rho)|\log(1-\rho)|, & s = 1/2, \\ (1-\rho)^{2s}, & s \in (0, 1/2), \end{cases}$$

for $\rho \in (0,1)$ close enough to 1. The constants involved depend on s, the diameter of I, the value $w(\zeta)$, and the constants in the relation $|w(\xi)-w(\zeta)| \sim |\xi - \zeta|^s$.

Let $\lambda \in \mathbb{D}$. For the absolutely continuous probability measure $\mu = \frac{1-|\lambda|^2}{|1-\lambda\xi|^2}\, dm$, we have

$$\mathcal{K}(\mu, z) = \log \operatorname{Re}\left(\frac{1+\lambda z}{1-\lambda z}\right) - \log\left(\frac{1-|\lambda|^2}{|1-\lambda z|^2}\right)$$
$$= \log \frac{1-|\lambda z|^2}{|1-\lambda z|^2} - \log\left(\frac{1-|\lambda|^2}{|1-\lambda z|^2}\right) = \log \frac{1-|\lambda z|^2}{1-|\lambda|^2},$$

due to the fact that integration against the Poisson kernel corresponds to harmonic continuation into the unit disk. We see that $\mathcal{K}(\mu, (1-1/n)\zeta) \sim 1/n$ as $n \to \infty$. Note that this agrees with bounds in Theorem 1.2 (we have $s = 1$ for this measure). By Theorem 1.1, we then have

$$\left|\frac{k_{\mu,n}(z_1, z_2)}{k_{\mu,n}(\zeta,\zeta)} - \frac{1-\overline{z_2}^n z_1^n}{n(1-\overline{z_2} z_1)}\right| \lesssim \frac{1}{\sqrt{n}}$$

for all z_1, z_2 in $B(\zeta, 1/n)$ and large enough $n \geqslant 0$, uniformly in $\zeta \in \mathbb{T}$. As we will see in Sect. 4, in fact

$$\sup_{|z_{1,2}-\zeta| \leqslant 1/n}\left|\frac{k_{\mu,n}(z_1, z_2)}{k_{\mu,n}(\zeta,\zeta)} - \frac{1-\overline{z_2}^n z_1^n}{n(1-\overline{z_2} z_1)}\right| \sim \frac{1}{n}.$$

This shows that the bound in Theorem 1.1 is not sharp for smooth measures. It seems, however, that this bound cannot be improved in the setting of the whole class $\mathrm{Sz}(\mathbb{T})$ of measures with finite logarithmic integral, i.e., there is a measure $\mu \in \mathrm{Sz}(\mathbb{T})$ such that

$$\sup_{|z_{1,2}-1| \leqslant 1/n}\left|\frac{k_{\mu,n}(z_1, z_2)}{k_{\mu,n}(1,1)} - \frac{1-\overline{z_2}^n z_1^n}{n(1-\overline{z_2} z_1)}\right| \gtrsim \sqrt{\mathcal{K}_\mu(1-1/n)},$$

for all $n \geqslant 1$. In Sect. 5, we consider the absolutely continuous measures $w_s\, dm$ such that $w_s(e^{i\theta}) = c_s e^{|\theta|^s}$, $\theta \in [-\pi, \pi]$, where the constant c_s is chosen so that $\int_{\mathbb{T}} w_s\, dm = 1$. By Theorem 1.2, we have

$$\mathcal{K}_{w_s}(1-1/n) \sim n^{-2s}, \quad s \in (0, 1/2).$$

We demonstrate numerically that for $x_n = 1 - n^{-1}$, and fixed $s \in (0, 1/2)$, we have

$$\left|\frac{k_{w_s,n}(x_n, x_n)}{k_{w_s,n}(1,1)} - \frac{1-|x_n|^{2n}}{n(1-|x_n|^2)}\right| \gtrsim n^{-s}.$$

In other words, for each $s \in (0, 1/2)$, we have

$$\sup_{|z_{1,2}-1|\leqslant 1/n} \left| \frac{k_{w_s,n}(z_1, z_2)}{k_{w_s,n}(1, 1)} - \frac{1 - \overline{z_2}^n z_1^n}{n(1 - \overline{z_2} z_1)} \right| \gtrsim \sqrt{\mathcal{K}_{w_s}(1 - 1/n)},$$

and estimate (1.5) in Theorem 1.1 is sharp on this class of examples. It remains an open problem to give a mathematical proof of this fact.

2. Proof of Theorem 1.1

Let μ be a probability measure supported on an infinite subset of $\mathbb{T}$, and let $\{\varphi_n\}_{n\geqslant 0}$ be the family of its orthonormal polynomials obtained by Gram–Schmidt orthogonalization of monomials z^n, $n \geqslant 0$, in $L^2(\mu)$. For a polynomial p of degree n, we set $p^*(z) = z^n \overline{p(1/\bar{z})}$. Note that p is also a polynomial of degree at most n. The polynomials $\{\varphi_n^*\}_{n\geqslant 0}$ are called reflected orthonormal polynomials. We have the following recurrence relation (see formula (1.5.25), page 58, in [14]):

$$\varphi_n = \frac{z\varphi_{n-1} - \overline{a_{n-1}}\varphi_{n-1}^*}{\sqrt{1 - |a_{n-1}|^2}}, \quad n \geqslant 1.$$

Here the recurrence coefficients, a_n, $n \geqslant 0$, belong to $\mathbb{D}$. It is also known (see Theorem 2.2.7, p. 124, in [14]) that the reproducing kernel in the n-dimensional space Hilbert space $(\mathcal{P}_n, (\cdot, \cdot)_{L^2(\mu)})$ at $z_2 \in \mathbb{C}$ is given by

$$k_{\mu,n}(z_1, z_2) = \sum_{k=0}^{n-1} \overline{\varphi_k(z_2)}\varphi_k(z_1) = \frac{\overline{\varphi_n^*(z_2)}\varphi_n^*(z_1) - \overline{\varphi_n(z_2)}\varphi_n(z_1)}{1 - \bar{z}_2 z_1}. \quad (2.1)$$

Note that $k_{\mu,n}(z_1, z_2)$ is indeed an element of $\mathcal{P}_n$, i.e., a polynomial with respect to z_1 of degree at most $n - 1$. It will be convenient to use a different representation of the reproducing kernel. Take $n \geqslant 1$, $a \in \mathbb{D}$, and define

$$\tilde{\varphi}_n = \frac{z\varphi_{n-1} - \bar{a}\varphi_{n-1}^*}{\sqrt{1 - |a|^2}}, \quad \tilde{\varphi}_n^* = \frac{\varphi_{n-1}^* - za\varphi_{n-1}}{\sqrt{1 - |a|^2}}. \quad (2.2)$$

Lemma 2.1. *For all $z_1, z_2 \in \mathbb{C}$, we have*

$$\overline{\tilde{\varphi}_n^*(z_2)}\tilde{\varphi}_n^*(z_1) - \overline{\tilde{\varphi}_n(z_2)}\tilde{\varphi}_n(z_1) = \overline{\varphi_n^*(z_2)}\varphi_n^*(z_1) - \overline{\varphi_n(z_2)}\varphi_n(z_1) \quad (2.3)$$

Proof. The proof is a direct computation. At first, let $z_1 = z_2 = z$. Then the left hand side in (2.3) is equal to

$$|\tilde{\varphi}_n^*(z)|^2 - |\tilde{\varphi}_n(z)|^2 = \frac{|\varphi_{n-1}^*(z) - za\varphi_{n-1}(z)|^2 - |z\varphi_{n-1}(z) - \bar{a}\varphi_{n-1}^*(z)|^2}{1 - |a|^2}$$
$$= |\varphi_{n-1}^*(z)|^2 - |z\varphi_{n-1}(z)|^2,$$

which does not depend on a. Taking a to be the recurrence coefficient a_{n-1}, we see that

$$|\tilde{\varphi}_n^*(z)|^2 - |\tilde{\varphi}_n(z)|^2 = |\varphi_n^*(z)|^2 - |\varphi_n(z)|^2. \quad (2.4)$$

This relation holds for all $z \in \mathbb{C}$. Since functions in (2.3) are analytic in z_1 and anti-analytic in z_2, the lemma follows. □

The following lemma is Corollary 4 in [4].

Lemma 2.2. *For every $\lambda \in \mathbb{D}$ there is $a \in \mathbb{D}$ such that the corresponding polynomial $\tilde{\varphi}_n^*$ in (2.2) defines a probability measure $\nu_{n,\lambda} = |\tilde{\varphi}_n^*|^{-2}\, dm$ on $\mathbb{T}$ such that*

$$\mathcal{K}_{\nu_{n,\lambda}}(\lambda) \leqslant \mathcal{K}_\mu(\lambda). \tag{2.5}$$

In the rest of the paper, we use notation $\tilde{\varphi}_n^*$ for the polynomial from Lemma 2.2, where the value of the parameter $\lambda \in \mathbb{D}$ will be clear from the context.

Lemma 2.3. *Let $\lambda \in \mathbb{D}$, and let $\tilde{\varphi}_n^*$ be the corresponding polynomial from Lemma 2.2. We have*

$$\int_{\mathbb{T}} \left|\frac{\tilde{\varphi}_n^*(\lambda)}{\tilde{\varphi}_n^*(\xi)} - 1\right|^2 \frac{1-|\lambda|^2}{|1-\bar{\xi}\lambda|^2}\, dm(\xi) = e^{\mathcal{K}_{\nu_{n,\lambda}}(\lambda)} - 1. \tag{2.6}$$

Proof. By (2.1) and Lemma 2.1, we have

$$\frac{|\tilde{\varphi}_n^*(z)|^2 - |\tilde{\varphi}_n(z)|^2}{1-|z|^2} = \sum_{k=0}^{n-1} |\varphi_k(z)|^2, \quad z \in \mathbb{C}. \tag{2.7}$$

It follows that the function $|\tilde{\varphi}_n^*|^2 - |\tilde{\varphi}_n|^2$ is positive in $\mathbb{D}$ and is comparable to $1-|z|$ when z approaches $\mathbb{T}$. Therefore, $\tilde{\varphi}_n^*$ has no zeroes in $\mathbb{D}$. In fact, it has no zeroes in $\overline{\mathbb{D}} = \mathbb{D} \cup \mathbb{T}$ (if $\tilde{\varphi}_n^*(z_0) = 0$ at some $z_0 \in \mathbb{T}$, then $1-|z| \lesssim |\tilde{\varphi}_n^*|^2 - |\tilde{\varphi}_n|^2 \lesssim |\tilde{\varphi}_n^*|^2 \lesssim |z-z_0|^2$ near z_0, leading to a contradiction). It follows that the function $z \mapsto \frac{\tilde{\varphi}_n^*(\lambda)}{\tilde{\varphi}_n^*(z)}$ is analytic in a neighbourhood of $\overline{\mathbb{D}}$. Then the Poisson formula

$$u(\lambda) = \int_{\mathbb{T}} u(\xi) \frac{1-|\lambda|^2}{|1-\bar{\xi}\lambda|^2}\, dm(\xi) \tag{2.8}$$

for harmonic functions implies

$$\int_{\mathbb{T}} \left|\frac{\tilde{\varphi}_n^*(\lambda)}{\tilde{\varphi}_n^*(\xi)} - 1\right|^2 \frac{1-|\lambda|^2}{|1-\bar{\xi}\lambda|^2}\, dm = \int_{\mathbb{T}} \left|\frac{\tilde{\varphi}_n^*(\lambda)}{\tilde{\varphi}_n^*(\xi)}\right|^2 \frac{1-|\lambda|^2}{|1-\bar{\xi}\lambda|^2}\, dm - 1, \tag{2.9}$$

after noting that the function $u = -2\,\mathrm{Re}\Big(\frac{\tilde{\varphi}_n^*(\lambda)}{\tilde{\varphi}_n^*}\Big) + 1$ is harmonic in a neighbourhood of $\overline{\mathbb{D}}$, $u(\lambda) = -1$. Observe that

$$\int_{\mathbb{T}} \left|\frac{\tilde{\varphi}_n^*(\lambda)}{\tilde{\varphi}_n^*(\xi)}\right|^2 \frac{1-|\lambda|^2}{|1-\bar{\xi}\lambda|^2}\, dm = |\tilde{\varphi}_n^*(\lambda)|^2 \int_{\mathbb{T}} \frac{1-|\lambda|^2}{|1-\bar{\xi}\lambda|^2}\, d\nu_{n,\lambda} = e^{\mathcal{K}_{\nu_{n,\lambda}}(\lambda)}, \tag{2.10}$$

because $|\tilde{\varphi}_n^*(\lambda)|^2 = \exp\big(\int_{\mathbb{T}} \log|\tilde{\varphi}_n^*(\xi)|^2 \frac{1-|\lambda|^2}{|1-\bar{\xi}\lambda|^2}\, dm\big)$ (we use again formula (2.8), this time – for the harmonic function $u = \log|\tilde{\varphi}_n^*|^2$). The lemma now follows from (2.9) and (2.10). □

Given two points $\xi_\pm \in \mathbb{T}$, $|\xi_+ - \xi_-| < 2$, and a number $r < 1$, we denote by $\Gamma(\xi_\pm, r)$ the path in $\mathbb{D}$ formed by the union of two line segments

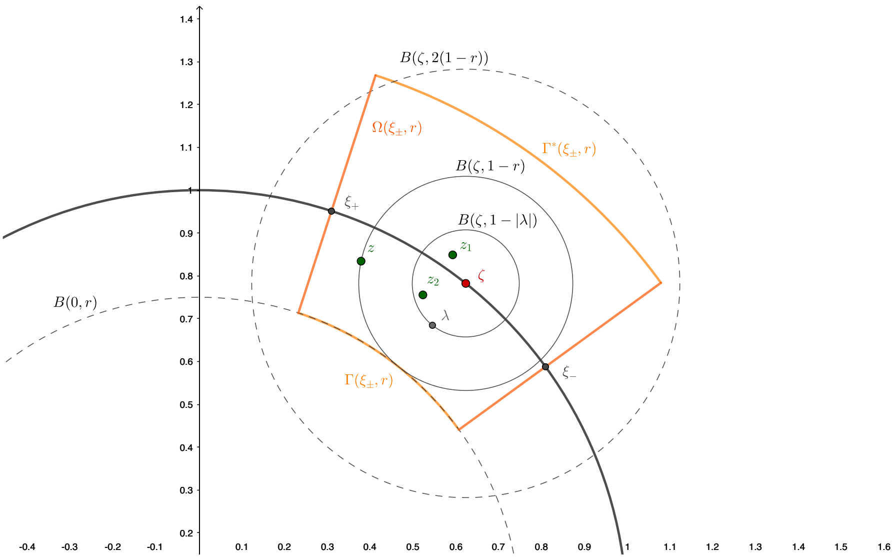

FIGURE 1. Objects that appear in the proof of Theorem 1.1. Here $\xi_- = e^{i\pi/5}$, $\xi_+ = e^{2i\pi/5}$, $r = 3/4$, $\zeta = e^{1.43\pi/5}$, $\lambda = (1+r)\zeta/2$.

$\{\rho\xi_\pm,\ \rho \in [r,1)\}$ and the smaller arc of the circle $|z| = r$ with endpoints $r\xi_-$, $r\xi_+$. We also let $z^* = 1/\bar{z}$ for $z \in \mathbb{C} \setminus \{0\}$, and

$$\Gamma^*(\xi_\pm, r) = \{z \in \mathbb{C} :\ z^* \in \Gamma(\xi_\pm, r)\}.$$

The union $\Gamma(\xi_\pm, r) \cup \{\xi_\pm\} \cup \Gamma^*(\xi_\pm, r)$ is then the boundary of a domain to be denoted by $\Omega(\xi_\pm, r)$. See Fig. 1.

Lemma 2.4. *Suppose that $h \in L^1(m)$, $\eta > 0$, and $\lambda \in \mathbb{D}\backslash B(0, 3/4)$ are such that*

$$\int_{\mathbb{T}} h(\xi)\frac{1-|\lambda|^2}{|1-\bar{\xi}\lambda|^2}\, dm(\xi) \leqslant \eta.$$

Then there are $\xi_\pm \in \mathbb{T}$ such that for every $f \in H^1$ satisfying $|f| \leqslant h$ on $\mathbb{T}$ we have $|f(z)| \lesssim \eta$ on $\Gamma(\xi_\pm, r)$, where r: $1 - r = 2(1 - |\lambda|)$. Moreover, $\xi_\pm$ are such that $\zeta = \lambda/|\lambda|$ belongs to the arc of $\mathbb{T}$ with endpoints $\xi_\pm$, and $1 - r \leqslant |\xi_\pm - \zeta| \leqslant 2(1 - r)$.

Proof. Let $\lambda \in \mathbb{D}\backslash B(0, 3/4)$, $r = 2|\lambda| - 1$, $\zeta = \lambda/|\lambda|$. Consider the arc of the unit circle $G = \mathbb{T} \cap B(\zeta, 2(1 - r))$, and define $m_G = \frac{\chi_G}{m(G)} dm$. We have

$$\int_{\mathbb{T}} h\, dm_G \lesssim \eta.$$

The set G is the metric space with respect to the usual distance in $\mathbb{C}$. The measure m_G has doubling property on this space:

$$m_G(B(\xi, 2\rho)) \leqslant 4m_G(B(\xi, \rho))$$

for every $\xi \in G$, $\rho > 0$. It follows that the weak norm of Hardy-Littlewood maximal operator on $L^1(G, m_G)$ is bounded by a constant that does not depend on G. In other words, for every $t > 0$, $g \in L^1(G, m_G)$ we have

$$m_G\left(\left\{\xi : \sup_{\rho>0} \frac{1}{m_G(B(\xi,\rho))}\int_{B(\xi,\rho)} g\, dm_G > t\right\}\right) \lesssim \frac{\|g\|_{L^1(G,m_G)}}{t},$$

where the constant involved does not depend on G, t, g. Taking $g = h$, $t = \varepsilon^{-1}\eta$ for some $\varepsilon > 0$, we obtain

$$m_G\left(\left\{\xi : \sup_{\rho>0} \frac{1}{m_G(B(\xi,\rho))}\int_{B(\xi,\rho)} h\, dm_G > \varepsilon^{-1}\eta\right\}\right) \lesssim \varepsilon.$$

It follows that

$$m_G\left(\left\{\xi : \sup_{\rho\in[0,1)} \int_G h(u)\frac{1-\rho^2}{|1-\rho\bar{u}\xi|^2}\, dm(u) > \varepsilon^{-1}\eta\right\}\right) \lesssim \varepsilon. \tag{2.11}$$

Indeed, this follows from the fact that for each $\xi \in \mathbb{T}$, $\rho \geqslant 0$, the Poisson kernel $u \mapsto \frac{1-\rho^2}{|1-\rho\bar{u}\xi|^2}$ can be uniformly approximated on $\mathbb{T}$ by positive convex combinations of functions of the form $\frac{\chi_{B(\xi,\delta)}}{m(B(\xi,\delta))}$, $\delta > 0$. Therefore, if $\xi \in G$ is such that for some $\rho \in [0, 1)$ we have

$$\int_G h(u)\frac{1-\rho^2}{|1-\rho\bar{u}\xi|^2}\, dm(u) > \varepsilon^{-1}\eta$$

then

$$\frac{1}{m(B(\xi,\delta))}\int_{B(\xi,\delta)} \chi_G(u)h(u)\, dm(u) > \varepsilon^{-1}\eta$$

for some $\delta > 0$, and so

$$\frac{1}{m_G(B(\xi,\delta))}\int_{B(\xi,\delta)} h(u)\, dm_G(u) > \varepsilon^{-1}\eta,$$

proving (2.11). Let us now take $\varepsilon \in (0, 1)$ so small that the left hand side of (2.11) does not exceed $1/10$. Then there are $\xi_\pm \in G$ such that ζ belongs to the arc of $\mathbb{T}$ with endpoints $\xi_\pm$, we have $|\xi_\pm - \zeta| \geqslant 1 - r$, and, moreover,

$$\sup_{\rho\in[r,1)} \int_G h\frac{1-\rho^2}{|1-\rho\bar{u}\xi_\pm|^2}\, dm \leqslant \varepsilon^{-1}\eta.$$

For $u \in \mathbb{T} \setminus G$, we have

$$\sup_{\rho\in[r,1)} \frac{1-\rho^2}{|1-\rho\bar{u}\xi_\pm|^2}\, dm \lesssim \frac{1-|\lambda|^2}{|1-\bar{u}\lambda|^2}.$$

It follows that

$$|f(\rho\xi_\pm)| \lesssim \int_{\mathbb{T}\setminus G} h\frac{1-|\lambda|^2}{|1-\bar{u}\lambda|^2}\, dm + \int_G h\frac{1-\rho^2}{|1-\rho\bar{\xi}\xi_\pm|^2}\, dm \lesssim (1+\varepsilon^{-1})\eta, \tag{2.12}$$

for every $\rho \in [r, 1)$ with absolute constants. We also note that

$$\sup_{\xi \in G} \frac{1 - r^2}{|1 - r\bar{u}\xi|^2} \lesssim \frac{1 - |\lambda|^2}{|1 - \bar{u}\lambda|^2}, \quad u \in \mathbb{T},$$

therefore,

$$|f(r\xi)| \lesssim \int_{\mathbb{T}} h(u) \frac{1 - |\lambda|^2}{|1 - \bar{u}\lambda|^2}\, dm(u) \lesssim \eta, \tag{2.13}$$

for $\xi \in G$. Collecting (2.12) and (2.13), we see that $|f(z)| \lesssim \eta$ for $z \in \Gamma(\xi_\pm, r)$. □

Define $\tilde{b}_n = \frac{\tilde{\varphi}_n}{\tilde{\varphi}_n^*}$. Formula (2.7) shows that $|\tilde{b}_n| \leqslant 1$ on $\mathbb{D}$. In fact, $\tilde{b}_n$ is a Blaschke product of order n.

Lemma 2.5. *Let $\lambda \in \mathbb{D}$, denote by $\eta = e^{\mathcal{K}_{\nu_{n,\lambda}}(\lambda)} - 1$ the number in the right hand side of (2.6). Set $\alpha = \overline{\tilde{\varphi}_n^*(\lambda)}/\tilde{\varphi}_n^*(\lambda)$. We have*

$$\int_{\mathbb{T}} |\tilde{b}_n(\xi) - \alpha\xi^n|^2 \frac{1 - |\lambda|^2}{|1 - \bar{\xi}\lambda|^2}\, dm \lesssim \eta.$$

Proof. Consider the sets

$$E_l = \left\{ \xi \in \mathbb{T} : \left| \frac{\tilde{\varphi}_n^*(\lambda)}{\tilde{\varphi}_n^*(\xi)} - 1 \right|^2 > 1/4 \right\}, \quad E_s = \mathbb{T} \setminus E_l.$$

On E_l, the difference $|\tilde{b}_n(\xi) - \alpha\xi^n| \leqslant 2$ could be large, but the measure of this set is small. Let us use Chebyshev's inequality and Lemma 2.3 to estimate the corresponding integral:

$$\int_{E_l} |\tilde{b}_n(\xi) - \alpha\xi^n|^2 \frac{1 - |\lambda|^2}{|1 - \bar{\xi}\lambda|^2} \leqslant 4 \int_{E_l} \frac{1 - |\lambda|^2}{|1 - \bar{\xi}\lambda|^2}\, dm \lesssim \eta.$$

On E_s, the difference $|\tilde{b}_n(\xi) - \alpha\xi^n|$ is small. Indeed, for $\xi \in E_s$, we write

$$|\tilde{b}_n - \alpha\xi^n| = \left| \frac{\bar{\xi}^n \tilde{\varphi}_n(\xi)}{\tilde{\varphi}_n^*(\xi)} - \alpha \right| = \left| \frac{\overline{\tilde{\varphi}_n^*(\xi)}}{\overline{\tilde{\varphi}_n^*(\lambda)}} \frac{\tilde{\varphi}_n^*(\lambda)}{\tilde{\varphi}_n^*(\xi)} - 1 \right|.$$

For $z \in \mathbb{C}$ such that $|1 - z| < 1/2$, we have $|z^{-1}| \leqslant 2$, so $|\tilde{\varphi}_n^*(\xi)/\tilde{\varphi}_n^*(\lambda)| \leqslant 2$ on E_s. Then

$$\left| \frac{\tilde{\varphi}_n^*(\xi)}{\tilde{\varphi}_n^*(\lambda)} - 1 \right| = \left| \frac{\tilde{\varphi}_n^*(\xi)}{\tilde{\varphi}_n^*(\lambda)} \right| \left| \frac{\tilde{\varphi}_n^*(\lambda)}{\tilde{\varphi}_n^*(\xi)} - 1 \right| \leqslant 2 \left| \frac{\tilde{\varphi}_n^*(\lambda)}{\tilde{\varphi}_n^*(\xi)} - 1 \right|.$$

Using $ab - 1 = a(b-1) + a - 1$ for $a = \overline{\varphi_n^*(\xi)/\varphi_n^*(\lambda)}$, $b = \varphi_n^*(\lambda)/\varphi_n^*(\xi)$, we see that

$$|\tilde{b}_n - \alpha\xi^n| \lesssim \left| \frac{\tilde{\varphi}_n^*(\lambda)}{\tilde{\varphi}_n^*(\xi)} - 1 \right|$$

on E_s. The claim now follow from Lemma 2.3. □

Lemma 2.6. *Let $\lambda \in \mathbb{D}$, $A \geqslant 1$, $\eta = e^{\mathcal{K}_{\nu_{n,\lambda}}(\lambda)} - 1$. Define r so that $1 - r = 2(1 - |\lambda|)$, and assume that $1/2 \leqslant 1 - A/n \leqslant r < 1$. There exists a number $\eta_0 \in (0, 1)$ depending only on A, such that if $\eta \leqslant \eta_0$, then there are $\xi_\pm \in \mathbb{T}$ such that*

$$|\tilde{b}_n(z) - \alpha z^n| \lesssim e^{4A}\sqrt{\eta}, \tag{2.14}$$

$$\left|\frac{\tilde{\varphi}_n^*(\lambda)}{\tilde{\varphi}_n^*(z)} - 1\right| \lesssim e^{2A}\sqrt{\eta}, \tag{2.15}$$

for all $z \in \Gamma(\xi_\pm, r) \cup \Gamma^*(\xi_\pm, r)$. Moreover, $\xi_\pm$ are such that $\zeta = \lambda/|\lambda|$ belongs to the arc of $\mathbb{T}$ with endpoints $\xi_\pm$, and $1 - r \leqslant |\xi_\pm - \zeta| \leqslant 2(1 - r)$.

Proof. Consider the function $h = |\tilde{b}_n - \alpha\xi^n|^2 + \left|\frac{\tilde{\varphi}_n^*(\lambda)}{\tilde{\varphi}_n^*} - 1\right|^2$ on $\mathbb{T}$. By Lemma 2.5 and Lemma 2.3, we have

$$\int_{\mathbb{T}} h(\xi)\frac{1 - |\lambda|^2}{|1 - \bar{\xi}\lambda|^2}dm(\xi) \lesssim \eta.$$

Then, by Lemma 2.4, applied to the functions $(\tilde{b}_n(z) - \alpha z^n)^2$, $(\frac{\tilde{\varphi}_n^*(\lambda)}{\tilde{\varphi}_n^*(z)} - 1)^2$, there exists a contour $\Gamma(\xi_\pm, r)$ such that

$$|\tilde{b}_n(z) - \alpha z^n|^2 \lesssim \eta, \qquad \left|\frac{\tilde{\varphi}_n^*(\lambda)}{\tilde{\varphi}_n^*(z)} - 1\right|^2 \lesssim \eta,$$

for all $z \in \Gamma(\xi_\pm, r)$. Moreover, $\xi_\pm$ are such that ζ belongs to the arc of $\mathbb{T}$ with endpoints $\xi_\pm$, and $1 - r \leqslant |\xi_\pm - \zeta| \leqslant 2(1 - r)$. Choosing $\eta_0 \in (0, 1)$ sufficiently small, one can guarantee that the left hand sides in these inequalities are smaller than $e^{-4A}/4$ for all $z \in \Gamma(\xi_\pm, r)$. In particular, for all $z \in \Gamma(\xi_\pm, r)$ we have

$$|z^n| \geqslant r^n \geqslant (1 - A/n)^n \geqslant e^{-2A},$$
$$|\tilde{b}_n(z)| \geqslant |z^n| - |\alpha z^n - \tilde{b}_n| \geqslant e^{-2A} - e^{-2A}/2 \geqslant e^{-2A}/2,$$

where we have used the elementary inequality $\log(1 - x) \geqslant -2x$, $x \in [0, 1/2]$. Then the identity

$$\theta(1/\bar{z}) = \overline{1/\theta(z)}, \quad z \in \mathbb{C},$$

for the inner functions $\theta = \tilde{b}_n$, $\theta = \alpha z^n$, gives (2.14) on $\Gamma^*(\xi_\pm, r)$:

$$|\tilde{b}_n(1/\bar{z}) - \alpha(1/\bar{z})^n| = \frac{|\tilde{b}_n(z) - \alpha z^n|}{|\tilde{b}_n(z)z^n|} \lesssim e^{4A}\sqrt{\eta}, \quad z \in \Gamma(\xi_\pm, r).$$

To estimate $\left|\frac{\tilde{\varphi}_n^*(\lambda)}{\tilde{\varphi}_n^*} - 1\right|$ on $\Gamma^*(\xi_\pm, r)$, we use relation

$$\frac{\tilde{\varphi}_n^*(\lambda)}{\tilde{\varphi}_n^*(1/\bar{z})} = \frac{\tilde{\varphi}_n^*(\lambda)}{\overline{\tilde{\varphi}_n(z)/z^n}} = \frac{\overline{z}^n}{\overline{\tilde{b}_n(z)}}\frac{\overline{\tilde{\varphi}_n^*(\lambda)}}{\overline{\tilde{\varphi}_n^*(z)}}\frac{\tilde{\varphi}_n^*(\lambda)}{\overline{\tilde{\varphi}_n^*(\lambda)}} = \overline{\left(\frac{\tilde{\varphi}_n^*(\lambda)}{\tilde{\varphi}_n^*(z)}\frac{\alpha z^n}{\tilde{b}_n(z)}\right)}.$$

Then, formula $\overline{ab} - 1 = \overline{a(b - 1) + a - 1}$ for $a = \frac{\tilde{\varphi}_n^*(\lambda)}{\tilde{\varphi}_n^*(z)}$, $b = \frac{\alpha z^n}{\tilde{b}_n(z)}$ implies (2.15) on $\Gamma^*(\xi_\pm, r)$. □

Lemma 2.7. *Let $\lambda, A, n, r, \zeta, \eta$ be as in Lemma 2.6 and let $z_{1,2} \in B(\zeta, 1-|\lambda|)$. Then*

$$\left|\overline{\tilde{b}_n(z_1)}\tilde{b}_n(z_2) - \overline{z_1}^n z_2^n\right| \lesssim e^{4A}\sqrt{\eta} \cdot |1 - \bar{z}_1 z_2| \cdot n.$$

Proof. By Lemma 2.6, relations (2.14), (2.15) hold on some contour $\Gamma(\xi_\pm, r) \cup \Gamma^*(\xi_\pm, r)$, where $\xi_\pm$ are such that $\zeta = \lambda/|\lambda|$ belongs to the arc of $\mathbb{T}$ with endpoints $\xi_\pm$, and $1 - r \leqslant |\xi_\pm - \zeta| \leqslant 2(1-r)$. By the maximum modulus principle, (2.14) holds in the bounded domain $\Omega(\xi_\pm, r)$ such that $\partial\Omega(\xi_\pm, r) = \Gamma(\xi_\pm, r) \cup \Gamma^*(\xi_\pm, r) \cup \{\xi_\pm\}$. In particular, (2.14) holds for all $z \in B(\zeta, 1-r)$. Now pick two points z_1, z such that

$$|z_1 - \zeta| \leqslant 1 - |\lambda|, \quad |z - \zeta| = 2(1 - |\lambda|) = 1 - r.$$

We have

$$\left|\frac{\overline{\tilde{b}_n(z_1)}\tilde{b}_n(z) - \overline{z_1}^n z^n}{1 - \bar{z}_1 z}\right| \lesssim \frac{\left|(\tilde{b}_n(z_1) - \alpha z_1^n)(\tilde{b}_n(z) - \alpha z^n)\right|}{1-r} + \frac{\left|\overline{z_1}^n(\tilde{b}_n(z) - \alpha z^n)\right|}{1-r} + \frac{\left|(\alpha z_1^n - \tilde{b}_n(z_1))z^n\right|}{1-r}.$$

Note that $|1 - \bar{z}_1 z| \gtrsim 1 - |\lambda| \geqslant \frac{1-r}{2} \gtrsim n^{-1}$, because $A \geqslant 1$. Recall that the maximum principle and (2.14) imply

$$|\tilde{b}_n(z_1) - \alpha z_1^n| \lesssim e^{2A}\sqrt{\eta}, \quad |\tilde{b}_n(z) - \alpha z^n| \lesssim e^{2A}\sqrt{\eta}.$$

Next, we have

$$\max(|z_1|^n, |z|^n) \leqslant (1 + 1 - r)^n \leqslant (1 + A/n)^n \leqslant e^A.$$

Since $\eta \leqslant \eta_0 \leqslant 1$, we can conclude that

$$\left|\frac{\overline{\tilde{b}_n(z_1)}\tilde{b}_n(z) - \overline{z_1}^n z^n}{1 - \bar{z}_1 z}\right| \lesssim e^{4A}\sqrt{\eta} \cdot n.$$

The function $z \mapsto \frac{\overline{\tilde{b}_n(z_1)}\tilde{b}_n - \overline{z_1}^n z^n}{1 - \bar{z}_1 z}$ is analytic, hence the same estimate holds for all $z \in B(\zeta, 1-r)$ by the maximum modulus principle. In particular, it holds for all points z_1, z_2 in $B(\zeta, 1 - |\lambda|)$. The lemma follows. □

Proof of Theorem 1.1. Let $\mu \in \mathrm{Sz}(\mathbb{T})$, $\zeta \in \mathbb{T}$, $A \geqslant 1$. Let us choose $\tilde{N}_0$ such that $A/\tilde{N}_0 < 1/4$ and

$$\sup_{\rho \in [1 - A/\tilde{N}_0, 1)} e^{\mathcal{K}_\mu(\rho\zeta)} - 1 < \eta_0, \tag{2.16}$$

where $\eta_0 \in (0, 1)$ is the number in Lemma 2.6. For $n \geqslant \tilde{N}_0$, consider $z_{1,2} \in B(\zeta, A/n)$ and let $\lambda \in \mathbb{D}$ be defined by

$$\lambda = (1 - \max_{k=1,2} |z_k - \zeta|)\zeta.$$

Define $\eta = e^{\mathcal{K}_{\nu_{n,\lambda}}(\lambda)} - 1$ and observe that $\eta < \eta_0$ by (2.16) and Lemma 2.2. We are in assumptions of Lemma 2.6. Estimate (2.15) and the maximum modulus principle give

$$\frac{\tilde{\varphi}_n^*(z)}{\tilde{\varphi}_n^*(\lambda)} = \frac{1}{1 + R_1(z)}, \quad |R_1(z)| \lesssim e^{2A}\sqrt{\eta},$$

for all $z \in B(\zeta, 1-|\lambda|)$. By Lemma 2.7, for all $z_1, z_2 \in B(\zeta, 1-|\lambda|)$ we have

$$\frac{1-\overline{\tilde{b}_n(z_1)}\tilde{b}_n(z_2)}{1-\overline{z_1}z_2} = \frac{1-\overline{z_1}^n z_2^n}{1-\overline{z_1}z_2} + R_2(z_1,z_2)\cdot n,$$

with $|R_2(z_1,z_2)| \lesssim e^{4A}\sqrt{\eta}$, so

$$k_{\mu,n}(z_1,z_2) = \left(\frac{1-\overline{z_1^n}z_2^n}{n(1-\overline{z_1}z_2)} + R_2(z_1,z_2)\right)\cdot\frac{|\tilde{\varphi}_n^*(\lambda)|^2\cdot n}{(1+R_1(z_1))\overline{(1+R_1(z_2))}}.$$

Taking $z_1 = z_2 = \zeta$, we get

$$k_{\mu,n}(\zeta,\zeta) = \frac{1+R_2(\zeta,\zeta)}{|1+R_1(\zeta)|^2}\cdot|\tilde{\varphi}_n^*(\lambda)|^2\cdot n.$$

It remains to write

$$\frac{k_{\mu,n}(z_1,z_2)}{k_{\mu,n}(\zeta,\zeta)} = \frac{\frac{1-\overline{z_1}^n z_2^n}{n(1-\overline{z_1}z_2)} + R_2(z_1,z_2)}{1+R_2(\zeta,\zeta)}\frac{|1+R_1(\zeta)|^2}{(1+R_1(z_1))\overline{(1+R_1(z_2))}}.$$

Note that

$$\max_{z\in B(\zeta,1-|\lambda|)}|R_1(z)| + \max_{z_{1,2}\in B(\zeta,1-|\lambda|)}|R_2(z_1,z_2)| \leqslant ce^{4A}\sqrt{\eta},$$

for an absolute constant c. We see that one can take $\varepsilon_0 \in (0,\eta_0)$ so that $ce^{4A}\sqrt{\varepsilon_0} \leqslant 1/10$, then the required estimate will hold. $\square$

3. Proof of Theorem 1.2

Proof of Theorem 1.2. Let I be an arc of $\mathbb{T}$ with center ζ such that $|w(\xi) - w(\zeta)| \sim |\xi-\zeta|^s$ for all $\xi \in I$ and some $s > 0$. By the assumption of the theorem, we have $w(\zeta) > 0$. Replacing I by a smaller interval, we can assume that

$$\delta < w(\xi) < 2\delta, \quad |\log w(\xi) - \log w(\zeta)| \sim |\xi-\zeta|^s,$$

for all $\xi \in I$ and some $\delta > 0$. Denote $u = \log w - \log w(\zeta)$. We have

$$\int_I e^{u(\xi)}\frac{1-\rho^2}{|1-\rho\bar{\xi}\zeta|^2}\,dm = \int_I\bigl(1+u(\xi)+O(u^2)\bigr)\frac{1-\rho^2}{|1-\rho\bar{\xi}\zeta|^2}\,dm.$$

We also have

$$\begin{aligned}\mathcal{K}_\mu(\rho\zeta) &= \log\left(\int_{\mathbb{T}} e^{u(\xi)}\frac{1-\rho^2}{|1-\rho\bar{\xi}\zeta|^2}\,dm\right) - \int_{\mathbb{T}} u(\xi)\frac{1-\rho^2}{|1-\rho\bar{\xi}\zeta|^2}\,dm\\ &\leqslant \int_{\mathbb{T}}(e^{u(\xi)}-1)\frac{1-\rho^2}{|1-\rho\bar{\xi}\zeta|^2}\,dm - \int_{\mathbb{T}} u(\xi)\frac{1-\rho^2}{|1-\rho\bar{\xi}\zeta|^2}\,dm\\ &\lesssim (1-\rho)\int_{\mathbb{T}\setminus I}(|e^{u(\xi)}-1| + |u(\xi)|)\,dm + \int_I u^2(\xi)\frac{1-\rho^2}{|1-\rho\bar{\xi}\zeta|^2}\,dm.\end{aligned}$$

By construction, we have

$$\int_I u^2(\xi)\frac{1-\rho^2}{|1-\rho\bar{\xi}\zeta|^2}\,dm \lesssim \int_I |\xi-\zeta|^{2s}\frac{1-\rho^2}{|1-\rho\bar{\xi}\zeta|^2}\,dm. \tag{3.1}$$

Let us set $\varepsilon = 1-\rho$, assume that $\varepsilon < m(I)/2$, and estimate

$$\int_{I\cap B(\zeta,\varepsilon)} |\xi-\zeta|^{2s}\frac{1-\rho^2}{|1-\rho\bar{\xi}\zeta|^2}\,dm \lesssim \frac{1}{\varepsilon}\int_{I\cap B(\zeta,\varepsilon)} |\xi-\zeta|^{2s}\,dm \lesssim \varepsilon^{2s},$$

and

$$\int_{I\setminus B(\zeta,\varepsilon)} |\xi-\zeta|^{2s}\frac{1-\rho^2}{|1-\rho\bar{\xi}\zeta|^2}\,dm \sim (1-\rho)\int_{I\setminus B(\zeta,\varepsilon)} |\xi-\zeta|^{2s-2}\,dm.$$

We have

$$\int_{I\setminus B(\zeta,\varepsilon)} |\xi-\zeta|^{2s-2}\,dm \sim \begin{cases} 1, & s\in(1/2,+\infty),\\ \log\frac{1}{\varepsilon}, & s=1/2,\\ \varepsilon^{2s-1}, & s\in(0,1/2), \end{cases} \tag{3.2}$$

as $\varepsilon\to 0$. The constants in these relations depend on s. We see that

$$\mathcal{K}_\mu(\rho\zeta) \lesssim \begin{cases} 1-\rho, & s\in(1/2,+\infty),\\ (1-\rho)|\log(1-\rho)|, & s=1/2,\\ (1-\rho)^{2s}, & s\in(0,1/2), \end{cases}$$

as $\rho\to 1$. On the other hand, for $\varepsilon = 1-\rho$ we have

$$\mathcal{K}_\mu(\rho\zeta) \gtrsim \int_{\mathbb{T}} |u(\xi)-c|^2\frac{1-\rho^2}{|1-\rho\bar{\xi}\zeta|^2}\,dm \tag{3.3}$$

Let c_1, c_2 be such that $c_1|\xi-\zeta|^s \leqslant u(\xi) \leqslant c_2|\xi-\zeta|^s$ for all $\xi\in I$. Choose $\delta\in(0,1)$ so small that

$$c_1(1-\delta)^s - c_2\delta^s > 0.$$

From (3.3) we see that there are $\xi_1\in B(\zeta,\delta\varepsilon)$, $\xi_2\in B(\zeta,\varepsilon)\setminus B(\zeta,(1-\delta)\varepsilon)$ such that

$$|u(\xi)-c|^2 \lesssim \mathcal{K}_\mu(\rho\zeta), \quad \xi=\xi_{1,2}.$$

Then

$$|u(\xi_1)-u(\xi_2)| \lesssim \sqrt{\mathcal{K}_\mu(\rho\zeta)},$$

and, simultaneously,

$$|u(\xi_1)-u(\xi_2)| \geqslant c_1((1-\delta)\varepsilon)^s - c_2(\delta\varepsilon)^s = \varepsilon^s(c_1(1-\delta)^s - c_2\delta^s) \gtrsim \varepsilon^s.$$

We see that $\mathcal{K}_\mu(\rho\zeta) \gtrsim (1-\rho)^{2s}$ for all ρ close enough to 1. A more accurate estimate is needed for $s\geqslant 1/2$. Let $\delta>0$ be such that

$$c_1(1+\delta)^s/2 - c_2 > 0.$$

Consider $j\in\mathbb{Z}$ satisfying $(1+\delta)^{j+3} < |I|/2$, and let ξ_1, ξ_2 be such that

$$(1+\delta)^j < |\xi_1-\zeta| < (1+\delta)^{j+1}, \quad (1+\delta)^{j+2} < |\xi_2-\zeta| < (1+\delta)^{j+3}.$$

We claim that either $|u(\xi_2)-c| \gtrsim |\xi_2-\zeta|^s$ or $|u(\xi_1)-c| \gtrsim |\xi_1-\zeta|^s$ for all such ξ_1, ξ_2, j. Indeed, if $|u(\xi_2)-c| \leqslant c_1|\xi_2-\zeta|^s/2$, then

$$\begin{aligned} |u(\xi_1)-c| &\geqslant |u(\xi_2)| - |u(\xi_1)| - |u(\xi_2)-c| \\ &\geqslant c_1|\xi_2-\zeta|^s - c_2|\xi_1-\zeta|^s - c_1|\xi_2-\zeta|^s/2 \\ &\gtrsim c_1(1+\delta)^{js+2s}/2 - c_2(1+\delta)^{js+s} \end{aligned}$$

$$\begin{aligned}&\gtrsim (c_1(1+\delta)^s/2 - c_2)(1+\delta)^{js+s}\\&\gtrsim (1+\delta)^{js+s} \gtrsim |\xi_1-\zeta|^s,\end{aligned}$$

proving the claim. We see that for each $j \in \mathbb{Z}$ such that $(1+\delta)^{j+3} < |I|/2$, we have $|u(\xi)-c| \gtrsim |\xi-\zeta|^s$ on a half of the set $\mathbb{T}\cap B(\zeta,(1+\delta)^{j+3})\setminus B(\zeta,(1+\delta)^j)$. It follows that

$$\mathcal{K}_\mu(\rho\zeta) \gtrsim \int_I |\xi-\zeta|^{2s}\frac{1-\rho^2}{|1-\rho\bar{\xi}\zeta|^2}\,dm,$$

and estimate (3.2) shows that

$$\mathcal{K}_\mu(\rho\zeta) \gtrsim \begin{cases}1-\rho, & s\in(1/2,+\infty),\\ (1-\rho)|\log(1-\rho)|, & s=1/2,\\ (1-\rho)^{2s}, & s\in(0,1/2).\end{cases} \qquad \square$$

4. Example: The Poisson Kernel

In the following example, the asymptotic behaviour of ratios of reproducing kernels could be explicitly analysed.

Example. Let $\lambda\in\mathbb{D}\setminus\{0\}$. Consider the probability measure $\mu = \frac{1-|\lambda|^2}{|1-\lambda\xi|^2}\,dm$ on the unit circle $\mathbb{T}$. For $\zeta\in\mathbb{T}$, $u,v\in\mathbb{R}$, $z_1=\zeta e^{u/n}$, $z_2=\zeta e^{v/n}$, we have

$$\frac{k_{\mu,n}(z_1,z_2)}{k_{\mu,n}(\zeta,\zeta)} = \frac{1-\overline{z_2}^n z_1^n}{n(1-\overline{z_2}z_1)} + \delta_n(u,v), \quad n\to\infty,$$

where $\sup_{|u|,\,|v|\leqslant 1}|\delta_n(u,v)|$ is comparable to $1/n$.

Proof. We have (see Sect. 1.6 in [14])

$$\varphi_n(z) = \frac{z^n-\bar{\lambda}z^{n-1}}{\sqrt{1-|\lambda|^2}}, \quad \varphi_n^*(z) = \frac{1-\lambda z}{\sqrt{1-|\lambda|^2}}.$$

It follows that

$$k_{\mu,n}(z,z) = \sum_0^{n-1}|\varphi_k(z)|^2 = \sum_0^{n-1}\frac{|z^k-\bar{\lambda}z^{k-1}|^2}{1-|\lambda|^2} = \frac{|z-\bar{\lambda}|^2}{|z|^2(1-|\lambda|^2)}\frac{1-|z|^{2n}}{1-|z|^2}.$$

Then

$$\frac{k_{\mu,n}(z,z)}{k_{\mu,n}(\zeta,\zeta)} = \frac{|z-\bar{\lambda}|^2}{|z|^2|\zeta-\bar{\lambda}|^2}\frac{1-|z|^{2n}}{n(1-|z|^2)},$$

and we see that

$$\frac{k_{\mu,n}(z,z)}{k_{\mu,n}(\zeta,\zeta)} - \frac{k_{m,n}(z,z)}{k_{m,n}(\zeta,\zeta)} = \frac{1-|z|^{2n}}{n(1-|z|^2)}\frac{|z-\bar{\lambda}|^2-|z|^2|\zeta-\bar{\lambda}|^2}{|z|^2|\zeta-\bar{\lambda}|^2}$$

is comparable to

$$\begin{aligned}|z-\bar{\lambda}|^2-|z|^2|\zeta-\bar{\lambda}|^2 &= (1-|z|^2)(|\lambda|^2+2\,\mathrm{Re}(\lambda(\zeta|z|^2-z)))\\&\gtrsim (1-|z|^2)|\lambda|^2 \gtrsim \frac{1}{n},\end{aligned}$$

if $\lambda \neq 0$, $|\zeta - z| \sim 1 - |z| \sim 1/n$, and n is large. It follows that

$$\sup_{|u|,\,|v|\leqslant 1} |\delta_n(u,v)| \gtrsim 1/n.$$

Now take $z_1 = \zeta e^{u/n}$, $z_2 = \zeta e^{v/n}$, with $|u|$, $|v| \leqslant 1$. We have

$$\begin{aligned} k_{\mu,n}(z_1,z_2) &= \frac{1}{1-|\lambda|^2} \frac{(1-\lambda z_1)\overline{(1-\lambda z_2)} - z_1^n \bar{z}_2^n (1-\bar{\lambda}/z_1)(1-\lambda/\bar{z}_2)}{1 - z_1 \bar{z}_2} \\ &= \frac{(1-\lambda z_1)\overline{(1-\lambda z_2)}}{1-|\lambda|^2} \frac{1 - z_1^n \bar{z}_2^n \theta_\lambda(z_1,z_2)}{1 - z_1\bar{z}_2}, \end{aligned}$$

where

$$\theta_\lambda(z_1,z_2) = \frac{(1-\bar{\lambda}/z_1)(1-\lambda/\bar{z}_2)}{\overline{(1-\lambda z_2)}(1-\lambda z_1)} = 1 + O(1/n).$$

In particular, we have

$$\frac{k_{\mu,n}(z_1,z_2)}{k_{\mu,n}(\zeta,\zeta)} = \frac{(1-\lambda z_1)\overline{(1-\lambda z_2)}}{|1-\lambda\zeta|^2} \frac{1 - z_1^n \bar{z}_2^n \theta_\lambda(z_1,z_2)}{n(1 - z_1\bar{z}_2)}.$$

Here

$$\frac{(1-\lambda z_1)\overline{(1-\lambda z_2)}}{|1-\lambda\zeta|^2} = 1 + O(1/n),$$

and

$$\frac{1 - z_1^n \bar{z}_2^n \theta_\lambda(z_1,z_2)}{n(1 - z_1\bar{z}_2)} = \frac{k_{m,n}(z_1,z_2)}{k_{m,n}(\zeta,\zeta)} + z_1^n \bar{z}_2^n \frac{\theta_\lambda(z_1,z_2) - 1}{n(1 - z_1\bar{z}_2)}.$$

Note that $\theta_\lambda(z_1,z_2) = 1$ in the case $z_1\bar{z}_2 = 1$. Considering z_1, z_2 such that $|1 - z_1\bar{z}_2| \sim 1/n$ and using maximum modulus principle for analytic functions, we see that

$$\left| z_1^n \bar{z}_2^n \frac{\theta_\lambda(z_1,z_2) - 1}{n(1 - z_1\bar{z}_2)} \right| \lesssim \frac{1}{n}$$

for all z_1, z_2 such that $|z_{1,2} - \zeta| \lesssim 1/n$. It follows that $\sup_{|u|,\,|v|\leqslant 1} |\delta_n(u,v)| \lesssim 1/n$.

5. Sharpness of Theorem 1.1

Sharpness of Theorem 1.1 is an open problem. As we have seen in the previous section, one can have

$$\sup_{|z_{1,2}-1|\leqslant 1/n} \left| \frac{k_{\mu,n}(z_1,z_2)}{k_{\mu,n}(1,1)} - \frac{1 - \overline{z_2}^n z_1^n}{n(1 - \overline{z_2} z_1)} \right| = o\left(\sqrt{\mathcal{K}_\mu(1-1/n)} \right),$$

if the measure μ is very regular. Our aim in this section is to present some numerical results for the measures of the form $w_s\, dm$, where $w_s(e^{i\theta}) = c_s e^{|\theta|^s}$, $\theta \in [-\pi,\pi]$, and the constant c_s is chosen so that $\int_{\mathbb{T}} w_s\, dm = 1$. We will see that

$$\sup_{|z_{1,2}-1|\leqslant 1/n} \left| \frac{k_{w_s,n}(z_1,z_2)}{k_{w_s,n}(1,1)} - \frac{1 - \overline{z_2}^n z_1^n}{n(1 - \overline{z_2} z_1)} \right| \gtrsim \sqrt{\mathcal{K}_{w_s}(1-1/n)},$$

in the case $s = 0.1$, $s = 0.2$, and $s = 0.4$ (similar results can be obtained for other values of $s \in (0, 1/2)$). In particular, the upper bound in Theorem 1.1 in these cases coincides with the lower bound up to a multiplicative factor comparable to 1. We use the following MATLAB code to produce our examples (note that the weight w_s and the orthogonal polynomials in the script are not normalized, because the normalization plays no role when we consider ratios of reproducing kernels).

```
clc
clear vars

N=8000;

step = 20;

s = 0.4;

diffold = 0;

MMNTS=zeros(N,1);
D = zeros(1,N/step);
alphaCand = zeros(1,N/step);
CalphaCand = zeros(1,N/step);

for j=0:1:N-1
fun = @(x) cos(j*x).*exp(abs(x).^s);
MMNTS(j+1) = integral(fun,-pi,pi);
end

for n=step:step:N
disp(['n = ', num2str(n)]);
xn = 1-1/n;
RepKer0atxnxn =((xn).^(0:1:n-2))*((xn).^(0:1:n-2))';
RepKer0at11 = n-1;
j=0:1:n-1;
moments = double(MMNTS(1:n));
T = toeplitz(moments);
eo = double(1:n == n)';
coefOP=((invToeplitz(T))*eo)';
coefRefOP = conj(flip(coefOP));
OPat1 = coefOP*ones(n,1);
RefOPat1 = coefRefOP*ones(n,1);
OPatxn = coefOP*((xn).^(0:1:n-1))';
RefOPatxn = coefRefOP*((xn).^(0:1:n-1))';
RepKerNumat1=coefRefOP*conj(RefOPat1)-coefOP*conj(OPat1);
RepKerDenat1 = [1, -1];
[RepKerat1,r1] = deconv(flip(RepKerNumat1),flip(RepKerDenat1));
```

```
RepKerNumatxn=coefRefOP*conj(RefOPatxn)-coefOP*conj(OPatxn);
RepKerDenatxn = [1, -conj(xn)];
[RepKeratxn,rxn]=deconv(flip(RepKerNumatxn), ...
flip(RepKerDenatxn));
RepKerat11 = flip(RepKerat1)*ones(n-1,1);
RepKeratxnxn = flip(RepKeratxn)*((xn).^(0:1:n-2))';

ratio0 = double(RepKer0atxnxn/RepKer0at11);
ratio1 = double(RepKeratxnxn/RepKerat11);
diff = ratio0-ratio1;
D(n/step) = abs(diff);
alphaCand(n/step) = (n/step).*(diffold./diff - 1);
CalphaCand(n/step) = diff.*(n.^alphaCand(n/step));
diffold = diff;
end
```

□

This script produces the array `D` consisting of differences

$$\texttt{D(n/20)} = \left| \frac{k_{w_s,n}(x_n,x_n)}{k_{w_s,n}(1,1)} - \frac{1-|x_n|^{2n}}{n(1-|x_n|^2)} \right|, \quad n = 20, 40, 60, \dots, 8000,$$

for the value $s = 0.4$ of Hölder continuity index. One can plot the values of this array (see below) and observe that it behaves like $C_\alpha n^{-\alpha}$ for certain values $\alpha > 0$, $C_\alpha > 0$. To find these values, we observe that

$$\frac{C_\alpha (n-20)^{-\alpha}}{C_\alpha n^{-\alpha}} = \left(1 - \frac{20}{n}\right)^{-\alpha} = 1 + \frac{20\alpha}{n} + O(n^{-2}),$$

which gives a reasonable recipe to compute the array `alphaCand` of candidates for the power index α:

$$\texttt{alphaCand(n/20)} = \frac{n}{20}\left(\frac{D(n/20-1)}{D(n/20)} - 1\right), \quad n = 20, 40, \dots, 8000.$$

When printed, the last 7 elements of this array (corresponding to values $n = 7880, 7900, \dots, 8000$) look as follows:

$$\dots \quad 0.39361 \quad 0.39362 \quad 0.39363 \quad 0.39363 \quad 0.39364 \quad 0.39365 \quad 0.39365$$

which is pretty close to $s = 0.4$. So it seems plausible that

$$\left| \frac{k_{w_s,n}(x_n,x_n)}{k_{w_s,n}(1,1)} - \frac{1-|x_n|^{2n}}{n(1-|x_n|^2)} \right| \geqslant \frac{C_{0.4}}{n^{0.4}}$$

for all n large enough. We can even suggest a candidate for the constant $C_{0.4}$ from our numerical data. For this we use approximation

$$C_{0.4} \sim C_\alpha = N^{0.39365} D(N/20) = 0.03791..., \quad N = 8000.$$

The plots of the resulting functions

$$f_1(n) = \left| \frac{k_{w_s,n}(x_n,x_n)}{k_{w_s,n}(1,1)} - \frac{1-|x_n|^{2n}}{n(1-|x_n|^2)} \right|, \quad f_2(n) = C_{0.4} n^{-0.4}, \qquad \square$$

are given on Fig. 2. We also consider the cases $s = 0.1$ and $s = 0.2$.

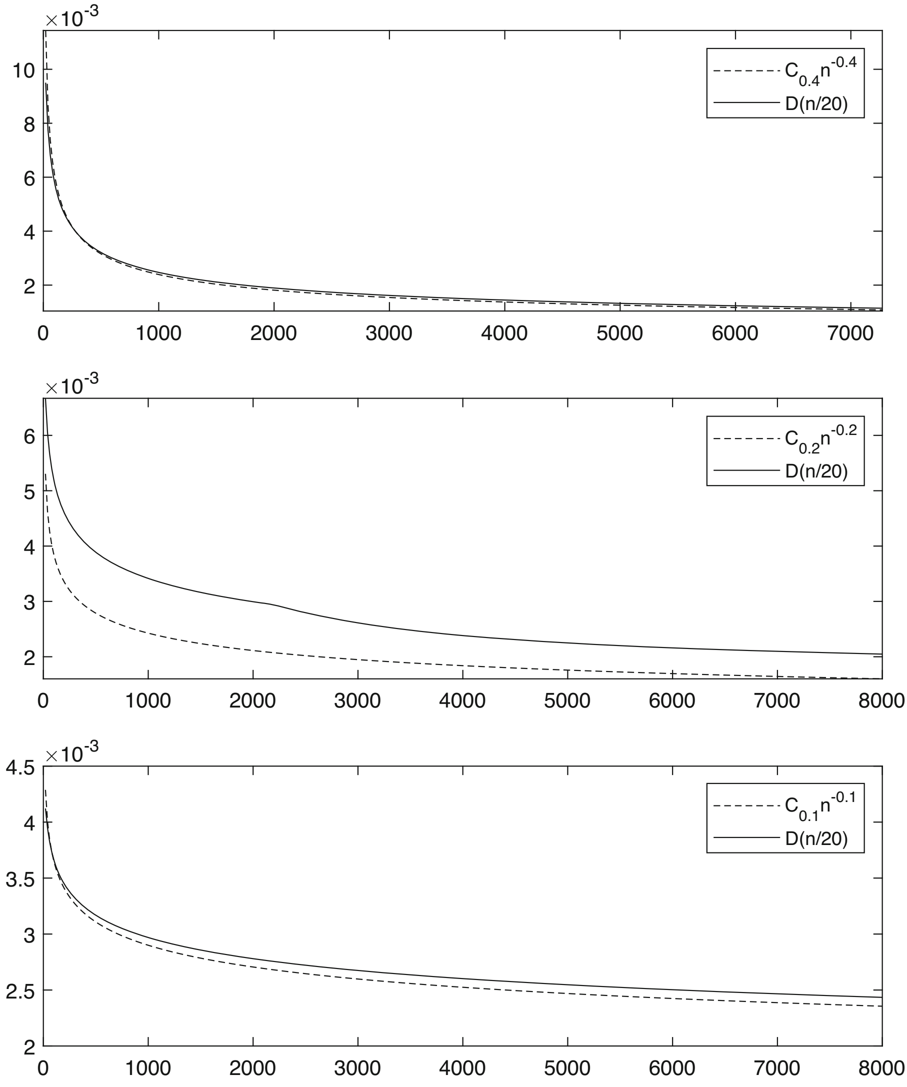

FIGURE 2. Graphs of functions f_1, f_2 for $s = 0.1$, $s = 0.2$, and $s = 0.4$

The graphs demonstrate the fact that $f_1 \geqslant f_2$ for $s \in \{0.1, 0.2, 0.4\}$ and large integers n. Similar numerical results can be obtained for other values $s \in (0, 1/2)$.

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